Stochastic Differential Equations Driven by Gaussian Processes with Dependent Increments and Related Market Models with Memory

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung
vorgelegt von

Daniel Schiemert

Hauptberichter: Priv.-Doz. Dr. J. Dippon
Mitberichter: Prof. Dr. C. Hesse
Tag der Einreichung: 07. Dez. 2006
Tag der mündlichen Prüfung: 22. Dez. 2006

Institut für Stochastik und Anwendungen
Erscheihnungsjahr 2007

Preface

If I only had the theorems!
Then I should find the proofs easily enough.

Riemann

Contents

Zusammenfassung

Zahlreiche Anwendungen zur fraktalen Brownschen Bewegung B_t^H wurden in den letzten zehn Jahren publiziert, siehe [Be], [GrNo], [HuOk],[HuOkSa], [So] [DeUs], [Oh] und [DuHuPa]. Die fraktale Brownsche Bewegung hat abhängige Zuwächse, so dass sie für Anwendungen in Finanzmathematik (siehe [HuOk] und [Be]) und in Netzwerk-Simulationen [No] interessant ist. Die Kovarianzfunktion der fraktalen Brownschen Bewegung lässt sich lediglich durch den Hurst-Parameter verändern. Insbesondere ist es nicht möglich, ein Kurzzeitgedächtnis durch eine Wahl des Hurst-Parameters zu bekommen. Verallgemeinernd wird eine Klasse von zentrierten Gaußschen Prozessen mit abhängigen Zuwächsen B_t^v definiert, welche die fraktale Brownsche Bewegung beinhaltet. In Anwendungen dieser Prozesse, insbesondere in der Finanzmathematik in geeigneten Black-Scholes-Modellen, braucht man ein stochastisches Integral, welches durch B_t^v getrieben wird. Wenn man das Integral mit dem Wick-Produkt definiert, so ist, wie im bereits bekannten Fall für die fraktale Brownsche Bewegung [HuOk],[Be], gesichert, dass der Erwartungswert des Integrals $\int_{\mathbb{R}} X_s dB_s^v$ 0 ist. Das Wick-Produkt ist jedoch nicht abgeschlossen in $L^2(\Omega)$, daher ist es hilfreich White-Noise-Theorie zu benutzen [HiKuPoSt],[Ku],[PoSt]. Mit der White-Noise-Theorie ist es weiterhin möglich, geeignete stochastische Prozesse abzuleiten [Ku],[La],[Be]. Insbesondere ist es gelungen, die bilineare stochastische Differentialgleichung mit fraktalen Brownschen Bewegungen zu lösen [HuOk]. Dieser Ansatz wird für stochastische Differentialgleichungen, getrieben durch Gaußsche Prozesse mit abhängigen Zuwächsen, formuliert.

Abschnitt 2.1 beginnt mit der Konstruktion des Schwartzschen Raumes und dessen Dualraum, welche später zur Erstellung der Hida-Test- und Hida-Distributionen-Räume verwendet werden. In dem darauffolgenden Abschnitt 2.2 wird die Klasse der Gaußschen Prozesse B_t^v eingeführt. Hierbei werden zahlreiche Eigenschaften dieser Klasse bewiesen und diskutiert. Der Abschnitt 2.3 ist den Hida-Test- und Hida-Distributionen-Räumen gewidmet. Im Abschnitt 2.4 werden die S-Transformation und das Wick-Produkt definiert. Diese sind für die Beweise in den folgenden Kapitel relevant. Mit der S-Transformation und dem Wick-Produkt ist es möglich, stochastische in deterministische Probleme umzuformulieren. Das macht Beweise kurz. Im Abschnitt 2.5 werden Charakterisierungs- und Konvergenztheoreme im Hida-Distributionen-Raum erläutert.

Das dritte Kapitel beginnt mit der Ableitung des Gaußschen Prozesses B_t^v im Hida-Distributionen-Sinn, welche durch die Ableitung im Sinne der temperierten Distributionen erklärt ist. Im Abschnitt 3.1 wird das stochastische Integral, getrieben durch den Gaußschen

Prozess mit abhängigen Zuwächsen, definiert. Dabei wird gezeigt, unter welchen Bedingungen sich das Integral durch Wick-Riemann-Summen approximieren lässt. Weiterhin werden typische Anwendungen des Integrals, wie partielle Integration und das Lösen der bilinearen stochastischen Differentialgleichung, gezeigt. Es ist auch ein Existenz- und Eindeutigkeitssatz für die Lösungen von stochastischen Differentialgleichungen formuliert, diese ist jedoch nicht auf viele Fälle anwendbar, so dass im Abschnitt 3.4 ein allgemeinerer Satz gezeigt und diskutiert wird. Im Abschnitt 3.2 werden allgemeinere Prozesse als der Gaußsche Prozess B_t^v betrachtet. Das korrespondierende Integral und die entsprechenden Eigenschaften werden diskutiert. Im Abschnitt 3.3 wird eine Ableitungsregel bewiesen, welche im Falle der Brownschen Bewegung mit der Itô-Regel identisch ist. Mehrere Version dieser Ableitungsregel werden vorgestellt.

Im vierten Kapitel wird das im vorigen Kapitel entwickelte Kalkül genutzt, um Optionen in einem geeigneten Black-Scholes-Markt und Zero-Coupon-Bonds in einem Vasicek-Modell zu bewerten. Da der Gaußsche Prozess B_t^v in den entsprechenden stochastischen Differentialgleichungen benutzt wird, ergeben sich abhängige Zuwächse in den beiden Modellen. Im Abschnitt 4.1 wird das Black-Scholes Markt mit Gedächtnis präsentiert. In der Veröffentlichung [BjHu] gibt es zahlreiche Kritiken zur ökonomischen Bedeutung des Wick-Portfolios, welches auch hier zur Bewertung der Optionen notwendig ist. Diese Kritik konnte aus dem Weg geräumt werden, indem gezeigt wird, dass es zu einem Wick-Portfolio ein eindeutiges äquivalentes gewöhnliches Portfolio gibt und umgekehrt. Danach wird unter einer schwachen Voraussetzung gezeigt, dass dieser Markt arbitragefrei ist. Die Bewertung erfolgt mittels eines replizierenden selbstfinanzierenden Portfolios. Ein Europäischer Call wird bewertet und dessen Greeks berechnet. In der Sektion 4.2 wird ein Zero-Coupon-Bond im Vasicek-Modell bewertet. Zuerst zeigt man die Äquivalenz der entsprechenden Portfolios und die Abwesenheit von Arbitrage. Die Bewertung erfolgt über den Marktpreis des Risikos. Dabei wird gezeigt, dass dieser unabhängig von der Laufzeit des Bonds ist. Beide Modelle stimmen mit den jeweiligen klassischen Modellen im Falle der Brownschen Bewegung überein.

Die Ergebnisse dieser Dissertation wurden bereits in den Preprints [DiSc] und [DiSc2] veröffentlicht.

1

Introduction

In the last ten years fractional Brownian motion B_t^H received a lot of attention (e.g. [Be], [GrNo], [HuOk],[HuOkSa], [So] [DeUs], [Oh] and [DuHuPa]). This process has dependent increments, which make it interesting for many applications such as finance (e.g. [HuOk], [Be]) and network simulations (e.g. [No]). However, B_t^H has a covariance function $E(B_t^H B_s^H)$, which depends only on the Hurst parameter $H \in (0,1)$. It follows for example that one cannot model a process with short range dependency with B_t^H. As a generalization a class of centered Gaussian processes with dependent increments B_t^v is defined. If B_t^v is used as noise process, it is often important to have an integral driven by this process.Several authors defined the stochastic integral driven by fractional Brownian motion $\int_{\mathbb{R}} X_s dB_s^H$. In order to use this integral to explain stochastic differential equations it is desirably that the stochastic integral driven by fractional Brownian motion has expectation value 0. If the integral is defined by use of the Wick product ([Be], [HuOk]) the expectation value of the stochastic integral driven by B_t^H is zero. Thus this definition is interesting for applications. In order to use the Wick product it was helpful to use white noise distribution theory, because the Wick product is not closed in $L^2(\Omega)$. This theory ([HiKuPoSt], [Ku], [PoSt]) offers also the possibility to derivate fractional Brownian motion in the Hida distribution sense. Further, it has a lot of advantages in the treatment of the Wick product, e.g. the Wick calculus is closed in the space of the Hida distributions [Be], [La]. So one can define stochastic differential equations by integrating the Wick product of the integrand with the derivative of the fractional Brownian motion and solve the bilinear equation (e.g. [HuOk]). This approach is formulated here for stochastic differential equations driven by B_t^v.

In the first chapter the auxiliary results of white noise theory are summarized. It begins in Section 2.1 with the construction of the Schwartz space and its dual, which is later used to define the Hida test and distribution space. In Section 2.2 the class of Gaussian processes B_t^v is defined and several properties are shown. The Section 2.3 introduces the Hida test and Hida distribution space. Their construction is using the chaos decomposition theorem and the definition of the Schwartz space. As mentioned before one can derivate certain stochastic processes in the Hida distribution space. This is based on the derivative of a deterministic function in the sense of tempered distributions. Further tools to examine the convergence in the Hida distribution space are part of the following section. There the S-transform and the Wick product are defined. The S-transform is a mapping, which allows to examine stochastic processes in a deterministic manner, as we will see in Chapter 3. At the end of Chapter 2, there are some characterization and convergence theorems referring to elements of the Hida distribution space.

In Chapter 3 the derivative of the Gaussian process B_t^v will be declared by a derivative in the sense of tempered distributions. In Section 3.1 we define the stochastic integral $\int_{\mathbb{R}} X_s \, dB_s^v$. Further, conditions are presented under which the integral can be approximated by Riemann sums. Several typical applications of the stochastic integrals are developped like partial integration or solving the bilinear stochastic differential equation driven by B_t^v. An existence and uniqueness theorem for solutions of stochastic differential equations is formulated a general one in Section 3.4. In Section 3.2 a stochastic integral driven by a in the Hida sense continuously differentiable stochastic process is discussed. As an example the related Ornstein-Uhlenbeck process is treated. In Section 3.3 we derive a chain formula and variants thereof for stochastic distribution processes, which concides with Itô's rule in the case of Brownian motion. By the S-transform and the Wick product it is possible to obtain the chain rule directly be the derivative of a deterministic function, which makes applications practical and proofs easy. As already mentioned, Section 3.4 presents several theorems on the existence and uniqueness of solutions of stochastic differential equations.

In Chapter 4 we use the stochastic integral calculus developped in Chapter 3 to price financial derivatives in a Black-Scholes market and to price a zero-coupon bond in a Vasicek model employing B_t^v as a noise process. In Section 4.1 the Black-Scholes market with memory is presented. We start with the presentation of the market assumptions. In order to be able to use the integral calculus above, the Wick product is used to define a certain kind of portfolios [HuOk]. There are some criticisms on the economical sense of a Wick portfolios by [BjHu]. To refute these arguments we show that there is an ordinary portfolio which is equal to a portfolio with Wick products and vice versa. The absence of arbitrage in the Black-Scholes market with memory is shown under weak assumptions. We price options in this market by a replicating portfolio. The price of a European call and its Greeks are calculated. In Section 4.2 the Vasicek model with memory is discussed. We show that the Vasicek model with memory is arbitragefree and derive that a corresponding market price of risk is independent of the maturity of the bond. Thus the bond is priced similar as in the memoryless Vasicek market. Both models coincide with the corresponding classical models in the case of the Brownian motion.

The results of this dissertation are already published in the preprints [DiSc] and [DiSc2].

Notations

- The space $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space, see Section 2.1.
- $|\cdot|_0$ is the norm of $L^2(\mathbb{R})$.
- The space $\mathcal{S}'(\mathbb{R})$ is the space of the tempered distributions, which is the dual of the Schwartz space, see Section 2.1.
- The stochastic process B_t^v is a centered Gaussian process with the covariance function $v(t,s)$, see Section 2.2.
- The stochastic process B_t^H is the fractional Brownian motion with the Hurst index H, see Section 2.2.
- The stochastic process B_t is the ordinary Brownian motion, see Section 2.2.
- The stochastic process B_t^s is a short range Gaussian process, see Section 2.2.
- The stochastic process B_t^{OU} is the Ornstein Uhlenbeck process, see Section 3.1.
- The stochastic process B_t^{GOU} is the generalized Ornstein Uhlenbeck process, which is driven by B_t^v, Section 3.1.
- The space $(\mathcal{S})$ is the Hida test function space, see Section 2.3.
- The space (L^2) is $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ with the norm $\|\cdot\|_0$, see Section 2.1.
- The space $(\mathcal{S})^*$ is the Hida distribution space, which is the dual of $(\mathcal{S})$, see Section 2.3.
- The form $:\omega^{\otimes n}:$ is the Wick tensor for $\omega \in \mathcal{S}'(\mathbb{R})$, see Section 2.3.
- The $\diamond$ is the Wick product, see Section 2.4.
- $(\cdot)^{\diamond n}$ is the n-th Wick power, see Section 2.4.
- The expression $S(\Phi)(\eta)$ is the S-transform of $\Phi \in (\mathcal{S})^*$ with $\eta \in \mathcal{S}(\mathbb{R})$, see Section 2.4.
- $\otimes$ is the tensor product, see Section 2.3.
- $\hat{\otimes}$ is the symmetrized tensor product, see Section 2.3.

2

White Noise Space

2.1 The construction of the white noise space

Let $|\cdot|_0$ denote the norm and $\langle\cdot,\cdot\rangle$ the inner product of $L^2(\mathbb{R})$. If $\eta \in C^\infty(\mathbb{R})$ and for all nonnegative integers n, k

$$|u^k \eta^{(n)}(u)| \longrightarrow 0, \qquad |u| \to \infty, \tag{2.1}$$

then η is called *rapidly decreasing*. The set $\mathcal{S}(\mathbb{R})$ denotes the set of rapidly decreasing functions and is called *Schwartz space*. By the inequation

$$|\eta(u)| \leq \frac{C}{|u|}$$

for sufficient large C and sufficient large $|u|$, which is a consequence of (2.1), it is established that $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. Let $\mathcal{S}'(\mathbb{R})$ denote the dual space of $\mathcal{S}(\mathbb{R})$, which is called the *space of tempered distributions*. As a result of the Hilbert space theory the inner product of $L^2(\mathbb{R})$ can be extended to the bilinear form $\langle\omega, \eta\rangle$, where $\omega \in \mathcal{S}'(\mathbb{R})$ and $\eta \in \mathcal{S}(\mathbb{R})$. There is another construction of the Schwartz space $\mathcal{S}(\mathbb{R})$ and its dual $\mathcal{S}'(\mathbb{R})$ by the use of the operator

$$A := -\frac{d^2}{dx^2} + x^2 + 1,$$

which is densely defined on $L^2(\mathbb{R})$. With the Hermite polynomial of degree n

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$$

let

$$e_n(x) := \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(x) e^{\frac{-x^2}{2}}.$$

The functions e_n are the eigenfunctions of A corresponding to the eigenvalue $\lambda_n := 2n+2$, $n \in \mathbb{N}_0$. The operator A^{-1} is bounded on $L^2(\mathbb{R})$, especially A^{-p} is a Hilbert-Schmidt operator for any $p > \frac{1}{2}$. Let for each $p \geq 0$, $|f|_p := |A^p f|_0$. This norm is given by use of the eigenvalues as

$$|f|_p = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} \langle f, e_n\rangle^2\right)^{1/2}.$$

Define

$$\mathcal{S}_p(\mathbb{R}) := \{f; f \in L^2(\mathbb{R}), |f|_p < \infty\}.$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ can be defined by $\mathcal{S}(\mathbb{R}) := \cap_{p\geq 0}\mathcal{S}_p(\mathbb{R})$, too. Let $\mathcal{S}'_p(\mathbb{R})$ with the norm $|\cdot|_{-p}$ be the dual of $\mathcal{S}_p(\mathbb{R})$. It follows that $\mathcal{S}'(\mathbb{R}) = \cup_{p\geq 0}\mathcal{S}'_p(\mathbb{R})$. By the properties of A it is ensured that $\mathcal{S}(\mathbb{R})$ is a nuclear space with a locally convex topology. In the weak topology on $\mathcal{S}'(\mathbb{R})$ the sequence $\{\omega_k\}_k \subset \mathcal{S}'(\mathbb{R})$ converges if and only if the sequence $\{\langle \omega_k, \eta\rangle\}_k$ converges for all $\eta \in \mathcal{S}(\mathbb{R})$. Therefore the Gel'fand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ is declared and the continuous mappings hold

$$\mathcal{S}(\mathbb{R}) \subset \mathcal{S}_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'_p(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

Suppose that the function $\zeta : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ is defined by

$$\zeta(\eta) := \exp\left(-\frac{|\eta|_0^2}{2}\right). \tag{2.2}$$

Lemma 2.1. *The function ζ is positive definite, i.e. for all $z_j \in \mathbb{C}$ and for all $\eta_j \in \mathcal{S}(\mathbb{R})$ with $j = 1, .., n$*

$$\sum_{i,j=1}^{n} z_j \zeta(\eta_j - \eta_i) z_i^* \geq 0, \tag{2.3}$$

where z^* denotes the conjugated complex number to z.

The proof is taken from [Ku] Chapter 3.1.

Proof. Let V denote the subspace of $\mathcal{S}(\mathbb{R})$ spanned by $\eta_1, .., \eta_n$. Suppose that μ_V is the standard Gaussian measure on V. The equation

$$\int_V \exp(i\langle \eta, y\rangle) d\mu_V(y) = \exp\left(-\frac{|\eta|_0^2}{2}\right)$$

is true for all $\eta \in V$. It holds that

$$\sum_{k,j=1}^{n} z_j \zeta(\eta_j - \eta_k) z_k^* = \textstyle\sum_{k,j=1}^{n} \int_V z_j \exp(i\langle \eta_j - \eta_k, y\rangle) z_k^* d\mu_V(y)$$
$$= \textstyle\int_V |\sum_{j=1}^{n} z_j \exp(i\langle \eta_j, y\rangle)|^2 d\mu_V(y) \geq 0.$$

□

Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathcal{S}'(\mathbb{R})$, i.e. the σ-algebra generated by the weak topology of $\mathcal{S}'(\mathbb{R})$.

Theorem 2.2 (Minlos theorem). *A complex-valued function ζ on $\mathcal{S}(\mathbb{R})$ is the characteristic function of a unique probability measure μ on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$, i.e.*

$$\zeta(\eta) = \int_{\mathcal{S}'(\mathbb{R})} \exp(i\langle \omega, \eta\rangle) d\mu(\omega) \tag{2.4}$$

with $\eta \in \mathcal{S}(\mathbb{R})$ if and only if the three conditions (1) $\zeta(0) = 1$, (2) ζ is continuous and (3) ζ is positiv definite in the sense of lemma 2.1 are satisfied.

Further details to this theorem are mentioned in the appendix of [HoOkUbZh]. By using Minlos theorem the following result is obtained.

Theorem 2.3 (Gaussian measure on $\mathcal{S}'(\mathbb{R})$). *There exists a unique Gaussian measure μ on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$ whose characteristic function is $\zeta(\eta)$ in equation (2.2)..*

The space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ is called *white noise space*. Furthermore, (L^2) denotes $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ with the norm $\|\cdot\|_0$ and inner product $\langle\langle \cdot, \cdot \rangle\rangle$.

Lemma 2.4. *The random variable $\langle \cdot, \eta \rangle$ is in (L^2) for all $\eta \in \mathcal{S}(\mathbb{R})$, where*

$$\int_{\mathcal{S}'(\mathbb{R})} \langle \omega, \eta \rangle \, d\mu(\omega) = 0 \tag{2.5}$$

and

$$\int_{\mathcal{S}'(\mathbb{R})} \langle \omega, \eta \rangle^2 \, d\mu(\omega) = |\eta|_0^2. \tag{2.6}$$

Proof. The statements are proved by expanding both sides in

$$\int_{\mathcal{S}'(\mathbb{R})} \exp(i \langle \omega, t\eta \rangle) d\mu(\omega) = \exp\left(-\frac{|t\eta|_0^2}{2}\right) \tag{2.7}$$

and comparison of the coefficients of t, $t \in \mathbb{R}$.
□

Lemma 2.5. *For all $f \in L^2(\mathbb{R})$ the bilinear form $\langle \omega, f \rangle$ with $f \in L^2(\mathbb{R})$ and $\omega \in \mathcal{S}'(\mathbb{R})$ is declared by the limit*

$$\lim_{k\to\infty} \langle \cdot, \eta_k \rangle = \langle \cdot, f \rangle$$

with convergence in (L^2), where $\eta_k \to f$ converging in $L^2(\mathbb{R})$ and $\{\eta_k\} \subset \mathcal{S}(\mathbb{R})$. Furthermore $E(\langle \cdot, f \rangle) = 0$ and $E(\langle \cdot, f \rangle^2) = |f|_0^2$.

The lemma is an immediate consequence of lemma 2.4.

2.2 A class of Gaussian processes with dependent increments

This section introduces a class of Gaussian processes with dependent increments B_t^v. The properties of B_t^v are based on a function $m(\cdot, \cdot)$. Suppose $m(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ such that, for all $t \in \mathbb{R}$, the function $m(\cdot, t) \in L^2(\mathbb{R})$. Define

$$v(s, t) = \int_{\mathbb{R}} m(u, t) m(u, s) \, du. \tag{2.8}$$

Definition 2.6. *The stochastic process B_t^v is defined by*

$$B_t^v := \langle \cdot, m(u, t) \rangle .$$

In the beginning of the following Chapters the class of Gaussian processes B_t^v will be restricted with respect to the applications in the Chapter.

Remark 2.7 (Notation). The notation $\langle \cdot, m(u,t)\rangle$ is meant in the following way. The $\cdot$ stands for an element ω, $\omega \in \mathcal{S}'(\mathbb{R})$, such that it is a tempered distribution with respect to the variable u. Hence $\langle \omega(u), m(u,t)\rangle$ depends on the parameter t only and represents, for a fixed ω, a path of the stochastic process B_t^v .

Proposition 2.8. *The stochastic process B_t^v is a Gaussian process with mean 0 and covariance function*

$$\int_{\mathcal{S}'(\mathbb{R})} B_t^v(\omega) B_s^v(\omega)\, d\mu(\omega) = v(t,s) \tag{2.9}$$

and

$$B_t^v = \int_{\mathbb{R}} m(u,t)\, dB_u \tag{2.10}$$

with the ordinary Brownian motion B_u on the real line.

