l-Groups and Bézout Domains

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Hiermit versichere ich an Eides Statt,
die vorliegende Arbeit selbständig
und mit den angegebenen Hilfsmitteln
angefertigt zu haben.

Abstract

We study the relationship between l-groups, Bézout domains, and MV-algebras. Our main motivation and starting point has been the Jaffard-Ohm correspondence between Abelian l-groups and Bézout domains

$$\begin{tikzcd}
& \boxed{\text{Bézout domains with non-zero unit radical}} \arrow[dl, dash] \arrow[dr, dash] & \\
\boxed{\text{Unital } l\text{-groups}} \arrow[rr, dash] & & \boxed{MV\text{-algebras}}
\end{tikzcd}$$

and Mundici's equivalence between Abelian unital l-groups and MV-algebras.

Using valuation theory, we give a positive answer to Conrad and Dauns' problem [41] whether a lattice-ordered skew-field with positive squares is linearly ordered. As a counterpart, we prove the existence of directed algebras with negative squares. For an arbitrary l-group, we give some characterizations of l-ideals in terms of absolute values, generalizing a similar result for o-ideals in Riesz spaces [1].

A number of ring-theoretical notions and properties are introduced for l-groups and MV-algebras. Using the correspondence between l-groups and MV-algebras, we answer a question of Belluce [9] on prime annihilators in MV-algebras in the negative.

For an Abelian l-group G, we construct a dense embedding $G \hookrightarrow E(G)$ into a laterally complete Abelian l-group $E(G)$ by means of sheaf theory. In case G is Archimedean, we prove that $E(G)$ is the lateral completion of G, while in general, this seems to be false. As a byproduct, we get a natural and elegant proof of Bernau's celebrated embedding theorem for Archimedean l-groups. If G is the group of divisibility of a Bézout domain D, we show that $Spec(D)$ is related to a quasi-compact topology on $Spec^*(G)$ which turns out to be the "Hochster dual" (see Section 5.2) of the spectral topology.

We study the C-topology [65] on Abelian l-groups and apply it to Bézout domains and MV-algebras. We correct two lemmas of Gusić [65] which leads to a generalization of one of his main results.

Finally, we answer a question of Dumitrescu, Lequain, Mott and Zafrullah [50] which shows that the Jaffard-Ohm correspondence does not hold for Abelian almost l-groups.

Contents

Chapter 0

Zusammenfassung

Wir untersuchen die Strukturtheorie abelscher l-Gruppen im Hinblick auf ihre Beziehungen zur Teilbarkeit in Bézout-Bereichen und zu MV-Algebren. Hauptmotivation für die Arbeit ist die Jaffard-Ohm-Korrespondenz zwischen abelschen l-Gruppen und Bézout-Bereichen, und D. Mundici's funktorielle Äquivalenz zwischen abelschen l-Gruppen mit starker Einheit und MV-Algebren.

Im folgenden erläutern wir kurz die Geschichte (nicht notwendig abelscher) l-Gruppen, die beiden oben genannten Korrespondenzen, sowie die Hauptergebnisse der vorliegenden Dissertation.

0.1 l-Gruppen - gestern und heute

Seit ihren ersten Anfangen stand die Theorie der l-Gruppen in einer fruchtbaren Wechselwirkung mit verschiedenen Gebieten der Mathematik, insbesondere der Gruppentheorie, Ringtheorie, Logik und Modelltheorie, Functionalanalysis, Topologie, Universellen Algebra und Verbandstheorie. Bereits 1901 hat Hölder gezeigt, dass jede archimedische angeordnete Gruppe isomorph zu einer l-Untergruppe der additiven Gruppe der reellen Zahlen ist. Hahn konstruierte eine Klasse angeordneter abelscher Gruppen derart, dass jede angeordnete abelsche Gruppe in ein Exemplar dieser Klasse eingebettet werden kann. Hilbert benutzte angeordnete Gruppen für seine Grundlagenforschung auf dem Gebiet der klassischen Geometrie.

Hilberts Theorie der Integralgleichungen führte zur Entwicklung wichtiger Teile der modernen Analysis, einschliesslich Operatorentheorie und Theorie der topologischen linearen Räume. Alle topologischen Vektorräume sind abelsche

l-Gruppen. Als solche finden sie sich in Arbeiten von Riesz, Freudenthal, Kantorovich, Artin, Schreier, Birkhoff, Stone und Yosida.

Durch die grundlegenden Arbeiten von Birkhoff und Nakano begann das Studium der l-Gruppen um ihrer selbst willen. Levi bewies, dass jeder torsionsfreien abelschen Gruppe die Struktur einer Verbandsgruppe gegeben werden kann. Lorenzen zeigte, dass jede abelsche l-Gruppe als subdirektes Produkt von angeordneten Gruppen dargestellt werden kann. Die allgemeine Theorie nahm bald Gestalt an, was vor allem dem Pionier Paul Conrad zuzuschreiben ist. Seine frühen Untersuchungen haben die Wichtigkeit des Verbands der konvexen l-Untergruppen einer l-Gruppe betont, worin das Wurzelsystem von besonderer Wichtigkeit ist. Ein Analogon des Hahnschen Einbettungssatzs für abelsche l-Gruppen wurde gefunden, wobei die linear angeordneten Indizesmengen durch Wurzelsysteme ersetzt wurden. Für beliebige l-Gruppen bewies Holland, dass jede l-Gruppe als Automorphismengruppe einer linear angeordneten Menge dargestellt werden kann.

0.2 Verwandte Strukturen

Die vorliegende Arbeit konzentriert sich auf Anwendungen von l-Gruppen auf Probleme von Bézout-Bereichen und Bewertungen einerseits, und MV-Algebren andererseits. MV-Algebren algebraisieren mehrwertige Logik im Sinne von Łukasiewicz.

Zu jedem (kommuativen) Integratätsbereich D gehört eine teilweise geordnete abelsche Gruppe $G(D)$, bestehend aus den von Null verschiedenen Bruchhauptidealen von D, die *Teilbarkeitsgruppe*. Ist D ein Bézout-Bereich, so ist $G(D)$ eine abelsche l-Gruppe. In diesem Fall ist also das Studium der Teilbarkeit in D gleichwertig zum Studium von $G(D)$ als abelsche l-Gruppe. Ward und Birkhoff haben zunächst gezeigt, dass D genau dann faktoriell ist, wenn die teilweise Ordnung von $G(D)^{\geq 0}$ artinsch ist (cf. [86]). In diesem Zusammenhang fanden Richard Dedekinds frühe Untersuchungen zur Teilbarkeit in einer (freien) abelschen l-Gruppe statt. Die Untersuchung allgemeiner Teilbarkeitengruppen wurde später von Krull, Ward, Lorenzen, Clifford, Dilworth, Nakayama, Jaffard, Ohm, Conrad und Mott, bis in die jüngste Zeit, verstärkt fortgesetzt.

Es ist bekannt, dass eine Teilbarkeitsgruppe gerichtet sein muss, aber diese Bedingung ist nicht genügend. Es gibt verschiedene Obstruktionen, die durch Beispiele in der Literatur (cf. [2, 8, 35, 46, 62, 71, 83, 86, 92]), auch in dieser

Dissertation (Kaptel 7), erfasst sind.

Wir haben erwähnt, dass jeder Bézout-Bereich D eine abelsche l-Gruppe $G(D)$ bestimmt. Andererseits kann gezeigt werden, dass jede abelsche l-Gruppe von dieser Form ist. Im angeordneten Fall bedeutet das, dass jede angeordnete abelsche Gruppe als Bewertungsgruppe eines Bewertungsbereichs (Krull [78]) auftritt. Im Allgemeinen hat Jaffard durch eine sinnreiche Konstruktion gezeigt (s. Theorem 2.2.1), dass jede abelsche l-Gruppe zur Teilbarkeitsgruppe eines GCD-Bereichs D ordnungsisomorph ist. Ohm (s. [3]) hat später beobachtet, dass Jaffards GCD-Bereich sogar ein Bézout-Bereich ist. Aufgrund dieser sog. Jaffard-Ohm-Korrespondenz gibt es zu jeder abelschen l-Gruppe eine Klasse zugehöriger Bézout-Bereiche, die keine Menge (im Sinne von Gödel-Bernays) ist.

Die Konstruktion einer abelschen l-Gruppe mit gegebenen Eigentschaften ist gewöhnlich viel leichter als die Herstellung eines entsprechenden Bézout-Bereichs. Deshalb ist die Jaffard-Ohm-Korrespondenz ein nützliches Werkzeug zur Lösung ringtheoretischer Probleme. Zum Beispiel hat Nakayama [90] diese Methode benutzt, um die berühmte Krull-Vermutung zu widerlegen [78][1].

Die Teilklasse der abelschen l-Gruppen mit starker Einheit spielt eine wichtige Rolle in der Theorie der C^*-Algebren. Die Existenz einer starken Einheit wird auch im Yosida-Fukamiya-Nakayamaschen Lehrsatz (s. [80, 107]) benötigt, ebenso für die C-Gruppen im Sinne von Gusić [65].

Mundici [88] hat überdies gezeigt, dass die Kategorie der abelschen l-Gruppen mit starker Einheit zur Kategorie der MV-Algebren äquivalent ist (s. Proposition 4.3.4). Obwohl die Axiome einer MV-Algebra seit dem grundlegenden Artikel von Chang [33] eine drastische Vereinfachung erlebt haben, scheint uns die Arbeit mit l-Gruppen immer noch leichter zu sein als der Umgang mit MV-Algebren. In der Tat, wir werden zeigen, dass sich die Beweise mehrerer Ergebnisse für MV-Algebren über die Mundici-Äquivalenz einfacher beweisen lassen (s. Chapter 4). Unter anderem werden wir eine offene Frage von Belluce zum Spektrum einer MV-Algebra mittels l-Gruppen beantworten.

0.3 Hauptresultate

In diesem Abschnitt geben wir einen kurzen Überblick über die Ergebnisse dieser Dissertation.

[1] Unabhängig davon gab Kaplansky ein Gegenbeispiel zu dieser Vermutung in seiner Dissertation, cf. [86].

Mit Hilfe der Bewertungstheorie erhalten wir eine positive Antwort zum Problem von Conrad und Dauns [41], ob ein Verbandsschiefkörper mit positiven Quadraten stets angeordnet ist. Als Gegenstück hierzu beweisen wir die Existenz gerichteter Algebren mit negativen Quadraten.

Für eine beliebige l-Gruppe verallgemeinern wir die Charakterisierung von o-Idealen in Rieszschen Räumen [1] auf l-Ideale. Mittels der Mundici-Korrespondenz zwischen l-Gruppen und MV-Algebren erhalten wir eine negative Entscheidung einer Frage von Belluce [9] über Prim-Annihilatoren in MV-Algebren.

Für eine abelsche l-Gruppe G konstruieren wir eine dichte Einbettung $G \hookrightarrow E(G)$ in eine lateral vollständige abelsche l-Gruppe $E(G)$ mittels Garbentheorie. Falls G archimedisch ist, beweisen wir, dass $E(G)$ die laterale Vervollständigung von G ist, während dies im Allgemeinen falsch zu sein scheint. Als Nebenprodukt erhalten wir einen eleganten Beweis des Bernauschen Einbettungssatzes für Archimedische l-Gruppen. Ist G die Teilbarkeitsgruppe eines Bézout-Bereichs D, so zeigen wir, dass $Spec(D)$ zum Spektrum $Spec^*(G)$ topologisch dual ist im Sinne von Hochster (s. Section 5.2).

Wir studieren die C-Topologie [65] abelscher l-Gruppen mit Blick auf Bézout-Bereiche und MV-Algebren. Dabei korrigieren wir zwei Lemmata von Gusić [65], was zu einer Verallgemeinerung eines seiner Hauptergebnisse führt.

Schliesslich beantworten wir eine Frage von Dumitrescu, Lequain, Mott und Zafrullah [50], indem wir zeigen, dass die Jaffard-Ohm Korrespondenz für abelsche fast-l-Gruppen nicht gilt.

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Chapter 1

Introduction and overview

This thesis studies the relationship between lattice-ordered groups (l-groups for short), Bézout domains and MV-algebras. The main motivation and starting point for our work has been the Jaffard-Ohm correspondence between Abelian l-groups and Bézout domains, and Mundici's functorial equivalence between Abelian l-groups with strong unit and MV-algebras.

In this introductory chapter, we start with a brief history of (not necessarily Abelian) l-group theory, as far as useful for our present concern. In Section 1.1, we give an informal explanation of the mentioned correspondences from the l-group theory point of view. The main results of the thesis, and an outline of its parts will be given in Section 1.2.

1.1 The ubiquity of l-groups

Over its history, l-group theory has benefited from contacts with many different areas at different times. There has been an increasing interaction with other branches of mathematics, including group theory, ring theory, algebra, logic, model theory, functional analysis, topology, universal algebra, and lattice theory. As early as in 1901, Hölder proved that every Archimedean linearly ordered group is isomorphic to an l-subgroup of the additive group of real numbers $\mathbb{R}$. More generally, Hahn constructed a class of linearly ordered Abelian groups, made up from the sequences of real numbers, indexed by an arbitrary totally ordered set, such that every totally ordered Abelian group can be embedded into some member of the class. Hilbert used linearly ordered groups for his foundational study of classical geometry.

At Hilbert's time, the study of integral equations led to the development of important parts of modern analysis, including the theory of operators and their various spaces. All those topological vector spaces are Abelian l-groups. They played an important role in the works of Riesz, Freudenthal, Kantorovich, Artin, Schreier, Birkhoff, Stone, Yosida and many others.

After the fundamental papers by Birkhoff and Nakano, l-groups became objects of study in their own right. Levi proved that every torsion-free Abelian group can be lattice-ordered. Lorenzen showed that each Abelian l-group can be represented as a subdirect product of totally ordered groups. The general theory emerged. In its present shape, it is built, to a large extent, on the work of Paul Conrad (although it seems that two papers on l-group theory of his advisor, R. Baer, have been forgotten by many mathematicians). His early investigations stressed the importance of the lattice of convex l-subgroups of a given l-group, and within it the root system of prime convex l-subgroups. An analogue of Hahn's theorem was established for Abelian l-groups, with totally ordered index sets replaced by root systems. For arbitrary l-groups, Holland proved that every l-group can be represented as a group of automorphisms of some linearly ordered set.

The present work intends to focus on applications of l-group theory to problems of Bézout domains and valuation theory on the one hand, and MV-algebras on the other hand. MV-algebras arose from many valued logic, and are of fundamental significance for logic theories without the *tertium non datur*.

Along with any (commutative) integral domain D, there comes the partially ordered Abelian group $G(D)$ of nonzero fractional principal ideals of D, with $aD \leq bD$ if and only if $aD \supseteq bD$, which is called the *group of divisibility* of D. Alternatively, if K^* denotes the multiplicative group of the quotient field K of D, and U_D denotes the group of units of D, then $G(D)$ is order isomorphic to K^*/U_D, where $aU_D \leq bU_D$ if and only if $ba^{-1} \in D$. If D is a Bézout domain, i. e. if every finitely generated ideal of D is principal, the poset structure of $G(D)$ is a lattice, and the multiplication with any group element is a lattice automorphism. In other words, $G(D)$ is an Abelian l-group. In this case, the study of divisibility in D essentially amounts to the study of $G(D)$ as an Abelian l-group. In this context, one of the earliest observations was made by Ward and Birkhoff. They proved that D is a unique factorization domain if and only if the Abelian l-group $G(D)$ satisfies the descending chain condition on positive elements (cf. [86]).

As a study of ideals and divisibility, the theory of l-groups was initiated by Dedekind in an attempt to restore unique factorization, with a side-view to Fermat's Last Theorem (which about 100 years later became a "theorem"). The investigation of divisibility groups was amplified later by Krull, Ward, Lorenzen, Clifford, Dilworth, Nakayama, Jaffard, Ohm, Conrad, Mott, until recent times.

It is known that a group of divisibility must be directed, but this condition is not sufficient unless $G(D)$ is lattice-ordered. There are several obstructions, exhibited by different examples in the literature (cf. [2, 8, 35, 46, 62, 71, 83, 86, 92]), and also in this thesis (see Chapter 7).

We mentioned that any Bézout domain D gives rise to an Abelian l-group $G(D)$. On the other hand, it can be shown that every Abelian l-group is of this form. In the totally ordered case, this means that every totally ordered Abelian group is the value group of a valuation domain (proved by Krull [78]). More generally, for any field k, the group ring kG of an Abelian l-group G is an integral domain since G is torsion-free. So there exists a quotient field K of kG. By an ingenious construction, Jaffard (see Theorem 2.2.1) showed that there exists a pseudo-Bézout domain (that is, a GCD-domain) D such that $G(D)$ is order-isomorphic to G. Ohm (see [3]) later observed that the GCD-domain constructed by Jaffard is actually a Bézout domain. So there is a one-to-many correspondence between Abelian l-groups and Bézout domains. (As the category of fields is not small, the class of Bézout domains corresponding to a fixed Abelian l-group is not a set.)

$$\begin{tikzcd}
 & & D_\alpha \arrow[dr, "\text{Jaffard}"] & & \\
 & (G,\leq) \arrow[ur, "\text{Jaffard}"] \arrow[r] \arrow[dr] & D_\beta \arrow[r] & G(D_\alpha) \cong_{l-g} G(D_\beta) \cong_{l-g} \cdots \cong_{l-g} (G,\leq) & \\
G \arrow[ur, "\text{Levi}"] \arrow[r] \arrow[dr] & \vdots & \vdots \arrow[ur] & \vdots & \\
 & \vdots & \vdots & \vdots & \\
\mathrm{Ab}_{tf} & \mathrm{Ab}_l & \mathcal{B} & \mathrm{Ab}_l &
\end{tikzcd}$$

Figure 1.1: Levi-Jaffard 1-many correspondence

By Levi's theorem, every torsion-free Abelian group can be equipped with the structure of an Abelian l-group (e. g., a totally ordered one). The two one-to-many correspondences between the categories of torsion-free Abelian groups AB_{tf}, Abelian l-groups Ab_l, and Bézout domains $\mathcal{B}$, are depicted in Figure 1.1.

As it is usually much easier to construct an Abelian l-group with given properties than to directly construct a Bézout domain, the Jaffard-Ohm correspondence has been and remains to be a useful tool for solving ring theoretic problems. For example, Nakayama [90] used this method to disprove a conjecture of Krull [78] (see the note after Corollary 2.2.9) on completely integrally closed domains[1].

An important class of l-groups is given by the category Ab_l^* of Abelian l-groups with strong unit, morphisms being the l-homomorphisms preserving the strong unit. These l-groups are important in the theory of C^*-algebras. The existence of a strong unit plays a role in the Yosida-Fukamiya-Nakayama theorem (see [80, 107]), and also for the C-groups in the sense of Gusić (see Section 6.1) ([65]).

$$\mathrm{Ab}_l^* \xrightarrow{\Gamma} \mathrm{MV}_{\mathrm{Alg}} \xrightarrow{\quad\Gamma^-\quad} \mathrm{Ab}_l^*$$

$$(G,u) \longmapsto \Gamma(G,u) \longmapsto \Gamma^-(\Gamma(G,u)) \cong_{l-g} (G,u)$$

Figure 1.2: Mundici's functor and its inverse

Mundici [88] showed that the category Ab_l^* of Abelian l-groups with strong unit is equivalent to the category MV_{Alg} of MV-algebras (see Proposition 4.3.4). Although the axioms of an MV-algebra have undergone a drastic simplification since Chang's basic article [33], it seems to be evident that l-groups are still easier handle than MV-algebras. In fact, we will show that the proofs of several results for MV-algebras can be simplified by means of l-group theory via Mundici's equivalence Γ: $Ab_l^* \to MV_{Alg}$ (see Chapter 4). We will answer an open question on prime ideals of MV-algebras via l-group theory in Section 4.4.

[1]Independently, Kaplansky gave an negative answer to the conjecture in his dissertation, cf. [86].

1.2 Main results

In this section, we give a brief overview of the results of this thesis. Chapter 2 starts with a review of basic concepts in l-group theory. In Section 2.2, we describe Abelian l-groups as divisibility groups of Bézout domains. In order to apply Abelian l-groups with a strong unit to ring theory, we introduce the *unit radical* $u(D)$ of an integral domain D and give a ring theoretic characterization of divisibility groups with strong unit (Theorem 2.2.7). Some important examples of l-groups will be provided in Section 2.3.

As already mentioned, l-ideals are of central importance for the structure theory of l-groups. In the special case of Riesz spaces, Alpay, Emel'yanov and Ercan [1] gave a characterization of convex l-subspaces in terms of absolute values. In Section 3.1, we generalize this result to l-ideals of arbitrary l-groups.

Section 3.2 contains a rather different application of absolute values. In [5], Artin and Schreier observed that a totally ordered commutative field cannot have negative squares, and Johnson [73] and Fuchs [56] extended this result to totally ordered domains with unit element. In 1969, Conrad and Dauns [41] raised the question (see Question 3.2.1 below)

whether a lattice-ordered skew-field F in which each square is positive must be totally ordered.

In case F is commutative, the statement is true and a well known theorem [25, 18, 94]. For arbitrary skew-fields, however, this remained to be an open problem. By use of some yoga with absolute values, we will prove the statement in general (see Theorem 3.2.3).

In a sense, this result shows that for an associative ring, the existence of a lattice-ordering is a strong property. If any pair of elements admits an upper bound (which need not be a supremum), the ordering is said to be *directed.* In this case, there also exist lower bounds for any finite subset. Birkhoff and Pierce [18] proved that the complex field $\mathbb{C}$ does not admit a lattice-ordering as an $\mathbb{R}$-algebra. Whether $\mathbb{C}$ itself can be made into an l-field is a famous open problem (cf. [18, 94]).

In contrast to Conrad and Dauns' situation, the field $\mathbb{C}$ has the property that -1 is a square. For many $\mathbb{R}$-algebras with negative squares, it is known that they cannot be endowed with a directed ordering (see [47, 104]). So it is natural to ask (see Question 3.3.1 below)

whether there exists any directed ring or directed $\mathbb{R}$-algebra with negative squares.

The answer seems to be unknown. Note that a negative answer would imply a negative solution to the Birkhoff-Pierce problem on $\mathbb{C}$. By another application of valuations, we prove the existence of directed rings (see Theorem 3.3.6) and algebras (see Proposition 3.3.12) with negative squares.

Generalizing the approach used in [104], we proceed with the study of directed orders on $K(i)$ with $i^2 = -1$ in some detail for a linearly ordered field K in Section 3.3.3. First we exhibit a particular class of l-group structures on $K(i)$ and show that they can be parameterized by positive elements of K (Theorem 3.3.4). By the same method, we establish a correspondence between directed orderings of the field $K(i)$ and absolutely convex subsets $V \subsetneq K$ which satisfy the implication (Theorem 3.3.15)

$$0 < a \notin V \Rightarrow \frac{1}{2}(a - a^{-1}) \notin V.$$

Then we characterize these $V \subsetneq K$ as convex additive subgroups with the property that for $a \in K^*$, either a or a^{-1} belongs to V (Theorem 3.3.19). Note that the later property holds for any valuation domain with quotient field K.

One of the intriguing features of l-groups is the fact that their underlying lattice must be distributive[2]. For an algebraist, this property suggests that their structure might be more or less trivial. The abundance of different examples of l-groups in various parts of analysis, however, soon leads to a quite different picture. Amazingly, there are nevertheless strong similarities to module theory. For example, if an l-group G has a strong unit, there is an analogue $Rad(G)$ of the Jacobson radical, such that Archimedean l-groups are similar to semi-simple modules. On the other hand, in contrast to semi-simple modules, the structure of Archimedean l-groups is much more complicated. We believe that the best way to understand Abelian l-groups is to use sheaf theory. Part of this programm was carried out by Keimel [76]. We show that the method leads to a deeper understanding of the Archimedean case (see Chapter 5). Moreover, the relationship between Abelian l-groups and Bézout domains reveals a new aspect of Hochster's duality between quasi-compact spectral topologies.

In Chapter 4, we investigate the relationship between MV-algebras and Bézout domains with non-zero unit radical. Moreover, we elaborate on the analogy between l-groups and modules. We define the *socle* $Soc(G)$ (resp. *radical* $Rad(G)$) of an l-group G to be the algebraic union (resp. intersection) of all minimal (resp. maximal) convex l-subgroups of G. In Section 4.1.2, we

[2]Note that [58] proves that a directoid group G is an l-group if and only if G satisfies the modular law (see Proposition 7.3.7 below).

study the analogue of small and large (= essential) submodules, and call them *small* and *large* l-subgroups, respectively. In Section 4.1.3, we show that the socle $Soc(G)$ (resp. radical $Rad(G)$) can be represented as the infimum (resp. supremum) of the large (resp. small) convex l-subgroups of G. In this vein, a number of ring-theoretic notions and properties carry over to l-groups and MV-algebras. For example, there is an analogue of "infinitely small", namely, the nilpotent elements of a ring are related to the bounded elements of an Abelian l-group with strong unit, and to the nilpotent elements in the corresponding MV-algebra (Proposition 4.3.12).

Precisely, an element g of an l-group G is said to be *bounded* or *infinitely small* if there exists an element $h \in G$ such that $n|g| \leq h$ holds for all $n \in \mathbb{N}$. We denote the set of bounded elements by $B(G)$. Thus G is Archimedean if and only if $B(G) = \{0\}$. Let $C(D)$ denote the complete integral closure of a Bézout domain D. Then it is known (see Mott [87], Corollary 2.6) that the group of divisibility of $C(D)$ is order-isomorphic to the factor group $G(D)/B(G(D))$. Note that the complete integral closure of a Bézout domain D need not be completely integrally closed. In other words, $G(D)/B(G(D))$ is not necessarily Archimedean, i. e. $B(G(D)/B(G(D)))$ need not be trivial. However, we will prove that the inclusion

$$B(G(D)) \subseteq Rad(G(D))$$

always holds, while $G/Rad(G)$ is known to be Archimedean for every Abelian l-group (Proposition 4.2.3). In case $u(D)$ is not trivial, we show that

$$B(G(D)) = Rad(G(D))$$

(Theorem 4.2.5). In particular, this means that the corresponding MV-algebra $\Gamma(G(D)/Rad(G(D)))$ is a representable MV-algebra (that is, an MV-algebra which is isomorphic to a subalgebra of a direct product of Archimedean linearly ordered MV-algebras). As a corollary, this implies the Yosida-Fukamiya-Nakayama theorem for Riesz spaces. In the presence of a strong unit, the relationship between Archimedean l-groups and the corresponding Bézout domains and MV-algebras is given by the following

Theorem *Let D be a Bézout domain with non-zero unit radical $u(D)$. Then the following statements are equivalent:*

(1) D is completely integrally closed.

(2) $G(D)$ is Archimedean.

(3) $Rad(G(D)) = \{0\}$.

(4) D is an intersection of rank 1 valuation overrings.

(5) $\Gamma(G(D))$ *is a representable* MV*-algebra.*

(6) $G(D)$ *is isomorphic to the* l*-group of all sections of a Hausdorff global sheaf of Archimedean linearly ordered groups over a compact space.*

Let g be an element of an l-group G. Recall that a convex l-subgroup of G is said to be a *regular* subgroup or a *value* of g if it is maximal among the subgroups which do not contain g. An element $g \in G$ is said to be *special* if g has a unique value M, and in this case, M is called a *special value*. G is said to be *finite-valued* if every element has only a finite number of values. For our purpose, we need a modification of this concept. We will say that G is *μ-valued* if every maximal convex l-subgroup of G is special. Thus every finite-valued l-group is μ-valued (see Lemma 4.2.20), but a μ-valued l-group need not be finite-valued. The l-group G will be called *semimaximal* if $Rad(G) = 0$. In Theorem 4.2.21 we give a characterization of semimaximal μ-valued l-groups.

Using Mundici's equivalence, we show in Section 4.3 that important properties of MV-algebras can be obtained by means of l-group theory. In 1986, Belluce [9] proved that for an MV-algebra A, every annihilator of a linearly ordered ideal of A is a prime ideal. The question whether the converse always holds remained open (see Question 4.4.2). Via l-groups, we show in Section 4.4 that the converse holds for exactly those MV-algebras for which every prime ideal is minimal and closed (Theorem 4.4.5). In particular, this implies a negative answer to Belluce's question.

Recall that a ring R which admits a full quotient ring Q is said to be an *Asano order* (see [66, 98]), if every integral ideal of R has an inverse in Q. Here an ideal of R is said to be integral if it contains a unit of Q. An ideal I of a ring R is $full$ if there is an invertible ideal $P \subseteq I$, and R is called an *Asano* ring [98]) if every full ideal of R is invertible. For a Noetherian ring, every invertible ideal is integral [98], and thus every Noetherian Asano order is an Asano ring. In Section 4.5 we shall characterize Asano and local Bézout domains via divisibility groups using the approach given in [98].

In Section 4.6, we compare the prime spectrum of l-groups and Bézout domains, and study properties concerning its partial order and topology.

In Chapter 5, we introduce the dual spectrum $Sp(G)$ of an Abelian l-group G and define a natural sheaf $\tilde{G}$ of fully ordered Abelian groups on $Sp(G)$, such that G consists of the global sections of $\tilde{G}$. We show that in case G is the divisibility group of a Bézout domain D, the space $Sp(G)$ is homeomorphic to $Spec(D)$, and $\tilde{G}$ is closely related to the affine scheme associated with D. On the other hand, the prime spectrum $Spec^*G$ of G turns out to be dual to $Sp(G)$ in the sense of Hochster [68]. In this context, the infinite prime G of $Spec^*(G)$ (which is discarded from the spectrum of G by many authors) corresponds to the infinite prime 0 of D. We call $Sp(G)$ the *Hochster dual* of $Spec^*(G)$. Accordingly, there is a dual of $\tilde{G}$, the *sheaf of germs* $\mathcal{G}(G)$ of G in the sense of Keimel [76]. While the stalks of $\tilde{G}$ are fully ordered, this is no longer true for the dual sheaf $\mathcal{G}(G)$. As the dense open subsets U of $Spec^*(G)$ are directed via "$\supseteq$", we can form the direct limit

$$E(G) := \varinjlim \Gamma(U, \mathcal{G}(G)).$$

This construction is quite natural and closely related to the quotient field of an integral domain in algebraic geometry. We show that G is dense in $E(G)$, and that $E(G)$ is laterally complete. We prove that if G is Archimedean, $E(G)$ is just its lateral completion. In this way, we obtain a very natural proof of Bernau's embedding theorem. In terms of sheafs, it states that G can be represented as a large l-subgroup of $E(\tilde{X})$, where $\tilde{X}$ is the Stone space of the algebra of polars of G, and $E(\tilde{X})$ is a kind of analogue to $E(G)$. As a laterally complete l-group, $E(G)$ admits a weak unit. Hence there is a dense compact-open subspace X of $Spec^*E(G)$. We show that $\tilde{X}$ can be obtained as the projective cover of X in the category of compact spaces (see [64]).

Recently, Gusić [65] showed that an Abelian l-group G can be equipped with a topology (which he calls C-topology) in several ways. He develops some interesting ideas with regard to G as a topological group. Unfortunately, there is a gap in Lemmas 5 and 6 of [65] which affected one of his main results (Theorem 1). In Chapter 5 we study the C-topology in some detail and apply it to Bézout domains and MV-algebras. We obtain correct versions of Lemmas 5 and 6 of [65], which leads to a generalization of Gusić's Theorem 1.

As mentioned earlier in Section 1.1, directed order is necessary for a partially ordered group to be a divisibility group. However, there are directed groups that are not groups of divisibility. Dumitrescu, Lequain, Mott and Zafrullah [50] proved that an integral domain D is $AGCD$ (i. e. for all $x, y \in D$, there is an integer $n \geq 1$ such that $(x^n) \cap (y^n)$ is principal) if and only if $G(D)$ is an *almost lattice-ordered group* (al-group for short), i. e. for each pair $x, y \in G$, there is an integer $n \geq 1$ for which

$$nx \vee ny$$

exists in G. They asked whether the Jaffard-Ohm correspondence holds in this more general context. Precisely, the problem (see Question 7.1.1 below) is

Is any al-group the divisibility group of an AGCD domain?

In Chapter 7, we discuss this problem in detail and show that the answer is negative. In particular, we give an example of an isolated Archimedean *al*-group which cannot arise as a divisibility group of an integral domain in Section 7.3.

Chapter 2

Preliminaries

Unless otherwise mentioned, we use the additive notation for the group operation of an arbitrary group. A domain will always mean a commutative ring with identity and without divisors of zero, that is, an integral domain. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of natural numbers, the rationals, the integers and the reals, respectively. "$\subset$" means strict inclusion in this thesis.

2.1 Review of l-group theory

In this section we will review some of the definition and elementary properties of l-groups. We start by defining the l-group.

A *partially ordered group* $(G, +, \leq)$ is both a (not necessarily Abelian) group $(G, +)$ (with binary operation $+$, identity element 0, where the inverse of $a \in G$ is denoted by $-a$), and a partially ordered set $(G, \leq)$ in which

$$a \geq b \text{ implies } d + a + c \geq d + b + c$$

for any $a, b, c, d \in G$. A partially ordered group (po-group for short) is called a *lattice-ordered group* (l-group for short) if the partial order is a lattice order, and it is called *totally ordered* or *linearly ordered* if the partial order is a linear order. If $(G, +, \leq)$ is an l-group and $a \in G$, then the *positive part* of a is $a^+ = a \vee 0 \geq 0$, *the negative part* of a is $a^- = (-a) \vee 0 \geq 0$, and *the absolute value* of a is $|a| = a \vee (-a) \geq 0$. We shall denote the *positive cone* of a po-group G by $G^{\geq 0} = \{g \in G \mid g \geq 0\}$ and the *strictly positive cone* by $G^{>0} = G^{\geq 0} \setminus \{0\}$. For instance, $\mathbb{N} = \mathbb{Z}^{>0}$. Furthermore, two elements a and b are incomparable in a partially ordered set is denoted by $a \parallel b$.

Theorem 2.1.1 (cf. [44]) *Let a, b, c be elements in an l-group G, and $n \in \mathbb{N}$. Then*

1) $a = a^+ - a^-$.

2) $|a| = a^+ + a^-$.

3) $a^+ \wedge a^- = 0$.

4) $|a| = 0$ if and only if $a = 0$.

5) $|a \vee b| \leq |a| \vee |b| \leq |a| + |b|$, $|a+b| \leq |a| + |b| + |a|$, and $|a+b| \leq |a| + |b|$ if and only if G is commutative.

6) $a + (b \vee c) = (a + b) \vee (a + c)$.

