Applications of Cartan and Tractor Calculus to Conformal and CR-Geometry

Von der Fakultät Mathematik und Physik der
Universität Stuttgart als Habilitationsschrift
genehmigte Abhandlung

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Fachbereich Mathematik
Oktober 2007

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Preface

The main object of this *Habilitationsschrift* is the geometric study of solutions of overdetermined conformally invariant differential equations via the use of Cartan and tractor calculus. This study fits into the broader research field of conformal and parabolic invariant theory. Parts of our investigations take special attention to conformal Lorentzian and spin geometry, which provides a link to the theories of modern physics.

The present text originated from a collection of research articles and other works of the author, which emerged since the year 2003. In order to make the text basically self contained with uniform notations and conventions I decided to prefix an extended introductory chapter. An English and German summary are included as well.

Over the years I have benefited and learnt from discussions with many mathematicians. Especially, I would like to thank Helga Baum, Florin Belgun, Andreas Čap, Jose Figueroa-O'Farrill, Rod Gover, Andreas Juhl, Wolfgang Kühnel, Thomas Neukirchner, Hans-Bert Rademacher and Gerd Schmalz. I'm also grateful to Jesse Alt, who proofread parts of this text.

Stuttgart, October 2007

2000 *Mathematics Subject Classification.* 53A30; 32V05; 53C27; 53C29; 53B30; 53C50; 53C25; 53C15; 57S20; 58J70; 83C20; 53B15.

Key words and phrases: Cartan and Parabolic Geometries; Conformal Geometry; CR-Geometry; Invariant Theory; Tractor Calculus; Fefferman-Graham Ambient Metric; Poincaré-Einstein Model; Fefferman Construction; Conformal Transformation Groups; Conformal Holonomy; Lorentzian Geometry; Spinor Field Equations; Twistor Spinors; Conformal Killing Forms.

Zusammenfassung (German Summary)

Die grundlegende Thematik dieser *Habilitationsschrift* ist die geometrische Untersuchung von Räumen, welche Lösungen von *überbestimmten, konform-invarianten Differentialgleichungen* zulassen. Dabei kommt in besonderem Maße der Verwendung von Methoden der *Cartan-Geometrie* und des *Tractor-Kalküls* Bedeutung zu. Unsere Untersuchungen sind im weiteren Kontext der konformen und parabolischen *Invariantentheorie* zu sehen. Teile davon nehmen auch Bezug auf die *Lorentzsche* und *spinorielle Geometrie*, was eine Beziehung zu den Theorien der modernen Physik herstellt. Spezielle Themen sind insbesondere die *konformen Killing-Formen, Twistorspinoren, konforme Holonomie, explizite Ambientmetriken nach Fefferman-Graham*, das *Poincaré-Einstein-Modell, wesentliche konforme Killingvektorfelder* und *Transformationsgruppen, partiell integrierbare CR-Strukturen* und ihre *Fefferman-Konstruktion, pseudo-Einsteinsche, transversal-symmetrische pseudo-Hermitische Strukturen* und *Fefferman-Einsteinmetriken.*

Die vorliegende *Habilitationsschrift* ist eine kumulative Abfassung verschiedener, eigener Forschungsarbeiten, die in den letzten Jahren seit meiner Promotion entstanden sind (vgl. [**106**]). Einige dieser Arbeiten sind bereits als Artikel in Fachzeitschriften erschienen. Andere Beiträge erscheinen als *Preprints* oder sind erst kürzlich neu entstanden. In jedem Fall haben wir versucht, die verschiedenen Quellen zu einer Einheit zusammenzufügen. Dabei kommt der ausführlichen Einleitung in Teil 1 (d.h. Kapitel 0), welcher aus 13 Abschnitten besteht, eine besondere Bedeutung zu. Zum einen sollen in diesem einführenden Kapitel die grundlegenden Themen und Prinzipien der *Differentialgeometrie* vorgestellt werden, welche für die Untersuchungen in Teil 2, den eigenen Forschungsergebnissen, zum Tragen kommen. Zum anderen legt die Einführung auch die Bezeichnungen und Konventionen fest, welche in den späteren Kapiteln durchgängig und einheitlich benutzt werden sollen. Die eigenen Forschungsergebnisse in Teil 2 sind dann in 6 Kapiteln zusammengefasst.

Wir beschreiben hier zunächst kurz den Inhalt der 13 einführenden Abschnitte. Die zentralen Themen darin sind *konforme, CR-* und *Spin-Geometrie.* Einige Themen und Prinzipien werden dabei ausführlich beschrieben, so dass man sie in weiterführenden und auch anderen Themengebieten der *parabolischen Geometrie* als Grundlage verwenden könnte, während andere Ausführungen eher kurz gehalten sind und nur das Wesentliche für unsere Zwecke bereithalten. Im Einzelnen: die ersten drei Abschnitte starten mit der Beschreibung sehr grundlegender Prinzipien und Methoden der Differentialgeometrie, welche auch in vielen Lehrbüchern nachgelesen werden können. Wir verweisen hier auf [**91**]. Insbesondere werden standardmäßig und auf grundlegendem Niveau *differenzierbare Mannigfaltigkeiten, Liesche Gruppen* und *Algebren*, ihre *Darstellungen, Faserbündel* mit *Zusammenhängen* und *Differentialoperatoren* eingeführt. In Abschnitt 0.4 erwähnen wir in kompakter Form die Grundlagen der Geometrie von *Riemannschen Metriken* mit beliebiger Signatur, wiederum auf grundlegendem Niveau mit besonderem Augenmerk auf *Lorentzscher Geometrie* und auch in Hinblick auf die *konforme Geometrie* (vgl. z.B. [**91, 128, 139**] und viele andere Quellen). Im darauffolgenden Abschnitt 0.5 wenden wir uns der *parabolischen Geometrie* im Kontext der *Cartan-Geometrie* zu. Bezüglich der allgemeinen Theorie von *Cartan-Zusammenhängen* verweisen wir auf [**144**]. Der beschriebene Zugang zur parabolischen Geometrie lehnt sich besonders an die Arbeiten [**145, 64, 41, 45, 42, 35**] an. Dabei ist eine parabolischen Geometrie modelliert auf einem *Hauptfaserbündel* mit Cartan-Zusammenhang zum Struktur-Paar (G, P), bestehend aus einer *halbeinfachen* Lieschen Gruppe G mit parabolischer Untergruppe P. In Abschnitt 0.6 führen wir dann die *konforme Geometrie* im Sinne der parabolischen Cartan-Geometrie ein (als Geometrie zweiter Ordnung), dass heißt als P-Hauptfaserbündel mit $\mathfrak{g}$-wertigem Cartan-Zusammenhang, wobei $\mathfrak{g}$ die Lie-Algebra der *Möbiusgruppe* $G = \mathrm{PO}(r+1, s+1)$ ist, und P beschreibt die Fixpunktgruppe des *Möbiusraumes* bei homogener G-Wirkung. Die zur Gruppe P gehörige Gradierung der Lie-Algebra $\mathfrak{g}$ hat die Länge 1. Die Beziehung von konformer Geometrie und *semi-Riemannscher* Geometrie wird ausführlich dargestellt (vgl. [**46, 90, 43, 44**]). Im darauffolgenden Abschnitt 0.7 führen wir das *Tractor-Kalkül* ein, welches auf T. Thomas (vgl. [**150**]) zurückgeht und in [**14**] grundlegend neu beleuchtet wurde (vgl. [**36, 64, 37**]). Das Tractor-Kalkül ist im Wesentlichen als ein äquivalentes Modell zur Cartan-Geometrie zu verstehen, wobei beide Zugänge (z.B. zur konformen Geometrie) ihre besonderen Stärken haben. In der Tat liegt bei unserer Einführung das Hauptaugenmerk auf dem konformen Tractor-Kalkül (vgl. [**14, 64**]). In Abschnitt 0.8 definieren wir die Grundlagen der semi-Riemannschen *Spin-Geometrie*, insbesondere *Spin-Strukturen, Spinorbündel, Clifford-Multiplikation, Spinorableitung*, den *Dirac*- und den *Penrose-Operator* (vgl. [**100, 15, 19, 139**]). Weiterhin entwickeln wir eine spinorielle Version des Tractor-Kalküls, was uns die Einführung von *Twistoren* und den Zugang zur *Twistorgleichung* für Spinoren bereitet. Abschnitt 0.9 stellt einen kurzen Diskurs zur Definition von *metrischen Kegeln* dar (vgl. [**62, 12**]). Wir werden später sehen, dass die metrische Kegelkonstruktion als ein erster einfacher Ansatz zur Theorie der *Ambientmetriken* im Sinne von *Fefferman* und *Graham* in Rahmen der konformen Geometrie gesehen werden kann. Die *CR-Geometrie vom Hyperflächentyp* wird dann in Grundzügen in Abschnitt 0.10 erklärt. Dabei beschreiben wir sowohl den Zugang mit Hilfe der parabolischen Geometrie für partiell integrierbare CR-Strukturen als auch den klassischen Zugang in Beziehung mit der *pseudo-Hermitischen Geometrie* (vgl. [**149, 152, 103, 86, 48, 41, 39**]). Die *Ambientmetrik* nach Fefferman-Graham ist ein sehr wichtiges Werkzeug für das Studium der *konformen* und *CR-Invariantentheorie*. Die Ambientmetrik-Konstruktion steht auch in direkter Beziehung zum *Poincaré-Einsteinschen Modell*, welches eine so wichtige Rolle bei der Untersuchung von *Branson's Q-Krümmung, Volumenrenomalisierung* und der *AdS/CFT-Korrespondenz* in der Physik spielt. Wir geben einen Überblick zu beiden Konstruktionen in Abschnitt 0.11 (vgl. [**53, 54, 55, 56, 74, 75, 76, 77, 78, 79**]). Abschließend behandeln wir die *Fefferman-Konstruktion*, welche eine enge Beziehung von CR-Geometrie und konformer

Geometrie herstellt (vgl. [**53, 31, 102, 146, 73, 33, 34, 39**]). In Abschnitt 0.13 fassen wir die Ergebnisse unserer Untersuchungen in Teil 2 dieser *Habilitationsschrift* zusammen. Das wollen wir nun auch hier als nächstes tun.

Die Hauptmotivation für unsere Untersuchungen in den Kapiteln 1 bis 6 (d.h. Teil 2) ist die *konforme Invariantentheorie*. Jedoch steht dabei nicht so sehr die Konstruktion (neuartiger) konformer Invarianten und ihre Diskussion im Vordergrund (was ein sehr interessantes und aktives Forschungsfeld ist mit dem wir uns auch in Zukunft verstärkt beschäftigen wollen), sondern viel mehr das Lösen von wohlbekannten, invarianten (partiellen) Differentialgleichungen und die Beschreibung der zugrundeliegenden Räume samt ihrer konformen Geometrie. Wie wir bereits erwähnt haben, interessieren wir uns dabei besonders für *überbestimmten Differentialgleichungen*, was in der Konsequenz dazu führt, dass die zugrundeliegenden konformen Räume im Allgemeinen gewisse *Symmetrien* besitzen. Das Studium der Existenz von Lösungen solcher konform-invarianten Differentialgleichungen und ihrer Symmetrien ist in den verschiedensten Gebieten der Geometrie und Physik von großem Interesse. In der Tat liegt ein Schwerpunkt unserer Untersuchungen im Bereich der Lorentzschen Geometrie, was eine Beziehung zu Supergravitationstheorien im Rahmen der mathematischen Physik herstellt. In diesem Zusammenhang sind auch das *Poincaré-Einstein-Modell* in Verbindung mit dem *holographischen Prinzip* und der *AdS/CFT-Korrespondenz* zu nennen. Um bereits ein konkretes Beispiel zu geben, ein Hauptgegenstand unserer Untersuchungen ist die Twistorgleichungen der Lorentzschen Spin-Geometrie, welche man in einem gewissen Sinn als die grundlegende invariante Differentialgleichung erster Ordnung der konformen Geometrie betrachten kann. Aber auch die konformen Killing-Formen und ganz einfach auch die konformen Killingvektorfelder sind sehr stark im Fokus des Interesses. Wir werden auch sehen, dass gewisse Symmetrien mit Hilfe einer konformen Holonomietheorie beschrieben werden können. In der Tat, der Begriff der konformen (Tractor)-Holonomie wird die verschiedenen Kapitel hindurch (abgesehen von Kapitel 4) die Rolle eines Leitmotivs spielen.

Im Einzelnen werden die folgenden Untersuchungen und Ergebnisse in den Kapiteln 1 bis 6 behandelt. In Kapitel 1 diskutieren wir die Gleichung $\nabla^{nor}\alpha = 0$ für $(p+1)$-Form-Tractoren α auf konformen Räumen (M^n, c) der Dimension n. Diese Diskussion basiert auf den Arbeiten [**110, 111**] (vgl. auch [**150, 88, 148, 14, 143, 37, 35, 51, 72, 80**]). Es stellt sich heraus, dass der projizierende Teil $\alpha_- = \Pi_H(\alpha)$ eine *konforme Killing-p-Form* mit Gewicht $(p+1)$ ist, d.h. eine Lösung der (tensoriellen) Gleichung

$$\nabla^g_X \alpha_- - \frac{1}{p+1}\iota_X d\alpha_- + \frac{1}{n-p+1} X^\flat \wedge d^*\alpha_- \;=\; 0$$

für alle $X \in TM$ bzgl. einer beliebigen, kompatiblen Metrik $g \in c$. Zusätzlich erfüllt diese konforme Killing-p-Form $\alpha_- \in \Omega^p(M)$ gewisse *Normalisierungsbedingungen*, welche wir auch bzgl. der Metrik g in der konformen Klasse c berechnen. (Dies läuft im Wesentlichen auf die Berechnung des *Splittingoperators* $\mathbf{S}$ in dieser speziellen Situation hinaus.) Außerdem berechnen wir die *Integrabilitätsbedingungen* für die Existenz von *normal*-konformen Killing-p-Formen. Auf konformen Einstein-Räumen sind wir in der Lage mit Hilfe der *metrischen Kegelkonstruktion* die zugrundeliegende konforme Geometrie von Räumen mit Lösungen α zu beschreiben. In der Tat, da in diesem Fall die Existenz einer normal-konformen Killing-p-Form auch die einer *speziellen* konformen Killing-Form impliziert, ergibt sich für vollständige Riemannsche Mannigfaltigkeiten eine Klassifikation mit Hilfe der speziellen Holonomiegruppen des Kegels (vgl. Theorem 4 und [**12, 143**]). Es ist auch wichtig zu erwähnen, dass die konforme Einstein-Bedingung selbst durch die Existenz von $\nabla^{\mathcal{T}}$-parallelen *Standardtractoren* beschrieben wird (vgl. [**28, 64, 93, 92, 6, 7, 65, 69, 122, 123, 3**]). Genauer gesagt, ein $\nabla^{\mathcal{T}}$-paralleler Standardtractor auf einer konformen Mannigfaltigkeiten (M^n, c) gibt in eindeutiger Weise eine Einsteinmetrik in c bis auf Singularitäten in M an. Falls alle Einstein-Skalierungen auf M Singularitäten aufweisen, sprechen wir von (konformen) *fast-Einsteinschen* Räumen.

Offensichtlich steht die Gleichung $\nabla^{nor}\alpha = 0$ in enger Beziehung mit der Holonomietheorie des *kanonischen* Zusammenhangs ω_{nor} (bzw. ∇^{nor}) der konformen Geometrie (vgl. auch [**6**]). Wir werden im weiteren Verlauf zeigen, dass diese Tatsache einige interessante Konsequenzen und Anwendungen nach sich zieht. Zunächst beweisen wir ein konformes Analogon des *Zerlegungssatzes von deRham* für semi-Riemannsche Räume (vgl. Theorem 6). Die entsprechende (lokale) Aussage in der konformen Geometrie besagt, dass die *konforme Holonomiegruppe* genau dann *zerlegbar* ist, wenn ein Paar von Einsteinmetriken existiert mit einer gewissen Relation zwischen den zugehörigen Skalarkrümmungen, deren Produkt in der konformen Klasse c liegt. Die notwendige Relation für die Skalarkrümmungen ist dabei gegeben durch

$$n_2(n_2-1)\cdot scal^1 \ = \ -n_1(n_1-1)\cdot scal^2 \ ,$$

wobei n_1 und n_2 die Dimensionen der Faktoren angeben (vgl. Proposition 6). Wir wollen an dieser Stelle auch hervorheben, dass insbesondere die konforme Einstein-Bedingung eine sehr nützliche Interpretation in Rahmen der konformen Holonomietheorie besitzt. Es ist nämlich einfach so, dass eine Mannigfaltigkeit mit konformer Struktur genau dann fast-Einsteinsch ist, falls die konforme Holonomiegruppe reduziert ist, so dass sie einen Standardtractor fixiert. Diese Holonomiebedingung kann in konkreten Situation sehr gut nachprüfbar sein. (Zumindest sollte diese Bedingung einfacher nachzuprüfen sein, als den Beweis zu führen, dass ein Standardtractor $\nabla^{\mathcal{T}}$-parallel ist!) Wir werden das in einer konkreten Situation in Kapitel 3 demonstrieren.

Ein *Twistorspinor* φ (oder auch *konformer Killingspinor* genannt) auf einer konformen *Spin-Mannigfaltigkeit* ist Lösung der *Twistorgleichung*

$$\nabla^{\mathcal{S}}_X\varphi + \frac{1}{n}X\cdot D^{\mathcal{S}}\varphi \ = \ 0$$

für all $X \in TM$. Eine bemerkenswerte Eigenschaft von Twistorspinoren ist, dass jeder zugehörige *Twistor* (d.h. ein Schnitt im spinoriellen Tractorbündel) in jedem Fall ∇^{nor}-parallel ist (vgl. [**137, 61, 19, 32**]). Mit Hilfe der *Spinorquadrate* eines Twistorspinors wird sofort deutlich, dass sich (in Abhängigkeit von Signatur und Dimension) verschiedene normal-konforme Killing-p-Formen auf einer konformen Spin-Mannigfaltigkeit (M, c) generieren lassen. Eine Besonderheit bei der konformen Lorentzschen Geometrie besteht darin, dass der *Dirac-Strom* (d.h. das Spinorquadrat in den Tangentialvektoren) dieselbe Nullstellenmenge besitzt wie der Spinor selbst. Da der Dirac-Strom zu einem Twistorspinor ein normal-konformes Killingvektorfeld ist, ist der zugehörige adjungierte Tractor ∇^{nor}-parallel, d.h. dieser adjungierte Tractor besitzt einen eindeutig bestimmten Orbittypen unter der Wirkung der *Möbiusgruppe* $O(2, n)$ (welcher auch über der Mannigfaltigkeit konstant ist). Mit Hilfe von Tafel 3 können wir eine Klassifikation von 2-Formen bzw. schiefadjungierten Endomorphismen auf dem pseudo-Euklidischen Raum $\mathbb{R}^{2,n}$ mit Signatur $(2, n)$ angeben. Das wesentliche Ergebnis unserer Klassifikation besagt, dass es exakt 4 generische Orbittypen (bzw. Normalformen) für 2-Formen auf $\mathbb{R}^{2,n}$ gibt, welche als spinorielles Quadrat entstehen können. Entsprechend dieser 4 generische Typen sind wir in der Lage in Theorem 10 eine vollständige geometrische Klassifikation für (konforme) Lorentzsche Spin-Mannigfaltigkeit anzugeben, welche Twistorspinoren ohne *Singularitäten* zulassen. Unter den beschriebenen möglichen Geometrien befinden sich *statische Monopole*, *Brinkmannsche Räume*, die *Fefferman-Räume*, die *Einsteinschen Sasaki-Räume* und gewisse (konforme) Produkträume (wie oben bereits erwähnt). Diese Klassifikation erweitert die bekannten Resultate aus [**117, 17, 106, 21**].

Die *metrische Kegelkonstruktion* für Einstein-Räume ist im Prinzip äquivalent zur *Ambientmetrik*-Konstruktion der zugörigen konformen Klasse nach *Fefferman* und *Graham*. In der Tat, wie man sogar einfach raten kann, ist die Ricci-flache Ambientmetrik einer konformen Einstein-Klasse durch das Produkt des Kegels und eines parallelen Linienelements gegeben. In Kapitel 2 verallgemeinern wir diese einfache Idee. Dieser Teil der Arbeit basiert auf [**68**], was eine Gemeinschaftsproduktion mit *Prof. A.R. Gover* von der *Universität Auckland* in Neuseeland ist. In Theorem 11 werden wir explizit eine Ricci-flache Ambientmetrik-Konstruktion nach Fefferman-Graham angeben für eine beliebige konforme Struktur mit zerlegbarer (konformer) Holonomie. Das Interessante an dieser Konstruktion ist, dass die zugrundeliegende konforme Struktur im Allgemeinen nicht Einsteinsch ist. Wir zeigen das, indem wir die konforme und die Holonomie der Ambientmetrik miteinander identifizieren (vgl. Theorem 15). (In allgemeineren Situation ist übrigens nicht zu erwarten, dass diese beiden Holonomiegruppen etwas gemein haben!)

Die Ambientmetrik nach Fefferman-Graham steht in enger Beziehung zum bekannten Poincaré-Einstein-Modell mit konformem Rand im Unendlichen. In der Tat kann das Poincaré-Einstein-Modell als Hyperfläche im Ambientmetrik-Raum realisiert werden. Auf der anderen Seite zeigen wir im Abschnitt 2.4 ganz allgemein, dass ausgehend vom Poincaré-Einstein-Modell mit Hilfe der Kegelkonstruktion eine *Ambientmetrik mit Rand* erzeugt werden kann. In diesem Sinn sind beide Modelle äquivalent. Die explizite Poincaré-Einsteinmetrik für einen konformen Rand mit zerlegbarer (konformer) Holonomie wird in Theorem 13 präsentiert. Wir geben dabei auch die explizite Taylorreihenentwicklung der Poincaré-Einsteinmetrik bzgl. der *speziellen definierenden Funktion* des Produktrandes an (vgl. [**74, 87**]). Man muß sicherlich sagen, dass unsere explizite Konstruktion nur für eine recht spezielle Klasse von konformen Strukturen funktioniert, wie die Taylorreihenentwicklung zeigt, welche bereits nach dem Term 4. Ordnung abbricht. Jedoch ist es unserem Wissen nach auch so, dass diese Konstruktion eine der wenigen bekannten ist, für Ränder die nicht konform-Einsteinsch sind (vgl. [**131**] für einen anderen bekannten Fall; vgl. auch [**77, 26, 104, 5, 125**] für existenzielle, nicht explizite Aussagen)! In Abschnitt 2.6 beschreiben wir dann noch eine Charakterisierung unserer expliziten Poincaré-Einsteinmetriken mit Hilfe der Existenz von gewissen speziellen Killing-p-Formen (vgl. [**143**]). Wir möchten hier nochmal erwähnen, dass das Poincaré-Einstein-Modell und natürlich auch die Ambientmetrik nach Fefferman-Graham von sehr großer Bedeutung sind in der Geometrie und Physik bei der Untersuchung von *konformen Invarianten*, wie z.B. *Branson's Q-Krümmung*, in Verbindung mit Fragen der *Volumenrenormaliserung* und ähnlicher Größen (vgl. [**1, 5, 47, 55, 74, 79, 87**]) oder auch der *AdS/CFT-Korrespondenz* (vgl. z.B. [**124, 78**]).

Nach den bisherigen Ausführungen sollte bereits deutlich geworden sein, dass die konforme Holonomietheorie eine zentrale Rolle bei unseren Untersuchungen spielt. In Kapitel 3 beabsichtigen wir den Beweis zu erbringen, dass eine konforme Holonomiegruppe in konkreten Situationen tatsächlich explizit berechenbar ist. Zu diesem Zweck entwickeln wir ein Kalkül zur Beschreibung des *kanonischen invarianten Cartan-Zusammenhangs* der konformen Geometrie von *bi-invarianten Metriken* auf Lieschen Gruppen. Tatsächlich sind Holonomieberechnungen dann durchaus handhabbar unter Benutzung dieses Kalküls und der klassischen iterativen Formel für Holonomiegruppen invarianter Zusammenhänge auf Hauptfaserbündeln (vgl. z.B. [**91**]). Konkret berechnen wir die konformen Holonomiegruppen der speziell orthogonalen Gruppen SO(3) und SO(4) mit bi-invarianten Metriken, welche in beiden Fällen durch die zugehörige Killingform der Lie-Algebra gegeben sind. Offensichtlich ist die bi-invariante Metrik auf SO(3) konform-flach und tatsächlich ergibt unsere Rechnung ein triviales Resultat für die konforme Holonomie. Auf der anderen Seite sieht man sofort ein, dass die bi-invariante Metrik auf SO(4) lokal das Produkt zweier runder Sphären ist, welches nicht konform-flach ist. In der Tat ergibt unsere Berechnung $\mathfrak{so}(7)$ für die konforme Holonomiealgebra von SO(4). Insbesondere zeigt dieses Resultat, dass die bi-invariante Metrik auf SO(4), welche von der Killingform kommt, bis auf einen konstanten konformen Faktor die einzige Einsteinmetrik in ihrer konformen Klasse darstellt. Der Inhalt der Ausführungen in Kapitel 3 ist der Arbeit [**109**] entnommen. Die hier vorgestellten Ideen können auch in sehr viel allgemeineren Situation homogener Räume mit parabolischen Geometrien angewandt werden. Das zeigt die Arbeit [**83**], welche in eleganter Art und Weise auch eine allgemeine Beziehung von *Automorphismengruppen* und invarianten Holonomiegruppen herstellt.

In Kapitel 4, welches auf den Quellen [**105, 108, 115**] beruht, wird unser Interesse an speziellen Lösungen konform-invarianter Differentialgleichungen besonders deutlich. Das Problem dieses Kapitels ist das Lösen der Twistorgleichung für Spinoren auf Lorentzschen Spin-Mannigfaltigkeiten mit *singulären Punkten*. Konkret heißt das hier, dass wir *Twistorspinoren mit Nullstellen* suchen. (Zur Erinnerung: in Kapitel 1 hatten wir eine geometrische Klassifikation im *nicht-singulären* Fall angegeben.) Wir wollen auch motivieren, warum Nullstellen von Twistorspinoren besonderes Interesse verdienen. In der Riemannschen Geometrie kann man mit klassischen Methoden einfach zeigen, dass Twistorspinoren mit Nullstelle außerhalb der Nullstelle konform-äquivalent zu parallelen Spinoren sein müssen (vgl. [**19**]). Auf kompakten Riemannschen Räumen existieren Twistorspinoren mit Nullstelle nur auf der runden Spähre S^n, welche bekanntermaßen konform-flach ist (vgl. [**120**]). Im nicht-kompakten Fall treten Nullstellen von Twistorspinoren typischer Weise im *unendlich fernen Punkt* von asymptotisch flachen Riemannschen Metriken mit irreduzibler (oder trivialer) Holonomiegruppe auf (vgl. [**96, 99**]).

Wie wir bereits zuvor im Kontext der Lorentzschen Geometrie erwähnt haben, ist der Dirac-Strom ein normal-konformes Killingvektorfeld, welches die gleiche Nullstellenmenge wie der Twistorspinor selbst hat. Die Existenz einer solchen Nullstelle eines Dirac-Stroms macht es dann auf einfache Art und Weise möglich, den Orbittypen des zugehörigen ∇^{nor}-parallelen adjungierten Tractors zu bestimmen (vgl. Tafel 3). Diese Untersuchung zeigt dann wiederum, dass ein Twistorspinor mit Nullstelle in der Lorentzschen Geometrie konform-äquivalent ist zu einem parallelen Spinor außerhalb der Nullstellenmenge (vgl. Theorem 19). Wir erhalten also dasselbe Resultat wie in der

Riemannschen Geometrie, jedoch scheint es so, als ob der Beweis in der Lorentzschen Geometrie ohne das Tractor-Kalkül nicht auskommt! Diese Bemerkung soll hier in einer nur ganz speziellen Situation verdeutlichen wie nützlich das Tractor-Kalkül ist.

Twistorspinoren mit Nullstellen in der Lorentzschen Spin-Geometrie sind auch aus folgendem Grund von Bedeutung. Es stellt sich nämlich durch eine einfache Überlegung heraus, dass der Dirac-Strom eines Twistorspinors mit Nullstelle ein *wesentliches* konformes Killingvektorfeld auf der zugrundeliegenden konformen Mannigfaltigkeit (M, c) ist, was bedeutet, dass der Dirac-Strom bezüglich *keiner* kompatiblen Metrik in der konformen Klasse c ein Killingvektorfeld ist, und somit als eine *echte* infinitesimale konforme Symmetrie von (M, c) zu verstehen ist (vgl. z.B. [**2, 4, 60**])! In der Riemannschen Geometrie kommen *wesentliche* konforme Transformationsgruppen nur sehr selten vor. In der Tat besagt ein wohlbekanntes Ergebnis, dass eine vollständige Riemannsche Mannigfaltigkeit mit wesentlicher konformer Transformationsgruppe entweder konform-äquivalent zur runden Sphäre ist (das ist der kompakte Fall) oder konform-äquivalent zum Euklidischen Raum ist (das ist der nicht-kompakte Fall). In beiden Fällen ist die konforme Geometrie flach (vgl. [**134, 116, 2**]). In der Lorentzschen Geometrie sind nicht konform-flache Räume mit wesentlicher Transformationsgruppe so gut wie unbekannt (vgl. [**106, 60**]).