Proof. It follows from lemma 2.5 that the process B_t^v has mean 0 and that $E((B_t^v)^2) = |m(\cdot,t)|_0^2$. The process B_t^v is therefore normally distributed. The Wiener integral $\int_{\mathbb{R}} m(u,t) dB_u$ is normally distributed with the same mean and variance as B_t^v, consequently

$$B_t^v = \int_{\mathbb{R}} m(u,t)\, dB_u \tag{2.11}$$

holds a.s. and for all $t \in \mathbb{R}$. The covariance function of $\int_{\mathbb{R}} m(u,t)\, dB_u$ is

$$\int_{\mathcal{S}'(\mathbb{R})} \int_{\mathbb{R}} m(u,t)\, dB_u \int_{\mathbb{R}} m(u,s)\, dB_u\, d\mu = \int_{\mathbb{R}} m(u,t) m(u,s) du = v(t,s) \tag{2.12}$$

by the use of the Itô isometry. □

In the following there are some special instances of B_t^v.

Example 2.9 (Ordinary Brownian motion). The stochastic process $B_t^v = B_t$ is the ordinary Brownian motion if $m(u,t) = 1_{[0,t]}(u)$, where $1_{[0,t]}$ is the indicator function of the intervall $[0,t]$. This process has then the covariance function $v(s,t) = \min(t,s)$. This example is further discussed in [Ku], Chapter 3.1.

Example 2.10 (Fractional Brownian motion). The following operator $M_\pm^H$ is used to introduce fractional Brownian motion and later on its derivative. Define $M_\pm^H$ with $H \in (0,1)$ for $\eta \in \mathcal{S}(\mathbb{R})$ as

$$(M_+^H \eta)(t) := \begin{cases} \frac{K_H}{\Gamma(H-1/2)} \int_{-\infty}^t \eta(s)(t-s)^{H-3/2}\, ds & \text{for } H > 1/2 \\ \eta(t) & \text{for } H = 1/2 \\ \frac{K_H(H-1/2)}{\Gamma(1/2-H)} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{\eta(t)-\eta(t-s)}{(s)^{3/2-H}}\, ds & \text{for } H < 1/2 \end{cases}$$

and

$$(M_-^H\eta)(t) := \begin{cases} \frac{K_H}{\Gamma(H-1/2)} \int_{-\infty}^t \eta(s)(t-s)^{H-3/2}\, ds & \text{for } H > 1/2 \\ \eta(t) & \text{for } H = 1/2 \\ \frac{K_H(H-1/2)}{\Gamma(1/2-H)} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{\eta(t)-\eta(t+s)}{(s)^{3/2-H}}\, ds & \text{for } H < 1/2 \end{cases}$$

with

$$K_H = \Gamma(H+1/2)\left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}\right)^{1/2}$$

The operator $M_\pm^H$ is essentially the Riemann-Liouville fractional integral for $H > 1/2$ and the Marchaud fractional derivative for $H < 1/2$. For further information about these operators see [SaKiMa], Chapter 6, and [Be], Chapter 1.6. The operator $M_\pm^H$ is just applied on functions $\eta \in \mathcal{S}(\mathbb{R})$, and for indicator functions $1_{[0,t]}$ for which $M_\pm^H$ is defined; see Chapter 1 of [Be]. For the fractional Brownian motion B_t^H it holds that

$$B_t^H = \int_{\mathbb{R}} (M_-^H 1_{[0,t]})(s)\, dB_s \tag{2.13}$$

up to modification. Another representation of the fractional Brownian motion up to modification is $B_t^H = \left\langle \cdot, M_-^H(1_{[0,t]})\right\rangle$. The covariance function is therefore $v(s,t) = 1/2(|t|^{2H} + |s|^{2H} - |s-t|^{2H})$. This example is further discussed in [Be].

Remark 2.11. Note that the fractional Brownian motion is not a semi-martingale for $H \neq 1/2$, so the class of Gaussian processes B_t^v has elements to which the calculus of semi-martingales does not suit.

Example 2.12 (A Gaussian process with short range dependency). Let B_t^v, $t \in \mathbb{R}$, be a centered Gaussian process with covariance function $v(s,t)$, such that $v(s,\cdot)$ has a global maximum and $\lim_{t\to\infty} v(s,t) = 0$ for all fixed $s \in \mathbb{R}$. Then B_t^v is said to be a short range Brownian motion. Let $m(u,t) = t^2\exp(-(u-t)^2)$, hence $v(s,t) = \sqrt{\pi}t^2s^2\exp(-(t-s)^2/2)$. This process B_t^v is a short range Brownian motion. Later this process will be denoted by B_t^s.

Our definition of B_t^v was formulated in terms of a function $m(u,t)$. Then the covariance function $v(t,s)$ was computed from $m(u,t)$. However, as the following proposition shows, it is also under certain circumstances possible to derive $m(u,t)$ from a given covariance function $v(t,s)$.

Proposition 2.13. *Let $v(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$, such that*

$$v(t,s) = \sum_{k=1}^\infty \alpha_k(t)\alpha_k(s) < \infty \tag{2.14}$$

for all $t,s \in \mathbb{R}$ with real-valued functions $\alpha_k(\cdot)$. Let further $\{\phi_k\}$ be a orthonormal base of $L^2(\mathbb{R})$. Then

$$m(u,t) = \sum_{k=1}^\infty \alpha_k(t)\phi_k(u) \tag{2.15}$$

with convergence in $L^2(\mathbb{R})$ with respect to u and pointwise in t.

Proof. The function $m(u,t)$ satisfies

$$\int_{\mathbb{R}} m(u,t)m(u,s)du = \sum_{j,k=1}^{\infty} \alpha_k(t)\alpha_j(s) \int_{\mathbb{R}} \phi(u)_j \phi_k(u)du$$
$$= \sum_{k=1}^{\infty} \alpha_k(t)\alpha_k(s) = v(t,s).$$

□

Note that the function $m(u,t)$ is with respect to u not unique in $L^2(\mathbb{R})$, it depends on the choice of the orthonormal base $\{\phi_k\}$. This is no further problem, because the centered Gaussian process B_t^v is uniquely determined by its covariance function $v(t,s)$.

Remark 2.14. The Gaussian process B_t^v is stationary if $v(t,s)$ depends only on $t-s$. E.g. let $m(u,t) = \exp(-(u-t)^2)$ so $v(t,s) = \sqrt{\pi}\exp(-(t-s)^2/2)$.

2.3 The construction of the Hida test and distribution space

In this Section we construct the Hida test function space and its dual, which is called the Hida distribution space. Their construction uses the chaos decomposition theorem and the properties of the Schwartz space and its dual. These space are later used to obtain the derivative of the Gaussian process B_t^v and to prove several theorems in the related stochastic integral calculus. The following can be found in [HiKuPoSt] or [Ku], too.

Definition 2.15. *The* trace operator τ, $\tau \in (\mathcal{S}'(\mathbb{R}))^{\otimes 2}$ *is given by*

$$\langle \tau, \eta_1 \otimes \eta_2 \rangle = \langle \eta_1, \eta_2 \rangle \tag{2.16}$$

for $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$.

Further comments on this operator are mentioned in [Ku] Chapter 5.1.

Definition 2.16. *The* Wick tensor $:\omega^{\otimes n}:$ *of an* $\omega \in \mathcal{S}'(\mathbb{R})$ *for* $n \in \mathbb{N}_0$ *is defined by*

$$:\omega^{\otimes n}: := \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!(-1)^k \omega^{\otimes(n-2k)} \hat{\otimes} \tau^{\otimes k}, \tag{2.17}$$

where the upper limit of the summation index is the integer part of $n/2$, $\hat{\otimes}$ *denotes the symmetrized tensor product, and* $(2k-1)!! = (2k-1)\cdot(2k-3)\cdots 3\cdot 1$ *with* $(-1)!! = 1$.

Thus the Wick tensor $:\omega^{\otimes k}:$ is an element of $(\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} k}$. The formula 2.17 can be transformed to

$$\omega^{\otimes n} := \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!! :\omega^{\otimes(n-2k)}: \hat{\otimes} \tau^{\otimes k}, \tag{2.18}$$

which can be seen in [Ku] Chapter 5.1, so $\omega^{\otimes n} \in (\mathcal{S}'(\mathbb{R}))^{\otimes n}$.

Lemma 2.17. *For all $\omega \in \mathcal{S}'(\mathbb{R})$ and for all $f \in L^2(\mathbb{R})$ it is true that*

$$\left\langle : \omega^{\otimes n} :, f^{\otimes n} \right\rangle = \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!(-|f|_0^2)^k \left\langle \omega, f \right\rangle^{n-2k} \tag{2.19}$$

and

$$\| \left\langle : \cdot^{\otimes n} :, f^{\otimes n} \right\rangle \|_0 = \sqrt{n!}|f|_0^n. \tag{2.20}$$

The proof can be obtained by using equation 2.17, see [Ku] Chapter 5.1. For $f \in L^2(\mathbb{R}^n)$ let the multiple Wiener Integral with respect to the ordinary Brownian motion

$$I_n(f) := \int_{\mathbb{R}^n} f(t_1, t_2, t_3, \dots, t_n)\, dB_{t_1} dB_{t_2} \dots dB_{t_n}$$

with $n \in \mathbb{N}_0$ and I_0 denotes a constant c. A function $f : \mathbb{R}^n \to \mathbb{C}$ with $n \in \mathbb{N}$ is said to be symmetric, if the function f is invariant under permutation of the variables. The function $\hat{f}$ denotes the symmetrization of the function f and is defined by

$$\hat{f}(t_1, .., t_n) := \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, ..., t_{\sigma(m)}), \tag{2.21}$$

where σ runs over all permutations of $\{1, .., n\}$. Let $\hat{L}^2(\mathbb{R}^n)$ be the space of symmetric complex-valued L^2 functions on $\mathbb{R}^n$.

Proposition 2.18. *Let $f \in \hat{L}^2(\mathbb{R}^n)$ then it holds a.s. that*

$$I_n(f)(\omega) = \left\langle : \omega^{\otimes n} :, f \right\rangle. \tag{2.22}$$

For a proof see Theorem 5.4 of [Ku].

The following outline of the construction of the Hida test and distribution space is taken essentially from [Ku] Chapter 3.3. The next proposition is known as the chaos decomposition of (L^2). Here with a slight abuse of the notation, the $|\cdot|_0$ denotes the norm in $L^2(\mathbb{R}^n)$, too .

Proposition 2.19. *For all $F \in (L^2)$ there is a unique sequence of $(f_n)_{n\in\mathbb{N}_0}$ such that f_n is symmetric and*

$$F = \sum_{n=0}^{\infty} \left\langle : \cdot^{\otimes n} :, f_n \right\rangle$$

with convergence in (L^2). Furthermore, it holds that

$$E(F^2) = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} f_n^2(s) ds = \sum_{n=0}^{\infty} n! |f_n|_0^2. \tag{2.23}$$

Proposition 2.18 allows another representation of the chaos decomposition Theorem by the use of the multiple Wiener integrals, which can found e.g. in [Nu] Theorem 1.1.2 .

Consider the operator $A = -d^2/dx^2 + x^2 + 1$ already mentioned before. Let $\Gamma(A)$ be an operator on $F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f_n \rangle \in (L^2)$ satisfying

$$\sum_{n=0}^{\infty} n! |A^{\otimes n} f_n|_0^2 < \infty \tag{2.24}$$

defined by

$$\Gamma(A)F = \sum_{n=0}^{\infty} \left\langle : \cdot^{\otimes n} :, A^{\otimes n} f_n \right\rangle .$$

The operator $\Gamma(A)$ has similar properties as the operator A, e.g. $\Gamma(A)$ has a set of eigenfunctions which forms an orthonormal basis for (L^2), $\Gamma(A)^{-1}$ is a bounded operator of (L^2) and for any $p > 1$ the operator $\Gamma(A)^{-p}$ is a Hilbert-Schmidt operator of (L^2), see [Ku] Chapter 3.3 . Let the space $(\mathcal{S})_p$ be for $p \geq 0$

$$(\mathcal{S})_p := \{\phi \in (L^2); \| \; \Gamma(A)^p \phi \; \|_0 < \infty\}.$$

The space $(\mathcal{S})_n$ is a Hilbert space with the norm $\| \; F \; \|_n := \| \; \Gamma(A)^n F \; \|$, too. As in the construction of the Schwartz space $\mathcal{S}(\mathbb{R})$ one can define a space $(\mathcal{S})$ as

$$(\mathcal{S}) := \cap_{p \geq 0} (\mathcal{S})_p. \tag{2.25}$$

The space $(\mathcal{S})$ is therefore a nuclear space and $(\mathcal{S}) \subset (L^2)$, and it is called the space of *Hida test function space*. The dual space $(\mathcal{S})^*$ of $(\mathcal{S})$ is called the *Hida distribution space*. For the dual space $(\mathcal{S})'_n := (\mathcal{S})_{-n}$ with the norm $\| \cdot \|_{-n}$, it holds that

$$(\mathcal{S})^* = \cup_{n \in \mathbb{N}} (\mathcal{S})_{-n}. \tag{2.26}$$

The topologies of $(\mathcal{S})$ and of $(\mathcal{S})^*$ are given by the projective limit topology and weak topology, respectively (see [Ku], Chapter 2.2). These spaces are a Gel'fand triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*. \tag{2.27}$$

The bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the inner product of (L^2), which is extended to the dual mapping $\langle\langle \Phi, \zeta \rangle\rangle$ with $\Phi \in (\mathcal{S})^*$ and $\zeta \in (\mathcal{S})$. So a sequence $\{\Phi_k\}$ converges in $(\mathcal{S})^*$ if and only if $\langle\langle \Phi_k, \zeta \rangle\rangle$ converges for all $\zeta \in (\mathcal{S})$. Similar to before there are continuous inclusion maps

$$(\mathcal{S}) \subset (\mathcal{S})_n \subset (L^2) \subset (\mathcal{S})_{-n} \subset (\mathcal{S})^*.$$

Remark 2.20. If $\eta \in \mathcal{S}(\mathbb{R})$ then it follows from the construction of $(\mathcal{S})$ that $\langle \cdot, \eta \rangle \in (\mathcal{S})$, because $|A^p \eta|_0^2 < \infty$ for all $p > 0$. Furthermore it will be shown in Section 2.5 Theorem 2.36, that all functions f_n in the chaos decomposition of an element

$$\phi = \sum_{n=0}^{\infty} \left\langle : \cdot^{\otimes n} :, f_n \right\rangle , \tag{2.28}$$

$\phi \in (\mathcal{S})$ are smooth. Thus it is possible to define a Hida distribution Φ by the use of the Delta distribution δ_0, $\delta_0 \in \mathcal{S}'(\mathbb{R})$ such that

$$\left\langle\!\!\left\langle \Phi, \sum_{n=0}^{\infty} \left\langle : \cdot^{\otimes n} :, f_n \right\rangle \right\rangle\!\!\right\rangle = \langle \delta_0, f_1 \rangle = f_1(0). \tag{2.29}$$

The notation that will be used for Φ is $\langle\cdot,\delta_0\rangle$, which is declared by 2.29. The expression $\langle\cdot,\delta_0\rangle$ is therefore a Hida distribution and is a mapping from $(\mathcal{S})\to\mathbb{C}$. If ϕ is given by (2.28), then $\langle\cdot,\delta_0\rangle(\phi)=\langle\delta_0,f_1\rangle$ as before. Thus we have an additional extension of the expression $\langle\cdot,\lambda\rangle$ with $\lambda\in\mathcal{S}'(\mathbb{R})$ which is a Hida distribution declared by $\langle\cdot,\lambda\rangle(\phi)=\langle\lambda,f_1\rangle$. This is notation is further discussed in [Ku] Chapter 3.

Proposition 2.21. *Suppose $\phi\in(\mathcal{S})$,*

$$\phi=\sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:,\eta_k\right\rangle \tag{2.30}$$

with $\eta_k\in(\mathcal{S}(\mathbb{R}))^{\hat{\otimes}k}$ and let $\Psi\in(\mathcal{S})^*$ such that

$$\Psi=\sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:,F_k\right\rangle \tag{2.31}$$

with $F_k\in(\mathcal{S}'(\mathbb{R}))^{\hat{\otimes}k}$. Then it holds that

$$\langle\langle\Psi,\phi\rangle\rangle=\sum_{k=0}^{\infty}k!\,\langle F_k,\eta_k\rangle\,. \tag{2.32}$$

The proof is in [Ku] Chapter 5.1.

2.4 S-transform and Wick product

In this Section we discuss the S-transform which is a tool for identifying Hida distributions. Due to the construction of the Hida test function space $(\mathcal{S})$ it follows that for any $\eta\in\mathcal{S}(\mathbb{R})$

$$\exp^{\diamond}(\langle\cdot,\eta\rangle):=\sum_{k=0}^{\infty}\frac{1}{k!}\left\langle :\cdot^{\otimes k}:,\eta^{\otimes k}\right\rangle \tag{2.33}$$

is an element of $(\mathcal{S})$. Therefore it is possible to evaluate any Hida distribution Φ on this random variables $\exp^{\diamond}(\langle\cdot,\eta\rangle)$ in the bilinear mapping:

Definition 2.22. *The* S-transform *of a Hida distribution $\Phi\in(\mathcal{S})^*$ is defined by*

$$S(\Phi)(\eta):=\langle\langle\Phi,\exp^{\diamond}(\langle\cdot,\eta\rangle)\rangle\rangle\,,\quad \eta\in\mathcal{S}(\mathbb{R}).$$

Proposition 2.23. *Let $\Phi\in(\mathcal{S})^*$ and $\Phi=\sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:,F_k\right\rangle$ with $F_k\in(\mathcal{S}'(\mathbb{R}))^{\hat{\otimes}k}$, then*

$$S(\Phi)(\eta)=\sum_{k=0}^{\infty}\left\langle F_k,\eta^{\otimes k}\right\rangle\,. \tag{2.34}$$

Proof. This is an immediate result of Proposition 2.21. □

Proposition 2.24. *Let $\Phi, \Psi \in (\mathcal{S})^*$. If $S(\Phi)(\eta) = S(\Psi)(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$, then $\Phi = \Psi$.*

The proof is taken from [Ku] page 39.

Proof. Because the S-transform $S(\Phi)(\eta)$ is linear in Φ it is sufficient to show that $S(\Phi)(\eta) = 0$ for all $\eta \in \mathcal{S}(\mathbb{R})$ implies $\Phi = 0$. Let $\Phi = \sum_{k=0}^{\infty} \left\langle :\cdot^{\otimes k}:, F_k \right\rangle$ then by Proposition 2.23 it follows that

$$S(\Phi)(\eta) = \sum_{k=0}^{\infty} \left\langle F_k, \eta^{\otimes k} \right\rangle . \tag{2.35}$$

For $t \in \mathbb{R}$ it holds that

$$S(\Phi)(t\eta) = \sum_{k=0}^{\infty} t^k \left\langle F_k, \eta^{\otimes k} \right\rangle = 0, \tag{2.36}$$

therefore it is possible to examine each summand particularly. Obviously it follows that $F_0 = 0$. Let $k \geq 1$, so $\left\langle F_k, \eta^{\otimes k} \right\rangle = 0$ for all $\eta \in \mathcal{S}(\mathbb{R})$. By the polarization identity it follows

$$\left\langle F_k, \eta_1 \hat{\otimes} \cdots \hat{\otimes} \eta_k \right\rangle = \frac{1}{k!} \sum_{l=1}^{k} (-1)^{k-l} \sum_{j_1 < \ldots < j_l} \left\langle F_k, (\eta_{j_1} + \ldots + \eta_{j_k})^{\otimes k} \right\rangle . \tag{2.37}$$

Therefore $F_k = 0$ for any $k \geq 1$, too and as a result it follows that $\Phi = 0$.
□

Further properties of the S-transfrom are the content of Section 2.5.

Corollary 2.25. *The set*

$$\{\exp^{\diamond}(\langle \cdot, \eta \rangle) : \eta \in \mathcal{S}(\mathbb{R})\}$$

is total in $(\mathcal{S})$.

This corollary is a consequence of Proposition 2.24

Example 2.26. The S-transform of the ordinary Brownian motion B_t is

$$\begin{aligned} S(B_t)(\eta) &= \langle\langle B_t, \exp^{\diamond}(\langle \cdot, \eta \rangle) \rangle\rangle \\ &= \left\langle\left\langle \left\langle \cdot, 1_{[0,t]} \right\rangle, \exp^{\diamond}(\langle \cdot, \eta \rangle) \right\rangle\right\rangle \\ &= \left\langle 1_{[0,t]}, \eta \right\rangle = \int_{\mathbb{R}} 1_{[0,t]}(u)\eta(u)\, du = \int_0^t \eta(u)\, du. \end{aligned}$$

Example 2.27. We calculate the S-transform of the fractional Brownian motion B_t^H.

$$\begin{aligned} S(B_t^H)(\eta) &= \left\langle\left\langle B_t^H, \exp^{\diamond}(\langle \cdot, \eta \rangle) \right\rangle\right\rangle \\ &= \int_{\mathcal{S}'(\mathbb{R})} B_t^H \exp^{\diamond}(\langle \cdot, \eta \rangle)\, d\mu \\ &= \int_{\mathcal{S}'(\mathbb{R})} \int_{\mathbb{R}} (M_-^H 1_{(0,t)})(s)\, dB_s \exp^{\diamond}(\langle \cdot, \eta \rangle)\, d\mu \\ &= \int_0^t (M_+^H \eta)(s)\, ds \end{aligned}$$

where several steps like fractional integration by parts and the fact that the integrals are well-defined are used (see [Be], Theorem 1.6.8).

Example 2.28. For the short range Brownian motion B_t^s we get

$$\begin{aligned} S(B_t^s)(\eta) &= \langle\langle B_t^s, \exp^{\diamond}(\langle\cdot,\eta\rangle)\rangle\rangle \\ &= \left\langle\left\langle\langle\cdot, t^2\exp(-(u-t)^2)\rangle\right., \exp^{\diamond}(\langle\cdot,\eta(u)\rangle)\right\rangle\rangle \\ &= \int_{\mathbb{R}} \eta(u)t^2\exp(-(u-t)^2)\,du. \end{aligned}$$

Definition 2.29. *The* Wick product $\Phi\diamond\Psi$ *of two Hida distributions* Φ *and* Ψ *in* $(\mathcal{S})^*$ *is the Hida distribution in* $(\mathcal{S})^*$ *such that* $S(\Phi\diamond\Psi)(\eta) = S(\Phi)(\eta)S(\Psi)(\eta)$ *for all* $\eta\in\mathcal{S}(\mathbb{R})$.