7) $-(a \vee b) = (-a) \wedge (-b)$.

8) G is torsion free and isolated (semiclosed).

9) $na^+ = (na)^+$, $n|a| = |na|$.

Then following Corollary 2.1.2 is directly obtained by Theorem 2.1.1 5).

Corollary 2.1.2 *An l-group G is Abelian if and only if $||a| - |b|| \leq |a + b| \leq |a| + |b|$ for all a, $b \in G$. Furthermore, $2(a \vee b) = a + b + |a - b|$ and $2(a \wedge b) = a + b - |a - b|$ for all a, b in an Abelian l-group.*

Let L be a lattice, a, b, $g \in L$ and $\{h_\lambda\}_\Lambda$ and $\{g_{ij}\}_{I,J}$ be subsets of L. L is *distributive* if $g \wedge (a \vee b) = (g \wedge a) \vee (g \wedge b)$ and dually. L is *Brouwerian* if whenever $\vee_\Lambda h_\lambda$ exists in L, then $g \wedge (\vee_\Lambda h_\lambda) = \vee_\Lambda (g \wedge h_\lambda)$. L is *dually Browerian* if the dual is true. L is *completely distributive* if whenever $\vee_{i \in I} \wedge_{j \in J} g_{ij}$ and $\wedge_{f \in J^I} \vee_{i \in I} g_{i,f(i)}$ exist, $\vee_{i \in I} \wedge_{j \in J} g_{ij} = \wedge_{f \in J^I} \vee_{i \in I} g_{i,f(i)}$, where J^I is the set of all functions with domain I and codomain J.

A subset S of a po-group T is *convex* if whenever s, $t \in S$ and $s \leq g \leq t$ in T, then $g \in T$. A subgroup A of an l-group G is an *l-subgroup* if A is also a sublattice of G. We denote a convex l-subgroup C of G by $C \leq G$, and denote the set of all convex l-subgroups of G by $\mathcal{C}(G)$. For a family $A_i \leq G$, $i \in I$, let $\wedge_{i \in I} A_i := \cap_{i \in I} A_i \leq G$ and $\vee_{i \in I} A_i := \cap \{B \mid \cup_{i \in I} A_i \subseteq B \leq G\}$. Then $\mathcal{C}(G)$ is a complete lattice (cf. [44]). $P \leq G$ is *prime* if whenever $a, b \in G^{\geq 0}$ and $a \wedge b \in P$, then $a \in P$ or $b \in P$. Note that $P = G$ is not excluded. $Sp(G)$ will denote the set of prime convex l-subgroups of G. An *l-ideal* of an l-group is a normal convex l-subgroup. $\mathcal{I}(G)$ will denote the set of l-ideals of G.

Theorem 2.1.3 (cf. [44], Corollary 3.16 and Proposition 7.10) *An l-group is distributive. The set of convex l-subgroups of an l-group is a distributive Brouwerian lattice.*

Let G and H be po-groups, $f: G \to H$ be a group homomorphism. Then f is called a *po-group homomorphism* if $f(G^{\geq 0}) \subseteq H^{\geq 0}$; it is called a *po-group epimorphism* if it is a surjection and $f(G^{\geq 0}) = H^{\geq 0}$; and it is called a *po-group isomorphism* (denoted by $\cong_{p-g}$) if it is a po-group epimorphism and a group isomorphism. If G and H are l-groups, f is called an *l-group homomorphism* if f is a lattice homomorphism. If l-groups G and H are l-isomorphic, then we denote it by $G \cong_{l-g} H$.

On the set of cosets of a po-group with respect to a convex l-subgroup, one can define a partial order.

Proposition 2.1.4 (cf. [44], pp. 44-46) *Let G be a po-group and S a convex l-subgroup of G. Let $\mathcal{R}(S)$ denote the set of right cosets of S. On $\mathcal{R}(S)$, define $S + x \geq S + y$ if there exist $s \in S$ such that $x \geq s + y$. Then $\geq$ is a partial order of $\mathcal{R}(S)$ and is called the coset order of $\mathcal{R}(S)$. Let L be an l-ideal of an l-group G. Then G/L under the coset order is an l-group and the natural homomorphism of G onto G/L is an l-group homomorphism. In the special case that $\mathcal{R}(L)$ becomes linearly ordered with this order, L is prime.*

Theorem 2.1.5 (cf. [44], Theorem 8.6, Isomorphism Theorems) *Let G and H be l-groups. Then*

a) if $f: G \to H$ a surjective l-group homomorphism, then $G/(Ker(f)) \cong_{l-g} H$.

b) let A be an l-subgroup of G and $B \in \mathcal{I}(G)$. Then $A + B$ is an l-subgroup of G and $(A + B)/B \cong_{l-g} A/(A \cap B)$.

c) let $A, B \in \mathcal{I}(G)$ with $A \subseteq B$. Then $B/A \in \mathcal{I}(G/A)$ and $G/B \cong_{l-g} (G/A)/(B/A)$.

Let G be an l-group. A convex l-subgroup of G maximal with respect to not containing an element g of G is called a *regular subgroup* and a *value* of g. $\Gamma(g)$ will denote the set of values of g. Actually, a convex l-subgroup M of G is regular if and only if $M \subset M^c = \cap\{C \leq G \mid M \subset C\}$. The convex l-subgroup M^c is called the *cover* of M, and M is a *normal value* if M is a normal subgroup of M^c.

Note that a maximal convex l-subgroup M of an l-group G must be a value for any element $x \in G \setminus M$. However, a value of an element need not be a maximal convex l-subgroup. For instance, each value of a bounded element is not a maximal convex l-subgroup by Proposition 4.1.22 below.

The following proposition will be used frequently in what follows.

Proposition 2.1.6 (cf. [44], Propositions 8.8, 9.11 and 10.6) *Let G be an l-group, A a subgroup of G, and $L \in \mathcal{I}(G)$ with $L \subseteq A$. Then*

a) A is an l-subgroup of G if and only if A/L is an l-subgroup of G/L.

b) $A \in \mathcal{C}(G)$ if and only if $A/L \in \mathcal{C}(G/L)$.

c) $A \in \mathcal{I}(G)$ if and only if $A/L \in \mathcal{I}(G/L)$.

d) $P \in Sp(G)$ if and only if $P/L \in Sp(G/L)$ for $P \in \mathcal{C}(G)$ with $L \subseteq P$.

e) $M \supseteq L \in \Gamma(g)$ if and only if $M/L \in \Gamma(g+L)$ in G/L for $g \in G \setminus L$.

It is clear that the intersection of a set of convex subgroups of a l-group is a convex subgroup and thus we have:

Proposition 2.1.7 (cf. [44], pp. 38-42) *If $S \subseteq G$ then there exists a minimal convex l-subgroup of G that contains S, which is denoted $G(S)$. For any $g \in G$, $G(g) = G(|g|) = \{h \in G \mid |h| \leq n|g| \text{ for some } n \in \mathbb{N}\}$ and is called a principal convex l-subgroup.*

Definition 2.1.8 *If $G(g) = G$ and $g > 0$, then g is called a strong unit.*

A po-group is said to be *Archimedean* if $na < b$ for all $n \in \mathbb{Z}$ implies $a \leq 0$. A po-group is called *completely integrally closed* if $na < b$ for all $n \in \mathbb{N}$ implies $a \leq 0$. Any completely integrally closed po-group is Archimedean but the converse implication does not hold in general, but it does in l-groups (cf. [55], p. 12)

Proposition 2.1.9 (cf. [16]) *An Archimedean totally ordered group is isomorphic to a subgroup of the additive group of the field of the reals. Any Archimedean directed group is commutative.*

Let A, B be l-groups, define the *componentwise sum* $A \oplus_c B$ of A and B to be the direct sum group and be an l-group such that $(a, b) \geq (0, 0)$ if and only if $a \geq 0$ and $b \geq 0$. In general, let $\{A_i \mid i \in I\}$ be a set of l-groups, the *cartesian product* $\Pi_{i \in I} A_i$ of l-groups A_i is the l-group which is the set of all functions $f,\ g:\ I \to \cup_{i \in I} A_i,\ f(i) \in A_i,\ (f+g)(i) = f(i) + g(i),\ f \leq g$ if and only if $f(i) \leq g(i)$ for all $i \in I$. The functions $f \in \Pi_{i \in I} A_i$ with finite support form an l-subgroup, the coproduct $\coprod_{i \in I} A_i$.

The following useful (see, for instance, the proof of Proposition 4.1.21) result is the Corollary 16.10 of [44].

Proposition 2.1.10 *Let G be an l-group. If A, B and $C \in \mathcal{C}(G)$ and $A \cap B = 0$, then $A \vee B = A \oplus_c B$ and $C \cap (A \vee B) \cong (C \cap A) \oplus_c (C \cap B)$, where $A \vee B$ is the minimal convex l-subgroup containing $A \cup B$.*

Definition 2.1.11 *Let G be an l-group, if there exists a set $\{N_i \mid i \in I\}$ of l-ideals in G such that $\cap_{i \in I} N_i = \{0\}$, then G is said to be a subdirect product of l-groups G/N_i, $i \in I$.*

Proposition 2.1.12 (cf. [4], p. 26) *An l-group is a subdirect product of l-groups $\{A_i\}_{i \in I}$ if there exists an injective l-group homomorphism f of G into $\Pi_{i \in I} A_i$ such that for each projection $p_\mu : \Pi_I A_i \to A_\mu$, $p_\mu f$ is surjective.*

An l-group is *representable* if it is l-isomorphic to a subdirect product of totally ordered groups. This concept is due to P. Lorenzen. Note that a representable l-group does not correspond to a *representable* MV-algebra which is defined in Chang [33] (see section 4.3).

Proposition 2.1.13 (cf. [4], Corollary 4.1.2) *Each Abelian l-group is representable.*

An element g of G is *special* if g has a unique value M, and in this case M is called a *special value*. We will use $\Gamma(G)$ to denote the set of regular subgroups of G. A subset S of $\Gamma(G)$ is *plenary* if $\cap S = \{0\}$, and $s \in S$, $t \in \Gamma(G)$ and $s \subseteq t$ implies that $t \in S$. G is a *special-valued* if the special values form a plenary subset of $\Gamma(G)$. G is *finite-valued* if every element has only a finite number of values. G is called *normal-valued* if each regular subgroup is a normal value. Note that a representable l-group is normal-valued (see [44], p. 303, Corollary 47.7). The following result shows that a finite-valued l-group is special-valued.

Proposition 2.1.14 (cf. [44], p. 297, Theorem 46.9) *For an l-group G the following statements are equivalent:*

1) G is finite-valued.

2) Every regular subgroup of G is special.

3) The complete lattice of convex l-subgroups of G is completely distributive.

Let G be an l-group. $0 < b \in G$ is *basic* if $\{g \in G \mid 0 \leq g \leq b\}$ is totally ordered. Two elements a and b are *disjoint* if $|a| \wedge |b| = 0$. A subset $\{s_\lambda\}$ is a basis of G if this set is a maximal disjoint set in G and each element is basic.

Proposition 2.1.15 (cf. [44], Propositions 27.3 and 27.4) *Every basic element is special and so every nonzero element of an totally ordered group is special. A special element is either positive or negative.*

2.2 Divisibility groups of Bézout domains

As was mentioned in the introduction, an Abelian l-group must be the divisibility group of some Bézout domain via Jaffard-Ohm's correspondence. In this section we present some basic relationships between divisibility groups and Bézout domains.

Let D be an integral domain with group of units U_D, quotient field K and as usual K^* the set of nonzero elements of K. Then the *group of divisibility* of D (denoted by $G(D)$) is the multiplicative group K^*/U_D. Furthermore, the partial order $\leq$ is defined on $G(D) = K^*/U_D$ by $aU_D \leq bU_D$ if and only if the relation "a divides b" (that is, $ba^{-1} \in D$) makes $G(D)$ into a partially ordered Abelian group with positive cone D^*/U_D, where $D^* = D \setminus \{0\}$.

A *semi-valuation* of a field K is a group epimorphism $\omega : K^* \to G$ into a po-group G such that $\omega(a+b) \in \mathcal{UL}\ \{\omega(a),\ \omega(b)\}$ for all $a,\ b \in K^*$, where $\mathcal{UL}\ \{\omega(a),\ \omega(b)\}$ denotes the set of upper bounds $\mathcal{U}$ of lower bounds $\mathcal{L}$ of $\{\omega(a),\ \omega(b)\}$. G is called the *semi-valuation group* of ω, and $R_\omega = \{x \in K^* \mid \omega(x) \geq 0\} \cup \{0\}$ is a subring of K which is called the *semi-valuation ring* of ω. A semi-valuation ω is said to be a *demi-valuation* if G is lattice-ordered, and $\omega(a+b) \geq \omega(a) \wedge \omega(b)$, the group G is called the *demi-value group*. A demi-valuation is called a *valuation* and G is called the value group if G is totally ordered. Note that $v(1) = v(-1) = 0$, $v(a^{-1}) = -v(a)$, and $v(-a) = v(a)$.

An important question is what po-groups G can occur as groups of divisibility. A necessary condition is that G is directed, but this condition is not sufficient as shown by several examples in this paper (see Chapter 7) and other examples in the literature (see [2, 8, 35, 46, 62, 71, 83, 86, 92]). On the contrary, some partial orders are sufficient for the existence of a domain D so that G is po-group isomorphic to $G(D)$. In fact, Krull [78] shown that any totally ordered Abelian group is the group of divisibility of a valuation ring. His result is a special case of the following more general Jaffard-Ohm theorem:

Theorem 2.2.1 (cf. [4, 59]) *An Abelian l-group is l-group isomorphic to a group of divisibility of a Bézout domain.*

Usually, the proof of Theorem 2.2.1 is given similarly to the one used by Krull as follows.

Let K be an arbitrary field and let $K[G]$ be the group algebra over K. Then $K[G]$ is an integral domain with identity since G is torsion-free. Let $F = K(G)$

be the quotient field of $K[G]$. Define $\omega : K[G] \to G$

$$\Sigma_{i=1}^{n} k_i g_i \mapsto \wedge_{i=1}^{n} g_i.$$

Extend ω to F by $\omega(f/g) = \omega(f) - \omega(g)$. Then ω is a semi-valuation on F. Moreover, if D is the semi-valuation ring of ω, then the group of divisibility of D is l-group isomorphic to G.

Note that the definition of $\omega : K[G] \to G$ is crucial in the proof, otherwise it is difficult to prove that ω is a semi-valuation. We remark that one can give a proof of Theorem 2.2.1 generalizing some exercises in Bourbaki [24].

The following theorem characterizes a completely integrally closed Bézout domain using its divisibility group.

Theorem 2.2.2 (cf. [4], Theorem 11.6) *A Bézout domain is completely integrally closed if and only if its group of divisibility is an Archimedean l-group.*

Proof. First suppose that D is completely integrally closed with $U = U_D$, that Ua, $Ub \in G(D)^{\geq 0}$, and that $n(Ua) \leq Ub$ for all $n \in \mathbb{N}$. Then $Ub + n(U(1/a) \geq 0$, which means that $b(1/a)^n \in D$ for all n. But $1/a \in D$, and so $Ua = 0$. Hence, $G(D)$ is Archimedean.

But then $Ua \geq nU(1/x)$ for all $n \in \mathbb{N}$, and so $U(1/x) \leq 0$, since $G(D)$ is Archimedean. Thus $Ux \geq 0$ and so $x \in D$, as required. □

Now we give the definition of the important concept of a *polar*.

Definition 2.2.3 *Let A be a subset of an l-group G. $A^{\perp} = \{g \in G \mid |g| \wedge |a| = 0$ for all $a \in A\}$ is called the polar of A. We call a convex l-subgroup C satisfying the equation $C = C^{\perp\perp}$ a polar subgroup of G and denote the collection of such by $\mathcal{P}(G)$. An element $g \in G$ is a weak unit if $\{g\}^{\perp} = \{0\}$.*

Proposition 2.2.4 (cf. [4, 16, 44]) *Let G be an l-group, A be a convex l-subgroup of G and $g \in G$. Then $g^{+} \in \{g^{-}\}^{\perp}$ and $A^{\perp}$ is a convex l-subgroup of G. $P \leq G$ is a minimal prime if and only if $P = \cup\{g^{\perp} \mid g \in G \setminus P\}$. Furthermore, each polar is an intersection of minimal primes and $\mathcal{P}(G)$ is a Boolean algebra. A strong unit is a weak unit.*

Consequently, we have

Corollary 2.2.5 *Let G be an l-group. Then 0 is prime if and only if G is totally ordered.*

The following theorem characterizes the elements that are weak units of a divisibility group by means of the Jacobson radical of its integral domain.

Theorem 2.2.6 *Let l-group $G(D)$ be the divisibility group of a Bézout domain D. Then $G(D)$ has a weak unit if and only if the Jacobson radical of D is non zero. In particular, if $G(D)$ has a strong unit, then $J(D) \neq \{0\}$, where $J(D)$ denotes the Jacobson radical of D.*

Proof. A nonzero element wD of the divisibility group $G(D)$ is a weak unit if and only if the gcd $(w, d) \notin U(D)$ for all $d \notin U(D)$. Since D is Bézout, this is equivalent to say that w is an nonzero element of the intersection of all maximal ideals of D, that is, $J(D) \setminus \{0\} = \{w \mid wD \text{ is a weak unit of } G(D)\}$. Furthermore, a strong unit is a weak unit by Proposition 2.2.4 above. □

Now we shall give a ring theoretic characterization for strong units of an Abelian l-group. Let R be an integral domain. Define the *unit radical*

$$u(R) = \cap\{P \lhd R \mid 0 \neq P \text{ prime}\},$$

where $P \lhd R$ means that P is an ideal of R. Then we have:

Theorem 2.2.7 *Let R be an integral domain, and $G(R)$ be its divisibility group. Then $G(R)$ has a strong unit if and only if $u(R) \neq 0$.*

Proof. "$\Rightarrow$:" If $u \in R$ is a strong unit and $0 \neq P$ is a prime ideal of R. Then for all $a \in P \setminus \{0\}$, there exists $n \in \mathbb{N}$ such that $u^n \in (a) \subseteq P$, which implies $u \in P$ since P is prime, and thus $u \in u(R)$.

"$\Leftarrow$:" Assume that $u \in u(R) \setminus \{0\}$. If u is not a strong unit, then there exists $0 \neq a \in R$ such that $(u^n) \nsubseteq (a)$ for all $n \in \mathbb{N}$. By Zorn's Lemma there exists $P \lhd R$ maximal with respect to $a \in P$ and

$$\{u^n \mid n \in \mathbb{N}\} \cap P = \varnothing.$$

This implies that P is prime. In fact, assume that there are two elements $x, y \in R \setminus P$ with $xy \in P$. Then

$$< P, x >_R \cap\{u^n \mid n \in \mathbb{N}\} \neq \varnothing$$

and

$$< P, y >_R \cap\{u^n \mid n \in \mathbb{N}\} \neq \varnothing,$$

where $< P, y >_R$ denotes the ideal of R generated by P and y. So there are natural numbers $m, n, k, l \in \mathbb{N}$ with

$$u^m = \Sigma_{i=1}^k a_i x^i + p_1$$

and

$$u^n = \Sigma_{j=1}^l b_j y^j + p_2,$$

where $a_i, b_j \in R$ $(i = 1, \cdots, k;\ j = 1, \cdots, l)$ and $p_1, p_2 \in P$. Hence we get $u^{n+m} = p \in P$. This contradicts the fact that $\{u^n \mid n \in \mathbb{N}\} \cap P = \varnothing$. Hence P is a prime ideal. But $u \notin P$, which is a contradiction. $\square$

Recall that an integral domain D is a *pseudo-Bézout domain* (i. e. a GCD domain) if any two elements in D have a greatest common divisor, and a *Bézout domain* is a pseudo-Bézout domain in which all finitely generated ideals are principal, that is, any greatest common divisor of two elements is a linear combination of the two elements.

The following correspondence theorem between overrings of a Bézout domain and the l-group homomorphic images of its group of divisibility is the crucial piece of machinery necessary to relate the theory of Abelian l-groups and Bézout domains. To describe that theorem we recall some terminologies of ring theory.

Let D be a domain and K its quotient field. Recall that a subring of K containing D is called an overring of D. A subset S of $D^* := D \setminus \{0\}$ is called a *multiplicative system* if S is closed under multiplication, and if a divides b and $b \in S$, then $a \in S$. The *localization* of D at S is then $D_S = \{d/s \in K \mid d \in D, s \in S\}$. For a Bézout domain all overrings are actually localizations (see [59]), that is, all overrings occur as D_S for a (uniquely determined) multiplicative system S. A case of particular interest is when the multiplicative system $S = D \setminus I$, where I is an ideal; this occurs precisely when the ideal is prime. We now state the correspondence theorem:

Theorem 2.2.8 (cf. [3, 4]) *For a Bézout domain D with unit group U, there are one-to-one correspondences between these sets:*

1) overrings of D;

2) multiplicative systems of D;

3) convex l-subgroups of $G(D)$;

4) l-group homomorphic images of $G(D)$.

The correspondences among 1), 2) and 3) are order-preserving, while the correspondence between 3) and 4) is order-reversing. Furthermore, $G(D_S) \cong_{l-g}$

$G(D)/H$, where $S = \{d \in D \mid Ud \in H\}$. Finally, the prime subgroups of $G(D)$ order-reversingly correspond to the prime ideals of D.

The correspondence between 1) and 2) is clearly order-preserving as described above (note that there is a typo in [3] typing order-preserving as "order-reversing"), while the correspondence between 3) and 4) is the order-reserving one between l-group homomorphic images and the kernels. It is straightforward to show that the statement at the end of the theorem describes the order-reversing correspondence between 2) and 4) (note there is another typo in [3] mixing order-reversing with "order-preserving"). In case the multiplicative system is the complement of a (necessarily) prime ideal, localization gives a valuation domain, and thus there exists an order-reserving correspondence between the prime ideals of the Bézout domain and the prime subgroups of the l-group.

As an application of Theorem 2.2.8, we get the following:

Corollary 2.2.9 (cf. [4], Corollary 11.5) *Any Bézout domain D is an intersection of overrings which are valuation domains.*

It is well known that a valuation ring of rank 1 is completely integrally closed. A conjecture of Krull [78] was that a completely integrally closed domain should be the intersection of rank 1 valuation overrings in its quotient field. Note that this conjecture translates into saying that an Archimedean l-group is a subdirect product of reals. By Nakayama [90], however, this is false.

The following concept will be used in Section 2.3.

Definition 2.2.10 *If G and H are two po-groups, then the antilexicographic sum $G \oplus_l H$ of G and H is defined by the positive cone $(G \oplus_l H)^+ = \{(a,\ b) \mid b > 0 \ or (b = 0 \text{ and } a \geq 0)\}$.*

Recall that a homomorphism f between two partially ordered sets P and Q is a function which preserves order: $x \geq y$ in $P \Rightarrow f(x) \geq f(y)$ in Q. An order-isomorphism is a one-one correspondence which preserves order. A biorder-isomorphism is a one-one correspondence which preserves both orders: $x \geq y$ in $P \Leftrightarrow f(x) \geq f(y)$ in Q. A homomorphism f between two semilattices S and T is a function which preserves join (or meet): $f(x \vee y) = f(x) \vee f(y)$ (*or* $f(x \wedge y) = f(x) \wedge f(y)$). A semilattice-isomorphism is an one-one correspondence which preserves join (or meet).

Proposition 2.2.11 *Let G and H be semilattices and $f : G \longrightarrow H$ be a bijection. The following conditions are equivalent:*

1) f is an semilattice isomorphism.

2) f^{-1} is an semilattice isomorphism.

3) Both f and f^{-1} are order-isomorphisms.

4) f is a biorder-isomorphism.

5) f^{-1} is a biorder-isomorphism.

Proof. Without loss of generality, let f be a semilattice isomorphism which preserves join.

1) $\Leftrightarrow$ 2): Let x, y be elements of H. Then $f(f^{-1}(x) \vee f^{-1}(y)) = x \vee y$. Hence $f^{-1}(x) \vee f^{-1}(y) = f^{-1}(x \vee y)$.

1) $\Leftrightarrow$ 4): $x \geq y \Rightarrow f(x \vee y) = f(x) = f(x) \vee f(y) \Rightarrow f(x) \geq f(y)$. On the other hand, $x \geq y \Rightarrow f^{-1}(x \vee y) = f^{-1}(x) = f^{-1}(x) \vee f^{-1}(y) \Rightarrow f^{-1}(x) \geq f^{-1}(y)$. Conversely, Let x, y be elements of G, then $f(x) \vee f(y) \leq f(x \vee y)$ and $f(x \vee y) = f(f^{-1}(f(x \vee y))) \leq f(f^{-1}(f(f^{-1}(f(x) \vee f(y))))) = f(x) \vee f(y)$.

The rest of the proof is sufficient to check that 4)$\Rightarrow$ 5). Let f be a biorder-isomorphism and $a, b \in H$. Then $f(f^{-1}(a) \vee f(f^{-1}(b)) = a \vee b = f(f^{-1}(a \vee b))$ implies $f^{-1}(a \vee b) \geq f^{-1}(a) \vee (f^{-1}(b)$. So f^{-1} is an order-isomorphism for $a \leq b \Rightarrow f^{-1}(b) = f^{-1}(a \vee b) \geq f^{-1}(a) \vee f^{-1}(b) \Rightarrow f^{-1}(a) \leq f^{-1}(b)$. On the other hand, $f^{-1}(a) \leq f^{-1}(b) \Rightarrow a = f(f^{-1}(a)) \leq f((f^{-1}(b)) = b$. Therefore, f^{-1} is also a biorder-isomorphism. $\square$

The following example shows that one can not weaken the hypothesis of Proposition 2.2.11 from a semilattice isomorphism to an order-isomorphism.

Example 2.2.12 *Let $G = \mathbb{Z}$ be the set of integer numbers with the order $a \leq b \Leftrightarrow ((b$ is even and a is less than b under the usual order) or $(a = b))$. Then G is a join semilattice. Define $f : G \to G$ by letting*

$$f(a) = \begin{cases} a & : \quad a = 2k+1 \in \mathbb{Z} \\ a+2 & : \quad \textit{otherwise} \end{cases}$$

Then it is easily to verify that f is a bijection of G onto itself, and it clearly preserves order. So f is an order-automorphism of semilattice G. But $f(1 \vee 3) = f(4) = 6$, while $f(1) \vee f(3) = 1 \vee 3 = 4$. Thus f is not a semilattice isomorphism.

2.3 Examples

In this section we provide a list of examples of l-groups.

Example 2.3.1 *The additive groups of integers $\mathbb{Z}$, rationals $\mathbb{Q}$, real numbers $\mathbb{R}$ are all Archimedean totally ordered groups under the usual order, and thus they are divisibility groups of some completely integrally closed valuation domains.*

Example 2.3.2 *Every torsion-free Abelian group G can be embedded (as a group) into a rational vector space V (cf. [54, 61]). Assume that $B := \{b_i\}_{i\in I}$ is a basis of V. Let $v = \Sigma_{i\in I_0} s_i b_i, w = \Sigma_{i\in I_0} t_i b_i$, where I_0 is a finite subset of I and $s_i, t_i \in \mathbb{Q}$. Totally order the basis B. Define $v < w$ if and only if $s_j < t_j$ (as rational numbers) where j is the greatest $i \in I_0$ such that $s_i \neq t_i$. Then both V and G are totally ordered Abelian l-groups.*

The l-groups of the following examples are not totally ordered.

Example 2.3.3 *Let G be any Abelian group equipped with the trivial order (that is, $f \leq g$ if and only if $f = g$) and H be any totally ordered Abelian group. Let $L = G \oplus H$. Define a partial order on L by: $(g_1, h_1) \leq (g_2, h_2)$ if and only if $h_1 < h_2$. Then L is a directed group. In the particular case that $G = C_2$ of 2 elements and H is the infinite cyclic group order-isomorphic to $\mathbb{Z}$, we get the so-called "infinite whalebone corset". It is a directed group that is not a divisibility group (see also Example 7.1.3 in Chapter 6 below).*

Example 2.3.4 *let X be an arbitrary topological space and $C(X)$ be the additive group of all continuous functions from X into $\mathbb{R}$ (where $\mathbb{R}$ is equipped with the usual topology); i. e. $(f + g)(x) = f(x) + g(x)$ for all $x \in X$. Then $C(X)$, under the pointwise order (i. e. $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$), is an Archimedean l-group.*

A related example is:

Example 2.3.5 *Let X be an extremally disconnected topological space (that is, a Hausdorff space in which the closure of every open set is open), and $D(X)$ be the set of all continuous functions f from X into $\mathbb{R} \cup \{\pm\infty\}$ such that $\{x \in X \mid f(x) \notin \mathbb{R}\}$ is nowhere dense. Then $D(X)$, under addition of functions and pointwise ordering, is an Abelian l-group, and $C(X)$ (see Example 2.3.4) is an l-ideal of $D(X)$.*

The following example shows that a subgroup of an l-group is a po-group but need not be an l-subgroup:

Example 2.3.6 *Let D be the group of all differentiable functions from $\mathbb{R}$ into $\mathbb{R}$. Then D is a subgroup of $C(\mathbb{R})$ (see Example 2.3.4). It is a directed group under the inherited pointwise order. However, it is not a sublattice of $C(\mathbb{R})$ because the supremum of the zero function and the differentiable function $y = x$ is not differentiable at $x = 0$.*

Example 2.3.7 *Let X be a μ-measurable space. Then the real Lebesgue space $L^p(X, \mu)$ is an Abelian l-group for all $1 \le p \le \infty$. (Note that L^p $(1 \le p \le \infty)$ is the class of measurable functions f for which $|f|^p$ is integrable; for $p = \infty$ the class is defined as the class of essentially bounded measurable functions, that is, those functions f for which there is a constant c such that $|f| \le c$ almost everywhere.) Similarly, most of the function spaces are l-groups (see [23, 77, 99]).*

Example 2.3.8 *Let Ω be a totally ordered set, and $Aut(\Omega, \le)$ be the group of all permutations of Ω that preserve the order. Then $Aut(\Omega, \le)$ is an l-group.*

To any l-group, a strong unit can be adjoined as follows.

Example 2.3.9 *Let G be an l-group. Then $G \oplus_l \mathbb{Z}$ is an l-group with strong unit* $(0, 1)$.

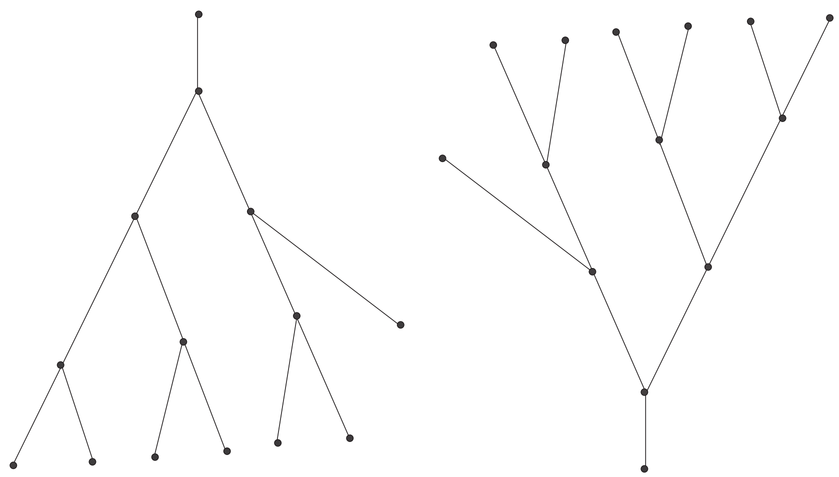


Figure 2.1: Root and tree systems

A partially ordered set Δ is called a *root system* if for each $\delta \in \Delta$, $\{\alpha \in \Delta \mid \alpha \ge \delta\}$ is totally ordered, or equivalently, noncomparable elements do not have a lower bound. For example, the left diagram of Figure 2.1 is a picture of a root system. The dual of a root system is a *tree system* (see the right diagram of Figure 2.1). The prime ideals of a Bézout domain form a tree. By Theorem 2.2.8, this corresponds to the fact that the prime l-ideals of an Abelian l-group form a root system.

The next example is known as *the full Hahn group on spine* Γ, which shows that any root system induces a spectral root system (for the definition of "spectral", see Section 4.6).

Example 2.3.10 *If Γ is an arbitrary root system, let $V = V(\Gamma,\ \mathbb{R})$ be the set of all functions g from Γ into $\mathbb{R}$ such that supp$(g) = \{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ is either empty or every totally ordered subset of supp(g) has a maximal element. Then V is an abelian l-group where the group operation is addition of functions, and $g > 0$ if $g(\delta) > 0$ for every maximal element δ of supp(g).*

Chapter 3

Applications of valuation theory

3.1 A characterization of l-ideals in l-groups

Since P. Conrad's fundamental contribution to the structure theory of Abelian l-groups, l-ideals play a central role in most investigations on l-groups. So it is useful to have different characterizations of this important notion. For the special case of o-ideals (that is, convex l-subspaces) in Riesz spaces, Alpay, Emel'yanov and Ercan [1] give a characterization (see the equivalence of (i), (iii) and (iv) in Corollary 3.1.2 below) without use of inequalities. Recall that a *Riesz space* is a lattice-ordered real linear space L as an Abelian group such that $pL^{\geq 0} \subseteq L^{\geq 0}$ for all positive real numbers p. We will show that their result can be generalized to characterize l-ideals in arbitrary l-groups (see Theorem 3.1.1 below).

Recall that a subgroup S of an l-group G is 2-*isolated*, if $2x \in S$ implies $x \in S$ for all $x \in G^{>0}$. Actually, an l-subgroup S of an l-group G is 2-isolated if and only if $2x \in S$ implies $x \in S$ for all $x \in G$.

It is clear that an l-ideal of an l-group is isolated (equivalently, semi-closed) and thus 2-isolated, but a 2-isolated subgroup may be not convex.

The following characterization of an l-ideal in an arbitrary l-group will be proved after Theorem 3.1.3 in this section.

Theorem 3.1.1 *Let N be a normal 2-isolated subgroup of an l-group G. Then the following statements are equivalent:*

a) N is an l-ideal.

b) $|a+b| - |a| - |b| \in N$ for all $a \in N$ and $b \in G$.

c) $|a+b| - |b| \in N$ for all $a \in N$ and $b \in G$.

d) $|a+b| - |a-b| \in N$ for all $a \in N$ and $b \in G$.

Note that Theorem 3.1.1 applies to other lattice-ordered algebraic systems. Especially, the following corollary holds since a vector subspace is 2-isolated and normal.