Das zweite Hauptresultat von Kapitel 4 ist dann auch die explizite Konstruktion einer Familie von nicht-kompakten Lorentzschen Mannigfaltigkeiten in der Dimension 5, welche Twistorspinoren mit isolierter Nullstelle besitzen, so dass die zugrundeliegende konforme Geometrie in jeder Umgebung der isolierten Nullstelle nicht konform-flach ist (vgl. Theorem 20). Insbesondere beweist dieses Resultat die Existenz von nicht konform-flachen Lorentzschen Räumen mit wesentlicher konformer Transformationsgruppe. (Wir möchten an dieser Stelle daran erinnern, dass eine bekannte *Vermutung von Lichnerowicz* besagt, dass kompakte Lorentzsche Räume mit wesentlicher Transformationsgruppe konform-flach sein müssen (vgl. z.B. [**4**]).) Unsere Konstruktion benutzt die *Eguchi-Hanson-Metrik* in Dimension 4 (welche eine *hyper-Kählersche Metrik* der Riemannschen Geometrie darstellt) und kann interpretiert werden als eine *konforme Vervollständigung* von Räumen, welche das asymptotische Verhalten des 5-dimensionalen Minkowskischen Raumes besitzen. Es muss erwähnt werden, dass die konstruierte Metrik nur *einmal* stetig differenzierbar ist. Jedoch ist es denkbar, dass die konforme Struktur glatt ist. Das bleibt an dieser Stelle unklar.

Bisher haben wir in dieser Zusammenfassung keine Resultate und Untersuchungen zur *CR-Geometrie* beschrieben. Das wird sich nun ändern, denn die beiden letzten Kapitel beschäftigen sich intensiv mit der CR-Geometrie und ihrer Verbindung zur konformen Geometrie. Die ursprüngliche Motivation für unser Interesse an der CR-Geometrie geht dabei wieder auf die Twistorgleichung der Lorentzschen Spin-Geometrie zurück. Die folgende Diskussion lässt sich auch sehr gut in das Thema der konformen Holonomietheorie integrieren. Der wesentliche Punkt in den Kapiteln 5 und 6 ist, dass mit Hilfe der (verallgemeinerten) *Fefferman-Konstruktion* aus CR-Geometrien konforme Räume entstehen, welche interessante Invarianzeigenschaften besitzen. (Auf der anderen Seite ist die Fefferman-Konstruktion natürlich auch ein sehr geeignetes Hilfsmittel zur Untersuchung der *CR-Invariantentheorie*.) Konkret werden wir festellen, dass die Fefferman-Konstruktion konforme Räume produziert, welche spezielle Lösungen von konform-invarianten Differentialgleichungen, wie der Twistorgleichung, zulassen und auch besondere Holonomieeigenschaften besitzen (vgl. [**117, 17, 40**]).

Das Kapitel 5 basiert auf der Arbeit [**112**]. Die grundlegende Annahme ist hierbei eine *partiell integrierbare CR-Struktur* vom *Hyperflächentyp*, welche zusätlich mit einer *pseudo-Hermitischen Struktur* versehen ist. Man beachte, dass die Annahme der partiellen Integrierbarkeit im Allgemeinen die Existenz eines nicht-trivialen *Nijenhuisschen Torsionstensor* $\mathcal{N}_J$ zur Folge hat. Zunächst stellt sich das Problem der Existenz eines ausgezeichneten linearen Zusammenhangs, welche durch die Wahl der pseudo-Hermitischen Struktur bestimmt ist. In der Tat können wir mit Hilfe der *Levi-Form* zur pseudo-Hermitischen Struktur und einer geeigneten Normalisierung der Torsion Bedingungen angeben, welche einen linearen Zusammenhang eindeutig bestimmen (vgl. Lemma 17). Wir nennen diesen Zusammenhang den *verallgemeinerten Tanaka-Webster-Zusammenhang*, da er im Fall integrabler CR-Strukturen tatsächlich mit dem klassischen Tanaka-Webster-Zusammenhang einer pseudo-Hermitischen Geometrie übereinstimmt (vgl. z.B. [**149, 152, 102, 103, 67, 48**]). Unsere explizite Konstruktion richtet sich nach Methoden, wie sie in [**17**] benutzt werden. In Abschnitt 5.3 konstruieren wir dann den Fefferman-Raum mit zugehöriger Metrik einer partiell integrierbaren CR-Mannigfaltigkeit mit pseudo-Hermitischer Struktur. Diese Konstruktion verläuft analog zur intrinsischen Fefferman-Konstruktion, welche auf J.M. Lee zurückgeht, d.h. mit Hilfe eines Weyl-Zusammenhangs auf dem kanonischen Kreisbündel des zugrundeliegenden CR-Raumes funktioniert (vgl. [**102**]). Es wird dann auch hier gezeigt, dass die konforme Klasse der Fefferman-Metrik nicht von der Wahl der pseudo-Hermitischen Struktur abhängt, und somit ist klar, dass die Fefferman-Konstruktion auch unter der allgemeineren Annahme der partiellen Integrierbarkeit eine CR-Invariante ist (vgl. Theorem 21).

Überraschender Weise stellen wir fest, dass die Benutzung des Weyl-Zusammenhangs (auch im speziellen integrablen Fall) gar nicht zwingend notwendig ist, um ein konforme Klasse zu konstruieren, welche nicht von der Wahl der pseudo-Hermitischen Form abhängt. Es ist eine Tatasche, dass man eigentlich sogar jeden beliebigen Zusammenhang im kanonischen Kreisbündel wählen kann, und am Ende einen Fefferman-Raum erhält, der nur von der CR-Struktur und einer *Eichform* ℓ abhängt. Wir nennen diese Erweiterung die *geeichte Fefferman-Konstruktion der partiell integrierbaren CR-Geometrie* (vgl. Definition 4). Ihr Nutzen, vom Standpunkt der CR-Geometrie aus gesehen, ist eher zweifelhaft, und vermutlich überhaupt gar nicht vorhanden! Trotzdem werden wir in Kapitel 6 sehen, dass die geeichte Fefferman-Konstruktion ein Phänomen im Reich der konformen Geometrie erklären kann. Auf jeden Fall berechnen wir in den folgenden Abschnitten 5.4 bis 5.6 Eigenschaften der Torsion, die Skalarkrümmung der geeichte Fefferman-Metriken (bzgl. der *Webster-Skalarkrümmung*; vgl. Theorem 22) und wenden den *Bochner-Laplace-Operator* auf das *fundamentale* Vektorfeld χ_K an, welches ein vertikales Killingvektorfeld in der geeichte Fefferman-Konstruktion darstellt (vgl. Proposition 16). All diese Berechnungen ermöglichen uns die Anwendung des Splittingoperators $\mathbf{S}$ auf das fundamentale Vektorfeld $2\chi_K$ zu berechnen, so dass wir das Resultat bzgl. der gegebenen Objekte explizit identifizieren können. Es sei bemerkt, dass wir zu diesem Zweck zunächst einmal auch überhaupt den Splittingoperator $\mathbf{S}$ bzgl. einer kompatiblen Metrik auf einem beliebigem konformen Raum (M, c) kennen müssen. Diese Berechnung wird in Abschnitt 5.7 durchgeführt. Insgesamt können wir am Ende feststellen, dass die Anwendung $\mathbf{S}(2\chi_K)$ im Rahmen der geeichten Fefferman-Konstruktion für partiell integrierbare

CR-Strukturen auf eine orthogonale komplexe Struktur $\mathcal{J}$ des Standardtractorbündels $\mathcal{T}$ führt, zumindest unter gewissen Bedingungen an die Eichform ℓ. Die abschließende Folgerung von Kapitel 5 (vgl. Theorem 23) ist dann der Nachweis der Konstruktion einer Familie von Lösungen $\mathcal{J}$ der konform-invarianten Tractorengleichung

$$\nabla^{nor}\mathcal{J} = -\Omega^{nor}(\Pi_H(\mathcal{J}),\cdot) \qquad \text{mit} \qquad \mathcal{J}\bullet\mathcal{J} = -id|_{\mathcal{T}} .$$

Abschließend werden in Kapitel 6 die folgenden drei Sachverhalte diskutiert. Zunächst betrachten wir Lösungen $\mathcal{J}$ der obigen Tractorengleichung auf einem beliebigen Raum (M,c) mit konformer Struktur. Es stellt sich heraus, dass allein mit Hilfe von $\mathcal{J}$ eine natürliche lokale Konstruktion existiert, welche in eindeutiger Art und Weise eine partiell integrierbare CR-Struktur vom Hyperflächentyp auf einer Quotienten-Mannigfaltigkeit erzeugt (wobei die Dimension des Quotientenraumes um eins reduziert ist; vgl. auch [**39**]). Weiterhin können wir dann zeigen, dass die konforme Struktur c auf M lokal äquivalent ist zu einer geeichten Fefferman-Metrik über dem induzierten CR-Quotienten bei geeigneter Wahl der Eichform ℓ. Zusammen mit den Resultaten aus Kapitel 6 haben wir dann also eine vollständige lokale Charakterisierung und ein Konstruktionsprinzip für Räume (M,c) mit Lösungen $\mathcal{J}$ der obigen Tractorengleichung gefunden (vgl. Theorem 24). Soviel zur Rechtfertigung unserer geeichten Fefferman-Konstruktion!

Das zweite Argument des Kapitel 6 betrifft die Holonomie-Charakterisierung der klassischen, intrinsischen Fefferman-Konstruktion für integrierbare CR-Strukturen nach J.M. Lee. Dieses charakterisierende Resultat wurde in unserer Arbeit [**113**] bewiesen (vgl. auch [**40**]). Die Aussage ist, dass ein konformer Raum (M,c), dessen konforme Tractor-Holonomie $Hol(\mathcal{T})$ soweit reduziert ist, so dass sie in der irreduziblen Untergruppe $\mathrm{U}(p+1,q+1)$ der Möbiusgruppe enthalten ist, zumindest lokal konform-äquivalent zu einem klassischen Fefferman-Raum über einer integrierbaren CR-Mannigfaltigkeit ist. Tatsächlich ist sogar eine stärkere Aussage wahr. Wir können nämlich im Tractor-Kalkül zeigen, dass wenn die konforme Holonomie-Algebra $\mathfrak{hol}(\mathcal{T})$ mindestens bis auf $\mathfrak{u}(p+1,q+1)$ reduziert ist, dann ist sie sogar mindestens bis zur speziell unitären Algebra $\mathfrak{su}(p+1,q+1)$ reduziert (vgl. Theorem 25). Dieses Resultat bedeutet dann auch, dass eine konforme Holonomiegruppe niemals identisch zur ganzen $\mathrm{U}(p+1,q+1)$ sein kann.

Die Holonomie-Charakterisierung der klassischen Fefferman-Konstruktion in Kombination mit der Holonomie-Charakterisierung der konformen Einstein-Bedingung lässt uns dann zum letzten Streich ausholen. Es ist wohlbekannt, dass eine klassische Fefferman-Metrik niemals eine Einsteinmetrik ist. Aber offensichtlich, wenn ein Raum eine konforme Holonomiegruppe in $\mathrm{U}(p+1,q+1)$ hat und außerdem die konforme Holonomie einen Standardtractor fixiert, dann sollte das die konforme Holonomiegruppe einer Fefferman-Metrik sein, die zumindest im konformen Sinne fast-Einsteinsch ist. Da es keinen Grund gibt anzunehmen, dass solche konformen Holonomiereduktion für einen Raum nicht möglich sind (abgesehen von der trivialen Holonomie), sollte es also tatsächlich möglich sein, solche Holonomie-reduzierten konformen Räume zu erzeugen. In der Tat, basierend auf unserer Arbeit [**114**], werden wir in den letzten Abschnitten des Kapitel 6 ein Konstruktionsprinzip für sogenannte *(TSPE)-Strukturen* entwickeln. Dabei ist mit (TSPE)-Struktur eine *pseudo-Einsteinsche Struktur* im Sinne von Lee (vgl. [**103**]) gemeint, welche simultan eine *transversale Symmetrie* in Form des zugehörigen *Reeb-Vektorfeldes* auf der zugrundeliegenden integrierbaren CR-Mannigfaltigkeit erzeugt. Eine solche (TSPE)-Struktur kann konkret mit Hilfe von *Kähler-Einstein-Räumen* konstruiert werden. Tatsächlich zeigen wir in Theorem 26, dass zumindest lokal jede (TSPE)-Struktur in natürlicher Weise von einer Kähler-Einsteinmetrik stammt. Da (TSPE)-Strukturen zu ∇^{nor}-parallelen Standardtractoren der CR-Geometrie korrespondieren, und der Lift solcher CR-Tractoren entlang der Fefferman-Konstruktion zu ∇^{nor}-parallelen Standardtractoren der konformen Geometrie führt, wird deutlich, dass jede Fefferman-Metrik, welche über einer (TSPE)-Struktur auf einem integrablen CR-Raum entsteht, konform fast-Einsteinsch sein muss! Tatsächlich sind wir in der Lage (lokal) einen konformen Faktor explizit zu bestimmen, der die Fefferman-Metrik einer beliebigen (TSPE)-Struktur auf eine Einsteinmetrik skaliert. Die resultierenden lokalen Ausdrücke für diese *Fefferman-Einsteinmetriken* werden in Theorem 27 angegeben. Global auf dem kanonischen Kreisbündel ist eine Fefferman-Metrik niemals konform Einsteinsch! Das zeigt, dass es Situationen geben kann, in denen neben den Einsteinmetriken auch andere kompatible Metriken einer konformen Klasse in besonders natürlicher Weise hervortreten, bzw. in denen die (fast)-Einsteinsche Eigenschaft im konformen Sinn eigentlich als die *natürlichste* Bedingung angesehen werden muss.

Part 1

The Basic Theory

CHAPTER 0

The Basic Theory

This is an introductory chapter consisting of 13 sections, whose aim is to provide essentially all the basic and necessary material of the theory, on which the topics and investigations in Part 2 (Chapter 1 to 6) of our work are based. The collection starts with the very common definitions of *differential geometry*, carries on with the theory of *parabolic Cartan geometries* and *tractor calculus* and finally comes to a brief discussion of the *ambient metric* and *Fefferman constructions*. Thereby, some of the discussed themes will be described in detail and are based on a rather broad footing, providing enough background for investigations that go beyond our aims, whereas some other parts of the theory are described briefly, providing just the necessary facts. The central topics are *conformal differential geometry* and *CR-geometry*, and other material in this introduction and later is developed in relation to these.

Section 0.1 to 0.3 are concerned with the standard facts of *differentiable manifolds, Lie groups* and *algebras, their representations, fibre bundles, principal connections* and *differential operators*. All these things can be found in many text books. We refer to [**91**]. Section 0.4 reviews the geometry of Riemannian metrics of arbitrary signature, again on a basic level with emphasis on features related to Lorentzian and conformal geometry (cf. e.g. [**91, 128, 139**] and many other sources). In the following section, we introduce *parabolic Cartan geometry*. As a general reference for the theory of Cartan connections we cite [**144**]. Our approach to parabolic geometry is essentially influenced by [**41, 64, 145, 45, 42, 35**]. In Section 0.6 we introduce conformal geometry in the framework of parabolic Cartan geometry. The relation with semi-Riemannian geometry is discussed as well (cf. [**46, 90, 43, 44**]). In the subsequent section we treat parabolic geometries from the viewpoint of *tractor calculus*, which can be seen as equivalent to the Cartan formulation. Tractor calculus was initially invented by T. Thomas in [**150**] and revived in [**14**]. The emphasis of our discussion is on conformal tractor calculus (cf. [**36, 64, 45, 37, 38**]). Further, in Section 0.8, we give a short explanation of *spin structures* in semi-Riemannian and conformal geometry (cf. [**100, 15, 19, 139**]). A spin version of conformal tractor calculus is implemented as well, which gives rise to the notion of *twistors*. This allows us to discuss the *twistor equation*, which is a fundamental topic for our studies. In Section 0.10 we discuss partially integrable CR-geometry of hypersurface type from the viewpoint of parabolic geometry and also via the classical approach in relation with pseudo-Hermitian structures (cf. [**149, 152, 103, 86, 48, 41, 39**]). The Fefferman-Graham ambient metric construction is a very useful tool for studying *conformal* and *CR-invariant theory*. The ambient construction is closely related to the *Poincaré-Einstein model*, which is of particular interest in relation with such research topics as *Q-curvature* and the *AdS/CFT correspondence* in physics. We review both constructions in Section 0.11 (cf. [**53, 54, 55, 56, 74, 75, 76, 77, 78, 79**]). The *metric cone* construction of Section 0.9 (vgl. [**62, 12**]) can be seen as a simplified version of the ambient metric construction for Einstein spaces. Finally, in Section 0.12, we discuss the Fefferman construction, which relates CR-geometry with conformal geometry (cf. [**53, 31, 102, 146, 73, 33, 34**]). This construction was originally invented by C. Fefferman in order to study geometric properties and the invariant theory of pseudoconvex boundaries. In Section 0.13 we give a summary of our results in Part 2 (Chapter 1 to 6).

1. Manifolds, Vector Bundles and Differential Operators

Let $\mathbb{R}^s$ denote the Euclidean vector space of dimension $s \geq 0$. A *topological manifold* is a Hausdorff topological space X, which admits a countable basis of open sets and is locally homeomorphic to $\mathbb{R}^n$, where $n = dim(X) \in \mathbb{N}$ is a fixed number, the dimension of X. A space X decomposes into disjoint connected components. If X consists of exactly one component then the space X is called *connected*. We denote the fundamental group of a connected space X by $\pi_1(X)$. A connected topological manifold X is called simply connected if $\pi_1(X)$ is trivial.

A *differentiable structure* $\mathcal{F}$ on a topological manifold X is given by a collection of coordinate systems (or charts), which cover the space X such that all coordinate changes on the overlap of any two charts give rise to diffeomorphisms. Then we call a pair

$$M := (X, \mathcal{F})$$

a (differentiable) manifold. The class of differentiability of M is determined by the differentiability of the coordinate changes. In this work all manifolds (without boundary) are assumed to be smooth, i.e., all coordinate changes are infinitely often differentiable and we denote this class by C^∞. A manifold is called real analytic if all coordinate changes are of class C^ω. The Cartesian product $M_1 \times M_2$ of two manifolds is in a natural way a manifold. A map

$$f : M \to N$$

between manifolds M and N is differentiable of class C^k if the induced maps with respect to any coordinate charts are k-times differentiable. Apart from some exceptions we will usually work with smooth maps.

Let E, M and F be manifolds with a (smooth) projection $\pi : E \to M$. The tuple

$$\mathcal{E} = (E, \pi, M; F)$$

is called a (locally trivial) *fibre bundle* over M with total space E and fibre type F if for any point $x \in M$ there exists an open neighbourhood $U(x)$ of x in M and a diffeomorphism

$$\Psi : \ \pi^{-1}(U(x)) \ \to \ U(x) \times F$$

such that the projection π is just the composition of Ψ and the natural projection onto the first factor of $U(x) \times F$. Usually, we will denote the total space of a fibre bundle $\mathcal{E}$ by $\mathcal{E}(M)$ or simply $\mathcal{E}$ again (via abuse of notion).

A *vector bundle* $\mathcal{E} \xrightarrow{\pi} M$ of rank $s \geq 0$ is a fibre bundle, where any preimage $\mathcal{E}_x := \pi^{-1}(x)$, $x \in M$, is a real vector space of dimension s and there exist charts $\Psi : \pi^{-1}(U(x)) \to U(x) \times \mathbb{R}^s$ for all $x \in M$ such that the natural projection at x onto the second factor $\mathbb{R}^s$ is a vector space isomorphism. Let $\mathcal{E} \xrightarrow{\pi} M$ and $\mathcal{E}' \xrightarrow{\pi'} M'$ be any two vector bundles. Then a map $f : \mathcal{E} \to \mathcal{E}'$ is called a homomorphism of vector bundles if $\pi' \circ f = l \circ \pi$ for some map $l : M \to M'$ and the restriction $f|_{\mathcal{E}_x}$ is a vector space homomorphism from $\mathcal{E}_x$ to $\mathcal{E}'_{l(x)}$ for all $x \in M$. A subset $\mathcal{V}$ of $\mathcal{E}$ is a subbundle in $\mathcal{E}$ if the inclusion is a vector bundle homomorphism, which induces the identity id_M on the base M. If M is a manifold and $l : M \to M'$ is a map to the base M' of a vector bundle $\mathcal{E}'$, then the pull-back along l induces a vector bundle $l^*\mathcal{E}'$ on M with fibre F'.

There are various other operations to produce vector bundles from given ones. For example, let $\mathcal{A}$ and $\mathcal{B}$ be vector bundles on M. Then we have the direct sum $\mathcal{A} \oplus \mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$, the dual bundle $\mathcal{A}^*$ and the vector bundles $\Lambda^k \mathcal{A}^* \otimes B$ of alternating k-forms on $\mathcal{A}$ over M with values in $\mathcal{B}$. If $\mathcal{V}$ is a vector subbundle of $\mathcal{A}$ then we have the quotient bundle $\mathcal{A}/\mathcal{V}$. Probably the most prominent vector bundles on a manifold M^n of dimension n are the *tangent bundle*, denoted by TM, and its dual T^*M. Moreover, we denote by

$$T^{(k,l)}M = \big(\otimes^k TM\big) \otimes \big(\otimes^l T^*M\big)$$

the bundle of (k,l)-tensors. This gives rise to the tensor algebra $\oplus_{k,l=0}^{\infty} T^{(k,l)}M$.

Moreover, we have the vector bundles $\Lambda^k T^*M$, $0 \le k \le n$, of exterior forms of degree k on M. The $\wedge$-product makes $\oplus_{k=0}^{n} \Lambda^k T^*M$ into an algebra. (Note that with our conventions, $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ for any 1-forms α and β.) We call a k-form α on M *simple* if it is a $\wedge$-product of 1-forms only. The insertion of a vector $X \in TM$ into an exterior form α is denoted by $\iota_X \alpha$. A differentiable map $f : M \to N$ between manifolds naturally induces the bundle homomorphism

$$f_* = df : TM \to TN \ ,$$

which is called the tangent map or differential of f. Exterior forms on N pull back to M via the differential f_*. There are also the bundles $S^k(T^*M)$ of symmetric tensors of degree $k \in \mathbb{N}$ on M. The symmetric product $\circ$ of two 1-forms α, β is defined by $\alpha \circ \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.

Now let $\mathcal{E} \xrightarrow{\pi} M$ be an arbitrary vector bundle. A (smooth) map $s : M \to \mathcal{E}$ is called a *section* of $\mathcal{E}$ if $\pi \circ s : M \to M$ is the identity map. (In case s is only defined on an open subset U of M we speak of a local section in $\mathcal{E}$.) The set of (global) sections in $\mathcal{E}$ over M is denoted by $\Gamma(\mathcal{E}; M)$ or just $\Gamma(\mathcal{E})$. A section in a tangent bundle TM is called a *vector field* on M and the space of vector fields is denoted by $\mathfrak{X}(M)$. Sections in $T^{(k,l)}M$ are called tensors of type (k,l) and sections in $\Lambda^k T^*M$ are called differential k-forms on M. The set of smooth differential k-forms is denoted by $\Omega^k(M)$. In particular, we have the space of smooth functions $C^\infty(M)$ on M. The differential df of a function $f \in C^\infty(M)$ is a 1-form. The derivative of f in direction of a vector field $X \in \mathfrak{X}(M)$ is again a smooth function on M, denoted by $Xf := df(X)$. More generally, the exterior derivative d maps a k-form $\alpha \in \Omega^k(M)$ to the $(k+1)$-form $d\alpha \in \Omega^{k+1}(M)$.

Let $X \in \mathfrak{X}(M)$ be a smooth vector field on a manifold M. A (differentiable) curve $\gamma(t)$ defined on an interval $I \subset \mathbb{R}$ is an *integral curve* of X if

$$\gamma'(t) = X(\gamma(t))$$

for all $t \in I$. To any point $p \in M$ there exists a unique integral curve $\gamma_p^X(t)$, which is defined on a maximal open interval $I \subset \mathbb{R}$ including $0 \in \mathbb{R}$ such that $\gamma_p^X(0) = p$. We set $X_t(p) := \gamma_p^X(t)$ and call the map $p \in M \mapsto X_t(p) \in M$ the flow of p along X at time t. For any $p \in M$ there exists a small $t > 0$ and a neighbourhood $U(p)$ such that the map $\tilde{p} \in U(p) \mapsto X_t(\tilde{p}) \in M$ is a diffeomorphism onto its image, i.e., the flow X_t is locally always defined for small times. The vector field X is called complete if the maximal integral curve γ_p^X is defined on $\mathbb{R}$ through any point $p \in M$. In this case the flow X_t of X is globally defined on M and forms a 1-parameter group of diffeomorphisms of M. The commutator $[X,Y] \in \mathfrak{X}(M)$ of two vector fields $X, Y \in \mathfrak{X}(M)$ is defined by

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}\left(Y - (X_t)_* Y\right) \ .$$

More generally, we have the *Lie derivative* $\mathfrak{L}$ of tensors fields. For example, the Lie derivative of a k-form α with respect to $X \in \mathfrak{X}(M)$ is denoted by $\mathfrak{L}_X \alpha$, which is again a k-form on M. (Note that with our conventions the exterior derivative of any 1-form α satisfies $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$.)

A map $f : M \to N$ between manifolds is called an *immersion* if the differential $df : TM \to TN$ is non-singular at every point $x \in M$. If, in addition, the map f is injective and a homeomorphism onto its image $f(M)$ in N, then we call f an *embedding* of M into N and the image $f(M)$ is a *submanifold* of N. A subbundle $\mathcal{E}$ with rank r of the tangent bundle TM over a manifold M is called a *distribution*. A distribution $\mathcal{E}$ is called involutive if $[X,Y] \in \Gamma(E)$ for all $X, Y \in \Gamma(E)$. The Frobenius' Theorem states that an involutive distribution can be integrated, i.e., there exists for any point $p \in M$ a (maximal) integral manifold $E_p \subset M$ of dimension r, which has the corresponding restriction of $\mathcal{E} \subset TM$ as its tangent space. Although an integral manifold is the image of an injective immersion, it might not be a submanifold. In such a case we call it a *weak* submanifold.

Next we want to introduce the notion of *differential operators*. So let $\mathbb{R}^n$ be the Euclidean space equipped with standard coordinates $(x^1, \cdots, x^n)$. The tangent space of $\mathbb{R}^n$ is spanned by the coordinate vectors $\{\partial/\partial x^1, \cdots, \partial/\partial x^n\}$. Now let us consider an open subset $U \subset \mathbb{R}^n$. The map

$$f \in C^\infty(U) \quad \mapsto \quad \partial f/\partial x^i = df(\partial/\partial x^i) \in C^\infty(U)$$

defines an $\mathbb{R}$-linear operator on functions. More generally, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers and let $|\alpha| = \sum_{i=1}^n \alpha_i$ be its length. Then we denote $x^\alpha := (x^1)^{\alpha_1} \cdot \ldots \cdot (x^n)^{\alpha_n}$ and $D^\alpha := (\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^n)^{\alpha_n}$. The operator D^α acts $\mathbb{R}$-linearly on smooth functions. Then let $x_o \in U$ be a base point. The *Taylor series expansion* of a smooth function f at $x_o \in U$ to order $k \geq 0$ is given by

$$\sum_{|\alpha|\leq k} \frac{(D^\alpha f)(x_o)}{|\alpha|!}(x - x_o)^\alpha \ + \ R_k(|x - x_o|) \ ,$$

where $|\alpha|! = \Pi_{i=1}^n(\alpha_i!)$ and $R_k(|x - x_o|)$ contains the higher order terms.

Two smooth functions f, g are called equivalent to order k at $x_o \in U$ if $D^\alpha(f - g)(x_o) = 0$ for all $|\alpha| \leq k$, i.e., the function $f - g$ vanishes to order k. In this case we write $j^k_{x_o} f = j^k_{x_o} g$ and $j^k_{x_o} f$ is called the k-jet of f at $x_o \in U$. More generally, we have the k-jet $j^k_{x_o} f$ at $x_o \in M$ for any smooth mapping $f : M \to N$ between manifolds, which contains the information of all partial derivatives of f to order k with respect to some coordinate charts around x_o in M and $f(x_o)$ in N. In particular, let $\mathcal{E} \xrightarrow{\pi} M$ be a vector bundle of rank r. A section $f \in \Gamma(\mathcal{E})$ gives rise to a k-jet $j^k_{x_o} f$ at $x_o \in M$ and the set of all k-jets of sections in $\mathcal{E}$ at a point forms a vector bundle over M, which is denoted by $J^k\mathcal{E} \xrightarrow{\pi} M$ and which is called the kth jet prolongation of $\mathcal{E}$. In particular, any section $f \in \Gamma(\mathcal{E})$ naturally gives rise to a section $j^k f \in \Gamma(J^k\mathcal{E})$ and we have $\pi_k \circ j^k f = j^{k-1} f$, where $\pi_k : J^k \mathcal{E} \to J^{k-1}\mathcal{E}$ denotes the natural bundle map, which forgets the information of all derivatives of order k of a section in $\mathcal{E}$.

Now let $\mathcal{V} \xrightarrow{\pi'} M$ be another vector bundle on M of rank s and let $D : J^k\mathcal{E} \to \mathcal{V}$ be a vector bundle homomorphism over M. Then the map

$$\mathcal{D} \ : \ f \in \Gamma(\mathcal{E}) \quad \mapsto \quad D(j^k f) \in \Gamma(\mathcal{V})$$

is given locally, with respect to coordinates $(x^1, \ldots, x^n)$ on $U \subset M$ and local frames in $\mathcal{E}|_U$ and $\mathcal{V}|_U$, by an expression of the form

$$\mathcal{D} = \sum_{|\alpha| \le k} A_\alpha(x) D^\alpha, \tag{1}$$

where the A_α's are $(r \times s)$-matrix valued functions on U . In general, a map $\mathcal{D}$ acting on (smooth) sections of a vector bundle, which is locally expressible in the above form (1) with respect to some coordinates, is called a *differential operator*. One can show that any differential operator $\mathcal{D} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{V})$ is induced by some bundle homomorphism $D : J^k\mathcal{E} \to \mathcal{V}$. If this map D does not factor through the projection π_k to a homomorphism $D' : J^{k-1}\mathcal{E} \to \mathcal{V}$, then we say that $\mathcal{D}$ is a differential operator of order k.