In Corollary 2.39 it is proved that the Wick product $\Phi\diamond\Psi$ of any two Hida distributions Φ and Ψ exists. By Proposition 2.24 it holds that the Wick product of two Hida distributions is unique. The following algebraic properties of the Wick product are obvious by regarding their S-transform

Lemma 2.30. *Let* $\Phi,\Psi,\Gamma\in(\mathcal{S})^*$*, then*

$$\begin{aligned} \Phi\diamond\Psi &= \Psi\diamond\Phi \\ \Gamma\diamond(\Phi\diamond\Psi) &= (\Gamma\diamond\Phi)\diamond\Psi \\ \Gamma\diamond(\Phi+\Psi) &= \Gamma\diamond\Phi+\Gamma\diamond\Psi. \end{aligned}$$

For $F,G\in(L^2)$ it may happen that $(F\diamond G)\notin(L^2)$, see [Be] Corollary 1.5.6. In general it holds that

$$T\diamond(G\cdot F)\neq(T\diamond G)\cdot F$$

with $T,G,F\in(L^2)$. Let $F,G\in(L^2)$, if $F\diamond G\in(L^2)$. Then the S-transform of F with $\eta\equiv 0$ is the expectation value of F, $\langle\langle F,1\rangle\rangle = E(F)$, hence

$$E(F\diamond G) = E(F)E(G).$$

Remark 2.31. There is another approach to Wick products using Malliavin Calculus. This concept is discussed e.g. in [Be] Chapter 1. Further information about Malliavin Calculus can be found in [Nu].

Proposition 2.32. *Let* $\langle :\cdot^{\otimes n}:, f_n\rangle$ *and* $\langle :\cdot^{\otimes m}:, g_m\rangle$ *be Hida distributions with* $f_n\in(\mathcal{S}'(\mathbb{R}))^{\hat{\otimes}n}$, $g_m\in(\mathcal{S}'(\mathbb{R}))^{\hat{\otimes}m}$*,* $n,m\in\mathbb{N}_0$*. Then*

$$\left\langle :\cdot^{\otimes n}:, f_n\right\rangle\diamond\left\langle :\cdot^{\otimes m}:, g_m\right\rangle = \left\langle :\cdot^{\otimes(n+m)}:, f_n\hat{\otimes}g_m\right\rangle. \tag{2.38}$$

See [Ku] Chapter 8 for a proof.

Example 2.33. Let $f\in L^2(\mathbb{R})$, then

$$\langle\cdot,f\rangle^{\diamond 2} = \left\langle :\cdot^{\otimes 2}:, f^{\otimes 2}\right\rangle \tag{2.39}$$

by the use of equation (2.17) it follows that

$$\left\langle :\cdot^{\otimes 2}:, f^{\otimes 2}\right\rangle = \langle\cdot,f\rangle^2 - |f|_0^2. \tag{2.40}$$

The right hand side is only well-defined if $f\in L^2(\mathbb{R})$, because the $|\cdot|_0$ is not declared for an element of $\mathcal{S}'(\mathbb{R})$.

Generalising this example the next lemma follows.

Lemma 2.34. *Let $f \in L^2(\mathbb{R})$, then the following equations hold with $n \in \mathbb{N}_0$*

$$\langle \cdot, f \rangle^{\diamond n} = \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!(-|f|_0^2)^k \langle \cdot, f \rangle^{n-2k} \tag{2.41}$$

and

$$\langle \cdot, f \rangle^{n} = \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!(|f|_0^2)^k \langle \cdot, f \rangle^{\diamond(n-2k)} \tag{2.42}$$

and

$$\exp(\langle \cdot, f \rangle - 1/2|f|_0^2) = \sum_{k=0}^{\infty} \frac{\langle \cdot, f \rangle^{\diamond k}}{k!} = \exp^{\diamond}(\langle \cdot, f \rangle). \tag{2.43}$$

Proof. The proof is obtained by equation (2.19) and Proposition 2.32 as one can see in [Ku] Chapter 5. □

The term $\exp^{\diamond}(\langle \cdot, f \rangle)$ is called the *Wick exponential* of $\langle \cdot, f \rangle$. This Lemma allows to switch between the Wick product and the ordinary product, which is used later in several proofs.

2.5 Characterization and convergence theorems

The next Theorem can be found with its proof in [Ku] Theorem 8.10.

Theorem 2.35 (Characterization of Hida test functions). *Let $\phi \in (\mathcal{S})$. Then its S-transform $F = S(\phi)$ satisfies the conditions*

- *For any η_1 and η_2 in $\mathcal{S}(\mathbb{R})$, the function $F(\eta_1 + z\eta_2)$ is an entire function of $z \in \mathbb{C}$.*
- *For any constants $a > 0$ and $p \geq 0$, there exists a constant $K > 0$ such that*

$$|F(\eta)| \leq K \exp(a|\eta|_{-p}^2) \tag{2.44}$$

for all $\eta \in \mathcal{S}(\mathbb{R})$.

Conversely suppose a function F defined on $\mathcal{S}(\mathbb{R})$ satisfies the two conditions above. Then there exists a unique $\phi \in (\mathcal{S})$ such that $F = S(\phi)$.

As stated before, there is a result about the analytical properties of $\phi \in (\mathcal{S})$. It is taken from [Ku] Theorem 6.5 and it is proven there.

Theorem 2.36. *Every test function ϕ in $(\mathcal{S})$ has a unique continuous version $\tilde{\phi}$, such that $\tilde{\phi} = \phi$ a.s.. Furthermore the continuous version $\tilde{\phi}$ is given by*

$$\tilde{\phi} = \sum_{n=0}^{\infty} \left\langle : \cdot^{\otimes n} :, f_n \right\rangle \tag{2.45}$$

with $f_n \in (\mathcal{S}(\mathbb{R}))^{\hat{\otimes} n}$.

This Theorem justifies the definition of a Hida distribution in Remark 2.20. The following result can be found with its proof in [Ku] Theorem 8.2.

Theorem 2.37 (Characterization of Hida distributions). *Let $\Phi \in (\mathcal{S})^*$. Then its S-transform $F = S(\Phi)$ satisfies the conditions*

- *For any η_1 and η_2 in $\mathcal{S}(\mathbb{R})$, the function $F(\eta_1 + z\eta_2)$ is an entire function of $z \in \mathbb{C}$.*
- *There exists nonnegative constanst K, a and p such that*

$$|F(\eta)| \leq K \exp(a|\eta|_p^2) \tag{2.46}$$

for all $\eta \in \mathcal{S}(\mathbb{R})$.

Conversely suppose a function F defined on $\mathcal{S}(\mathbb{R})$ satisfies the two conditions above. Then there exists a unique $\Phi \in (\mathcal{S})^*$ such that $F = S(\Phi)$.

In order to show where conditions of Theorem 2.37 may fail, here is an example.

Example 2.38. Let $h \in L^2(\mathbb{R})$. Then $\exp^\diamond(\langle \cdot, h\rangle^{\diamond k}) \notin (\mathcal{S})^*$ for $k > 2$. Calculate the S-transform and get

$$S\left(\sum_{n=0}^{\infty} \frac{(\langle \cdot, h\rangle)^{\diamond(kn)}}{n!}\right)(\eta) = \sum_{n=0}^{\infty} \frac{(\langle h, \eta\rangle^k)^n}{n!} = \exp(\langle h, \eta\rangle^k).$$

Now choose $h = \eta$, then condition (2.46) fails.

Corollary 2.39. *The Wick product $\Phi \diamond \Psi$ of two Hida distributions $\Phi, \Psi \in (\mathcal{S})^*$ exists and is unique.*

Proof. It holds that $S(\Phi \diamond \Psi) = S(\Phi)S(\Psi)$ satisfies the conditions of Theorem 2.37. Therefore there exists a $\Gamma \in (S)^*$ such that $S(\Gamma) = S(\Phi)S(\Psi)$. □

Theorem 2.40. *Let $\{\Phi_n\}$ be a sequence in $(\mathcal{S})^*$ and let $F_n = S(\Phi_n)$. Then Φ_n converges in the weak topology of $(\mathcal{S})^*$ if and only if the following conditions are satisfied:*

- $\lim_{n\to\infty} F_n(\eta)$ *exists for each $\eta \in \mathcal{S}(\mathbb{R})$.*
- *There exist nonnegative constants K, a and p, indepedent of n such that*

$$|F_n(\eta)| \leq K \exp(a|\eta|_p^2), \tag{2.47}$$

for all $n \in \mathbb{N}$ and for all $\eta \in \mathcal{S}(\mathbb{R})$.

This Theorem can be proved by Theorem 2.37, this can be seen in [Ku] Theorem 8.6.

Proposition 2.41. *The Wick product as mapping from $(\mathcal{S})^* \times (\mathcal{S})^* \to (\mathcal{S})^*$ is continuous in the weak topology.*

The proof is contained in the proof of Theorem 8.12 in [Ku].

2.6 Notes

Remark 2.42. In this work only the Hida test and distribution spaces are used, although in [Ku] several other spaces are introduced, such that the Hida test and distribution spaces are a special case.

Remark 2.43. In [HuOk] they constructed for each $H \in (0,1)$ of the fractional Brownian motion B_t^H a Hida test and distribution space depending on H. Their construction is similar to the considerations here.

3

Stochastic Differential Equations Driven by Gaussian Processes with Dependent Increments

In this Chapter it will be assumed that the function $m(u,t)$ satisfies the following property.

Assumption
The function $m(u,t)$ shall satisfy for all $t \in \mathbb{R}$ that

$$\frac{d\,m(\cdot,t)}{dt} \in \mathcal{S}'(\mathbb{R}), \tag{3.1}$$

i.e. the function $m(u,t)$ is differentiable in the *Schwartz distributional sense* with respect to t. This will be necessary to construct the derivative of the process B_t^v in the $(\mathcal{S})^*$-sense.

3.1 Stochastic integrals driven by Gaussian processes with dependent increments

3.1.1 The white noise W_t^v

Definition 3.1. *Let I be an interval in $\mathbb{R}$. A mapping $X : I \to (\mathcal{S})^*$ is called a* stochastic distribution process. *A stochastic distribution process is called* differentiable *in the $(\mathcal{S})^*$-sense, if*

$$\lim_{h\to 0} \frac{X_{t+h} - X_t}{h}$$

exists in $(\mathcal{S})^*$.

The next theorem presents a criterion for differentiability (see [Be], Theorem 5.3.9). Let I be as above. A function $F : I \to \mathcal{S}'(\mathbb{R})$ is differentiable, if the limit $(F(t+h) - F(t))/h$ exists in $\mathcal{S}'(\mathbb{R})$ with respect to the weak topology of $\mathcal{S}'(\mathbb{R})$.

Theorem 3.2. *Let $F : I \to \mathcal{S}'(\mathbb{R})$ be differentiable. Then $\langle \cdot, F(t)\rangle$ is a differentiable stochastic distribution process and*

$$\frac{d}{dt}\langle \cdot, F(t)\rangle = \left\langle \cdot, \frac{d}{dt}F(t)\right\rangle .$$

Proof. Let $\{h_n\}$ be a sequence that tends to zero. It is true that

$$\left\langle \frac{F(t+h_n)-F(t)}{h_n}, \eta \right\rangle = \frac{\langle F(t+h_n),\eta\rangle - \langle F(t),\eta\rangle}{h_n} = \frac{S(\langle \cdot, F(t+h)\rangle)(\eta) - S(\langle \cdot, F(t)\rangle)(\eta)}{h_n},$$

for each $\eta \in \mathcal{S}(\mathbb{R})$, where the second equality follows from Proposition 2.21 considering the single term with $k=1$. Letting $n \to \infty$ the assertion is proved. □

Theorem 3.3. *The stochastic distribution process B_t^v is differentiable in the $(\mathcal{S})^*$-sense and satisfies*

$$W_t^v := \frac{d}{dt} B_t^v = \left\langle \cdot, \frac{d}{dt} m(u,t) \right\rangle;$$

its derivative is called white noise of B_t^v. The S-transform of W_t^v equals

$$S(W_t^v)(\eta) = \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle.$$

Proof. The proof is given by Theorem 3.2. □

Example 3.4 (Ordinary Brownian motion). To compute the derivate of the ordinary Brownian motion B_t, it will be used that $\frac{d}{dt} 1_{[0,t]} = \delta_t$ in $\mathcal{S}'(\mathbb{R})$, hence

$$\begin{aligned}\frac{d}{dt} \left\langle\left\langle \left\langle \cdot, 1_{[0,t]} \right\rangle, \exp(\langle \cdot, \eta\rangle - 1/2|\eta|_0^2)\right\rangle\right\rangle &= \frac{d}{dt} \left\langle 1_{[0,t]}(u), \eta \right\rangle \\ &= \frac{d}{dt} \int_0^t \eta(u) du = \eta(t) = \langle \delta_t, \eta \rangle \\ &= \left\langle\left\langle \langle \cdot, \delta_t \rangle, \exp(\langle \cdot, \eta\rangle - 1/2|\eta|_0^2)\right\rangle\right\rangle.\end{aligned}$$

Therefore is $dB_t/dt =: W_t = \langle \cdot, \delta_t \rangle$ and $S(W_t)(\eta) = \eta(t)$ (see [Ku], Chapter 3.1).

Example 3.5 (Fractional Brownian motion). For $H \in (0,1)$ the fractional Brownian motion $B_\cdot^H : \mathbb{R} \to (\mathcal{S})^*$ is differentiable in the $(\mathcal{S})^*$-sense and

$$W_t^H := \frac{d}{dt} B_t^H = \left\langle \cdot, \delta_t \circ M_+^H \right\rangle$$

and for all $\eta \in \mathcal{S}(\mathbb{R})$

$$\left\langle \delta_t \circ M_+^H, \eta \right\rangle = (M_+^H \eta)(t).$$

The S-transform of W_t^H is given by

$$S(W_t^H)(\eta) = \left\langle\left\langle \frac{d}{dt} B_t^H, \exp(\langle \cdot, \eta\rangle - 1/2|\eta|_0^2) \right\rangle\right\rangle = \frac{d}{dt} S(B_t^H)(\eta) = (M_+^H \eta)(t).$$

This example is presented in [Be], Chapter 5.

Example 3.6 (Short range Brownian motion). The derivative of B_t^s is obviously given by

$$\left\langle \cdot, \frac{d}{dt} t^2 \exp(-(t-u)^2) \right\rangle = \left\langle \cdot, 2t \exp(-(t-u)^2) + t^2(-2(u-t)) \exp(-(u-t)^2) \right\rangle.$$

3.1.2 White noise integral and stochastic differential equations driven by B_t^v

Definition and properties of the white noise integral will be sketched only. A more detailly discussed can be found in [Ku] Chapter 13. Suppose that X_t is a mapping from $\mathbb{R} \to (\mathcal{S})^*$.

Definition 3.7. *The stochastic distribution process X_t is* white noise integrable*, if there is $\Psi \in (\mathcal{S})^*$ such that, for all $\eta \in \mathcal{S}(\mathbb{R})$, $(SX.)(\eta) \in L^1(\mathbb{R})$ and*

$$(S\Psi)(\eta) = \int_{\mathbb{R}} (SX_t)(\eta)\, dt.$$

This definition makes sense as the S-transform is injective.

Definition 3.8. *Let the stochastic distribution process X_t be such that $X_t \diamond W_t^v$ is white noise integrable, then X_t is* integrable with respect to B_t^v *and $\int_{\mathbb{R}} X_t\, dB_t^v$ is defined by*

$$\int_{\mathbb{R}} X_t\, dB_t^v := \int_{\mathbb{R}} X_t \diamond W_t^v\, dt.$$

This definition coincides in the case of a fractional Brownian motion with the definition of the fractional Itô integral (see Bender ([Be]), Øksendal and Hu [HuOk]). Therefore it is a generalization of the Itô integral, too. This is further discussed in [Ku] Chapter, 13.

This definition can be extended to more general integrators:

Definition 3.9. *Let X_t be a differentiable stochastic distribution process. If $Y_t \diamond (dX_t/dt)$ is white noise integrable, then Y_t* is integrable with respect to the process X_t*, and*

$$\int_{\mathbb{R}} Y_t\, dX_t := \int_{\mathbb{R}} Y_t \diamond \frac{dX_t}{dt}\, dt.$$

The following theorem is inspired by Bender's theorem for fractional Ito integrals (see [Be], Chapter 5).

Theorem 3.10. *Let $a, b \in \mathbb{R}$ and let $X. : [a,b] \to (\mathcal{S})_{-p}$ be continuous for some $p \in \mathbb{N}$. Further, let $W_\cdot^v : \mathbb{R} \to (\mathcal{S})_{-q}$ be continuous for some $q \in \mathbb{N}$. Then $\int_a^b X_t\, dB_t^v$ exists. Additionally for any sequence of tagged partitions $\tau_n = (\pi_k^{(n)}, t_k^{(n)})$ of $[a,b]$ with $\lim_{n\to\infty} \max\{|\pi_k - \pi_{k-1}|; k = 1,..,n\} = 0$, it follows that*

$$\lim_{n\to\infty} \sum_{k=0}^{n} X_{t_{k-1}^{(n)}} \diamond \left(B^v_{\pi_k^{(n)}} - B^v_{\pi_{k-1}^{(n)}} \right) = \int_a^b X_t\, dB_t^v$$

with limit in $(\mathcal{S})^*$.

Proof. Concerning integrability:

$$
\begin{aligned}
S\left(\int_a^b X_s \, dB_s^v\right)(\eta) &= \int_a^b S(X_s \diamond W_s^v)(\eta)\, ds \\
&= \int_a^b S(X_s)(\eta) S(W_s^v)(\eta)\, ds \\
&\le \max_s \{|S(X_s)(\eta)|\} \int_a^b S(W_s^v)(\eta)\, ds \\
&\le K \exp(a|\eta|_p^2) \int_a^b S(W_s^v)(\eta)\, ds < \infty
\end{aligned}
$$

By continuity of W_t^v and

$$
\begin{aligned}
\left|\left\langle \frac{d}{dt} m(u,t) - \frac{d}{dt} m(u,t_0), \eta(u) \right\rangle\right| &\le |\eta(u)|_q \left|\frac{d}{dt} m(u,t) - \frac{d}{dt} m(u,t_0)\right|_{-q} \\
&= |\eta(u)|_q \|W_t^v - W_{t_0}^v\|_{-q}
\end{aligned}
$$

with $t, t_0 \in [a,b]$ it follows that $\langle \frac{d}{dt} m(u,t), \eta(u)\rangle$ is a continuous function in t. So the function $\langle m(u,t), \eta(u)\rangle$ is continuously differentiable on the interval $[a,b]$. It follows by a well-known result in real analysis that for all $\eta \in \mathcal{S}(\mathbb{R})$ the continuous function $S(X_t)(\eta)$ is Riemann-Stieltjes integrable with respect to $\langle m(u,t), \eta(u)\rangle$, so the approximation is proved. □

The S-transform of $\int_{\mathbb{R}} X_t \, dB_t^v$ is therefore given by

$$
S\left(\int_{\mathbb{R}} X_t \, dB_t^v\right)(\eta) = \int_{\mathbb{R}} S(X_t \diamond W_t^v)(\eta)\, dt = \int_{\mathbb{R}} (SX_t)(\eta) \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle dt.
$$

Corollary 3.11. *Let* $\int_a^b X_s \, dB_s^v$ *exist and* $\int_a^b X_s \, dB_s^v \in (L^2)$ *than* $E(\int_a^b X_s \, dB_s^v) = 0$.

Proof. Let $\eta \equiv 0$, so $S(B_t^v)(0)$ is the expectation value of B_t^v and

$$
\begin{aligned}
S\left(\int_a^b X_s dB_s^v\right)(0) &= \int_a^b S(X_s)(0) \left\langle \frac{dm(u,s)}{ds}, 0 \right\rangle ds \\
&= \int_a^b S(X_s)(0) \frac{d}{ds} \langle m(u,s), 0\rangle \, ds \\
&= 0,
\end{aligned}
$$

which proves the claim. □

Remark 3.12. If the Gaussian process B_t^v shall serve as a model for stock prices, the zero expectation property is highly desirable. Since in this case the stochastic part in the related stochastic differential equation has expectation value zero, too. Otherwise the stochastic part of the stochastic differential equation has a drift and that might cause an arbitrage opportunity.

Example 3.13. Calculate

$$S(\int_0^t B_s^v \, dB_s^v)(\eta) = \int_0^t S(B_s^v)(\eta) S(W_s^v)(\eta)\, ds$$
$$= \int_0^t S(B_s^v)(\eta) \frac{d}{dt} S(B_t^v)(\eta)\, ds = \frac{1}{2}\left(S(B_t^v)(\eta)\right)^2 = \frac{1}{2} S((B_t^v)^{\diamond 2})(\eta)$$

where $(\cdot)^{\diamond 2}$ is the Wick square. By equation (2.40) it follows that

$$\frac{1}{2}\left(B_t^v\right)^{\diamond 2} = \frac{1}{2}\left(B_t^v\right)^2 - \frac{1}{2} |m(\cdot, t)|_0^2.$$

Proposition 3.14. *Let X_t and Y_t be differentiable stochastic distribution processes. Then the Wick product rule*

$$\frac{d(X_t \diamond Y_t)}{dt} = \frac{dX_t}{dt} \diamond Y_t + X_t \diamond \frac{dY_t}{dt} \tag{3.2}$$

holds in the $(\mathcal{S})^*$- sense.

Proof. It holds for the S-transform of $X_t \diamond Y_t$ that

$$\frac{d(S(X_t \diamond Y_t)(\eta))}{dt} = \frac{d(S(X_t)(\eta) S(Y_t)(\eta))}{dt} = \frac{dS(X_t)(\eta)}{dt} S(Y_t)(\eta) + S(X_t)(\eta) \frac{dS(Y_t)(\eta)}{dt}.$$

□

From the proposition above one can derive the formula for partial integration

$$\int_{\mathbb{R}} Y_t dX_t = Y_t \diamond X_t - Y_0 \diamond X_0 - \int_{\mathbb{R}} X_t dY_t \tag{3.3}$$

in the $(\mathcal{S})^*$- sense.