Corollary 3.1.2 *Let F be a vector subspace of a Riesz space E. Then the following statements are equivalent:*
(i) F is an order ideal.
(ii) $|a+b|-(|a|+|b|) \in F$ for all $a \in F$ and $b \in E$.
(iii) $|a+b|-|b| \in F$ for all $a \in F$ and $b \in E$.
(iv) $|a+b|-|a-b| \in F$ for all $a \in F$ and $b \in E$.

The equivalence of (i), (iii) and (iv) in Corollary 3.1.2 is due to Alpay, Emel'yanov and Ercan [1]. For an Abelian l-group G, Theorem 3.1.1 is easy to prove. Actually, Theorem 3.1.3 below is a generalization of Theorem 2 of [1]. Note that the condition of 2-isolation is necessary. For instance, $H := 2\mathbb{Z}$ is a non-ideal l-subgroup of $\mathbb{Z}$ with respect to the usual order, furthermore, while H satisfies the condition 2)[1] of Theorem 3.1.3 below.

Theorem 3.1.3 *Let N be a 2-isolated subgroup of an Abelian l-group G, and let a and b be an arbitrary element in N and G, respectively. Then the following statements are equivalent:*
1) N is an l-ideal.
2) $|a+b|-|b| \in N$.
3) $|a+b|-|b|-|a| \in N$.
4) $|a+b|-|a-b| \in N$.

Proof. 1)⇒2): By the Busulini-Kalman inequality (see Theorem 2.1.1, 5)) in Abelian l-groups we have $||a+b|-|b|| = ||a+b|-|-b|| \leq |a|$ for all $a \in N$ and $b \in G$, which implies 2).

2)⇒3): Let $a \in N$ and $b \in G$. Then $|a+b|-|b|-|a| = (|a+b|-|b|)-|a| \in N$.

3)⇒4): Let $a \in N$ and $b \in G$. Then $|a+b|-|a-b| = (|a+b|-|b|-|a|) - (|a+(-b)|-|-b|-|a|) \in N$.

4)⇒1): Let $a \in N$. Then by Theorem 2.1.1 we have $|a+a|-|a-a| = |2a| = 2|a| \in N$ which implies that $|a| \in N$ since N is 2-isolated, and thus $|a|+a = 2a \vee 0 = (2a)^+ = 2(a^+) \in N$ and hence $a^+ \in N$. This implies that N is an l-subgroup. Let $0 \leq a \leq b \in N$ with $a \in G$, then we have $|b+a|-|b-a| = 2a \in N$ which implies that $a \in N$ since N is 2-isolated. □

In order to prove Theorem 3.1.1 in general, we need the following

[1]Which is a bicondition for an ideal of an MV-algebra (cf. [52], Lemma 2.2.19).

Lemma 3.1.4 *Let G be an l-group, and let a and b be arbitrary elements in G. Then $-|a|+|b|-|a| \leq |a+b| \leq |a|+|b|+|a|$.*

Proof. By Theorem 2.1.1 it suffices to prove $-|a|+|b|-|a| \leq |a+b|$ for all $a,\ b \in G$. Let $c = a+b$ and $b = -a+c$ in the inequality $|a+b| \leq |a|+|b|+|a|$ we obtain $|c| \leq |a|+|-a+c|+|a|$, and it follows that $|a+b| = |-(-a)+b| \geq -|a|+|b|-|a|$. □

Now we give the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. a)⇒b): By Lemma 3.1.4 we have

$$(-|a|+|b|-|a|)-(|b|+|a|) \leq |a+b|-(|b|+|a|) \leq |a|$$

for all $a \in N$ and $b \in G$. Then $|a+b|-(|b|+|a|) \in N$ since N is a convex l-subgroup and

$$|a+b|-(|b|+|a|) \geq (-|a|+|b|-|a|)-(|b|+|a|) = -|a|+(|b|-2|a|-|b|) \in N,$$

which follows from the fact that N is normal.

b)⇒c): Let $a \in N$ and $b \in G$. Then $|a+b|-|b| = (|a+b|-(|b|+|a|)) + (|b|+|a|-|b|) \in N$ since N is a normal subgroup.

c)⇒d): Let $a \in N$ and $b \in G$. Then $|a+b|-|a-b| = (|a+b|-|b|)-(|a+(-b)|-|-b|) \in N$.

d)⇒a): Let $a \in N$. Then we have $|a+a|-|a-a| = 2|a| \in N$ which implies that $|a| \in N$ since N is 2-isolated, and thus $|a|+a = 2(a^+) \in N$ and hence $a^+ \in N$. This implies that N is an l-subgroup. Let $0 \leq x \leq b \in N$ with $x \in G$, then we have $|b+x|-|b-x| = b+(2x)-b \in N$ which implies $x \in N$ since N is 2-isolated. □

3.2 Lattice-ordered skew-fields without negative squares

Recall that a ring R whose additive group is an l-group and in which $a \geq 0$ and $b \geq 0$ implies $ab \geq 0$ is called an *l-ring*. If R is a division ring (resp. a field), then R is called a *lattice-ordered division ring* (resp. an *l-field*).

In [5], Artin and Schreier observed that a totally ordered commutative field cannot have negative squares, and Johnson [73] and Fuchs [56] extended this result to totally ordered domains with unit element. In [100], Schwartz showed that an Archimedean lattice-ordered (commutative) field that has $1 > 0$ and that is algebraic over its maximal totally ordered subfield cannot have negative squares, and in [47], DeMarr and Steger showed that in a partially ordered finite dimensional real linear algebra no square can be the negative of a strong unit. Furthermore, in [104] (see Section 3.3 below), we prove that there exist directed commutative fields with negative squares. In 1969, Conrad and Dauns [41] raised the following problem (this is Question (b) of their list in [41], p. 397]).

Question 3.2.1 *Is it true that a lattice-ordered skew-field F in which each square is positive must be totally ordered?*

In case F is commutative, an affirmative answer can be found in Bourbaki [25] (Chapitre VI, p. 43) or in Birkhoff and Pierce [18] (p. 59). The same statement was proved in 1975 by Redfield (see [94], p. 124).

In this section, we give a positive answer to the general problem of Conrad and Dauns (see our forthcoming paper [105]). As a consequence, the "lattice-ordered skew-fields" in Brumfiel [29] are in fact totally ordered since each square of an element is positive according to his definition (see [29], p. 32). Furthermore, we note that every lattice order determined by a *pre-positive cone* P on a skew-field F is linearly ordered since $F^2 \subseteq P$ (see Prestel [93]).

Lemma 3.2.2 *If R is an l-ring in which each strictly positive element of R is invertible and in which $a^2 \geq 0$ for every a in R, then $a^+|a|^{-1}a^- = 0$ holds for all a in $R \setminus \{0\}$.*

Proof. Let $a \in R$, $a \neq 0$, and define $b = a^+|a|^{-1}a^+$. By hypothesis, $(|a|^{-1})^2|a| = |a|^{-1} > 0$ and so

$$b \geq 0.$$

From $a = a^+ + a^-,\ |a| = a^+ - a^-$ we obtain that

$$b - a = a^+|a|^{-1}a^+ - (a^+ - a^-) = (-a^-)|a|^{-1}(-a^-) \geq 0.$$

That is, $b \geq a$, and hence

$$b \geq a^+.$$

On the other hand, we have

$$a^+ - b = a^+ - a^+|a|^{-1}a^+ = a^+|a|^{-1}a^- \geq 0.$$

This means that

$$a^+ \geq b,$$

and thus it shows that

$$b = a^+.$$

Consequently, the last inequality implies

$$a^+|a|^{-1}a^- = 0. \qquad \square$$

The following theorem characterizes lattice-ordered rings which are totally ordered skew-fields and provides a solution to Conrad and Dauns' problem.

Theorem 3.2.3 *If R is an l-ring, then the following statements are equivalent:*

(a) each strictly positive element in R has a multiplicative inverse and $a^2 \geq 0$ for every a in R;

(b) R is a totally ordered skew-field.

Proof. It suffices to prove (a)$\Rightarrow$(b): Lemma 3.2.2 above shows that $a^+|a|^{-1}a^- = 0$ holds for each nonzero element a in R. Hence $a^+ = 0$ or $a^- = 0$ (i. e. R is totally ordered). Hence R is a skew-field. $\square$

For a lattice-ordered skew-field, we extract from Theorem 3.2.3 the following information:

Corollary 3.2.4 *Let R be a lattice-ordered division ring. Then the following statements are equivalent:*

(α) $a^2 \geq 0$ for all $a \in R$.

(β) R is order division-closed (that is, for all $a,\ b \in R$, $ab > 0$ and one of $a,\ b$ is > 0, then so is the other).

(γ) R is totally ordered.

(δ) if $a \in R$, then there exists $n_a \in \mathbb{N}$ such that $a^{n_a} \geq 0$.

(ϵ) R is an f-ring (that is, $a \wedge b = 0$ and $c \geq 0$ implies $ca \wedge b = ac \wedge b = 0$).

(ζ) R is an almost f-ring (that is, $a \wedge b = 0$ implies $ab = 0$).

(λ) the additive group of R is the group of divisibility of a valuation domain.

Proof. (γ)$\Leftrightarrow$ (α) is clear. (γ)$\Leftrightarrow$ (λ) follows from the well known theorem of Krull. For the rest of the proof, it suffices to prove ζ) $\Leftrightarrow$ ϵ), but this is a directed corollary of Lemma 1 (that is, in any f-ring, $a \wedge b = 0$ implies $ab = 0$) on p. 404 of [16] and the definitions of almost f-rings and division rings. $\square$

Especially, for a lattice-ordered (commutative) field, we recover [18, 25, 94]:

Corollary 3.2.5 (Bourbaki, Birkhoff and Pierce, Redfield) *If R is a lattice-ordered (commutative) field in which every square is positive, then R is totally ordered.*

Using Proposition 4.1.17 below, we prove that the real field $\mathbb{R}$ admits a unique lattice-order such that each square is positive, while there are $2^{2^{\aleph_0}}$ Archimedean lattice orders for $\mathbb{R}$. In particular, there is only one way to make $\mathbb{R}$ into a linearly ordered field.

Corollary 3.2.6 *The field $\mathbb{R}$ of reals has a unique lattice-order such that each square is positive.*

Proof. By Corollary 3.2.4 we know that the order must be a total order, and thus each positive element of the usual order on $\mathbb{R}$ is positive in that order since it is a square of some element of $\mathbb{R}$. This implies that any strictly positive element $r \in \mathbb{R}$ is a strong unit, whence the order is the same Archimedean total order as the usual one on $\mathbb{R}$ by Proposition 4.1.17 below. □

Moreover, it is easily seen that the ring of integers and the field of rationals have a unique lattice-order. The question of whether $\mathbb{R}$ can be made into an l-field in any non-standard way was raised by Birkhoff and Pierce [18]. This problem was solved by Wilson [103], who produced uncountably many distinct lattice orders on $\mathbb{R}$. We note that $\mathbb{R}$ can be given $2^{2^{\aleph_0}}$ Archimedean lattice orders, and hence its additive group can be considered as the divisibility group of some completely integrally closed Bézout domains that are not valuation domains.

Corollary 3.2.7 *The field $\mathbb{R}$ of the reals admits exactly $2^{2^{\aleph_0}}$ Archimedean lattice orders, and the field $\mathbb{A}$ of the real algebraic numbers admits exactly $2^{\aleph_0}$ Archimedean lattice orders. Furthermore, all those Archimedean lattice orders constructed in [103] can be extended to the unique Archimedean total order on $\mathbb{R}$ or $\mathbb{A}$, respectively.*

Proof. Using [103, 100], this follows from Corollaries 3.2.4, 3.2.6 and the Archimedean property of the usual order on $\mathbb{R}$. □

Note also that the additive group of $\mathbb{R}$ can be non-Archimedean totally ordered, as a vector space over the field of the rationals. So it serves as the divisibility group of some valuation domain which is not completely integrally closed.

3.3 Directed rings and algebras with $i^2 < 0$

The possibility of obtaining directed linear algebras with negative square was first systematically studied by DeMarr and Steger [47]. They proved that there are infinitely many finite dimensional *real* algebras that do not admit a directed order. This generalized the well known theorem of Birkhoff and Pierce [18] which states that the field of complex numbers cannot be *lattice*−ordered regarded as a 2-dimensional real linear algebra. Since then no further general result on directed algebras with negative squares seems to have been appeared in the literature. Especially, the following seems to be unknown:

Question 3.3.1 *Do there exist directed rings or algebras with negative squares?*

This question is related to the well known open problem of Birkhoff and Pierce [18] (see also [94, 100]) whether there exists a partial order on $\mathbb{C}$ with respect to which $\mathbb{C}$ becomes an l-field. Actually, the nonexistence of directed algebras with negative squares would imply a negative answer to the problem of Birkhoff and Pierce.

Using valuations, we prove in [104] the existence of directed rings and algebras that have negative squares. In this section we will generalize the method used in [104] and proceed with the study of directed orders on $K(i)$ with $i^2 = -1$ in some detail for a linearly ordered field K. First we exhibit a particular class of l-group structures on $K(i)$ and show that they can be parameterized by positive elements of K (see Theorem 3.3.4). By the same method, we establish a correspondence between directed orderings of the field $K(i)$ and absolutely convex subsets $V \subsetneq K$ which satisfy the implication (see Theorem 3.3.15)

$$0 < a \notin V \Rightarrow \frac{1}{2}(a - a^{-1}) \notin V.$$

Then we characterize these $V \subsetneq K$ as convex additive subgroups with the property that for $a \in K^*$, either a or a^{-1} belongs to V (Theorem 3.3.19). Note that the later property holds for any valuation domain with quotient field K.

Let K be a linearly ordered field and $i^2 = -1$. A subset V of K will be called a *segment* of K, if V satisfies

(1) $0 \in V \subset K$, and

(2) $|a| \leq |b|,\ b \in V$ implies $a \in V$.

For any segment $V \subset K$, define

$$P_V = \{a + bi \in K(i) \mid a, b \geq 0; b > 0 \Rightarrow (a > 0, ab^{-1} \notin V)\}.$$

Lemma 3.3.2 *The P_V defined above satisfies:*

(a) $P_V \cap (-P_V) = \{0\}$*;*
(b) $P_V + P_V \subseteq P_V$*;*
(c) $P_V - P_V = K(i)$*.*

Proof. (a) Trivial.

(b) Assume that $a + bi, c + di \in P_V$. Then $a, b, c, d \geq 0$, without loss of generality, suppose that $b > 0$, which implies $a > 0$ and $a/b \notin V$, so $a + c > 0$. If $d = 0$, then $\frac{a+c}{b+d} \geq \frac{a}{b} \notin V$, which implies $\frac{a+c}{b+d} \notin V$. There is no loss of generality in assuming that $\frac{a}{b} \leq \frac{c}{d}$. It follows that $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$, and thus $\frac{a+c}{b+d} \notin V$.

(c) For all $a + bi \in K(i)$. Choose $y > 0$ and $y > b$. There exists $z > 0$ with $z \in K \setminus V$ since $V \neq K$. Choose $x > 0$, $x > a$, $x \geq yz$ and $x \geq a + (y - b)z$ since K is totally ordered. Then $x - a, y - b > 0$; $\frac{x}{y} \geq z$ and $\frac{x-a}{y-b} \geq z$, hence $\frac{x}{y}, \frac{x-a}{y-b} \notin V$. □

Theorem 3.3.3 *$V \mapsto P_V$ defines a bijection between segments $V \subset K$ and partial orders $\leq$ on $K(i)$ such that $K^{\geq 0} \subseteq K(i)^{\geq 0}$ and $K(i)$ is a directed additive group with*
(1) $0 \not\leq bi \not\leq a$ for all $a \in K$ and $0 < b \in K$;
(2) there exists $a > 0$ in K with $-a < i$.

Proof. By Lemma 3.3.2 the positive cone P_V defines a directed order on $K(i)$ via $\alpha \leq \beta \Leftrightarrow \beta - \alpha \in P_V$ for $\alpha, \beta \in K(i)$. Suppose that $bi \leq a \in K$ for some $0 < b \in K$ and $a \geq 0$, then $a - bi \in P_V$, a contradiction, which implies (1). $V \neq K$ implies that there is an $a \in K^{\geq 0} \setminus V$, so $a + i \in P_V$ which implies (2).

Conversely, if $(K(i), +, \leq)$ is a directed group with the properties above. Define
$$V := \{ab^{-1} \in K \mid |a| + |b|i \ngeq 0\}.$$
By (2) there exists $0 < a \in K$ with $a + i > 0$, which implies $V \neq K$. The condition $i \ngeq 0$ implies $0 \in V$. Furthermore, $|a| \leq |b|$ and $b \in V$ implies $|b| + i \ngeq 0$ and $a \in V$ since $|b| + i \geq |a| + i \geq 0$ is impossible. So V is a segment. We next to show that the positive cone of $K(i)$ is P_V, that is,
$$a + bi \geq 0 \Leftrightarrow (a, b \geq 0; b > 0 \Rightarrow a > 0, ab^{-1} \notin V).$$

"$\Rightarrow$": Assume $b < 0$. Then $-bi \leq a$, which is a contradiction and thus $b \geq 0$. Suppose $a < 0$. Then $b > 0$, so $bi \geq -a > 0$, which contradicts (1), and hence $a \geq 0$. If $b > 0$, then $a > 0$, and $ab^{-1} \notin V$.

"$\Leftarrow$": For $b = 0$ and $a \geq 0$, we get $a + bi = a \geq 0$ since $K^{\geq 0} \subseteq K(i)^{\geq 0}$. For $b > 0$, we have $a > 0$ and $ab^{-1} \notin V$ which implies $|a| + |b|i = a + bi \geq 0$.

Finally, it is trivial that $V = \{ab^{-1} \in K \mid |a| + |b|i \notin P_V\}$. □

We call a segment V *rational* if it has the form $V_\alpha := \{a \in K \mid |a| < \alpha\}$, where $\alpha \in K^{>0}$.

Theorem 3.3.4 *The order in Theorem 3.3.3 makes $K(i)$ into an l-group if and only if V is rational.*

Proof. Assume that $V = V_\alpha$ with $0 < \alpha \in K$. To prove $0 \vee (a + bi) \in K(i)$ for all $a + bi \in K(i)$. For $b = 0$ it is trivial.

Case I. $b > 0$: Then $0 \vee (a + bi) = c + bi$ with $c = \max\{a, b\alpha\}$.

Case II. $b < 0$: Then $0 \vee (a + bi) = max\{0, a - b\alpha\}$.

Conversely, suppose that $K(i)$ is an l-group, but V is not rational. Let $0 \vee i = a + bi$. Then $b \geq 1$, $a > 0$ and $ab^{-1} \notin V$. Thus there exists $\alpha \notin V$ with $0 < \alpha < ab^{-1}$. However, $b\alpha + bi \geq 0$ and $b\alpha + bi \geq i$ are plain for $b = 1$. Furthermore, if $b > 1$, then $\alpha \leq \frac{b\alpha}{b-1} \notin V$ which implies $b\alpha + bi \geq i$. Therefore, $a + bi - (b\alpha + bi) = a - b\alpha > 0$, a contradiction. □

Lemma 3.3.5 *A segment $V \subset K$ defines a partial order such that $K^{\geq 0} \subseteq K(i)^{\geq 0}$ and $K(i)$ is a directed field if and only if*

(i) $1 \in V$, and

(ii) $a, b \notin V, a > 0, b > 0$ implies $\frac{ab-1}{a+b} \notin V$.

Proof. If $K(i)$ is a directed field, that is, $P_V P_V \subseteq P_V$. Assume $1 \notin V$, then $1 + i \in P_V$ implies $(1 + i)^2 = 2i \in P_V$, in contradiction with Theorem 3.3.3 (1) and thus (i) holds. Let $a, b \in K^{>0} \setminus V$, then $a + i, b + i \in P_V$ and hence $(a + i)(b + i) = (ab - 1) + (a + b)i \in P_V$, which implies $\frac{ab-1}{a+b} \notin V$.

Conversely, suppose that (i) and (ii) hold. Let $a + bi, c + di \in P_V$. Then $a, b, c, d \in K^{\geq 0}$.

Case I: $bd = 0$. Then $ac - bd \geq 0$. Without loss of generality, assume that $d = 0$. If $ad + bc > 0$, then $b, c > 0$ which implies $a > 0$ and $ab^{-1} \notin V$. So $ac - bd = ac > 0$ and $\frac{ac-bd}{ad+bc} = \frac{ac}{bc} = \frac{a}{b} \notin V$.

Case II: $bd > 0$ implies that $a, c > 0$ and $ab^{-1}, cd^{-1} \notin V$. Thus $ab^{-1}, cd^{-1} > 1$ since $1 \in V$, and hence $ac - bd > 0$. (ii) implies that $\frac{\frac{a}{b}\frac{c}{d}-1}{\frac{a}{b}+\frac{c}{d}} \notin V$, which implies $\frac{ac-bd}{ad+bc} \notin V$. □

Let F be a ring with unit element 1, let G be a nontrivial, totally ordered, additive, Abelian group, and let $G_{-\infty} = G \cup \{-\infty\}$, where $-\infty + (-\infty) =$

$-\infty = -\infty + a = a + (-\infty) < a$ for all $a \in G$. A function $v: \ F \to G_{-\infty}$ is called a *negative valuation* (cf. [104]) if v is onto and the following statements hold for all $a,\ b \in F$: (1) $v(a) = -\infty$ if and only if $a = 0$; (2) $v(ab) = v(a)+v(b)$; and (3) $v(a+b) \leq max\{v(a),\ v(b)\}$.

For instance, the degree of polynomials or rational fractions is a negative valuation (cf. [27], p. A.IV. 20). For later use, note that by (2), $v(1) = 0 = v(-1)$ and hence $v(-a) = v(a)$ for all $a \in F$.

The following result of [104] can be inferred from the above discussion.

Theorem 3.3.6 *Let F be a totally ordered field, and let $F(i)$ be an extension of F by an element i, where $i^2 = -1$. If there exists a nontrivial totally ordered Abelian group and a negative valuation $v: \ F \to G_{-\infty}$ such that $v(a+b) = max\{v(a),\ v(b)\}$ for all $0 \leq a, b \in F$, then there exists a partial order on $F(i)$ with respect to which $F(i)$ is a directed field.*

Note that $P = \{a + bi \mid a \geq 0, b \geq 0, \text{ and if } b \neq 0, \text{ then } v(a) > v(b)\}$ is a positive cone with respect to which $F(i)$ a directed field.

Corollary 3.3.7 (cf. [104]) *Let F be a totally ordered field and let Q denote the quotient field of the polynomial ring $F[x]$. If i is a solution of $x^2 + 1 = 0$, then there exists a partial order on $Q(i)$ with respect to which $Q(i)$ is a directed field.*

Artin and Schreier [5] showed that any proper subfield F which is of finite co-dimension in $R(i)$ must be real closed using Galois theory, where R is real closed and $i^2 = -1$. Furthermore, we note that any real closed field cannot be obtained by adjunction of a transcendental element to a proper subfield.

Proposition 3.3.8 *Let F be a proper subfield of a real closed field R, and t is a transcendental element over F, then $F(t) \ncong R$.*

Proof. Assume that $(R,\ \leq)$ is a linear ordered real closed field. If $F(t) \cong R$, then there exists a field isomorphisms $h: F(t) \to R$.Without loss of generality, assume that $h(t) > 0,\ t > 0$, then $\sqrt{h(t)}$ has no inverse image in $F(t)$. In fact, if $h(x) = \sqrt{h(t)}$, then $(h(x))^2 = h(t) = h(x^2)$ implies (without loss of generality) that $x = \sqrt{t}$, however, $\sqrt{t}$ is not in $F(t)$, otherwise, assume that $\sqrt{t} = h(t)/g(t)$, then $tg^2(t) = h^2(t)$, and it follows that

$$deg\ (tg^2(t)) = 2m + 1 = deg\ (h^2(t)) = 2n,\ m,\ n \in \mathbb{N}$$

a contradiction. □

Moreover, note that for a proper subfield F of a real closed field R, $F(x)(i)$ is directed orderable by Corollary 3.3.7 or Theorem 3.3.6. However, since we have not shown that the order defined in the proof of Theorem 3.3.6 is a lattice-order, we have not answered the mentioned question of Birkhoff and Pierce. In view of the work above, a more general question would be the following. Note that since an l-field is directed, a negative answer to this question would yield a negative answer to the question of Birkhoff and Pierce.

Question 3.3.9 *Can the field $\mathbb{C}$ be made into a directed field?*

Let T be a totally ordered ring. Recall (see [47]) that a *directed T-algebra* is an algebra A over T with a partial order that makes it into a directed ring with the following compatibility property: if $0 \leq t \in T$ and $0 \leq a \in A$, then $0 \leq td$.

In [18], Birkhoff and Pierce showed that the field $\mathbb{Q}(i)$ admits no partial order with respect to which it is an l-field and that $\mathbb{C}$ admits no partial order with respect to which it is an l-algebra over $\mathbb{R}$. In [100], Schwartz proved that the field of algebraic numbers is not an l-field under any order. And in [47], DeMarr and Steger proved that if A is a finite dimensional nontrivial algebra over $\mathbb{R}$ whose center contains a square root of -1, then A cannot be partially ordered such that A becomes a directed real linear algebra. We first note that the proof of DeMarr and Steger may be easily generalized to prove the following result.

Proposition 3.3.10 *Let $(R, \leq_R, 0_R, 1_R)$ be a linearly ordered ring whose identity 1_R is a strong unit such that the map $x \mapsto x + x$ is onto, and let A be a finite dimensional nontrivial R-algebra whose center contains a square root of -1. Then A is not a directed R-algebra under any order.*

The proof of the theorem above is similar in spirit to that of [47] and is therefore omitted. However, the following corollary is interesting.

Corollary 3.3.11 *(1) The field $\mathbb{Q}(i)$ admits no directed order such that $\mathbb{Q}(i)$ becomes a directed algebra over $\mathbb{Q}$.*

(2) The field $\mathbb{A}(i)$ of algebraic numbers cannot be partially ordered such that $\mathbb{A}(i)$ becomes a directed algebra over the field $\mathbb{A}$ of real algebraic numbers.

(3) There exists no partial order on $\mathbb{C}$ such that $\mathbb{C}$ is a directed algebra over $\mathbb{R}$.

Note that (1) and (3) in the corollary above generalize one result of Birkhoff and Pierce which shows that $\mathbb{Q}(i)$ and $\mathbb{C}$ cannot be lattice-ordered as vector space over $\mathbb{Q}$ and $\mathbb{R}$, respectively. Since the field of algebraic numbers is algebraic over $\mathbb{Q}$, (2) generalizes, in some sense, the interesting theorem of Schwartz [100] which states that the field of algebraic numbers cannot be lattice-ordered. Furthermore, we note that Ma [81] proves that the matrix algebras of quaternions over a subfield of $\mathbb{R}$ cannot be lattice-ordered.

On the other hand, since the proof of Theorem 3.3.6 remains unchanged if the ring is viewed as an algebra over a totally ordered ring F, we have the following result for directed algebras.

Proposition 3.3.12 *Let F be a totally ordered field, and let $F(i)$ be an extension of F by an element i, where $i^2 = -1$. If there exists a nontrivial totally ordered Abelian group and a negative valuation $v : F \to G_{-\infty}$ such that $v(a) > v(b)$ implies $a > b$ for all $0 \leq a, b \in F$, then there exists a partial order on $F(i)$ with respect to which $F(i)$ is a directed F-algebra. In particular, if Q is the quotient field of the polynomial ring $F[x]$, then there exists a partial order on $Q(i)$ with respect to which $Q(i)$ is a directed Q-algebra.*

Remark 3.3.13 *Note that the order constructed in the proof of Theorem 3.3.6 is an order that can be defined by a segment. It is straightforward to verify that the order defined in the proof of Theorem 3.3.6 satisfies conditions (1) and (2) of Theorem 3.3.3 above. In fact, (1) follows from the definition of the positive cone, and the non-triviality of the negative valuation and $v(1) = 0$ implies (2). So there exists a segment V corresponding to the order defined in the proof of Theorem 3.3.6. Furthermore, $1 \in V$ since $1 + i \not\geq 0$ which implies (i) of Lemma 3.3.5, and (ii) of Lemma 3.3.5 follows from the existence of the directed order on $F(i)$.*

By the condition (i) of Lemma 3.3.5 and the definition of the segment we have following interesting corollary which is also a corollary of Theorem 3.3.10.

Corollary 3.3.14 *If 1 is a strong unit in K, then $K(i)$ cannot be partially ordered such that $K(i)$ is a directed field for every segment V of K.*

Especially, if K is an l-subfield of the Archimedean totally ordered field $\mathbb{R}$, then the corollary above is applicable. Equivalently, the field K in Theorem 3.3.15 below can be considered as a non-archimedean linearly ordered one.

Let us call a segment $V \subset K$ *multiplicative* if it satisfies

(iii) $0 < a \notin V$ implies $\frac{1}{2}(a - a^{-1}) \notin V$.

Theorem 3.3.15 *$V \mapsto P_V$ defines a bijection between multiplicative segments V and partially orders $\leq$ on $K(i)$ such that $K(i)$ is a directed field with $bi \not\leq a$ for all $a \in K$, $b \in K^{>0}$, and $k < i$ for some $k \in K$.*

Proof. $(ii) \Rightarrow (iii)$: Set $a = b$, then $\frac{a^2-1}{2a} = \frac{1}{2}(a - a^{-1}) \notin V$.

$(iii) \Rightarrow (ii)$: Without loss of generality, suppose that $0 < a < b$. We want to show that $\frac{1}{2}(a - a^{-1}) \leq \frac{ab-1}{a+b}$, which is equivalent to show $b(a^2+1) \geq a(a^2+1)$.

$(iii) \Rightarrow (i)$: Suppose that $1 \notin V$, so $\frac{1}{2}(1 - 1^{-1}) \notin V$, a contradiction.

The conditions (1) and (2) in Theorem 3.3.3 are simplified: If $i \geq 0$, then $i^2 = -1 \geq 0$, which is impossible. $\square$

Remark 3.3.16 *Let $a' = \frac{1}{2}(a - a^{-1})$ for $a > 0$. Then (iii) is equivalent to*

$$0 < a \notin V \Rightarrow 0 < a' \notin V.$$

In fact, $0 < a \notin V$ implies that $a > 1$ and thus $a > a^{-1}$, which implies $a' > 0$. Furthermore, for $a > 0$ we have $i > -a \Rightarrow i > -a'$ since $i + a > 0$ gives that $(i+a)^2 > 0$ which gives that $(2a)i > 1 - a^2$, and thus $i > -\frac{1}{2}(a - a^{-1})$.

Corollary 3.3.17 *If $K(i)$ is an l-field with K as a linearly ordered subfield. Then $i \parallel a$ for all $a \in K$, or there exist $a, b \in K$ such that $a < i < b$, or for all $a \in K$ there exists $b \in K$ such that $a \not< i \leq b$.*

Proof. It suffices to prove that every directed order in Theorem 3.3.15 cannot make $K(i)$ into an l-field. Assume that $V = V_\alpha$. Then $\alpha \notin V$ and $\alpha > 1$, and thus $0 < \alpha' < \alpha$, so $\alpha' \in V$. In contradiction with the definition of a multiplicative segment. $\square$

Remark 3.3.18 *If on the field $\mathbb{C}$ there exists a partial order such that $\mathbb{C}$ is a directed field with respect to Theorem 3.3.15, then there exists $0 > a \in \mathbb{R}$ with $i > a$. Thus*

$$i > \frac{1}{2}(a - a^{-1}) > \frac{a}{2},$$

which implies

$$i > \frac{a}{2^n}$$

for all $n \in \mathbb{N}$ by induction. Hence $\mathbb{C}$ cannot be partially ordered in such a way that it becomes an Archimedean l-field, which is an interesting result similar to Lemma 7, Theorems 7, 8 and 10 in Schwartz [100]. However, we cannot get $i \geq 0$ in general if the order is not Archimedean (see also Theorem 3.3.10, Corollary 3.3.14 above and Propositions 6.1.3 and 6.1.4 below).

Theorem 3.3.19 *A segment $V \subset K$ is multiplicative if and only if is satisfies the following conditions:*
(!) V is an additive proper subgroup;
(!!) V is convex;
(!!!) $\forall a \in K \setminus \{0\}$: $a \in V$ or $a^{-1} \in V$.

Proof. If V is a multiplicative segment and $0 < a < b$ in V, then $a < \frac{a+b}{2} < b$ implies $\frac{a+b}{2} \in V$. Assume that $a+b \notin V$. Then $\frac{a+b}{2} \geq \frac{1}{2}((a+b)-(a+b^{-1})) \notin V$, a contradiction. Thus $a+b \in V$. Now let $a, b \in V$ be arbitrary. Then $|a|+|b| \in V$ and $|a+b| \leq |a|+|b|$ implies $a+b \in V$. Hence (!) is proved.

(!!) is clear.

(!!!): If $\neq 0$ and $a, a^{-1} \notin V$. Then $|a|, |a|^{-1} \notin V$ implies a contradiction: $0 = \frac{|a||a|^{-1}-1}{|a|+|a|^{-1}} \notin V$.

Conversely, suppose that V satisfies (!), (!!) and (!!!). Then V is a segment is plain. Let $0 < a \notin V$. By (!!!) it follows that $a^{-1} \in V$. Assume that $\frac{1}{2}(a-a-1) \in V$. Then $a - a^{-1} \in V$ and $a = (a - a^{-1}) + +a^{-1} \in V$, a contradiction. □

Corollary 3.3.20 *Every convex valuation subdomain $V \neq K$ in its quotient field K (non-archimedean linearly ordered) is a multiplicative segment.*

Proof. The conditions (!), (!!) and (!!!) are clearly satisfied. By Corollary 3.3.14 1 is not a strong unit, which implies that K is non-archimedean. □

Remark 3.3.21 *The condition (!!!) in Theorem 3.3.19 can be substituted by*

$$1 \in V.$$

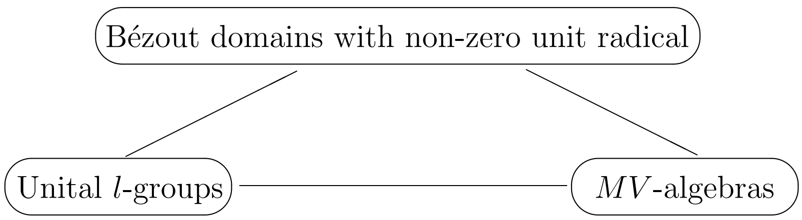
Actually, (!!!) implies $1 = 1^{-1} \in V$. Conversely, assume that $a \notin V$, then $|a| \notin V$ implies $|a| > 1$, and thus $1 > |a|^{-1} = |\frac{1}{a}| > 0$, which implies $\frac{1}{a} \in V$.