Finally, we define here manifolds with *boundary*. For this purpose, let us consider the half space

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \ : \ x^1 \ge 0\} \ .$$

The boundary $\partial\mathbb{R}^n_+$ of $\mathbb{R}^n_+$ is the set $\{0\} \times \mathbb{R}^n$. We call a function f defined on an open subset U of $\mathbb{R}^n_+$ smooth (resp. differentiable of class C^k) if in a neighbourhood of any point $p \in U$, f is the restriction of some smooth (resp. C^k-) function on $\mathbb{R}^n$. A Hausdorff topological space X of dimension n equipped with a differentiable structure $\mathcal{F}_b$, which allows coordinate maps to the half space $\mathbb{R}^n_+$, gives rise to a manifold $M^n = (X, \mathcal{F}_b)$ with boundary. The boundary is denoted by ∂M and is a manifold of dimension $n-1$. The interior $M \smallsetminus \partial M$ is a manifold without boundary, which we sometimes call the *bulk* of M. Again, the differentiability of M depends on the coordinate changes in $\mathcal{F}_b$. Usually, the bulk and the boundary (considered as manifolds) are assumed to be smooth, whereas the class of differentiability at the boundary in M needs to be specified from case to case. Often the boundary of a space M is (locally) given by the zero set of a *defining function* r, which is differentiable up to a certain order (cf. Section 0.11). In this case any object on M (e.g. functions or tensor fields) can be expanded into a Taylor series with respect to r. For example, the expansion of a function $f : M \to \mathbb{R}$ with respect to r at the boundary $\{r = 0\}$ to order k is given by an expression of the form

$$f(r) \ = \ \sum_{j=0}^{k} \frac{1}{j!} \cdot \frac{\partial^j f}{\partial r^j} \cdot r^j \ + \ o(r^k) \ ,$$

where $o(r^k)$ contains the expansion terms of higher order.

2. Lie Groups, Representations and Principal Bundles

A group $(G, \cdot)$ is called a *Lie group* if G is a smooth manifold (of finite dimension n) and the group operation $(a, b) \in G \times G \mapsto a \cdot b^{-1} \in G$ is a smooth map. A (continuous) group homomorphism $\rho : G \to G'$ between Lie groups G and G' is called a Lie group homomorphism. A closed subgroup H of a Lie group G is again a Lie group with induced differentiable (submanifold) structure. Also the quotient G/H admits a natural differentiable structure such that the canonical projection is smooth. Moreover, to any Lie group G there exists a smooth universal covering space $\tilde{G}$ equipped with a group multiplication such that the canonical projection is a Lie group homomorphism. If G is connected then the Lie group $\tilde{G}$ is simply connected and unique up to isomorphism. It is called the universal covering group of G.

A smooth map

$$\begin{array}{rcccc} \beta & : & G \times M & \to & M \ , \\ & & (g,p) & \mapsto & \beta(g,p) = g \cdot p \end{array}$$

such that $g_1 \cdot (g_2 \cdot p) = (g_1 \cdot g_2) \cdot p$ for all $g_1, g_2 \in G$ and $p \in M$, where G is a Lie group and M a manifold, is called a Lie group action by G from the left on M (resp. G is called a (left) transformation group on M). Transformation groups from the right are similarly defined. If $\beta : G \times M \to M$ is a transitive group action of G on M then $M \cong G/H_p$ as smooth manifold, where H_p is the isotropy group (or stabiliser) at $p \in M$. In this case we call M a homogeneous space with transitive G-action. For example, let G be a Lie group. Then the map $L_a : G \to G, g \mapsto a \cdot g$, where $a \in G$ is a fixed group element, is a group action of G on itself, which is called left translation by a. Similarly, we have the right translations R_a. A Lie group G acts also from the left on itself by inner automorphisms:

$$\begin{array}{rcccc} \alpha & : & G \times G & \to & G \ , \\ & & (a,g) & \mapsto & \alpha_a(g) = aga^{-1} \ . \end{array}$$

A real vector space V (of arbitrary dimension) equipped with an $\mathbb{R}$-bilinear and skew-symmetric product $[\cdot,\cdot] : V \times V \to V$, which satisfies the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \text{for all } X,Y,Z \in V \ ,$$

is called a *Lie algebra* and $[\cdot,\cdot]$ is the Lie bracket on V. A subset W of V is called an ideal of $(V,[\cdot,\cdot]_V)$ if $[X,Y]_V \in W$ for all $X \in W$, $Y \in V$ and a subset S of V is a Lie subalgebra if $[X,Y]_V \in S$ for all $X,Y \in S$. A linear map $\psi : (V,[\cdot,\cdot]_V) \to (W,[\cdot,\cdot]_W)$ between Lie algebras such that $[\psi(a),\psi(b)]_W = \psi([a,b]_V)$ for all $a,b \in V$ is called a Lie algebra homomorphism. For example, the commutator of vector fields on a manifold M produces a Lie algebra structure on the infinite dimensional real vector space $\mathfrak{X}(M)$.

A vector field X on a Lie group G is called left invariant if $(L_a)_*X = X$ for all $a \in G$. The vector field X is uniquely determined by its value X_e at the identity element $e \in G$. On the other hand, a vector X_e at $e \in G$ gives rise to a unique smooth left invariant vector field X on G by left translation. We denote the space of left invariant vector fields on a Lie group G by $\mathfrak{g}$. It has the same finite dimension as G, and since the commutator of left invariant vector fields is again left invariant, the space $\mathfrak{g}$ is naturally equipped with a Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}}$. We call $\mathfrak{g}$ (with induced Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$) the Lie algebra of G. The differential $d\rho : TG \to TG'$ of a Lie group homomorphism $\rho : G \to G'$ gives naturally rise to a Lie algebra homomorphism, which we denote by $\rho_* : \mathfrak{g} \to \mathfrak{g}'$. The Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ of a Lie group G and its universal covering $\tilde{G}$ are canonically isomorphic. On the other hand, any Lie algebra $\mathfrak{g}$ (of finite dimension) has a (simply connected) Lie group G, whose Lie algebra is $\mathfrak{g}$.

For example, let V be a real vector space of dimension n. A vector space isomorphism $V \cong \mathbb{R}^n$ induces a smooth differentiable structure on V (which does not depend on the chosen isomorphism). Using this differentiable structure the group $(\mathrm{Gl}(V), \circ)$ of linear automorphisms of V is in a natural way a Lie group, which is diffeomorphic to the group $\mathrm{Gl}(n, \mathbb{R}) = \mathrm{Gl}(n)$ of invertible $(n \times n)$-matrices. The corresponding Lie algebra is $\mathfrak{gl}(V)$, the space of endomorphisms of V with Lie bracket $[A, B] = A \circ B - B \circ A$ for $A, B \in \mathfrak{gl}(V)$. The algebra $\mathfrak{gl}(V)$ is isomorphic to the algebra $\mathfrak{gl}(n)$ of $(n \times n)$-matrices with matrix multiplication.

A *representation* of a Lie group G on a real vector space $\mathbb{V}^n$ of finite dimension n is given by a Lie group homomorphism $\rho : G \to \mathrm{Gl}(\mathbb{V})$. In this case the group G acts on $\mathbb{V}$ via ρ by $(g, v) \in G \times \mathbb{V} \mapsto g \cdot v \in \mathbb{V}$ and $\mathbb{V}$ is called a representation space of G or simply a (real) *G-module*. A representation of a Lie group G on $\mathbb{V}$ is called *irreducible* if there exists no proper subspace $\mathbb{W} \neq \{0\}$ in $\mathbb{V}$, which is invariant under the group action of G. The differential of a representation ρ induces a homomorphism $\rho_* : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$ of Lie algebras, and in particular, a representation of $\mathfrak{g}$ on $\mathbb{V}$, which is called the infinitesimal representation of G.

For example, any inner automorphisms α_g, $g \in G$, fixes the identity on a Lie group G and the corresponding differentials induce linear Lie algebra isomorphisms $(\alpha_g)_*$ on $\mathfrak{g}$. This construction gives rise to the adjoint representation of G on its Lie algebra,

$$\begin{array}{rcccl} Ad_G & : & G & \to & \mathrm{Gl}(\mathfrak{g}) \; , \\ & & g & \mapsto & (\alpha_g)_* \quad . \end{array}$$

Furthermore, we denote the differential of the adjoint representation by $ad_{\mathfrak{g}}$. This is a representation of $\mathfrak{g}$ on itself given by the Lie bracket

$$ad_{\mathfrak{g}}(X)(Y) = [X, Y], \qquad X, Y \in \mathfrak{g} \; .$$

A Lie algebra $\mathfrak{g}$ is called nilpotent (resp. solvable) if its lower central (resp. derived) series becomes trivial after a finite number of steps. Any Lie algebra $\mathfrak{g}$ admits a maximal solvable ideal, which is called the radical of $\mathfrak{g}$. If the radical of $\mathfrak{g}$ is trivial, then the Lie algebra $\mathfrak{g}$ is called *semisimple*. A Lie group G with semisimple Lie algebra $\mathfrak{g}$ is called semisimple as well. Any finite dimensional representation space $\mathbb{V}$ of a semisimple group G decomposes into a direct sum $\bigoplus \mathbb{V}_i$ of irreducible G-modules $\mathbb{V}_i$. A Lie algebra $\mathfrak{g}$, which is the direct sum of its centre $\mathfrak{g}_c$ and a semisimple part $\mathfrak{g}_0$ is called a *reductive* Lie algebra. For example, $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{R}$ is reductive, where the semisimple part $\mathfrak{sl}(n)$ denotes the Lie subalgebra of tracefree matrices in $\mathfrak{gl}(n)$. Now let $\mathfrak{g}$ be an arbitrary Lie algebra of finite dimension n. We define

$$B_{\mathfrak{g}}(X, Y) := tr(adX \circ adY)$$

to be the trace of the $\mathfrak{g}$-endomorphism $adX \circ adY$. Then $B_{\mathfrak{g}}$ is a symmetric bilinear form on $\mathfrak{g}$, which is called the *Killing form*. The Killing form $B_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant, i.e., $B_{\mathfrak{g}}([X, Y], Z) = B_{\mathfrak{g}}(X, [Y, Z])$ for all $X, Y, Z \in \mathfrak{g}$. The Killing form $B_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple.

Let $\mathbb{C}$ be the complex number field. We denote the real, resp., imaginary part of a complex number z by $Re(z)$ and $Im(z) \in \mathbb{R}$. With $\mathbb{C}^n$ we denote the complex n-space, which is a vector space of dimension n over $\mathbb{C}$. A complex Lie algebra $\mathfrak{g}$ is a complex vector space equipped with a $\mathbb{C}$-bilinear Lie bracket $[\cdot, \cdot]$. A real vector space V is complexified by $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Correspondingly, a real Lie algebra $\mathfrak{g}$ can be complexified to $\mathfrak{g}_{\mathbb{C}}$, where the Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ is given by complex bilinear extension of $[\cdot, \cdot]_{\mathfrak{g}}$. This construction gives rise to a complex Lie algebra. A complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is semisimple (as complex Lie algebra) if and only if the real Lie algebra $\mathfrak{g}$ is semisimple.

A left invariant vector field $X \in \mathfrak{g}$ on a Lie group G is always complete. In particular, the integral curve to X through the identity element is defined on $\mathbb{R}$. We denote this integral curve by $\gamma_e^X = \exp(tX)$. Then the mapping

$$\exp : \ X \in \mathfrak{g} \mapsto \exp(X) \in G$$

is called the *exponential map* of the Lie group G. Now let $\beta : M \times G \to M$ be a group action from the right on a manifold M. For any $p \in M$ and $A \in \mathfrak{g}$ the mapping $t \in \mathbb{R} \mapsto p \cdot \exp(tA)$ defines a curve in M and the differential of this curve at p gives rise to a vector denoted by $\chi_A(p)$. Then the mapping $p \in M \mapsto \chi_A(p) \in T_pM$ defines a smooth vector field, which we call the *fundamental vector field* (from the right) to $A \in \mathfrak{g}$ on M. There is a similar construction for fundamental vector fields coming from left group actions.

Next let us consider a (locally trivial) fibre bundle $\mathcal{G} = (E, \pi, M; H)$ over a manifold M with fibre a Lie group H. If H acts on E from the right preserving the fibres and acting simply transitive on each fibre of M, then we call $\mathcal{G}$ a principal bundle with structure group H. The right action of H on $\mathcal{G}$ generates fundamental vector fields χ_A for any $A \in \mathfrak{h}$. These vector fields project by π_* to the trivial vector on M, i.e., they are *vertical vectors* in $\mathcal{G}$. In fact, the fundamental vector fields span pointwise the subbundle $T^v\mathcal{G}$ of vertical vectors in $T\mathcal{G}$. A basic example for a principal fibre bundle is the frame bundle $Gl(M)$ of a manifold M. The bundle $Gl(M)$ consists of all linear bases of TM and it inherits a differentiable structure and a $\mathrm{Gl}(n)$-action on the right from its definition via the tangent bundle, i.e., the frame bundle $Gl(M)$ is a principal $\mathrm{Gl}(n)$-bundle.

Now let G be another Lie group, $\lambda : G \to H$ a Lie group homomorphism and let $\mathcal{K} \xrightarrow{\pi'} M$ be a principal G-bundle over M. A map $\Phi : \mathcal{K} \to \mathcal{G}$, which preserves the fibres over M and commutes with the Lie group actions, i.e., $\pi \circ \Phi = \pi'$ and $\Phi(p \cdot g) = \Phi(p) \cdot \lambda(g)$ for all $p \in \mathcal{K}$ and $g \in G$, is called a λ-reduction of $\mathcal{G}$. If the principal G-bundle $\mathcal{K}$ is a ι-reduction, where $\iota : G \to H$ is an inclusion of G, then we call $\mathcal{K}$ simply a G-reduction of $\mathcal{G}$. A G-reduction of the general linear frame bundle $GL(M)$ over a manifold M is called a *G-structure* on M.

Furthermore, let $\mathcal{G} \xrightarrow{\pi} M$ be a principal H-bundle and let $\nu : H \times P \to P$ be a left action of H on a manifold P. The group H acts on the product space $\mathcal{G} \times P$ by $h \cdot (g, p) = (g \cdot h, h^{-1} \cdot p)$, where $h \in H$ and $(g, p) \in \mathcal{G} \times P$. The quotient of $\mathcal{G} \times P$ through the orbits of this H-action is denoted by

$$\mathcal{R} := \mathcal{G} \times_\nu P \ .$$

The (locally trivial) bundle $\mathcal{R} \to M$ is the *ν-associated fibre bundle* to $\mathcal{G}$ over M with fibre P. If ν is a group homomorphism into a Lie group P, we call the fibre bundle $\mathcal{R}$ the ν-extension of $\mathcal{G}$ by the group P. This ν-extension $\mathcal{R}$ is in a natural way a principal P-bundle, which admits $\mathcal{G}$ as a natural subbundle. In fact, the corresponding embedding is given by

$$\begin{array}{rccl} \iota & : & \mathcal{G} & \to & \mathcal{R} \ , \\ & & g & \mapsto & [g, e] \ , \end{array}$$

where $[g, e] \in \mathcal{R}$ denotes the H-orbit through $(g, e) \in \mathcal{G} \times P$.

3. Principal Connections and Holonomy

Let H be a Lie group with Lie algebra $\mathfrak{h}$ and let $\mathcal{G} \xrightarrow{\pi} M$ be an H-principal bundle over a manifold M^n of dimension n. A 1-form $\omega : T\mathcal{G} \to \mathfrak{h}$ with the properties

(1) $\omega(\chi_A) = A$ for all $A \in \mathfrak{h}$ and
(2) $R_h^*\omega = Ad_H(h^{-1}) \circ \omega$ for all $h \in H$,

where R_h denotes the right action by $h \in H$ on $\mathcal{G}$, is called a principal connection (1-form) on $\mathcal{G}$. The kernel of ω generates a distribution $T^h\mathcal{G}$ of rank n in $T\mathcal{G}$. The intersection of $T^h\mathcal{G}$ with any vertical vector in $T\mathcal{G}$ is trivial. Thus the tangent space $T\mathcal{G}$ has the direct sum decomposition $T^v\mathcal{G} \oplus T^h\mathcal{G}$ with projections π_v and π_h. The vector bundle $T^h\mathcal{G}$ is called the horizontal distribution in $T\mathcal{G}$ with respect to the connection ω. The curvature of the connection ω is defined by

$$\Omega := d\omega \circ \pi_h \ .$$

Obviously, the curvature Ω is a 2-form on $\mathcal{G}$ and insertion of vertical vectors produces zero, i.e., $\iota_X\Omega = 0$ for all $X \in T^v\mathcal{G}$. The curvature Ω can be seen as measure for the integrability of the horizontal distribution $T^h\mathcal{G}$. In particular, $\Omega \equiv 0$ holds iff the distribution $T^h\mathcal{G}$ is integrable. The structure equation for the connection ω is

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X),\omega(Y)]_{\mathfrak{h}} \qquad \text{for all } X,Y \in T\mathcal{G} \ ,$$

and the Bianchi identity says that

$$d\Omega \circ \pi_h = 0 \ .$$

By use of a (local) section $s : U \subset M \to \mathcal{G}$ the connection ω and its curvature Ω can be pulled back to the base manifold M and we obtain local connection and curvature forms

$$\omega^s := \omega \circ ds \qquad \text{and} \qquad \Omega^s := \Omega \circ ds \ .$$

The frame bundle $Gl(M) \xrightarrow{\pi} M$ of a manifold is equipped with the canonical (soldering) form

$$\theta : TGl(M) \to \mathbb{R}^n \ ,$$

which maps a vector X at $p \in Gl(M)$ to the coordinates of $d\pi(X) \in T_{\pi(p)}M$ with respect to the frame p at $\pi(p) \in M$. A connection form ω on $Gl(M)$ is called a *linear connection*. The structure equations for a linear connection ω can be written as

$$d\theta = -\omega \wedge \theta + \Phi \ ,$$

where Φ is called the torsion 2-form of ω, and

$$d\omega = -\omega \wedge \omega + \Omega$$

with $\omega \wedge \omega(X,Y) = [\omega(X),\omega(Y)]_{\mathfrak{h}}$ for $X,Y \in T\mathcal{G}$ (cf. [**91**]). The first and second Bianchi identity for the curvature form Ω of a linear connection ω are given by

$$d\Phi \circ \pi_h = \Omega \wedge \theta \qquad \text{and} \qquad d\Omega \circ \pi_h = 0 \ .$$

Next let us consider an arbitrary principal H-bundle $(\mathcal{G},\omega)$ equipped with a connection form $\omega : T\mathcal{G} \to \mathfrak{h}$ over a manifold M^n of dimension n and let λ be an H-representation on a vector space $\mathbb{V}$. The associated bundle $\mathcal{V} := \mathcal{G} \times_\lambda \mathbb{V}$ is a vector bundle over M with structure group H. Let $\Lambda^k T^*\mathcal{G} \otimes \mathbb{V}$ denote the bundle of k-forms on $\mathcal{G}$ with values in $\mathbb{V}$. We call a section ϕ in $\Lambda^k T^*\mathcal{G} \otimes \mathbb{V}$ tensorial if $R_h^*\phi = \lambda(h^{-1}) \cdot \phi$ for all $h \in H$ (i.e., ϕ is H-equivariant) and $\iota_X\phi = 0$ for all vertical vectors $X \in T^v\mathcal{G}$.

It is a matter of fact that the space of tensorial k-forms with values in $\mathbb{V}$, denoted by $\Omega^k_h(\mathcal{G};\mathbb{V})^\lambda$, is canonically identified with $\Omega^k(M;\mathcal{V})$, the space of k-forms on M with values in the associated vector bundle $\mathcal{V}$. With the aid of the connection form ω on $\mathcal{G}$ we obtain the exterior covariant derivative d^ω given by

$$\phi \in \Gamma(\Lambda^k T^*\mathcal{G} \otimes \mathbb{V}) \quad \mapsto \quad d^\omega\phi := d\phi \circ \pi_h \in \Gamma(\Lambda^{k+1}T^*\mathcal{G} \otimes \mathbb{V}) \ .$$

And, via the identifications $\Omega^k_h(\mathcal{G};V)^\lambda \cong \Omega^k(M;\mathcal{V})$, $0 \le k \le n$, we thus obtain first order differential operators acting on $\Omega^k(M;\mathcal{V})$. In particular, if $k = 0$ we have $C^\infty(\mathcal{G};\mathbb{V})^\lambda \cong \Gamma(\mathcal{V})$ and the corresponding operator acting on sections of $\mathcal{V}$ is denoted by

$$\begin{array}{ccccccc} \nabla^\omega & : & \mathfrak{X}(M) & \otimes & \Gamma(\mathcal{V}) & \to & \Gamma(\mathcal{V}) \ , \\ & & (\ X & , & \phi \ ) & \mapsto & \nabla^\omega_X\phi \quad . \end{array}$$

The map ∇^ω is a so-called *covariant derivative*, i.e., we have $\nabla^\omega_\cdot\phi \in \Gamma(T^*M \otimes \mathcal{V})$ for any section $\phi \in \Gamma(\mathcal{V})$ and the Leibnitz rule

$$\nabla^\omega_X(f \cdot \phi) = f \cdot \nabla^\omega_X\phi + X(f) \cdot \phi$$

is satisfied for all $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Locally, with respect to a section s in $\mathcal{G}$ we can express the covariant derivative on $\mathcal{V}$ by

$$\nabla^\omega_X\phi \ = \ [\ s \ , \ d\phi(X) + \lambda_*(\omega^s(X)) \cdot \phi \ ] \ .$$

Finally, we introduce here the notion of *holonomy* for principal connections (and covariant derivatives). Let $(\mathcal{G},\omega) \xrightarrow{\pi} M$ be a principal H-bundle with connection ω over a connected base space M^n of dimension n. Let $x \in M$ be an arbitrary point and let $C(x)$ denote the set of continuously differentiable curves in M, which start and end at $x \in M$. For any curve (or loop) $\gamma \in C(x)$ and any point $p \in \pi^{-1}(x)$ in the fibre of $\mathcal{G}$ over $x \in M$, there exists a uniquely defined curve γ^*_p starting at p such that $\pi \circ \gamma^*_p = \gamma$ and the tangent vectors to γ^*_p (at any point of the curve) are horizontal. The curve γ^*_p is called the horizontal lift of γ through p. Now, if we map any $p \in \pi^{-1}(x)$ to the end point of γ^*_p, which is again a point in the fibre $\pi^{-1}(x)$, then we obtain a map

$$\Gamma^\gamma : \pi^{-1}(x) \to \pi^{-1}(x) \ ,$$

which is called the parallel displacement of the fibre $\pi^{-1}(x)$ onto itself with respect to ω along the curve γ. One can prove that every parallel displacement Γ^γ, $\gamma \in C(x)$, acts as a diffeomorphism on the fibre $\pi^{-1}(x)$, and, in fact, the set of all such displacements forms a Lie group, which is called the holonomy group of $(\mathcal{G},\omega)$ with respect to the base point $x \in M$.

The holonomy group with respect to $x \in M$ can be realised as a subgroup of H. To see this, we fix a particular $p \in \pi^{-1}(x)$ and compare for any $\gamma \in C(x)$ the reference point p with the end point $\Gamma^\gamma(p) \in \pi^{-1}(x)$ of the horizontal lift γ^*_p. Then $\Gamma^\gamma(p) = p \cdot b^\gamma$ for some uniquely determined group element $b^\gamma \in H$. The collection

$$H_p := \{ \ b^\gamma \ : \ \gamma \in C(x) \ \}$$

of these group elements forms a (weak) Lie subgroup of H, which we call the holonomy group of the connection ω with respect to the reference point $p \in \mathcal{G}$. It is a matter of fact that for any two points $p, q \in \mathcal{G}$ the holonomy groups H_p and H_q are conjugated subgroups in H, i.e., the isomorphism class of the H_p, $p \in \mathcal{G}$, is a uniquely defined Lie group. We denote this Lie group by $Hol(\mathcal{G},\omega)$ and call it the *holonomy group* of $(\mathcal{G}, \omega) \xrightarrow{\pi} M$. The corresponding Lie algebra is denoted by $\mathfrak{hol}(\mathcal{G}, \omega)$ or simply $\mathfrak{hol}(\omega)$. The restricted holonomy group $Hol^0(\mathcal{G}, \omega)$ is by definition generated via parallel displacement along lifts of null homotopic curves only. It turns out that the restricted holonomy group is just the identity component of $Hol(\mathcal{G}, \omega)$.

The set $\mathcal{G}(p)$ of points in $\mathcal{G}$ which can be reached by a horizontal curve starting at the fixed base point $p \in \mathcal{G}$, is in a natural way a smooth principal H_p-bundle and a H_p-reduction of $\mathcal{G}$. We call $\mathcal{G}(p)$ the holonomy bundle of $(\mathcal{G}, \omega)$ with respect to the base point $p \in \mathcal{G}$. Moreover, the connection ω on $\mathcal{G}$, which defines the holonomy bundle $\mathcal{G}(p)$, pulls back to a principal H_p-connection form $\omega(p)$ on $\mathcal{G}(p)$. The holonomy theorem of *Ambrose* and *Singer* states that the Lie algebra $\mathfrak{h}_p$ of the holonomy group H_p with respect to the reference point $p \in \mathcal{G}$ is the linear span in $\mathfrak{h}$ of the curvature values $\Omega(X, Y)$, where X, Y are arbitrary tangent vectors (at any point) of the holonomy bundle $\mathcal{G}(p)$.

Now let $\mathcal{V} \xrightarrow{\pi} M$ be a vector bundle equipped with a covariant derivative ∇ over a connected manifold M. For any curve $\gamma(t)$ in M running from some x to $y \in M$ and any vector $V_x \in \mathcal{V}_x$, there exists a uniquely defined vector field V along the curve γ with values in $\mathcal{V}$ such that $\nabla_{\dot\gamma(t)} V \equiv 0$. The vector field V is called the parallel displacement of V_x along γ. In particular, for any $V_x \in \mathcal{V}_x$ we have a uniquely defined parallel displacement vector $V_{\gamma(t)} \in \mathcal{V}_{\gamma(t)}$ along γ to the end point $y \in M$. This parallel displacement gives rise to a vector space isomorphism

$$\Gamma^\gamma : \mathcal{V}_x \to \mathcal{V}_y \ .$$

In particular, if $\gamma \in C(x)$ is a loop we obtain an automorphism Γ^γ of $\mathcal{V}_x$. The collection of all these automorphisms Γ^γ for loops $\gamma \in C(x)$ is a subgroup of $\mathrm{Gl}(\mathcal{V}_x)$. We denote this group by $Hol_x(\mathcal{V}, \nabla)$ and call it the holonomy group of $(\mathcal{V}, \nabla)$ at $x \in M$. One can show that the holonomy groups $Hol_x(\mathcal{V}, \nabla)$ and $Hol_y(\mathcal{V}, \nabla)$ to any two base points $x, y \in M$ are isomorphic. Thus the corresponding isomorphism class is uniquely defined. We denote it by $Hol(\mathcal{V}, \nabla)$ and call this the holonomy group of $(\mathcal{V}, \nabla)$. In case $\mathcal{V} \xrightarrow{\pi} M$ is the associated vector bundle to a principal H-bundle $\mathcal{G}$ on M with representation $\lambda : H \to \mathrm{Gl}(\mathbb{V})$ and the covariant derivative ∇ on $\mathcal{V}$ is induced by a connection ω on $\mathcal{G}$ the relation $\lambda(Hol(\mathcal{G}, \omega)) = Hol(\mathcal{V}, \nabla)$ holds.

4. Semi-Riemannian Geometry

We start with some basic linear algebra. Let V^n be a real vector space of dimension n equipped with a symmetric and non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

and let $\mathcal{B}^e := \{\, e_1, \ldots, e_n\}$ denote a (pseudo)-orthonormal basis for $\langle \cdot, \cdot \rangle$, i.e.,

$$\varepsilon_{ij} := \langle e_i, e_j \rangle = \begin{cases} \quad 0 & \text{if} \quad i \neq j \\ -1 \text{ or } 1 & \text{if} \quad i = j \end{cases} \ .$$

We set $\varepsilon_i := \varepsilon_{ii}$, for $i = 1, \ldots, n$, and denote by r the number of basis vectors e_i in $\mathcal{B}^e$ with $\varepsilon_i = -1$, whereas s denotes the number of those e_i's with $\varepsilon_i = 1$. We have $r+s = n$ and the ordered pair $sig(\langle \cdot, \cdot \rangle) := (r, s)$ is independent of the choice of orthonormal basis $\mathcal{B}^e$. We call (r, s) the signature of $\langle \cdot, \cdot \rangle$ on V. In case $r, s \geq 1$ the scalar product $\langle \cdot, \cdot \rangle$ on V is called indefinite, and otherwise definite. In particular, if the signature of $\langle \cdot, \cdot \rangle$ is $(0, n)$ we call $(V, \langle \cdot, \cdot \rangle)$ a Euclidean vector space. In this case we set $\|x\| := \sqrt{\langle x, x \rangle}$

for $x \in V$. Then $\|x\| = 0$ iff $x = 0$. The standard Euclidean scalar product on $\mathbb{R}^n$ is given by $\langle x, y\rangle_n := \sum_{i=1}^n x^i y^i$, where $x = (x^1, \dots, x^n)^\top$ and $y = (y^1, \dots, y^n)^\top \in \mathbb{R}^n$. The standard orthonormal basis of $\langle\cdot,\cdot\rangle_n$ on $\mathbb{R}^n$ is $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$.