Proposition 3.15. *With $b > a$ let $f(\cdot) : [a, b] \to \mathbb{C}$ be a continuously differentiable function. Then it holds that*

$$\| \int_a^b f(s) dB_s^v \|_0^2 = (f(b))^2 v(b, b) - 2 f(b) f(a) v(b, a) + (f(a))^2 v(a, a)$$
$$- 2 \int_a^b f(b) \frac{df(s)}{ds} v(b, s) ds + 2 \int_a^b f(a) \frac{df(s)}{ds} v(a, s) ds + \int_a^b \int_a^b \frac{df(s)}{ds} \frac{df(t)}{dt} v(s, t) ds dt.$$

Proof. Use the formula for partial integration (3.3) and derive

$$\int_a^b f(s) dB_s^v = f(b) B_b^v - f(a) B_a^v - \int_a^b \frac{df(s)}{ds} B_s^v ds.$$

Now regard the bilinear mapping and use Fubinis Theoem

$$\left\langle\left\langle f(b)B_b^v - f(a)B_a^v - \int_a^b \frac{df(s)}{ds} B_s^v ds, f(b)B_b^v - f(a)B_a^v - \int_a^b \frac{df(s)}{ds} B_s^v ds \right\rangle\right\rangle$$
$$= (f(b))^2 v(b,b) - 2f(b)f(a)v(b,a) + (f(a))^2 v(a,a) - 2\int_a^b f(b)\frac{df(s)}{ds} v(b,s)ds$$
$$+ 2\int_a^b f(a)\frac{df(s)}{ds} v(a,s)ds + \int_a^b \int_a^b \frac{df(s)}{ds}\frac{df(t)}{dt} v(s,t)dsdt.$$

□

Example 3.16. Consider the equation

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t^v,$$

with constants μ and σ. This expression is declared by the integral equation

$$\begin{aligned} X_t &= \int_0^t \mu X_s \, ds + \int_0^t \sigma X_s \, dB_s^v. \\ &= \int_0^t (\mu X_s + \sigma X_s \diamond W_s^v) \, ds \\ &= \int_0^t X_s \diamond (\mu + \sigma W_s^v) \, ds, \end{aligned}$$

and the solution is given by

$$X_t = X(0) \exp^{\diamond}(\mu t + \sigma B_t^v) = X(0) \exp(\mu t + \sigma B_t^v - \frac{1}{2}\sigma^2 |m(u,t)|_0^2).$$

Let $a(\cdot,\cdot), b(\cdot,\cdot) : \mathbb{R} \times (\mathcal{S})^* \to (\mathcal{S})^*$ and regard the stochastic differential equation on $[0,T]$ $(T > 0)$

$$dX_t = a(t, X_t) \, dt + b(t, X_t) \, dB_t^v.$$

It is defined in terms of the corresponding integral equation with $t \in [0,T]$

$$X_t = X_0 + \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dB_s^v.$$

If this equation is regarded in the $(\mathcal{S})^*$-sense, it can be transformed to

$$X_t = X_0 + \int_0^t (a(s, X_s) + b(s, X_s) \diamond W_s^v) \, ds.$$

Set $(a(s, X_s) + b(s, X_s) \diamond W_s^v) = f(s, X_s)$, thus it follows an integral equation of the form

$$X_t = X_0 + \int_0^t f(s, X_s) \, ds$$

as a white noise integral equation, where $f(\cdot,\cdot)$ is a mapping $\mathbb{R} \times (\mathcal{S})^* \to (\mathcal{S})^*$. A stochastic distribution process X_t is called *weakly measurable* if $S(X_t)(\eta)$ is measurable for all $\eta \in \mathcal{S}(\mathbb{R})$.

Definition 3.17. *A stochastic distribution process X_t is called a* weak solution *of the white noise integral equation on $[0,T]$, if it satisfies the following conditions:*

(a)X is weakly measurable.
(b)The stochastic distribution process $f(t,X_t)$ is white noise integrable.
(c)For each $\eta \in \mathcal{S}(\mathbb{R})$ the equality holds for almost all $t \in [0,T]$

$$S(X_t)(\eta) = S(X_0)(\eta) + \int_0^t S(f(s,X_s))(\eta)\,ds.$$

The following theorem on existence and uniqueness of solutions of white noise integral equations is from [Ku], Theorem 13.43.

Theorem 3.18. *Suppose f is function from $[0,T]\times(\mathcal{S})^*$ into $(\mathcal{S})^*$ satisfying the following conditions:*

(a)(Measurability condition) The function $f(s,X_s)$ $s \in [0,T]$, is weakly measurable for any weakly measurable function $X : [0,T] \to (\mathcal{S})^*$.
(b)(Lipschitz condition) For almost all $t \in [0,T]$,

$$|S(f(t,\Phi))(\eta) - S(f(t,\Psi))(\eta)| \le L(t,\eta)|S(\Phi)(\eta) - S(\Psi)(\eta)|,$$

for all $\eta \in \mathcal{S}(\mathbb{R})$ and $\Psi, \Phi \in (\mathcal{S})^*$, where L is nonnegative and

$$\int_0^T L(t,\eta)\,dt \le K(1 + |\eta|_p^2)$$

for some $K, p \ge 0$.
(c)(Growth condition) For almost all $t \in [0,T]$

$$|S(f(t,\Phi))(\eta)| \le \rho(t,\eta)(1 + |S(\Phi)(\eta)|),$$

for all $\eta \in S(\mathbb{R})$ and $\Phi \in (\mathcal{S})^*$, where ρ is nonnegative and

$$\int_0^T \rho(t,\eta)\,dt \le K \exp\left(c|\eta|_p^2\right),$$

where K, p are the same as above and $c \ge 0$.

Then for any $X_0 \in (\mathcal{S})^*$ the equation

$$X_t = X_0 + \int_0^t f(s,X_s)\,ds$$

has a unique weak solution X such that for all $\eta \in \mathcal{S}(\mathbb{R})$

$$\operatorname*{ess\,sup}_{t\in[0,T]} |SX_t(\eta)| < \infty.$$

The last theorem will be used to prove the following theorem on existence and uniqueness of solutions of stochastic integral equations driven by Gaussian processes with dependent increments.

Theorem 3.19 (Existence and Uniqueness Theorem for bilinear SDE's). *Let $\sigma, \mu \in C([0,T])$ and $X_0 \in (\mathcal{S})^*$ and $t \in [0,T]$. Further let $m(u,t)$ such that there exist $K, p > 0$ satisfying*

$$\int_0^T \left| \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle \right| dt \leq K(1 + |A^p \eta|_0^2)$$

for all $\eta \in \mathcal{S}(\mathbb{R})$. Then there exists a unique weak solution of

$$X_t = X_0 + \int_0^t (\mu(s) X_s + \sigma(s) X_s \diamond W_s^v)\, ds$$

which is given by

$$X_t = X_0 \exp^{\diamond}\Big(\int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dB_s^v\Big).$$

Proof. The conditions of the existence and uniqueness Theorem 3.18 have to be shown. The measurability condition is met due to continuity of the Wick product and measurability of $dm(u,t)/dt$. For the Lipschitz condition look at for almost all $t \in [0,T]$ and for all $\Phi, \Psi \in (\mathcal{S})^*$ and $\eta \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}
&\Big| S(\mu(t)\Phi)(\eta) + S(\sigma(t)\Phi)(\eta) \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle \\
&\quad - S(\mu(t)\Psi)(\eta) - S(\sigma(t)\Psi)(\eta) \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle \Big| \\
&\leq (|\mu(t)| + |\sigma(t) \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle |) |S(\Phi)(\eta) - S(\Psi)(\eta)|.
\end{aligned}$$

Thus it holds that

$$(|\mu(t)| + |\sigma(t) \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle |) = L(t,\eta)$$

and the upper bound for $\int_0^T L(t,\eta)\, dt$ is

$$\begin{aligned}
&\int_0^T (|\mu(t)| + |\sigma(t) \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle |)\, dt \\
&\leq \max\{|\mu(t)|\} T + \max\{|\sigma(t)|\} \int_0^T | \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle |\, dt.
\end{aligned}$$

This proves the Lipschitz condition. In order to check the growth condition proceed in a similar manner

$$
\begin{aligned}
&|S(\mu(t)\Phi)(\eta) + S(\sigma(t)\Phi)(\eta)\left\langle \frac{d}{dt}m(u,t),\eta(u)\right\rangle| \\
&\leq \left(\max_t\{|\mu(t)|\} + \max_t\{|\sigma_t|\}|\left\langle \frac{d}{dt}m(u,t),\eta(u)\right\rangle|\right)|S(\Phi)(\eta)| \\
&\leq \left(\max_t\{|\mu(t)|\} + \max_t\{|\sigma_t|\}|\left\langle \frac{d}{dt}m(u,t),\eta(u)\right\rangle|\right)(1+|S(\Phi)(\eta)|).
\end{aligned}
$$

Then $\rho(t,\eta) := \big(\max_t\{|\mu(t)|\} + \max_t\{|\sigma_t|\}|\left\langle \frac{d}{dt}m(u,t),\eta(u)\right\rangle|\big)$ and the condition for $\rho(t,\eta)$ is obviously satisfied. The solution is calculated by

$$
\begin{aligned}
X_t &= \int_0^t (\mu(s) + \sigma(s)W_s^v)\diamond X_s\,ds \\
X_t &= \exp^\diamond(\int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dB_s^v).
\end{aligned}
$$

□

Remark 3.20. It is natural to ask why the bilinear case is considered only, whereas the existence and uniqueness theorem of Kuo is formulated even for a nonlinear situation. In the authors' point of view the theorem of Kuo is only applicable in the bilinear case. Note that the function f is a function from $\mathbb{R}\times(\mathcal{S})^* \to (\mathcal{S})^*$. But the motivation of the authors is to take a real valued function $g:\mathbb{R}^2\to\mathbb{R}$ with $|g(t,x)|^2 \leq K(1+|x|^2)$, and to define the stochastic differential equation with this function. One has to explain such a real-valued function with elements of $(\mathcal{S})^*$. A further argument is that the Lipschitz condition fails with a nonlinear function g. Suppose $F,G\in(L^2)$ and a $\eta\in\mathcal{S}(\mathbb{R})$ such that $|S(F-G)(\eta)|=0$ and beside $|S(g(t,F)-g(t,G))(\eta)|\neq 0$. This is possible if $F-G$ is linearly independent to $g(t,F)-g(t,G)$. If the linear dependence is demanded above, one gets with the property of a real valued function only the linear or constant case for g in x.

Remark 3.21. A more general type of stochastic differential equations is discussed in Section 2.5.

Now the ordinary Ornstein-Uhlenbeck process in the $(\mathcal{S})^*$-sense and a generalized Ornstein-Uhlenbeck process are presented.

Example 3.22 (Ordinary Ornstein-Uhlenbeck process). The centered Ornstein-Uhlenbeck process B_t^{OU} is a stationary centered Gaussian process with covariance function $v(s,t) = (\sigma^2/2\alpha)\exp(-\alpha|t-s|)$ with positiv constants α,σ, if the initial vaule has a normal B_0^{OU} distribution with mean zero and variance $(\sigma^2/2\alpha)$ (see Karatzas and Shreve [KaSh], Chapter 5.6). So

$$B_t^{OU} := B_0^{OU}\exp(-\alpha t) + \left\langle \cdot, \sigma 1_{[0,t]}\exp(-\alpha(t-u))\right\rangle,$$

so the derivative of the Ornstein-Uhlenbeck process in the $(\mathcal{S})^*$ is

$$\frac{d}{dt}B_t^{OU} := B_0^{OU}(-\alpha)\exp(-\alpha t) + \left\langle \cdot, \sigma\frac{d}{dt}(1_{[0,t]}\exp(-\alpha(t-u)))\right\rangle.$$

The integral and the stochastic differential equation with respect to the Ornstein Uhlenbeck process can be defined similarly as before. Note that now it is possible to derivate the Ornstein-Uhlenbeck process.

Example 3.23 (Generalized Ornstein-Uhlenbeck process). Let $\alpha, \sigma > 0$ and B_t^v be as in Theorem 3.10. The generalized Ornstein-Uhlenbeck process is the solution of the stochastic differential equation

$$dX_t = -\alpha X_t \, dt + \sigma \, dB_t^v,$$

which is the same as

$$\frac{dX_t}{dt} = -\alpha X_t + \sigma W_t^v$$

in the $(\mathcal{S})^*$-sense. Note that by the existence and uniqueness theorem one can show with slight modifications that the solution is unique. Solve the corresponding non-disturbed equation $dX_t = -\alpha X_t \, dt$ and get the solution $X_t = C \exp(-\alpha t)$, assume that the C is a stochastic distribution process and by the product rule 3.14 the derivative becomes to

$$\frac{d(C_t \exp(-\alpha t))}{dt} = \frac{dC_t}{dt} \exp(-\alpha t) - \alpha C_t \exp(-\alpha t).$$

Comparing this with the stochastic differential equation one deduces

$$\frac{dC_t}{dt} \exp(-\alpha t) = \sigma W_t^v$$

and so $C_t = C_0 + \int_0^t \exp(\alpha s) \sigma W_s^v \, ds = C_0 + \sigma \int_0^t \exp(\alpha s) \, dB_s^v$. Thus the solution is

$$B_t^{GOU} := X_t = C_0 \exp(-\alpha t) + \sigma \int_0^t \exp(-\alpha(t-s)) \, dB_s^v,$$

where C_0 is the initial random variable of the process B_t^{GOU}. By corollary 3.11 the expectation value of B_t^{GOU} is $S(B^{GOU})(0) = E(C_0) \exp(-\alpha t)$, and its covariance function is equal to

$$\begin{aligned}
&E\left((B_t^{GOU} - E(C_0)\exp(-\alpha t))(B_s^{GOU} - E(C_0)\exp(-\alpha s))\right) \\
&\quad = \exp(-\alpha(t+s))(E(C_0^2) - E(C_0)^2) + E(C_0 \exp(-\alpha t)\sigma \int_0^s \exp(-\alpha(s-u)) \, dB_s^v) \\
&\qquad + E(C_0 \exp(-\alpha s)\sigma \int_0^t \exp(-\alpha(t-u)) \, dB_s^v) \\
&\qquad + E(\sigma^2 \int_0^t \exp(-\alpha(t-u)) \, dB_s^v \int_0^s \exp(-\alpha(s-u)) \, dB_s^v).
\end{aligned}$$

Finally a process that can be considered as an extension of the Brownian Bridge is presented.

Example 3.24 (Brownian bridge with B_t^v). An ordinary Brownian bridge is given for $0 < a < b$ and $t \in [0, T]$ by

$$B_t^{a \to b} := a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + \left(B_t - \frac{t}{T} B_T\right).$$

This definition motivates the extension to the generalized Brownian Bridge given by

$$B_t^{v,a\to b} := a\left(1-\frac{t}{T}\right) + b\frac{t}{T} + \left(B_t^v - \frac{t}{T}B_T^v\right).$$

The expectation value is $E(B_t^{v,a\to b}) = E(B_t^{a\to b}) = a + (b-a)t/T$. The covariance function of $B_t^{v,a\to b}$ is equal to

$$E\left(\left(B_s^v - \frac{s}{T}B_T^v\right)\left(B_t^v - \frac{t}{T}B_T^v\right)\right) = v(s,t) - \frac{t}{T}v(s,T) - \frac{s}{T}v(t,T) + \frac{st}{T}v(T,T).$$

3.2 Stochastic integrals driven by stochastic processes in $(\mathcal{S})^*$

In this Section the definition of the stochastic integral driven by a stochastic distribution in $(\mathcal{S})^*$ will be further discussed. According to the chaos decomposition theorem a stochastic distribution process $X(t)$ can be represented as

$$X(t) = \sum_{k=0}^{\infty} \left\langle : \cdot^{\otimes k} :, d_k(u,t) \right\rangle, \tag{3.4}$$

where $d_k(u,\cdot) : \mathbb{R} \to (\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} k}$ with respect to u for $k \geq 1$ and $d_0(t)$ a function from $\mathbb{R} \to \mathbb{C}$. As in Theorem 3.2 one can prove that for a differentiable $F_k(\cdot) : \mathbb{R} \to (\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} k}$ it holds that

$$\frac{d\left\langle : \cdot^{\otimes k} :, F_k(t) \right\rangle}{dt} = \left\langle : \cdot^{\otimes k} :, \frac{dF_k(t)}{dt} \right\rangle.$$

This will be used in the proof of the following Theorem.

Theorem 3.25. *Let $b > a$ and $d_k(u,\cdot) : [a,b] \to (\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} k}$ be continuously differentiable for $k \in \mathbb{N}$, such that there exists $q, p > 0$ and a positiv bounded function $L(t)$ on $[a,b]$ such that*

$$\left| \left\langle \frac{d(d_k(u,t))}{dt}, \eta^{\otimes k}(u) \right\rangle \right| \leq \frac{L(t) q^k |\eta|_p^{2k}}{k!} \tag{3.5}$$

for all $\eta \in \mathcal{S}(\mathbb{R})$ and for all $t \in [0,T]$ and for all $k \geq 1$. Further let $d_0(\cdot)$ be continuously differentiable on $[a,b]$ in the classical sense. Then

$$X(t) = \sum_{k=0}^{\infty} \left\langle : \cdot^{\otimes k} :, d_k(u,t) \right\rangle$$

is differentiable stochastic distribution process and

$$\frac{d(X(t))}{dt} = \frac{d(d_0(t))}{dt} + \sum_{k=1}^{\infty} \left\langle : \cdot^{\otimes k} :, \frac{d(d_k(u,t))}{dt} \right\rangle. \tag{3.6}$$

Proof. Define

$$g_\eta^n(t) := d_0(t) + \sum_{k=1}^{n} \left\langle d_k(u,t), (\eta(u))^{\otimes k} \right\rangle .$$

The function $g_\eta^n(t)$ is therefore continuously differentiable and

$$\begin{aligned} \frac{dg_\eta^n(t)}{dt} &:= \frac{d(d_0(t))}{dt} + \sum_{k=1}^{n} \left\langle \frac{d(d_k(u,t))}{dt}, (\eta(u))^{\otimes k} \right\rangle \\ &= \frac{d}{dt}\left(d_0(t) + \sum_{k=1}^{n} \left\langle d_k(u,t), (\eta(u))^{\otimes k} \right\rangle \right). \end{aligned}$$

Let $n \to \infty$ and one obtains

$$\frac{d(d_0(t))}{dt} + \sum_{k=1}^{\infty} \left\langle \frac{d(d_k(u,t))}{dt}, (\eta(u))^{\otimes k} \right\rangle = \frac{d}{dt}\left(d_0(t) + \sum_{k=1}^{\infty} \left\langle d_k(u,t), (\eta(u))^{\otimes k} \right\rangle \right).$$

Due to equation (3.5) the left hand side converges and the order of differentation and summation can be changed. Furthermore, this expression is an element of $(\mathcal{S})^*$ as it satisfies the conditions of Theorem 2.37. The right hand side is the derivative of $S(X(t))(\eta)$ with respect to t. So $X(t)$ is a continuously differentiable stochastic distribution process. □

The definition of the integral $\int_{\mathbb{R}} Y_t \, dX_t$ is given by Definiton 3.9.

Example 3.26. It holds that

$$\int_a^b X(t)\, dX(t) = \frac{1}{2}(X(b))^{\diamond 2} - \frac{1}{2}(X(a))^{\diamond 2}.$$

which is obvious by regarding the S-transforms

$$\begin{aligned} \int_a^b S(X(t))(\eta) S\left(\frac{dX(t)}{dt}\right)(\eta)\, dt &= \int_a^b S(X(t))(\eta) \frac{dS(X(t))(\eta)}{dt}\, dt \\ &= \frac{1}{2}(S(X(b))(\eta))^2 - \frac{1}{2}(S(X(a))(\eta))^2. \end{aligned}$$

Example 3.27 (Associated Ornstein-Uhlenbeck process). Consider the stochastic differential equation with the constants $\alpha, \sigma \in \mathbb{R}$

$$dY(t) = -\alpha Y(t)\, dt + \sigma\, dX(t). \tag{3.7}$$

The solution of the undisturbed equation is $C\exp(-\alpha t)$, supposing C is a stochastic distribution process it follows by the Proposition 3.14 that

$$d(\exp(-\alpha t)C(t)) = -\alpha \exp(-\alpha t) C(t)\, dt + \exp(-\alpha t)\, dC(t).$$

Comparing this to equation (3.7) it follows that $\sigma\, dX(t) = \exp(-\alpha t)\, dC(t)$, therefore the solution of (3.7) equals

$$Y(t) = \exp(-\alpha t)C_0 + \sigma \int_0^t \exp(-\alpha(t-s))\, dX(s) \tag{3.8}$$

with a stochastic distribution C_0. For fixed $\eta \in \mathcal{S}(\mathbb{R})$ the S-transform of equation (3.7) is an ordinary differential equation in t, which has a unique solution, which is given by the S-transform of the same η by equation (3.8). Therefore it is shown that equation (3.7) has a unique solution.

Example 3.28 (The failing bilinear stochastic differential equation driven by $X(t)$). It is natural to ask if the bilinear stochastic differential equation with constants $\mu, \sigma \in \mathbb{R}$

$$dY(t) = \mu Y(t)\, dt + \sigma Y(t)\, dX(t)$$

can be considered and solved in $(\mathcal{S})^*$. It will be shown, that with $X(t)$ of the form (3.4) will fail if there exists a $k > 1$ with $d_k(u,t) \not\equiv 0$ with respect to u in $L^2(\mathbb{R}^k)$ for all $t \in \mathbb{R}$. To obtain this regard the S-transform with an any fixed η

$$\frac{dS(Y(t))(\eta)}{dt} = \mu S(Y(t))(\eta) + \sigma S(Y(t))(\eta) \frac{dS(X(t))(\eta)}{dt}.$$

This ordinary differential equation in t has a unique solution of the form

$$S(Y(t))(\eta) = S(Y(0))(\eta) \exp(\mu t + \sigma S(X(t))(\eta)), \tag{3.9}$$

if there exists a $d_k(u,t) \not\equiv 0$ with $k > 1$ then there exists η such that the growth condition of Theorem 2.37 is not satisfied, therefore equation (3.9) does not describe an element in $(\mathcal{S})^*$.