Chapter 4

MV-algebras, l-groups and Bézout domains with non-zero unit radical

Similar to the radical of an MV-algebra defined in [10], [82, 106] defined a *radical* $Rad(G)$ (that is, the intersection of maximal convex l-subgroups) of an l-group G. There seems to be no relationship to other known radicals in l-group theory, neither to the Conrad radical $R(G)$, nor to the distributive radical $D(G)$, nor to Conrad's ideal radical $L(G)$. In this chapter, we use the "Jacobson" radical $Rad(G)$ to relate properties of l-groups to those of Bézout domains, using valuation theory. Theorem 4.2.10 proves that the quotient group $G/Rad(G)$ of a μ-normal-valued l-group G is the divisibility group of some completely integrally closed Bézout domain, hence of an intersection of rank 1 valuation domains. Corollary 4.2.6 shows that an Abelian l-group with strong unit is Archimedean if and only if $Rad(G) = 0$. For an arbitrary Bézout domain D, the radical of $G(D)$ gives rise to a completely integrally closed overring of D obtained by localization. For Abelian l-group G with strong unit, $Rad(G)$ consists of the elements of G which are infinitely small (Theorem 4.2.5). Corollary 4.2.12 gives a class of l-groups which are Archimedean if and only if they are completely distributive.

Then, we consider Mundici's functorial equivalence between MV-algebras and Abelian l-groups with strong unit, which allows us to translate properties from any of the three classes, the class of MV-algebras, the class of Bézout domains with nonzero unit radical, and the class of Abelian l-groups with



strong unit, to the other two classes. In Section 4.4 we use this correspondence to answer a problem on prime ideals of MV-algebras via l-group theory.

4.1 Radicals of an l-group

4.1.1 The Conrad radical and the distributive radical

For the investigation of the completely distributive property of l-groups, three kinds of radical have been introduced by Conrad [36], and Byrd and Lloyd [31], respectively.

Let g be a strictly positive element of an l-group G, and $R_g = \vee\{V \leq G \mid V$ is a value of $g\}$. Then the *Conrad radical* is $R(G) := \cap\{R_g \mid 0 < g \in G\}$. The following result is well known. Recall that a convex subgroup A of an l-group G is said to be *closed* if whenever $\{a_i \mid i \in I\} \subseteq A$ and either $\vee a_i$ or $\wedge a_i$ exists in G, then $\vee a_i$ or $\wedge a_i$ is an element of A.

Proposition 4.1.1 *([44], p. 135) Let G be an l-group, and $R(G)$ is its Conrad radical. Then*

(1) G is completely distributive if $R(G) = 0$;

(2) $R(G)$ is a closed l-ideal of G;

(3) The set $\Gamma(G)$ of regular subgroups of G has a minimal plenary subset if and only if $R(G) = 0$;

(4) If $C \in \mathcal{C}(G)$, then $R(C) = C \cap R(G)$;

(5) $R(R(G)) = R(G)$.

Conrad [36] (see Byrd [30]) introduces the *ideal radical* $L(G)$. For a non-zero element g of an l-group G, let $L_g = \vee\{L \leq G \mid L$ is a convex l-ideal not containing $g\}$. Then the *ideal radical* is $L(G) := \cap\{L_g \mid 0 \neq g \in G\}$.

The *distributive radical* is defined by Byrd and Lloyd in [31]. Let G be an l-group and $A \leq G$. A is prime if $a \wedge b \in A$ implies $a \in A$ or $b \in A$ for all elements $a,\ b \in G$. The *distributive radical* $D(G)$ is the intersection of all closed prime subgroups of G. It has the following properties.

Proposition 4.1.2 *([44], pp. 129-131) Let G be an l-group, $D(G)$ be its distributive radical. Then*

(1) $D(G)$ is the intersection of all closed regular subgroups of G;

(2) $D(G)$ is a closed l-ideal of G;

(3) If $C \in \mathcal{C}(G)$, then $D(C) = C \cap D(G)$;

(4) $D(D(G)) = D(G)$;

(5) $D(G) = 0$ if and only if $D(G(g)) = 0$ for all $g \in G$;

(6) $D(G) = 0$ if and only if there exists a set of closed primes such that 0 is the only l-ideal contained in their intersection;

(7) $D(G) = 0$ if and only if G is completely distributive.

The "radicals" $R(G)$ and $D(G)$ are quite different from the radicals encountered in module theory. In fact, there are non-trivial examples of l-groups G with $D(G) = R(G)$ and $R(G/R(G)) = G/R(G)$ (see [44], pp. 129-135). For other differences, see Remark 4.1.27 below.

In general, the inclusion $L(G) \subseteq D(G) \subseteq R(G)$ holds. For a representable l-group G, the three radicals coincide: $L(G) = D(G) = R(G)$ (cf. [31]). By Proposition 4.1.2 an l-group G is completely distributive if and only if its distributive radical $D(G)$ is zero (see [31]). However, it seems that these radicals have no relationship to the Archimedean property of an l-group.

4.1.2 Small and large l-subgroups

Although the theory of po-groups and module theory are rather different, there are still remarkable analogies. In this section, we study the analogue of small and large (=essential) submodules, and call them *small* and *large* l-subgroups, respectively.

Definition 4.1.3 *Let G be an l-group, and let A be an l-subgroup. A is said to be small (denoted by $A \ll G$) if $A \vee W = G$ implies $W = G$ for all $W \leq G$. A is said to be large (denoted by $A \blacktriangleleft G$) if $A \wedge W = 0$ implies $W = 0$ for all $W \leq G$.*

The following equivalent characterizations of large convex l-subgroup is clear.

Proposition 4.1.4 *Let G be an l-group and $A \leq G$. The following conditions are equivalent:*

(1) $A \blacktriangleleft G$;

(2) $A^{\perp} = 0$;
(3) $A^{\perp\perp} = G$;
(4) if $K \cap A = 0$ *for any closed* l*-subgroup* $K \leq G$*, then* $K = 0$;
(5) if $P \cap A = 0$ *for any* $P \in \mathcal{P}(G)$*, then* $P = 0$;
(6) if $M \cap A = 0$ *for any minimal* l*-subgroup* $M \leq G$*, then* $M = 0$;
(7) if $N \cap A = 0$ *for any prime* l*-subgroup* $N \leq G$*, then* $N = 0$;
(8) if $V \cap A = 0$ *for any* $V \in \Gamma(G)$*, then* $V = 0$.

Note that an l-subgroup A of G is large is equivalent to asserting that for all $g \in G^{\geq 0}$, there exists $a \in A^{\geq 0}$ and $n \in \mathbb{N}$ for which $a \leq ng$. In module theory, every given module M admits an injective hull $M \hookrightarrow I$, i. e. a maximal supermodule I in which M is large. By contrast, l-groups show a quite different behavior. In fact, if an l-group G is large in H, it is also large in any lexicographical extension of H by a fully ordered group. Therefore, a non-zero non-archimedean l-group has no large extension which admits no proper large extension (cf. [40]).

If $G(D)$ is the group of divisibility of a Bézout domain D, and $\mathcal{D}(D)$ is the set of all divisorial fractional ideals of D, then $G(D)$ is a large l-subgroup of $\mathcal{D}(D)$ under the order dual to inclusion.

Remark 4.1.5 *Recall that an* l*-subgroup* H *of* G *is dense if* $0 < g \in G$ *implies* $0 < h \leq g$ *for some* $h \in H$*. Clearly a dense* l*-subgroup is large. Furthermore, let* $G^{\wedge}$ *denote the Dedekind MacNeille completion,* G^{l} *the lateral completion, and* G^{d} *the divisible hull of an Archimedean* l*-group* G*. Then* G *is dense in both* $G^{\wedge}$ *and* G^{l}*, and large in* G^{d}*. By Bernau's theorem [13], there is an* l*-embedding* $G \hookrightarrow D(X)$ *with a Stone space* X *(see Example 2.3.5) such that* $D(X)$ *does not arise as a large* l*-subgroup of any Archimedean* l*-group* $H \ncong_{l-g} D(X)$*. By Conrad's theorem [40],* $D(X)$ *is* l*-isomorphic to* $((G^{d})^{\wedge})^{l}$*.*

Definition 4.1.6 *Let* G *and* H *be* l*-groups. An injective* l*-group homomorphism* f *from* G *to* H *is said to be essential in case* $Im(f) \blacktriangleleft H$*. A surjective* l*-group homomorphism* g *from* G *to* H *is said to be superfluous in case* $Ker(g) \ll G$*.*

The following properties of large resp. small l-subgroups in Abelian l-groups are reminiscent of module theory.

Proposition 4.1.7 *Let* G*,* H *be Abelian* l*-groups. Then*

(1) an injective l-group homomorphism $f : H \to G$ is large if and only if, for all l-group homomorphisms s, if sf is injective, then s is injective;

(2) a surjective l-group homomorphism $g : G \to H$ is small if and only if, for all l-group homomorphisms h, if gh is surjective, then h is surjective.

Proof. (1) Let $L = Im(f)$. Then $v : \ L \to H$ with $v(l) = h$ if $f(h) = l$ for $l \in L$ is an l-isomorphism such that $fv = i_L$. Thus sf is injective if and only if $sfv = si_L$ is injective. But the latter condition holds if and only if $Ker(s) \cap L = 0$. It follows that $Ker(s) = 0$ and s is injective for all s if and only if f is $large$.

(2) Assume that g is superfluous. If gh is surjective, then $g(Im(h)) = H = g(G) \cong_l G/K$ and thus $G/K \cong_{l-g} Im(h)/K$, which implies $Im(h) \vee Ker(g) = G$ and hence h is surjective. Conversely, let the condition in (2) be satisfied. Suppose that $Ker(g) \vee C = G$ holds for some $C \in \mathcal{C}(G)$. Then $Ker(g) + C = G$ by the Riesz' Lemma (cf. Theorem 3.11 of [44]). Therefore, the inclusion $i : C \hookrightarrow G$ has the property that gi is surjective. Thus $C = G$. □

Proposition 4.1.8 *Let G be an l-group with l-subgroups $K \subseteq N \subseteq G$ and $H \subseteq G$. Then*

(1) $K \blacktriangleleft G$ if and only if $K \blacktriangleleft N$ and $N \blacktriangleleft G$;

(2) $H \cap K \blacktriangleleft G$ if and only if $H \blacktriangleleft G$ and $K \blacktriangleleft G$.

Proof. (1) Let $K \blacktriangleleft G$ and suppose $\{0\} \neq L \leq G$, then $L \cap K \neq \{0\}$. In particular this is true if $L \leq N$, so $K \blacktriangleleft N$. But also $K \leq N$ so $L \cap N \neq \{0\}$ whence $N \blacktriangleleft G$.

Conversely, if $K \blacktriangleleft N$ and $N \blacktriangleleft G$ and $L \leq G$, then $L \cap K = \{0\}$ implies that $K \cap (L \cap N) = \{0\}$. Hence $L \cap N = \{0\}$, and thus $L = \{0\}$.

(2) One implication follows at once from (1) above. For the other, suppose $H \blacktriangleleft G$ and $K \blacktriangleleft G$. If $L \leq G$ with $L \cap H \cap K = \{0\}$, then $L = \{0\}$. □

Concerning small convex l-subgroups we have:

Proposition 4.1.9 *Let G be an l-group with convex l-subgroups $K \leq N \leq G$ and $H \leq G$. Then*

(1) $K \ll G$ and $N/K \ll G/K$ implies $N \ll G$:

(2) $H \vee K \ll G$ if and only if $H \ll G$ and $K \ll G$.

Proof. (1) Let $M \leq G$ and $M \vee N = G$, then $(M \vee N)/K = ((M \vee K)/K) \vee (N/K) = G/K$ implies $(M \vee K)/K = G/K$. Therefore, $M \vee K = G$, whence $M = G$.

(2) If $H \vee K \ll G$ with $M \leq G$ and $H \vee M = G$ (or $K \vee M = G$), then $(H \vee K) \vee M = G$,. whence $M = G$. Conversely, if $H \ll G$ and $K \ll G$, and $M \leq G$ with $M \vee H \vee K = G$, then $M = G$. □

Corollary 4.1.10 *Let G be an Abelian l-group with l-subgroups $K \leq N \leq G$ and $H \leq G$. Then $N \ll G$ if and only if $K \ll G$ and $N/K \ll G/K$.*

Proof. By Proposition 4.1.9, it suffices to prove that $N \ll G$ implies $K \ll G$ and $N/K \ll G/K$. It is clear that $K \ll G$. Suppose that $M/K \leq G/K$ and $(N/K) \vee (M/K) = G/K$. Then $(N \vee M)/K = G/K$, i. e. $N \vee M = G$. So we get $M = G$, whence $M/K = G/K$. □

Proposition 4.1.11 *Let G, H be l-groups, and $f : G \to H$ an l-group homomorphism.*

(1) If $K \ll G$ then $f(K) \ll H$. Especially, if $K \ll G \leq H$ then $K \ll H$.

(2) If $N \blacktriangleleft H$ then $f^{-1}(N) \blacktriangleleft G$. Especially, if $K \leq G \leq H$ and $K \blacktriangleleft H$ then $G \blacktriangleleft H$.

Proof. (1) Let $L \leq H$ and assume $L \vee f(K) = H$. Then $f^{-1}(L) \vee K = G$. $K \ll G$ implies $K \leq G = f^{-1}(L)$, so $f(K) \leq L$ and $L \vee f(K) = L = H$.

(2) Let $M \leq G$ and $M \cap f^{-1}(N) = \{0\}$, then $f(M) \cap N = \{0\}$ implies $f(M) = \{0\}$. So $M \subseteq Ker(f) \subseteq f^{-1}(N)$, whence $M \cap f^{-1}(N) = \{0\} = M$. □

The following corollary generalizes Proposition 2.1.6.

Corollary 4.1.12 *Let G and H be l-groups, $f : G \to H$ a superfluous surjective l-group homomorphism. Then a convex l-subgroup C of H is small if and only if $f^{-1}(C)$ is small in G.*

Proof. By Theorem 2.1.5, $H \cong_{l-g} G/Ker(f)$. By Proposition 4.1.11 it suffices to prove that $f^{-1}(C)$ is small in G if $C \ll G/ker(f)$. This follows immediate by Proposition 4.1.9 (1). □

Proposition 4.1.13 *Let G, H be l-groups. Then*

(1) $A \blacktriangleleft G$ and $B \blacktriangleleft H$ if and only if $A \oplus_c B \blacktriangleleft G \oplus_c H$.

(2) $A \ll G$ and $B \ll H$ if and only if $A \oplus_c B \ll G \oplus_c H$.

Proof. (1) ⇒:" It suffices to show $A \oplus_c B \blacktriangleleft A \oplus_c H$. Let p_H be the projection of $A \oplus_c H$ on H. Then $B \blacktriangleleft H$ implies

$$(p_H)^{-1}(B) = A \oplus_c B \blacktriangleleft A \oplus_c H$$

follows by Proposition 4.1.11.

Conversely, if $A \oplus_c B \blacktriangleleft G \oplus_c H$, then

$$A = (A \oplus_c B) \cap G = (id_G)^{-1}(A \oplus_c B) \blacktriangleleft G \oplus_c H$$

follows by Proposition 4.1.11, where $id_G : G \hookrightarrow G \oplus_c H$.

(2) "$\Rightarrow$:" Let $U \leq G \oplus_c H$ with $U \vee (A \oplus_c B) = G \oplus_c H$, then

$$[U \vee (A \oplus_c B)] \cap G = G = A \vee [(U \vee B) \cap G]$$

implies $(U \vee B) \cap G = G$ which implies $G \subseteq U \vee B = G \oplus_c H$, and hence $H = (U \vee B) \cap H = B \vee (U \cap H)$ which implies $U \cap H = H$, whence $H \subseteq U = U \vee B = G \oplus_c H$.

Conversely, let p_G be the projection of $G \oplus_c H$ on G. Then $A = p_G(A \oplus_c B) \ll G$. ☐

The following proposition shows that the Boolean algebra of polars remains the same in large extensions.

Proposition 4.1.14 (cf. [4], Theorem 8.1.1) *Let G and H be l-groups and $G \blacktriangleleft H$, we denote the polar operations of G and H by $\perp$ and $'$ respectively, and define $\Phi(P) = P \cap G$, $\Psi(Q) = (Q^{\perp})'$ for all $P \in \mathcal{P}(H)$ and $Q \in \mathcal{P}(G)$ (see Definition 2.2.3). Then the lattices $\mathcal{P}(G)$ and $\mathcal{P}(H)$ are isomorphic under the maps Φ and Ψ.*

If G and H are l-groups and $f : G \to H$ a superfluous surjective l-group homomorphism, Corollary 4.1.12 shows that $f^{-1}(C)$ is a small convex l-subgroup of G for any $C \ll H$. However, the following example shows that $\mathcal{P}(G)$ and $\mathcal{P}(H)$ need not be isomorphic under the map f.

Example 4.1.15 *Let $G = (\mathbb{Z} \oplus_c \mathbb{Z}) \oplus_l \mathbb{Z}$, $H = (\mathbb{Z} \oplus_c 0) \oplus_l \mathbb{Z}$, and f be the canonical epimorphism from G to H. Then $ker(f) = (0 \oplus_c \mathbb{Z}) \oplus_l 0 \ll G$, $(0 \oplus_c 0) \oplus_l 0 = G^{\perp}$ and $(0 \oplus_c \mathbb{Z}) \oplus_l 0 = ((\mathbb{Z} \oplus_c 0) \oplus_l 0)^{\perp}$. However, $f((0 \oplus_c 0) \oplus_l 0) = f((0 \oplus_c \mathbb{Z}) \oplus_l 0)$.*

Another useful property of large extensions is that they preserve all (infinite) meets and joins of the large l-subgroup.

Proposition 4.1.16 (cf. [4], Theorem 8.1.2) *Suppose $G \blacktriangleleft H$ and $A \subseteq G$ which has $g = \vee A \in G$, then $g = \vee A \in H$. A similar statement holds for meets.*

4.1.3 The socle and the radical of an l-group

For an l-group G, an element $g \in G$ is said to be *bounded* (or *infinitely small*) if there is an element $h \in G$ such that $n|g| \leq h$ for all $n \in \mathbb{N}$. We denote by $B(G)$ the set of all bounded elements of an l-group G. It is clear that an l-group G is Archimedean if and only if $B(G) = \{0\}$.

Let us call an l-group G *simple* if $|\mathcal{C}(G)| = 2$. We call G *semisimple* if every convex l-subgroup A admits a *complement* $B \leq G$, that is, $A \vee B = G$ and $A \cap B = 0$. G is called *1st Archimedean* if, for $x,\ y \in G$ with $0 < x < y$, there exists a natural number $n > 1$ such that $nx \geq y$. A 1st Archimedean l-group G is *strong* if, for all strictly positive elements $x,\ y \in G$, there exists a natural number n such that $nx \geq y$.

Note that a non-zero convex l-subgroup G of an l-group H is simple if and only if G is minimal. It is known (see [16, 22, 58]) that a *totally ordered* group is Archimedean if and only if it is strongly 1st Archimedean (cf. [58]). Note that strongly 1st Archimedean property is called *Archimedean* and *strongly Archimedean* by Jaffard and Dubreil-Jocotin, respectively. Jaffard proved that strongly 1st Archimedean implies Archimedean for l-groups but not in general (cf. [58]). More generally, we have

Proposition 4.1.17 *Let G be a non-trivial l-group. Then the following statements are equivalent:*

1) G is simple.

2) G is a non-zero l-subgroup of $\mathbb{R}$.

3) G is 1st Archimedean.

4) G is strongly 1st Archimedean.

5) G is the group of divisibility of a completely integrally closed valuation domain.

Proof. It is plain that 2) $\Leftrightarrow$ 5), 2)$\Rightarrow$1), and 2)$\Rightarrow$4)$\Rightarrow$3).

1)$\Rightarrow$2): Let G be simple. Suppose that $a \wedge b = 0$ for some $a, b \in G$. Then $a \wedge nb = 0$ holds for all $n \in \mathbb{N}$. Hence, there is a value P of a with $b \in P$. Since G is simple, we infer that $P = 0$, whence $b = 0$. This shows that G is fully ordered. Moreover, we get $B(G) = 0$, i. e. G is Archimedean. By Hölder's theorem (cf. Proposition 2.1.9), G can be embedded into $\mathbb{R}$.

3)$\Rightarrow$2): Assume that $a \wedge b = 0$ with $a, b \in G^{>0}$. Then $0 < a < a + b$, and so there exists an integer $n > 1$ with $a + b \leq na$. Hence $b \leq (n-1)a$. On the other hand, $(n-1)a \wedge b = 0$ by Riesz's Lemma. Thus $b = 0$, a contradiction. Therefore, we infer that G is totally ordered and Archimedean. $\square$

Like in module theory, semisimple l-group can be characterized as follows.

Proposition 4.1.18 *For an l-group G, the following conditions are equivalent.*

(a) G is semisimple.

(b) $G = \vee S_i$ with $S_i \in \mathcal{C}(G)$ simple.

(c) $G \cong \coprod S_i$ with $S_i \in \mathcal{C}(G)$ simple.

Proof. (a)⇒(b): It suffices to prove that $G(g)$ has a simple convex l-subgroup whenever $g \neq 0$. In fact, if $G = \vee\{S \leq G \mid S \text{ simple}\} \oplus_c C$ with $C \neq 0$, we find $g \neq 0$ in C, which leads to a contradiction. Now let D be a value of g in $G(g)$, and $G = D \oplus_c E$. Then $G(g) = D \oplus_c (G(g) \cap E)$, which implies that $D = 0$ since $g \in G(g) \cap E \leq G(g)$ is a strong unit of $G(g)$. Hence, $G(g)$ itself is simple.

(b)⇒(c): Assume that $G = \vee_{i \in I} S_i$ with $S_i \in \mathcal{C}(G)$ simple. By Zorn's lemma, there is a maximal subset J of I with $\vee_{i \in J} S_i = \coprod_{i \in J} S_i$. For any $i \in I$, this implies that $S_i \cap (\vee_{j \in J} S_j) = S_i$, whence $G \cong \coprod S_i$.

(c)⇒(a): The assumption states that $G = \vee_{i \in I} S_i \cong \coprod_{i \in I} S_i$ with $S_i \in \mathcal{C}(G)$ simple. For a given $H \in \mathcal{C}(G)$, Zorn's lemma yields a maximal $J \subseteq I$ with $H \cap (\vee_{j \in J} S_j) = 0$. Then $S_i \subseteq H \vee (\vee_{j \in J} S_j)$ holds for all $i \in I \setminus J$. Hence $G \cong H \oplus_c \vee_{j \in J} S_j$. □

Corollary 4.1.19 *Every semisimple l-group G is the divisibility group of some Bézout domain which is an intersection of completely integrally closed rank 1 valuation domains. Especially, G is Abelian.*

Proof. If $G \cong \coprod S_i$ with $S_i \in \mathcal{C}(G)$ simple, then G is Archimedean and the intersection of maximal convex l-subgroups of G is zero by Proposition 4.1.17. Hence Propositions 2.1.9 and 2.1.12, Theorems 2.2.1 and 2.2.2, and Corollary 2.2.9 finish the proof. □

Definition 4.1.20 *Let G be an l-group. We define the socle of G to be the convex l-subgroup $Soc(G) := \vee\{S \leq G \mid S \text{ simple}\}$, and dually, define the radical of G to be $Rad(G) := \cap\{B \leq G \mid G \neq B,\ B \text{ maximal}\}$.*

Similar to module theory and MV-algebra theory, the socle (resp. radical) of an l-group coincides with the intersection of all large (resp. maximal) convex l-subgroups.

Proposition 4.1.21 *For an l-group G, $Soc(G) = \wedge\{A \leq G \mid A \blacktriangleleft G\}$.*

Proof. For $S \leq G$ simple and $A \in \mathcal{C}(G)$ large, we have $S \cap A \neq 0$, whence $S \subseteq A$. Therefore, $Soc(G) \subseteq H := \wedge\{A \leq G \mid A \blacktriangleleft G\}$. By Proposition 4.1.18, it remains to be shown that H is semisimple. Thus let $B \leq H$ be given. Chosen $C \in \mathcal{C}(G)$ maximal with $B \cap C = 0$. Then $B \vee C$ is large in G since $(B \vee C) \cap M \cong (B \cap M) \oplus_c (C \cap M)$ by Proposition 2.1.10, and therefore, $H \subseteq B \vee C$. Hence, by Proposition 2.1.10 again, we get $H = H \cap (B \vee C) = (H \cap B) \vee (H \cap C) \cong B \oplus_c (H \cap C)$. □

Dually, we get

Proposition 4.1.22 ([106]) *For an l-group G, $Rad(G) = \vee\{B \mid B \ll G\}$.*

Corollary 4.1.23 *Let G, H be l-groups and $f : G \to H$ an l-group homomorphism. Then $f(Rad(G)) \leq Rad(H)$ and $f(Soc(G)) \leq Soc(H)$.*

Proof. This follows from Propositions 4.1.22, 4.1.11 and 4.1.21. □

Definition 4.1.24 *We call an l-group G semimaximal if $Rad(G) = \{0\}$.*

For instance, every semisimple l-group is semimaximal by Corollary 4.1.19. However, an Archimedean semimaximal l-group is not necessary semisimple by Remark 4.2.18 below. In contrast to the Conrad radical and related radicals mentioned at the beginning of Section 4.1.1, $Rad(G)$ shares the fundamental property $Rad(G/Rad(G)) = 0$ with radicals in module theory.

Proposition 4.1.25 *For an l-group G, $Rad(G)$ is a convex l-ideal of G. Furthermore, $G/Rad(G)$ is a semimaximal l-group.*

Proof. To the first part, it is suffices to prove $Rad(G)$ is normal. For any $g \in G$, the map $x \mapsto g + x - g$ is an l-automorphism of G. Hence $A \ll G$ implies $g + A - g \ll G$ by Proposition 4.1.11. Thus $Rad(G)$ is normal. The semimaximality of $G/Rad(G)$ follows by Definition 4.1.20. □

Conversely, we have:

Proposition 4.1.26 *Let G be an l-group and C an l-ideal of G, such that $Rad(G/C) = \{0\}$. Then $Rad(G) \leq C$.*

Proof. It follows from Propositions 4.1.11 and 2.1.4. □

Note that for an l-ideal I of G, it is not necessary that $Rad(I) = Rad(G) \cap I$ as showed in Remark 4.1.27. We also note that for a convex l-subgroup C of a representable l group G, it is not necessary that $Rad(C) = Rad(G) \cap C$, which holds for the distributive radical, the Conrad radical and the ideal radical (see Propositions 4.1.1 and 4.1.2).

Remark 4.1.27 *$Rad(Rad(G)) = Rad(G)$ does not hold for the radical of an l-group, though $D(D(G)) = D(G)$ and $R(R(G)) = R(G)$ (see Propositions 4.1.1 and 4.1.2). For example, let $G = \mathbb{R} \oplus_l \mathbb{R}$, where $\mathbb{R}$ is the additive group of the real field under the usual order. Then $Rad(Rad(G)) = 0 \neq Rad(G) = \mathbb{R}$.*

For the socle of convex l-subgroups, we have the following property which is similar to module theory.

Proposition 4.1.28 *Let G be an l-group and $C \leq G$. Then $Soc(C) = Soc(G) \cap C$. In particular, $Soc(Soc(G)) = Soc(G)$.*

Proof. The inclusion $Soc(C) \subseteq C \cap Soc(G)$ follows by Corollary 4.1.23. Since $C \cap Soc(G)$ is semisimple, $Soc(C) = C \cap Soc(G)$. □

4.2 Divisibility groups of Bézout domains with non-zero unit radical

By Theorem 2.2.7, the divisibility group $G(D)$ of a Bézout domain D has a strong unit if and only if the unit radical $u(D)$ is non-trivial. The divisibility group of such a Bézout domain has even stronger analogies with the theory of rings. Like the nilpotent elements in algebraic geometry, the bounded elements are precisely the elements common to all maximal convex l-subgroups and thus constitute the radical of $G(D)$ (cf. Theorem 4.2.5). In fact, Proposition 4.3.12 will show that the bounded elements in an Abelian l-group with strong unit are "precisely" the nilpotent elements in the corresponding MV-algebra.

If $C(D)$ is the complete integral closure of a Bézout domain D, and G is the divisibility group of D, then it can be shown (see Mott [87], Corollary 2.6) that $G(C(D)) \cong_{p-g} G/B(G)$. However, it is well known that a complete integral closure of a Bézout domain need not be completely integrally closed, that is, $G/B(G)$ is not necessarily Archimedean, or equivalently, $B(G/B(G))$ need not be trivial. However, we have:

Lemma 4.2.1 *For any l-group G, $B(G)$ is an l-ideal. Furthermore, if G is an l-group with strong unit, then $Rad(G) \subseteq B(G)$. In particular, $G/B(G)$ is semimaximal.*

Note that we shall also use "$\ll$" to denote the infinitely small relation between two positive elements, that is, $0 < a \ll b$ if and only if $na < b$ for all $n \in \mathbb{N}$. This is justified since for $a, b \in G^{>0}$ in an l-group G, $a \ll b$ if and only if $G(a) \ll G(b)$ by Corollary 4.2.2 below.

Proof. For given $a,\ b \in B(G)$, assume that there exist $g,\ h \in G$ such that $|a| \ll g$ and $|b| \ll h$. Then $||a| - |b|| \leq |a| + |b| + |a| \ll g + h + g$, and it follows that $B(G)$ is an l-subgroup. Furthermore, it is clear that $B(G)$ is convex and for any $x \in G$, $b \in B(G)$ with $b \ll h \in G$, we have $x + b - x \ll x + h - x$, whence $B(G)$ is an l-ideal.

If G has a strong unit u, then for all $0 \leq b \in Rad(G)$, we have $b \ll u$. In fact, assume that there exists $N \in \mathbb{N}$ such that $Nb \parallel u$. By Proposition 2.2.4 it follows that $u > Nb \wedge u > 0$ and $Nb \wedge u < Nb$. Then $(u - Nb \wedge u) \wedge (Nb - Nb \wedge u) = 0$. Furthermore, $G(u - Nb \wedge u) \vee G(Nb \wedge u) = G$ and $G(Nb \wedge u) \subseteq G(b) \ll G$ by Proposition 4.1.22, which implies that $G(u - Nb \wedge u) = G$, so there exists $m \in \mathbb{N}$ with $m(u - Nb \wedge u) \geq Nb - Nb \wedge u$. However, $0 \leq m(u - Nb \wedge u) \wedge (Nb - Nb \wedge u) = Nb - Nb \wedge u \leq m[(u - Nb \wedge u) \wedge (Nb - Nb \wedge u)] = 0$ by Riesz property of l-groups (cf. Theorems 3.11 and 3.12 of [44]), which is a contradiction. Hence we have $Rad(G) \subseteq B(G)$. Especially, $Rad(G/B(G)) = \{\bar{0}\}$ follows by Proposition 4.1.25. □

As a consequence of the proof of the lemma above, we get

Corollary 4.2.2 *Let G be an l-group, $a, b \in G^{\geq 0}$. Then*

(1) $a \ll b$ if and only if $G(a) \ll G(b)$;

(2) G is Archimedean if and only if each finitely generated convex l-subgroup of G is Archimedean.

We note that for an Abelian l-group G with strong unit u, Lemma 3.5 in Bleier and Conrad [19] states that $Rad(G)$ is a closed l-ideal generated by the set $\{x \in G \mid 0 \leq x \ll u\}$, and implies that $G/Rad(G)$ is Archimedean. Furthermore, Conrad [39] Lemma 3.3 shows that G/C is Archimedean if G is Archimedean and C is a polar of G. Then he and McAlister ([43], Proposition 2.6) prove that $G/R(G)$ is Archimedean and $R(G/R(G)) = \{0\}$, where G is an Archimedean l-group and $R(G)$ is the Conrad radical of G. Actually, it is easily seen that $G/L(G) = G/R(G) = G/D(G)$ is Archimedean from Propositions 4.1.1 and 4.1.2 and the remark after them in this case. However, the semimaximality of $G/Rad(G)$ holds for any l-group G by Proposition 4.1.25, and we will prove that every μ-normal-valued l-group (see Definition 4.2.9) G has the property that $G/R(G)$ is Archimedean (see Theorem 4.2.10). We first show that any Abelian l-group G satisfies

Proposition 4.2.3 *Let G be an Abelian l-group, and $a \in B(G)$. Then $G(a) \ll G$. Consequently, $B(G) \subseteq Rad(G)$ and $G/Rad(G)$ is an Archimedean l-group which is a subdirect product of simple l-groups.*

Proof. Let $g \in G^{>0}$ and $M \leq G$ such that $0 \leq a \ll g$ and $M \vee G(a) = G$. Suppose that $2g \leq x + na$ (cf. Theorem 7.4 and Corollary 7.6 of [44]), where $x \in M$ and $n \in \mathbb{N}$, then $0 \leq g \leq g + (g - na) \leq x \in M$, whence $g \in M$ and $M = G$. Consequently, $B(G) \subseteq Rad(G)$ and $G/Rad(G)$ is an Archimedean l-group, which is a subdirect product of l-subgroups of $\mathbb{R}$, follows by Proposition 4.1.25 and the fact of $B(G) \subseteq Rad(G)$ proved above. $\square$

The following corollary shows that the radical gives rise to a completely integrally closed overring for an arbitrary Bézout domain.

Corollary 4.2.4 *Let G be an Abelian l-group which is given as the group of divisibility of a Bézout domain D. The localization D_S is a completely integrally closed overring, where $S = \{d \in D \mid U_D d \in Rad(G)\}$.*

Proof. By Proposition 4.2.3 it follows that $G/Rad(G)$ is Archimedean. By Theorem 2.2.8 and Theorem 2.2.2 it follows that D_S is a completely integrally closed overring of D. $\square$

By Lemma 4.2.1 and Proposition 4.2.3, we get

Theorem 4.2.5 *Let G be an Abelian l-group with strong unit u. Then $Rad(G) = B(G)$.*

The following corollary of Theorem 4.2.3 generalizes Yosida-Fukamiya's theorem on Riesz spaces (see [80]), can be proved by use of Theorem 4.2.5 and Proposition 4.1.25.

Corollary 4.2.6 *Let G be an Abelian l-group with strong unit u. Then G is Archimedean if and only if G is semimaximal. Especially, $G/Rad(G)$ is always Archimedean.*

Moreover, we have following result. For a further generalization, see Theorem 4.2.10 below.