Now let us consider a space V^n with indefinite scalar product $\langle\cdot,\cdot\rangle$. Then we call a vector $0 \neq x \in V$ *spacelike* if $\langle x, x\rangle > 0$, *timelike* if $\langle x, x\rangle < 0$ and *lightlike* or *null* if $\langle x, x\rangle = 0$. Usually, we order an orthonormal basis $\mathcal{B}^e$ for $(V, \langle\cdot,\cdot\rangle)$ such that the first r basis vectors are timelike and the last s basis vectors are spacelike. Then we set $\mathbb{J}_{r,s} = (\varepsilon_{ij})_{i,j=1,\dots n}$, which is a diagonal $(n \times n)$-matrix. A subspace W of V is called non-degenerate if the restriction of $\langle\cdot,\cdot\rangle$ is a scalar product on W. The space $\mathbb{R}^n$ furnished with the scalar product $\langle x, y\rangle_{r,s} := x^\top \mathbb{J}_{r,s} y$ is the standard pseudo-Euclidean space of signature (r, s). We denote this space by $\mathbb{R}^{r,s} = (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{r,s})$. For later use, we accent from time to time the case of a scalar product $\langle\cdot,\cdot\rangle$ of signature $(1, n-1)$ on a vector space V^n. In this case $(V, \langle\cdot,\cdot\rangle)$ is called a Minkowski space of dimension n and any orthonormal basis of $(V, \langle\cdot,\cdot\rangle)$ has exactly one timelike vector. The set

$$L := \{\ x \in V \ :\ \langle x, x\rangle_{1,n-1} \leq 0\ \}$$

is called the *causal cone* of V. Non-trivial elements in L are called causal vectors. The set

$$L_o := \{\ x \in V \ :\ \langle x, x\rangle_{1,n-1} = 0\ \}$$

is called the *null cone* of V.

An isometry of an arbitrary scalar product space $(V, \langle\cdot,\cdot\rangle)$ is a linear map $A : V \to V$ such that $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in V$. The group of isometries acting on $\mathbb{R}^{r,s}$ (with arbitrary signature (r, s)) is denoted by $\mathrm{O}(r, s)$ and is called the (pseudo)-orthogonal group of signature (r, s). The special orthogonal group is denoted by $\mathrm{SO}(r, s)$. The latter group consists of the matrices A, for which $A^\top \mathbb{J}_{r,s} A = \mathbb{J}_{r,s}$ and $\det A = 1$ holds. The Lie algebra of $\mathrm{SO}(r, s)$ (and also of $\mathrm{O}(r, s)$) is denoted by $\mathfrak{so}(r, s)$. The group of isometric motions on $\mathbb{R}^{r,s}$ is the semidirect product

$$Euc(r, s) \ := \ \mathrm{O}(r, s) \ltimes \mathbb{R}^n \ ,$$

where the $\mathbb{R}^n$-part acts by translation. The dimension of $Euc(r, s)$ is $n(n + 1)/2$ and the isotropy subgroup of the $Euc(r, s)$-action on $\mathbb{R}^{r,s}$ at the origin is $\mathrm{O}(r, s)$. Thus $\mathbb{R}^{r,s}$ is naturally identified with the homogeneous space $Euc(r, s)/\mathrm{O}(r, s)$.

Now, in general, let $\mathbb{V}^m$ be a finite dimensional representation space of $\mathrm{O}(r, s)$. Then $\mathbb{V}$ admits also the induced infinitesimal action of the Lie algebra $\mathfrak{so}(r, s)$. The $\mathrm{O}(r, s)$-module $\mathbb{V}$ is called irreducible if any proper, $\mathrm{O}(r, s)$-invariant subspace is trivial. In general, any $\mathrm{O}(r, s)$-module $\mathbb{V}^m$ is (in a unique way) the direct sum of irreducible $\mathrm{O}(r, s)$-representations, i.e., $\mathbb{V} = \oplus_{i \in \mathbb{I}} \mathbb{V}_i$ with projections π_i. It is well known that any finite dimensional $\mathrm{O}(r, s)$-module (resp. $\mathrm{SO}(r, s)$-module) $\mathbb{V}$ can be realised as a submodule of the tensor algebra $\oplus_{k,l=0}^\infty T^{(k,l)}\mathbb{R}^n$.

We call the action of $\mathrm{O}(r, s)$ by matrix multiplication on $\mathbb{R}^n$ the *standard* representation of $\mathrm{O}(r, s)$. The standard module $\mathbb{R}^n$ (resp. $\mathbb{R}^{r,s}$) is irreducible and canonically identified with the dual representation $\mathbb{R}^{n*}$ via the scalar product $\langle\cdot,\cdot\rangle_{r,s}$. If $x \in \mathbb{R}^n$ is a vector then we denote by $x^\flat := \langle x, \cdot\rangle_{r,s}$ the metric dual in $\mathbb{R}^{n*}$. And for $\alpha \in \mathbb{R}^{n*}$ we denote by $\alpha^\sharp$ the vector in $\mathbb{R}^n$, whose metric dual is α. The space $End(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^{n*} \otimes \mathbb{R}^n$ as $\mathrm{O}(r, s)$-module and the trace

$$tr \ : \ End(\mathbb{R}^n) \ \to \ \mathbb{R}$$

is a homomorphism of $\mathrm{O}(r,s)$-modules (where $\mathbb{R}$ denotes the trivial representation). With respect to an arbitrary orthonormal basis $\mathcal{B}^e$ of $\mathbb{R}^{r,s}$ the trace of $\alpha \in \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ is given by $tr(\alpha) = \sum_{i=1}^n \varepsilon_i \cdot \alpha(e_i, e_i)$. More generally, via the identifications of $End(\mathbb{R}^n)$ with $\mathbb{R}^n \otimes \mathbb{R}^n$ and $\mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$, we can apply the trace to any pair of indices (a,b) of a (k,l)-tensor. This gives rise to $\mathrm{O}(r,s)$-module homomorphisms

$$tr_{ab} \;:\; T^{(k,l)}\mathbb{R}^n \;\to\; T^{(\tilde{k},\tilde{l})}\mathbb{R}^n \;,$$

for some appropriate $(\tilde{k},\tilde{l})$ with $\tilde{k} + \tilde{l} + 2 = k + l$. The operation tr_{ab} is also called *contraction*. A complete contraction of a (k,l)-tensor is a real number, which is produced by applying trace operations a maximal number of times. Note that the module $End(\mathbb{R}^n)$ is not $\mathrm{O}(r,s)$-irreducible. In fact, $End(\mathbb{R}^n)$ decomposes into the direct sum $\Lambda^2\mathbb{R}^{n*} \oplus S^2_o(\mathbb{R}^{n*}) \oplus \mathbb{R}$ of irreducible $\mathrm{O}(r,s)$-modules, where $S^2_o(\mathbb{R}^{n*})$ denotes the tracefree part of the symmetric $(0,2)$-tensors, i.e., $S^2_o(\mathbb{R}^{n*})$ is the kernel of $tr_{12} : S^2(\mathbb{R}^{n*}) \to \mathbb{R}$.

In general, a subgroup H of $\mathrm{O}(r,s)$ is called *irreducible* if the induced action of H on the standard module $\mathbb{R}^{r,s}$ admits no proper, non-trivial invariant subspace. If there exist degenerate invariant subspaces we call the subgroup H *weakly irreducible*. This can happen only in the indefinite signature case, since any subspace of the Euclidean space $\mathbb{R}^n$ is non-degenerate. In general, an irreducible $\mathrm{O}(r,s)$-module $\mathbb{V}^m$ will decompose as direct sum into weakly irreducible subspaces with respect to the action of some subgroup H of $\mathrm{O}(r,s)$.

We are prepared now to consider *semi-Riemannian metrics*. So let M^n be a n-dimensional (connected and differentiable) manifold. A metric g on M is a differentiable section in $S^2(T^*M)$ such that

$$g_p = g_p(\cdot,\cdot) \;:\; T_pM \times T_pM \;\to\; \mathbb{R}$$

is non-degenerate at every point $p \in M$. Since g is continuous and M is connected, the signature (r,s) of g is constant on M. If $sig(g)$ is equal to $(0,n)$ we call g a Riemannian metric and (M,g) a Riemannian manifold. If the signature is indefinite, (M,g) is a *pseudo*-Riemannian space. In particular, if the signature of g is $(1,n-1)$ we call (M,g) a *Lorentzian manifold* with Lorentzian metric g. If we do not specify the signature of a metric, then we speak of a semi-Riemannian metric, resp., semi-Riemannian space. In this work, any metric g on a manifold M is usually assumed to be smooth. However, there will be a few exception as we will indicate when appropriate.

Let us fix a (smooth) metric g of signature (r,s) on a connected manifold M. The metric g determines at every point p of M the orthonormal bases of T_pM. This gives rise to a smooth reduction of the general linear frame bundle $Gl(M)$ to the structure group $\mathrm{O}(r,s)$, i.e., the metric g naturally gives rise to a $\mathrm{O}(r,s)$-structure on M, which we denote by $O(M)$. Moreover, the metric g induces a scalar product on any tensor bundle $T^{(k,l)}M$. Usually, we denote these scalar products by g again. The manifold M is *orientable* if the frame bundle $O(M)$ allows for a further reduction to the structure group $\mathrm{SO}(r,s)$. We denote such a reduction by $SO(M)$. A bundle $SO(M)$ determines the oriented orthonormal frames on M and a *volume form* dM, which is a global section of length square 1 in the determinant bundle Λ^nT^*M. With respect to a local orthonormal and oriented frame $s = \{s_1,\dots,s_n\}$ on M the volume form is given by $dM = s_1 \wedge \cdots \wedge s_n$. Moreover, an orientation on M defines the *Hodge star operator*

$$\star \;:\; \Lambda^p(T^*M) \;\to\; \Lambda^{n-p}(T^*M)$$

for any $0 \le p \le n$ by the relation $\eta \wedge \star\xi = g(\eta,\xi)dM$ for all $\eta, \xi \in \Lambda^p(T^*M)$.

The canonical form θ on $Gl(M)$ restricts to the bundle $O(M)$ of orthonormal frames on (M,g). It is a basic fact of semi-Riemannian geometry that there exists a uniquely determined principal connection ω^g on $O(M)$, which has no torsion, i.e.,

$$\Phi = d\theta + \omega^g \wedge \theta = 0 \ .$$

The connection form ω^g on $O(M)$ is called the *Levi-Civita connection* of (M,g). In case (M,g) is oriented the Levi-Civita connection ω^g reduces also to the principal bundle $SO(M)$. The tangent bundle TM is naturally identified with the associated bundle $O(M) \times_{\mathrm{O}(r,s)} \mathbb{R}^n$ coming from the standard representation and the Levi-Civita connection gives rise to a canonical covariant derivative on vector fields, which is metric, torsion-free and denoted by

$$\begin{array}{ccccccc} \nabla^g & : & \mathfrak{X}(M) & \otimes & \mathfrak{X}(M) & \to & \mathfrak{X}(M) \ , \\ & & (\ X & , & Y \ ) & \mapsto & \nabla^g_X Y \quad . \end{array}$$

More generally, let $\mathbb{V}$ be any $\mathrm{O}(r,s)$-module with irreducible decomposition $\oplus_{i\in\mathbb{I}}\mathbb{V}_i$. Then, accordingly, the associated vector bundle $\mathcal{V} \xrightarrow{\pi} M$ decomposes into a direct sum $\oplus_{i\in\mathbb{I}}\mathcal{V}_i$ of irreducible $\mathrm{O}(r,s)$-associated vector bundles $\mathcal{V}_i$, $i \in \mathbb{I}$, and $\mathcal{V}$ is equipped with a natural covariant derivative ∇^g, which preserves the decomposition $\oplus_{i\in\mathbb{I}}\mathcal{V}_i$. In particular, any $\mathrm{O}(r,s)$-subbundle of the tensor algebra $\oplus_{k,l=0}^{\infty} T^{(k,l)}M$ is equipped with a natural covariant derivative ∇^g.

The condition that ∇^g is metric and torsion-free is expressed by $\nabla^g g = 0$ and $\nabla^g_X Y - \nabla^g_Y X = [X,Y]$ for all $X, Y \in \mathfrak{X}(M)$. The *Riemannian curvature operator* R^g of ∇^g on TM is defined by

$$R^g(X,Y)Z \ := \ \nabla^g_X\nabla^g_Y Z - \nabla^g_Y\nabla^g_X Z - \nabla^g_{[X,Y]}Z \ ,$$

where $X, Y, Z \in TM$. We also have the Riemannian curvature tensor, which we denote by R^g as well, and which is given by

$$R^g(X,Y,Z,W) := g(\nabla^g_X\nabla^g_Y Z - \nabla^g_Y\nabla^g_X Z - \nabla^g_{[X,Y]}Z, W)$$

with $W \in TM$. The Riemannian curvature tensor R^g has the symmetry properties

$$R^g(X,Y,Z,W) = -R^g(Y,X,Z,W) = -R^g(X,Y,W,Z) = R^g(Z,W,X,Y) \qquad (2)$$

and it satisfies the Bianchi identities

$$R^g(X,Y,Z,W) + R^g(Y,Z,X,W) + R^g(Z,X,Y,W) = 0 \ ,$$
$$(\nabla^g_X R^g)(Y,Z) + (\nabla^g_Y R^g)(Z,X) + (\nabla^g_Z R^g)(X,Y) = 0 \ .$$

A semi-Riemannian space (M,g) is called (locally) flat iff $R^g = 0$ everywhere on M. The *Ricci curvature* and the *scalar curvature* of (M,g) are defined as contractions of the Riemannian curvature tensor R^g by

$$Ric^g \ := \ tr^g_{23}R^g \qquad \text{and} \qquad scal^g \ := \ tr^g_{12}Ric^g \ .$$

The Schouten tensor

$$\mathsf{P}^g \ := \ \frac{1}{n-2}\left(\frac{scal^g}{2(n-1)} - Ric^g\right)$$

is by definition a scalar normalisation of the Ricci-curvature.

The Kulkarni-Nomizu product $*$ of two symmetric $(0,2)$-tensors h and k is given by

$$\begin{aligned}(h*k)(X,Y,Z,W) \;&:=\; -h(X,W)k(Y,Z)-h(Y,Z)k(X,W)\\ &\qquad h(X,Z)k(Y,W)+h(Y,W)k(X,Z)\ .\end{aligned}$$

The tensor $h*k$ is a $(0,4)$-tensor, which has the symmetry properties (2) and satisfies the first Bianchi identity. The *Weyl tensor* of (M,g) is defined as the tracefree part of the Riemannian curvature tensor R^g and is given explicitly by

$$W^g \;=\; R^g - \mathsf{P}^g * g\ .$$

Moreover, we have the *Cotton tensor*

$$C^g(X,Y,Z) \;:=\; (\nabla^g_X \mathsf{P}^g)(Y,Z) \;-\; (\nabla^g_Y \mathsf{P}^g)(X,Z)$$

and the *Bach tensor* (cf. [**11**])

$$B^g(X,Y) \;:=\; \sum_{i=1}^n \varepsilon_i\cdot\nabla_{s_i}C(Y,s_i,X) - \sum_{i=1}^n \varepsilon_i\cdot W(\mathsf{P}(s_i),X,Y,s_i),$$

All curvature quantities for (M,g) that we have defined so far are so-called *Riemannian invariants*. We want to explain this notion in some detail. In general, a semi-Riemannian metric g on a space M is locally given with respect to coordinates $(x^1,\dots,x^n)$ by $g=\sum_{i,j=1}^n g_{ij}dx^idx^j$, where $g_{ij}:=g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^i})$. We set

$$g_{ij,\alpha} := D^\alpha g_{ij}\ ,$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ can be any n-tuple of non-negative integers and D^α denotes the corresponding partial derivative of order $|\alpha|$ (cf. Section 0.1). A tensor-valued Riemannian invariant $Q(g)$ is a tensor on (M,g), whose components with respect to a coordinate system are polynomials in the variables $g_{ij,\alpha}$, $|\alpha|\geq 0$, and $det(g_{ij})^{-1}$. Thereby, the polynomial expression is required to be coordinate free in the sense that it looks formally the same for any choice of coordinates, and universal in the sense that it constitutes an invariant tensor for any semi-Riemannian metric g on a space M (regardless of any property of g). In particular, if $Q(g)$ is given by exactly one coordinate free, universal polynomial in above variables (i.e., $Q(g)$ gives rise to an $\mathbb{R}$-valued function with respect to any g on M), then this object is called a *scalar Riemannian invariant*. For example, the Riemannian curvature tensor R^g is given (in coordinates) by the universal expression

$$R_{ijkl} \;=\; \frac{1}{2}\left(\, g_{ik,jl}+g_{jl,ik}-g_{jk,il}-g_{il,jk}\,\right) \;+\; \sum_{p,q} g_{pq}\cdot\left(\,\Gamma^p_{ik}\Gamma^q_{jl}-\Gamma^p_{il}\Gamma^q_{jk}\,\right)\ , \qquad (3)$$

where $\Gamma^k_{ij} := \frac{1}{2}(g_{kl})^{-1}\big(g_{il,j}+g_{jl,i}-g_{ij,l}\big)$ are the Christoffel symbols. A complete contraction of an expression of the form $(\nabla^g)^{l_1}R^g\otimes\cdots\otimes(\nabla^g)^{l_r}R^g$ with $l_1,\dots,l_r$ some non-negative integers (i.e., a tensor product of covariant derivatives of the Riemannian curvature tensor R^g) is called a *Weyl invariant*. Classical invariant theory shows that any scalar Riemannian invariant is a linear combination of Weyl invariants. The simplest scalar Riemannian invariant is the scalar curvature $scal^g$. In fact, we obtain from (3) through a complete contraction the universal polynomial, which represents the scalar curvature $scal^g$.

More generally, a metric invariant differential operator $\mathcal{D}$ acting on tensors over a space (M,g) is given locally, with respect to coordinates, by an expression of the form $\sum_{|\alpha|\leq k} A_\alpha(x) D^\alpha$ (cf. (1)), which is coordinate free (in above sense) and universal, where the $A_\alpha(x)$'s are matrix-valued polynomials in the variables $g_{ij,\alpha}$ and $det(g_{ij})^{-1}$. In general, such invariant differential operators can be produced as follows. Let $\mathcal{V}(M)$ be the associated vector bundle over M to an $\mathrm{O}(r,s)$-module $\mathbb{V}$. Application of the covariant derivative ∇^g to sections of $\mathcal{V}(M)$ produces a first order invariant differential operator

$$\nabla^g : \Gamma(\mathcal{V}) \to \Gamma(T^*M \otimes \mathcal{V}) \ .$$

Moreover, for any vector bundle homomorphism $\phi : T^*M \otimes \mathcal{V} \to \mathcal{V}'$, the superposition $\phi \circ \nabla^g : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$ is an invariant operator. In particular, the projections π_i to the irreducible components of $T^*M \otimes \mathcal{V}$ give rise to first order invariant differential operators $\pi_i \circ \nabla^g$. More generally, iterated applications of ∇^g combined with vector bundle homomorphisms give rise to higher order invariant operators. In fact, application k times of the covariant derivative ∇^g induces uniquely a bundle homomorphism $\nabla^k : J^k\mathcal{V} \to \otimes^k T^*M \otimes \mathcal{V}$ of the k-jet bundle of $\mathcal{V}$ (cf. Section 0.1). An invariant differential operator then acts on sections $f \in \Gamma(\mathcal{V})$ by superposition of the k-jet prolongation $j^k f$ with ∇^k and some homomorphism $\phi : \otimes^k T^*M \otimes \mathcal{V} \to \mathcal{V}'$.

In particular, we have the following standard differential operators for semi-Riemannian spaces (M,g). There is the gradient

$$f \in C^\infty(M) \quad \mapsto \quad grad^g(f) := df^\sharp \in \mathfrak{X}(M)$$

of first order and the Hessian

$$f \in C^\infty(M) \quad \mapsto \quad Hess^g(f) = g(\nabla^g_\cdot grad^g(f), \cdot) \in \Gamma(S^2(T^*M))$$

of second order, which both act on functions. The trace of the Hessian gives rise to the Laplace operator $\Delta^g_{tr} f = tr_{12} Hess(f) \in C^\infty(M)$. More generally, we have the Bochner Laplacian for any associated vector bundle $\mathcal{V}$ on (M,g):

$$\Delta^g_{tr} := tr \circ (\nabla^g)^2 : \ \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}) \ .$$

The divergence of a $(0,s)$-tensor α is defined by $div^g_l(\alpha) := tr_{1,l+1}(\nabla^g \alpha)$, with $1 \leq l \leq s$ determines the contraction. On p-forms $\alpha \in \Omega^p(M)$ we have the codifferential $d^*\alpha := -div^g_1 \alpha$, which is given with respect to a local orthonormal frame $\mathcal{B}^s = \{s_1, \dots, s_n\}$ by $d^* = -\sum_{i=1}^n \iota_{s_i} \nabla^g_{s_i}$. The p-form Laplacian is defined as

$$\Delta^g_p := dd^* + d^*d \ .$$

There are other invariants for semi-Riemannian spaces, e.g. the holonomy. Since the Levi-Civita connection ω^g on the orthonormal frame bundle $O(M)$ over a space (M^n, g) of dimension n is uniquely determined by the metric, we can define

$$Hol(M,g) \ := \ Hol(O(M), \omega^g) \ .$$

We call $Hol(M,g)$ the holonomy (group) of the semi-Riemannian space (M,g). In general, $Hol(M,g)$ is a (weak) Lie subgroup of $\mathrm{O}(r,s)$. The Lie algebra of $Hol(M,g)$ is denoted by $\mathfrak{hol}(M,g)$. The *Ambrose-Singer Theorem* states that the holonomy algebra $\mathfrak{hol}(M,g)$ is spanned (with respect to a base point $p \in M$) by all parallel displacements along any curve from any x_o to $p \in M$ of any curvature endomorphism $R^g_{x_o}(X,Y)$, $X, Y \in T_{x_o}M$. It follows that a space (M,g) is flat iff $\mathfrak{hol}(M,g)$ is trivial. The deRham decomposition Theorem states that a metric g on a space M is locally isometric (see below) to a Riemannian product metric $h \times k$ if $Hol(M,g)$ acts decomposable on the standard module $\mathbb{R}^n$. If $Hol(M,g)$ is not decomposable then the space (M,g) does not split (locally) (cf. [**141**]). In this case, $Hol(M,g)$ in $\mathrm{O}(r,s)$ is either irreducible or weakly irreducible. The latter case can only occur in pseudo-Riemannian geometry. A complete classification of irreducible holonomy groups $Hol(M,g)$ for Riemannian spaces (M,g) is well known (cf. e.g. [**25, 29**]).

A diffeomorphism $\phi : (M,g) \to (N,h)$ between semi-Riemannian spaces is called an isometry if $\phi^* h = g$. The set $Aut(M,g)$ of isometries on a space (M,g) is in a canonical way a Lie group, which acts smoothly on (M,g), and is called the *isometry group* of (M,g). In case the isometry group $Aut(M,g)$ acts transitively, the underlying space (M,g) is called a homogeneous Riemannian space. A special instance of a homogeneous space is a Lie group with left or bi-invariant metric (cf. Chapter 3). An infinitesimal automorphism of (M,g) is a vector field $X \in \mathfrak{X}(M)$, for which $\mathcal{L}_X g = 0$. Such an X is called a *Killing vector field*. The local flow X_t along a Killing vector field X consists of (local) isometries on (M,g). The set of all Killing vector fields gives rise to a finite dimensional Lie algebra $\mathfrak{iso}(M,g)$. The dimension of $\mathfrak{iso}(M,g)$ is bounded by $n(n+1)/2$, which is the dimension of $Euc(r,s)$. It is known that a space (M,g) is flat if $dim(\mathfrak{iso}(M,g)) = n(n+1)/2$ is maximal. We note that the condition $\mathcal{L}_X g = 0$ for a vector field X is equivalent to

$$g(\nabla^g_Y X, Z) = -g(\nabla^g_Z X, Y) \qquad \text{for all } Y, Z \in TM\ ,$$

i.e., the operator $\nabla^g X$ is skew-adjoint with respect to g on M. In general, a Killing vector field X has no divergence, i.e., $div^g X = 0$, and $\Delta^g_{tr} X = -Ric^g(X)$. The dual $X^\flat := g(X,\cdot)$ of a Killing vector field satisfies the equation $\nabla^g X^\flat = \frac{1}{2} dX^\flat$. This equation has an obvious generalisation to p-forms $\alpha \in \Omega^p(M)$, namely

$$\nabla^g \alpha \;=\; \frac{1}{p+1} d\alpha\ .$$

Solutions of this equation are called *Killing p-forms* (cf. e.g. [**147, 89, 143**]). The equation for Killing p-forms is invariant and describes the kernel of an invariant differential operator. In addition, if $\alpha \in \Omega^p(M)$ is Killing and satisfies the equation

$$\nabla^g_X d\alpha \;=\; c \cdot X^\flat \wedge \alpha$$

for all $X \in TM$ with some fixed Killing constant $c \in \mathbb{R}$, then we call α a *special* Killing p-form (cf. Section 0.9 and [**143**]).

The equations of geodesic motion are metric invariant as well. A smooth curve $\gamma : I \to M$ is called a geodesic of a semi-Riemannian space (M,g) if its tangent vector $\gamma'(t)$ is parallel along γ, i.e., $\nabla^g_{\gamma'(t)}\gamma'(t) = 0$. The theory of ordinary differential equations guarantees for any $p \in M$ and any $V_p \in T_pM$ the existence of a unique maximal geodesic $\zeta^V_p(t)$ with $\zeta^V_p(0) = p$ and $(\zeta^V_p)'(0) = V_p$. A semi-Riemannian space (M,g) is called geodesically complete if the maximal geodesics exist for all times $t \in \mathbb{R}$ on (M,g). Now let us fix an arbitrary point $p \in M$. The set of vectors $V_p \in T_pM$ such that $\zeta^V_p(1) \in M$ exists gives rise to an open subset U_p of T_pM. Then we can define the exponential map

$$exp_p \;:\quad V \in U_p \;\mapsto\; \exp_p(V) := \zeta^V_p(1)\ .$$

at any point $p \in M$. The exponential map exp_p is locally on a suitable neighbourhood of the origin in T_pM a diffeomorphism onto its image, and gives rise to so-called *normal* *coordinates* around $p \in M$. A local neighbourhood W_p of a point $p \in M$ is called geodesically convex if any two points in W_p are connected by a unique geodesic in W_p. Convex neighbourhoods always exist around any point of a semi-Riemannian space. In case the metric g on M is indefinite we call a geodesic $\zeta_p^V(t)$ lightlike (timelike) if its tangent vector V at p is lightlike (timelike). In particular, we have, on a Lorentzian space (M, g) with respect to the exponential map at a point p, the geodesic light cone $L_{o,p}$ and the causal cone $L_p \subset M$, which are defined as images of the cones L_o, resp., L in T_pM (intersected with U_p).

Finally, we note that a submersion $\pi : (M, g) \to (N, h)$ between semi-Riemannian spaces is called a Riemannian submersion if the restriction of the differential $d\pi$ to the orthogonal complement of its kernel in TM is a pointwise isometry onto the target. There are well known formulae for the relationship of the curvatures of (M, g) and (N, h) in a Riemannian submersion, which we will use in Chapter 6. We omit these formulae here. They can be found in [**127**]. Accordingly, a Riemannian immersion $\iota : (M, g) \to (N, h)$ is an immersion such that $\iota^* h = g$. Again, there are basic formulae relating the curvatures. The metric cone construction and the Fefferman-Graham ambient metric construction provide common examples for Riemannian submersion (and immersion) spaces (cf. Section 0.9 and 0.11).

5. Parabolic Cartan Geometry

Let G be a Lie group and H a closed subgroup. We call a pair (G, H) a *Klein geometry*. The corresponding pair of Lie algebras is denoted by $(\mathfrak{g}, \mathfrak{h})$. The coset space $M = G/H$ is in a natural way a manifold with smooth projection $\pi : G \to M$ and we call M the (flat) homogeneous model of (G, H). Obviously, the group G acts transitively from the left on M and, in general, one can expect that some (geometric) structure on M is preserved by the G-action. For example, if $G = \mathbb{R}^n \rtimes \mathrm{O}(n)$ is the Euclidean group of motions and $H = \mathrm{O}(n)$ the orthogonal group in dimension n, then the homogeneous model $M = G/H$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$ and the action of the Euclidean group is isometric with respect to the standard inner product $\langle \cdot, \cdot \rangle_n$ on $\mathbb{R}^n$.

Now let M be a smooth manifold of dimension n and let (G, H) be a Klein geometry with corresponding Lie algebras $(\mathfrak{g}, \mathfrak{h})$. A Cartan connection ω on a principal H-bundle $\mathcal{P} \xrightarrow{\pi} M$ over a manifold M is a smooth 1-form with values in $\mathfrak{g}$ such that

(1) $\omega(\chi_A) = A$ for all fundamental fields χ_A, $A \in \mathfrak{h}$,
(2) $R_h^* \omega = Ad(h^{-1}) \circ \omega$ for all $h \in H$ and
(3) $\omega|_{T_u \mathcal{P}(M)} : T_u \mathcal{P}(M) \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}(M)$.