If $d_k(u,t) \in \hat{L}^2(\mathbb{R}^k)$ then the process $X(t)$ has the following properties. The expectation value of $X(t)$ is

$$E(X(t)) = S(X(t))(0) = d_0(t) + \sum_{k=1}^{\infty} \langle d_k(u,t), 0 \rangle = d_0(t). \tag{3.10}$$

The covariance of $X(t)$ is

$$\begin{aligned} E((X(t)-d_0(t))(X(s)-d_0(s)) &= \left\langle\!\!\left\langle \sum_{k=1}^{\infty} \left\langle :\cdot^{\otimes k}:, d_k(u,t) \right\rangle, \sum_{k=1}^{\infty} \left\langle :\cdot^{\otimes k}:, d_k(u,t) \right\rangle \right\rangle\!\!\right\rangle \\ &= \sum_{k=1}^{\infty} \langle d_k(u,t), d_k(u,s) \rangle . \end{aligned}$$

3.3 A chain formula for stochastic distribution processes

In this section a chain formula for a certain class of stochastic distribution processes is given. It includes Itô's rule in the case of the Brownian motion.

Theorem 3.29. *Let $f(\cdot,\cdot) : \mathbb{R} \times (\mathcal{S})^* \to (\mathcal{S})^*$, such that $f(t, X) = \sum_{k=0}^{n} a_k(t) X^{\diamond k}$ with continuously differentiable functions $a_k(t)$. Then the chain formula*

$$\begin{aligned} f(b, B_b^v) - f(a, B_a^v) &= \int_a^b \left(\frac{\partial}{\partial t} f(t, B_t^v) + \frac{\partial}{\partial x} f(t, B_t^v) \diamond W_t^v \right) dt \\ &= \int_a^b \frac{\partial}{\partial t} f(t, B_t^v)\, dt + \int_a^b \frac{\partial}{\partial x} f(t, B_t^v)\, dB_t^v \end{aligned}$$

holds.

Proof. The S-transform of $f(t, B_t^v)$ is given by $F_t(\eta) := \sum_{k=0}^{n} a_k(t)(S(B_t^v)(\eta))^k$. It follows that the derivative of the function $F_t(\eta)$ equals

$$\frac{dF_t(\eta)}{dt} = \sum_{k=0}^{n} \left(\frac{da_k(t)}{dt} (S(B_t^v)(\eta))^k + a_k(t) k (S(B_t^v)(\eta))^{k-1} S(W_t^v)(\eta) \right),$$

by use of Theorems 3.2 and 3.3. Integrating both sides with respect to t, the S-transform of the chain formula follows. □

In the next step the chain rule for $\lim_{n\to\infty} \sum_{k=0}^{n} a(t)(B_t^v)^{\diamond k}$ will be considered. The idea is to approximate functions by these Wick polynomials and to define a class of functions to which the chain rule applies. One of these functions is the already known Wick exponential $\exp^{\diamond}(\langle \cdot, f \rangle) = \sum_{k=0}^{\infty} (\langle \cdot, f \rangle)^{\diamond k}/k!$ with $f \in \mathcal{S}'(\mathbb{R})$.

Definition 3.30. *Let D be a subset of $(\mathcal{S})^*$. A function $f : \mathbb{R} \times D \to (\mathcal{S})^*$ admits a* Wick representation *with respect to D, if there exists a sequence of Wick polynomials $\{a_k(t) X^{\diamond k}\}_{k=0}^{\infty}$ with continuously differentiable $a_k(t)$, such that for all $X \in D$*

$$f(t, X) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(t) X^{\diamond k}$$

with convergence in $(\mathcal{S})^*$.

The following proposition contains a sufficient condition for a function f to admit a Wick representation.

Corollary 3.31. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ has a power series representation with $g(x) = \sum_{k=0}^{\infty} g_k x^k$ for all $x \in \mathbb{R}$, and that there exist positive constants C, p and a such that for $X \in (\mathcal{S})^*$ and for all $\eta \in \mathcal{S}(\mathbb{R})$*

$$|g(S(X)(\eta))| \leq C \exp(a|\eta|_p^2). \tag{3.11}$$

Then

$$g^{\diamond}(X) := \left(\sum_{k=0}^{\infty} g_k X^{\diamond k} \right) \in (\mathcal{S})^*.$$

Proof. We have to check the first condition of Theorem 2.37. The convergence of $F_n(\eta) := \sum_{k=0}^n g_k(S(X)(\eta))^k$ follows from the convergence of $\sum_{k=0}^n g_k x^k$ for all $x \in \mathbb{R}$. □

Theorem 3.32. *Let $D := \{\langle \cdot, h\rangle, h \in \mathcal{S}'(\mathbb{R})\}$. Suppose that $f : \mathbb{R} \times D \to (\mathcal{S})^*$ admits a Wick representation in D, then the chain rule for f holds with $b > a$*

$$f(b, B_b^v) - f(a, B_a^v) = \int_a^b \left(\frac{\partial f(t, B_t^v)}{\partial t} + \frac{\partial f(t, B_t^v)}{\partial x} \diamond W_t^v \right) dt.$$

Proof. Observe that $f(t, B_t^v) = \lim_{n\to\infty} \sum_{k=0}^n a_k(t)(B_t^v)^{\diamond k}$ and

$$\sum_{k=0}^n \left(a_k(b)(B_b^v)^{\diamond k} - a_k(a)(B_a^v)^{\diamond k} \right) = \int_a^b \sum_{k=0}^n \left(\frac{da_k(u)}{du}(B_u^v)^{\diamond k} + a_k(u)k(B_u^v)^{\diamond(k-1)} \diamond W_u^v \right) du$$

Taking limits $n \to \infty$ on both sides proves the assertion. □

Theorem 3.33. *Let D and f be as in Theorem 3.32 and let*

$$f(t, \langle \cdot, h\rangle) = \lim_{n\to\infty} \sum_{k=0}^n a_k(t)(\langle \cdot, h\rangle)^k$$

converge in (L^2) for $h \in L^2(\mathbb{R})$. If B_t^v is the ordinary Brownian motion B_t, then the chain rule as given in Theorem 3.32 and Itô's rule coincide.

Proof. Due to the Wick calculus in Lemma 2.34 the function $f(t, B_t)$ has two representations, one with Wick product, and the other with ordinary product. The chain rule is declared with respect to the representation with the Wick product and Itô's rule is declared with respect to the representation with the ordinary product. Both will be calculated and by comparing the summands it will be shown, that they are equal. As the chain rule and Itô's rule are linear mappings, it is sufficient to check the assertion for $a(t)(B_t)^n$ with any continuously differentiable $a(t)$ and any $n \in \mathbb{N}$ and then to proceed to infinite sums. First regard the representation with the Wick product of $a(t)(B_t)^n$ by use of the Wick calculus and the chain rule

$$\begin{aligned}
a(t)(B_t)^n &= a(t) \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!\, t^k (B_t)^{\diamond(n-2k)} \\
&= \int_0^t \frac{da(u)}{du} \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!\, u^k (B_u)^{\diamond(n-2k)}\, du \\
&\quad + \int_0^t a(u) \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!\, k u^{k-1} (B_u)^{\diamond(n-2k)}\, du \\
&\quad + \int_0^t a(u) \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!\, u^k (n-2k) (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&=: I_1 + I_2 + I_3
\end{aligned}$$

Applying Itô's rule to $a(t)(B_t)^n$ and then using the Wick representation it follows

$$\begin{aligned}
a(t)(B_t)^n &= \int_0^t \frac{da(u)}{du}(B_u)^n\, du + \int_0^t a(u)n(B_u)^{n-1}\, dBu + \frac{1}{2}\int_0^t a(u)n(n-1)(B_u)^{n-2}\, du \\
&= \int_0^t \frac{da(u)}{du}\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k)}\, du \\
&\quad + \int_0^t a(u)n \sum_{k=0}^{(n-1)/2}\binom{(n-1)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&\quad + \frac{1}{2}\int_0^t a(u)n(n-1)\sum_{k=0}^{(n-2)/2}\binom{(n-2)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-2)}\, du \\
&=: I_4 + I_5 + I_6.
\end{aligned}$$

The term I_1 is equal to I_4. The integral I_2 is equal to I_6, because with $k+1 =: m$ in I_6 one gets

$$\begin{aligned}
I_6 &= \frac{1}{2}\int_0^t a(u)n(n-1)\sum_{k=0}^{(n-2)/2}\binom{(n-2)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-2)}\, du \\
&= \int_0^t a(u)\sum_{m=1}^{n/2}\frac{(n-2)!\, n\,(n-1)\, 2m\,(2m-1)}{2(n-2m)!(2m)!}(2m-3)!!\, u^{m-1}(B_u)^{\diamond(n-2m)}\, du \\
&= \int_0^t a(u)\sum_{m=1}^{n/2}\binom{n}{2m}(2m-1)!!\, m u^{m-1}(B_u)^{\diamond(n-2m)}\, du \\
&= I_2.
\end{aligned}$$

The integral I_3 is equal to I_5, note that both sums have the same largest k if n is odd, and if n is even, then the last term in I_3 vanishes because of the factor $(n-2k)$, hence both sums end again at the same k. In order to show that both integrals are equal, we calculate

$$\begin{aligned}
I_5 &= \int_0^t a(u)n \sum_{k=0}^{(n-1)/2}\binom{(n-1)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&= \int_0^t a(u)\sum_{k=0}^{(n-1)/2}\frac{n!}{(n-1-2k)!(2k)!}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&= \int_0^t a(u)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!(n-2k)\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&= I_3.
\end{aligned}$$

So Itô's rule and the chain formula are identical in the case of the Brownian motion for finite polynoms $\sum_{k=0}^{n} a_k(t)(B_t)^n$, now let $n \to \infty$ and the sum converges in (L^2). The theorem is proved. □

Replacing the Wick product in $a_k(t)(B_t^v)^{\diamond k}$ by the usual product, the chain rule appears as follows.

Proposition 3.34. *Let $f^o : \mathbb{R} \times D \to (\mathcal{S})^*$ admit a Wick representation with respect to $D := \{\langle \cdot, g \rangle, g \in L^2(\mathbb{R})\}$, and let $f^o(t, B_t^v) = \sum_{k=0}^{\infty} f_k^o(t)(B_t^v)^k$ converge in (L^2). Then the chain rule has also the representation*

$$df^o(t, B_t^v) = f_t^o(t, B_t^v)dt + f_x^o(t, B_t^v)dB_t^v + \frac{1}{2}\frac{d|m(\cdot,t)|_0^2}{dt} f_{xx}^o(t, B_t^v)dt. \tag{3.12}$$

Proof. As the chain rule is a linear mapping, regard the chain rule only for $a_n(t)(B_t^v)^n$, and by Lemma 2.34

$$\begin{aligned}
d(a_n(t)(B_t^v)^n &= d(a_n(t)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!(|m(\cdot,t)|_0^2)^k(B_t^v)^{\diamond(n-2k)})\\
&= \frac{da_n(t)}{dt}\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!(|m(\cdot,t)|_0^2)^k(B_t^v)^{\diamond(n-2k)})\\
&\quad + a_n(t)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!k\frac{d|m(\cdot,t)|_0^2}{dt}(|m(\cdot,t)|_0^2)^{k-1}(B_t^v)^{\diamond(n-2k)})\\
&\quad + a_n(t)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!(|m(\cdot,t)|_0^2)^k(n-2k-1)(B_t^v)^{\diamond(n-2k)})dB_t^v\\
&= f_t^o(t, B_t^v)dt + \frac{1}{2}\frac{d|m(\cdot,t)|_0^2}{dt} f_{xx}(t, B_t^v)dt + f_x^o(t, B_t^v)dB_t^v
\end{aligned}$$

as in the proof of Theorem 3.33. □

This representation is already known for the fractional Brownian motion (see [Be] Chapter 5). As with Itô's chain rule it is helpful to have a chain rule which applies on functions of solutions of stochastic differential equations, too.

Theorem 3.35. *Suppose $f : \mathbb{R} \times D \to (\mathcal{S})^*$ admits a Wick representation with respect to D, and let the stochastic distribution process $X. : \mathbb{R} \to D$ be given by*

$$X(t) - X(0) = \int_0^t \left(\mu(u) + \sigma(u) \diamond W_u^v\right) du$$

with two stochastic distribution processes μ and σ such that the integral exists. Then the chain rule for $f(t, X_t)$ holds with $b > a$

$$f(b, X_b) - f(a, X_a) = \int_a^b \left(\frac{\partial f(u, X_u)}{\partial t} + \frac{\partial f(u, X_u)}{\partial x} \diamond (\mu(u) + \sigma(u) \diamond W_u^v)\right) du.$$

Proof. The proof is obvious by taking the S-transform on both sides, and using the Wick representation of f and the integral representation of X_t. Fubini's theorem is applied to change the order of integration and S-transformation. □

Theorem 3.36 (Chain rule). *Suppose $f : \mathbb{R} \times D \to (\mathcal{S})^*$ admits a Wick representation and let $X. : \mathbb{R} \to D$ be a continuously differentiable stochastic distribution process. Then for $b > a$ it holds*

$$f(b, X_b) - f(a, X_a) = \int_a^b \frac{\partial f(u, X_u)}{\partial t} + \frac{\partial f(u, X_u)}{\partial x} \diamond \frac{dX_u}{du}\, du.$$

Proof. The proof is given by calculating the S-transform as above. □

3.4 Existence and uniqueness of solutions of stochastic differential equations

Let $a(t, X) := \sum_{k=0}^n a_k(t) X^{\diamond k}$ and $b(t, X) := \sum_{k=0}^n b_k(t) X^{\diamond k}$ for all $X \in (\mathcal{S})^*$ and with continuously differentiable functions $a_k(\cdot), b_k(\cdot) : \mathbb{R} \to \mathbb{R}$ with $k \in \{0, .., n\}$.

Lemma 3.37. *Let $\Psi \in (\mathcal{S})^*$ and $\Psi = \sum_{k=0}^\infty \left\langle : \cdot^{\otimes k} :, F_k(u) \right\rangle$, then it holds that*

$$\Psi^{\diamond k} = \sum_{l=0}^\infty \sum_{j_1 + .. + j_k = l} \left\langle : \cdot^{\otimes l} :, F_{j_1}(u) \hat{\otimes} ... \hat{\otimes} F_{j_k}(u) \right\rangle . \tag{3.13}$$

Proof. The proof is obtained by induction, the assertion is satisfied for $k = 2$ by

$$\Psi^{\diamond 2} = \sum_{l=0}^\infty \sum_{j_1 + j_2 = l} \left\langle : \cdot^{\otimes l} :, F_{j_1}(u) \hat{\otimes} F_{j_2}(u) \right\rangle .$$

The convergence in $(\mathcal{S})^*$ is granted by the existence of the Wick product. If the claim is satisfied for k then

$$\begin{aligned}\Psi^{\diamond (k+1)} &= \sum_{l=0}^\infty \sum_{j_1 + .. + j_k = l} \left\langle : \cdot^{\otimes l} :, F_{j_1}(u) \hat{\otimes} ... \hat{\otimes} F_{j_k}(u) \right\rangle \diamond \Big(\sum_{k=0}^\infty \left\langle : \cdot^{\otimes k} :, F_k(u) \right\rangle\Big) \\ &= \sum_{l=0}^\infty \sum_{j_1 + .. + j_{k+1} = l} \left\langle : \cdot^{\otimes l} :, F_{j_1}(u) \hat{\otimes} ... \hat{\otimes} F_{j_{k+1}}(u) \right\rangle .\end{aligned}$$

Again the convergence is attained in $(\mathcal{S})^*$ by the existence of the Wick product, so the proof is complete. □

Consider the stochastic differential equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB_t^v = (a(t, X(t)) + b(t, X(t)) \diamond W_t^v)dt. \tag{3.14}$$

If the solution of this equation exists, then it has the form

$$X(t) = \sum_{k=0}^\infty \left\langle : \cdot^{\otimes k} :, d_k(u, t) \right\rangle = d_0(t) + \sum_{k=1}^\infty \left\langle : \cdot^{\otimes k} :, d_k(u, t) \right\rangle \tag{3.15}$$

due to the chaos decomposition theorem. If this representation of $X(t)$ is applied to the stochastic differential equation (3.14), then one gets with Theorem 3.25

$$\frac{d\sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle}{dt} = a(t, \sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle) + b(t, \sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle) \diamond W_t^v$$

$$\frac{d\sum_{k=0}^{\infty}\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle}{dt} = \sum_{k=0}^{n}(a_k(t)\sum_{l=0}^{\infty}\sum_{j_1+..+j_k=l}\left\langle :\cdot^{\otimes l}:, d_{j_1}(u,t)\hat{\otimes}...\hat{\otimes}d_{j_k}(u,t)\right\rangle$$
$$+ b_k(t)\sum_{l=0}^{\infty}\sum_{j_1+..+j_k=l-1}\left\langle :\cdot^{\otimes l}:, d_{j_1}(u,t)\hat{\otimes}...\hat{\otimes}d_{j_k}(u,t)\hat{\otimes}\frac{dm(u,t)}{dt}\right\rangle.$$

The summands with different l are linearly independent. Furthermore they are orthogonal to each other if both belong to (L^2). Thus for each l

$$\frac{d\left\langle :\cdot^{\otimes l}:, d_l(u,t)\right\rangle}{dt} = \sum_{k=0}^{n}a_k(t)\sum_{j_1+..+j_k=l}\left\langle :\cdot^{\otimes l}:, d_{j_1}(u,t)\hat{\otimes}...\hat{\otimes}d_{j_k}(u,t)\right\rangle$$
$$+ b_k(t)\sum_{j_1+..+j_k=l-1}\left\langle :\cdot^{\otimes l}:, d_{j_1}(u,t)\hat{\otimes}...\hat{\otimes}d_{j_k}(u,t)\hat{\otimes}\frac{dm(u,t)}{dt}\right\rangle.$$

this equation is an inhomogen linear stochastic differential equation for $\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle$, thus it is of the shape with a deterministic function $\alpha(t)$ and a function $g(t) : \mathbb{R} \to (\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} k}$

$$\frac{d\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle}{dt} = \alpha(t)\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle + \left\langle :\cdot^{\otimes k}:, g(t)\right\rangle.$$

Lemma 3.38. *The equation*

$$\frac{d\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle}{dt} = \alpha(t)\left\langle :\cdot^{\otimes k}:, d_k(u,t)\right\rangle + \left\langle :\cdot^{\otimes k}:, g(t)\right\rangle. \tag{3.16}$$

with $t \in [0,T]$ and with a continuously differentiable function $\alpha(t)$ and a function $g(t) :\to (\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} k}$ has a unique solution in $(\mathcal{S})^*$ for each $\left\langle :\cdot^{\otimes k}:, d_k(u,0)\right\rangle$.

Proof. Apply the S-transfrom on both sides of equation (3.16)

$$\frac{d\left\langle d_k(u,t), (\eta(u))^{\otimes k}\right\rangle}{dt} = \left\langle d_k(u,t), (\eta(u))^{\otimes k}\right\rangle + \left\langle g(t), (\eta(u))^{\otimes k}\right\rangle,$$

this is a deterministic inhomogen linear differential equation, which has a unique solution $F_t(\eta)$ depending on $\eta \in \mathcal{S}(\mathbb{R})$. The function $F_t(\eta)$ is multilinear in η and thus it is analytic in η. Further there exists nonnegative constants K, p and $a >$ such that

$$|f_\eta(t)| \leq K\exp(a|\eta|_p^2).$$

for all $t \in [0,T]$ So the function $F_t(\eta)$ satisfy the conditions of Theorem 2.37 and therefore it is the S-transform of a stochastic distribution process, which solves equation (3.16).

□

Theorem 3.39. *Let $a(t,X)$ and $b(t,X)$ be finite Wick polynomials as above. Then the stochastic differential equation*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB_t^v = (a(t, X(t)) + b(t, X(t)) \diamond W_t^v)dt \tag{3.17}$$

has a unique solution in $(\mathcal{S})^*$ for all $X(0) \in (\mathcal{S})^*$.

Proof. By the considerations before, one can split the differential equation into equations of the kind of (3.16). Due to Lemma (3.38) they have a unique solution. Adding up these solutions one obtains the solution of (3.17). □

Definition 3.40. *Let $P_k(\cdot) : (\mathcal{S})^* \to (\mathcal{S})^*$ be an operator defined by*

$$P_k(\sum_{n=0}^{\infty} \left\langle : \cdot^{\otimes n} :, f_n \right\rangle) := \left\langle : \cdot^{\otimes k} :, f_k \right\rangle . \tag{3.18}$$

It obviously satisfies $(P_k)^2 = P_k$ and $P_n P_m = 0$ if $n \neq m$, thus by the operator P_k it is possible to modify the stochastic process $X(t)$ in single subspaces, e.g. let $X(t) = \sum_{k=0}^{3} \left\langle : \cdot^{\otimes k} :, f_k(u,t) \right\rangle$. Then consider

$$\tilde{X}(t) := X(t) + P_2(X(t)) \diamond B_t^v = \sum_{k=0}^{3} \left\langle : \cdot^{\otimes k} :, f_k(u,t) \right\rangle + \left\langle : \cdot^{\otimes 3} :, f_2(u,t) \hat{\otimes} m(u,t) \right\rangle .$$

The process B_t^v is plugged in without changing $\sum_{k=0}^{2} \left\langle : \cdot^{\otimes k} :, f_k(u,t) \right\rangle$. Such an effect cannot be realised by $X(t) \diamond B_t^v$. The covariance function of $\tilde{X}(t)$ is

$$\begin{aligned}
&E((\tilde{X}(t) - f_0(t))(\tilde{X}(s) - f_0(s)) \\
&= \sum_{k=0}^{2} \langle f_k(u,t), f_k(u,s) \rangle + \left\langle f_3(u,t) + f_2(u,t) \hat{\otimes} m(u,t), f_3(u,s) + f_2(u,s) \hat{\otimes} m(u,s) \right\rangle \\
&= \sum_{k=0}^{3} \langle f_k(u,t), f_k(u,s) \rangle + \left\langle f_3(u,t), f_2(u,s) \hat{\otimes} m(u,s) \right\rangle \\
&\quad + \left\langle f_2(u,t) \hat{\otimes} m(u,t), f_3(u,s) \right\rangle + \langle f_2(u,t), f_2(u,s) \rangle \langle m(u,t), m(u,s) \rangle .
\end{aligned}$$

If this existence and unqiuenes theorem should be extended to functions a and b, which are infinite Wick polynoms, one has to face the problem, that the infinite Wick polynoms don't need to be defined on the whole $(\mathcal{S})^*$ as seen in Example 2.38. One can use the operator P_1 to get a wide class of functions $a, b : \mathbb{R} \times D \to (\mathcal{S})^*$ for which the stochastic differential equation 3.17 with $D := \{\langle \cdot, f \rangle, f \in \mathcal{S}'(\mathbb{R})\}$ is declared. Suppose that a, b admit a Wick representation with respect to D, then the stochastic differential equation on $[0, T]$

$$dX(t) = a(t, P_1(X(t)))dt + b(t, P_1(X(t)))dB_t^v \tag{3.19}$$

is declared for all $X(0) \in (\mathcal{S})^*$.