Proposition 4.2.7 *Let G be a representable l-group with strong unit u. Then G is Archimedean if and only if G is semimaximal. Especially, $G/Rad(G)$ is always Abelian.*

Proof. If G is Archimedean, then $Rad(G) = 0$ since $\{0\} \subseteq Rad(G) \subseteq B(G) = \{0\}$ by Lemma 4.2.1. Conversely, suppose that G is semimaximal, we want to show that $B(G) = 0$. By Corollaries III and IV on p. 1.23 of [37], it follows that any maximal convex l-subgroup M of G is normal, and thus G/M is an Archimedean totally ordered group (see Proposition 4.1.17 below), and thus G is Archimedean. Especially, $Rad(G/Rad(G)) = \{\bar{0}\}$ implies that $G/Rad(G)$ is Abelian from the fact that an Archimedean l-group is Abelian.

Then Proposition 4.1.25 completes the proof. □

Recall that a *Riesz space* is a lattice-ordered real linear space L as an Abelian group which satisfies $pL^{\geq 0} \subseteq L^{\geq 0}$ for all positive real numbers p. The theory of Abelian l-groups clearly applies to Riesz spaces. In particular, the analogue of Theorem 4.2.5 for Riesz spaces, Theorem 27.5 of [80], can be obtained as a corollary. The following Corollary 4.2.8 gives two necessary and sufficient conditions for a Riesz space with strong unit to be Archimedean.

Corollary 4.2.8 *For a Riesz space L, let B be a Bézout domain whose divisibility group is po-group isomorphic to L, and $u(B) \neq 0$, where $u(B)$ is the unit radical of B. The following statements are equivalent:*

1) L is Archimedean.

2) $Rad(L) = \{0\}$.

3) L is isomorphic to a subdirect product of copies of $\mathbb{R}$.

Proof. 1)⇔2) follows by Corollary 4.2.6 and Theorem 2.2.7.

2)⇔3) is a consequence of the Yosida-Fukamiya-Nakayama Theorem (see [80, 107]). □

Definition 4.2.9 *An l-group G is μ-normal-valued if any maximal convex l-subgroup of G is normal.*

It is plain that both normal-valued and representable l-groups G are μ-normal-valued since the cover of every maximal convex l-subgroup is G. However, a value of an element in G may be not maximal. The following theorem generalizes Corollary 4.2.6 and Proposition 4.2.7 above.

Theorem 4.2.10 *Let G be a μ-normal-valued l-group. Then*

(1) G is a subdirect product of simple l-groups if and only if $Rad(G) = \{0\}$.

(2) $G/Rad(G)$ is a subdirect product of simple l-groups. In particular, $G/Rad(G)$ is Abelian.

Proof. (1): $\Leftarrow$: For each maximal convex l-subgroup M, it is clear that G/M is an Archimedean totally ordered group by Proposition 4.1.17. Therefore, $Rad(G) = \{0\}$ implies that G is a subdirect product of Archimedean totally ordered groups. $\Rightarrow$: If G is a subdirect product of Archimedean totally ordered groups, it is well known that $Rad(G) = \{0\}$.

(2) follows from (1), Proposition 4.1.25 and Proposition 2.1.9. □

For a normal-valued l-group G, Lemmas 2.5 and 2.6 in Bleier and Conrad [20] imply that $Rad(G)$ is a closed l-ideal of G which is generated by the set $\{0 < x \in G \mid x \ll g\}$, and $G/Rad(G)$ is Archimedean. Proposition 53.20 of [44] is a special case of Theorem 4.2.10 in case of a normal-valued l-group.

It is well known that a representable l-group is normal-valued ((see [44], p. 303, Corollary 47.7)). Thus Theorem 4.2.10 can be used for representable l-groups (see Proposition 4.2.7).

Remark 4.2.11 *It is known that any Archimedean l-group can be l-embedded as a large l-subgroup into a complete real vector lattice with a weak unit (cf. [44], Section 54). However, by the above discussion and Proposition 4.1.11 it follows that not all Archimedean l-groups can be embedded as a convex l-subgroup into an Archimedean l-group with strong unit.*

In order to state the next corollary succinctly, we recall the following terminologies. Let D be an integral domain, A an algebra over D. An element x of A is called *integral* over D if the subalgebra $A[x]$ of A is a finitely generated D-module. A is called *integral* over D if every element of A is integral over D. The sub-D-algebra $D^{'}$ of A consisting of the elements of A integral over D is called *integral closure* of D in A. If $D^{'}$ is equal to the canonical image of D in A, D is called *integrally closed* in A. D is said to be *integrally closed* if it is integrally closed in its quotient field. If K is the quotient field of D then an element x of K is said to be *almost integral* over D if there exists a nonzero element y of D such that yx^n is an element of D for each positive integer n. The set $D^{'}_K$ of elements of K almost integral over D is called the *completely integrally closure* of D and D is said to be *completely itegrally closed* if $D^{'}_K = D$.

If G is an l-group, $b \in G^{>0}$ is called *basic* if the set $\{x \in G \mid 0 \leq x \leq b\}$ is totally ordered. A *basis* for G is a maximal (pairwise) disjoint subset

$$\{b_\lambda \mid \lambda \in \Lambda,\ \lambda \text{ is basic}\}.$$

Corollary 4.2.12 *Let D be a Bézout domain such that its divisibility group $G(D)$ has a strong unit. The following statements are equivalent:*

(a) $G(D)$ is a subdirect product of l-subgroups of $\mathbb{R}$.

(b) D is integrally closed and every $a \in D \setminus U_D$ lies in a minimal prime ideal.

(c) $G(D)$ is Archimedean and has a basis.

(d) $Rad(G(D)) = L(G(D)) = D(G(D)) = R(G(D)) = B(G(D)) = \{0\}$.

(e) G is completely distributive.

(f) G is Archimedean.

Proof. (a)⇔(b) follows from [91].

$(a) \Leftrightarrow (c) \Leftrightarrow (d)$ follows from Theorem 5.7 of [42], Theorem 7.3 of [38] and Theorem 4.2.5.

By Theorem 53.17 of [44] it follows that (e)⇔(f). Finally, (a)⇔(f) follows by Corollary 4.2.6 □

As a consequence, we get an explicit description of the minimal completely integrally closed overring for a Bézout domain with non-zero unit radical.

Corollary 4.2.13 *Let G be an Abelian l-group with strong unit, and G be the group of divisibility of a Bézout domain D, then D_S is the minimal completely integrally closed overring $\tilde{D}$, where $S = \{d \in D \mid U_D d \in Rad(G)\}$.*

Proof. It follows from Theorem 2.2.8 and Proposition 4.1.26. □

Note that the minimal completely integrally closed overring $\tilde{D} \neq D_S$ in general as shown by Proposition 4.2.16 below.

If D is an integral domain with quotient field K, we use $D[[x]]$ to denote the ring of formal power series over D in a single indeterminate x. Consider following statements that are known to be valid for a Noetherian integrally closed domain D:

(i) D is an intersection of rank 1 valuation rings,

(ii) D is completely integrally closed,

(iii) $D[[x]]$ is integrally closed,

(iv) D is integrally closed and $\cap_{i=0}^{\infty} a^i D = 0$ for every non-unit $a \in D$.

(v) D is integrally closed and every non-unit of D is in a minimal prime ideal.

It is known that all these conditions are equivalent for a finite intersection of valuation rings (see Ohm [91]), Corollary 1.9). In general (see Ohm [91]), (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) and (v)$\Rightarrow$(iv). Ohm [91] constructs examples to show that (iv)$\not\Rightarrow$(iii)$\not\Rightarrow$(ii)$\not\Rightarrow$(i), (i)$\not\Rightarrow$(v)$\not\Rightarrow$(iii). For Bézout domain with non-zero unit radical, we have

Corollary 4.2.14 *Let D be a Bézout domain such that its divisibility group $G(D)$ has a strong unit and $Rad(G(D)) = \{0\}$, then*

1) $D[[x]]$ is integrally closed, where x is an indeterminate;

2) D is integrally closed and $\cap_{i=0}^{\infty} a^i D = \{0\}$ for any $a \in D \setminus U_D$.

Proof. This follows from Ohm's theorem 0.2 of [91] since D is completely integrally closed. □

From Lemma 4.2.1 we know that the principal convex l-subgroup (of an Abelian l-group) generated by a bounded element must be small. However, we note that a non-zero element of a small convex l-subgroup of an l-group which lacks a strong unit need not be bounded, and thus in general bounded elements are not so simply characterized. Actually, the importance of the existence of the strong unit in Corollary 4.2.6 can be shown by many examples. The Archimedean l-group given below has no maximal l-ideals at all, and so $Rad(G) = G$ (see [44], Example 54.1).

Example 4.2.15 *Let H be the l-group of all integer-valued functions on the interval $[0, 1]$ and let $L = \{h \in H \mid supp(h)$ is countable $\}$. Let $G = H/L$.*

If $B(G) \neq 0$, then there exist $f,\ g \in H$ with $L < L + g \ll L + f$ in G. For each $n \in \mathbb{N}$, let $I_n = \{x \in [0, 1] \mid ng(x) \geq f(x)\}$. It is easy seen that $|I_n| \leq \aleph_0$ since $L + g \leq L + f$. So $|J := \cup_{n=1}^{\infty} I_n| \leq \aleph_0$. Let $r = f|_{[0,1]\setminus J}$ and $s = g|_{[0,1]\setminus J}$. Then $L + r$ and $L + s$ both are non-zero since f and g are not in L. But then $s \ll r$, which is absurd. This shows that $B(G) = 0$ and thus G is Archimedean.

If G has a maximal l-ideal $N = M/L$ for some maximal l-ideal M of H. But $\Sigma_{[0,1]}\mathbb{Z}$ is the basis subgroup of H and is contained in L. So M contains each basic element and thus cannot be maximal. Hence $Rad(G) = G$.

Proposition 4.2.16 *Let G be the l-group in the Example 4.2.15, And D be a Bézout domain with $G(D) \cong_{p-g} G$. Then $\tilde{D} \neq D_S$, where $S = \{d \in D \mid U_D d \in Rad(G)\}$.*

Proof. If $\tilde{D} \neq D_S$, then $\tilde{D}$ is the quotient field K_D of D since $S = \{d \in D \mid U_D d \in Rad(G) = G\} = D$. On the other hand, $D = \tilde{D}$ since G is Archimedean. Therefore, $D = K_D$ and thus D is a completely integrally closed valuation domains, which implies that $G(D)$ is an l-subgroup of $\mathbb{R}$. A contradiction. □

Another example is due to Kaplansky (see [61], Example 5.5.1) which shows that $Rad(D(X))$ is not 0.

Example 4.2.17 *Let X be a quasi-compact extremally disconnected space and $G = D(X)$ (see Example 2.3.5). Then $C(X)$ (see Example 2.3.4) is an l-ideal of G. Moreover, if $z \in X$ and $G_z = \{g \in G \mid g(z) = 0\}$, then G_z is an l-ideal of G. Note that $C(X) + G_z$ comprises all elements of G that have finite value at z. Let U_z be a meagre (a subset S of X is meagre in X if and only if S is the union of a countable family of nowhere dense sets. A subset S of a topological space is nowhere dense if and only if the interior of the closure of S is void) open subset of X containing z. Let f be any function from $X \setminus U_z$ into $\mathbb{R}$, and $f(x) = \infty$ if $x \in U_z$. Then $f \in G$ but $f \notin C(X) + G_z$. So if M is any maximal l-ideal of G, then for each $z \in X$, either there is $f_z \in M$ such that $f_z(z) = \infty$ or $M \subseteq C(X) + G_z$ (and so $M = C(X) + G_z$). Hence, in either case, there are $f_z \in M$ and the open subset I_z of X containing z such that $f_z(x) > 1$ for all $x \in I_z$. Since X is quasi-compact, there is a finite open subcover $\{I_{z_1}, \cdots, I_{z_n}\}$ of X. Let $f = f_{z_1} \vee \cdots \vee f_{z_n}$. Then $f \in M$ and f exceeds the constant function $\underline{1}$. Therefore $\underline{1} \in Rad(G) \neq 0$.*

Remark 4.2.18 *Note that a semimaximal l-group is not necessary semisimple. Otherwise, $G(g)$ is semimaximal by Corollary 4.2.6 for all $g \in D(X)$, and thus $G(g)$ is semisimple. Then $\vee\{G(g) \mid g \in D(X)\} = D(X)$ is semisimple by Proposition 4.1.18, and hence $Rad(D(X)) = 0$ by Corollary 4.1.19. A contradiction.*

Specifically, examples 4.2.15 and 4.2.17 show that there exist completely integrally closed domains which are not an intersection of valuation rings of rank ≤ 1. Note that an example due to Nakayama (see [107], pp. 481-482) shows that the existence of the strong unit is also important even for Archimedean Riesz spaces.

Let G be an l-group. Recall that a convex l-subgroup of G maximal with respect to not containing an element g of G is called a regular subgroup and a *value* of g. Actually, a convex l-subgroup M of G is regular if and only if

$M \subset M^c = \cap\{C \leq G \mid M \subset C\}$. The convex l-subgroup M^c is called the *cover* of M, and M is a *normal value* if M is a normal subgroup of M^c. An element g of G is *special* if g has a unique value M, and in this case M is called a *special value*. G is *finite-valued* if every element has only a finite number of values. Now we give the definition of a μ-valued l-group which is a generalization of a finite-valued l-group.

Definition 4.2.19 *An l-group G is called to be μ-valued if every maximal convex l-subgroup of G is special.*

It is clear that a μ-valued l-group need not be finite-valued, since a value may be not a maximal convex l-subgroup. However, we have following:

Lemma 4.2.20 *A finite-valued l-group is μ-valued.*

Proof. It follows from Proposition 2.1.14 and the fact that a maximal convex l-subgroup is a value of some element. □

By definition, an Abelian l-group G is semimaximal if and only if G is a subdirect product of simple l-groups. For μ-valued l-groups, we have

Theorem 4.2.21 *Let G be a μ-valued l-group. Then G is semimaximal if and only if G is l-isomorphic to a subdirect product of semimaximal l-groups without proper l-ideal.*

Proof. "⇒": Let $In(G) = \{f \mid f \text{ is an inner automorphism of } G\}$ and

$$\mathbb{M} = \{M \leq G \mid M \text{ maximal}\}.$$

Define a relation $\backsim$ on $\mathbb{M}$: for all $M_1,\ M_2 \in \mathbb{M}$, $M_1 \backsim M_2$ if there exists $f \in In(G)$ such that $M_2 = f(M_1)$. Then $\backsim$ is an equivalence relation on $\mathbb{M}$, and thus the class $\{C_i \mid i \in I\}$ of all equivalence classes C_i forms a partition of $\mathbb{M}$. For all C_i, it is clear that $N_i = \cap C_i$ is an l-ideal of G and

$$\cap_{i\in I} N_i = Rad(G) = \{0\}.$$

Let $X \neq G/N_i$ be an arbitrary l-ideal of G/N_i. Then X corresponds to an l-ideal $N_i \leq \mathcal{X} \neq G$. Furthermore, there exists an $M \in C_i \subseteq \mathbb{M}$ such that

$$M \supseteq \mathcal{X} \supseteq N_i$$

and M is the special value of any element $a \notin \mathcal{X}$. Also, $C_i = \{f(M) \mid f \in In(G)\}$ and $f(M) \supseteq f(\mathcal{X}) = \mathcal{X}$, which implies

$$\mathcal{X} \subseteq \cap_{f \in In(G)} f(M) = \cap_{M \in C_i} M = N_i,$$

whence $N_i = \mathcal{X}$. Moreover, it is easily seen that there is a one-to-one correspondence between the set $\{\mathcal{X} \in \mathbb{M} \mid N_i \subseteq \mathcal{X}\}$ and the set $\{X \mid X$ is maximal convex l-subgroups of $G/N_i\}$ by Proposition 2.1.6, and hence

$$Rad(G/N_i) = \cap_{N_i \subseteq \mathcal{X} \in \mathbb{M}} \mathcal{X}/N_i = (\cap_{N_i \subseteq \mathcal{X} \in \mathbb{M}} \mathcal{X})/N_i = \{0\}.$$

Therefore, $\cap_{i \in I} N_i = \{0\}$ and $Rad(G/N_i) = \{0\}$ and G is l-isomorphic to a subdirect product of semimaximal l-groups without proper l-ideal by Definition 2.1.11.

"$\Leftarrow$": Assume that G is l-isomorphic to subdirect product $\Pi_{i \in I} G_i$ of semimaximal l-groups G_i, $i \in I$, without proper l-ideal. Let α be an injective l-group homomorphism from G to $\Pi_{i \in I} G_i$ Then $\alpha_i = p_i \alpha$ is a surjective l-group homomorphism from G to G_i, where p_i is the projection from $\Pi_{i \in I} G_i$ onto G_i. Corollary 4.1.23 implies that

$$\alpha_i(Rad(G)) \subseteq Rad(G_i) = \{0\},$$

and thus $Rad(G) \subseteq Ker(\alpha_i)$ implies

$$Rad(G) \subseteq \cap_{i \in I} Ker(\alpha_i).$$

For all $a \in \cap_{i \in I} Ker(\alpha_i)$, $\alpha_i(a) = 0$ implies $\alpha(a) = 0$ implies $a = 0$, whence $Rad(G) = \{0\}$. □

Corollary 4.2.22 ([106]) *A finite-valued l-group G is semimaximal if and only if G is l-isomorphic to a subdirect product of semimaximal l-groups without proper l-ideal.*

Proof. It follows from Lemma 4.2.20 and Theorem 4.2.21. □

Corollary 4.2.23 *Let a Bézout domain D be a finite intersection of valuation rings. The following statements are equivalent:*

1) D is an intersection of rank 1 valuation rings.

2) D is completely integrally closed.

3) $D[[x]]$ is integrally closed, where x is an indeterminate.

4) $\cap_{i=0}^{\infty} a^i D = \{0\}$ for any $a \in D \setminus U_D$.

5) D is integrally closed and every $a \in D \setminus U_D$ is in a minimal prime ideal.

6) $G(D)$ is Archimedean.

7) $Rad(G(D)) = \{0\}$.

Proof. Theorem 2.2.8 implies that $G(D)$ has only finitely many prime ideals and thus finite-valued since any value is prime. By Corollary 4.2.22 it follows that $Rad(G(D)) = \{0\}$ if and only if $G(D)$ is l-isomorphic to a subdirect product of finitely many semisimple l-groups. Thus 1)⇔7) and 7)⇒6). Furthermore, 2)⇔6) holds by Theorem 2.2.2. 2)⇔3)⇔4)⇔5) follows by Corollary 1.9 of [91]. 1)⇒2) is clear.

2)⇒1): For any minimal prime ideal P of D, the quotient ring $D_{D\setminus P}$ of D with respect to the multiplicative system $D \setminus P$ is a rank one valuation ring, by Theorem 2.2.8 and the well known result of Krull. It is easy to see that

$$\cap D_{D\setminus P} := \cap_{\{P \lhd D \ \mid \ P \text{ minimal prime}\}} D_{D\setminus P} = D_{D\setminus \cup P} := D_{D\setminus \cup\{P \ \mid \ P \text{ minimal prime ideal}\}}$$

since $D \setminus \cup\{P\} = \cap\{D \setminus P\}$. Therefore, 5) implies that $D = D_{D\setminus \cup P} = \cap D_{D\setminus P}$, and hence 1) holds. □

Proposition 4.2.24 *Let G be an l-group with no proper l-ideals, and with strong unit. Then G is semimaximal.*

Proof. $Rad(G) \neq G$ since a value of a strong unit is maximal. □

Recall that an l-group is said to be *hyperarchimedean* if all of its l-group homomorphic images are Archimedean. Note that for an Archimedean finite-valued l-group G, Theorem 55.6 of [44] shows that every minimal prime subgroup is closed, and thus G is hyperarchimedean by Theorem 53.17 of [44], since G is a hyperarchimedean l-group if and only if every proper prime l-ideal of G is maximal.

Recall that an l-group G satisfies *the ascending chain condition* if, every nonempty set of convex l-subgroups of G satisfies the ascending chain condition. The following theorem of Conrad [37] (see also [63]) completely characterizes l-groups which satisfies the ascending chain condition.

Proposition 4.2.25 *An l-group G satisfies the ascending chain condition if and only if G is a lex-sum of finitely many linearly ordered groups, and each linearly ordered group used in the construction satisfies the ascending chain condition as well.*

As a consequence, we get

Corollary 4.2.26 *Let G be an l-group with the ascending chain condition. Then $Rad(G) = \{0\}$ if, and only if, G is a subdirect product of finitely many simple l-groups with strong unit.*

Proof. By Theorem 5.3.4 on p. 122 of [63] and the Corollary 4.2.22 above it follows that G is l-isomorphic to a subdirect product of semimaximal simple l-groups G_i ($i \in I$). For all $a,\ b \in G_i$, $G_i(a) \leq G_i(a) \vee G_i(b) = G_i(a \vee b)$ implies that there exists a finite ascending chain $G_i(a_1) < G_i(a_2) < \cdots < G_i(a_n) = G_i(a_{n+1}) = G_i$, that is, a_n is a strong unit. □

Corollary 4.2.27 *Let G be an Abelian l-group with the ascending chain condition. The following statements are equivalent:*

(1) $Rad(G) = \{0\}$.

(2) G is a subdirect product of finitely many l-subgroups of $\mathbb{R}$.

(3) G is the group of divisibility of a completely integrally closed Bézout domain which is an intersection of rank 1 valuation domains.

Proof. It follows from Corollaries 4.2.26 and the existence of strong units in G from the proof of Corollary 4.2.26. □

4.3 MV-algebras

As a vehicle to study algebraically the infinite-valued logic of Łukasiewiez, the theory of MV-algebras has been developed considerably since Chang's pioneering paper [33], where MV is supposed to suggest $many - valued$ logics. Equivalent algebraic systems are found in the literature under various names, including *bounded commutative BCK-algebra, Bosbach's bricks, Buff's S-algebra, Komori's CN-algebra, Rodriguez's Wajsberg algebras*. MV-algebras also provide an interesting example of "*quantum structures*" (see [52, 96]). Furthermore, one can generalize measure theory on MV-algebras, and use MV-algebras to interpret AF C^*-algebras (see [32, 33, 88]).

Recall that an *MV-algebra* in the sense of Gispert and Mundici [60] is an Abelian monoid $(A, +, 0)$ with an involution map $^* : A \to A$ (i. e. $(x^*)^* = x$ for all $x \in A$), such that the following axioms are satisfied for all $x, y \in A$.

(0) $x + 0^* = 0^*$, and

(1) $y + (y + x^*)^* = x + (x + y^*)^*$ (Łukasiewiez axiom).

We remark that these axioms admit a further simplification since (0) is redundant as shown by the proposition below.

Proposition 4.3.1 *An Abelian monoid $(A, +, 0)$ with an involution map $* : A \to A$ is an MV-algebra if and only if the Łukasiewiez axiom is satisfied.*

Proof. It suffices to prove that the axiom (1) implies the axiom (0). Let $y = 0$ in (1), we get $0 + (0 + x^*)^* = x + (x + 0^*)^*$, that is, $x = x + (x + 0^*)^*$ for all $x \in A$. Hence $(x + 0^*)^* = 0$ by the well known uniqueness of the identity element in a monoid. Therefore, $x + 0^* = ((x + 0^*)^*)^* = 0^*$, as desired. $\square$

For any $x, y \in A$ we write

$$x \leq y \text{ if and only if } x^* + y = 1 := 0^*.$$

Then $\leq$ induces a partial order relation ([33]), called *natural order* of A. Specifically, the natural order endows A with a bounded distributive lattice structure, where the join $x \vee y$ and the meet $x \wedge y$ can be defined by

$$x \vee y := x + (x + y^*)^*, \ x \wedge y := (x^* \vee y^*)^*$$

respectively. Then the condition (1), the so-called Łukasiewiez axiom, states that $x \vee y = y \vee x$. Furthermore, it is natural to introduce a multiplication by

$$x \cdot y := (x^* + y^*)^*.$$

For all $n \in \mathbb{N}$ and $x \in A$, the MV-operations nx and x^n are inductively defined by

$$0x := 0, (n + 1)x := x + nx \text{ and } x^0 := 1, x^{n+1} := x \cdot x^n.$$

The operation x^n takes precedence over any other operation, * takes precedence over $\cdot$, and $\cdot$ takes precedence over $+$. For more details of MV-algebras, the reader is referred to [34, 52, 60, 88].

It is known that the classical two-valued logic gives rise to the study of Boolean algebras and, as can be expected, every Boolean algebra will be an MV-algebra whereas the converse does not hold. Actually, from an axiomatic point of view,

$$\textit{MV-algebra + idempotency = Boolean algebra.}$$

In other words, the distinguishing feature between an MV-algebra and a Boolean algebra is the lack of the idempotent law $x + x = x$ (see Chang [33]). We mention also that, while various generalizations of the Boolean algebras which do not satisfy the law of the excluded middle are known (e.g., all kinds of lattices), there are very few generalizations of the Boolean algebra where the idempotent law does not hold. In fact, the system $(\mathcal{B}(A) = \{x \in A \mid x + x = x\}, +, \cdot, ^*, 0, 1)$ is not only a subalgebra of A but is also the largest subalgebra of A which is at the same time a Boolean algebra with respect to the same operations of A.

4.3.1 Examples and Mundici's functorial equivalence

The first and most important example of an MV-algebra is the algebra L obtained by considering the $\aleph_0$-valued (Łukasiewiez) propositional calculus (see [33]).

Another class of examples of MV-algebras is obtained by considering the set of real numbers between 0 and 1:

Example 4.3.2 *The real unit interval* $[0, 1]$ *equipped with* $x^* := 1 - x$ *and addition* $(x, y) \mapsto min\{1, x + y\}$ *(for all* $x, y \in [0, 1]$*) is an MV-algebra.*

Note that any subalgebra, homomorphic image, or direct product of MV-algebras will again be an MV-algebra. The most general example of an MV-algebra is

Example 4.3.3 *Let* (G, u) *be an Abelian* l*-group with strong unit* u *(unital* l*-group for short). Let* $[0, u] := \{x \in G \mid 0_G \leq x \leq u\}$, $0 := 0_G$, $x^* := u - x$, *and* $x + y = u \wedge_G (x +_G y)$.

Then $\Gamma(G, u) = ([0, u], 0, +,^*)$ *is an MV-algebra (see [88]). Furthermore, given a unital* l*-group homomorphism* $\theta : (G, u) \to (H, v)$*, the restriction* $\Gamma(\theta)$ *of* θ *to* $[0, u]$ *is an MV-homomorphism (i. e.* $\Gamma(\theta)$ *respects the operations* $+$, * *and the element* 0. *Note that* $a \vee b = (a \vee_G b) \wedge_G u$*, and* $a \wedge b = a \wedge_G b$ *for all* $a, b \in [0, u]$.

$$\begin{array}{ccccc} \mathrm{Ab}_l^* & \xrightarrow{\Gamma} & \mathrm{MV}_{\mathrm{Alg}} & \xrightarrow{\qquad\Gamma^-\qquad} & \mathrm{Ab}_l^* \\ (G,u) & \longmapsto & \Gamma(G,u) & \longmapsto & \Gamma^-(\Gamma(G,u)) \cong_{l-g} (G,u) \end{array}$$

Figure 4.1: Mundici's functor and its inverse

Actually, Γ in the example above is just Mundici's functorial equivalence [89] from the category of Abelian l-groups with strong unit, with l-group homomorphisms preserving strong unit as arrows, onto the category of MV-algebras, with MV-homomorphisms as arrows.

Proposition 4.3.4 ([88], Theorems 2.5 and 3.8) *Let* Ab_l^* *be the category of Abelian* l*-groups with strong unit, and let MV-Alg be the category of MV-algebras. Then the map* Γ *is a functorial equivalence from* Ab_l^* *onto MV-Alg.*

For the case of linearly ordered MV-algebras, the above proposition has been proved by Chang [33].

Although the axioms of an MV-algebra have undergone a drastic simplification since Chang's basic article [33] (with 20 axioms), it seems that unital l-groups are still easier to handle than MV-algebras. Actually, in this section and the next two sections we will show that the proofs of several results for MV-algebras can be simplified by means of l-group theory via Mundici's equivalence Γ: $Ab_l^* \to MV_{Alg}$

Let $\{A_\lambda\}_\Lambda$ be a set of po-algebras. On the cartesian product $\times A_\lambda$, place the componentwise operations. The resulting po-algebra $\Pi_\Lambda A_\lambda$ is called the direct product of $\{A_\lambda\}_\Lambda$. A po-algebra A is a *subdirect product* of po-algebras $\{A_\lambda\}_\Lambda$ if there exists an injective po-homomorphism σ of A into $\Pi_\Lambda A_\lambda$ such that for each projection ρ_μ: $\Pi_\Lambda A_\lambda \to A_\mu$, $\rho_\mu\sigma$ is surjective. Then the following theorem (see [88], Theorem 2.3; [32], Lemma 3) is an obvious corollary from the theory of l-groups. Note that (ii) generalizes the corresponding result in [53] which holds only for "finitely subdirectly irreducible".

Theorem 4.3.5 *(i) Every MV-algebra is a subdirect product of totally ordered MV-algebras.*

(ii) Any MV-algebra is a distributive lattice.

Proof. ii) follows by i), and (i) follows from Propositions 4.3.4 and 2.1.13. □

4.3.2 Ideal correspondence of Cignoli and Torrens

Recall that a subset I of an MV-algebra A is said to be an *ideal* of A if

(i) $0 \in I$,

(ii) if $x, y \in I$, then $x + y \in I$, and

(iii) if $x \in I$ and $y \leq x$, then $y \in I$.

The *distance function* of A is defined as follows:

$$d(x, y) = (x^* \cdot y) + (y^* \cdot x)$$

for all x and y in A. If I is an ideal, then the relation defined by xRy if and only if $d(x, y) \in I$ is a congruence relation (see Theorems 3.14 and 4.3 of [33]), and $A/I = A/R$ where R is the unique congruence relation associated with I. Equivalently, by an *ideal* I of an MV-algebra A we mean the kernel I of a homomorphism h of A into some MV-algebra B. An ideal I of A is *prime* provided that $I \neq A$ and $a \wedge b \in I$ implies $a \in I$ or $b \in I$. An ideal M of A is *maximal* if and only if M is a maximal proper ideal.

Lemma 4.3.6 ([34], Theorem 1.2, Corollary 1.3) *Let (G, u) be an Abelian l-group with strong unit u. The correspondence*

$$\phi : J \mapsto \{x \in G \mid |x| \wedge u \in J\}$$

defines an isomorphism from the poset of ideals of $A = \Gamma(G, u)$ onto the poset of l-ideals of G. The inverse isomorphism is given by the correspondence

$$\psi : H \mapsto H \cap [0, u].$$

Furthermore, let Spec(G) be the set of all proper prime l-ideals of G, and let Spec($\Gamma(G, u)$) be the set of prime ideals of $\Gamma(G, u)$. Then

$$(Spec(G, u), \subseteq) \cong_o Spec(\Gamma(G, u), \subseteq).$$

One can borrow from ring theory the concepts of essential submodule and socle, superfluous submodule and radical, transferring them profitably into the context of an MV-algebra A. Then the socle is the least upper bound of the minimals and the greatest lower bound of the essentials (see [12]), the radical is the least upper bound of the superfluous and the greatest lower bound of the maximals, in the lattice of ideals of A. These results are direct consequences of Proposition 4.1.21, Lemmas 4.3.6 and 4.1.22.

Let A be an MV-algebra. The *order* of an element $x \in A$, in symbols $ord(x)$, is the least integer m such that $\overbrace{x + \cdots + x}^{m} = 1$. If no such integer exists then $ord(x) := \infty$. We call an MV-algebra A *locally finite* if and only if every element of A different from 0 has a finite order. An MV-algebra is *representable* if and only if it is isomorphic to a subalgebra of a direct product of locally finite MV-algebras.

We will see below that locally-finite MV-algebras correspond to simple l-groups.

Theorem 4.3.7 *Let A be an MV-algebra, I be an ideal of A, and let (G, u) and I_G be the Abelian l-group and the l-ideal corresponding to A and I respectively (Proposition 4.3.4). Moreover, let B be any Bézout domain whose divisibility group is l-isomorphic to G, and let $S \subseteq B$ denote the multiplicative system corresponding to I_G. Then the following statements are equivalent:*

(a) I is a prime ideal of A.

(b) A/I is a linearly ordered MV-algebra.

(c) I_G is a prime l-ideal of G.

(d) G/I_G is a linearly ordered Abelian l-group with strong unit.
(e) $B \setminus S$ is a prime ideal of B.
(f) The localization B_S is to G/I_G is a valuation domain.

Proof. $(a) \Leftrightarrow (c)$ follows by Lemma 4.3.6. $(c) \Leftrightarrow (e) \Leftrightarrow (d) \Leftrightarrow (f)$ follows by Theorem 2.2.8. Furthermore, $(a) \Rightarrow (b)$ follows by Lemma 1 of [32], and $(b) \Rightarrow (a)$ follows from the definitions. □

By a well known result of Abelian l-group theory, we know that for all $g \neq 0$ in G, there exists a prime l-ideal P_G of G such that $g \notin P_G$. Thus if a is a nonzero element in an MV-algebra A, then there exists also a prime ideal P of A such that $a \notin P$. Especially, $\cap Spec(A) = \{0\}$.

For maximal ideals, we have the following theorem which shows that for a locally finite MV-algebra A, the l-group G with $\Gamma(G, u) \cong A$ is simple, and vice versa.

Theorem 4.3.8 *Let A be an MV-algebra, I be an ideal of A, and let (G, u) and I_G be the Abelian l-group and the l-ideal corresponding to A and I respectively (Proposition 4.3.4). Moreover, let B be any Bézout domain whose divisibility group is l-isomorphic to G, and let $S \subseteq B$ be the multiplicative system corresponding to I_G. Then the following statements are equivalent:*
(a) I is maximal.
(c) A/I is locally finite.
(b) I_G is maximal.
(d) G/I_G is simple.
(e) $B \setminus S$ is a minimal prime ideal.
(f) B_S is a completely integrally closed valuation domain.

Note that [52] states that (d) is only a "partial converse " of (b) (see Theorems 2.3.38 and 2.2.16 of [52]). However, we prove that (d) and (b) are equivalent.