Basically, these properties mean that ω is an H-equivariant absolute parallelism on $\mathcal{P}(M)$. A *Cartan geometry* on M of type (G, H) is a pair $(\mathcal{P}, \omega)$, which consists of a principal H-bundle with Cartan connection ω. Clearly, in this case $\dim(M) = \dim(G) - dim(H)$. For example, let ω_G be the Maurer-Cartan form on a Lie group G. Then for any closed subgroup H the group G is a principal H-bundle over $M = G/H$ and the Maurer-Cartan form ω_G is a Cartan connection on $G \xrightarrow{\pi} M$. Furthermore, if (M, g) is a Riemannian manifold we can define $\mathcal{P}(M)$ to be the set of orthonormal frames in the tangent space TM and a Cartan connection is given by $\theta \oplus \omega^g : T\mathcal{P}(M) \to \mathbb{R}^n \rtimes \mathfrak{so}(n)$, where θ is the soldering form.

Let us fix an arbitrary Cartan geometry $(\mathcal{P},\omega) \xrightarrow{\pi} M$. The curvature 2-form Ω of ω is defined by

$$\Omega \;=\; d\omega + \frac{1}{2}[\omega,\omega]\ ,$$

where $[\cdot,\cdot]$ denotes the bracket of the corresponding differential graded Lie algebra (cf. e.g. [**144**], p. 61). Then $\iota_{\chi_A}\Omega = 0$ for all $A \in \mathfrak{h}$ and Ω satisfies the Bianchi identity $d\Omega = [\Omega,\omega]$. The corresponding $Ad(H)$-equivariant curvature function

$$\kappa:\ \mathcal{P}(M)\ \rightarrow\ Hom(\Lambda^2\mathfrak{g},\mathfrak{g})$$

to Ω is defined pointwise by

$$\kappa(u)(X,Y) := d\omega(\omega_u^{-1}(X),\omega_u^{-1}(Y)) + [X,Y], \qquad u\in\mathcal{P}(M),\quad X,Y\in\mathfrak{g}\ .$$

Since the curvature Ω is vertically trivial, we can view κ as a function with values in $Hom(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$. If the curvature function κ takes only values in the H-submodule $Hom(\Lambda^2\mathfrak{g},\mathfrak{h})$ then the Cartan geometry $(\mathcal{P},\omega) \xrightarrow{\pi} M$ is called *torsion-free*. If $\kappa = 0$ (resp. $\Omega = 0$) on $\mathcal{P}(M)$ then the Cartan connection ω is called *flat*. For example, the Maurer-Cartan form ω_G is flat for any $M = G/H$. The curvature Ω of $\theta\oplus\omega^g$ on $\mathcal{P}(M)$ over a Riemannian space (M,g) is torsion-free and Ω is just (the right translation by $Euc(n)$ of) the curvature of the Levi-Civita connection ω^g on $O(M)$.

Now let $\mathbb{V}$ be an H-representation space. This gives rise to an associated vector bundle $\mathcal{V} := \mathcal{P}(M)\times_H\mathbb{V}$ over M. A function $f\in C^\infty(\mathcal{P},\mathbb{V})$ admits an invariant differential defined by

$$\begin{array}{cccc}\nabla^\omega & : & C^\infty(\mathcal{P},\mathbb{V}) & \rightarrow & C^\infty(\mathcal{P},\mathfrak{g}^*\otimes\mathbb{V})\\ & & f & \mapsto & \mathcal{L}_{\omega^{-1}(\cdot)}f\end{array}$$

with respect to ω on $\mathcal{P}(M)$. It is a matter of fact that the invariant differential of an H-equivariant function f on $\mathcal{P}$ with values in $\mathbb{V}$ is an H-equivariant function with values in $\mathfrak{g}^*\otimes\mathbb{V}$. Thus, we obtain an invariant differential operator $D^\omega:\Gamma(\mathcal{V})\rightarrow\Gamma(\mathcal{A}^*\otimes\mathcal{V})$, where $\mathcal{A}^*$ denotes the dual of the associated bundle to the adjoint representation of H on $\mathfrak{g}$. However, note that this operator D^ω does not give the derivatives in the direction of tangent vectors on M. This problem can be resolved if $\mathbb{V}$ is a G-module, as we will see next.

In general, a Cartan geometry $(\mathcal{P},\omega) \xrightarrow{\pi} M$ of type (G,H) admits a natural extension by G in the following manner. Let $\mathcal{G}(M) := \mathcal{P}(M)\times_H G$ be the principal G-bundle induced by extension via the inclusion of H in G and let $\tilde\omega$ on $\mathcal{G}(M)$ be the right translation of ω by G. The 1-form $\tilde\omega$ is a principal G-bundle connection. If we assume that $\mathbb{V}$ is a G-module then the associated bundle $\mathcal{V} = \mathcal{G}\times_G\mathbb{V}$ is equipped with a covariant derivative $\nabla^{\tilde\omega}$ induced by $\tilde\omega$ on $\mathcal{G}(M)$. Moreover, we have the covariant exterior derivatives $d^{\tilde\omega}$ acting on k-forms with values in $\mathcal{V}$ (cf. Section 0.3). The extension $(\mathcal{G}(M),\tilde\omega)$ of a Cartan geometry $(\mathcal{P}(M),\omega)$ also makes it possible to introduce the notion of *holonomy for Cartan geometries.* In fact, we define the holonomy group $Hol(\omega)$ as the holonomy group of the principal G-connection $\tilde\omega$ on the extended bundle $\mathcal{G}(M)$ in the usual way via parallel translations (cf. Section 0.3). The holonomy algebra $\mathfrak{hol}(\omega)$ is the Lie algebra of the holonomy group $Hol(\omega)$. We remark that there is a direct way of defining the holonomy group for a Cartan connection without using the G-extension $\tilde\omega$ (cf. [**144**]).

We want to specialise the situation to parabolic Cartan geometries. First, we do some preparation. Let V be a finite dimensional vector space. A filtration on V consists of a sequence of subspaces $V^i \subset V$, $i \in \mathbb{Z}$, such that $V^i \supset V^{i+1}$ for all $i \in \mathbb{Z}$ and $V^a = V$, $V^b = \{0\}$ for some $a < b \in \mathbb{Z}$. We call V a *filtered vector space*. The associated graded $\mathrm{gr}(V)$ is defined to be the sum of the quotients $\mathrm{gr}_i(V) := V^i/V^{i+1}$, $i \in \mathbb{Z}$, i.e.,

$$\mathrm{gr}(V) \;=\; \bigoplus_{i=a}^{b-1} \mathrm{gr}_i(V) \;.$$

Now let $\mathfrak{g}$ be a Lie algebra. A direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_a \oplus \cdots \oplus \mathfrak{g}_b$$

with $a < b \in \mathbb{Z}$, which is compatible with the Lie bracket in the sense that for all $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ the commutator $[X,Y]_{\mathfrak{g}}$ is an element of $\mathfrak{g}_{i+j}$, is called a grading of the Lie algebra $\mathfrak{g}$. We set $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_b$ for all $a \le i \le b$. This defines a filtration of the vector space $\mathfrak{g}$, which is compatible with the Lie bracket, and an associated graded $\mathrm{gr}(\mathfrak{g})$ is induced as well. We denote the group of automorphisms of $\mathfrak{g}$ preserving the grading by $Aut_{\mathrm{gr}}(\mathfrak{g})$.

The notion of filtration (and associated graded) for vector spaces extends in an obvious way to vector bundles in the form of sequences of smooth subbundles. In particular, we can consider a smooth filtration of the tangent bundle TM of a manifold M given by

$$TM = TM^{-k} \supset \cdots \supset TM^{-1} \supset TM^0 = M \;,$$

where $k > 0$. The associated graded is denoted by $\mathrm{gr}(TM) = \bigoplus \mathrm{gr}_i(TM)$ and $\mathfrak{X}(M, T^iM)$ denotes the space of sections in T^iM over M. If the Lie bracket of vector fields is compatible with the filtration on TM, i.e., for all $\xi_1 \in \mathfrak{X}(M, T^iM)$ and $\xi_2 \in \mathfrak{X}(M, T^jM)$ the commutator $[\xi_1, \xi_2]$ is an element of $\mathfrak{X}(M, T^{i+j}M)$, then we speak of a filtered manifold M. In this case the Lie bracket $[\cdot,\cdot]$ of vector fields generates pointwise bilinear maps

$$\mathcal{Z}_{ij} : \mathrm{gr}_i(TM)_x \times \mathrm{gr}_j(TM)_x \to \mathrm{gr}_{i+j}(TM)_x$$

for all $x \in M$ and (all relevant) $i, j < 0$. The collection of all $\mathcal{Z}_{ij}$ is denoted by $\mathcal{Z} \in \Lambda^2(\mathrm{gr}(TM)^*) \otimes \mathrm{gr}(TM)$ and we call it the (generalised) *Levi-bracket*. The Levi-bracket satisfies the Jacobi identity and thus $\mathrm{gr}(TM)_x$ is a nilpotent graded Lie algebra for every $x \in M$. If $\mathrm{gr}(TM)$ is locally trivial with (constant) fibre type, some nilpotent graded Lie algebra $\mathfrak{a}$, then there exists a natural frame bundle with structure group $Aut_{\mathrm{gr}}(\mathfrak{a})$, which we denote by $Aut_{\mathrm{gr}}(TM)$.

In the following, we are interested in the case of semisimple Lie algebras $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$, which are graded by

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$$

for some $k > 0$. We call this a $|k|$-grading of $\mathfrak{g}$. In order to avoid unintentional side effects we always assume that $\mathfrak{g}_0$ does not contain any simple ideal of $\mathfrak{g}$. Then we denote $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, which are both nilpotent subalgebras of $\mathfrak{g}$. Moreover, we set $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{p}_+$. The $\mathfrak{g}^i$, $i \in \{-k, \dots, k\}$, define a filtration. The associated graded $\mathrm{gr}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$ as a $\mathfrak{g}_0$-module, but not as a $\mathfrak{p}$-module, in general. There exists a unique element $E \in \mathfrak{g}_0$ such that

$$[E, X]_{\mathfrak{g}} = jX \qquad \text{for all } X \in \mathfrak{g}_j \quad \text{and} \quad j \in \{-k, \dots, k\} \;,$$

which is called the grading element of $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$. Moreover, let $B_{\mathfrak{g}}$ denote the Killing form of the Lie algebra $\mathfrak{g}$. Then $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$. In the latter case, the Killing form $B_{\mathfrak{g}}$ induces an isomorphism $\mathfrak{g}_i^* \cong \mathfrak{g}_{-i}$ of $\mathfrak{g}_0$-modules for all $i = -k, \dots, k$.

In case $\mathfrak{g}$ is a complex semisimple Lie algebra equipped with some $|k|$-grading, the corresponding subalgebra $\mathfrak{p}$ is a parabolic subalgebra, which means that $\mathfrak{p}$ contains a maximal solvable subalgebra (i.e., a Borel subalgebra) of $\mathfrak{g}$. On the other hand, any parabolic subalgebra $\mathfrak{p}$ of a complex semisimple Lie algebra $\mathfrak{g}$ naturally gives rise to a $|k|$-grading of $\mathfrak{g}$. Thus a $|k|$-grading of a complex semisimple $\mathfrak{g}$ is nothing, but a choice of parabolic $\mathfrak{p}$ in $\mathfrak{g}$. A subalgebra $\mathfrak{p}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called a parabolic if its complexification in $\mathfrak{g}_{\mathbb{C}}$ is parabolic.

In general, the cohomology groups of a Lie algebra $\mathfrak{h}$ with values in an $\mathfrak{h}$-module $\mathbb{V}$ are defined as follows. Let $Hom(\Lambda^l \mathfrak{h}, \mathbb{V})$, $l \in \mathbb{N}$, be the chain groups with differential

$$\partial : Hom(\Lambda^l \mathfrak{h}, \mathbb{V}) \to Hom(\Lambda^{l+1} \mathfrak{h}, \mathbb{V})$$

given by

$$\begin{aligned} \partial\varphi(X_0, \dots, X_l) \quad := \quad & \sum_{i=0}^{l} (-1)^i X_i \cdot \varphi(X_0, \dots, \hat{X}_i, \dots, X_l) \\ & + \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_l), \end{aligned}$$

where $X_i \in \mathfrak{h}$ for $i \in \{0, \dots, l\}$ and the *hat* denotes omission. The differential ∂ computes the cohomology groups $H^l(\mathfrak{h}, \mathbb{V})$ of $\mathfrak{h}$ with values in $\mathbb{V}$. In particular, if $\mathfrak{g}$ is a $|k|$-graded semisimple Lie algebra, then the nilpotent subalgebra $\mathfrak{g}_-$ acts on $\mathfrak{g}$ by the adjoint representation and we have the cohomology groups $H^l(\mathfrak{g}_-, \mathfrak{g})$. Note that ∂ is in this case a $\mathfrak{g}_0$-homomorphism and the cohomology groups are in a natural way $\mathfrak{g}_0$-modules. By definition, an element $\phi \in Hom(\Lambda^l \mathfrak{g}_-, \mathfrak{g})$ has homogeneity degree s if $\phi(X_1, \dots, X_l) \in \mathfrak{g}_{i_1 + \dots + i_l + s}$ where $X_j \in \mathfrak{g}_{i_j}$ for all $j \in \{1, \dots, l\}$. The homogeneity degree gives rise to a bigrading $H^l(\mathfrak{g}_-, \mathfrak{g}) = \oplus_s H^l_s(\mathfrak{g}_-, \mathfrak{g})$ of the cohomology by $\mathfrak{g}_0$-modules.

Now let $\mathbb{V}$ be any $\mathfrak{g}$-module of a $|k|$-graded semisimple $\mathfrak{g}$. Via the Killing form $B_{\mathfrak{g}}$ the chain group $Hom(\Lambda^l \mathfrak{g}_-, \mathbb{V})$ is dual to $Hom(\Lambda^l \mathfrak{p}_+, \mathbb{V}^*)$ for all $l \in \mathbb{N}$. In particular, the negative of the dual map of the Lie algebra differential $\partial : Hom(\Lambda^r \mathfrak{p}_+, \mathbb{V}^*) \to Hom(\Lambda^{r+1} \mathfrak{p}_+, \mathbb{V}^*)$ is a linear map $\partial^* : Hom(\Lambda^{r+1} \mathfrak{g}_-, \mathbb{V}) \to Hom(\Lambda^r \mathfrak{g}_-, \mathbb{V})$, which is called the codifferential and satisfies $\partial^* \circ \partial^* = 0$. For the special case when $\mathbb{V} = \mathfrak{g}$ we obtain the *Kostant-codifferential*

$$\partial^* : Hom(\Lambda^r \mathfrak{g}_-, \mathfrak{g}) \to Hom(\Lambda^{r-1} \mathfrak{g}_-, \mathfrak{g}),$$

which is a $\mathfrak{p}$-module homomorphism (via the identification $\mathfrak{g}_- \cong \mathfrak{g}/\mathfrak{p}$). An explicit formula for the codifferential $\partial^* : Hom(\Lambda^2 \mathfrak{g}_-, \mathfrak{g}) \to Hom(\Lambda^1 \mathfrak{g}_-, \mathfrak{g})$ is given by

$$\partial^* \phi(X) = \sum_{\alpha} [\eta_\alpha, \phi(\xi_\alpha, X)] \; + \; \frac{1}{2} \sum_{\alpha} \phi([\eta_\alpha, X]_-, \xi_\alpha) \; ,$$

where $\{\xi_\alpha\}$ is a basis of $\mathfrak{g}_-$ and $\{\eta_\alpha\}$ denotes the dual basis of $\mathfrak{p}_+$. In general, the construction shows that the Lie algebra cohomology $H^l(\mathfrak{g}_-, \mathbb{V})$ is dual as a $\mathfrak{g}_0$-module to $H^l(\mathfrak{p}_+, \mathbb{V}^*)$.

It is a matter of fact that on any $|k|$-graded semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there exists a positive definite inner product B^*, which is defined via the Killing form $B_{\mathfrak{g}}$ and certain involutive automorphisms on the simple ideals of $\mathfrak{g}$. The differential ∂ and the codifferential ∂^* are adjoint with respect to this inner product B^*, i.e., $B^*(\partial\phi, \psi) = B^*(\phi, \partial^*\psi)$ for all $\phi \in Hom(\Lambda^l \mathfrak{g}_-, \mathfrak{g})$ and $\psi \in Hom(\Lambda^{l+1}\mathfrak{g}_-, \mathfrak{g})$. In particular, the spaces $Hom(\Lambda^l \mathfrak{g}_-, \mathfrak{g})$, $l \in \mathbb{N}$, split as direct sum into the image $Im(\partial)$ and the kernel $Ker(\partial^*)$, i.e., any cohomology class contains a ∂-closed and ∂^*-closed representative, the so-called harmonic representative.

Let us consider now the situation at the group level. Let G be a semisimple Lie group with $|k|$-graded Lie algebra $\mathfrak{g}$ and corresponding parabolic $\mathfrak{p}$. Then, in general, the Lie subgroup P in G, which consists of all elements in G preserving the filtration of $\mathfrak{g}$ under the adjoint action, has Lie algebra $\mathfrak{p}$ and is called a parabolic subgroup of G. The subgroup G_0 of G preserving the grading of $\mathfrak{g}$ is a reductive Lie group with Lie algebra $\mathfrak{g}_0$. The exponential map on $\mathfrak{g}$ restricted to the nilpotent subalgebra $\mathfrak{p}_+$ gives rise to a diffeomorphism between $\mathfrak{p}_+$ and a normal vector Lie subgroup P_+ in P. The parabolic subgroup P is isomorphic to the semidirect product $P = G_0 \ltimes \exp(\mathfrak{p}_+)$ and $P/P_+ = G_0$. We call such a pair (G, P) a *parabolic Klein geometry.*

Now let M be a smooth manifold equipped with a Cartan geometry $(\mathcal{P}, \omega) \xrightarrow{\pi} M$ modelled on a parabolic Klein geometry (G, P) with Lie algebras $(\mathfrak{g}, \mathfrak{p})$. We call $(\mathcal{P}(M), \omega)$ a *parabolic geometry* of type (G, P) on M. Note that for any such $(\mathcal{P}, \omega)$ the associated bundle $\mathcal{P}(M) \times_P \mathfrak{g}/\mathfrak{p}$ is canonically isomorphic to the tangent bundle TM. And the Cartan connection ω induces for any P-module $\mathbb{V}$ a differential $\nabla^\omega|_{\mathfrak{g}_-^*} : C^\infty(\mathcal{P}; \mathbb{V}) \to C^\infty(\mathcal{P}; \mathfrak{g}_-^* \otimes \mathbb{V})$. However, even if $f \in C^\infty(\mathcal{P}; \mathbb{V})$ is assumed to be a P-equivariant function, the function $\nabla^\omega f|_{\mathfrak{g}_-^*}$ is not P-equivariant, in general.

The corresponding curvature function $\kappa : \mathcal{P} \to \Lambda^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ to a parabolic geometry $(\mathcal{P}, \omega) \xrightarrow{\pi} M$ takes values in the 2-chains of the complex computing the Lie algebra cohomology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$. The curvature function κ splits with respect to the grading of $\mathfrak{g}$ into the direct sum $\kappa = \kappa_- + \kappa_0 + \kappa_+$ and with respect to the homogeneity degree it splits into $\kappa = \sum_{i=-k+2}^{3k} \kappa^{(i)}$.

Definition 1. *Let $(\mathcal{P}, \omega)$ be a parabolic geometry on a manifold M^n and let κ denote the curvature function of the Cartan connection ω. Then the parabolic geometry $(\mathcal{P}(M), \omega) \xrightarrow{\pi} M$ is called*

(1) *normal if $\partial^*\kappa = 0$,*
(2) *regular if $\kappa^{(i)} = 0$ for all $i \le 0$,*
(3) *torsion-free if $\kappa_- = 0$,*
(4) *flat if $\kappa = 0$.*

The *regular normal* Cartan connections play a central in the theory of parabolic geometries. To briefly explain this, let us consider the following reduction procedure for an arbitrary parabolic geometry $(\mathcal{P}(M), \omega) \xrightarrow{\pi} M$. Since the tangent bundle TM is canonically isomorphic to $\mathcal{P}(M) \times_P \mathfrak{g}/\mathfrak{p}$, the P-invariant filtration on $\mathfrak{g}/\mathfrak{p}$ induces a canonical filtration of TM with associated graded $\mathrm{gr}(TM) = \mathcal{P}(M) \times_P \mathrm{gr}(\mathfrak{g}/\mathfrak{p})$. Moreover, the space $\mathcal{P}_0(M) := \mathcal{P}/P_+$ is a principal G_0-bundle over M, and since P_+ acts trivially on $\mathrm{gr}(\mathfrak{g}/\mathfrak{p})$, we can factor through the P_+-action and write the associated graded as

$$\mathrm{gr}(TM) = \mathcal{P}_0(M) \times_{G_0} \mathrm{gr}(\mathfrak{g}/\mathfrak{p}) \ .$$

Thereby, the bundle $\mathcal{P}_0(M) \xrightarrow{\pi} M$ is a reduction of the frame bundle $Aut_{\mathrm{gr}}(TM)$ of $\mathrm{gr}(TM)$ to the structure group G_0. Since $\mathrm{gr}(\mathfrak{g}/\mathfrak{p}) \cong \mathfrak{g}_-$ as G_0-modules, we see that the associated graded $\mathrm{gr}(TM)$ is a locally trivial bundle of nilpotent Lie algebras with fibre type $\mathfrak{g}_-$. In particular, we have an algebraic bracket $\{\cdot,\cdot\}$ on $\mathrm{gr}(TM)$ induced by the G_0-invariant Lie bracket on $\mathfrak{g}_-$. Now, if the curvatures $\kappa^{(i)}$ vanish for all $i < 0$, one can show that the induced filtration of TM makes M into a filtered manifold. And, if in addition $\kappa^{(0)} = 0$, i.e., $(\mathcal{P}, \omega)$ is regular, then the algebraic bracket $\{\cdot,\cdot\}$ coincides with the generalised Levi-bracket $\mathcal{Z}$.

On the other hand, let (G, P) be a parabolic Klein geometry with the property that the cohomology groups $H^1_k(\mathfrak{g}_-, \mathfrak{g})$ vanish for all $k > 0$ and let M be a filtered manifold such that the associated graded $\mathrm{gr}(TM)_x$ equipped with Levi bracket $\mathcal{Z}_x$ is isomorphic to $\mathfrak{g}_-$ for all $x \in M$ (and $\mathrm{gr}(TM)$ is locally trivial). Then any reduction of $\mathrm{gr}(TM)$ to the structure group G_0, for which the induced algebraic bracket $\{\cdot,\cdot\}$ on $\mathrm{gr}(TM)$ coincides with $\mathcal{Z}$, admits a unique prolongation procedure, which results in a regular normal parabolic geometry $(\mathcal{P}, \omega_{nor}) \xrightarrow{\pi} M$. Thereby, the reduction procedure for $(\mathcal{P}, \omega_{nor})$ as described in the previous paragraph reproduces the given filtration on M (cf. [**41**]).

Finally, we introduce the notion of Weyl structures for parabolic geometries in the sense of [**42**]. Let $(\mathcal{P}, \omega) \xrightarrow{\pi} M$ be a parabolic geometry of type (G, P) on a space M. Then we have $\mathcal{P}_0(M) := \mathcal{P}/P_+$, which is a principal G_0-bundle over M, and $\mathcal{P}$ is a principal P_+-bundle over $\mathcal{P}_0$. A smooth section $\sigma : \mathcal{P}_0(M) \to \mathcal{P}(M)$ considered over the base $\mathcal{P}_0(M)$, which is G_0-equivariant with respect to the G_0-actions on $\mathcal{P}_0(M)$ and $\mathcal{P}(M)$, is called a *Weyl structure* for the parabolic geometry $(\mathcal{P}, \omega) \xrightarrow{\pi} M$. In this situation the pull-back $\sigma^*\omega_-$ of the $\mathfrak{g}_-$-part of the Cartan connection ω to $\mathcal{P}_0(M)$ is a soldering form and gives rise to an embedding of $\mathcal{P}_0(M)$ into the general linear frame bundle $GL(M)$. We conclude that the choice of a Weyl structure σ gives rise to G_0-structure on M. Moreover, the pull-back $\sigma^*\omega_0$ of the $\mathfrak{g}_0$-part of ω is a G_0-equivariant 1-form on $\mathcal{P}_0(M)$ with values in $\mathfrak{g}_0$, which induces the tautological map on fundamental vector fields in vertical directions. This means that $\sigma^*\omega_0$ is a principal connection form on the G_0-reduction $\mathcal{P}_0(M)$, which is called the Weyl connection ω^σ to the Weyl structure σ. The pull-back of the $\mathfrak{p}_+$-part of ω is tensorial on M and gives rise to the famous Rho-tensor P^σ !

6. Conformal Geometry

Conformal geometry is a classical object of interest in geometry and physics. There are different approaches for its definition and treatment. We aim to introduce conformal geometry in the framework of parabolic geometry as discussed in the previous section. The structure group of conformal geometry is the *Möbius group* $G = \mathrm{PO}(r+1, s+1)$, which acts on the flat homogeneous model $S^{r,s}$, the Möbius space of signature (r, s). The corresponding parabolic P generates a $|1|$-grading on the structure algebra $\mathfrak{g}$ and $\mathfrak{g}_0$ acts irreducibly on $\mathfrak{g}_{-1}$. This means that a conformal Cartan geometry induces a trivial filtration on the underlying space M. However, the structure group of $\mathrm{gr}(TM) = TM$ is reduced to $G_0 = \mathrm{CO}(r, s)$.

First, we describe the flat model of conformal geometry on the Lie algebra level. Let $\mathbb{R}^{r,s}$ be the Euclidean space of dimension $n = r + s \geq 3$ equipped with the scalar product $\langle\cdot,\cdot\rangle_{r,s}$ of signature (r,s) given by the matrix

$$\mathbb{J}_{r,s} = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix} .$$

We denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{so}(r+1,s+1)$ of the orthogonal group $\mathrm{O}(r+1,s+1)$, which acts via the standard representation on the Euclidean space $\mathbb{R}^{r+1,s+1}$ of dimension $n+2$ equipped with coordinates $(x^-,x^1,\dots,x^n,x^+)$ and indefinite scalar product

$$\langle x,y\rangle \ = \ x^-y^+ + x^+y^- + (x^1,\dots,x^n)\ \mathbb{J}_{r,s}\ (y^1,\dots,y^n)^\top .$$

The Lie algebra $\mathfrak{g}=\mathfrak{so}(r+1,s+1)$ is $|1|$-graded by

$$\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1 ,$$

where $\mathfrak{g}_0=\mathfrak{co}(r,s)$, $\mathfrak{g}_{-1}=\mathbb{R}^n$ and $\mathfrak{g}_1=\mathbb{R}^{n*}$. The 0-part $\mathfrak{g}_0$ is reductive and decomposes further into the centre $\mathbb{R}$ and the semisimple part $\mathfrak{so}(r,s)$, which is the Lie algebra of the isometry group of the Euclidean space $\mathbb{R}^{r,s}$. We realise the subspaces $\mathfrak{g}_0,\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ by matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ m & 0 & 0 \\ 0 & -m^\top\mathbb{J}_{r,s} & 0 \end{pmatrix}\in\mathfrak{g}_{-1}\ ,\quad \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}\in\mathfrak{g}_0\ ,\quad \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & -\mathbb{J}_{r,s}\ l^\top \\ 0 & 0 & 0 \end{pmatrix}\in\mathfrak{g}_1\ .$$

The commutators with respect to these matrices are given by

$$\begin{aligned}
&[\ ,\ ]:\ \mathfrak{g}_0\times\mathfrak{g}_0\to\mathfrak{g}_0\ , && [(A,a),(A',a')] = (AA'-A'A,0)\\
&[\ ,\ ]:\ \mathfrak{g}_0\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}\ , && [(A,a),m] = Am+am\\
&[\ ,\ ]:\ \mathfrak{g}_1\times\mathfrak{g}_0\to\mathfrak{g}_1\ , && [l,(A,a)] = lA+al\\
&[\ ,\ ]:\ \mathfrak{g}_{-1}\times\mathfrak{g}_1\to\mathfrak{g}_0\ , && [m,l] = (ml-\mathbb{J}_{r,s}\ (ml)^\top\mathbb{J}_{r,s},lm)\ ,
\end{aligned}$$

where (A,a), $(A',a')\in\mathfrak{so}(r,s)\oplus\mathbb{R}$, $m\in\mathbb{R}^n$, $l\in\mathbb{R}^{n*}$.

As before we denote $\mathfrak{p}:=\mathfrak{g}_0\oplus\mathfrak{g}_1$ and $\mathfrak{p}_+:=\mathfrak{g}_1$. The filtration

$$\mathfrak{g}\ \supset\ \mathfrak{p}\ \supset\ \mathfrak{p}_+$$

is $\mathfrak{p}$-invariant. The 0-part $\mathfrak{g}_0$ acts via the adjoint representation, preserving the grading of $\mathfrak{g}$. In particular, $\mathfrak{g}_0$ acts irreducibly on $\mathfrak{g}_{-1}$ and we obtain a natural inclusion of $\mathfrak{g}_0$ into $\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_{-1}$ (cf. [**135, 44**]). The *first prolongation* of the Lie algebra $\mathfrak{g}_0=\mathfrak{co}(r,s)$ is by definition the intersection of $\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_0$ with $S^2\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_{-1}$. This intersection is given by the kernel $ker(\partial)$ of the differential $\partial:\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_0\to\Lambda^2\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_{-1}$. Since $H^1_1(\mathfrak{g}_-,\mathfrak{g})$ is trivial, the kernel $ker(\partial)$ is the injective image of $\partial:\mathfrak{p}_+\to\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_0$, which shows that the nilpotent radical $\mathfrak{p}_+$ is the first prolongation $\mathfrak{co}(r,s)^{(1)}$. The second prolongation of $\mathfrak{co}(r,s)$ vanishes.