Theorem 3.41. *Let*

$$a(t, \langle \cdot, f\rangle) = \sum_{k=0}^{\infty} a_k(t) \langle \cdot, f\rangle^{\diamond k}$$
$$b(t, \langle \cdot, f\rangle) = \sum_{k=0}^{\infty} b_k(t) \langle \cdot, f\rangle^{\diamond k}$$

with $f \in \mathcal{S}'(\mathbb{R})$, these a, b are admitting a Wick representation with respect to $D := \{\langle \cdot, f\rangle, f \in \mathcal{S}'(\mathbb{R})\}$. Then the equation

$$dX(t) = a(t, P_1(X(t)))dt + b(t, P_1(X(t)))dB_t^v \tag{3.20}$$

has a unique solution on $[0,T]$ for all $X(0) \in (\mathcal{S})^*$ in $(\mathcal{S})^*$.

Proof. Equation (3.20) can be written as

$$\sum_{k=0}^{\infty} \frac{d\left\langle : \cdot^{\otimes k} :, d_k(u,t)\right\rangle}{dt} = \sum_{k=0}^{\infty} a_k(t)\left\langle : \cdot^{\otimes k} :, d_1(u,t)^{\otimes k}\right\rangle + b_k(t)\left\langle : \cdot^{\otimes k} :, d_1(u,t)^{\otimes k} \hat{\otimes} \frac{dm(u,t)}{dt}\right\rangle,$$

separating this equation into equations indexed by k we obtain

$$\frac{d\left\langle : \cdot^{\otimes k} :, d_k(u,t)\right\rangle}{dt} = a_k(t)\left\langle : \cdot^{\otimes k} :, d_1(u,t)^{\otimes k}\right\rangle + b_{k-1}(t)\left\langle : \cdot^{\otimes k} :, d_1(u,t)^{\otimes (k-1)} \hat{\otimes} \frac{dm(u,t)}{dt}\right\rangle$$

follows. Using the S-transfrom on both sides, one gets a deterministic equations, which have a unique solution $F_t^k(\eta)$. As $F_t^k(\eta)$ is multilinear in η it is analytic in η and satisfies the boundary condition of Theorem 2.37, so it is the S-transform of a stochastic distribution process, therefore there is unique solution, the sum $\sum_{k=0}^{\infty} F_t^k(\eta)$ is therefore the S-transform of the solution. □

Example 3.42. Let $\mu, \sigma \in \mathbb{R}$, consider the equation

$$dX(t) = \mu P_1(X(t))dt + \sigma P_1(X(t)) \diamond W_t^v dt,$$

thus the equations

$$\left\langle \cdot, \frac{dd_1(u,t)}{dt}\right\rangle = \mu \langle \cdot, d_1(u,t)\rangle, \qquad \left\langle : \cdot^{\otimes 2} :, \frac{dd_2(u,t)}{dt}\right\rangle = \left\langle : \cdot^{\otimes 2} :, d_1(u,t) \hat{\otimes} \frac{dm(u,t)}{dt}\right\rangle,$$

have unique solutions. Therefore

$$X(t) = \exp(\mu t)\langle \cdot, h(u)\rangle + \langle \cdot, h(u)\rangle \diamond (\int_0^t \exp(\mu t) dB_t^v).$$

Example 3.43. Let

$$\sin^{\diamond}(\langle \cdot, f\rangle) := \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (\langle \cdot, f\rangle)^{\diamond(2k+1)}$$

and

$$\cos^{\diamond}(\langle \cdot, f\rangle) := \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} (\langle \cdot, f\rangle)^{\diamond(2k)}$$

with $f \in \mathcal{S}'(\mathbb{R})$. Then the stochastic differential equation

$$\frac{dX(t)}{dt} = \sin^{\diamond}(P_1(X(t))) + \cos^{\diamond}(P_1(X(t))) \diamond W_t^v \tag{3.21}$$

is declared and the solution can be calculated by regarding the corresponding linear stochastic differential equations as in the proof before. Let $X(t) = \sum_{k=0}^{\infty} \left\langle : \cdot^{\otimes k} :, d_k(u,t) \right\rangle$ such that for $2k+1$

$$\begin{aligned} \frac{d\left\langle : \cdot^{\otimes(2k+1)} :, d_{2k+1}(u,t) \right\rangle}{dt} &= \frac{(-1)^{k+1}}{(2k+1)!} \left\langle : \cdot^{\otimes(2k+1)} :, d_1(u,t)^{\otimes(2k+1)} \right\rangle \\ &\quad + \frac{(-1)^{k}}{(2k)!} \left\langle : \cdot^{\otimes(2k+1)} :, (d_1(u,t))^{\otimes(2k)} \hat{\otimes} \frac{dm(u,t)}{dt} \right\rangle, \end{aligned}$$

the equations for $2k$ are

$$\frac{d\left\langle : \cdot^{\otimes(2k)} :, d_{2k}(u,t) \right\rangle}{dt} = 0.$$

The equation for $2k+1 = 1$

$$\left\langle \cdot, \frac{dd_1(u,t)}{dt} \right\rangle = \langle \cdot, d_1(u,t)\rangle + \left\langle \cdot, \frac{dm(u,t)}{dt} \right\rangle$$

is the stochastic differential equation for the Ornstein-Uhlenbeck process, which has the solution

$$\langle \cdot, d_1(u,t)\rangle = C_0 \exp(t) + \int_0^t \exp(t-s) dB_t^v. \tag{3.22}$$

the remaining the solutions for $2k+1 > 1$ are

$$\begin{aligned} \left\langle : \cdot^{\otimes(2k+1)} :, d_{2k+1}(u,t) \right\rangle &= \int_0^t \frac{(-1)^{k+1}}{(2k+1)!} \left\langle : \cdot^{\otimes(2k+1)} :, d_1(u,t)^{\otimes(2k+1)} \right\rangle \\ &\quad + \frac{(-1)^{k}}{(2k)!} \left\langle : \cdot^{\otimes(2k+1)} :, (d_1(u,t))^{\otimes(2k)} \hat{\otimes} \frac{dm(u,t)}{dt} \right\rangle dt, \end{aligned}$$

the solution for $2k$ is $\left\langle : \cdot^{\otimes(2k)} :, f_{2k} \right\rangle$ with $f_{2k} \in (\mathcal{S}'(\mathbb{R}))^{\otimes(2k)}$.

Example 3.44. Many other types of stochastic differential equations may be treated in a similar manner as above. As a last example, the SDE contains the projection $P_1(X(t))$ and the process $X(t)$ itself:

$$dX(t) = (\sin^{\diamond}(P_1(X(t))) \diamond W_t^v + X(t))\,dt.$$

This problem can be treated by solving the following sequence of differential equations

$$\begin{aligned}\left\langle :\cdot^{\otimes(2k+1)}:, \frac{d\,d_{2k+1}(u,t)}{dt}\right\rangle &= \left\langle :\cdot^{\otimes(2k+1)}:, d_{2k+1}(u,t)\right\rangle \\ \left\langle :\cdot^{\otimes(2k)}:, \frac{d\,d_{2k}(u,t)}{dt}\right\rangle &= \frac{(-1)^{k+1}}{(2k-1)!}\left\langle :\cdot^{\otimes(2k)}:, d_{2k-1}(u,t)\hat{\otimes}\frac{dm(u,t)}{dt}\right\rangle \\ &\quad + \left\langle :\cdot^{\otimes(2k)}:, d_{2k}(u,t)\right\rangle\end{aligned}$$

Their solutions are given by

$$\left\langle :\cdot^{\otimes(2k+1)}:, d_{2k+1}(u,t)\right\rangle = \exp(t)\left\langle :\cdot^{\otimes(2k+1)}:, f_{2k+1}(u)\right\rangle$$

and

$$\begin{aligned}\left\langle :\cdot^{\otimes(2k)}:, d_{2k}(u,t)\right\rangle &= C(0)\exp(t)\left\langle :\cdot^{\otimes(2k)}:, f_{2k}(u)\right\rangle \\ &\quad + \int_0^t \exp(t-s)\frac{(-1)^{k+1}}{(2k-1)!}\left\langle :\cdot^{\otimes(2k)}:, d_{2k-1}(u,s)\hat{\otimes}\frac{dm(u,s)}{ds}\right\rangle ds.\end{aligned}$$

with $f_i(u) \in (\mathcal{S}'(\mathbb{R}))^{\hat{\otimes} i}$.

Remark 3.45. For the classical existence and uniqueness theorem for stochastic differential equations driven by Brownian motion it is required that the coefficient functions are bounded by a quadratic function. This not necessary here.

3.5 Notes

Remark 3.46. The conditions to the function $m(u,t)$ are weak, so one has a large class of functions which satisfy these properties.

Remark 3.47. In the paper of [AlMaNu] the authors discuss a stochastic calculus with Gaussian processes. They use the Malliavin Calculus and their class of Gaussian processes and their techniques are different. The approach here is more general as their approach. In the paper of [NuTa] a Wick-Itô rule for Gaussian processes is proved. Their proof uses some properties of the variation of the Gaussian process. Their approach differs to the approach in the present paper. Furthermore their Wick-Itô rule is applicable for the fractional Brownian motion B_t^H for $H \in (1/4, 1)$, where the approach in the present paper allows $H \in (0,1)$. The calculus, which the authors of [NuTa] use, is based on the Malliavin calculus presented in [Nu].

4

Related Market Models with Memory

Let $m(u,t)$ satisfy the conditions in the beginning of Chaper 3 and the following assumptions. Let $|m(\cdot,t)|_0^2$ be a continuous differentiable function with respect to the variable t, $t \in [0,T]$. It will be assumed that there is an $\eta \in \mathcal{S}(\mathbb{R})$ such that

$$1_{[0,T]}(t) = \left\langle \frac{dm(u,t)}{dt}, \eta(u) \right\rangle \tag{4.1}$$

for almost all $t \in [0,T]$. This is met in many cases.

Example 4.1. In the case of the Brownian motion, it follows $\langle \delta_t, \eta \rangle = \eta(t) = 1_{[0,T]}(t)$ with $t \in [0,T]$ can be satisfied on $[0,T]$ with an η, which is 1 on $[0,T]$ and decays outside of this intervall smoothly to 0. The equation is also satisfied for the fractional Brownian motion. The function $1_{[0,T]}(t)$ can be extended to the domain $t \in \mathbb{R}$ smoothly such that it has compact support, and it is therefore in the domain of the operator $M_{\pm}^H$. By using the inverse operator the existence of $\eta \in \mathcal{S}(\mathbb{R})$ is obtained (see [Be] Chapter 1 especially Proposition 1.6.9 for technical details).

A discussion of equations of the type (4.1) can be found in [Ho].

4.1 Pricing in a Black-Scholes market with memory

The aim of this section is to derive a pricing formula for options. To begin with we discuss the properties of our market model. Its formulation is similar to [HuOk].

4.1.1 Market assumptions

Suppose the market offers two types of investment assets. Fix some $T > 0$. Firstly, a bond $A(t)$ with constant interest rate r follows

$$\frac{dA(t)}{dt} = r\,A(t) \tag{4.2}$$

with $t \in [0,T]$ and $A(0) = 1$. Secondly, consider a stock whose price process $S(t)$ is the solution of the stochastic differential equation

$$\frac{dS(t)}{dt} = \mu S(t) + \sigma S(t) \diamond W_t^v, \tag{4.3}$$

with the constants $\mu \in \mathbb{R}$ and $\sigma > 0$ and $t \in [0,T]$. Its unique solution is given by

$$S(t) = S(0)\exp(\mu t + \sigma B_t^v - 1/2\sigma^2 |m(\cdot,t)|_0^2).$$

A portfolio or a trading strategy is given by a two dimensional process $\theta(t,S(t)) = (g(t,S(t)), h(t,S(t)))$, where g and h are the quantities of bonds and stocks held at time t, respectively. In preceeding papers about fractional Brownian motion in finance at least two possibilities to model the value of a portfolio are suggested (see [BjHu]). One of them says that the wealth process of the *Wick portfolio* is

$$V^w(t,\theta) = g^w(t,S(t)) \diamond A(t) + h^w(t,S(t)) \diamond S(t) = g^w(t,S(t)) \cdot A(t) + h^w(t,S(t)) \diamond S(t).$$

Here the Wick product at $A(t)$ is changed to the ordinary product because $A(t)$ is deterministic. The second possibility models the wealth process of the portfolio with ordinary product in both assets, thus

$$V^o(t,\theta) = g^o(t,S(t)) \cdot A(t) + h^o(t,S(t)) \cdot S(t).$$

In [BjHu] there were several critical comments on the wealth process of the Wick portfolio. These arguments would normally force to deny the Wick portfolio. But we will show that under certain circumstances one can deduce from a given portfolio with ordinary products a Wick portfolio, such that both portfolios are equal in (L^2). Let

$$D := \{S(0)\exp(\mu t + \sigma B_t^v - 1/2\sigma^2 |m(u,t)|_0^2) : t \in [0,T]\}. \tag{4.4}$$

Here D is the set of random variables describing the stock prices, which can occur in $[0,T]$. Let $g^o, h^o : [0,T] \times \mathbb{R} \to \mathbb{R}$, such that $g^o(t,x) = \sum_{k=0}^{\infty} g_k^o(t)x^k$ and $h^o(t,x) = \sum_{k=0}^{\infty} h_k^o(t)x^k$ for all $x \in \mathbb{R}$ and continuously differentiable $g_k^o(t)$ and $h_k^o(t)$ and suppose $g^o(t,X), h^o(t,X) \in (L^2)$ for all $X \in D$ and for all $t \in [0,T]$.

Theorem 4.2. *Let D, g^o and h^o be as above. Then there exists a unique Wick portfolio wealth process $V^w(t,\theta)$ such that for all $t \in [0,T]$ it holds a.s. that*

$$\begin{aligned} V^o(t,\theta) &= g^o(t,S(t)) \cdot A(t) + h^o(t,S(t)) \cdot S(t) \\ &= g^w(t,S(t)) \diamond A(t) + h^w(t,S(t)) \diamond S(t) = V^w(t,\theta), \end{aligned}$$

where g^w, h^w admit a Wick representation in D. Conversely for a Wick portfolio wealth process $V^w(t,\theta)$, where g^w and h^w admit a Wick representation in D, there is a unique wealth process of a portfolio with ordinary products $V^o(t,\theta)$, such that $V^o(t,\theta) = V^w(t,\theta)$ a.s. and for all $t \in [0,T]$.

Proof. Calculate for $n \in \mathbb{N}_0$, where $X^{\diamond 0} := 1$ with $X \in D$,

$$\begin{aligned}
(S(t))^{\diamond n} &= (S(0))^n (\exp^{\diamond}(\mu t + \sigma B_t^v))^{\diamond n} \\
&= (S(0))^n \exp(n\mu t)(\exp^{\diamond}(\langle \cdot, \sigma m(u,t)\rangle))^{\diamond n} \\
&= (S(0))^n \exp(n\mu t) \exp^{\diamond}(\langle \cdot, n\sigma m(u,t)\rangle) \\
&= (S(0))^n \exp(n\mu t + n\sigma B_t^v - \frac{1}{2} n^2 \sigma^2 |m(\cdot,t)|_0^2) \\
&= (S(0))^n \exp(n\mu t + n\sigma B_t^v - \frac{1}{2} n \sigma^2 |m(\cdot,t)|_0^2) \exp(\frac{1}{2}(n - n^2)\sigma^2 |m(\cdot,t)|_0^2) \\
&= (S(0))^n (\exp(\mu t + \sigma B_t^v - \frac{1}{2} \sigma^2 |m(\cdot,t)|_0^2))^n \exp(\frac{1}{2}(n - n^2)\sigma^2 |m(\cdot,t)|_0^2) \\
&= (S(t))^n \exp(\frac{1}{2}(n - n^2)\sigma^2 |m(\cdot,t)|_0^2).
\end{aligned}$$

Applying this identity to the portfolio with ordinary products

$$\begin{aligned}
V^o(t,\theta) &= g^o(t,S(t)) \cdot A(t) + h^o(t,S(t)) \cdot S(t) \\
&= \sum_{k=0}^{\infty} A(t) g_k^o(t) (S(t))^k + \sum_{k=0}^{\infty} h_k^o(t) (S(t))^{k+1} \\
&= \sum_{k=0}^{\infty} A(t) g_k^o(t) \exp(\frac{1}{2}(k^2 - k)\sigma^2 |m(u,t)|_0^2)(S(t))^{\diamond k} \\
&\quad + \sum_{k=0}^{\infty} h_k^o(t) \exp(\frac{1}{2}((k+1)^2 - (k+1))\sigma^2 |m(\cdot,t)|_0^2)(S(t))^{\diamond (k+1)} \\
&= \left(\sum_{k=0}^{\infty} g_k^w(t) (S(t))^{\diamond k} \right) A(t) + \left(\sum_{k=0}^{\infty} h_k^w(t) (S(t))^{\diamond k} \right) \diamond S(t) \\
&= g^w(t,S(t)) \diamond A(t) + h^w(t,S(t)) \diamond S(t) = V^w(t,\theta),
\end{aligned}$$

where

$$g_k^w(t) := g_k^o(t) \exp\left(\frac{1}{2}(k^2 - k)\sigma^2 |m(\cdot,t)|_0^2 \right)$$

and

$$h_k^w(t) := h_k^o(t) \exp\left(\frac{1}{2}((k+1)^2 - (k+1))\sigma^2 |m(\cdot,t)|_0^2 \right).$$

The uniqueness of the Wick representation follows by the fact that the Wick product of two Hida distributions is unique. □

This justifies the use of the Wick portfolio and we assume the wealth process $V(t,\theta) := V^w(t,\theta)$ to have the representation of the Wick portfolio, where g, h admit a Wick representation in D.

Definition 4.3. *A trading strategy is* self-financing *if*

$$V(t,\theta) - V(0,\theta) = \int_0^t g(t,S(t))\, dA(t) + \int_0^t h(t,S(t))\, dS(t)$$

where integrals are defined in the $(\mathcal{S})^*$*-sense.*

Definition 4.4. *The portfolio θ is called an* arbitrage *for the market given by (4.2) and (4.3), if it is self-financing and $V(0,\theta)=0$, $V(T,\theta)\geq 0$ and $\mu(\{\omega; V(T,\theta)(\omega)>0\})>0$. Let Θ be the class of self-financing trading strategies in this market.*

Theorem 4.5. *There is no arbitrage in the class of self-financing trading strategies Θ.*

Proof. The wealth process $V(t,\theta)$ of a self-financing trading strategy θ satisfies with $v(t,S(t)) = g(t,S(t))A(t) + h(t,S(t)) \diamond S(t)$

$$\begin{aligned} V(T,\theta) &= V(0,\theta) + \int_0^T g(u,S(u))rA(u)\,du \\ &\quad + \int_0^T \mu\, h(u,S(u)) \diamond S(u)\,du + \int_0^T \sigma\, h(u,S(u)) \diamond S(u)\,dB_u^v \\ &= V(0,\theta) + \int_0^T v(u,S(u))r\,du \\ &\quad + \int_0^T (\mu - r)h(u,S(u)) \diamond S(u)\,du + \int_0^T \sigma\, h(u,S(u)) \diamond S(u)\,dB_u^v. \end{aligned}$$

According to (4.1) there exists a $\tilde{\eta} \in \mathcal{S}(\mathbb{R})$ such that

$$\left\langle \frac{dm(u,t)}{dt}, \tilde{\eta}(u) \right\rangle = \frac{r-\mu}{\sigma} 1_{[0,T]}(t).$$

Evaluating the S-transform of above with $\tilde{\eta}$ we get

$$S(V(T,\theta))(\tilde{\eta}) = V(0,\theta) + \int_0^T rS(V(u,\theta))(\tilde{\eta})\,du.$$

The solution of this integral equation is

$$S(V(T,\theta))(\tilde{\eta}) = V(0,\theta)\exp(rT).$$

Suppose that $V(0,\theta)=0$, then $S(V(T,\theta))(\tilde{\eta})=0$. Because the measures μ and $\mu_\eta(\cdot) := S(1(\cdot))(\eta)$, the latter induced by the S-transform, are equivalent (for further details see [Be], Chapters 2 and 6), it follows that $V(T,\theta)=0$ with respect to the measure μ. □

Now we consider the class of path independent contingent claims $\mathcal{X}$ with expiry date T consisting of those contingent claims $X(T)$, whose payoff functions admit a representation $p(T,S(T)) = \sum_{k=0}^{\infty} p_k (S(T))^{\diamond k}$ and satisfy $p(T,S(T)) \in (L^2)$. Due to arguments above one can also deduce a power series representation of $p(T,S(T))$ in $S(T)$ with ordinary products. This market with (4.2) and (4.3) is called the *Black-Scholes market with memory*, where the memory is determined by the covariance function $v(s,t)$. As in the risk neutral valuation approach the no arbitrage price of a contingent claim at time 0 is explained by the value of the replicating portfolio at time 0. The latter value is examined in the next subsection.

4.1.2 Pricing of contingent claims in the Black-Scholes market with memory

A contingent claim in the market is *attainable*, if there exists a self-financing trading strategy θ that replicates the contingent claim X.

Theorem 4.6 (Pricing of contingent claims). *The contingent claim $X \in \mathcal{X}$ is attainable in the Black-Scholes market with memory. The value process $v(t, S(t)) = V(t, \theta)$ of the portfolio θ replicating the contingent claim X satisfies the following stochastic partial differential equation*

$$rv_s(t, S(t)) \diamond S(t) - rv(t, S(t)) + v_t(t, S(t)) = 0 \tag{4.5}$$

with boundary condition $v(T, S(T)) = p(T, S(T))$.