Proof. $(a) \Leftrightarrow (c)$ is a consequence of Lemma 4.3.6. The equivalences $(c) \Leftrightarrow (e) \Leftrightarrow (d) \Leftrightarrow (f)$ follow from Theorem 2.2.8 and Proposition 4.1.17. Furthermore, the equivalence $(c) \Leftrightarrow (d)$ follows by Theorem 4.3.7, Proposition 4.1.17 and the fact that the strong unit in Proposition 4.3.4 can be arbitrarily chosen. □

An MV-algebra A is *Archimedean* in Dvurečenskij's sense[1] if for each $x \in A$, $nx \leq x^*$ for all $n \in \mathbb{N}$ entails $x = 0$. Note that representable MV-

[1]Archimedean in Belluce's sense: for all $x, y \in A$, $nx \leq y (\forall n \in \mathbb{N})$ implies $x \cdot y = x$.

algebras correspond to unital Abelian l-groups that are Archimedean. Thus we can define the radical $Rad(A)$ (the intersection of maximal ideals) of an MV-algebra A. Then $Rad(A) = 0$ if and only if A is Archimedean.

Theorem 4.3.9 *Let A be an MV-algebra, and let (G, u) be the Abelian l-group corresponding to A (Proposition 4.3.4). Moreover, let B be any Bézout domain whose divisibility group is l-isomorphic to G. Then the following statements are equivalent:*

(a) A is representable.

(b) $Rad(A) := \cap\{M \mid M \text{ is a maximal ideal of } A\} = \{0\}$.

(c) $Rad(G) = \{0\}$.

(d) G is a subdirect product of the reals.

(e) B is the intersection of rank ≤ 1 valuation overrings in its quotient field.

(f) A is Archimedean.

(g) G is Archimedean.

Proof. By the definition of a representable MV-algebra, Theorem 4.3.7, and Lemma 4.3.6, it follows that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$. The equivalences between (d) and (e) follow by Corollary 4.2.6. The equivalence $(b) \Leftrightarrow (f)$ follows from the definition and Proposition 4.3.12. Finally, $(d) \Leftrightarrow (g)$ is trivial since there exists a strong unit in G. □

Remark 4.3.10 *Note that the definitions of an Archimedean MV-algebra in Belluce [10] and Dvurečenskij [51] are equivalent by Theorem 4.3.9.*

The following corollary is a direct consequence of Proposition 4.1.25 and 4.3.9 above, and Theorem 4.3 of [33] which implies that $A/Rad(A)$ is always a representable MV-algebra.

Corollary 4.3.11 *Let A be an MV-algebra. Then $A/Rad(A)$ is a representable MV-algebra.*

Belluce [9] has shown that $(Rad(A))^2 = 0$ holds for an MV-algebra. More precisely, we have

Proposition 4.3.12 *If A is an MV-algebra, then the following statements are equivalent:*

(a) $x \in Rad(A)$.

(b) $nx \leq x^*$ for every natural number n.

(c) $(nx)^2 = 0$ for every natural number n.

The property that $Rad(A)$ is a closed ideal is a relaxed form of the well known fact in l-groups that $Rad(G)$ for any unital l-group G is a closed l-ideal (see the note before Proposition 4.2.3).

Proof. Let (G, u) be the corresponding l-group to A, where $0 < u$ is a strong unit.

$(a) \Rightarrow (b)$: $x \in Rad(A)$ implies (see Proposition 4.1.22) that

$$G(x) = G(nx) \ll G = G(u) = G(u - x)$$

for every natural number n, which gives $nx \leq x^*$.

$(b) \Rightarrow (c)$: For every natural number n, $(nx)^2 = ((nx)^* + (nx)^*)^* = u - ((nx)^* + (nx)^*) \wedge u = u - (2u - 2nx) \wedge u = u - u = 0$.

$(c) \Rightarrow (a)$: $(nx)^2 = 0$ implies $u - (2u - 2nx) \wedge u = 0$ for every natural number n. It follows that $x \ll u$ and thus $x \in Rad(G)$ by Proposition 4.1.22. Hence, $x \in Rad(A)$. $\square$

Actually, for an l-group with strong unit u, Lemma 3.5 in Bleier and Conrad [19] shows that $Rad(G)$ is generated by the set of infinitely small (or bounded) elements of G, and thus the radical $Rad(A)$ of $A := \Gamma(G, u)$ is the same as $Rad(G)$, which implies Proposition 4.3.12. However, we note that the set of infinite order elements in A is larger than $Rad(A)$, in general.

Belluce [11] defines an MV-algebra A to be *hyperarchimedean* (it is also called *quasi-locally finite* in [10]) if for each $x \in A$ there is an $n \in \mathbb{N}$ such that $nx + nx = nx$. Belluce ([10], Theorem 4) shows that A is hyperarchimedean if and only if all prime ideals are maximal. On the other hand, Theorem 2.4 of [37] (see also Lemma 5.8.2 of [61] or Theorem 55.1 of [44]) states that an l-group G is hyperarchimedean if and only if every prime subgroup is maximal. Hence, we have

Corollary 4.3.13 *An MV-algebra A is quasi-locally finite if and only if the corresponding l-group G with $\Gamma(G) \cong A$ is hyperarchimedean.*

As a further example, we reprove a result on hyperarchimedean MV-algebras using l-group theory.

Corollary 4.3.14 *If an MV-algebra A is hyperarchimedean then A is Archimedean.*

Proof. The corresponding l-group G must have zero radical by Corollary 4.2.6 since it is Archimedean with strong unit. Thus Theorem 4.3.9 completes the proof. $\square$

Note that the above corollary is equivalent to Theorem 3 of [10]. Moreover we note that the proof of Theorem 4 of [10] requires the following result which is in fact a consequence of [102].

Proposition 4.3.15 *For arbitrary a, b in an MV-algebra we have $ma \wedge nb \leq mn(a \wedge b)$.*

Proof. By [102] (I, p. 11) we know that $ma \wedge nb \leq (m+n-1)(a \wedge b)$ for all positive elements a, b and $n, m \in \mathbb{N}$. Thus the lemma follows. □

4.4 A problem of annihilators and primes in MV-algebras

Let A be an MV-algebra, $\varnothing \neq X \subseteq A$. Recall that the *annihilator* of X is

$$X^{\perp} = \{a \in A \mid a \wedge x = 0, \text{ for all } x \in X\}.$$

Theorem 25 of [9] states that each annihilator of A is an ideal. Furthermore, Belluce shows that

Theorem 4.4.1 ([9], Theorem 26) *The annihilator of a linearly ordered ideal is a prime ideal.*

The question whether the converse always holds remained open (see Belluce [9], p. 1368, notice under the proof of Theorem 26):

Question 4.4.2 *Is it true that every prime ideal of an MV-algebra A can be obtained as an annihilator of a linearly ordered ideal of A?*

Via l-group theory, we show in this section that this holds for exactly those MV-algebras for which every prime ideal is minimal and closed. In particular, this implies a negative answer to Belluce's question.

Lemma 4.4.3 *Let $\mathcal{P}(G)$ be the po-set of polars of a unital l-group (G, u), and let $\mathcal{A}(\Gamma(G))$ be the po-set of annihilators of the corresponding MV-algebra $\Gamma(G, u)$. Then there exists an order-preserving one-to-one correspondence between $\mathcal{P}(G)$ and $\mathcal{A}(\Gamma(G))$.*

Proof. Recall that $S_G^\perp = \{g \in G \mid |g| \wedge |s| = 0\ \forall s \in S\}$ and $T_{\Gamma(G)}^\perp = \{a \in \Gamma(G) \mid a \wedge t = 0\ \forall t \in T\}$ for all subsets S and T of G and $\Gamma(G)$, respectively. Define the maps

$$\psi : S^\perp \mapsto S^\perp \cap [0,u]$$

and

$$\phi : T^\perp \mapsto \{g \in G \mid |g| \wedge t \in T^\perp\}.$$

Then ψ and ϕ are order-preserving, and
$\psi(S_G^\perp)$
$= S_G^\perp \cap [0,u]$
$= \{g \in \Gamma(G) \mid g \wedge |s| = 0\ \forall s \in S\}$
$= \{g \in \Gamma(G) \mid g \wedge |s| \wedge u = 0\ \forall s \in S\}$
$= (|S| \cap [0,u])^\perp$
$= |S|_{\Gamma(G)}^\perp \in \mathcal{A}(\Gamma(G))$ since u is a strong unit, where $|S| = \{|s| \mid s \in S\}$. Analogously, we can prove that $\phi(T_{\Gamma(G)}^\perp) = T_G^\perp \in \mathcal{P}(G)$. Furthermore, it is straightforward to verify that $\psi\phi(T_{\Gamma(G)}^\perp) = T_{\Gamma(G)}^\perp$ and $\phi\psi(S_G^\perp) = S_G^\perp$. □

The following theorem is a generalization of Belluce's Theorem 4.4.1 above.

Theorem 4.4.4 *A convex sub-MV-algebra I of an MV-algebra A is totally ordered if and only if $I^\perp$ is a prime ideal.*

Proof. It follows from Lemmas 4.3.6 and 4.4.3 above, and Theorem 19.1 (that is, a convex l-subgroup S of an l-group is totally ordered if and only if $S^\perp$ is a prime subgroup) of [44] or Theorem 5.2.1 of [63]. □

Theorem 4.4.5 *Each prime ideal of an MV-algebra A can be obtained as an annihilator of a linearly ordered ideal of A if and only if A is hyperarchimedean and any element in Spec(A) is closed.*

Proof. By Lemmas 4.3.6 and 4.4.3 above, and [44] Theorem 53.6 (that is, an l-group is Archimedean if and only if every closed convex l-subgroup is a polar), it suffices to prove the necessity. By Theorem 26 of [9] it follows that the annihilator of a linearly ordered ideal is a prime ideal. By Lemma 4.3.6 above, Theorems 13.1 (that is, every polar is a convex l-subgroup), 19.1 and 53.6, Proposition 21.11 (that is, every polar is closed) of [44] we infer that each prime ideal of A is minimal and closed, as required. □

Remark 4.4.6 *Note that there are many unital l-groups which are not hyperarchimedean. Therefore, there exists at least one MV-algebra A with a prime ideal which is not an annihilator of any linearly ordered ideal of A. In addition, the existence of an MV-algebra (see, for instance, [11], p. 342) with non-quasi-compact Min-Spectrum $Min(A) = \{P \in Spec(A) \mid P \text{ minimal}\}$, and Lemma 4.4.5 also implies that the problem of Belluce has a negative answer in general.*

Let α be a cardinal number. Recall that a lattice A is *α-complete* if $\vee_{i\in I} a_i \in A$ and $\wedge_{i\in I} a_i \in A$ for any index set I with $|I| \leq \alpha$. We call an MV-algebra A *α-complete* if as a lattice, A is α-complete. We call A *complete* if it is $|A|$-complete, where $|A|$ is the cardinality of A.

Remark 4.4.7 *Another problem which is unanswered in Belluce [9] asks whether or not a complete MV-algebra is Archimedean. It has been answered several years later in his paper [10]. However, this problem can be transferred (see Proposition 4.4.8) to l-group theory, since it is well known that a (conditionally) complete l-group is Archimedean (see [16], p. 313, Corollary 2). This implies that the corresponding l-group is semimaximal by Corollary 4.2.6, and thus any complete MV-algebra must be Archimedean by Theorem 4.3.9.*

Proposition 4.4.8 *([7], Theorem 5.15) An MV-algebra is complete if and only if $\Gamma^{-}(A)$ is complete.*

As an another example, we answer an open problem of Belluce, Di Nola and Sessa [11] in two special cases.

Call an MV-algebra A *Boolean dominated* if for each $x \in A$ with $ord(x) = \infty$ there is an $e \in \mathcal{B}(A) = \{x \in A \mid x + x = x\}$ such that $x \leq e < 1$. An open problem in [11] is: *Whether any Boolean dominated MV-algebra A is hyperarchimedean?* We shall give an affirmative answer to this problem for two special cases. We call A is *properly Boolean dominated* if $\mathcal{B}(A) \setminus \{1\}$ is closed with respect to $+$.

Proposition 4.4.9 *Every properly Boolean dominated MV-algebra A is hyperarchimedean.*

Proof. Suppose A is a properly Boolean dominated MV-algebra. Let P be an element of $Spec(A)$, M a maximal ideal with $P \subseteq M$. We want to show that $P = M$.

We prove that M contains every majorizing Boolean element e_x for all $x \in M$ such that $x \leq e_x < 1$. In fact, for all $x, y \in M$ we have

$$x + e_x \leq e_x + e_x = e_x < 1,\ x + e_y \leq e_x + e_y < 1.$$

Thus for each $x \in M$, $e_x \in P$ and $x \in P$. So $P = M$. □

Proposition 4.4.10 *Every Boolean dominated linearly ordered MV-algebra A is hyperarchimedean.*

Proof. Let $\Gamma^-(A) = G$. Then G is linearly ordered. By Proposition 4.4.9 it suffices to prove that A is properly Boolean dominated. Indeed, for all $x, y \in A \setminus \{1\}$ and $e_x, e_y \in \mathcal{B}(A) \setminus \{1\}$ such that $x \leq e_x < 1$, $y \leq e_y < 1$, we have $e_x = e_y = 0$. Otherwise, there is no loss of generality in assuming that $e_x > 0$. Then we have $(e_x +_G e_x) \wedge_G 1 = e_x \in \mathcal{B}(A) \setminus \{1\}$. On the other hand, we have $e_x < (e_x +_G e_x) \wedge 1 = 1$ or $2e_x$ since G is linearly ordered. A contradiction. Hence $e_x = 0$. □

Finally, we note that Di Nola and Lettieri [49] prove that the category of perfect MV-algebras is equivalent to the category of Abelian l-groups. This is due to the fact that for an Abelian l-group G, the lexicographical product $G \oplus_l \mathbb{Z}$ admits a natural strong unit (see Example 2.3.9). Dvurečenskij [51] shows that the category of arbitrary l-groups with strong unit is equivalent to the category of pseudo MV-algebras; Galatos and Tsinakis obtained that the categories of generalized MV-algebras and arbitrary l-groups are equivalent. We believe that the research of these MV-algebra categories will be further stimulated by its interaction with l-group theory.

4.5 Asano and local Bézout domains

Recall that a ring R which admits a full quotient ring Q is said to be an *Asano order* (see [66]). if every integral ideal of R has an inverse in Q. Here an ideal of R is said to be integral if it contains a unit of Q. An ideal I of a ring R is $full$ if there is an invertible ideal $P \subseteq I$, and R is called an *Asano* ring if every full ideal of R is invertible (see [97, 98]). W. Rump [98] proves that every invertible ideal is integral for a Noetherian ring. Furthermore, he gives several characterizations of Asano rings. We define a ring R to be *quasi-Asano* if every ascending chain of invertible ideals in R becomes stationary. It is clear that an Asano ring is quasi-Asano, but the converse does not hold in general. We call an l-group G *Asano* if $S \leq p$ implies that $\wedge S$ exists for all subsets $S \subseteq G^{\geq 0}$ with $p \in G^{>0}$.

In this section we shall characterize Asano and local Bézout domains via divisibility groups. We first give a characterization of Asano Bézout domains.

Theorem 4.5.1 *Let D be a Bézout domain. Then the following statements are equivalent:*

(1) D is quasi-Asano.
(2) $G(D)$ is a cardinal sum of copies of $\mathbb{Z}$.
(3) D is factorial.
(4) D is Asano.
(5) D is Noetherian.
(6) D is a principal ideal domain.
(7) $G(D)$ is Asano.

Proof. By Theorem 4.2.2 of [86] it follows (2) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (2) : By the definition of a divisibility group it follows that a Bézout domain D is quasi-Asano if and only if $G(D)$ satisfies the descending chain condition on positive elements, since an invertible ideal of a commutative ring is finitely generated by Lemma 2 and the Corollary of [98] (see also Theorem 58 of [74]), and thus a principal ideal. Hence, $G(D)$ is isomorphic to a cardinal sum of copies of $\mathbb{Z}$ from the well known theorem of Ward and Birkhoff.

(5) $\Leftrightarrow$ (6), (6) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (4) : By Proposition 3 of [98] and the proof of (1) $\Leftrightarrow$ (2) above we infer that a quasi-Asano Bézout domain is Asano.

(4) $\Rightarrow$ (6) : Let I be an nonzero ideal of D, then there exists $0 \neq a \in I$. Thus $(a) \subseteq I$ and I is invertible since (a) is invertible, which implies that I is a principal ideal domain.

(4) $\Leftrightarrow$ (7) : By the definition of the Asano rings and divisibility groups we obtain that D is Asano if and only if an ideal I of D is full if and only if I is principal. In fact, By [74] (P. 42, Exercise 15) we have that every invertible ideal of a pseudo-Bézout domain is principal. Furthermore, it is clear that a principal ideal is invertible. Thus D is Asano if and only if every full ideal I of D is principal, i. e. I corresponds to a positive element in $G(D)$. By the definition of a full ideal I, this means that $aD \subseteq I$ implies $I = bD$ for some $b \in D$. $\square$

Remark 4.5.2 *It is known that the distinction between Bézout and pseudo-Bézout domains is lost upon passage to divisibility groups. Therefore, Theorem*

4.5.1 holds also for pseudo-Bézout domains. Moreover, we note that divisibility groups of quasi-Asano Bézout and pseudo-Bézout domains are free.

Recall that an element $s \in D^* \setminus U(D)$ for an integral domain D is said to be *irreducible* or *nonfactorable* if s cannot be written as a product of two nonunits of D. Clearly, $s \in D$ is irreducible if and only if sD is a minimal positive element (atom) of the divisibility group $G(D)$. Thus, it is appropriate to designate irreducible elements as *atoms* of D, and then to say that D is *atomic* if each nonzero nonunit of D is a finite product of atoms. It is clear that a unique factorization domain (UFD) is atomic. For an l-group G, $b \in G^{>0}$ is called *basic* if the set $\{x \in G \mid 0 \leq x \leq b\}$ is totally ordered. We know that a *basis* for G is a maximal (pairwise) disjoint subset $\{b_\lambda \mid \lambda \in \Lambda,\ \lambda \text{ is basic}\}$. Hence, an atomic Bézout domain has a basis.

Corollary 4.5.3 *Let D be a quasi-Asano Bézout domain. Then $G(D)$ has a basis. Especially, if D is completely integrally closed, then $G(D)$ is completely distributive. Consequently, $Rad(G(D)) = D(G(D)) = 0$ if $G(D) = \Gamma^-(A)$ for some MV-algebra A.*

Proof. That $G(D)$ has a basis follows from Theorem 4.5.1 and the note above. If D is completely integrally closed, then $G(D)$ is completely distributive by Theorem 53.18 (that is, an Archimedean l-group is completely distributive if and only if it has a basis) of [44]. Furthermore, if $G(D) = \Gamma^-(A)$ for some MV-algebra A then $u(D) \neq 0$, i. e. there exists a strong unit in $G(D)$. $\square$

Now we characterize local Bézout domains.

Theorem 4.5.4 *Let D be a Bézout domain. Then the following conditions are equivalent:*

(1) D is local.

(2) 0 is the unique minimal prime l-ideal of $G(D)$.

(3) $G(D)$ is linearly ordered.

(4) D is a valuation domain.

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) are clear from Corollary 2.2.5, Theorem 2.2.8 and the well known result of Krull.

(4) $\Rightarrow$ (1) is also well known (see [24]).

(2) $\Rightarrow$ (4) : Obvious. $\square$

4.6 Prime spectrum of Bézout domains

Let $Spec(R)$ denote the set of prime ideals of a commutative ring R with identity, considered as a partially ordered set under set inclusion. Kaplansky [74] notes the following two properties of $Spec(R)$:

(K_1) *every chain in Spec R has a least upper bound and a greatest lower bound;*

(K_2) *If the inclusion $P \subset Q$ holds for distinct elements P, Q of Spec R, then there are distinct elements P_1 and Q_1 of Spec R with $P \subseteq P_1 \subset Q_1 \subseteq Q$ such that there is no element of Spec R properly between P_1 and Q_1.*

We call a partially ordered set (poset for short) *spectral* if it satisfies (K_1) and (K_2). In general, a spectral poset is not necessarily isomorphic (as a poset) to $Spec(R)$, for some ring R (see [79]). However, Cignoli and Torrens [34] show that a poset is order isomorphic to the proper prime spectrum of an l-group with strong unit if and only if it is a spectral root system. This fact will be exploited for the spectrum of a Bézout domain using some well known results of valuation theory.

4.6.1 The poset of prime ideals

It is well known that for each Abelian l-group G, the poset (under set inclusion) of prime l-ideals of G is a spectral root system having a unique greatest element. Accordingly, for a Bézout domain D, $Spec(D)$ is a spectral tree system with a unique minimal element (see Močkoř [85], p. 154).

Theorem 4.6.1 *If $(X, \leq)$ is a root system, then the following statements are equivalent:*

(1) X is spectral.

(2) X is order isomorphic to the poset $Spec(H)$ of proper prime l-ideals of an Abelian l-group H with a strong unit.

(3) X is order isomorphic to the poset $Spec(G)$ of an Abelian l-group G.

(4) X is order isomorphic to $Spec(A)$ of an MV-algebra A.

Proof. (1) $\Leftrightarrow$ (2) follows by [34]. (2) $\Rightarrow$ (3) is trivial. (2) $\Leftrightarrow$ (4) follows from Lemma 4.3.6.

(3) $\Rightarrow$ (1): $Spec(G)$ of an Abelian l-group satisfies the condition (K_1) since it is plain that $P \subseteq C \in \mathcal{C}(G)$ implies C is prime for all prime l-subgroup P of G, and the intersection of a chain of prime l-subgroups is a prime l-subgroup

as well. On the other hand, let $P \subset Q$ be distinct prime l-ideals in G. Insert (Zorn's lemma) a maximal chain $\mathbb{L}$ of prime l-ideals between P and Q. Take any element $g \in Q \setminus P$. Define $Q_1 = \cap\{P_i \in \mathbb{L} \mid g \in P_i\}$, $P_1 = \cup\{P_i \in \mathbb{L} \mid g \notin P_i\}$. By the argument above, P_1 and Q_1 are prime. Obviously the condition (K_2) holds since no prime l-ideal at all can lie between P_1 and Q_1 by the maximality of $\mathbb{L}$. $\square$

Furthermore, it is clear that a root system with a unique maximal element is spectral if and only if it is order isomorphic to the poset $Sp(H) = Spec(H) \cup \{H\}$ of primes for some Abelian l-group with strong unit.

Using Theorem 4.6.1 and Theorem 2.2.8, we get

Corollary 4.6.2 *If $(X, \leq)$ is a tree system, then the following statements are equivalent:*

(1) X is spectral.

(2) X is order isomorphic to $Spec(R) \setminus \{0\}$ of a Bézout domain R with $u(R) \neq 0$.

(3) X is order isomorphic to $Spec(D) \setminus \{0\}$ of a Bézout domain D. Consequently, a tree system with a unique minimal element is spectral if and only if it is order isomorphic to $Spec(R)$ for some Bézout domain R with a strong unit. That is, for a Bézout domain D, there exists a Bézout domain R with strong unit and an MV-algebra A such that the following diagram is commutative, where $\cong_{o.r}$ denotes an order-reversing isomorphism, $Sp(G(D))$ (resp.

$$\begin{tikzcd}
\mathrm{Spec}(D) \arrow[r, "\cong_o"] & \mathrm{Spec}(R) \arrow[r, "\cong_{o,r}"] & \mathrm{Spec}(A) \cup \{A\} \arrow[d, equal] \\
Sp(G(D)) \arrow[u, "\cong_{o,r}"'] \arrow[r, "\cong_o"] & Sp(G(R)) \arrow[u, "\cong_{o,r}"'] \arrow[r, "\cong_o"] & \mathrm{Spec}(A) \cup \{A\}
\end{tikzcd}$$

Figure 4.2: Posets of prime ideals

$Sp(G(R))$) denotes the set of prime l-ideals of $G(D)$ (resp. $G(R)$), and the maps $Sp(G(D)) \to Spec(D)$ and $Sp(G(R)) \to Spec(R)$ are the well known order-reversing isomorphisms in Theorem 2.2.8.

For an MV-algebra A, Belluce ([10], Theorem 1) shows that $Rad(A)$ is a nilpotent ideal of A that contains all other nilideals of A (see also Proposition 4.3.12), which is similar to Proposition 4.1.22 (see also the note under the proof of Lemma 4.2.1). Thus we obtain the following corollary which is a generalization of Lemma 4.3.6.

Corollary 4.6.3 *Let $Small(G)$ be the set of all small l-ideals of G, and let $Nil(\Gamma(G,u))$ be the set of nilideals of $\Gamma(G,u)$. Then $(Small(G,u),\subseteq) \cong_o Nil(\Gamma(G,u),\subseteq)$.*

4.6.2 Topological properties of $Spec(G)$

Let G be an l-group, $Spec(G)$ will denote the set of *proper* prime subgroups of G. For $A \subseteq G$, define

$$S(A) = \{P \in Spec(G) \mid A \not\subseteq P\}.$$

Then it is easy to verify that the $S(C)$, $C \in \mathcal{C}(G)$ are the open sets of a topology, called the *spectral topology*, on $Spec(G)$. The set $\{S(g) \mid g \in G\}$ forms a basis of $Spec(G)$. Moreover, it is easy to verify that each $S(g)$ is quasi-compact and every quasi-compact open subset is of this form (see [44], pp. 318-319). Furthermore, $Spec(G)$ is quasi-compact if and only if G has a formal unit (that is, an element u which is not contained in any proper prime subgroup). In particular, $Spec(G)$ is quasi-compact if G has a strong unit.

One problem concerning the spectral topology is that it is usually not Hausdorff, which is generally desired. This difficulty can be overcome if one considers instead the topology inherited by a subset of pairwise incomparable primes of $Spec(G)$. If $X \subseteq Spec(G)$ consists of mutually incomparable prime subgroups, then the induced topology on X is Hausdorff. In particular, the space of all minimal primes and the space of all maximal primes, are Hausdorff spaces. By Theorem 4.6.1, we get

Proposition 4.6.4 *If G is an Abelian l-group, then there exists an Abelian l-group H with strong unit such that the quasi-compact spectral space $Spec(H)$ is order-isomorphic to $Spec(G)$ regarded as posets under set inclusion.*

Note that Proposition 4.6.4 cannot be translated directly to the prime spectrum of Bézout domains since $G(D) \notin Spec(G(D))$ but $0 \in Spec(D)$ for a Bézout domain D. Therefore, if G is an Abelian l-group, we set $Spec^*(G) = Spec(G) \cup \{G\}$ with the topology whose only open sets are the those of $Spec(G)$, and $Spec^*(G)$. Then $Spec^*(G)$ is quasi-compact.

Chapter 5

Sheaf representations

Recall that a *sheaf* is a triple $\mathcal{S}= (E,\ f,\ X)$, where E and X are topological spaces and $f : E \to X$ is a local homeomorphism. $\mathcal{S}$ is a sheaf of l-groups if for each $x \in X$ the fiber $E_x = f^{-1}(x)$ is an l-group, and letting $E\Delta E = \cup_{x\in X}(E_x \times E_x)$ with the induced topology from $E \times E$, all operations $\vee,\ \wedge,\ +,$ and $-$ are continuous from $E\Delta E$ to E. A *section* of $\mathcal{S}$ is a continuous map $\sigma :\ X \to E$ such that $f\sigma = id_X$; we denote by $\Gamma\mathcal{S}$ or $\Gamma(X, E)$ the set of all sections of $\mathcal{S}$. Then $\mathcal{S}$ is *global* if for all $a \in E$, there exists a section σ and $x \in X$ such that $\sigma(x) = a$. $\mathcal{S}$ is *Hausdorff* if E is so.

Grothendiek has shown that corresponding to each commutative ring R with identity, there is a sheaf of local rings over $Spec(R)$ (with Zariski topology), and an isomorphism of R upon the ring of all global sections of the sheaf (see [67], Proposition 2.2, p. 71). For sheaf-theoretic representations of rings and modules as global sections in a sheaf, the reader is referred to the work of Hofmann [69]. A natural problem is whether the group of divisibility of a Bézout domain has a sheaf-theoretic representation. Keimel [76] gives many representation theorems for l-rings and Abelian l-groups with strong unit by sections in sheaves.

In this chapter, we associate a sheaf $\tilde{G}$ on a quasi-compact space $Sp(G)$ to any Abelian l-group G such that the stalks of $\tilde{G}$ are linearly ordered, and G coincides with the l-group of global sections of $\tilde{G}$. As a set, $Sp(G)$ consists of the prime l-ideals of G (including G itself). Thus $Sp(G) = Spec^*(G)$ as a set. We show that the quasi-compact spaces $Sp(G)$ and $Spec^*(G)$ are dual in the sense of Hochster [68]. It is known that the sheaf of germs $\mathcal{G}(G)$ on $Spec^*(G)$ also admits a representation of G by global sections. If $G = G(D)$ for a Bézout domain D, we show that the dual sheaf $Sp(G)$ is homeomorphic to $Spec(D)$.

Using sheaf theory, we give a careful analysis of Archimedean l-groups G which leads to a very simple construction of the lateral completion G^l and a quite natural proof of Bernau's embedding theorem [13].

5.1 The sheaf representation

Let G be an Abelian l-group. A subset $F \neq \varnothing$ of $G^{\geq 0}$ is said to be a *filter* if $a, b \in F$ implies $a \wedge b \in F$ and $c \in F$ for all $c \geq a$. Every $a \in G^{\geq 0}$ defines a *principal filter*

$$[a] := \{b \in G \mid b \geq a\},$$

and every filter is a union of principal filters. In particular, $G^{\geq 0} = [0]$.

For example, every prime l-ideal P of G gives rise to a filter $F := G^{\geq 0} \setminus P$, and P then coincides with the l-ideal generated by $G^{\geq 0} \setminus F$ (cp. Theorem 2.2.8, and [101], Theorem 2.2). For any filter F of G, we set

$$D(F) := \{P \in Sp(G) \mid P \cap F \neq \varnothing\}.$$

It is easily verified that the following holds for any set of filters F_i:

$$D(\cup F_i) = \cup D(F_i);\ \ D(F_i \cap F_2) = D(F_1) \cap D(F_2).$$

Moreover,

$$D(G^{\geq 0}) = Sp(G), \text{ and } \ D(\varnothing) = \varnothing.$$

Therefore, the $D(F)$ are the open sets of a topology on $Sp(G)$. Endowed with this topology, we call $Sp(G)$ the *dual spectrum* of G. Note that the topology on $Sp(G)$ is in a sense opposite to the one considered in [21, 76]. The precise relationship will be explained below.

Proposition 5.1.1 *For any Abelian l-group G, the dual spectrum $Sp(G)$ is quasi-compact.*

Proof. Let $Sp(G) = \cup_{i \in I} D_i$ be a covering by non-empty open sets D_i. To exhibit a finite subcovering, we may assume that the D_i are of the form $D_i = D([a_i])$ with $a_i \in G^{\geq 0},\ i \in I$. Then the a_i form a set $A := \{a_i \mid i \in I\}$ such that every $P \in Sp(G)$ meets A, and we have to show that there is a finite subset $A' \subseteq A$ with this property. Let $F \subseteq G^{\geq 0}$ be the filter generated by A. Suppose that $F \neq G^{\geq 0}$. Then there exists an ultrafilter F' with $F \subseteq F' \subset G^{\geq 0}$. By [21], Theorem 3.4.10, this implies that there is a minimal prime l-ideal P

with $F' = G^{\geq 0} \backslash P$. Hence $P \cap A = \varnothing$, a contradiction. Thus we obtain $F = G^{\geq 0}$, which means that there are $a_1, \cdots, a_n \in A$ with $a_1 \wedge \cdots \wedge a_n = 0$. So every $P \in Sp(G)$ meets $\{a_1, \cdots, a_n\}$. $\square$

Now we define a sheaf $\mathcal{S} := (\tilde{G}, p, Sp(G))$ of fully ordered Abelian groups on $Sp(G)$. For the stalks, we simply set

$$\tilde{G}_P := G/P$$

for any $P \in Sp(G)$. Let

$$\tilde{G} = \cup_{P \in Sp(G)} G/P,$$

and let p be the obvious projection

$$p : \tilde{G} \to Sp(G).$$

For an element $g \in G$ and a filter $F \subseteq G^{\geq 0}$, let $U_{g,F}$ denote the set of residue classes $g + P \in G/P$ with $P \in D(F)$. We define the topology on $\tilde{G}$ by the $U_{g,F}$ as a basis of open sets. By $g_{D(F)}$ we denote the corresponding section in $\Gamma(D(F), \tilde{G})$. If H is another filter on $G^{\geq 0}$, and $h \in G$, then

$$U_{g,F} \cap U_{h,H} = U_{g, F \cap H \cap [|g-h|]},$$

which shows that the $U_{g,F}$ form indeed a basis. Furthermore, this equation shows that $p : \tilde{G} \to Sp(G)$ is a local homeomorphism. Since

$$U_{g,F} + U_{h,F} \subseteq U_{g+h,F}; \; U_{g,F} \vee U_{h,F} \subseteq U_{g \vee h,F}$$

holds for all filters $F \subseteq G^{\geq 0}$ and $g, h \in G$, it follows that $\mathcal{S}$ is a sheaf. There is a natural l-group homomorphism

$$\epsilon : \; G \to \Gamma(Sp(G), \tilde{G})$$

which maps $g \in G$ to $g_{Sp(G)}$.

Proposition 5.1.2 *For an Abelian l-group G, the above homomorphism ϵ is an isomorphism of l-groups.*

Proof. The injectivity of ϵ follows since $\cap Sp(G) = 0$. Now let $s \in \Gamma(Sp(G), \tilde{G})$ be given. Since $Sp(G)$ is quasi-compact, there is a finite covering $Sp(G) = \cup_{i=1}^{n} D([a_i])$, and there are $g_1, \cdots, g_n \in G$ with $s(P) = g_i + P$

for all $P \in D([a_i])$ and $i \in \{1, \cdots, n\}$. Thus $g_i - g_j \in P$ holds for all $P \in D([a_i]) \cap D([a_j])$ and $i \neq j$. Consequently, we get

$$g_i - g_j \in G(a_i \vee a_j)$$

for all $i \neq j$. By the “théorème chinois” (see [21], Lemma 10.6.3), we find $g \in G$ with

$$g - g_i \in G(a_i)$$

for all i, i. e. $s(P) = g + P$ for all $P \in Sp(G)$. Hence ϵ is surjective. □

Proposition 5.1.2 shows that every Abelian l-group G (not necessarily with strong unit) can be represented by global sections of a sheaf of fully ordered Abelian groups over a quasi-compact space. This allows us to localize G at any open set $D(F) \subset Sp(G)$, where F denotes a filter on $G^{\geq 0}$. Consider the Abelian l-groups

$$G^{(F)} := \{f : F \to G \mid \forall a, b \in F : \; f(a) - f(b) \in G(a \vee b)\}$$

and

$$G_{(F)} := \{f : F \to G \mid \forall a \in F : \; f(a) \in G(a)\} \subseteq G^{(F)}.$$

By [21], 1.3.18, these are in fact l-groups. Then the definition of $\tilde{G}$ yields the following identification

$$\tilde{G}(F) := \Gamma(D(F), \tilde{G}) = G^{(F)}/G_{(F)}.$$

In fact, a section in $\tilde{G}(F)$ is given by a collection of compatible sections $f(a)_{[a]}$, $a \in F$, where the compatibility just means that $f(a) + P = f(b) + P$ for all $P \in D([a]) \cap D([b]) = D([a \vee b])$. Here, f defined the trivial section if and only if $f \in G_{(F)}$. For a principal filter $[a]$, we have, in particular,

$$\tilde{G}([a]) \cong G/G(a)$$

in a natural way. Note that $D([a]) \cong Sp(G/G(a))$ is again quasi-compact.