The Kostant-codifferential ∂^* is given on 2-chains by

$$\begin{aligned}
\partial^*:\ &Hom(\Lambda^2\mathfrak{g}_{-1},\mathfrak{g}) \ \to\ Hom(\mathfrak{g}_{-1},\mathfrak{g})\ ,\\
&\psi \ \mapsto\ \partial^*\psi\ =\ (\ X\in\mathfrak{g}_{-1}\ \mapsto\ \textstyle\sum_{i=1}^n[\eta_i,\psi(\xi_i,X)]\ )\ ,
\end{aligned}$$

where $\{\xi_i\ :\ i=1,\dots,n\}$ is some basis of $\mathfrak{g}_{-1}$ and $\{\eta_i\ :\ i=1,\dots,n\}$ is the corresponding dual basis of $\mathfrak{p}_+$ (with respect to the Killing form of $\mathfrak{g}$). On 1-chains we have

$$\begin{aligned}
\partial^*:\ &Hom(\mathfrak{g}_{-1},\mathfrak{g}) \ \to\ \mathfrak{g}\ ,\\
&\psi \ \mapsto\ \partial^*\psi\ =\ \textstyle\sum_{i=1}^n[\eta_i,\psi(\xi_i)]\ .
\end{aligned}$$

The projective orthogonal group $\mathrm{PO}(r+1,s+1)$ is defined as $\mathrm{O}(r+1,s+1)$ divided by its centre $\mathbb{Z}_2$. We set $G := \mathrm{PO}(r+1,s+1)$ and call it the Möbius group of signature (r,s). The subgroup P of the Möbius group G, which consists of those elements whose adjoint action on $\mathfrak{g}$ preserve the filtration, is a parabolic subgroup with Lie algebra $\mathfrak{p}$. The subgroup G_0, which preserves the grading of $\mathfrak{g}$, is isomorphic to the group $\mathrm{CO}(r,s) = \mathrm{O}(r,s) \times \mathbb{R}_+$ with Lie algebra $\mathfrak{g}_0 = \mathfrak{co}(r,s)$, where $\mathbb{R}_+$ denotes the positive real numbers. Moreover, the exponential map of $\mathfrak{g}$ restricts to a diffeomorphism from $\mathfrak{p}_+$ onto a normal vector subgroup $P_+ \cong \mathbb{R}^n$ of P. Then $P = \mathrm{CO}(r,s) \ltimes \mathbb{R}^n$ and $P/P_+ = \mathrm{CO}(r,s)$. As (lifted) subgroup of $\mathrm{O}(r+1,s+1)$ the parabolic P is given by the set of matrices

$$\left\{ \begin{pmatrix} a^{-1} & v & b \\ 0 & A & r \\ 0 & 0 & a \end{pmatrix} \;\middle|\; \begin{array}{c} A \in \mathrm{O}(r,s),\ a \in \mathbb{R}_+,\ v \in \mathbb{R}^{n*}, \\ r = -aA\mathbb{J}_{r,s}v^\top,\ b = -\frac{a}{2}v\mathbb{J}_{r,s}v^\top \end{array} \right\} .$$

The vector group P_+ is given by

$$\left\{ \begin{pmatrix} 1 & v & b \\ 0 & I_n & -\mathbb{J}_{r,s}v^\top \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; v \in \mathbb{R}^{n*},\ b = -\frac{1}{2}v\mathbb{J}_{r,s}v^\top \right\} .$$

The pair (G,P) represents the Klein pair on which conformal geometry is modelled. The projective representation of G on $\mathbb{R}^{r+1,s+1}$ gives rise to a transitive action on the (projective) null cone $\mathbb{P}L_o$ of $\mathbb{R}^{r+1,s+1}$ and the parabolic P is the stabiliser of the real null line $o = [1:0:\cdots:0]$. The homogeneous space $S^{r,s} = G/P$ is the flat model of conformal geometry and we call $S^{r,s}$ the *Möbius space* of signature (r,s). The normal subgroup P_+ is the kernel of the isotropy representation on the tangent space $T_oS^{r,s}$. In the Riemannian case the Möbius space is diffeomorphic to the sphere S^n, which is the compactification of the Euclidean space $\mathbb{R}^n$ by adding a point at infinity. (The round metric on S^n represents the flat conformal structure.) In the Lorentzian case the space $S^{1,n-1}$ is the compactification of the Minkowski space $\mathbb{R}^{1,n-1}$ by a light cone at infinity. The space $S^{1,n-1}$ is diffeomorphic to $(S^1 \times S^{n-1})/\mathbb{Z}_2$.

Now let M^n be a C^∞-manifold of dimension $n \geq 3$ and let $(\mathcal{P},\omega) \xrightarrow{\pi} M$ be a Cartan geometry on M of conformal type (G,P) with curvature Ω. The curvature function

$$\kappa:\ \mathcal{P}(M)\ \to\ \mathit{Hom}(\Lambda^2\mathfrak{g}_{-1},\mathfrak{g})$$

decomposes to $\kappa_{-1}+\kappa_0+\kappa_1$ according to the grading of $\mathfrak{g}$. Thereby, the 0-part κ_0 can be seen as a function on $\mathcal{P}(M)$, which takes values in $\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}$. The (generalised) Bianchi identity for a Cartan connection ω on $\mathcal{P}(M)$ is expressed by

$$-\partial\kappa(X,Y,Z)\ =\ \sum_{cycl}\big(\ \nabla^\omega_Z(\kappa(X,Y))\ +\ \kappa(\kappa_{-1}(X,Y),Z)\ \big)\ ,$$

where $\sum_{cycl}$ denotes the sum over all cyclic permutations of the arguments.

The normalisation condition $\partial^*\circ\kappa = 0$ for a Cartan connection ω on $\mathcal{P}(M)$ of type (G,P) is equivalent to

$$\kappa_{-1} = 0 \qquad \text{and} \qquad tr\kappa_0 := \sum_{i=1}^n \kappa_0(\xi_i,\cdot)(\cdot)(\eta_i) = 0\ ,$$

where $\{\xi_i\}$ is a basis of $\mathfrak{g}_{-1}$ with dual basis $\{\eta_i\}$, i.e., a normal Cartan connection ω of type (G,P) is torsion-free and the 0-part of the curvature function is traceless. For normal Cartan connections ω the Bianchi identity simplifies to

$$\begin{aligned} \sum_{cycl}[\kappa_0(X,Y),Z] &= 0\,, \qquad \sum_{cycl}\nabla^\omega_Z(\kappa_1(X,Y)) \;=\; 0 \qquad \text{and} \\ \sum_{cycl}\nabla^\omega_Z(\kappa_0(X,Y)) &= \sum_{cycl}[\kappa_1(X,Y),Z]\,. \end{aligned}$$

Let $(\mathcal{P},\omega)\stackrel{\pi}{\to} M$ be an arbitrary Cartan geometry of conformal type (G,P). We observe that the bundle $\mathcal{G}_0 := \mathcal{P}/P_+$ is a principal G_0-bundle on M and the (-1)-part ω_{-1} of the Cartan connection ω induces a soldering form θ_{-1} on $\mathcal{G}_0$. Thus the bundle $\mathcal{G}_0$ is a G_0-structure on M, which determines orthogonal frames at every point of M. We call $(\mathcal{G}_0,\theta_{-1})$ a *conformal structure* (of first order) on M. In general, different Cartan connections on $\mathcal{P}$ can induce the same first order conformal structure $(\mathcal{G}_0,\theta_{-1})$. However, there exists a canonical Cartan connection on $\mathcal{P}(M)$, which induces $(\mathcal{G}_0,\theta_{-1})$. To see this, we describe briefly the natural prolongation procedure for $(\mathcal{G}_0,\theta_{-1})$ (cf. Section 0.5 and [**44**]). The resulting canonical Cartan connection is uniquely determined by the condition $\partial^*\kappa = 0$, i.e., it is the unique normal one. We will denote the canonical connection by ω_{nor}.

So let $(\mathcal{G}_0,\theta_{-1})\stackrel{\pi}{\to} M$ be a $\mathrm{CO}(r,s)$-structure on M with θ_{-1}, the soldering form. Then let us consider a linear map $\phi:\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\to T_u\mathcal{G}_0(M)$ for $u\in\mathcal{G}_0(M)$ such that $\phi(0,A)=\chi_A$ for all $A\in\mathfrak{g}_0$ and $\theta_{-1}\circ\phi(X,A)=X$ for all $X\in\mathfrak{g}_{-1}$. The map ϕ is uniquely determined by the image $\phi(\mathfrak{g}_{-1})$ in $T_u\mathcal{G}_0(M)$, which is a transverse complement to the vertical space, i.e., a horizontal subspace. The exterior derivative $d\theta_{-1}$ gives rise to an element $t_\phi\in\Lambda^2\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_{-1}$, which sends $X\wedge Y\in\Lambda^2\mathfrak{g}_{-1}$ to $d\theta_{-1}(\phi(X,0),\phi(Y,0))$ and which is called the torsion of ϕ. We set

$$\mathcal{P} \;:=\; \{\ \phi \text{ horizontal in } T\mathcal{G}_0(M) \text{ and } \partial^* t_\phi = 0\ \}\ .$$

The set $\mathcal{P}$ is seen to be non-empty and for any two $\phi_1,\phi_2\in\mathcal{P}_u$, $u\in\mathcal{G}_0$, there exists a unique α in the prolongation $\mathfrak{p}_+=\mathfrak{co}(r,s)^{(1)}$ such that $\phi_2(X,A)-\phi_1(X,A)=\chi_{\partial\alpha(X)}(u)$ for all $(X,A)\in\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$. In fact, the parabolic P acts on $\mathcal{P}$ by $\phi\in\mathcal{P}_u\mapsto R^*_{l_0}\circ\phi\circ Ad(l)\in\mathcal{P}_{u\cdot l_0}$, where l_0 denotes the class in $G_0=P/P_+$ of $l\in P$. This action is free and transitive on the fibres of $\mathcal{P}\stackrel{\pi}{\to} M$ and makes $\mathcal{P}$ into a principal P-bundle over M. Furthermore, let ϕ be any point in $\mathcal{P}$ and let $\xi\in T_\phi\mathcal{P}$ be a vector at ϕ. Then we can take the projection of ξ to $T\mathcal{G}_0$ and apply to it ϕ^{-1}. This gives rise to a soldering form $\theta:=\theta_{-1}\oplus\theta_0 : T\mathcal{P}\to\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$. The bundle $(\mathcal{P},\theta)\to M$ with soldering form θ is called the first prolongation of the conformal structure $(\mathcal{G}_0,\theta_{-1})\stackrel{\pi}{\to} M$. Since the torsion $T_\theta := d\theta_{-1}+[\theta_0,\theta_{-1}]$ of θ vanishes (in case of conformal geometry), it can be shown that $\mathcal{P}$ is via θ a P-reduction of the second order frame bundle $Gl^{(2)}(M)$ (the bundle of 2-jets of coordinates at points in M).

In the next step, we consider certain linear maps $\Phi:\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\to T_u\mathcal{P}$ at $u\in\mathcal{P}$ and again we can define a torsion t_Φ as before by use of the soldering form θ. This time, since the cohomology $H^1_2(\mathfrak{g}_{-1},\mathfrak{g})$ is trivial (in case of conformal geometry), we can conclude that there exists a unique $\tilde\Phi$ with $\partial^* t_{\tilde\Phi}=0$ at each $u\in\mathcal{P}$. In this sense the second prolongation of $(\mathcal{G}_0(M),\theta_{-1})$ is trivial. The collection of the inverse $\tilde\Phi^{-1}$

at each $u \in \mathcal{P}$ gives rise to a smooth 1-form $\omega_{nor} : T\mathcal{P} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, which is by construction a *normal Cartan connection* on $\mathcal{P} \xrightarrow{\pi} M$ of type (G, P).

The described prolongation procedure establishes a bijective correspondence between first order conformal structures and normal conformal Cartan geometries (or second order conformal structures) on a space M. In particular, the canonical connection ω_{nor} solves the equivalence problem of conformal geometry, i.e., two conformal (first order) structures are equivalent if and only if their prolongations $(\mathcal{P}, \omega_{nor})$ are isomorphic as Cartan geometries. The existence of the canonical connection in conformal geometry was discovered by E. Cartan (cf. [**46**]). Note that there exists no conformally covariant linear connection on the first order structure $(\mathcal{G}_0, \theta_{-1})$.

The canonical connection ω_{nor} is a useful tool for the construction of invariants for conformal geometries $(\mathcal{G}_0, \theta_{-1}) \xrightarrow{\pi} M$. For example, we have the conformal holonomy group $Hol(\omega_{nor})$, which is uniquely defined as the holonomy group of the G-connection $\tilde{\omega}_{nor}$ on the G-extension $\mathcal{G} \xrightarrow{\pi} M$ of $\mathcal{P}$ (cf. Section 0.5). In general, the group $Hol(\omega_{nor})$ is a subgroup of the Möbius group $G = \mathrm{PO}(r+1, s+1)$. We denote the corresponding holonomy algebra by $\mathfrak{hol}(\omega_{nor})$. The holonomy algebra $\mathfrak{hol}(\omega_{nor})$ is trivial iff the conformal structure is flat. The curvature Ω of ω_{nor} is a basic invariant as well, which we will discuss in more detail below.

A *Weyl structure* for a normal conformal Cartan geometry $(\mathcal{P}, \omega_{nor}) \xrightarrow{\pi} M$ is a G_0-equivariant section σ in $\mathcal{P}$ considered as a P_+-bundle over the bundle $\mathcal{G}_0(M) = \mathcal{P}/P_+$ of first order. Since the torsion T_θ vanishes, the pull-back $\omega^\sigma := \sigma^* \circ \theta_0$ with respect to such a Weyl structure is a torsion-free linear connection on the G_0-structure $\mathcal{G}_0(M)$. On the other hand, any torsion-free connection γ on $\mathcal{G}_0(M)$ gives rise to a Weyl structure $\sigma^\gamma : \mathcal{G}_0 \to \mathcal{P}$ via the γ-horizontal subspaces in $T\mathcal{G}_0(M)$. This shows that there is a natural and bijective correspondence between Weyl structures and torsion-free linear G_0-connections in conformal geometry. We call the torsion-free connections on $\mathcal{G}_0(M)$ the *Weyl connections*.

Now let us consider the standard inclusion $\iota : \mathrm{O}(r, s) \hookrightarrow \mathrm{CO}(r, s)$ of the semisimple part of the reductive structure group G_0 of first order. A ι-reduction $O(M)$ of the conformal structure $(\mathcal{G}_0(M), \theta_{-1})$ on M is nothing, but the choice of a semi-Riemannian metric g on M. We call such a metric g *compatible* with the given conformal structure $(\mathcal{G}_0, \theta_{-1})$ on M. Since the Levi-Civita connection ω^g of a compatible metric g is uniquely determined and torsion-free, it gives rise to a Weyl structure $\sigma^g : \mathcal{G}_0(M) \to \mathcal{P}(M)$. However, note that not every Weyl structure σ comes from a compatible metric.

Obviously, whenever g is compatible for a given $(\mathcal{G}_0, \theta_{-1}) \xrightarrow{\pi} M$, then $\tilde{g} = e^{2f} g$ is compatible as well for any function $f : M \to \mathbb{R}$. In fact, any two compatible metrics g_1, g_2 are related by $g_2 = e^{2f} g_1$ for some function f and we call the metrics g_1 and g_2 *conformally equivalent* in this case. Now, if a compatible metric g for some $(\mathcal{G}_0, \theta_{-1})$ on M does exist, then we denote by $c := [g]$ the equivalence class of conformally related (i.e. compatible) metrics. The class c is uniquely determined by the conformal structure $(\mathcal{G}_0, \theta_{-1})$. On the other hand, any semi-Riemannian metric g on a space M determines a class $c = [g]$ of conformally equivalent metrics and, in particular, the metric g, resp., its conformal class c defines uniquely a conformal structure $(\mathcal{G}_0, \theta_{-1})$ on M. For that reason, we call a class c of conformally related metrics on a space M also a conformal structure. Note that, in general, a compatible metric for a given $(\mathcal{G}_0, \theta_{-1})$ need not exist on M. However, in Riemannian signature it is clear (by use of a partition of unity) that any $(\mathcal{G}_0, \theta_{-1})$ admits a compatible metric, i.e., there is a natural bijective correspondence between conformal classes c and G_0-reductions $(\mathcal{G}_0, \theta_{-1})$.

In the following, we will usually assume that a compatible metric does exist for a given conformal structure $(\mathcal{G}_0, \theta_{-1})$ on M or that $(\mathcal{G}_0, \theta_{-1})$ is actually given by a metric g, resp., its conformal class $c = [g]$. This assumption guarantees that the ray subbundle $\mathcal{Q}$ of pointwise conformally related metrics to g in $S^2(T^*M)$ is a trivial principal $\mathbb{R}_+$-bundle over M, where $s \in \mathbb{R}_+$ acts on g by $g \cdot s^2$. Then the representations $\rho^w : a \in \mathbb{R}_+ \mapsto a^{-w/2} \in Aut(\mathbb{R})$, $w \in \mathbb{R}$, give rise to real line bundles

$$\mathcal{E}[w] := \mathcal{Q} \times_{\rho^w} \mathbb{R} \ ,$$

which are trivial on M. For any $w \in \mathbb{R}$ the bundle $\mathcal{E}[w]$ is called the density bundle of conformal weight w on (M, c).

Now let g be a semi-Riemannian metric of arbitrary signature (r, s) on a manifold M of dimension $n \geq 3$ and let $c := [g]$ (resp., $(\mathcal{G}_0, \theta_{-1})$) be the corresponding conformal structure on M. The natural prolongation uniquely gives rise to the normal conformal Cartan geometry $(\mathcal{P}(M), \omega_{nor})$, where $\omega_{nor} = \omega_{-1} + \omega_0 + \omega_1$, and we have the induced Weyl structure $\sigma^g : \mathcal{G}_0 \to \mathcal{P}$. In this situation the pull-back $\sigma^{g*}\omega_0$ is the Levi-Civita connection to g on $\mathcal{G}_0$. Moreover, the pull-back $\sigma^{g*}\omega_1$ is tensorial on $\mathcal{G}_0$ and projects to the Schouten tensor $\mathsf{P}^g = \frac{1}{n-2}\Big(\frac{scal^g}{2(n-1)} - Ric^g \Big)$ on M. The curvature Ω of ω_{nor} decomposes to $\Omega_0 + \Omega_{-1}$ and $\sigma^{g*}\Omega_0$ gives rise to the Weyl tensor W^g on (M, g). The $\mathfrak{p}_+$-part $\sigma^{g*}\Omega_1$ corresponds to the Cotton tensor C^g (cf. Section 0.7).

We introduced in Section 0.4 the notion of metric invariants $Q(g)$ and differential operators $\mathcal{D}(g)$ for semi-Riemannian spaces (M, g) via universal polynomial formulae with respect to components of the metric given in coordinates. Now let us consider conformal spaces (M, c) and let $Q(g)$ be an invariant for some compatible metric $g \in c$. Then we call $Q(g)$ a conformal invariant of (M, c) if

$$Q(e^{2\phi} g) \ = \ e^{-b\phi} \cdot Q(g)$$

for any $\tilde{g} = e^{2\phi} g \in c$ with some fixed number $b \in \mathbb{R}$. For example, it is well known that the Weyl tensor W^g is such a conformal invariant. In dimension $n = 3$, the Cotton tensor C^g is a conformal invariant, since the Weyl tensor vanishes. (In general, we have $C^g = (3 - n) \cdot div_4^g W^g$.) In dimension 4, the Bach tensor B^g is a conformal invariant (cf. Section 0.4 and 0.11). Accordingly, a metric invariant differential operator $\mathcal{D}(g)$ is called conformally covariant of bidegree (a, b) if $\mathcal{D}(e^{2\phi} g) = e^{-b\phi} \circ \mathcal{D}(g) \circ e^{a\phi}$ for all $\phi \in C^\infty(M)$ with some fixed numbers $a, b \in \mathbb{R}$. A well-known example for a conformally covariant differential operator is the conformal Laplacian

$$P^g_{2,n} \ := \ \Delta^g_{tr} + \frac{n-2}{4(n-1)} scal^g$$

of bidegree $(\frac{n}{2} - 1, \frac{n}{2} + 1)$.

Finally, in this section we want to discuss symmetries of conformal structures and related covariant differential equations. First, let $(\mathcal{P}, \omega_{nor}) \xrightarrow{\pi} M$ be a normal conformal Cartan geometry. An automorphism Φ of $(\mathcal{P}(M), \omega_{nor})$ is a principal bundle automorphism of $\mathcal{P}(M)$ such that $\Phi^*\omega_{nor} = \omega_{nor}$. The group of such automorphisms is denoted by $Aut(\omega_{nor})$. Since $Aut(\omega_{nor})$ preserves an absolute parallelism on $\mathcal{P}(M)$, it is clear that $Aut(\omega_{nor})$ acts as a finite dimensional Lie group on $\mathcal{P}(M)$, whose dimension is bounded by $dim(G) = (n + 1)(n + 2)/2$. An infinitesimal automorphism of a normal conformal Cartan geometry $(\mathcal{P}(M), \omega_{nor})$ is a P-invariant vector field X on $\mathcal{P}(M)$ such that $\mathcal{L}_X \omega_{nor} = 0$. The Lie algebra of infinitesimal automorphisms is denoted by $\mathfrak{inf}(\omega_{nor})$ and $dim(\mathfrak{inf}(\omega_{nor})) \leq dim(G)$ with equality only if $(\mathcal{P}(M), \omega_{nor})$ is a flat Cartan geometry.

Of course, there is also an equivalent first order formulation for conformal automorphisms. In first order we say, a map $\phi : (M, c) \to (M, c)$ is a conformal transformation (or automorphism) if ϕ is a diffeomorphism and $\phi^* g = l \cdot g$ for some (hence any) metric $g \in c$, where $l \in C^\infty(M)$ is an appropriate function depending on g. Accordingly, we call a vector field V on (M, c) a *conformal Killing vector field* iff $\mathcal{L}_V g = \lambda g$ for some metric $g \in c$ with appropriate $\lambda \in C^\infty(M)$. (This generalises the notion of Killing vector fields for semi-Riemannian spaces.) We denote the set of conformal Killing vector fields on a space (M, c) by $\mathfrak{inf}(M, c)$. The local flow of any $V \in \mathfrak{inf}(M, c)$ is a local 1-parameter group of conformal transformations on (M, c). In particular, the local flow of V determines the flow of the first order frames in $\mathcal{G}_0(M) \subset Gl(M)$ and also of the second order frames in $\mathcal{P}(M) \subset Gl^{(2)}(M)$. In fact, the 2-jet $j^2(V)$ of V is a P-invariant vector field on $\mathcal{P}(M)$ with $\mathcal{L}_{j^2(V)} \omega_{nor} = \omega_{nor}$, i.e., $j^2(V)$ is an infinitesimal automorphism of $(\mathcal{P}, \omega_{nor})$. On the other hand, any element $X \in \mathfrak{inf}(\omega_{nor})$ projects down to a conformal Killing vector field $V := \pi_*(X)$ on (M, c). This establishes a canonical Lie algebra isomorphism between $\mathfrak{inf}(M, c)$ and $\mathfrak{inf}(\omega_{nor})$. The analogous statement for automorphisms of (M, c) and $(\mathcal{P}(M), \omega_{nor})$ is also true. In particular, it is clear now that a conformal transformation of a connected space (M, c) is uniquely determined by its 2-jet at a single point $p \in M$ (cf. [**90**]).

We highlight some properties and further definitions for conformal Killing vector fields V. So let us consider such a V on a semi-Riemannian space (M, g). Then, in general, $\mathcal{L}_V g = \lambda g$ with $\lambda = \frac{2}{n} div^g(V)$ and $\nabla^g V = \frac{1}{n} div^g(V) \cdot id|_{TM} + A$, where A is some skew-symmetric endomorphism on TM. Moreover, using a Weitzenböck formula we obtain

$$\Delta^g_{tr} V \;=\; -Ric^g(V) \;+\; \frac{2-n}{n} grad^g(div^g V)\ . \tag{4}$$

If $div^g(V) \neq 0$ is constant on (M, g), we call V a *homothetic* Killing vector. A conformal gradient field is a conformal vector field V, which is the gradient of some function, i.e., $V = grad^g f$. Both these notions are just metric invariant. Homothetic gradient fields are closely related to the metric cone construction (cf. Section 0.9).

A conformal Killing vector field V is called *essential* if $div^g(V) \not\equiv 0$ for all $g \in c$ on M. This definition means that an essential conformal Killing vector field is not a Killing vector for any metric g in a given conformal class c on M. Locally, an essential conformal Killing vector V field needs to have a zero. Accordingly, the conformal transformation group $Aut(M, c)$ of a space (M, c) is called essential if there exists a $\phi \in Aut(M, c)$, which is not an isometry with respect to any metric $g \in c$. In Riemannian geometry, $Aut(M, c)$ is non-compact if and only if $Aut(M, c)$ is essential. In this case it is also known that $Aut(M, c)$ is essential only for the Euclidean space $\mathbb{R}^n$ and for the round sphere (S^n, g_o) (up to conformal equivalence), which are both conformally flat (cf. [**133, 134, 116, 2, 153**]). The investigation of pseudo-Riemannian conformal spaces with essential transformation group is much more involved (cf. e.g. [**4, 60**]). In Chapter 4 we will present examples of essential conformal Killing vectors on non-compact Lorentzian spaces, which are not conformally flat!

The dual 1-form $\alpha_V := V^\flat$ to a conformal Killing vector field V on a space (M, g) satisfies the equation

$$\nabla^g_X \alpha_V - \frac{1}{2}\iota_X d\alpha_V + \frac{1}{n} X^\flat \wedge d^*\alpha_V \;=\; 0 \qquad \text{for all } X \in TM\ .$$

More generally, let us consider the vector bundle $T^*M \otimes \Lambda^p T^*M$. This bundle decomposes into the direct sum $\Lambda^{p-1}(T^*M) \oplus \Lambda^{p+1}(T^*M) \oplus C_p(M)$, where $C_p(M)$ denotes the intersection of the kernels of $\wedge$-product and insertion ι applied to elements of $T^*M \otimes \Lambda^p T^*M$. The composition $pr_C \circ \nabla^g : \Omega^p(M) \to \Gamma(C_p(M))$ is a first order differential operator, which is actually conformally covariant. The kernel of $pr_C \circ \nabla^g$ on a semi-Riemannian space (M, g) is described by the conformally covariant partial differential equation

$$\nabla^g_X \alpha - \frac{1}{p+1}\iota_X d\alpha + \frac{1}{n-p+1} X^\flat \wedge d^*\alpha \;=\; 0\ , \qquad X \in TM\ , \tag{5}$$

for p-forms α. We call a solution $\alpha \in \Omega^p(M)$ of this equation a conformal Killing p-form (cf. Section 0.7; [**88, 148, 143, 72**]). Conformal Killing p-forms are related to higher symmetries of the Laplacian (cf. [**51**]).

7. Tractor Calculus

We start by introducing tractor bundles for parabolic geometries, in general. Later in this section we will specialise to conformal geometry. In Section 0.8 and 0.10 we will also meet a spin version of conformal tractor calculus and furthermore CR-tractors.

Let $(\mathcal{P}, \omega) \xrightarrow{\pi} M$ be a parabolic geometry on M of arbitrary type (G, P) with Lie algebras $(\mathfrak{g}, \mathfrak{p})$. The adjoint representation of P on $\mathfrak{g}$ gives rise to an associated vector bundle $\mathcal{A}(M) := \mathcal{P} \times_P \mathfrak{g}$, which is called the *adjoint tractor bundle* to $\mathcal{P}(M)$. This vector bundle $\mathcal{A}$ admits a (locally trivial) invariant filtration $\mathcal{A}^{-k} \supset \cdots \supset \mathcal{A}^k$ and an algebraic bracket $\{\cdot,\cdot\}_{\mathcal{A}}$ on each fibre induced by $\mathfrak{g}$ and its $|k|$-grading. More generally, let $\mathbb{V}$ be a finite dimensional G-module with effective action ρ. The space $\mathbb{V}$ decomposes as direct sum $\oplus_i \mathbb{V}_i$ into G_0-modules, where a summand $\mathbb{V}_j$ denotes the eigenspace to an eigenvalue j with respect to the action of the grading element $E \in \mathfrak{g}_0$. Then the action of the component $\mathfrak{g}_i$ of $\mathfrak{g}$ maps $\mathbb{V}_j$ to $\mathbb{V}_{i+j}$ and the splitting $\oplus_i \mathbb{V}_i$ gives rise to a P-invariant filtration $\mathbb{V}^a \supset \cdots \supset \mathbb{V}^b$ of $\mathbb{V}$. (Note that we allow here a and b to be real numbers.) We call the associated vector bundle

$$\mathcal{T}_{\mathbb{V}} := \mathcal{P}(M) \times_{\rho(P)} \mathbb{V}$$

the $\mathbb{V}$-tractor bundle for $\mathcal{P}(M)$ over M. The infinitesimal $\mathfrak{g}$-action ρ_* on $\mathbb{V}$ is compatible with the P-action and this gives rise to a bundle homomorphism $\mathcal{A} \otimes \mathcal{T}_{\mathbb{V}} \to \mathcal{T}_{\mathbb{V}}$, which makes $\mathcal{T}_{\mathbb{V}}$ into a bundle of modules over $\mathcal{A}$. We denote the action of this module structure by $(A, t) \mapsto A \bullet t$. Moreover, $\mathcal{T}_{\mathbb{V}}$ is equipped with a locally trivial filtration $\mathcal{T}^a_{\mathbb{V}} \supset \cdots \supset \mathcal{T}^b_{\mathbb{V}}$, which is invariant under the action $\bullet$ of $\mathcal{A}^0$ on $\mathcal{T}_{\mathbb{V}}$.