Proof. The wealth process is defined by $v(t, S(t)) = g(t, S(t))A(t) + h(t, S(t)) \diamond S(t)$. Apply the chain rule from Theorem 3.36 with D as in (4.4) to the wealth process and get

$$v(t, S(t)) - v(0, S(0)) = \int_0^t v_t(u, S(u))du + \int_0^t v_s(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)du.$$

Because the replicating portfolio is self-financing it follows

$$v(t, S(t)) - v(0, S(0)) = \int_0^t g(u, S(u))rA(u)\, du + \int_0^t h(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)\, du.$$

Comparing both results one gets

$$\begin{aligned}&\int_0^t v_t(u, S(u))\, du + \int_0^t v_s(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)\, du \\ &= \int_0^t g(u, S(u))rA(u)\, du + \int_0^t h(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)\, du\end{aligned}$$

and this becomes to

$$\begin{aligned}&\int_0^t (v_t(u, S(u)) + \mu v_s(u, S(u)) \diamond S(u) - g(u, S(u))rA(u) - \mu h(u, S(u)) \diamond S(u))\; du \\ &+ \int_0^t \sigma(v_s(u, S(u)) \diamond S(u) - h(u, S(u)) \diamond S(u))\, dB_t^v = 0.\end{aligned}$$

By regarding the S-transform one can deduce that a.s. and for each $u \in [0, T]$

$$v_s(u, S(u)) - h(u, S(u)) = 0 \tag{4.6}$$

and

$$v_t(u, S(u)) + \mu v_s(u, S(u)) \diamond S(u) - g(u, S(u))rA(u) - \mu h(u, S(u)) \diamond S(u) = 0.$$

By applying $-rv(t, S(t)) + rh(t, S(t)) \diamond S(t) = -rg(t, S(t))A(t)$ it holds

$$v_t(u, S(u)) - rv(t, S(t)) + rv_s(u, S(u)) \diamond S(t) = 0,$$

thus the pricing formula holds. In order to show that the contingent claim is attainable, investigate the payoff function $p(T,S(T)) = \sum_{k=0}^{\infty} p_k (S(T))^{\diamond k}$. The wealth process $v(t,S(t))$ also admits a Wick representation $\sum_{k=0}^{\infty} v_k(t)(S(t))^{\diamond k}$. Therefore it follows

$$\sum_{k=0}^{\infty} v_k(T)(S(T))^{\diamond k} = \sum_{k=0}^{\infty} p_k (S(T))^{\diamond k},$$

hence $v_k(T) = p_k$. If one plugs $\sum_{k=0}^{\infty} v_k(t)(S(t))^{\diamond k}$ in the pricing formula and compare the coefficients of $(S(t))^{\diamond k}$, the equation for $k \in \mathbb{N}_0$

$$\frac{dv_k(t)}{dt} = (r - rk)v_k(t)$$

follows. So $v_k(t) = c_k \exp((r - rk)t)$ and $p_k = c_k \exp((r - rk)T)$ with the constants c_k. This shows that the contingent claim can be replicated. The self-financing replicating portfolio is given by (4.6) and by

$$g(t,S(t)) = \frac{v(t,S(t)) - v_s(t,S(t)) \diamond S(t)}{A(t)}.$$

□

Due to considerations above this stochastic partial differential equation (4.5) admits a representation with ordinary products.

Theorem 4.7 (The pricing partial differential equation). *The stochastic partial differential equation with boundary condition in Theorem 4.6 can be transformed into the deterministic partial differential equation*

$$v_t^o(t,s) + rsv_s^o(t,s) + \frac{1}{2}\sigma^2 \frac{d|m(u,t)|_0^2}{dt} s^2 v_{ss}^o(t,s) = rv^o(t,s), \tag{4.7}$$

where $v^o(T,s) = p(T,s)$ and $v^o(t,S(t))$ denotes the representation of the wealth process with ordinary product.

Proof. The wealth process admits a Wick representation, so

$$v(t,S(t)) = \sum_{k=0}^{\infty} a_k(t)(S(t))^{\diamond k}.$$

Now regard the partial derivatives in the stochastic partial differential equation involving the Wick representation

$$v_t(t,S(t)) = \sum_{k=0}^{\infty} \frac{da_k(t)}{dt}(S(t))^{\diamond k}$$

$$v_s(t,S(t)) = \sum_{k=0}^{\infty} a_k(t)k(S(t))^{\diamond(k-1)}.$$

The derivative with respect to t of the representation with ordinary product as in the proof of Theorem 4.2

$$v^o(t,S(t)) = \sum_{k=0}^{\infty} a_k(t) \exp(1/2(k-k^2)\sigma^2 |m(\cdot,t)|_0^2)(S(t))^k$$

is

$$\begin{aligned}
v_t^o(t,S(t)) = &\sum_{k=0}^{\infty} \frac{da_k(t)}{dt} \exp\left(\frac{k-k^2}{2}\sigma^2|m(\cdot,t)|_0^2\right)(S(t))^k \\
&- \sum_{k=0}^{\infty} a_k(t)\frac{1}{2}(k-1)k\sigma^2 \frac{d|m(\cdot,t)|_0^2}{dt} \exp\left(\frac{k-k^2}{2}\sigma^2|m(\cdot,t)|_0^2\right)(S(t))^k \\
= &\sum_{k=0}^{\infty} \frac{da_k(t)}{dt} \exp\left(\frac{k-k^2}{2}\sigma^2|m(\cdot,t)|_0^2\right)(S(t))^k \\
&- (S(t))^2 \frac{\sigma^2}{2}\frac{d|m(\cdot,t)|_0^2}{dt} \sum_{k=0}^{\infty} a_k(t)(k-1)k \exp\left(\frac{k-k^2}{2}\sigma^2|m(\cdot,t)|_0^2\right)(S(t))^{k-2} \\
= &\, v_t(t,S(t)) - (S(t))^2 \frac{\sigma^2}{2}\frac{d|m(\cdot,t)|_0^2}{dt} v_{ss}^o(t,S(t)).
\end{aligned}$$

It holds that $S(t) \cdot v_s^o(t,S(t)) = S(t) \diamond v_s(t,S(t))$. By substituting all terms one gets the partial differential equation with the same boundary condition. □

Remark 4.8. This pricing equation involves the classical case with ordinary Brownian motion, because

$$\frac{d|1_{[0,t]}|_0^2}{dt} = \frac{d\,t}{dt} = 1.$$

Remark 4.9. Some contingent claims have a payoff function $p(T,S(T))$ which is not smooth with respect to the stock price $S(T)$. On first view this may be in conflict with smoothness requirements on the wealth process. However this problem can be overcome as in the classical case, where the partial differential equation is solved on $[0,T)$ having the solution $v(t,S(t))$, if $\lim_{t\to T} v(t,S(t)) = p(T,S(T))$, which is met in many cases, the boundary condition can be satisfied. So the payoff function $p(T,S(T))$ does not need to satisfy smoothness properties.

An European call option EC_K is a contingent claim, whose payoff function is $p(T,S(T)) = \max\{(S(T)-K),0\}$. Instead of trying to investigate whether EC_K belongs to the set of admissible contingent claims $\mathcal{X}$, one can price the European call option by solving the partial differential equation. But it is necessary that the wealth process $v_{EC_K}^o(t,S(t))$ admits a Wick representation in $S(t)$, so the stochastic partial differential equation (4.5) can be satisfied, too. The wealth process of the European call is derived by solving the deterministic partial differential equation. Let $N(x) := (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-z^2/2)dz$.

Theorem 4.10. *Let $|m(\cdot,t)|_0^2$ additionally to the conditions at the begin of this Chapter be strictly monotone increasing and let*

$$\lim_{t\to T}\frac{(T-t)}{\sqrt{|m(\cdot,T)|_0^2-|m(\cdot,t)|_0^2}}<\infty.$$

Then the value process of the European call option in the Black-Scholes market with memory is

$$v^o(t,s)=sN(d_1(t,s))-K\exp(-r(T-t))N(d_2(t,s)),$$

where

$$d_1(t,s):=\frac{\ln(s/K)+r(T-t)+1/2\sigma^2(|m(\cdot,T)|_0^2-|m(u,t)|_0^2)}{\sigma\sqrt{(|m(\cdot,T)|_0^2-|m(\cdot,t)|_0^2)}}$$

and

$$d_2(t,s):=\frac{\ln(s/K)+r(T-t)-1/2\sigma^2(|m(\cdot,T)|_0^2-|m(\cdot,t)|_0^2)}{\sigma\sqrt{(|m(\cdot,T)|_0^2-|m(\cdot,t)|_0^2)}}$$

Proof. The solution satisfies the partial differential equation with the boundary condition. Note that

$$rv^o(t,s)=rsN(d_1(t,s))-rK\exp(-r(T-t))N(d_2(t,s))=:D_1+D_2,$$

With $(dN(t))/(dt)=:\phi(t)$ and $\beta(t):=\sigma\sqrt{|m(\cdot,T)|_0^2-|m(\cdot,t)|_0^2}$ the derivatives turn out to be

$$\begin{aligned}
&v_t^o(t,s)\\
&=s\phi(d_1(t,s))\frac{\partial d_1(t,s)}{\partial t}-rK\exp(-r(T-t))N(d_2(t,s))\\
&\quad-K\exp(-r(T-t))\phi(d_2(t,s))\frac{\partial d_2(t,s)}{\partial t}\\
&=s\phi(d_1(t,s))\left(\frac{-r}{\beta(t)}-\frac{\frac{1}{2}\sigma^2\frac{d|m(\cdot,t)|_0^2}{dt}}{\beta(t)}+\sigma^2\frac{(\ln(s/k)+r(T-t)+\frac{1}{2}(\beta(t))^2)\frac{1}{2}\frac{d|m(\cdot,t)|_0^2}{dt}}{(\beta(t))^3}\right)\\
&\quad-rK\exp(-r(T-t))N(d_2(t,s))\\
&\quad-K\exp(-r(T-t))\phi(d_2(t,s))\cdot\\
&\qquad\cdot\left(\frac{-r}{\beta(t)}+\frac{\frac{1}{2}\sigma^2\frac{d|m(u,t)|_0^2}{dt}}{\beta(t)}+\sigma^2\frac{(\ln(s/k)+r(T-t)-\frac{1}{2}(\beta(t))^2)\frac{1}{2}\frac{d|m(\cdot,t)|_0^2}{dt}}{(\beta(t))^3}\right)\\
&=:D_3+D_4+D_5+D_6+D_7+D_8+D_9,
\end{aligned}$$

where $D_3=s\phi(d_1(t,s))(-r)/\beta(t)$ and the same way for D_7. The term in (4.7) including the partial derivative with respect to s can be computed as follows

$$\begin{aligned}
&rsv_s^o(t,s)\\
&=rsN(d_1(t,s))+rs^2\phi(d_1(t,s))\frac{1}{s\beta(t)}-rsK\exp(-r(T-t))\phi(d_2(t,s))\frac{1}{s\beta(t)}\\
&=:D_{10}+D_{11}+D_{12}.
\end{aligned}$$

The term in (4.7) including the second partial derivative with respect to s equals

$$\begin{aligned}
&\frac{1}{2}\sigma^2 s^2 \frac{d|m(\cdot,t)|_0^2}{dt} v_{ss}^o(t,s)\\
&= \sigma^2 s^2 \frac{d|m(\cdot,t)|_0^2}{dt}\phi(d_1(t,s))\frac{1}{s\beta(t)}\\
&\quad -\frac{1}{2}\sigma^2 s^3 \frac{d|m(\cdot,t)|_0^2}{dt}\phi(d_1(t,s))d_1(t,s)\frac{1}{s^2(\beta(t))^2}\\
&\quad -\frac{1}{2}\sigma^2 s^3 \frac{d|m(u,t)|_0^2}{dt}\phi(d_1(t,s))\frac{1}{s^2\beta(t)}\\
&\quad +\frac{1}{2}\sigma^2 s^2 \frac{d|m(\cdot,t)|_0^2}{dt}K\exp(-r(T-t))\phi(d_2(t,s))d_2(t,s)\frac{1}{s^2(\beta(t))^2}\\
&\quad +\frac{1}{2}\sigma^2 s^2 \frac{d|m(\cdot,t)|_0^2}{dt}K\exp(-r(T-t))\phi(d_2(t,s))\frac{1}{s^2\beta(t)}\\
&=: D_{13}+D_{14}+D_{15}+D_{16}+D_{17}.
\end{aligned}$$

Now compare the summands and notice that $D_1 = D_{10}$, $D_2 = D_6$, $D_3 = D_{11}$, $D_4 + D_{15} = D_{13}$, $D_5 = D_{14}$, $D_7 = D_{12}$, $D_8 = D_{17}$ and $D_9 = D_{16}$.

The wealth process v^o admits a Wick representation, because it has a representation with power series in s with convergence radius equal to infinity with each $t \in [0,T)$ and the process satisfies $v(t,S(t)) \in (L^2)$, so the Wick representation of the wealth process solves the stochastic partial differential equation in Theorem 4.6. Therefore it is the wealth process of the replicating portfolio. □

Remark 4.11. It was assumed that μ, σ and r are constants. But without effort one can formulate the results with deterministic and continuously differentiable $\mu(t), \sigma(t)$ and $r(t)$, where $\sigma(t) > c$ with a positiv constant c.

The Greeks for the price of the European call option are the Delta Δ, which is the partial derivative of $v^o(t,s)$ with respect to s with the notation of the proof of Theorem 4.10

$$\begin{aligned}
\Delta &= \frac{\partial v^o(t,s)}{\partial s}\\
&= N(d_1(t,s)) + s\phi(d_1(t,s))\frac{1}{s\beta(t)} - K\exp(-r(T-t))\phi(d_2(t,s))\frac{1}{s\beta(t)}.
\end{aligned}$$

The second partial derivative Gamma Γ of $v^o(t,s)$ with respect to s is

$$\begin{aligned}
\Gamma = v_{ss}^o(t,s) &= 2\phi(d_1(t,s))\frac{1}{s\beta(t)} - s\phi(d_1(t,s))d_1(t,s)\frac{1}{s^2(\beta(t))^2}\\
&\quad - s\phi(d_1(t,s))\frac{1}{s^2\beta(t)} + K\exp(-r(T-t))\phi(d_2(t,s))d_2(t,s)\frac{1}{s^2(\beta(t))^2}\\
&\quad + K\exp(-r(T-t))\phi(d_2(t,s))\frac{1}{s^2\beta(t)}.
\end{aligned}$$

The derivative vega ν cannot sensibly defined as the partial derivative of $v^o(t,s)$ with respect to σ, because the process B_t^v is not further specified, it can contain also the σ. Furthermore the constant σ could be a part of B_t^v, such that it does not appear in the formulas of Theorem 4.10.

4.2 Bond pricing in short rate models with memory

One can find a lot of short rate models. Many of them could be generalized to a short rate model with memory if the Brownian motion is replaced by a Gaussian process with dependent increments B_t^v and if the equation is regarded in the $(\mathcal{S})^*$-sense. In the following Subsection the Vasicek Model with memory will be presented and a zero coupon bond with this short rate will be priced. Further information about bond pricing can be found in [BrMe].

4.2.1 Vasicek model with memory

As already mentioned the stochastic differential equation for the short rate in the Vasicek model is given by

Definition 4.12. *The* short rate $r(t)$ *is the unique solution of*

$$\frac{dr(t)}{dt} = \kappa(\theta - r(t)) + \sigma W_t^v \tag{4.8}$$

with the constant initial value $r(0)$ *and* $t \in [0,T]$.

The solution of equation (4.8) is

$$r(t) = r(0)\exp(-\kappa t) + \theta(1 - \exp(-\kappa t)) + \int_0^t \sigma \exp(-\kappa(t-s))\, dB_s^v,$$

the solution follows from before. Thus the expectation value of the short rate is

$$S(r(t))(0) = E(r(t)) = r(0)\exp(-\kappa t) + \theta(1 - \exp(-\kappa t))$$

and the variance of the short rate is with equation (3.15)

$$E(r(t) - E(r))^2 = E\left(\int_0^t \sigma \exp(-\kappa(t-s)) dB_s^v\right)^2 \tag{4.9}$$

$$= \sigma^2 v(t,t) - 2\sigma^2 \int_0^t \kappa \exp(-\kappa(t-s)) v(t,s) ds \tag{4.10}$$

$$+ \sigma^2 \int_0^t \int_0^t \kappa^2 \exp(-2\kappa t + \kappa(s+u)) v(u,s) ds du. \tag{4.11}$$

It is desirable that the wealth process of an account is

$$A(t) = A(0)\exp\left(\int_0^t r(s) ds\right),$$

this wealth process has a representation with the Wick exponential of the form

$$A(t) = A(0)\exp\left(\int_0^t r(s)ds\right) = A(0)\exp^{\diamond}\left(\int_0^t r(s)ds + \frac{1}{2} \parallel \int_0^t \int_0^s \sigma\exp(-\kappa(s-u))dB_u^v ds \parallel_0^2\right),$$

set $\gamma(t) := \frac{1}{2} \parallel \int_0^t \int_0^s \sigma\exp(-\kappa(s-u))dB_u^v ds \parallel_0^2$. Let $F^T(t,r(t))$ be the wealth process of a zero coupon bond with maturity T, thus it holds that $F^T(T,r(T)) = 1$.

Definition 4.13. *Let $t \in [0,T]$ with $T > 0$. The wealth process $V^o(t)$ of an* ordinary portfolio *$(\phi_A^o(t), \phi_T^o(t))$ consisting of an account $A(t)$ and of a zero coupon bond F^T is with $\phi_A^o(t), \phi_T^o(t) \in (L^2)$ defined by*

$$V^o(t) := \phi_A^o(t)\cdot A(t) + \phi_T^o(t)\cdot F^T(t,r(t)),$$

such that $V^o(t) \in (L^2)$. The wealth process $V^w(t)$ of a Wick portfolio *$(\phi_A^w(t), \phi_T^w(t))$ consisting of an account $A(t)$ and of a zero coupon bond F^T is with $\phi_A^w(t), \phi_T^w(t) \in (L^2)$ defined by*

$$V^w(t) = \phi_A^w(t)\diamond A(t) + \phi_T^w(t)\diamond F^T(t,r(t)),$$

such that $V^w(t) \in (L^2)$.

It is without loss of generality possible to assume that $F^T(t,r(t))$ is exponentially affin linear with respect to the short rate $r(t)$, thus the F^T is of the shape

$$F^T(t,r(t)) = \exp(a^o(t) + b^o(t)r(t)),$$

with two deterministic functions a^o, b^o, this can be written as a Wick exponential by

$$F^T(t,r(t)) = \exp^{\diamond}(a^o(t) + b^o(t)r(t) + \frac{(b^o(t))^2}{2} \parallel \int_0^t \sigma\exp(-\kappa(t-s))dB_s^v \parallel_0^2).$$

In the following the notation $F^T(t,r(t)) = \exp^{\diamond}(a(t)+b(t)r(t))$, where $a(t) := a^o(t) + \frac{(b^o(t))^2}{2} \parallel \int_0^t \sigma\exp(-\kappa(t-s))dB_s^v \parallel_0^2$ and $b(t) := b^o(t)$, will be used. Thus it is possible to define the Wick inverse of $F^T(t,r(t))$ defined by $(F^T(t,r(t)))^{-\diamond} := \exp^{\diamond}(-a(t) - b(t)r(t))$, the necessary algebraic properties are obtained if one regards the S-transform, so $(F^T(t,r(t)))^{-\diamond}\diamond F^T(t,(t)) = 1$. In the following it will be proved that

$$b(t) = \frac{\exp(-k(T-t)) - 1}{k} \tag{4.12}$$

and

$$a(t) = (\zeta - \kappa\theta)\frac{1/k(\exp(-k(T-t)) - 1) + T - t}{k} \tag{4.13}$$

$$+ \frac{\parallel \int_0^t \int_0^s \sigma\exp(-\kappa(s-u))\, dB_u^v\, ds \parallel_0^2}{2} - \frac{\parallel \int_0^T \int_0^s \sigma\exp(-\kappa(s-u))\, dB_u^v\, ds \parallel_0^2}{2} \tag{4.14}$$

$$= (\frac{\zeta}{\kappa} - \theta)(b(t) + T - t) + \frac{\parallel \int_0^t \int_0^s \sigma\exp(-\kappa(s-u))\, dB_u^v\, ds \parallel_0^2}{2} \tag{4.15}$$

$$- \frac{\parallel \int_0^T \int_0^s \sigma\exp(-\kappa(s-u))\, dB_u^v\, ds \parallel_0^2}{2}. \tag{4.16}$$

with a not specified constant ζ. The functions a and b satisfy the differential equations

$$\frac{db(t)}{dt} = 1 + \kappa b(t)$$

with the condition $b(T) = 0$ and

$$\frac{da(t)}{dt} = (\zeta - \kappa\theta)b(t) + \frac{d\gamma(t)}{dt}$$

with the condition $a(T) = 0$.

Theorem 4.14. *For an ordinary portfolio* $(\phi_A^o(t), \phi_T^o(t))$ *there exists a Wick portfolio* $(\phi_A^w(t), \phi_T^w(t))$ *such that*

$$V^w(t) = V^o(t), \tag{4.17}$$

a.s. and for all $t \in [0,T]$. *For a Wick portfolio* $(\phi_A^w(t), \phi_T^w(t))$ *there exists an ordinary portfolio* $(\phi_A^o(t), \phi_T^o(t))$ *such that equation (4.17) holds a.s. and for all* $t \in [0,T]$.