Now we consider the relationship to Bézout domains. More generally, let D be a GCD domain with group of divisibility $G = G(D)$. Every prime ideal $P \in Spec(D)$ corresponds to a multiplicative system $S := D \setminus P$ with $U_D \subseteq S$, hence to a prime ideal SS^{-1}/U_D of G. Note that $SS^{-1} \cap D = S$. An open set in $Spec(D)$ is of the form

$$D(I) := \{P \in Spec(D) \mid I \not\subseteq P\}$$

with $I \subseteq D$ which can be assumed to be an ideal of D. Then $F := I/U_D$ is a filter in $G^{\geq 0} = D/U_D$, and the prime ideals in $D(I)$ correspond to the elements of $D(F)$. Hence there is a natural homeomorphism of topological spaces

$$Spec(D) \cong Sp(G(D)).$$

As is well known, there is a ring isomorphism

$$D \cong \Gamma(Spec(D), \tilde{D}),$$

where $\tilde{D}$ is the affine scheme associated to D. This representation of D is in a sense parallel to that of $G(D)$, although the existence of the latter is not a mere consequence of the sheaf representation of D.

Next we show that the lattice $\mathcal{C}(G)$ of convex l-subgroups and the lattice $\mathcal{P}(G)$ of polars for an Abelian l-group G can be determined in terms of $Sp(G)$. First, let X be an arbitrary topological space. We define $\mathcal{L}(X)$ to be the class of open sets U which coincide with the kernel $\overset{\circ}{\overline{U}}$ of its closure $\overline{U}$. The following properties seems to be well known. By lack of a reference, we provide a proof.

Proposition 5.1.3 *Let X be a topological space. Then $\mathcal{L}(X)$ is closed with respect to finite intersection. As a partially ordered set with respect to inclusion, $\mathcal{L}(X)$ is a complete Boolean algebra.*

Proof. Let $U, V \in \mathcal{L}(X)$ be given. Then $U \cap V \subseteq \overset{\circ}{\overline{U \cap V}} \subseteq \overset{\circ}{\overline{U}} = U$. Hence $U \cap V \in \mathcal{L}(X)$. Moreover, we have $\varnothing, X \in \mathcal{L}(X)$. For a family $U_i \in \mathcal{L}(X)$, the union $\cup U_i$ is open. Hence $\cup U_i \subseteq \overset{\circ}{\overline{\cup U_i}} \subseteq \overline{\cup U_i}$, which shows that $\overset{\circ}{\overline{\cup U_i}}$ is the supremum of the U_i in $\mathcal{L}(X)$. Furthermore, every $U \in \mathcal{L}(X)$ has a complement $X \setminus \overline{U}$ in $\mathcal{L}(X)$. By duality, this shows that $\mathcal{L}(X)$ is a complete lattice. It remains to prove that $\mathcal{L}(X)$ is distributive. Thus let $U, V, W \in \mathcal{L}(X)$ be given. We have to show that the inclusion

$$\overset{\circ}{\overline{U \cup V}} \cap \overset{\circ}{\overline{U \cup W}} \subseteq \overset{\circ}{\overline{U \cup (V \cap W)}}.$$

The left-hand side being open, it suffices to prove

$$\overset{\circ}{\overline{U \cup V}} \cap \overset{\circ}{\overline{U \cup W}} \subseteq \overline{U \cup (V \cap W)}.$$

Assume that $x \in \overset{\circ}{\overline{U \cup V}} \cap \overset{\circ}{\overline{U \cup W}}$. If $x \in \overline{U}$, then $x \in \overline{U \cup (V \cap W)}$. Therefore, let us assume that $x \notin \overline{U}$. Then $x \in \overset{\circ}{\overline{U \cup V}} \setminus \overline{U} \subseteq \overline{V}$. Since $\overset{\circ}{\overline{U \cup V}} \setminus \overline{U}$ is open, it is contained in $\overset{\circ}{\overline{V}} = V$. Hence $x \in V$. Similarly, we get $x \in W$, and thus $x \in \overline{U \cup (V \cap W)}$. $\square$

Since the dual spectrum *Sp*(G) of an Abelian l-group G is isomorphic to *Spec*(D) for a Bézout domain D, we infer that *Sp*(G) is a *spectral space* in the sense of Hochster [68]. This means that *Sp*(G) is a T_0-space X such that the class $\mathcal{K}(X)$ of quasi-compact open sets is closed under finite intersection (in particular, the empty intersection $X = \cap\varnothing$ is quasi-compact) and form a basis, and every closed irreducible set $A \subseteq X$ has a generic point x (i. e. $A = \overline{\{x\}}$). Hochster [68] has shown that the topology of a spectral space X admits a dual topology which can be obtained as follows. Let $\mathcal{K}'(X)$ be the class of complements of sets in $\mathcal{K}(X)$. Then $\mathcal{K}'(X)$ is a basis of open sets for a topology on X. It is quite remarkable that the sets in $\mathcal{K}'(X)$ are just the quasi-compact open sets in the Hochster dual. For an Abelian l-group G, the Hochster dual of *Sp*(G) coincides with

$$\mathit{Spec}^*(G) = \mathit{Spec}(G) \cup \{G\},$$

the Alexandroff (quasi)-compactification of *Spec*(G). Due to this connnection with *Sp*(G), the full spectrum *Spec*$^*G$ seems to be a more natural concept than *Spec*(G).

For any T_0-space X, there is a natural poset structure on X, given by

$$x \le y :\Leftrightarrow y \in \overline{\{x\}}.$$

In the special case $X = \mathit{Sp}(G)$, we have $\overline{\{P\}} = \{Q \in \mathit{Sp}(G) \mid Q \subseteq P\}$, and thus

$$P \le Q \Leftrightarrow P \supseteq Q,$$

while in the Hochster dual *Spec*$^*G$,

$$P \le Q \Leftrightarrow P \subseteq Q$$

for prime l-ideals P, Q of G. So the partial order in *Sp*(G) coincides with the divisibility relation.

Jaffard's correspondence and Hochster's duality relate the topology of *Spec*$^*(G)$ for an Abelian l-group G to the spectrum of a Bézout domain. Define a *tree space* to be a spectral space with the property that any pair of closed irreducible sets A, B is either disjoint or comparable (i e. $A \subseteq B$ or $B \subseteq A$). Accordingly, the Hochster dual of a tree space will be called a *root space.* Thus a spectral space X is a root space if and only if two points $x, y \in X$ such that x has an open neighborhood U with $y \notin U$, and y has an open neighborhood V with $x \notin V$, cannot be in the closure of a third point. Note that a strong converse of this property holds for any spectral space, namely, if x and y are not in the closure of a third point, they can be separated by disjoint open neighborhoods.

By Jaffard's correspondence, it follows that the spectrum $Spec^*(G)$ of an Abelian l-group G is an irreducible root space, while $Spec(G)$ need not be irreducible. To obtain a characterization of the root spaces arising in this way, a search for suitable Bézout domains along the lines of Hochster's construction [68] might be useful. The reason why the truncated spectrum $Spec(G)$ appears in the literature (e. g. [76]) comes from the first part of the following

Proposition 5.1.4 *Let G be an Abelian l-group. The lattice $\mathcal{C}(G)$ of l-ideals is isomorphic to the lattice of proper open sets in $Spec^*(G)$, the lattice $\mathcal{P}(G)$ of polars of G is isomorphic to $\mathcal{L}(Spec^*(G))$.*

Proof. An l-ideal A of G is given by the intersection of all prime ideals in the closed set $\cap_{a\in A} D([a]) \subseteq Spec^*(G)$. As this correspondence is order reversing. we get a one-to-one correspondence

$$A \mapsto S(A) := Spec(G) \setminus \cap_{a\in A} D([a])$$

between $\mathcal{C}(G)$ and the open sets in $Spec(G)$.

For the polar of A, we have (cf. [21], Proposition 3.4.1, or [44], Theorem 14.1)

$$A^\perp = \cap S(A),$$

which yields $S(A^\perp) = Spec^*(G) \setminus \overline{S(A)}$. Hence $S(A^{\perp\perp}) = \overset{\circ}{\overline{S(A)}}$. ☐

Remark 5.1.5 *The importance of the lattice $\mathcal{C}(G)$ results from the early work of Paul Conrad. Apparently, it has been so prevalent that it determined his successors to remove the infinite prime G from the spectrum $Spec^*(G)$, so that $Spec(G)$ is no longer quasi-compact, in general. Note that by Hochster duality, the prime G corresponds to the zero ideal of a corresponding Bézout domain. From the view point of algebraic geometry, however, it is quite natural to include the infinite prime(s) to the spectrum.*

5.2 The dual sheaf

We have seen that an Abelian l-group G can be identified with $\Gamma(Sp(G), \tilde{G})$. Keimel has shown ([76], Corollary 3.8) that there is a similar representation on $Spec^*(G)$. To analyze the relationship between both sheaf representations, we first show

Proposition 5.2.1 *A non-empty open set in $Sp(G)$ is quasi-compact if and only if it is of the form $D([a])$ for some $a \in G^{\geq 0}$.*

Proof. Since $D([a]) = Sp(G/G(a))$, the open sets $D([a])$ are quasi-compact. Conversely, let $\varnothing \neq U \subseteq Sp(G)$ be open and quasi-compact. Then $U = \cup_{b \in B} D([b])$ for some $B \subseteq G^{\geq 0}$. Hence there is a finite subset $\{b_1, \cdots, b_n\}$ of B with $U = D([b_1]) \cup \cdots \cup D([b_n]) = D([b_1 \cap \cdots \cap b_n])$. $\square$

Note that a similar result for $Spec(G)$ was proved by Keimel ([76], Proposition 1.21). This is equivalent to Proposition 5.2.1 via Hochster's duality. So the quasi-compact open sets in $Spec(G)$ are exactly the sets

$$S(a) := Sp(G) \setminus D([a])$$

with $a \in G^{\geq 0}$. More generally, the open sets in $Spec(G)$ are of the form (cf. Section 4.6.2)

$$S(A) := \{P \in Spec(G) \mid A \not\subseteq P\},$$

where A can be chosen in $\mathcal{C}(G)$, and vice versa. In $Spec^*(G)$, there is just one additional open set, the whole space $Spec^*(G)$, which corresponds to the empty set in $Sp(G)$.

We have seen that the sections of $\tilde{G}$ on a quasi-compact open set $D([a])$ are induced by elements in G, and thus

$$\Gamma(D([a]), \tilde{G}) = G/G(a).$$

Therefore, the stalks $\tilde{G}_P$ are given by the global sections $g \in G$ modulo those g which vanish in a neighborhood of P. Passing to the Hochster dual $Spec^*(G)$ of $Sp(G)$, it is natural to consider the *dual sheaf* $\mathcal{G}(G)$, where the stalks at P are the global sections $g \in G$ modulo those which vanish in a neighborhood of P with respect to the topology of $Spec^*(G)$. Thus

$$\mathcal{G}(G)_P = \varinjlim_{a \notin P} G/a^{\perp} \cong G/o(a),$$

where $o(a) := \cup_{a \notin P} a^{\perp}$ denotes the *germinal l-ideal* of P (see, e.g. [76]). Therefore, $\mathcal{G}(P)$ is also called the *sheaf of germs*. It gives rise to the above mentioned dual sheaf representation

$$G \cong \Gamma(Spec^*(G), \mathcal{G}(G)).$$

Recall that an Abelian l-group G is said to be *laterally complete* if each set $S \subseteq G^{\geq 0}$ of pairwise orthogonal elements admits a supremum $\vee S \in G$. The existence of a lateral completion G^l was proved by Conrad [39] (for representable l-groups) and Bernau [14, 15] (for arbitrary l-groups). These proofs are extremely technical. Precisely, an l-group G^l with G as an l-subgroup is said to be a *lateral completion* of G if the following conditions are satisfied:

(1) G^l is laterally complete.

(2) G is *dense* in G^l, i. e. for each $g \in G^l$ with $g > 0$, there exists $h \in G$ with $0 < h \leq g$.

(3) There is no laterally complete l-subgroup H of G^l with $G \subset H$.

Note that a dense l-subgroup is always large. We will show that a dense embedding of any Abelian l-group into a laterally complete group can be obtained in a simple way by means of sheaf theory. As the intersection of two dense open subsets of a topological space is again dense, the dense open sets $U \subseteq Spec^*G$ form a lower directed system which yields an Abelian l-group

$$E(G) := \varinjlim \Gamma(U, \mathcal{G}(G)).$$

For $t \in E(G)$, the largest (dense) open set $U \subseteq Spec^*G$ where t is defined will be denoted by $\mathrm{def}(t)$. The support

$$supp(t) := \{P \in \mathrm{def}(t) \mid t(P) \neq 0\}$$

of t is then open, too.

Lemma 5.2.2 *Two elements $x, y \in E(G)$ are orthogonal if and only if $supp(x) \cap supp(y) = \varnothing$.*

Proof. We may assume without loss of generality that $x, y \geq 0$. Consider the dense open set $U := \mathrm{def}(x) \cap \mathrm{def}(y)$ in $Spec^*(G)$. Let $S(a) \subseteq U$ be an open set such that the restrictions of x and y to $S(a)$ are represented by $f, g \in G$, respectively. Then $x \wedge y = 0$ says that $f \wedge g$ is zero on $S(a)$, i. e. $f \wedge g \wedge a = 0$, which is tantamount to $S(f) \cap S(g) \cap S(a) = \varnothing$. This means that $supp(x) \cap supp(y) = \varnothing$. Since U covered by such $S(a)$, the assertion follows.$\square$

Theorem 5.2.3 *Let G be an Abelian l-group. There is a dense embedding into the l-group $E(G)$ which is laterally complete.*

Proof. There is a natural morphism

$$\epsilon : \; G \to E(G)$$

of l-groups which maps $g \in G$ to the corresponding global section. Assume that $\epsilon(g) = 0$. Then $g|_U = 0$ for some dense open set $U \subseteq Spec^*(G)$. Moreover, there is an l-ideal $A \in \mathcal{G}$ with $U = S(A)$ and $A^\perp = 0$. Hence $g \in \cap S(A) = A^\perp = 0$. This shows that ϵ is an embedding of l-groups. To show that ϵ is dense, let $x \in E(G)$ with $x > 0$ be given. Then there is some $S(a) \subseteq \operatorname{def}(x)$ such that $x|_{S(a)}$ can be represented by some $f \in G$ with $f > 0$ on $S(a)$, i. e. $g := f \wedge |a| > 0$. Hence $supp(g) \subseteq S(a)$, and thus $0 < g \leq x$.

Now let $E \subseteq E(G)^{\geq 0}$ be a set of pairwise orthogonal elements in $E(G)$. By Lemma 5.2.2, the supports of the $e \in E$ are pairwise disjoint. We set $U := \cup_{e \in E} supp(e)$ and $V := Spec^*(G) \setminus \overline{U}$. Then $U \cup V$ is open and dense in $Spec^*(G)$, and there is an element $x \in E(G)$ with $x|_{supp(e)} = e$ for all $e \in E$, and $x|_V = 0$. So we have $\vee E = x$ in $E(G)$. ☐

It is well known that a laterally complete Abelian l-group G admits a *weak unit,* i. e. an element e with $\overline{S(e)} = Spec^*(G)$ or equivalently, $e^\perp = 0$.

Proposition 5.2.4 *Let G be a laterally complete Abelian l-group. Then G has a weak unit.*

Proof. By Zorn's lemma, there is a maximal set $E \subseteq G^{\geq 0}$ of pairwise orthogonal elements. Hence $e := \vee E$ exists, and $e^\perp \subseteq E^\perp = 0$. ☐

5.3 The Archimedean property

Our next result shows that Archimedean l-groups admit a simple characterization in terms of their spectrum.

Proposition 5.3.1 *For an Abelian l-group G, the following statements are equivalent.*

(a) G is Archimedean.

(b) For each quasi-compact open set $U \subset Spec^*(G)$, the points which are closed in U form a dense subset of U.

Proof. G is Archimedean if and only if $G(a)$ is Archimedean for all $a \in G$ by Corollary 4.2.2. Since $G(a)$ has a strong unit, Corollary 4.2.6 implies that $G(a)$ is Archimedean if and only if $Rad(G(a)) = 0$. Now every quasi-compact open set $U \subset Spec^*(G)$ is of the form $U = S(a)$, and the $P \in S(a)$ which are closed in $S(a)$ are exactly the values of a. Therefore, condition (b) states that the intersection of the values of a is contained in $\cap S(a) = a^\perp$. Since every value of a intersects $G(a)$ in a maximal ideal of $G(a)$, and each maximal ideal of $G(a)$ is obtained in this way, condition (b) reduces to $Rad(G(a)) = 0$ for all $a \in G$. □

Proposition 5.3.2 *If G is an Archimedean l-group, then $E(G)$ is Archimedean.*

Proof. Let $x, y \in E(G)$ be given, and $nx \leq y$ for all $n \in \mathbb{N}$. Suppose that $x \not\leq 0$. Then we find some $S(a) \subseteq \mathrm{def}(x) \cap \mathrm{def}(y)$ where x, y is represented by $f, g \in G$ such that $f \not\leq 0$ on $S(a)$, i. e. $(f \vee 0) \wedge |a| > 0$. Hence $(f \wedge |a|) \vee (0 \wedge |a|) > 0$, i. e. $f \wedge |a| \not\leq 0$. If G is Archimedean, this implies that $n(f \wedge |a|) \not\leq g$ for some $n \in \mathbb{N}$. Thus $nx \wedge n|a| = n(x \wedge |a|) \not\leq y$, in contract to the assumption. Hence $x \leq 0$. □

One of the most powerful structure theorems for Archimedean l-groups is Bernau's description of the essential closure in terms of almost finite functions. Using Theorem 5.2.3, we are able to give a natural and elegant proof.

For a topological space X, consider the Abelian l-group (cf. [21], Chapitre 13)

$$E(X) := \varinjlim C(U),$$

where U runs through the dense open subsets of X. If X is *extremally disconnected,* i. e. X is a Hausdorff space in which $\overline{U}$ is open for every open subset $U \subseteq X$, then $E(X)$ can be identified with $D(X)$ (see Example 2.3.5), while $D(X)$ is not an l-group in general. Recall that a totally disconnected compact space is said to be a *Stone space.*

Proposition 5.3.3 *Let G be an Abelian l-group. For every $a \in G^{\geq 0}$, the subspace $val(a) \subseteq Spec^*(G)$ of values of a is a Stone space. If $b \in G$, then $S(b) \cap val(a)$ is open and closed in $val(a)$.*

Proof. $val(a)$ consists of the closed points in $S(a)$. Let $val(a) \subseteq \cup_{i \in I} U_i$ be a covering by open sets $U_i \subseteq S(a)$. For each $P \in S(a)$, there is a value $Q \in val(a)$ with $P \subseteq Q$, i. e. $Q \in \overline{P}$. Hence $Q \in U_i$ implies that $P \in U_i$.

Therefore, we get $\cup_{i\in I}U_i = S(a)$. Since $S(a)$ is quasi-compact, this shows that $val(a)$ is quasi-compact. To show that $val(a)$ is Hausdorff, let P and Q be different values of a. Then there are $b, c \in G^{\geq 0}$ with $b \in P \setminus Q$ and $c \in Q \setminus P$. hence $x := b - b \wedge c \in P \setminus Q$ and $y := c - b \wedge c \in Q \setminus P$. Since $x \wedge y = 0$, we get $P \in S(y)$, $Q \in S(x)$, and $S(x) \cap S(y) = \varnothing$. Thus $val(a)$ is compact. Now let $b \in G$ be given. Then $S(b) \cap val(a) = val(a \wedge |b|)$ is open and compact in $val(a)$, hence closed and open. □

The next proposition can be obtained as a special case of [21], 13.2.4 ff. The presence of a weak unit, which we get via Theorem 5.2.3 and Proposition 5.2.4, allows us to give a short proof.

Proposition 5.3.4 *Let G be an Archimedean l-group with a weak unit $e > 0$, and let $X := val(e)$ be the associated Stone space. Then G admits an embedding*

$$\epsilon : G \hookrightarrow E(X)$$

as a large l-subgroup. If e is a strong unit, the image of ϵ lies in $C(X)$.

Proof. For $f \in G$, consider the set

$$\mathrm{def}(f) := \{P \in X \mid f \in P^c\}$$

where $P^c = P + G(e)$ is the minimal l-ideal containing P. Since P^c/P is simple, there is a unique embedding of l-groups $\epsilon_P : P^c/P \hookrightarrow \mathbb{R}$ with $\epsilon_P(e + P) = 1$. In particular, $P \in \mathrm{def}(f)$ implies that $f + P > ne + P$ holds for some $n \in \mathbb{Z}$, i. e. $(ne - f) \vee 0 \notin P$. Therefore,

$$\mathrm{def}(f) = \cup_{n\in\mathbb{Z}} S((ne - f) \vee 0) \cap X,$$

which shows that $\mathrm{def}(f)$ is open. Furthermore, $P \in \mathrm{def}(f)$ is equivalent to $|f| \vee e \in P + G(e)$, which gives

$$\mathrm{def}(f) = val(|f| \vee e) \cap X.$$

By Proposition 5.3.1, this implies that $\mathrm{def}(f)$ is dense in X. Now we define a function $\epsilon(f) : \mathrm{def}(f) \to \mathbb{R}$ by

$$\epsilon(f)(P) := \epsilon_P(f + P).$$

As we have seen above,

$$\epsilon(f) > n \Leftrightarrow P \in S((ne - f) \vee 0).$$

Since $f + P < \frac{n}{m}$ can be expressed by $mf \lesseqgtr n$ if $0 \lesseqgtr m$, we get

$$\epsilon(f)(P) < \frac{n}{m} \Leftrightarrow P \in S((ne - mf) \vee 0)$$

$$\epsilon(f)(P) > \frac{n}{m} \Leftrightarrow P \in S((mf - ne) \vee 0).$$

This shows that $\epsilon(f) \in C(\mathrm{def}(f)) \subseteq E(X)$. The injectivity of ϵ follows since $\cap \mathrm{def}(f) = 0$. Moreover, the definition of ϵ shows that ϵ is an l-homomorphism.

If e is a strong unit, then $G(e) = G$, and thus $\mathrm{def}(f) = X$ for all $f \in G$. So it remains to be shown that G is large in $E(X)$. Let $f > 0$ in $E(X)$ be given. Then f is defined on some $S(a) \neq \varnothing$ with $a \in G^{\geq 0}$ and $f(P) > 0$ for all $P \in S(a)$. Hence $(f \wedge a \wedge e)(P) > 0$ for all $P \in S(a \wedge e)$, and zero otherwise. Since $S(a \wedge e)$ is compact by Proposition 5.3.3, and $a \wedge e$ is defined on $S(a \wedge e)$, there is some $n \in \mathbb{N}$ with $0 \neq a \wedge e \leq n(f \wedge a \wedge e)$. Therefore, the l-ideal generated by f intersects G non-trivially. $\square$

As an immediate consequence of Theorem 5.2.3 and Proposition 5.3.4, we get the following version of Bernau's theorem [15].

Corollary 5.3.5 *Every Archimedean l-group G is a large l-subgroup of $E(X)$ for some Stone space X.*

Explicitly, our embedding ϵ is obtained in two steps

$$\epsilon : \ G \hookrightarrow E(G) \hookrightarrow E(X),$$

where $X = val(e)$ for some weak unit e of $E(G)$. In Bernau's theorem, the "Stone space" X corresponds to the complete Boolean algebra $\mathcal{P}(G)$ of polars of G. So it remains to clarify the role of X for a given G and the extent to which X is unique. Let us first recall that the Boolean algebra of polars remains the same in large extensions (see Proposition 4.1.14). By Stone's representation theorem, there is a duality between the category of Stone spaces and the category of Boolean algebras, which associates the Boolean algebra $co(X)$ of compact open sets to each Stone space X.

Proposition 5.3.6 *Let X be a Stone space. Then $\mathcal{L}(X)$ is the completion of $co(X)$. Moreover, $\mathcal{L}(X)$ is isomorphic to $\mathcal{P}(E(X))$.*

Proof. The first assertion follows since the compact open sets form a basis of open sets in X. For any $f \in E(X)$, defined on a dense open set $U \subseteq X$, let

$s(f) := \{x \in U \mid f(x) \neq 0\}$ be its support. Then $|f| \wedge |g| = 0$ holds in $E(X)$ if and only if $s(f) \cap s(g) = \varnothing$. Hence $f^{\perp} = \{g \in E(X) \mid s(g) \subset X \subset s(f)\}$. Since $s(f)$ is open, each polar $A \in \mathcal{P}(E(X))$ is of form

$$A = \{g \in E(X) \mid s(g) \subset U\}$$

with $U \subseteq X$ open and $U = \frac{o}{U}$. Thus $A \mapsto U$ defines the required isomorphism $\mathcal{P}(E(X)) \to \mathcal{L}(X)$. □

Proposition 5.3.6 shows, in particular, that every Stone space X gives rise to a continuous map of Stone spaces

$$p: \ \tilde{X} \to X$$

induced by the embedding $co(X) \hookrightarrow \mathcal{L}(X)$. Since $\mathcal{L}(X)$ is complete, $\tilde{X}$ is extremally disconnected. By [64], $\tilde{X}$ can be regarded as a projective cover in the category of compact spaces.

Proposition 5.3.7 *The map p induces an embedding $p^* : \ E(X) \hookrightarrow E(\tilde{X})$ of $E(X)$ as a large l-subgroup of $E(\tilde{X})$.*

Proof. By Stone duality, the points of $\tilde{X}$ can be regarded as maximal ideals I of $\mathcal{L}(X)$. Consider the map $i: \ X \to \tilde{X}$ given by

$$i(x) := \{U \in \mathcal{L}(X) \mid x \notin U\}.$$

Then $pi = i_X$. For $f \in E(X)$, define $p^*(f) := fp$. To show that p^* is well-defined, we have to verify that the inverse image $p^{-1}(U)$ of a dense open set $U \subseteq X$ is dense in $\tilde{X}$. Since $\mathcal{L}(X)$ is a basis of $\tilde{X}$, it suffices to show that $V \cap p^{-1}(U) \neq \varnothing$ for each non-empty $V \in \mathcal{L}(X)$. So there is some $x \in U \cap V$. Hence $V \notin i(x)$, and thus $i(x) \in V \subset \tilde{X}$. This gives $i(x) \in V \cap p^{-1}(U)$. Since p is surjective, p^* makes $E(X)$ into an l-subgroup of $E(\tilde{X})$. It remains to prove that $E(X) \blacktriangleleft E(\tilde{X})$, i. e. $E(X)^{\perp} = 0$ in $E(\tilde{X})$ (see Proposition 4.1.4). As the constant function $1 \in E(X)$ is a weak unit in $E(\tilde{X})$, the proof is complete. □

Corollary 5.3.8 (Bernau [15]) *Let G be an Archimedean l-group, and let $\tilde{X}$ denote the Stone space of $\mathcal{P}(G)$. Then G is isomorphic to a large l-subgroup of $E(\tilde{X})$.*

Proof. By Corollary 5.3.5, $G \blacktriangleleft E(X)$ for some Stone space X. Proposition 5.3.7 gives $E(X) \blacktriangleleft E(\tilde{X})$, where $\tilde{X}$ is the Stone space of $\mathcal{L}(X)$. Now Propositions 5.3.6 and 4.1.14 imply that $\mathcal{L}(X) \cong \mathcal{P}(G)$. □

Remark 5.3.9 *Conrad [40] has shown that an Archimedean l-group G is of the form $D(X) = E(X)$ for an extremally disconnected compact space X if and only if G is laterally complete, divisible and complete. Recall that G is said to be complete if G is closed with respect to infinite joins of bounded subsets, and G is said to be divisible if its underlying Abelian group is divisible (hence a $\mathbb{Q}$-vector space).*

Now let us return to Theorem 5.2.3. Using Proposition 5.3.4, we can show that $E(G)$ is the lateral completion of G in the Archimedean case.

Theorem 5.3.10 *Let G be an Archimedean l-group. Then $E(G)$ is the lateral completion of G.*

Proof. By Theorem 5.2.3, it suffices to show that every $s \in E(G)^{\geq 0}$ can be obtained as a join of pairwise orthogonal elements of G. Assume that s is defined on the dense open set $U \subseteq Spec^*(G)$. By Zorn's lemma, there is a maximal set $A \subseteq G^{\geq 0}$ of pairwise orthogonal elements such that for every $a \in A$, we have $S(a) \subseteq U$, and $s|_{S(a)}$ can be represented by some $fa \in G^{\geq 0}$ which satisfies $fa - na \in a^\perp$ for a suitable integer $n = n(a) \in \mathbb{N}$. Let us show that

$$\overline{\cup_{a\in A} S(a)} = Spec^*(G).$$

If this is not true, we find some $b \in G^{\geq 0} \setminus A$ with $\varnothing \neq S(b) \subseteq U \setminus \cup_{a\in A} S(a)$. Furthermore, we can assume that $s|_{S(b)}$ is represented by $f \in G^{\geq 0}$. Now $spec^* G/b^\perp = \overline{S(b)}$, and b is a weak unit in $G/b^\perp$. Therefore, Proposition 5.3.4 yields an embedding of l-groups

$$G/b^\perp \hookrightarrow E(X)$$

with $X := val(b)$. In particular, the residue class $f + b^\perp \in G/b^\perp$ gives rise to some element $f' \in E(X)$. Hence there is an open subset $S(c) \cap X$ of X with $0 < c \leq b$ where f' is defined. (We need not require $S(c) \cap X$ to be dense.) Since $S(c) \cap X$ is compact in X, we find an $n \in \mathbb{N}$ with $f'|_{S(c)\cap X} \leq nc$. (We regard c as an element of $C(S(c) \cap X)$.) Therefore, we get an $s|_{S(c)} = f|_{S(c)} \leq nc|_{S(c)}$, i. e. $f - nc \in c^\perp$. Thus if we set $fc := f$, we get a contradiction since $c \in A^\perp$. So we have proved that $\cup_{a\in A} S(a)$ is dense in $Spec^*(G)$. By construction, $fa \leq na$ holds on $S(a)$ for all $a \in A$. Therefore, $fa \wedge na$ vanishes outside $S(a)$, which yields $s|_{S(a)} = fa \wedge na$. Hence $s = \vee_{a\in A}(fa \wedge na)$, and the $fa \wedge na$ are pairwise orthogonal. $\square$

Remark 5.3.11 *By Mundici's functor and Cignoli and Torrens' ideal correspondence, both the sheaf representation theory discussed above and the results of [76] can be applied to MV-algebras. Note that in case of MV-algebras, we deal with Abelian l-groups possessing a strong unit, which amounts to a strong simplification. On the other hand, since a strong unit can be adjoined to any Abelian l-group (see Example 2.3.9), the category of Abelian l-groups is equivalent to the full subcategory of perfect MV-algebras, i. e., those with*

$$A = (\cap Max(A)) \cup (\cap Max(A))^*,$$

where $(\cap Max(A))^* = \{x^* \mid x \in \cap Max(A)\}$*. Therefore, an application of the preceding theory to MV-algebras with regard to the two-fold categorical relationship should be of some interest and a starting point for further investigation.*

Chapter 6

The C-topology

Many authors (for instance, Ball [6] and Redfield [95]) have investigated topologies on l-groups. Recently, Gusić [65] shows that an Abelian l-group (and hence the set of the principal fractional ideals of every Bézout domain) can be equipped with a topology (C-topology) with respect to an admissible subset C of positive elements, and he presents some ideas and results on the topological group properties of Abelian l-groups. However, the argument of [65] has a gap in Lemmas 5 and 6 which affected one main result (Theorem 1 of [65]) of the paper. In this Chapter we shall present corrected versions of Lemmas 5 and 6 and Theorem 1 of [65]; it leads to a more general result.

Actually, we show that any C-topology makes a 2-divisible Abelian l-group A into a topological lattice, and thus a topological l-group. Then, we prove that A is Archimedean if and only if it is Hausdorff with respect to $C = \cup_{n\in\mathbb{N}} A_{a,n}$ for any $a \in A^{>0}$, which gives a topological characterization for a Bézout domain with 2-divisible divisibility group to be completely integrally closed. Furthermore, we apply the results to MV-algebras. After that, we prove that every Archimedean lattice-ordered vector space is a Hausdorff topological lattice-ordered vector space under the C-topology, and give a counterexample for Lemmas 5 and 6 and Theorem 1 of [65].

Throughout this Chapter, l-group will always stand for an Abelian l-group, and we shall confine our attention to such groups.

6.1 C-topology on l-groups and MV-algebras

Let A be a 2-divisible (that is, $\frac{x}{2} \in A$ for all $x \in A$) l-group, a filter C of A is an *admissible* subset of A if $x \in C$ implies $x/2 \in C$.

Let A be a 2-divisible l-group with an admissible subset C. Then the *open C-ball of radius $r \in C$, with the center $x_0 \in A$*, is the set

$$U_{x_0,r} = \{x \in A \mid r - |x - x_0| \in C\}.$$

For a given admissible $C \subseteq A$, we say that A is *C-Archimedean* (or equivalently A is called a *C-group*) if

$$\forall x, y \in C,\ \exists n \in \mathbb{N},\ ny > x.$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in an l-group A is said to *converge in norm* to x if

$$\forall \epsilon \in A^{>0},\ \exists N \in \mathbb{N},\ |x_n - x| < \epsilon$$

for all $n \geq N$. A decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in A *converges* to x if $x = \wedge_n x_n$ exists, and denoted by $lim(x_n) = x$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in A is said to *converge in order* to x if there exists a decreasing sequence p_n with $lim(p_n) = 0$ such that

$$|x_n - x| \leq p_n$$

holds for all n. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in A *C-converges* to x if

$$\forall \epsilon \in C,\ \exists N \in \mathbb{N},\ |x_n - x| < \epsilon$$

for all $n \geq N$, and denoted by $lim_C(x_n) = x$.