Now let $(\mathcal{G}(M), \tilde{\omega})$ be the G-extension of $(\mathcal{P}(M), \omega)$ over M and $\mathbb{V}$ an effective G-module. The principal connection $\tilde{\omega}$ gives rise to a covariant derivative

$$\nabla^{\tilde{\omega}} : \Gamma(\mathcal{T}_{\mathbb{V}}) \to \Gamma(T^*M \otimes \mathcal{T}_{\mathbb{V}})\ ,$$

on the $\mathbb{V}$-tractor bundle (cf. Section 0.5). We call $\nabla^{\tilde{\omega}}$ a tractor connection. We also denote by $\nabla^{\mathcal{A}}$ the tractor connection on the adjoint tractor bundle $\mathcal{A}$ (in case ω is a canonical choice). In general, the action of $\nabla^{\tilde{\omega}}$ on $t \in \Gamma(\mathcal{T}_{\mathbb{V}})$ is given by $u^{-1}(\nabla^{\tilde{\omega}}_X t) =$

$\tilde{X}(\tilde{t})(u) + \rho_*(\omega(\tilde{X}))\tilde{t}(u)$, where $\tilde{t} \in C^\infty(\mathcal{P};\mathbb{V})$ denotes the P-equivariant lift of the section t, the isomorphism $u^{-1} : (\mathcal{T}_\mathbb{V})_{\pi(u)} \to \mathbb{V}$ is given for $u \in \mathcal{P}(M)$ by $[u,\tilde{s}] \mapsto \tilde{s}$ and $\tilde{X}$ is an arbitrary lift of $X \in T_{\pi(u)}M$ to $T_u\mathcal{P}(M)$. Such a tractor connection $\nabla^{\tilde{\omega}}$ is distinguished by the following properties. For any tangent vector $\xi \in T_u\mathcal{P}(M)$, $u \in \mathcal{P}(M)$, the map $\Phi_u(\xi) := (\ \tilde{t}(u) \in \mathbb{V} \mapsto u^{-1} \circ \nabla^{\tilde{\omega}}_{\pi_*(\xi)} t - \xi(\tilde{t})(u) \in \mathbb{V}\ )$ is the action of some element in $\mathfrak{g}$, and $\nabla^{\tilde{\omega}}$ is non-degenerate in the sense that for any $p \in M$ and any tangent vector $X \in T_pM$, there exists a number i and a local smooth section $t \in \Gamma(\mathcal{T}^i_\mathbb{V})$ such that $\nabla^{\tilde{\omega}}_X t(p) \notin (\mathcal{T}^i_\mathbb{V})_p$. We call a connection $\nabla^{\tilde{\omega}}$ which satisfies these two properties a non-degenerate $\mathfrak{g}$-connection on $\mathcal{T}_\mathbb{V}$.

We remark that our definition of tractor bundles $\mathcal{T}_\mathbb{V}$ with derivative $\nabla^{\tilde{\omega}}$ makes use of a parabolic Cartan geometry $(\mathcal{P},\omega)$ on M. However, tractor calculus can also be defined without the input of Cartan geometry. In fact, given a locally trivial bundle of filtered Lie algebras modelled on some semisimple $|k|$-graded $\mathfrak{g}$, any non-degenerate $\mathfrak{g}$-connection induces a Cartan connection for some principal bundle with parabolic structure group $P \subset Aut(\mathfrak{g})$. In this respect, tractor calculus and parabolic Cartan geometry are on an equal footing (cf. [**150, 14, 37**]).

Now let us consider the principal G_0-bundle $\mathcal{G}_0(M) = \mathcal{P}(M)/P_+$ for a parabolic geometry over M. The associated graded $\mathrm{gr}(\mathbb{V})$ of a G-module $\mathbb{V}$ admits a natural G_0-action. Via this action $\mathrm{gr}(\mathcal{T}_\mathbb{V})$ becomes an associated vector bundle to $\mathcal{G}_0(M)$. However, there is no natural identification of $\mathrm{gr}(\mathcal{T}_\mathbb{V})$ and $\mathcal{T}_\mathbb{V}$ unless we introduce a Weyl structure. So let $\sigma : \mathcal{G}_0(M) \to \mathcal{P}(M)$ be a Weyl structure. The Weyl structure σ give rise to a G_0-structure $(\mathcal{G}_0,\theta_-)$ on M equipped with connection $\sigma^*\omega_0$. Moreover, we obtain a splitting of $\mathcal{T}_\mathbb{V}$ via σ into a direct sum $\oplus_i \mathcal{T}_{\mathbb{V}_i}$ of G_0-associated bundles $\mathcal{T}_{\mathbb{V}_i}$. In particular, we have an identification of $\mathrm{gr}(\mathcal{T}_\mathbb{V})$ and $\mathcal{T}_\mathbb{V}$ via σ and the Weyl connection $\sigma^*\omega_0$ on $\mathcal{G}_0(M)$ induces covariant derivatives $\nabla^{\mathbb{V}_i}$ on each $\mathcal{T}_{\mathbb{V}_i}$. The difference between $\nabla^{\tilde{\omega}}$ and the direct sum of the $\nabla^{\mathbb{V}_i}$ on $\oplus_i \mathcal{T}_{\mathbb{V}_i}$ is described by the action of the Rho-tensor $\mathsf{P}^\sigma = \sigma^*\omega_+$.

We specialise the situation now to tractor calculus in conformal geometry. So let M^n be a space of dimension $n \geq 3$ equipped with a conformal structure c, which is given by the conformal class of some metric g on M. Then we have the trivial density bundles $\mathcal{E}[w]$ of conformal weight $w \in \mathbb{R}$ over M (cf. Section 0.6). We denote by $(\mathcal{P},\omega_{nor}) \xrightarrow{\pi} M$ the corresponding normal Cartan geometry on (M,c) of conformal type (G,P) with $G = \mathrm{PO}(r+1,s+1)$. Since $\mathfrak{g}$ is $|1|$-graded (in conformal geometry), the principal bundle $\mathcal{G}_0(M) = \mathcal{P}(M)/P_+$ is already a G_0-structure on M. Then, if $\mathcal{B}$ denotes an arbitrary vector bundle associated to $\mathcal{G}_0(M)$, we denote by $\mathcal{B}[w]$ the conformally weighted bundle $\mathcal{B} \otimes \mathcal{E}[w]$. For example, $TM = T^*M[2]$, which means that vector fields on M are identified via the conformal structure c with 1-forms of conformal weight 2.

The adjoint tractor bundle $\mathcal{A}(M)$ for a conformal Cartan geometry of type (G,P) is given by $\mathcal{P}(M) \times_P \mathfrak{so}(r+1,s+1)$ and is filtered by

$$\mathcal{A} \supset \mathcal{A}^0 \supset \mathcal{A}^1\ ,$$

where $\mathcal{A}^0 = \mathcal{P} \times_P \mathfrak{p}$ and $\mathcal{A}^1 = \mathcal{P} \times_P \mathfrak{p}_+$. Obviously, the subbundle $\mathcal{A}^1$ is canonically isomorphic to T^*M, i.e., the dual tangent bundle T^*M is canonically embedded in $\mathcal{A}(M)$. However, in order to be able to resort to some more assortment of tractors we extend the group G by the centre $\mathbb{Z}_2$ to $\tilde{G} = \mathrm{O}(r+1,s+1)$. This gives rise to the extended Klein geometry $(\tilde{G}, \tilde{P})$ (of conformal type), where $\tilde{P}$ is the stabiliser of a null line in $\mathbb{R}^{r+1,s+1}$ under the standard action of $\tilde{G}$. We set $\tilde{G}_0 := G_0$ (in abuse of our conventions so far) and consider it as a (lifted) subgroup in $\tilde{G}$. Now, since we assume the existence of a compatible metric g on M, the ray bundle $\mathcal{Q} \subset S^2(T^*M)$ is trivial. This ensures that the trivial $\mathbb{Z}_2$-covering $\tilde{\mathcal{P}}(M)$ over $\mathcal{P}(M)$ is a $\tilde{P}$-reduction of $\mathcal{P}(M)$ as principal fibre bundle over M, which is then endowed with the lift of the normal Cartan connection ω_{nor}. Thus we obtain a normal conformal Cartan geometry $(\tilde{\mathcal{P}}, \omega_{nor}) \xrightarrow{\pi} M$ of type $(\tilde{G}, \tilde{P})$.

The construction of $(\tilde{\mathcal{P}}, \omega_{nor})$ allows us to define the *standard tractor bundle* of conformal geometry on (M, c) by

$$\mathcal{T}(M) \ := \ \tilde{\mathcal{P}} \times_{\tilde{P}} \mathbb{R}^{r+1,s+1} \ ,$$

which is a vector bundle of rank $n+2$. The standard tractor bundle is naturally endowed with the invariant scalar product $\langle \cdot, \cdot \rangle_{\mathcal{T}}$, which is induced by the $\tilde{G}$-invariant product on $\mathbb{R}^{r+1,s+1}$. The dual standard tractor bundle is denoted by $\mathcal{T}^*(M)$. In general, a conformal tractor bundle $\mathcal{T}_{\mathbb{V}}$ is defined as vector bundle over (M, c) associated to some (effective) $\tilde{G}$-module $\mathbb{V}$. Any tractor bundle $\mathcal{T}_{\mathbb{V}}$ comes with an invariant filtration. In case of the standard tractor bundle $\mathcal{T}$ this filtration is given by $\mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1$, where $\mathcal{T}^1$ corresponds to the $\tilde{P}$-stabilised null line in $\mathbb{R}^{r+1,s+1}$ and $\mathcal{T}^0$ is the $\langle \cdot, \cdot \rangle_{\mathcal{T}}$-orthogonal complement of $\mathcal{T}^1$. In fact, the real line $\mathcal{T}^1$ can be canonically identified with the density bundle $\mathcal{E}[-1]$.

We remark that the standard tractor bundle $\mathcal{T}(M)$ on a space (M, c) with conformal structure $c = [g]$ can be constructed directly as follows. Let us consider the 2-jet prolongation $J^2(\mathcal{E}[1])$ of the density bundle $\mathcal{E}[1]$ on (M, c). By definition of the prolongation, we have the exact sequence

$$0 \to S^2(T^*M)[1] \to J^2(\mathcal{E}[1]) \to J^1(\mathcal{E}[1]) \to 0 \ .$$

The conformal structure c on M gives rise to a splitting of $S^2(T^*M)[1]$ into $S^2_0(T^*M)[1] \oplus \mathcal{E}[-1]$, where $S^2_0(T^*M)[1]$ denotes the trace-free part, and we can see that $S^2_0(T^*M)[1]$ sits as a smooth subbundle in $J^2(\mathcal{E}[1])$. The corresponding quotient bundle is isomorphic to the standard tractor bundle $\mathcal{T}(M)$ (cf. [**36**]).

All tractor bundles $\mathcal{T}_{\mathbb{V}}$ on (M, c) are equipped with a *canonical covariant derivative* ∇^{nor} induced by ω_{nor} (via $\tilde{G}$-extension). In particular, the canonical covariant derivative on $\mathcal{T}(M)$ is called the standard tractor connection, which we denote by $\nabla^{\mathcal{T}}$. The standard tractor connection $\nabla^{\mathcal{T}}$ has a holonomy group $Hol(\mathcal{T})$ with algebra $\mathfrak{hol}(\mathcal{T})$, which is defined as usual via parallel translations of frames in $\mathcal{T}$ along loops in M (cf. Section 0.3). We call $Hol(\mathcal{T})$ the *conformal tractor holonomy group* of (M, c). In general, the tractor holonomy $Hol(\mathcal{T})$ differs from $Hol(\omega_{nor})$ (by a $\mathbb{Z}_2$-covering). However, the conformal holonomy algebras $\mathfrak{hol}(\mathcal{T})$ and $\mathfrak{hol}(\omega_{nor})$ coincide canonically.

Next we consider gradings of conformal tractor bundles with respect to Weyl structures. So let $\sigma : \mathcal{G}_0 \to \mathcal{P}$ be a Weyl structure on M (which we can trivially lift to the $\mathbb{Z}_2$-cover $\tilde{\mathcal{P}}(M)$). Any conformal tractor bundle $\mathcal{T}_{\mathbb{V}}$ on (M, c) is identified via σ with its associated graded $\mathrm{gr}(\mathcal{T}_{\mathbb{V}}) = \oplus_i \mathrm{gr}_i(\mathcal{T}_{\mathbb{V}})$. Basic representation theory of the reductive group $\tilde{G}_0 = \mathrm{CO}(r, s)$ shows that any summand $\mathrm{gr}_i(\mathcal{T}_{\mathbb{V}})$ can be realised as a subbundle of the tensor algebra $\oplus_{k,l} T^{(k,l)}M$ equipped with a certain conformal weight. We illustrate this for the *p-form tractor bundles* $\Lambda^p \mathcal{T}^*(M)$ on (M, c), which are defined as

$\tilde{P}(M) \times_{\tilde{P}} \Lambda^p \mathbb{R}^{r+1,s+1*}$. The p-form tractors are filtered by $\Lambda^p \mathcal{T}^* \supset (\Lambda^p \mathcal{T}^*)^0 \supset (\Lambda^p \mathcal{T}^*)^1$, where the 1-part is naturally identified with the $(p-1)$-forms $\Lambda^{p-1} T^*M[p-2]$ of conformal weight $p-2$. The associated graded is given by

$$\Lambda^{p-1} T^*M[p] \ \oplus \ \Big(\ \Lambda^p T^*M[p] \ \oplus \ \Lambda^{p-2} T^*M[p-2] \ \Big) \ \oplus \ \Lambda^{p-1} T^*M[p-2] \ ,$$

which is naturally identified with $\Lambda^p \mathcal{T}^*(M)$ only via σ. In particular, for $p=2$ we have $\Lambda^2 \mathcal{T}^* M \cong \mathcal{A}$, which is graded by $TM \oplus \mathfrak{co}(TM) \oplus T^*M$, where $\mathfrak{co}(TM)$ denotes the skew-symmetric endomorphisms on TM (with respect to any metric in c) plus multiples of the identity map $id|_{TM}$. The standard tractor bundle $\mathcal{T}$ is graded via σ by

$$\mathcal{E}[1] \oplus TM[-1] \oplus \mathcal{E}[-1] \ .$$

Now we choose a metric $g \in c$, which gives rise to a $\mathrm{O}(r,s)$-structure $\mathcal{G}_s(M) = O(M,g)$ on M. This reduction allows a natural identification of all density bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$, with the trivial real line bundle $\mathbb{R}$ on M. In particular, any tensor bundle $\mathcal{B}[w]$ on (M,c) looses its conformal weight. For example, $\mathcal{T}$ admits via g a natural identification with the direct sum $\mathbb{R} \oplus TM \oplus \mathbb{R}$ and, more generally, the p-form tractors split into $\Lambda^{p-1} T^*M \oplus \Lambda^p T^*M \oplus \ \Lambda^{p-2} T^*M \oplus \Lambda^{p-1} T^*M$. In particular, we can write any adjoint tractor $A \in \mathcal{A}$ with respect to a metric g as a triple $(\xi, \phi, \rho) \in TM \otimes \mathfrak{co}(TM) \otimes T^*M$, where $\phi = \phi_c + \phi_s$ splits further into a trace and a skew-symmetric part. A standard tractor $t \in \mathcal{T}$ is given by a triple (f, ψ, h), where f, h are real functions and ψ is a vector field on (M,c). The algebraic bracket $\{\cdot,\cdot\}_{\mathcal{A}}$ on $\mathcal{A}$ is given for (ξ_1, ϕ_1, ρ_1) and (ξ_2, ϕ_2, ρ_2) with respect to g by

$$\begin{gathered} \{ \ (\xi_1, \phi_1, \rho_1) \ , \ (\xi_2, \phi_2, \rho_2) \ \}_{\mathcal{A}} \ = \\ (\ \phi_1(\xi_2) - \phi_2(\xi_1) \ , \ \phi_1 \circ \phi_2 - \phi_2 \circ \phi_1 + \{\xi_1, \rho_2\} + \{\rho_1, \xi_2\} \ , \ \rho_1 \circ \phi_2 - \rho_2 \circ \phi_1 \ ) \ , \end{gathered}$$

where $\{\xi_1, \rho_2\} = \xi_1 \otimes \rho_2 - \rho_2^\sharp \otimes \xi_1^\flat + \rho_2(\xi_1) id|_{TM}$. The action $\bullet$ of $\mathcal{A}(M)$ on $\mathcal{T}(M)$ is given by

$$(\xi, \phi, \rho) \bullet (f, \psi, h) = (\ \phi_c \cdot f - g(\xi, \psi) \ , \ -f \cdot \rho^\sharp + \phi_s(\psi) + h \cdot \xi \ , \ \rho(\psi) - \phi_c \cdot h \ ) \ .$$

Alternatively, we can express tractors $A = (\xi, \phi, \rho)$ and $t = (f, \psi, h)$ in matrix form. The action $\bullet$ is then given by the matrix product

$$A \bullet t \ = \ \begin{pmatrix} -\phi_c & \rho & 0 \\ \xi & \phi_s & -\rho^\sharp \\ 0 & -g(\xi, \cdot) & \phi_c \end{pmatrix} \cdot \begin{pmatrix} h \\ \psi \\ f \end{pmatrix} \ .$$

The canonical covariant derivative ∇^{nor} acts on sections $\gamma \in \Gamma(\mathcal{T}_{\mathbb{V}})$ of any tractor bundle $\mathcal{T}_{\mathbb{V}}$ with respect to a metric $g \in c$ and a vector $X \in TM$ by

$$\nabla^{nor}_X \gamma \ = \ \nabla^g_X \gamma \ + \ (\ X \ , \ 0 \ , \ \mathsf{P}^g(X) \ ) \bullet \gamma \ .$$

In particular, on sections of $\mathcal{A}$ we have

$$\nabla^{\mathcal{A}}_X (\xi, \phi, \rho) \ = \ (\ \nabla^g_X \xi \ , \ \nabla^g_X \phi \ , \ \nabla^g_X \rho \ ) + \{ \ (X, 0, \mathsf{P}^g(X)) \ , \ (\xi, \phi, \rho) \ \} \ .$$

On standard tractors $t = (f, \psi, h) \in \Gamma(\mathcal{T})$ the action of the tractor connection $\nabla^{\mathcal{T}}$ is expressed in matrix notation by

$$\nabla^{\mathcal{T}}_X t \ = \ \begin{pmatrix} Xh \\ \nabla^g_X \psi \\ Xf \end{pmatrix} + \begin{pmatrix} 0 & \mathsf{P}^g(X, \cdot) & 0 \\ X & 0 & -\mathsf{P}^g(X, \cdot)^\sharp \\ 0 & -g(X, \cdot) & 0 \end{pmatrix} \cdot \begin{pmatrix} h \\ \psi \\ f \end{pmatrix} \ . \qquad (6)$$

The action of ∇^{nor} on p-form tractors in terms of a compatible metric will be explained in Chapter 1. The curvature of $\nabla^{\mathcal{T}}$ is given as adjoint tractor in matrix form by

$$\begin{pmatrix} 0 & \Omega_1 & 0 \\ 0 & \Omega_0 & -\Omega_1^\sharp \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C^g & 0 \\ 0 & W^g & -C^{g\sharp} \\ 0 & 0 & 0 \end{pmatrix} , \qquad (7)$$

where W^g is the Weyl tensor and C^g is the Cotton tensor of g (cf. Section 0.4), and acts on any tractor bundle $\mathcal{T}_{\mathbb{V}}$ via the action $\bullet$.

The tractor connection $\nabla^{nor} : \Gamma(\mathcal{T}_{\mathbb{V}}) \to \Gamma(T^*M \otimes \mathcal{T}_{\mathbb{V}})$ is a linear conformally invariant first order differential operator acting on sections of any tractor bundle $\mathcal{T}_{\mathbb{V}}$ associated to a $\tilde{G}$-module $\mathbb{V}$. More generally, the extended principal connection $\tilde{\omega}_{nor}$ on $\tilde{\mathcal{G}}(M)$ gives rise to the covariant exterior derivatives $d^{\tilde{\omega}} : \Omega^p(M, \mathcal{T}_{\mathbb{V}}) \to \Omega^{p+1}(M, \mathcal{T}_{\mathbb{V}})$ acting on p-forms with values in $\mathcal{T}_{\mathbb{V}}$. However, we are now interested in the construction of conformally covariant linear differential operators

$$\mathcal{D} : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{U}) ,$$

where $\mathcal{V}$ and $\mathcal{U}$ are associated vector bundles to the $\mathrm{CO}(r, s)$-structure $\mathcal{G}_0(M)$, i.e., $\mathcal{V}$ and $\mathcal{U}$ are *tensor bundles*, but not tractor bundles on (M, c). The operators $\mathcal{D}$ that we will construct in the following occur naturally as part of the so-called *BGG sequences*. Their construction is based on the exterior covariant derivatives $d^{\tilde{\omega}}$ and a certain splitting operator $\mathbf{S}$ as invented in [**45**] (cf. also [**32, 72**]).

To start with, let us consider in general a parabolic geometry $(\mathcal{P}, \omega) \xrightarrow{\pi} M$ of arbitrary type (G, P). Let $\mathcal{T}_{\mathbb{V}}$ be a tractor bundle and let $d^{\tilde{\omega}} : \Omega^p(M, \mathcal{T}_{\mathbb{V}}) \to \Omega^{p+1}(M, \mathcal{T}_{\mathbb{V}})$ be the corresponding exterior covariant derivative. The codifferential $\partial^* : \Lambda^p\mathfrak{p}_+ \otimes \mathbb{V} \to \Lambda^{p-1}\mathfrak{p}_+ \otimes \mathbb{V}$ is P-equivariant and thus induces bundle maps

$$\partial^* : \Lambda^p T^*M \otimes \mathcal{T}_{\mathbb{V}} \to \Lambda^{p-1}T^*M \otimes \mathcal{T}_{\mathbb{V}}$$

for any $1 \leq p \leq n$. The kernels and images of ∂^* are invariant subbundles of the $\Lambda^p T^*M \otimes \mathcal{T}_{\mathbb{V}}$ and the corresponding quotients $Ker(\partial^*)/Im(\partial^*)$ are P-associated vector bundles to $\mathcal{P}(M)$ for any $0 \leq p \leq n$, which we denote by

$$H^p_{\mathbb{V}}M = \mathcal{P}(M) \times_P \mathcal{H}^k_{\mathbb{V}} ,$$

where $\mathcal{H}^p_{\mathbb{V}} = H^p(\mathfrak{g}_-, \mathbb{V})$ is the pth Lie algebra cohomology of $\mathfrak{g}_-$ with values in the $\mathfrak{g}$-module $\mathbb{V}$. We denote the natural projections by $\Pi_H : Ker(\partial^*) \to H^p_{\mathbb{V}}M$. Since the action of P_+ maps $Ker(\partial^*)$ to $Im(\partial^*)$, it follows that P_+ acts trivially on the cohomology groups $\mathcal{H}^p_{\mathbb{V}}$. Thus any bundle $H^p_{\mathbb{V}}M$ can be naturally identified with $\mathcal{G}_0(M) \times_{G_0} \mathcal{H}^p_{\mathbb{V}}$, which allows a direct interpretation in terms of the underlying graded G_0-structure on TM. Actually, with respect to a G_0-reduction of $\mathcal{P}(M)$ the p-form bundles $\Lambda^p T^*M \otimes \mathcal{T}_{\mathbb{V}}$ split into the direct sum $Im(\partial) \oplus Ker(\Box) \oplus Im(\partial^*)$, where $\Box = \partial \circ \partial^* + \partial^* \circ \partial$ is the Laplacian of the corresponding harmonic Hodge theory. Any bundle $H^p_{\mathbb{V}}M$ can thus be identified with $Ker(\Box)$.

There exists a certain sequence of differential operators

$$\mathcal{D}^{\mathbb{V}} : \Gamma(M; H^p_{\mathbb{V}}M) \to \Gamma(M; H^{p+1}_{\mathbb{V}}M), \qquad p \in \{0, \dots, n-1\} .$$

To construct the $\mathcal{D}^{\mathbb{V}}$, we use the so-called splitting operators

$$\mathbf{S} \ : \ \Gamma(M; H^p_{\mathbb{V}}M) \to \Omega^p(M; \mathcal{T}_{\mathbb{V}}) ,$$

whose action on a section s is uniquely determined by the properties

(1) $\partial^* \circ \mathbf{S}(s) = 0$ and $\Pi_H \circ \mathbf{S}(s) = s$,
(2) $\partial^* d^{\tilde\omega} \mathbf{S}(s) = 0$.

We set $\mathcal{D}^{\mathbb{V}} := \Pi_H \circ d^{\tilde\omega} \circ \mathbf{S}$, which gives rise to a linear differential operator for any $0 \le p \le n-1$ of a certain order $k \ge 0$. The operators $\mathcal{D}^{\mathbb{V}}$ generate a so-called *BGG sequence*

$$0 \longrightarrow \Gamma(M; H^0_{\mathbb{V}}M) \xrightarrow{\mathcal{D}^{\mathbb{V}}} \Gamma(M; H^1_{\mathbb{V}}M) \xrightarrow{\mathcal{D}^{\mathbb{V}}} \dots \xrightarrow{\mathcal{D}^{\mathbb{V}}} \Gamma(M; H^n_{\mathbb{V}}M) \longrightarrow 0$$

for every G-representation $\mathbb{V}$. This sequence is a complex if $(\mathcal{P}, \omega)$ is a flat parabolic geometry. In general, the BGG sequences are not exact.

In particular, let $(\tilde{\mathcal{P}}, \omega_{nor})$ be a normal conformal Cartan geometry of type $(\tilde G, \tilde P)$ on a conformal space (M, c). Since $\mathcal{G}_0(M) = \tilde{\mathcal{P}}/\tilde P_+$ is already a G_0-structure on M, the cohomology bundles $H^p_{\mathbb{V}}M$, $p \in \mathbb{N}$, for any $\tilde G$-module $\mathbb{V}$ can be interpreted as certain subbundles of the tensor algebra. Let us concentrate on the $\tilde G$-modules $\mathbb{V} = \Lambda^{k+1}\mathbb{R}^{r+1,s+1*}$, which give rise to the $(k+1)$-form tractor bundles $\Lambda^{k+1}\mathcal{T}^*(M)$. In this case the image of ∂^* in $\Lambda^{k+1}\mathcal{T}^*$ is $(\Lambda^{k+1}\mathcal{T}^*)^0$ and $Ker(\partial^*) = \Lambda^{k+1}\mathcal{T}^*(M)$. Thus we obtain $H^0_{\mathbb{V}}M = \Lambda^k T^*M[k+1]$. The covariant exterior derivative $d^{\tilde\omega}$ is given on 0-forms with values in $\Lambda^{k+1}\mathcal{T}^*(M)$ by the canonical tractor connection, i.e., $d^{\tilde\omega} = \nabla^{nor}$. The splitting operator is a map $\mathbf{S} : \Omega^k T^*M[k+1] \to \Omega^{k+1}\mathcal{T}^*(M)$, whose defining properties are $\Pi_H \circ \mathbf{S}(s) = s$ and $\partial^* \nabla^{nor} \mathbf{S}(s) = 0$ for any k-form s. Altogether, this gives rise to the first differential operator

$$\mathcal{D}_0 := \Pi_H \circ \nabla^{nor} \circ \mathbf{S} \;:\; \Omega^k(M) \to \Gamma(H^1_{\mathbb{V}}M)$$

in a BGG sequence. Note that the differential operator $\mathcal{D}_0$ is natural and conformally covariant, since we used the canonical connection ω_{nor} for its definition.

The bundle $H^1_{\mathbb{V}}M$ of first cohomologies for the $(k+1)$-form tractors $\Lambda^{k+1}\mathcal{T}^*(M)$ is isomorphic to $C_k(M)$ (cf. Section 0.6). In fact, computing the splitting operator $\mathbf{S}$ with respect to some $g \in c$ and applying ∇^{nor} to $\mathbf{S}(\alpha)$ for $\alpha \in \Omega^k(M)$ shows that

$$\Pi_H(\nabla^{nor}_X \mathbf{S}(\alpha)) = \nabla^g_X \alpha - \frac{1}{k+1}\iota_X d\alpha + \frac{1}{n-k+1} X^\flat \wedge d^*\alpha$$

for any $X \in TM$. We conclude that $\alpha \in \Omega^k(M)$ is a conformal Killing k-form if and only if it is an element in the kernel of $\mathcal{D}_0 : \Omega^k(M) \to \Gamma(C_k(M))$. For $k = 1$ we have $H^0_{\mathbb{V}}M = T^*M[2] = TM$ and thus the splitting operator $\mathbf{S}$ maps vector fields to adjoint tractors on M. The corresponding conformally covariant differential operator $\mathcal{D}_0$ is given by

$$\begin{array}{rcccl} \mathcal{D}_0 & : & \mathfrak{X}(M) & \to & S^2_0(T^*M)[2] \;, \\ & & X & \mapsto & \mathcal{L}_X \mathbf{g} \;. \end{array}$$

This is the so-called conformal Killing operator, where $\mathbf{g}$ denotes the canonical section in $S^2_0(T^*M)[2] = C_1(M)$, which determines the conformal structure $c = [g]$ on M.