Proof. As the set $\{\exp^\diamond(\langle\cdot, f(u)\rangle, f \in L^2(\mathbb{R})\}$ is total in (L^2) it follows that

$$\phi_A^o(t) = \sum_{k=0}^{\infty} a_k(t) \exp^\diamond(\langle\cdot, f_k(u,t)\rangle)$$

with $a_k(\cdot) : [0,T] \to \mathbb{C}$ and $f_k(\cdot, t) \in L^2(\mathbb{R})$ for all $k \geq 0$ and for all $t \in [0,T]$. Similarly for $\phi_T^o(t)$ it holds that

$$\phi_T^o(t) = \sum_{k=0}^{\infty} b_k(t) \exp^\diamond(\langle\cdot, g_k(u,t)\rangle)$$

with $b_k(\cdot) : [0,T] \to \mathbb{C}$ and $g_k(\cdot, t) \in L^2(\mathbb{R})$ for all $k \geq 0$ and for all $t \in [0,T]$. Regarding the wealth process $V^o(t)$ it follows

$$\begin{aligned}
V^o(t) &= \phi_A^o(t) \cdot A(t) + \phi_T^o(t) \cdot F^T(t, r(t)) \\
&= \sum_{k=0}^{\infty} a_k(t) \exp^\diamond(\langle\cdot, f_k(u,t)\rangle) \cdot \exp^\diamond\left(\int_0^t r(s)ds + \gamma(t)\right) \\
&\quad + \sum_{k=0}^{\infty} b_k(t) \exp^\diamond(\langle\cdot, g_k(u,t)\rangle) \cdot \exp^\diamond(a(t) + b(t)r(t)) \\
&= \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(-|f_k(u,t)|_0^2 - 2\gamma(t) + \left|f_k(u,t) + \int_0^t \int_0^s \sigma \exp(-\kappa(s-w)) \frac{dm(u,w)}{dw} dwds\right|_0^2\right)\right) \\
&\quad a_k(t) \exp^\diamond(\langle\cdot, f_k(u,t)\rangle) \diamond \exp^\diamond\left(\int_0^t r(s)ds + \gamma(t)\right) \\
&\quad + \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(-|g_k(u,t)|_0^2\right.\right. \\
&\quad \left.\left. - \left|b(t)\int_0^t \exp(-\kappa(t-s))\frac{dm(u,s)}{ds} ds\right|_0^2 + \left|f_k(u,t) + \int_0^t b(t)\sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds\right|_0^2\right)\right) \\
&\quad b_k(t) \exp^\diamond(\langle\cdot, g_k(u,t)\rangle) \diamond \exp^\diamond(a(t) + b(t)r(t))
\end{aligned}$$

So set

$$\phi_A^w(t)$$

$$:= \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(-|f_k(u,t)|_0^2 - 2\gamma(t) + \left| f_k(u,t) + \int_0^t \int_0^s \sigma \exp(-\kappa(s-w)) \frac{dm(u,w)}{dw} dw ds \right|_0^2\right)\right)$$

$$a_k(t) \exp^{\diamond}(\langle \cdot, f_k(u,t) \rangle)$$

and

$$\phi_T^w(t) := \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(-|g_k(u,t)|_0^2 - |b(t) \int_0^t \sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds|_0^2 + \left| f_k(u,t) + b(t) \int_0^t \sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds \right|_0^2\right)\right) b_k(t) \exp^{\diamond}(\langle \cdot, g_k(u,t) \rangle)$$

and the Wick portfolio is derived. The ordinary portfolio is obtained from a Wick portfolio in the following way. Let

$$\phi_A^w(t) := \sum_{k=0}^{\infty} a_k(t) \exp^{\diamond}(\langle \cdot, f_k(u,t) \rangle)$$

and

$$\phi_T^o(t) := \sum_{k=0}^{\infty} b_k(t) \exp^{\diamond}(\langle \cdot, g_k(u,t) \rangle).$$

Set

$$\phi_A^o(t) :=$$

$$\sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(|f_k(u,t)|_0^2 + 2\gamma(t) - \left| f_k(u,t) + \int_0^t \int_0^s \sigma \exp(-\kappa(s-w)) \frac{dm(u,w)}{dw} dw ds \right|_0^2\right)\right)$$

$$a_k(t) \exp^{\diamond}(\langle \cdot, f_k(u,t) \rangle)$$

and

$$\phi_T^o(t) := \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(|g_k(u,t)|_0^2 + |b(t) \int_0^t \sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds|_0^2 - \left| f_k(u,t) + \int_0^t b(t) \sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds \right|_0^2\right)\right) b_k(t) \exp^{\diamond}(\langle \cdot, g_k(u,t) \rangle),$$

in the calculus before one derives the ordinary portfolio. □

Definition 4.15. *A portfolio* (ϕ_A, ϕ_T) *is* self-financing *if*

$$V(t) - V(0) = \int_0^t \phi_A(t) dA(t) + \int_0^t \phi_T(t) dF^T \tag{4.18}$$

holds a.s. and for all $t \in [0, T]$.

Definition 4.16. *A self-financing portfolio* (ϕ_A, ϕ_T) *is called an* arbitrage *if* $V(0) = 0$ *and* $V(T) \geq 0$ *a.s. and* $E(V(T)) > 0$.

Theorem 4.17. *Assume that equation (4.1) holds and that* $F^T(t, r(t))$ *is given by (4.12) and (4.13), then there is no arbitrage in the class of self-financing portfolios.*

Proof. Apply equation

$$V(t) = \phi^A(t) \diamond A(t) + \phi^T(t) \diamond F^T(t, r(t))$$

in equation (4.18) and the chain rule for F^T and for $A(t)$, it is obtained

$$\begin{aligned}
&\int_0^T r(t) \diamond V(t, r(t)) + V(t, r(t))\psi(t) - r(t) \diamond \phi^T \diamond F^T - (\phi^T \diamond F^T)\psi(t) \\
&+ \phi^T \diamond (F_t^T + F_r^T \diamond (\kappa(\theta - r(t)))dt + \int_0^T \phi_T \diamond F_r^T dB_t^v \\
&= \int_0^T r(t) \diamond V(t, r(t)) + V(t, r(t))\psi(t)dt \\
&\quad + \int_0^T \phi^T \diamond (F_r^T \diamond (\kappa(\theta - r(t))) - r(t) \diamond F^T - \psi(t)F^T + F_t^T + \sigma F_r^T \diamond W_t^v)dt
\end{aligned}$$

If there exists an $\eta \in \mathcal{S}(\mathbb{R})$ such that

$$S\left(F_r^T \diamond (\kappa(\theta - r(t))) - r(t) \diamond F^T - \psi(t)F^T + F_t^T + \sigma F_r^T \diamond W_t^v\right)(\eta) = 0 \tag{4.19}$$

then it follows that by applying the change of S-transform and integration

$$S(V(T))(\eta) - S(V(0))(\eta) = \int_0^T S(V(t, r(t))(\eta) S(r(t) + \psi(t))(\eta))dt$$

this is an integral equation for $S(V(t, r(t)))(\eta)$ and has the solution

$$S(V(T, r(T))(\eta) = S(V(0, r(0)))(\eta) \exp\left(S(\int_0^T r(t)dt + \int_0^T \psi(t)dt)(\eta)\right)$$

so

$$V(T, r(T)) = V(0, r(0)) \exp^\diamond \left(\int_0^T r(t)dt + \int_0^T \psi(t)dt\right) = V(0, r(0)) \exp\left(\int_0^T r(t)dt\right).$$

Define the measure $\mu_\eta(\mathcal{B}) = S(1_\mathcal{B})(\eta)$. As the measure μ of the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ and the measure μ_η, which is induced by the S-transfrom are equivalent, it follows that if $V(0) = 0$ in the measure μ, so $V(0) = 0$ in the measure μ_η, then $V(T) = 0$ in the measure the μ_η and so $V(0) = 0$ in the measure μ. Regarding the partial derivatives of $F^T(t, r(t))$

$$\begin{aligned}
F_t^T(t, r(t)) &= F^T(t, r(t)) \diamond \left(\frac{da(t)}{dt} + \frac{b(t)}{dt} r(t)\right) \\
F_r^T(t, r(t)) &= F^T(t, r(t))b(t)
\end{aligned}$$

Now plugging these partial derivatives into the equation (4.19) and the differential equations for a and b

$$\begin{aligned}
&S(F_r^T)(\eta)S(\kappa(\theta - r(t)))(\eta) - S(r(t))(\eta)S(F^T)(\eta) - S(\psi(t)F^T)(\eta) + S(F_t^T)(\eta) \\
&= -\sigma S(F_r^T)(\eta)\left\langle \frac{dm(u,t)}{dt}, \eta(u)\right\rangle, \\
&S(F^T)(\eta)(b(t)\kappa(\theta - S(r(t))(\eta)) - S(r(t))(\eta) - \psi(t) + \frac{da(t)}{dt} + \frac{b(t)}{dt}S(r(t))(\eta)) \\
&= -\sigma b(t)S(F^T)(\eta)\left\langle \frac{dm(u,t)}{dt}, \eta(u)\right\rangle
\end{aligned}$$

thus

$$\begin{aligned}
S(r(t))(\eta)(\frac{db(t)}{dt} - \kappa b(t) - 1) + b(t)\kappa\theta - \psi(t) + \frac{da(t)}{dt} &= -b(t)\sigma\left\langle \frac{dm(u,t)}{dt}, \eta(u)\right\rangle, \\
b(t)\kappa\theta - \psi(t) + (\zeta - \kappa\theta)b(t) + \psi(t) &= -b(t)\left\langle \frac{dm(u,t)}{dt}, \eta(u)\right\rangle, \\
\zeta b(t) &= -b(t)\sigma\left\langle \frac{dm(u,t)}{dt}, \eta(u)\right\rangle, \\
-\zeta &= \left\langle \frac{dm(u,t)}{dt}, \sigma\eta(u)\right\rangle,
\end{aligned}$$

due to equation (4.1) there exists an η such that the last equation is satisfied.
□

The chain rule for $F^T(t, r(t))$ is

$$dF^T = (F_t^T + F_r^T \diamond (\kappa(\theta - r(t))))dt + F_r^T \sigma\, dB_t^v$$

setting

$$\alpha^T(t, r(t)) := (F_t^T + F_r^T \diamond (\kappa(\theta - r(t)))) \diamond (F^T)^{-\diamond}$$

and

$$\sigma^T(t, r(t)) := \sigma F_r^T \diamond (F^T)^{-\diamond}$$

it follows

$$dF^T = \alpha^T \diamond F^T dt + \sigma^T \diamond F^T dB_t^v. \tag{4.20}$$

Define $\frac{d}{2dt} \parallel \int_0^t \int_0^s \sigma \exp(-\kappa(s-u)\, dB_u^v\, ds \parallel_0^2 =: \psi(t)$.

Definition 4.18. *The unique process $\lambda(t)$ in the equation*

$$\lambda(t) \diamond \sigma^T(t, r(t))) := \alpha^T(t, r(t)) - (r(t) + \psi(t))$$

is called market price of risk.

Due to the case of the classical short rate modelling, it is considerable that the arbitragefree market chooses the $\lambda(t)$. It is assumed that λ is a constant with respect to the time t, $t \in [0, T]$.

Theorem 4.19. *From the absence of arbitrage it follows that the market price of risk λ is independent of the maturity of the zero coupon bond F^T.*

Proof. Let the bonds F^T and F^S have two different λ^T and λ^S with $\lambda^T = \lambda^S + p$, where $p > 0$. Regard the self-financing portfolio

$$V(t) = \phi^T \diamond F^T + \phi^S \diamond F^S$$

applying this equation and equation (4.20) and the property that it is a self-financing strategy one obtains

$$dV = \left(\phi^T \diamond F^T \diamond (\alpha^T + \sigma^T \diamond W_t^v) + \phi^S \diamond F^S \diamond (\alpha^S + \sigma^S \diamond W_t^v)\right) dt,$$

insert the term for λ^T and the term λ^S and the difference of them

$$\begin{aligned} dV &= (\phi^T \diamond F^T \diamond (\lambda^T \diamond \sigma^T + r(t) + \psi(t) + \sigma^T \diamond W_t^v) \\ &\quad + \phi^S \diamond F^S \diamond (\lambda^S \diamond \sigma^S + r(t) + \psi(t) + \sigma^S \diamond W_t^v))dt \\ &= ((r(t) + \psi(t)) \diamond (\phi^T \diamond F^T + \phi^S \diamond F^S) \\ &\quad + ((\lambda^S + W_t^v) \diamond (\phi^T \diamond F^T \diamond \sigma^T + \phi^S \diamond F^S \diamond \sigma^S)) + p\,\sigma b(t)\phi^T \diamond F^T)dt \\ &= (r(t) + \psi(t)) \diamond V dt + \left((\lambda^S + W_t^v) \diamond (\phi^T \diamond F^T \diamond \sigma^T + \phi^S \diamond F^S \diamond \sigma^S)\right) dt + p\,\sigma b(t)\phi^T \diamond F^T dt \end{aligned}$$

For the term $\lambda^S + W_t^v$ it holds that

$$S(\lambda^S + W_t^v)(\eta) = 0$$

with the η of equation (4.19). Thus it holds with the same η in the integral representation of the differential forms that

$$S(V(T) - V(0))(\eta) = \int_0^T S((r(t) + \psi(t)))(\eta) S(V(t))(\eta) dt + \int_0^T \sigma b(t) p\, S(\phi^T)(\eta) S(F^T)(\eta) dt.$$

Take a trading strategy such that $(\sigma b(t)\phi^T \diamond F^T) > 0$, then there is an arbitrage. Thus $p = 0$. □

Theorem 4.20 (Term structure equation). *The price of a zero coupon bond F^T with the maturity T satisfies the equation in $[0,T]$*

$$(\lambda\sigma^T + r(t) + \psi(t)) \diamond F^T = F_t^T + F_r^T \diamond (\kappa(\theta - r(t)))$$

with $F^T(T, r(T)) = 1$, so the bond price is price of risk

$$\begin{aligned} F^T(t, r(t)) &= \exp^{\diamond}(a(t) + b(t)r(t)) \\ &= \exp\left(a(t) + b(t)r(t) - \frac{1}{2}b^2(t) \parallel \int_0^t \sigma \exp(-\kappa(t-s)) dB_s^v \parallel_0^2\right) \end{aligned}$$

where

$$b(t) = \frac{\exp(-k(T-t)) - 1}{k}$$

and with the constant market price of risk λ

$$\begin{aligned} a(t) &= (\lambda\sigma - \kappa\theta)\frac{1/k(\exp(-k(T-t)) - 1) + T - t}{k} + \\ &+ \frac{1}{2}\int_0^t \int_0^t \sigma^2 \exp(-2\kappa t + \kappa(u+s))v(u,s)\,du\,ds \\ &- \frac{1}{2}\int_0^T \int_0^T \sigma^2 \exp(-2\kappa T + \kappa(s+u))v(u,s)\,du\,ds \\ &= (\frac{\lambda\sigma}{\kappa} - \theta)(b(t) + T - t) + \frac{1}{2}\int_0^t \int_0^t \sigma^2 \exp(-2\kappa t + \kappa(u+s))v(u,s)\,du\,ds \\ &- \frac{1}{2}\int_0^T \int_0^T \sigma^2 \exp(-2\kappa T + \kappa(s+u))v(u,s)\,du\,ds, \end{aligned}$$

the term $\| \int_0^t \sigma \exp(-\kappa(t-s)) dB_s^v \|_0^2$ *is given by (4.9).*

Proof. Regard

$$dF^T(t, r(t)) = F_t^T + F_r^T \diamond (\kappa(\theta - r(t))) + \sigma F_r^T \diamond W_t^v dt$$

and

$$dF^t(t, r(t)) = \alpha^T \diamond F^T + \sigma^T \diamond F^T \diamond W_t^v dt,$$

subtracting the equations and using the market price of risk λ, one receives

$$(\lambda(t) \diamond \sigma^T + r(t) + \psi(t)) \diamond F^T = F_t^T + F_r^T \diamond (\kappa(\theta - r(t)))$$

Insert that $F^T(t, r(t)) = \exp^\diamond(a(t) + b(t)r(t))$ and get the partial derivatives of F^T

$$\begin{aligned} F_t^T &= F^T \diamond (a'(t) + b'(t)r(t)), \\ F_r^T &= F^T b(t). \end{aligned}$$

These results and the definiton of σ^T leads to

$$\begin{aligned} &(\lambda(t) \diamond \sigma^T + r(t) + \psi(t)) \diamond F^T = F^T \diamond (a'(t) + b'(t)r(t)) + F^T \diamond (\kappa(\theta - r(t)))b(t), \\ &F^T \diamond \big(\lambda(t) \diamond \sigma^T + r(t) + \psi(t) - a'(t) - b'(t)r(t) - \kappa(\theta - r(t))b(t)\big) = 0 \\ &F^T \diamond (\lambda(t)\sigma b(t) + r(t) + \psi(t) - a'(t) - b'(t)r(t) - \kappa(\theta - r(t))b(t)) = 0 \end{aligned}$$

Now set $\lambda(t) =: \lambda$ constant and knowing that F^T is positiv a.s. it is concludable that

$$\begin{aligned} (\lambda\sigma b(t) + r(t) + \psi(t) - a'(t) - b'(t)r(t) - \kappa(\theta - r(t))b(t)) = 0 \\ \lambda\sigma b(t) + \psi(t) - a'(t) - \kappa\theta b(t) + r(t)(1 - b'(t) + \kappa b(t)) = 0. \end{aligned}$$

From this equation the ODEs with the conditions $b(T) = 0$ and $a(T) = 0$ are

$$b'(t) = 1 + \kappa b(t)$$

and

$$a'(t) = (\lambda\sigma - \kappa\theta)b(t) + \psi(t)$$

the solutions are

$$b(t) = \frac{\exp(-k(T-t)) - 1}{k}$$

and

$$\begin{aligned} a(t) &= (\lambda\sigma - \kappa\theta)\frac{1/k(\exp(-k(T-t)) - 1) + T - t}{k} \\ &\quad + \frac{\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u))dB_u^v ds \|_0^2}{2} - \frac{\| \int_0^T \int_0^s \sigma \exp(-\kappa(s-u))dB_u^v ds \|_0^2}{2} \\ &= (\frac{\lambda\sigma}{\kappa} - \theta)(b(t) + T - t) + \frac{\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u))dB_u^v ds \|_0^2}{2} \\ &\quad - \frac{\| \int_0^T \int_0^s \sigma \exp(-\kappa(s-u))dB_u^v ds \|_0^2}{2}. \end{aligned}$$

The representation holds

$$F^T = \exp^\diamond(a(t) + b(t)r(t)) = \exp\left(a(t) + b(t)r(t) - \frac{1}{2}b^2(t) \| \int_0^t \sigma \exp(-\kappa(t-s))dB_s^v \|_0^2\right).$$

The expression above in the function a can be further calculated

$$\begin{aligned} \frac{\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u))dB_u^v ds \|_0^2}{2} &= \frac{\| \int_0^t \int_u^t \sigma \exp(-\kappa(s-u))ds dB_u^v \|_0^2}{2} \\ &= \frac{\| \int_0^t \sigma \frac{-1}{k}(\exp(-\kappa(t-u)) - 1)dB_u^v \|_0^2}{2} \\ &= \frac{1}{2}\int_0^t \int_0^t \sigma^2 \exp(-2\kappa t + \kappa(u+s))v(u,s)\, du\, ds, \end{aligned}$$

by the use of equation (3.15). Setting $\lambda\sigma = \zeta$ one has proven the equations (4.12) and (4.13). □

Remark 4.21. A multi factor Vasicek model with memory can be constructed similarly.

References

[AlMaNu] Alos E., Mazet O. and Nualart D. Stochastic Calculus with respect ot Gaussian processes, Ann. Probab. **29**, p. 766-801

[Be] Bender C. *Integration with Respect to a Fractional Brownian Motion and Related Market Models*, Dissertation, August 2003, University of Konstanz

[BjHu] Björk T., and Hult H. A note on Wick products and the fractional Black-Scholes model, *Finance Stochast.*, **9**(2005), 197–209

[BrMe] Brigo D., and Mercurio F. *Interest Rate Models-Theory and Practice*, 2006, Springer Verlag, Heidelberg

[DeUs] Decreusefond L. and Üstünel A.S. Stochastic analysis of the fractional Brownian motion, Pot. Anal., **10**, 1999,p. 177-214

[DiSc] Dippon J. and Schiemert D. Stochastic differential equations driven by Gaussian processes with dependent increments, Preprint 2006, Universität Stuttgart

[DiSc2] Dippon J. and Schiemert D. Option Pricing in a Black-Scholes market with memory, Preprint 2006, Universität Stuttgart

[DuHuPa] Duncan T.E.,Hu Y. and Pasok-Duncan B. Stochastic Calculus for fractional Brownian motion, I. Theory SICON, **38**, 2000, p. 582-612

[GrNo] Gripenberg G., and Norros I. On the prediction of fractional Brownian motion. *J. Appl. Prob.*, **33**(1996), 400–410

[HiKuPoSt] Hida T., Kuo H.-H., Potthoff J., and Streit L. *White Noise: An Infinite Dimensional Calculus*, 1993, Dordrecht: Kluwer Academic Publishers

[Ho] Hochstad H. *Integral Equations*, A Wiley Interscience Publication, 1973, USA

[HoOkUbZh] Holden H., Øksendal B., Ubøe J. and Zhang T. *Stochastic Partial Differential Equations*, Birkhäuser 1996, Boston

[HuOk] Hu Y., and Øksendal B. *Fractional White Noise Calculus and Applications to Finance*, *IDAQP*, **6**(2003), 1–32

[HuOkSa] Hu Y., Øksendal B., and Salopek D.M. Weighted local time for fractional Brownian motion and applications to finance, *Stoch. Appl. Anal.*, **23**(2005), 15–30

[KaSh] Karatzas I., and Shreve S.E. *Brownian Motion and Stochastic Calculus*, 1991, Springer, New York

[Ku] Kuo H.-H. *White Noise Distribution Theory*, CRC Press, 1996, Boca Raton

[La] Lanconelli A. Translated Brownian motions and associated Wick products, Stoch. Anal. and Appl. **24** No. 4, 2006, p. 795-812

[No] Norros I. On the use of the fractional Brownian motion in the theory of connectionless networks, *IEEE J. Sel. Areas Commun.*, **13**(1995), 953–962

[Nu] Nualart D. *The Malliavin Calculus and Related Topics*, Springer Verlag 1995, Heidelberg

[NuTa] Nualaret D. and Taqqu M.S. Wick-Itô Formula for Gaussian Processes, *Stoch. Anal. and Appl.*, **24**, 3, 2006, 599-614

[Oh] Ohashi A. M. F. Stochastic evolution equations driven by a fractional white noise, **24** No 4, 2006, p. 555-578

[PoSt] Potthoff J., and Streit L. A characterization of Hida distributions, *J. Funct. Anal.*, **101**(1991), 212–229

[SaKiMa] Samko S.G., Kilbas A.A., and Marichev O.I. *Fractional Integrals and Derivatives: Theory and Applications*, 1993, Gordon and Breach

[So] Sottinen T. Fractional Brownian motion, random walks and binary market models, *Finance Stochast.*, **5**(2001), 343–355

Erklärung

Hiermit erkläre ich diese Dissertation selbständig verfasst und keine anderen Quellen oder Hilfsmittel als die angegebenen benutzt zu haben.
Daniel Schiemert