Remark 6.1.1 *It is easily seen that the condition "C-Archimedean" is equivalent to say that 0 is a C-limit of each sequence $\{2^{-n}x\}_{n\in\mathbb{N}} \subseteq A^{\geq 0}$ or*

$$(\forall x \in A^{\geq 0})(\forall y \in C)(\exists n \in \mathbb{N})(ny > x).$$

An l-group is simple if and only if (see Proposition 4.1.17)

$$(\forall x \in A^{\geq 0})(\forall y \in A^{>0})(\exists n \in \mathbb{N})(ny > x).$$

Note that a C-Archimedean l-group need not be Archimedean (see Example 6.1.7 and Theorem 6.1.9), an Archimedean l-group is not necessarily C-Archimedean (see Examples 6.2.3, 4.2.17 and 4.2.15), either.

Note that an l-group is a subgroup of $\mathbb{R}$ if and only if the order-convergence, the norm-convergence and the C-convergence coincide. Suppose that A is a

2-divisible l-group, and that $a \in A^{>0}$. Put $A_a = \{x \in A \ : \ x \geq a\}$ and denote $A_{a,n} = \frac{1}{2^{n-1}} A_a$ for $n \in \mathbb{N}$. Then we have $A_{a,n+1} \supseteq A_{a,n}$ for every $n \in \mathbb{N}$. Let

$$C = \cup_{n \in \mathbb{N}} A_{a,n}.$$

Then C is a set of admissible elements. This is the minimal set of admissible elements containing a. It is clear that a C-convergent sequence is order convergent in a 2-divisible Archimedean l-group if the set of admissible elements is the minimal set of admissible elements with respect to containing a fixed element $a \in A^{>0}$. Moreover, it is easy to see that if A is a C-group then $C = \cup_{n \in \mathbb{N}} A_{a,n}$ for any $a \in C$.

Furthermore, by Remark 6.1.1 we get

Proposition 6.1.2 *Let A be a 2-divisible l-group. Then the following conditions are equivalent:*
(1) There is an admissible subset C such that A is a C-group.
(2) There is a strong unit in A.

For norm convergence, we have

Proposition 6.1.3 *If A is an Archimedean 2-divisible l-group, the following statements are equivalent:*
1) Every sequence $\{\frac{p}{2^{n-1}}\}_{n \in \mathbb{N}}$ in $A^{\geq 0}$ in norm converges to 0.
2) A is simple.

Proof. If there exists a sequence $\{\frac{p}{2^{n-1}}\}_{n \in \mathbb{N}}$ in $A^{\geq 0}$ does not norm-converge to 0. there exists an $\epsilon > 0$ with $2^{-n}p \not< p$ for infinitely many $n \in \mathbb{N}$. If A would be totally ordered, this yields $2^n \epsilon \leq p$ for almost all n. Thus $\epsilon = 0$, a contradiction. Conversely, If A is not simple, then there exists an element $a \in A$ which is noncomparable with 0 and $a^+ \wedge a^- = 0$. Then $\{\frac{a^-}{2^n}\}$ does not norm converges to 0 (see the proof of Theorem 4.1.17). □

For order convergence, we have

Proposition 6.1.4 *Let A be a 2-divisible l-group. Then A is Archimedean if and only if every sequence $\{\frac{p}{2^n}\}_{n \in \mathbb{N}}$ in $A^{\geq 0}$ order converges to 0.*

Proof. "⇒:" Without loss of generality, we can assume that $p > 0$. Then it suffices to prove that any lower bound l of the sequence $\{\frac{p}{2^n}\}$ satisfies $l \leq 0$. In fact, let l be such a lower bound, then $s := 0 \leq l \vee 0$ is still a lower bound of the sequence, so $0 \leq 2^n s \leq p$ for all $n \in \mathbb{N}$ and thus $0 \leq ns \leq p$ for all $n \in \mathbb{N}$. It follows that $s = 0$ since A is Archimedean, and thus $l \leq 0$.

"$\Leftarrow$:" Let $v \in A$ and assume that $nv \leq p \in A^{\geq 0}$ for all $n \in \mathbb{N}$. Then $v \leq \frac{p}{2^n}$ for all $n \in \mathbb{N}$. Since $\wedge_n \frac{p}{2^n} = 0$ implies that $v \leq 0$, A is Archimedean. $\square$

The C-*closure* of a subset S of A is defined to be the set

$$CL(S) := \{lim_C x_n \in A \mid x_n \in S\}.$$

Accordingly, we say that S is C-closed if and only if $S = CL(S)$. For C-convergence, we have

Proposition 6.1.5 *Let A be an Archimedean 2-divisible l-group. Then $A^{\geq 0}$ is the C-closure of C with respect to $C = \cup_{n \in \mathbb{N}} A_{a,n}$ for any $a \in A^{>0}$.*

Proof. Assume that $C = \cup_{n \in \mathbb{N}} A_{a,n}$ for a strictly positive fixed element $a \in A$. Then

$$C = \cup_{n \in \mathbb{N}} \{\frac{a}{2^{n-1}} + p \ : p \in A^{\geq 0}\}.$$

Especially, it follows that 0 is a C-limit of the sequence $\{\frac{a}{2^{n-1}}\}_{n \in \mathbb{N}}$. Suppose that x is another C-limit of the sequence $\{\frac{a}{2^{n-1}}\}_{n \in \mathbb{N}}$, then we get $|x| < \epsilon$ for all $\epsilon \in C$, especially, $|x| < \frac{a}{2^{n-1}}$ for all $n \in \mathbb{N}$ implies $|x| = 0$ since A is Archimedean and thus $x = 0$ implies that the C-limit of the sequence $\{\frac{a}{2^{n-1}}\}_{n \in \mathbb{N}}$ is unique. Note that analogously, we may also prove that the C-limit for any sequence $(x_n)_{n \in \mathbb{N}}$ in A is unique using the Archimedean property of A. Furthermore, for all $0 \leq p \in A$, it is clear that $lim_n(\frac{a}{2^{n-1}} + p) = p$ and $\frac{a}{2^{n-1}} + p \in C$ for all $n \in \mathbb{N}$. Thus we have shown that the positive cone of A is a subset of the closure of C.

On the other hand, let x be a C-limit of a sequence $\{x_n\}_{n \in \mathbb{N}}$ in A. Then the continuity of the absolute value (see Proof 2 of Lemma 6.1.12) in A implies that the sequence $\{|x_n|\}_{n \in \mathbb{N}}$ C-converges to $|x|$. Hence a limit x of a sequence $\{x_n\}_{n \in \mathbb{N}}$ in C must be positive. Therefore, the closure of C is a subset of the positive cone of A. $\square$

Recall that a *topological group* is a group and a topological space such that the addition map and the inversion map are both continuous. Note that a T_0-group is already Hausdorff (see [95], p. 105, Theorem B).

Gusić [65] proves the following important result:

Theorem 6.1.6 *Let A be an l-group (not necessarily 2-divisible), and let C be an admissible subset of A. Then open C-balls form a base of a topology τ on A (C-topology). If A is 2-divisible, then A is a topological group.*

The following statement is Theorem 1 of [65]:

Statement *Every C-group is a Hausdorff space.*

In order to prove the above statement, Gusić [65] tried to prove the following statements that are Lemmas 5 and 6 in [65].

Statements *Let A be a lattice-ordered 2-divisible C-Archimedean Abelian group. Then*

(a) $A^{\geq 0} = CL(C)$.

(b) Suppose that $x \geq 0$ and that $x < c$, for every $c \in C$. Then $x = 0$.

However, Lemmas 5 and 6, and Theorem 1 in [65] are *not correct* as shown by the following example:

Example 6.1.7 *Let $A = \mathbb{R} \oplus_l \mathbb{R}$. Then A is a linearly ordered divisible group. Let $C = \cup_{n\in\mathbb{N}} A_{(0,1),n}$. Then A is a C-group. However, A is not T_2 under the C-topology. For instance, $(1,0)$ and $(2,0)$ cannot be separated by C-balls.*

In fact, it is easy to prove that any C-topological group A, not only a C-group, is Hausdorff if $A^{\geq 0}$ is C-closed.

Corollary 6.1.8 *Let A be an l-group, and let C be an admissible subset of A. If $A^{\geq 0}$ is C-closed, then the C-topology on A is Hausdorff.*

Proof. By Theorem 6.1.6 it follows that A is a topological group. $A^{\geq 0}$ is closed and so is $-A^{\geq 0}$, thus $\{0\} = A^{\geq 0} \cap (-A^{\geq 0})$, which implies A is a Hausdorff space by the above remark. □

Note that if A is not Archimedean, then the C-limit of a sequence in A may be not unique (see Example 6.1.7 below). The following theorem might be of interest in itself.

Theorem 6.1.9 *Let A be a 2-divisible l-group. Then the following statements are equivalent:*

1) A is Archimedean.

2) $A^{\geq 0}$ is closed with respect to any $C = \cup_{n\in\mathbb{N}} A_{a,n}$ for any $a \in A^{>0}$.

3) A is Hausdorff with respect to any $C = \cup_{n\in\mathbb{N}} A_{a,n}$ for any $a \in A^{>0}$.

Proof. 1)⇒2) ⇒3): This is clear by Proposition 6.1.5 and Corollary 6.1.8.

3)⇒1): Suppose that A is not Archimedean. Then there exist $a, b \in A^{>0}$ such that $b \ll a$. Let $C = \cup_{n\in\mathbb{N}} A_{a,n}$, then the elements b and $2b$ can not be separated in the C-topology, and hence A is not T_2. □

Remark 6.1.10 *Recall that a T_1-group is an arbitrary group (not necessarily Abelian) and a T_1-space, with the inverse continuous and group addition separately continuous. Furthermore, a C-group in terms of Kaplansky [74] is a T_1-group where the mapping $a \mapsto -a + x + a$ (x fixed) is continuous. So, by Theorem 6.1.9 and the remark before Theorem 6.1.6 we can conclude that an Archimedean 2-divisible l-group with the C-topology is a C-group in terms of Kaplansky. However, we shall give a counterexample in the Example 6.2.3 below to show that this is not true for a C-group in terms of Gusić [65].*

Corollary 6.1.11 *Let A be an Archimedean 2-divisible l-group with the C-topology. Then the closure of a subgroup of A is a subgroup.*

Proof. It follows from the remark above and Theorem 8.1 of [74]. □

Lemma 6.1.12 *Let A be a 2-divisible l-group. Then A is a topological lattice with respect to any admissible subset C of A.*

Proof. By Theorem 6.1.6 and Corollary 2.1.2 it suffices to prove that the absolute value

$$|-|: A \to A,\ x \mapsto |x|$$

is continuous. For all $\epsilon \in C$, let $U_{|a|,\ \epsilon}$ be an open C-ball of radius ϵ around $|a|$, and let $U_{a,\epsilon}$ be an open C-balls of radius ϵ around a. Taken $x \in U_{a,\epsilon}$. By Corollary 2.1.2 we have

$$\epsilon - ||x| - |a|| \geq \epsilon - |x - a| = \epsilon - (\epsilon - c_x) = c_x \in C,$$

where $\epsilon - |x - a| = c_x \in C$. Since x can be chosen arbitrary, it follows that

$$|-|(U_{a,\epsilon}) \subseteq U_{|a|,\ \epsilon},$$

so the lattice continuity is proved. □

By Theorem 6.1.6 and Lemma 6.1.12 we have

Theorem 6.1.13 *Every C-topology on a 2-divisible l-group A makes A into a topological l-group.*

Concerning MV-algebras, the concept of an admissible subset of A can be defined similarly to the corresponding one for an l-group, and we have

Theorem 6.1.14 *Let A be an MV-algebra (not necessarily 2-divisible). Then open C-balls form a base of a topology τ on A (C-topology). If A is 2-divisible, then A is a topological MV-algebra. If $A^{\geq 0}$ is C-closed, then the C-topology on A is Hausdorff.*

6.2 C-topology on lattice-ordered vector spaces

In this section, we will show that any Archimedean lattice-ordered vector space is a T_2 topological lattice-ordered vector space under the C-topology. The C-topology theory of l-groups developed in [65] and above, clearly applies to Riesz spaces. Furthermore, note that one can also discuss the C-topology on lattice-ordered rings and lattice-ordered fields, etc..

In particular, we have the following result from Theorem 6.1.13 and Corollary 6.1.8.

Proposition 6.2.1 *Let L be a Riesz Space. Then each C-topology τ makes L into a topological l-group. If L is a Riesz C-space (that is, the l-group is C-Archimedean) in which L^+ is closed, then L is Archimedean. Furthermore, any Riesz Archimedean C-space is an Archimedean function lattice.*

For Archimedean lattice-ordered vector spaces over an Archimedean totally ordered field, we have

Theorem 6.2.2 *The C-topology on any Archimedean vector space V over an Archimedean totally ordered field F makes V into a T_2 topological vector space with respect to $C = \cup_{n\in\mathbb{N}} A_{v,n}$ for any $v \in V^{>0}$.*

Proof. By Proposition 6.2.1 it suffices to prove that the map

$$m: \ F \times V \to V, \ (\lambda, x) \mapsto \lambda x$$

is continuous. For all $\epsilon \in C$,

$$\epsilon - |\lambda x - \lambda_0 x_0| \geq \epsilon - ||\lambda|| \cdot |x - x_0| + |x| \cdot ||\lambda - \lambda_0||,$$

where $||-||$ is the absolute value of the Archimedean totally ordered field F.

Hence, if we choose $N \in \mathbb{N}$, δ_{λ_0} and δ_{x_0} in $F^{>0}$ and C, respectively, such that $\delta_{\lambda_0} < \frac{1}{2^{3+N}}$, $\delta_{x_0} < \frac{\epsilon}{2^{3+N}}$, $\frac{|x_0|}{2^N} < \frac{\epsilon}{8}$ and $\frac{||\lambda_0||}{2^N} < \frac{1}{8}$, then for all $\lambda \in U_{\lambda_0,\, \delta_{\lambda_0}}$ and $x \in U_{x_0, \delta_{x_0}}$, we have

$\epsilon - |\lambda x - \lambda_0 x_0|$
$\geq \epsilon - (||\lambda||\delta_{x_0} + |x|\delta_{\lambda_0})$
$\geq \epsilon - ((\delta_{\lambda_0} + ||\lambda_0||)\delta_{x_0} + (\delta_{x_0} + |x_0|)\delta_{\lambda_0})$
$\geq \epsilon - ((\frac{1}{2^{3+N}} + \frac{2^N}{8})\frac{\epsilon}{2^{3+N}} + (\frac{\epsilon}{2^{3+N}} + \frac{2^N\epsilon}{8})\frac{1}{2^{3+N}})$

$\geq \epsilon - \frac{\epsilon}{2}$
$= \frac{\epsilon}{2} \in C.$ □

Now we will give an example to show that not all of the T_2 topological lattice-ordered vector spaces are C-Archimedean, and thus not all of the T_2 topological l-groups are C-Archimedean, either.

By Corollary 3.2.7 we know that any lattice-order $\leq$ on $\mathbb{R}$ constructed by Wilson [103] is an Archimedean one. Furthermore, $\mathbb{R}$ has a largest Archimedean totally ordered subfield F with respect to the Wilson order $\leq$. Clearly, $\mathbb{R}$ is a lattice-ordered vector space over F.

Example 6.2.3 *Let $\leq$ be a Wilson order on $\mathbb{R}$ with the largest Archimedean totally ordered subfield F, and $C = \cup_{n\in\mathbb{N}} A_{r,n}$ for any $r \in \mathbb{R}^{>0}$ with respect to the Wilson order $\leq$. Then the C-topology τ makes $\mathbb{R}$ into a T_2 topological Archimedean vector space which is not C-Archimedean. Actually, by Artin-Schreier theory, there is no strong unit in $\mathbb{R}$ with such orders constructed in [103] since the construction gives just infinitely many successive simple field extensions, and thus any strictly positive element in $\mathbb{R}$ is not a strong unit in the next extension step any more. Hence, $\mathbb{R}$ is not C-Archimedean.*

Chapter 7

Almost l-groups

An integral domain D is called an *almost GCD domain* (*AGCD* domain for short) or an *almost pseudo*-Bézout domain if, for all $x, y \in D$, there is an integer $n = n(x, y) \geq 1$ such that $(x^n) \cap (y^n)$ is principal; and a directed partially ordered Abelian group G is called an *almost l-group (al-group for short)* if, for each pair x and y in G there exists an integer $n = n(x, y) \geq 1$ such that $nx \vee ny$ exists in G.

It is known that an Abelian l-group is the group of divisibility of a Bézout domain (cf. [3, 4, 59, 85]), and the divisibility group of a Bézout domain is lattice-ordered. Furthermore, a necessary condition for a po-group G to occur as the divisibility group of a domain is that G is directed. But this condition is not sufficient as is shown by several examples in the literature (cf. [2, 8, 35, 46, 62, 71, 83, 86, 92]). Dumitrescu, Lequain, Mott, and Zafrullah [50] proved that AGCD domains can be characterized by the property that their divisibility group is an al-group ([50], Proposition 5.2). They raised the problem whether the converse holds. The aim of this Chapter is to answer the question in the negative.

7.1 An almost l-group need not be torsion free

It is known that an integral domain D is an AGCD domain if and only if $G(D)$ is an al-group (see [50], Proposition 5.2). Conversely, Dumitrescu, Lequain, Mott, and Zafrullah [50] raised the following problem.

Question 7.1.1 *Is every al-group the divisibility group of an AGCD domain?*

The group of divisibility of a Bézout domain is an l-group, and thus it is torsion free (that is, $a \neq 0$ implies $na \neq 0$ for all $n \in \mathbb{N}$) (cf. [16]). However, for an al-group, this need not be the case. Indeed, we shall give an example which shows that a 1st strongly Archimedean al-group need not be torsion free. This exhibits a striking difference between an al-group and an l-group.

As the positive cone of any po-group is torsion free, we construct an al-group G that is generated by a submonoid M which properly contains the positive cone $G^{\geq 0}$ of G and, in M, any two elements admit an *lcm*[1] according to Dehornoy's theory [45]:

Proposition 7.1.2 ([45], Proposition 1 and Corollary 2) *Let G be a group and M be a submonoid of G such that M generates G and, in M, any two elements admit a right lcm. Then G is torsion free if and only if M is torsion free.*

However, Proposition 7.1.2 does not tell us whether an al-group is torsion-free. In the following example (see also the so-called "infinite whalebone corset" in Example 2.3.3), we show that an al-group need not be torsion free.

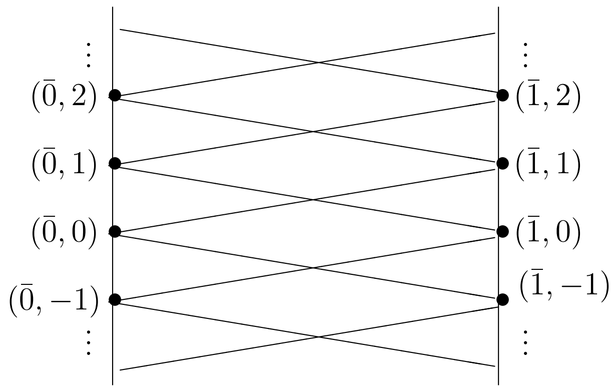


Figure 7.1

Example 7.1.3 *Let $G = ((\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}, \leq)$ (see Figure 7.1): $(\bar{0}, 0) \leq (\bar{1}, 1)$, $(\bar{1}, 0) \leq (\bar{0}, 1)$, $(\bar{0}, n) \leq (\bar{0}, m)$ and $(\bar{1}, n) \leq (\bar{1}, m)$ if and only if $n \leq m$ under the usual order on $\mathbb{Z}$. Then it is easily verified that G is an al-group, but it is not torsion-free or Archimedean. In fact, for all elements $(\bar{x_1}, y_1)$ and*

[1]For $x,\ y$ in a (not necessarily commutative) monoid $(M,\ +,\ 0)$, we say that y is a *right multiple* of x and if $y = x + z$ holds for some z in M; we say that z is a right least common multiple, or *right lcm*, of x and y if z is a right multiple of x and y and any common right multiple of x and y is a right multiple of z. Note that a right lcm need not be unique (cf. [45]).

$(\bar{x_2}, y_2)$ in G, $2(\bar{x_1}, y_1) \vee 2(\bar{x_2}, y_2) = (\bar{0}, 2y_1) \vee (\bar{0}, 2y_2)$ exists in G, moreover, $n(\bar{x}, y) > (\bar{0}, 0)$ implies $(\bar{x}, y) > (\bar{0}, 0)$. However, $(\bar{0}, 0) \neq (\bar{1}, 0)$, and $(\bar{0}, 0) = 2(\bar{1}, 0)$, which shows that G is not torsion free. In fact, $M = G^{\geq 0} \cup \{(\bar{1}, 0)\}$ is a submonoid that satisfies the hypotheses in Proposition 7.1.2, but M is not torsion-free. Furthermore, $(\bar{0}, 0) \neq (\bar{1}, 0) \ll (\bar{0}, 1)$, which shows that the order is not Archimedean.

In [86], Mott gives the following "box condition" theorem, which provides a useful tool for verifying the group of divisibility structure of a po-group.

Lemma 7.1.4 (cf. [86], Theorem 4.4.1) *Suppose that ω is a semi-valuation on a field K with semivalue group G. Suppose moreover that $x,\ y \in K^*$ are such that $\omega(x)$ and $\omega(y)$ are not comparable under the order in G. If $x + y \in K^*$, then $\omega(x + y) \in \mathcal{UL}\ \{\omega(x),\ \omega(y)\} \setminus (\mathcal{U}\ \{\omega(x)\} \cup \mathcal{U}\{\omega(y)\})$.*

Clearly any po-group that violates the above "box condition" of Lemma 7.1.4 cannot be a group of divisibility.

Proposition 7.1.5 *A strongly 1st Archimedean al-group need not be torsion free, Archimedean or a group of divisibility.*

Proof. We verify that the al-group in Example 7.1.3 violates the condition of Lemma 7.1.4. In fact, $\omega(x) = 0 := (\bar{0}, 0)$ and $\omega(y) = e := (\bar{1}, 0)$ violate the "box condition" since $\mathcal{UL}\ \{\omega(x) = 0,\ \omega(y) = e\} = \mathcal{U}\ \{\omega(x) = 0\} \cup \mathcal{U}\{\omega(y) = e\}$. □

7.2 An almost l-group need not be isolated

Recall that a po-group G is *isolated* if, for all $n \in \mathbb{N}$ and $x \in G$ the implication $nx \geq 0 \Rightarrow x \geq 0$ holds. A po-group is *weakly* isolated if the implication $nx > 0 \Rightarrow x > 0$ holds for all $n \in \mathbb{N}$ and $x \in G$.

It is well known that an l-group is isolated [16], however, Example 7.1.3 above shows that this property cannot be generalized to an al-group:

Proposition 7.2.1 *A weakly isolated strongly 1st Archimedean al-group need not be torsion free or isolated.*

We start with the observation that an isolated al-group is torsion free. In fact, we have:

Proposition 7.2.2 *A group G is torsion free if and only if it is isomorphic to an isolated po-group.*

Proof. "$\Rightarrow$": The trivial order on G makes G into an isolated po-group.
"$\Leftarrow$": Assume that there exists $0 < m \in \mathbb{Z}$ such that $mx = 0$, then $mx \geq 0$ and thus $x \geq 0$, which implies $x = 0$. $\square$

Theorem 7.2.3 ([46]) *Let $0 < n \in \mathbb{Z}$. Then, up to isomorphism, the positive cones of divisibility orders (i. e. orderings that give rise to a divisibility group) of $\mathbb{Z}^n$ can all be effectively determined and described in terms of the Archimedean total orders of $\mathbb{Z}^i$, $1 \leq i \leq n$. In particular, $\mathbb{Z}$ has only one divisibility order, up to isomorphism. For $\mathbb{Z}^2$, there is, besides the total orders and the cardinal orders, just one more divisibility order. The positive cone of that order can be given as $(\mathbb{Z}^{>0} \times \mathbb{Z}^{>0}) \cup \{(0,\ 0)\}$. For $\mathbb{Z}^3$ (respectively, $\mathbb{Z}^4$ and $\mathbb{Z}^5$), there are, besides the lattice orders, exactly five (respectively, 27 and 105) types of divisibility orders.*

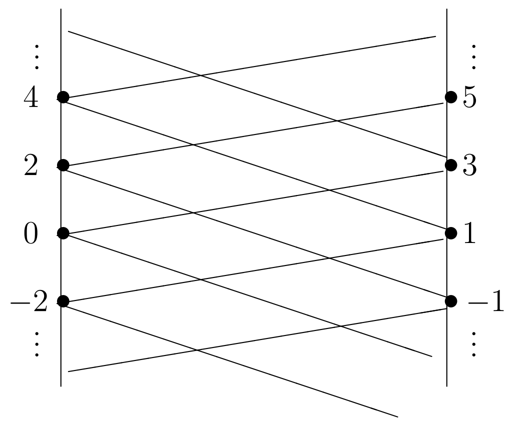


Figure 7.2

Therefore, the unique divisibility order on the additive group of $\mathbb{Z}$ is exactly the total Archimedean order, and thus any other order cannot make $\mathbb{Z}$ into a group of divisibility. However, Theorem 7.2.3 does not tell us whether an al-group is isolated. We shall use $\mathbb{Z}$ to construct an example of a torsion free strongly 1st Archimedean al-group which is not isolated:

Example 7.2.4 *Let $G = (\mathbb{Z},\ \leq)$ with the order $\leq$ defined as in Figure 7.2. Then G is a torsion free directed group. Furthermore, it is strongly 1st Archimedean, and for all $x,\ y$ in G, $2x \vee 2y$ exists in G. However, $0 \vee 1$ does not exist in G.*

Thus Example 7.2.4 implies the following result by Lemma 7.1.4 again.

Proposition 7.2.5 *A torsion free strongly 1st Archimedean al-group need not be isolated.*

7.3 Are isolated Archimedean almost l-groups lattice-ordered?

In this section we shall prove that an isolated Archimedean al-group (iAal-group for short) need not be an l-group or a group of divisibility. We first give an order on $\mathbb{Z}^2$ such that it is an iAal-group which is not lattice-ordered. Actually, by Theorem 7.2.3 we know that the positive cone of such an order should violate the condition of Theorem 7.2.3.

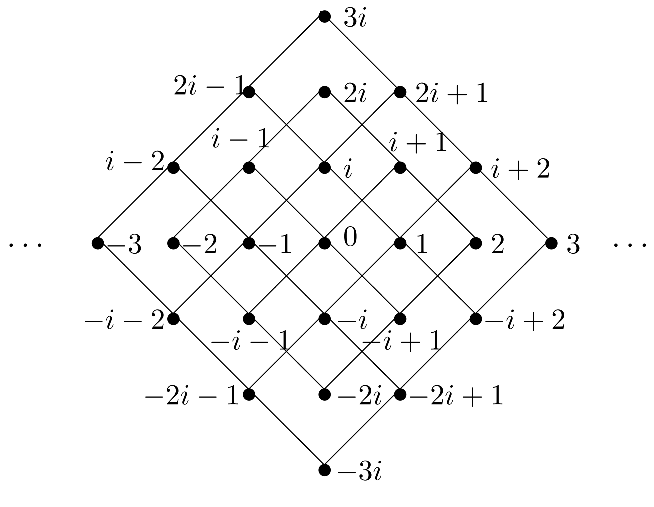


Figure 7.3

Example 7.3.1 *Let $G = (\mathbb{Z} + i\mathbb{Z}, \ \leq)$, the order $\leq$ defined as in Figure 7.3. Then it is straightforward to show that G is a torsion free directed group. Furthermore, it is Archimedean, and $a+bi \geq 0$ if and only if $|a| \leq b$ under the usual order on $\mathbb{Z}$, where $|a|$ is the absolute value of a. Thus $nx = n(a+bi) \geq 0$ if and only if $|na| = n|a| \leq nb$ if and only if $|a| \leq b$, which implies $a+bi \geq 0$. Furthermore, for all $x = a+bi$ in G, $0 \vee 2x = 2a + (-2a) \vee ((2b) \vee (-2b))$ exists* *in G which shows that G is an al-group. However, $0 \vee 1$ does not exist in G. Furthermore, it is easily seen that G can be order embedded in an Abelian l-group H (for instance, the divisible hull of G) in which G is a partially ordered subgroup. But G is not an l-subgroup of H, this implies again that the 2-isolated condition of Theorem 3.1.1 is necessary.*

Therefore, by Lemma 7.1.4 again, we get

Proposition 7.3.2 *An iAal-group need not be an l-group or a group of divisibility.*

For an Archimedean directed group G, Lemma 2.14 of [58] characterizes the condition of G to be linear.

Proposition 7.3.3 ([58]) *An Archimedean directed group G is linearly ordered if and only if there exists a strictly positive element which is comparable with all positive elements.*

Applying Proposition 7.3.3 to *al*-groups, we obtain

Corollary 7.3.4 *An Archimedean al-group is the divisibility group of a valuation domain if and only if there exists a strictly positive element which is comparable with all positive elements.*

For an Abelian directed group with strong unit, we have

Proposition 7.3.5 *Let (G, u) be an Abelian directed group with strong unit u. Then G is linearly ordered if and only if u is comparable with all positive elements.*

Proof. It suffices to prove the sufficiency. Assume that u is comparable with all positive elements, and that $a \parallel b$. Then we have either $a, b > u$ or $a, b < u$. Suppose $a, b < u$. Then $u - b + a > 0$, so $u - b + a$ and u are comparable. But then a and b are comparable since $(u - b + a) - u = a - b$ is comparable with 0. From this contradiction we conclude that $a, b > u$. Hence $a - u \parallel b - u$ and $a - u, b - u > 0$. As above $a - u, b - u > u$, i. e. $a, b > 2u$. Repetitions of this argument show that $a > nu$ and $b > nu$ for all $n \in \mathbb{N}$. But this is impossible as u is a strong unit. This finishes the proof. $\square$

Applying Proposition 7.3.5 to *al*-groups we get

Corollary 7.3.6 *Let (G, u) be an Abelian al-group with strong unit u. Then G is the divisibility group of a valuation domain if and only if u is comparable with any positive elements.*

Recall that a *directoid* (cf. [58, 72]) is a groupoid with a binary operation $\sqcup$ satisfying the identities (i) $x \sqcup x = x$, (ii) $(x \sqcup y) \sqcup x = x \sqcup y$, (iii) $y \sqcup (x \sqcup y) = x \sqcup y$, and (iv) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

A *directoid group* G is defined to be an Abelian group with a commutative directoid operation $\sqcup$ satisfying $p + (q \sqcup r) = (p + q) \sqcup (p + r)$ for all $p, q, r \in G$. As noted in [58], directoid groups must be 2-torsion-free. If we define $a \leq b$ if and only if $a \sqcup b = b$, then [72] shows that a groupoid $(A, \sqcup)$ with a binary operation $\sqcup$ is a directoid if and only if $(A, \leq)$ is an up-directed partially ordered set satisfying

(1) $a \leq ab$ and $b \leq ab$ for all $a, b \in a$, and

(2) if $a \leq b$, then $a \sqcup b = b \sqcup a = b$.

[58] gives the following result which characterizes the condition of directoid group to be an l-group.

Proposition 7.3.7 ([58], Propositions 2.3 and 2.5) *Let G be a directoid group. Then the following conditions are equivalent.*

(a) G satisfies the distributive law.

(b) G satisfies the modular law.

(c) $G^{\geq 0}$ is a semilattice.

(d) G is an l-group.

Note that (c)$\Leftrightarrow$(d) and (d)$\Rightarrow$(a)$\Rightarrow$(b) are well known, for the proof of (b)$\Leftrightarrow$(d), the reader is referred to [58]. For Abelian *al*-group, we have

Corollary 7.3.8 *An Abelian directoid al-group G is an l-group if and only if G satisfies the modular law.*

Note that for a torsion-free Abelian group G, by Levi's theorem (cf. [16]) we know that G can be totally ordered and thus can be lattice-ordered. From the well known result of Krull we know that the totally ordered Abelian group G is the group of divisibility of a valuation ring, and from the Jaffard-Ohm theorem we know that any Abelian l-group G is the group of divisibility of a Bézout domain (see Theorem 2.2.1). Recall that a domain D is *completely integrally closed* if every element of its quotient field K almost integral over D

belongs to D (i. e. for a and b in K with $ab^n \in D$ implies $b \in D$). A po-group G is called *completely integrally closed* if $na < b$ $(n \in \mathbb{N})$ implies $a \leq 0$. Any completely integrally closed po-group is Archimedean. The converse implication does not hold in general (cf. [55]). And it is well known that a Bézout domain is completely integrally closed if and only if $G(D)$ is an Archimedean l-group (see Theorem 2.2.2). For a directed group, we have the following result which generalizes Theorem 2.2.2.

Proposition 7.3.9 *Let D be an integral domain and $G(D)$ be its group of divisibility. Then $G(D)$ is completely integrally closed if and only if D is completely integrally closed.*

Proof. Let Ua, Ub be two element of $G(D)$, and suppose that $nUa \leq Ub$, for all $n \in \mathbb{N}$. Then $Ub + n(U(1/a)) \geq 0$, which means that $b(1/a)^n$ in D for all positive n. But then $1/a \in D$, and so $Ua \leq 0$. Hence, $G(D)$ is completely integrally closed.

Conversely, assume that $Ua \geq nU(1/b)$ for all $n \in \mathbb{N}$. Then $U(1/b) \leq 0$ since $G(D)$ is completely integrally closed. Thus $Ub \geq 0$ and so $b \in D$, as claimed. □

By Proposition 7.3.9 and [50] Proposition 5.2 we have

Corollary 7.3.10 *An AGCD domain D is completely integrally closed if and only if $G(D)$ is completely integrally closed.*

Moreover, from [54] we know that for any torsion free Abelian group G there exists a non-trivial group homomorphism θ from G to the additive group of the field of rationals. Therefore, $P = \{g \in G \mid \theta(g) > 0\} \cup \{0\}$ is a positive cone of a directed order on G which makes G into an isolated strongly 1st Archimedean al-group (but not necessarily Archimedean). However, the results above show that:

Proposition 7.3.11 *Neither a torsion free strongly 1st Archimedean al-group nor an iAal-group need be the group of divisibility of a completely integrally closed domain.*

Therefore, the Jaffard-Ohm correspondence theorem does not hold in almost Abelian l-groups, which answers the problem of Dumitrescu, Lequain, Mott and Zafrullah [50], mentioned at the begin of this chapter, in the negative.

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