Finally, we note that the space $\Gamma(\mathcal{A})$ of adjoint tractors on a conformal space (M, c) is naturally identified via the normal Cartan connection ω_{nor} with the space $\mathfrak{X}(\mathcal{P}(M))^{\tilde P}$ of $\tilde P$-invariant vector fields on $\tilde{\mathcal{P}}(M)$. In fact, an adjoint tractor $A \in \Gamma(\mathcal{A})$ is equivalently given by a $\tilde P$-equivariant function $A : \tilde{\mathcal{P}}(M) \to \mathfrak{g}$, and $Q_A(u) := \omega_{nor}^{-1} \circ A(u)$, $u \in \tilde{\mathcal{P}}(M)$, is the pointwise definition for the corresponding $\tilde P$-invariant smooth vector field on $\tilde{\mathcal{P}}(M)$. One can show that $Q_A \in \mathfrak{inf}(\omega_{nor})$ iff

$$\nabla^{\mathcal{A}}_X A = -\Omega(\Pi_H(A), X) \qquad \text{for all } X \in TM \;. \tag{8}$$

In this situation the projection $V_A := \pi_*(Q_A) = \Pi_H(A) \in \mathfrak{X}(M)$ is a conformal Killing vector field on (M,c). In return, the adjoint tractor A_V, which corresponds to the 2-jet $j^2(V)$ of a conformal Killing vector V on (M,c) satisfies (8), and we have $A_V = \mathbf{S}(V)$. We conclude that for any $V \in \mathfrak{inf}(M,c)$ the relation $\nabla^{\mathcal{A}}\mathbf{S}(V) = -\Omega(V,\cdot)$ holds. Hence,

$$\tilde{\nabla}^{\mathcal{A}}_X := \nabla^{\mathcal{A}}_X - \Omega(X, \Pi_H(\cdot)) \ , \qquad X \in TM \ ,$$

defines a covariant derivative on $\mathcal{A}$, whose parallel sections correspond uniquely to conformal Killing vectors. Moreover, one can show that if $\nabla^{\mathcal{A}}A = 0$, i.e., A is a $\nabla^{\mathcal{A}}$-parallel adjoint tractor on (M,c), then $\Omega(\Pi_H(A),\cdot) = 0$ (in the setting of conformal geometry), and therefore, $\Pi_H(A)$ has to be a conformal Killing vector field (cf. [**35, 111**]). We will see in Chapter 1 that the analogous statement is true in general for ∇^{nor}-parallel k-form tractors on (M,c).

8. Spinors and Twistors

The theory of orthogonal representations shows that any finite-dimensional (irreducible) $\mathrm{SO}(r,s)$-module occurs as submodule of the tensor algebra. However, the orthogonal Lie algebra $\mathfrak{so}(r,s)$ admits spin representations, which are not tensorial (in the sense that they can be realised in the tensor algebra of $\mathbb{R}^n$), i.e., a spin representation is not infinitesimally induced by a $\mathrm{SO}(r,s)$-representation. In fact, it is induced by a representation of the spin group $\mathrm{Spin}(r,s)$, which is by definition a 2-fold covering group of $\mathrm{SO}(r,s)$ (and in most cases the universal covering). In the following, we will realise the *complex spin representations* of $\mathrm{Spin}(r,s)$ by use of *Clifford algebras*. Under certain topological assumptions, spin representations give rise to spinor bundles on semi-Riemannian spaces (M^n,g), and via the spinor derivative, geometric differential operators for spinors, like the Dirac operator and the Penrose twistor operator, can be defined. Moreover, we will describe how spin geometry carries over to conformal geometry. This leads us to the notion of *twistor calculus*, which is the spin version of tractor calculus.

The *Clifford algebra* $Cl_{r,s}$ of the Euclidean space $\mathbb{R}^{r,s} = (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{r,s})$ with scalar product of signature (r,s) is generated multiplicatively by the vectors of the orthonormal standard basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^{r,s}$ subject to the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\langle e_i, e_j\rangle_{r,s} \ .$$

The Clifford algebra $Cl_{r,s}$ is as vector space isomorphic to the exterior algebra $\Lambda^*\mathbb{R}^n$. We denote the complexification of $Cl_{r,s}$ by $Cl^{\mathbb{C}}_{r,s}$. In case $n = 2m$ is even, $Cl^{\mathbb{C}}_{r,s}$ is isomorphic to the algebra $M(2^m,\mathbb{C})$ of complex $(2^m \times 2^m)$-matrices. We set $\Delta_{r,s} = \mathbb{C}^{2^m}$ and the real Clifford algebra $Cl_{r,s}$ acts on $\Delta_{r,s}$ via the choice of an identification $\Psi : Cl^{\mathbb{C}}_{r,s} \cong M(2^m,\mathbb{C})$ and matrix multiplication. In case $n = 2m+1$ is odd, $Cl^{\mathbb{C}}_{r,s}$ is isomorphic to $M(2^m,\mathbb{C}) \oplus M(2^m,\mathbb{C})$. Again, we set $\Delta_{r,s} = \mathbb{C}^{2^m}$ and $Cl_{r,s}$ acts on $\Delta_{r,s}$ via the homomorphism $\Psi_1 : Cl^{\mathbb{C}}_{r,s} \to M(2^m,\mathbb{C})$, which is the projection onto the first factor of an identification $\Psi : Cl^{\mathbb{C}}_{r,s} \cong M(2^m,\mathbb{C}) \oplus M(2^m,\mathbb{C})$. The *spin group* $\mathrm{Spin}(r,s)$ is realised in $Cl_{r,s}$ by

$$\mathrm{Spin}(r,s) \ = \ \{x_1 \cdot \ldots \cdot x_{2l} \in Cl_{r,s} \mid x_j \in \mathbb{R}^{r,s}, \ \langle x_j, x_j\rangle_{r,s} = \pm 1, \ l \in \mathbb{N} \ \}$$

and the Lie algebra $\mathfrak{spin}(r,s)$ of the spin group is the subspace

$$Span\{e_k \cdot e_l \in Cl_{r,s} | k < l\} \quad \subset \quad Cl_{r,s} \ .$$

The map

$$\begin{array}{cccc} \lambda : & \mathrm{Spin}(r,s) & \longrightarrow & \mathrm{SO}(r,s)\ , \\ & u & \longmapsto & (x \in \mathbb{R}^{r,s} \mapsto u \cdot x \cdot u^{-1} \in \mathbb{R}^{r,s}) \end{array}$$

is a 2-fold covering and a Lie group homomorphism. If $n \geq 3$, $r = 0$ (Riemannian case) or $n \geq 4$, $r = 1$ (Lorentzian case) the fundamental group of $SO(r,s)$ is $\mathbb{Z}_2$ and $\lambda : \mathrm{Spin}(r,s) \to SO(r,s)$ is a universal covering. The differential $\lambda_* : \mathfrak{spin}(r,s) \to \mathfrak{so}(r,s)$ is given by $\lambda_*(e_k \cdot e_l) = 2E_{kl}$, where $E_{kl} := \varepsilon_k e_l \cdot e_k^\top - \varepsilon_l e_k \cdot e_l^\top$.

The *spinor representation* $\rho_{r,s}$ of $\mathrm{Spin}(r,s)$ is given by restriction of the action of the Clifford algebra $Cl_{r,s}$ on $\mathbb{\Delta}_{r,s}$. We call $\mathbb{\Delta}_{r,s}$ the complex spinor module. In case $n = r+s$ is odd the spinor representation is irreducible. For $n = 2m$ the spinor module $\mathbb{\Delta}_{r,s}$ splits into two (in general, non-equivalent) irreducible $\mathrm{Spin}(r,s)$-modules $\mathbb{\Delta}^+_{r,s}$ and $\mathbb{\Delta}^-_{r,s}$. This decomposition is given by the eigenspaces of the action of the volume form of $\mathbb{R}^{r,s}$ (with a choice of orientation). In the following we refer to elements of $\mathbb{\Delta}^+_{r,s}$ and $\mathbb{\Delta}^-_{r,s}$ as positive, resp., negative *Weyl spinors*. (Sometimes we just call them *half spinors*.) Note that the spin representation on $\mathbb{R}^{r,s} \otimes \mathbb{\Delta}_{r,s}$ decomposes into $\mathbb{\Delta}_{r,s} \oplus \mathbb{W}_{ker}$, where $\mathbb{W}_{ker}$ denotes the kernel of the Clifford multiplication.

Next let $\mathrm{Spin}_o(r,s)$ denote the identity component of the spin group. There exists a $\mathrm{Spin}_o(r,s)$-invariant, non-degenerate Hermitian scalar product $\langle\cdot,\cdot\rangle_{\mathbb{\Delta}}$ on the spinor modules $\mathbb{\Delta}_{r,s}$. In the Riemannian case this product $\langle\cdot,\cdot\rangle_{\mathbb{\Delta}}$ is definite. Otherwise, in the pseudo-Riemannian case it is indefinite. Depending on the signature (r,s) the Clifford multiplication of $\mathbb{R}^{r,s}$ on $\mathbb{\Delta}_{r,s}$ is self-adjoint or skew-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathbb{\Delta}}$. Moreover, depending on dimension and signature there exist real or quaternionic structures on the spinor modules $\mathbb{\Delta}_{r,s}$. We note that for $n = 2m+1$ odd the tensor product $\mathbb{\Delta}_{r,s} \otimes \mathbb{\Delta}_{r,s}$ is isomorphic as $\mathrm{Spin}(r,s)$-module to $\Lambda^*_{ev}\mathbb{R}^{n*} \otimes \mathbb{C}$, the complexification of the exterior algebra of $\mathbb{R}^{n*}$ of even degree. For n even the module $\mathbb{\Delta}_{r,s} \otimes \mathbb{\Delta}_{r,s}$ is isomorphic to $\Lambda^*\mathbb{R}^{n*} \otimes \mathbb{C}$.

The latter identifications give rise to the so-called spinor square (or pairing). Let (ϕ, ψ) be a pair of spinors. Then

$$\varsigma(\phi,\psi) \;=\; \oplus_{i=0}^n \varsigma_i(\phi,\psi)$$

denotes a uniquely determined element of the exterior algebra $\Lambda^*\mathbb{R}^{n*} \otimes \mathbb{C}$, where the summands $\varsigma_i(\phi,\psi)$ are certain exterior forms of degree i. Especially, in Lorentzian geometry the square of a spinor ϕ gives rise to a vector $v_\phi := \varsigma_1(\phi,\phi)^\sharp \in \mathbb{R}^{1,n-1}$. The interesting point of this remark in Lorentzian geometry is that $v_\phi = 0$ iff $\phi = 0$ and any $v_\phi \neq 0$ is causal and points to the future (which means here in direction of the standard basis vector e_1) (cf. [**106, 21**]).

Now let (M^n, g) be an oriented semi-Riemannian space and let $SO(M)$ denote the $SO(r,s)$-bundle of oriented orthonormal frames on M. Furthermore, let $(Spin(M), \pi)$ be a reduction of $SO(M)$ over M with respect to the homomorphism $\lambda : \mathrm{Spin}(r,s) \to SO(r,s)$. We call the pair $(Spin(M), \pi)$ a *spin structure* on (M,g) and we refer to points in $Spin(M)$ as spinor frames. In general, a spin structure need not exist on a space (M,g). And if a spin structure exists, it need not be unique. If (M,g) admits a spin structure we call (M,g) a *spin manifold*. We state the following standard criteria for the existence of spin structures: If (M,g) is space- and time-orientable (i.e. $SO(M)$ admits a reduction to the identity component $SO_o(r,s)$) a spin structure exists if and only if the *second Stiefel-Whitney class* $w_2(M)$ of the underlying manifold M vanishes.

In particular, an orientable Riemannian space (M, g) admits a spin structure if and only if $w_2(M) = 0$.

In the following, we assume that (M^n, g) is a time-oriented spin manifold. So let $(Spin(M), \pi)$ be a spin structure structure on (M, g). We define the *complex spinor bundle*

$$\mathcal{S} = Spin(M) \times_{\rho_{r,s}} \Delta_{r,s}$$

as associated vector bundle to $Spin(M)$ via $\rho_{r,s}$, which admits an invariant Hermitian scalar product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$. In case n is even we have the positive and negative *Weyl spinor* bundles $\mathcal{S}^+$ and $\mathcal{S}^-$. The exterior algebra $\Lambda^*(TM)$ acts by Clifford multiplication on $\mathcal{S}$ and we have a spinor square mapping, which sends spinors $\phi, \psi \in \Gamma(\mathcal{S})$ to differential forms $\varsigma_i(\phi, \psi)$ of degree i. Since the Levi-Civita connection ω^g on $SO(M)$ lifts uniquely to a connection on $Spin(M)$, we obtain a canonical covariant derivative on sections of $\mathcal{S}$, the so-called *spinor derivative*

$$\nabla^{\mathcal{S}} : \Gamma(\mathcal{S}) \to \Omega^1(M) \otimes \Gamma(\mathcal{S}) \ .$$

The spinor derivative $\nabla^{\mathcal{S}}$ is metric with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$. The curvature tensor $R^{\mathcal{S}}$ of $\nabla^{\mathcal{S}}$ acts on a spinor ψ with respect to a local orthonormal frame $\{s_1, \dots, s_n\}$ by

$$R^{\mathcal{S}}(X,Y)\psi = \frac{1}{2} \cdot \sum_{1 \le k < l \le n} \varepsilon_k \varepsilon_l \cdot R^g(X, Y, s_k, s_l) s_k \cdot s_l \cdot \psi$$

for all $X, Y \in TM$. Note that a spinor field $\phi \in \Gamma(\mathcal{S})$ on a semi-Riemannian space (M, g) is called parallel if $\nabla^{\mathcal{S}} \phi = 0$. This is the most basic spinor field equation.

The decomposition $T^*M \otimes \mathcal{S} = \mathcal{S} \oplus \mathcal{W}_{ker}$ with orthogonal projections π_1 onto $\mathcal{S}$ and π_2 onto the kernel of the Clifford multiplication gives rise to first order differential operators acting on spinors. Namely, we obtain the *Dirac operator* $D^{\mathcal{S}} := \pi_1 \circ \nabla^{\mathcal{S}}$ and the *Penrose twistor operator* $P^{\mathcal{S}} := \pi_2 \circ \nabla^{\mathcal{S}}$. With respect to a local orthonormal frame $\{s_1, \dots, s_n\}$ the Dirac operator $D^{\mathcal{S}}$ acts on a spinor ϕ by

$$D^{\mathcal{S}} \phi = \sum_{i=1}^{n} \varepsilon_i s_i \cdot \nabla^{\mathcal{S}}_{s_i} \phi \ .$$

Our next aim is to introduce spin geometry in the context of conformal geometry. For this purpose, let us consider the Klein pair (G, P), where $G = \mathrm{SO}(r+1, s+1)$ is the special orthogonal group with Lie algebra $\mathfrak{g} = \mathfrak{so}(r+1, s+1)$ and P is the parabolic in G, which preserves the filtration on $\mathfrak{g}$ coming from the $|1|$-grading $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The pair (G, P) is the Klein model of conformal geometry equipped with an orientation (and a square root for density bundles). Now let $\lambda : \tilde{G} \to G$ be the 2-fold spin covering and let $\tilde{P}$ be the parabolic in $\tilde{G}$, which is the preimage of P under λ. The parabolic $\tilde{P}$ is isomorphic to the semidirect product $\widetilde{\mathrm{CSpin}}(r, s) \ltimes \mathbb{R}^n$, where $\widetilde{\mathrm{CSpin}}(r, s) = \mathrm{Spin}(r, s) \times (\mathbb{R} \setminus \{0\})$. The pair $(\tilde{G}, \tilde{P})$ is the Klein model of conformal spin geometry and the homogeneous space $\tilde{G}/\tilde{P}$ is the Möbius space equipped with a spin structure (and a square root for densities).

The group $\tilde{G}_0 := \mathrm{CSpin}(r, s) = \mathrm{Spin}(r, s) \times \mathbb{R}_+$ acts by $\rho^w \circ (\lambda \otimes id)$ on $\mathbb{R}[w]$ with conformal weight w. As standard representation of $\tilde{G}_0$ on $\Delta_{r,s}$ we choose $\rho_{r,s} \otimes id$. The spin representation of $\tilde{G}$ is denoted by $\mathbb{W} := (\Delta_{r+1,s+1}, \rho_{r+1,s+1})$ and we call $\mathbb{W}$ the (standard) *twistor module*. The twistor module $\mathbb{W}$ is irreducible for n odd and for $n = 2m$ even it decomposes into an irreducible positive part $\mathbb{W}^+$ and a negative part $\mathbb{W}^-$. Further, we denote by $\mathbb{W}^{1/2}$ the set of twistors in $\mathbb{W}$, which are annihilated by Clifford multiplication with any standard tractor in the $\mathfrak{p}$-invariant null line of $\mathbb{R}^{r+1,s+1}$. The space $\mathbb{W}^{1/2}$ is isomorphic as $\mathfrak{g}_0$-module to $\Delta_{r,s}[-3/2]$, the spinor module of conformal weight $-3/2$. The sequence $\mathbb{W} \supset \mathbb{W}^{1/2}$ is a $\mathfrak{p}$-invariant filtration of $\mathbb{W}$ and the quotient $\mathbb{W}/\mathbb{W}^{1/2}$ is as $\mathfrak{g}_0$-module isomorphic to $\Delta_{r,s}[-1/2]$. The associated $\mathfrak{g}_0$-invariant grading of $\mathbb{W}$ is then given by $\Delta_{r,s}[-1/2] \oplus \Delta_{r,s}[-3/2]$. By restricting the $\mathfrak{g}_0$-action on the associated graded to the semisimple part $\mathfrak{spin}(r,s)$ we simply obtain the splitting $\Delta_{r,s} \oplus \Delta_{r,s}$ of $\mathbb{W}$.

Now let us consider a normal conformal Cartan geometry $(\mathcal{P}(M), \omega_{nor})$ of type $(\mathrm{SO}(r+1, s+1), P)$ on an oriented and time-oriented space M with conformal structure c. A λ-reduction $(\tilde{\mathcal{P}}(M), \pi)$ of $\mathcal{P}(M)$ over M is by definition a conformal spin structure. We note that (M, c) admits a *conformal spin structure* if and only if some metric $g \in c$ (hence any compatible metric) allows for a semi-Riemannian spin structure. In particular, a conformal Riemannian space (M, c) admits a spin structure iff $w_2(M) = 0$. The *standard spin tractor bundle* for $(\tilde{\mathcal{P}}(M), \omega_{nor})$ is defined as

$$\mathcal{W} \ := \ \tilde{\mathcal{P}} \times_{\rho_{r+1,s+1}} \mathbb{W} \ ,$$

which is filtered by $\mathcal{W} \supset \mathcal{W}^{1/2}$. (We omit the name *twistor bundle*, since this notion is reserved for something else (cf. [**10**]). Nevertheless, we call the vectors in $\mathcal{W}$ *(standard) twistors*.) The spin tractor bundle $\mathcal{W}$ is also endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ and we have the *twistor pairings* $\varsigma_p : \mathcal{W} \otimes \mathcal{W} \to \Lambda^p(\mathcal{T}^*M)$, $0 \le p \le n+2$, with values in the p-form tractors. More generally, any $\tilde{P}$-module $\mathbb{U}$ (which does not reduce to a P-module) gives rise to a spin tractor bundle $\mathcal{W}_{\mathbb{U}}$. All spin tractor bundles $\mathcal{W}_{\mathbb{U}}$ are equipped with a canonical covariant derivative ∇^{nor} coming from the normal Cartan connection ω_{nor}. For standard twistors we denote the canonical covariant derivative by $\nabla^{\mathcal{W}}$, which is a metric covariant derivative with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$.

Now let $\Phi \in \Gamma(\mathcal{W})$ be a $\nabla^{\mathcal{W}}$-parallel standard twistor, i.e., $\nabla^{\mathcal{W}}\Phi = 0$. Then the quotient $\Pi_H(\Phi) \in \Gamma(\mathcal{S}[-1/2])$ is a so-called *conformal Killing spinor* on (M, c). (We will also use the expression *twistor spinor*, since this is a standard notion in the literature.) Let us discuss the equation $\nabla^{\mathcal{W}}\Phi = 0$ with respect to the choice of compatible metric g. Then the spin tractor bundle $\mathcal{W}$ splits as $\mathcal{S} \oplus \mathcal{S}$ and, accordingly, the twistor Φ decomposes to a pair (ϕ, ψ) of spinors on (M, g). The equation $\nabla^{\mathcal{W}}\Phi = 0$ translates with respect to g into the system

$$\nabla^{\mathcal{S}}_X \phi + \frac{1}{n} X \cdot \psi = 0 \qquad \text{and} \qquad \nabla^{\mathcal{S}}_X \psi = \frac{n}{2} \mathsf{P}^g(X) \cdot \phi \ , \qquad X \in TM \ ,$$

of partial differential equations for ϕ and ψ (cf. [**106**] for an explicit calculation). Computing the splitting operator $\mathbf{S}$ involved in the construction of the corresponding spinorial BGG sequence in terms of the compatible metric g results in

$$\phi \in \Gamma(\mathcal{S}) \ \mapsto \ \mathbf{S}(\phi) = \big(\ \phi \ , \ D^{\mathcal{S}}\phi \ \big) \ .$$

It follows that the first equation above is equivalent to the so-called *twistor equation*

$$\nabla^{\mathcal{S}}_X \phi + \frac{1}{n} X \cdot D^{\mathcal{S}}\phi = 0 \qquad \text{for all } X \in TM \ .$$

And a simple calculation shows now that the second equation of the above system is a consequence of the twistor equation. We conclude that ∇^{nor}-parallel twistors

Φ correspond uniquely via the splitting operator $\mathbf{S}$ and a metric $g \in c$ to solutions $\phi \in \Gamma(\mathcal{S})$ of the twistor equation. It is well known that the twistor equation describes the kernel of the Penrose operator $P^{\mathcal{S}}$ (cf. [**137, 61, 138, 19**]). In fact, the kernel $\mathcal{W}_{ker}$ of the Clifford multiplication is isomorphic to the bundle $H^2_{\mathbb{W}}M$ of twistor valued second cohomology classes of $\mathfrak{g}_-$ on M, and the first differential operator $\mathcal{D}_0^{\mathbb{W}} := \Pi_H \circ \nabla^{\mathcal{W}} \circ \mathbf{S} : \Gamma(\mathcal{S}[-1/2]) \to \Gamma(\mathcal{W}_{ker})$ in the spinorial BGG sequence is the Penrose operator. This means that for the twistor representation $\mathcal{W}$ the space of ∇^{nor}-parallel sections is identical to the kernel of the differential operator $\mathcal{D}_0^{\mathbb{W}}$ in the corresponding BGG sequence!

In particular, we understand now that the twistor equation is a conformally covariant first order PDE for spinors. In fact, the Penrose operator $P^{\mathcal{S}}$ rescales with respect to conformally equivalent metrics g and $\tilde{g} = e^{2\sigma} \in c$ by

$$P^{\tilde{g}}\tilde{\phi} = e^{-\sigma/2} \cdot \left(P^g(e^{-\sigma/2}\tilde{\phi}) \right) \ ,$$

where $\tilde{\phi} = e^{\sigma/2} \cdot \phi$ for $\phi \in \Gamma(\mathcal{S}^g)$, i.e., the bidegree of $P^{\mathcal{S}}$ is $(-1/2, 1/2)$. (*Attention*! The notation $\tilde{\phi}$ is a bit misleading here, since $\tilde{\phi}$ and ϕ denote the same spinor fields in $\Gamma(\mathcal{S}[-1/2])$. Nevertheless, we use this notation sometimes when we work with a conformal change of metrics!) Since the twistor equation is an overdetermined PDE, the existence of solutions ϕ is obstructed by integrability conditions, which are expressed in terms of curvature properties. The basic integrability conditions are implied by application of the curvature operator of $\nabla^{\mathcal{W}}$ to the $\nabla^{\mathcal{W}}$-parallel twistor $\Phi = \mathbf{S}(\phi)$. In terms of a metric g this is expressed by

$$W^g(Y,Z) \cdot \phi = 0 \qquad \text{and} \qquad W^g(Y,Z) \cdot D^{\mathcal{S}}\phi = n \cdot C^g(Y,Z) \cdot \phi \ .$$

Further, one can deduce the condition

$$(\nabla^g_X W^g)(Y,Z) \cdot \phi = X \cdot C^g(Y,Z) \cdot \phi + \frac{2}{n}(\iota_X W^g(Y,Z)) \cdot D^{\mathcal{S}}\varphi$$

for all $X, Y, Z \in TM$. A conformal Killing spinor ϕ satisfies also $(D^{\mathcal{S}})^2\phi = \frac{n \cdot scal^g}{4(n-1)}\phi$. We remark that the Dirac operator $D^{\mathcal{S}}$ is conformally covariant as well with bidegree $(\frac{n-1}{2}, \frac{n+1}{2})$.

In the past, integrability conditions for twistor spinors have been intensely studied in geometry and physics and various structure results are known (cf. e.g. [**118, 119, 84, 121, 120, 81, 19, 117, 94, 82, 96, 99, 17, 106, 21**]). We state here an interesting structure result concerning certain twistor spinors in Lorentzian geometry. First, recall that any spinor $\varphi \in \Gamma(\mathcal{S})$ on a Lorentzian spin space (M,g) gives rise to the spinor square $\varsigma_1(\varphi)$. The dual vector field V_φ to $\varsigma_1(\varphi)$ is determined by the relation

$$g(V_\varphi, X) \ = \ -\langle X \cdot \varphi, \varphi \rangle_{\mathcal{S}} \qquad \text{for all } X \in TM \ .$$

We call V_φ the *Dirac current* of the spinor φ. Note that the zero set of a spinor φ and its Dirac current V_φ always coincide and off the zero set V_φ is causal (cf. Chapter 4; [**18, 106**]). Moreover, if φ is a twistor spinor then V_φ is a conformal Killing vector field (cf. Chapter 1; [**18, 19**]).

Theorem 1. *(cf.* [**117, 17, 106, 21, 40**]*) Let (M,g) be a Lorentzian spin manifold admitting a conformal Killing spinor φ such that the spinor square V_φ is a lightlike Killing vector. Then the function $\mathrm{Ric}^g(V_\varphi, V_\varphi)$ is constant and non-negative on M. In particular,*

(1) $\mathrm{Ric}^g(V_\varphi, V_\varphi) > 0$ *if and only if* (M, g) *is locally isometric to a* Fefferman space*. In this case the Dirac current is* twisting *and the dimension of* M *is even.*
(2) $\mathrm{Ric}^g(V_\varphi, V_\varphi) = 0$ *if and only if* (M, g) *is locally conformal equivalent to a* Brinkmann space *with parallel spinors. In this case the Dirac current is* non-twisting.

We add some explanations to Theorem 1. Obviously, the first part of the result is closely linked to the Fefferman construction, which plays an important role for our investigations. We will explain the Fefferman construction in Section 0.11 and further discuss it in Chapter 5 and 6. The second part of Theorem 1 is related to the geometry of spaces admitting parallel spinors. In Riemannian geometry, it is well known that parallel spinors occur only on Ricci-flat spaces with special holonomy (cf. [**151**]; see also [**20, 30**] for the pseudo-Riemannian case). In Lorentzian geometry, there exist basically two types of spaces with parallel spinors. We have the *static monopole* solutions, which are in general defined as the product of a Ricci-flat Riemannian metric with a time axis. On the other hand, we have the *Brinkmann spaces*, which admit by definition a parallel lightlike vector field V. A special class of Brinkmann spaces are the *pp-waves* (M, g), which satisfy the curvature condition $tr_{(3,5),(4,6)} R^g \otimes R^g = 0$. Parallel vectors do not have *twist*. In general, the twist of a vector V is given by $\alpha_V \wedge d\alpha_V$, where $\alpha_V := g(V, \cdot)$ denotes the dual 1-form. If $\alpha_V \wedge d\alpha_V \neq 0$ we say that the vector V is twisting. Also, note that the assumption in Theorem 1 for V_φ to be Killing is not essential (for the conformal geometry), since this can be achieved (locally) by rescaling the metric g in the conformal class. The condition only indicates when $Ric^g(V_\varphi, V_\varphi)$ is constant. The type of conformal geometry is actually determined by the twist of the lightlike Dirac current V_φ.

Finally, note that a twistor spinor $\phi \in \Gamma(\mathbb{S})$ on a semi-Riemannian space (M^n, g) of dimension n, which is also an eigenspinor of the Dirac operator $D^{\mathbb{S}}$, is called a *Killing spinor*. Killing spinors ϕ solve the equation

$$\nabla^{\mathbb{S}}_X \phi = \mu X \cdot \phi$$

for any $X \in TM$ with some fixed Killing number $\mu \in \mathbb{C}$, which implies $D^{\mathbb{S}}\phi = -n\mu\phi$. (In case $\mu = 0$, the spinor ϕ is parallel.) It is a matter of fact that the Killing spinor equation is a metric invariant, but not conformally covariant (due to the different conformal weights of the Penrose and the Dirac operator). In Riemannian signature, the derived integrability conditions for the Killing spinor equation imply that the underlying space has to be Einstein. Also, note that the existence of a Killing spinor does not force the base space to be Einstein in the general pseudo-Riemannian case. The relation $scal^g = 4n(n-1)\mu^2$ always holds, which implies that the Killing number μ is either real or purely imaginary. In the next section, we will meet the Einstein-Sasaki spaces, which are typical examples for spaces admitting Killing spinors (see also Chapter 1). Killing spinors play a very important role in supergravity theories with supersymmetry (cf. e.g. [**49, 50, 57, 58**]).

9. Metric Cone Constructions

The *metric cone* is a standard construction for the investigation of various problems in semi-Riemannian geometry. Since we will apply the metric cone frequently in Chapter 1 and 2, we dedicate a separate introductory section to its construction. In general, the metric cone of a semi-Riemannian space is defined as a certain warped-product. In the special case of the standard sphere S^n the metric cone is isometric to the flat Euclidean space $\mathbb{R}^{n+1}$ of dimension $n+1$ (with deleted origin). In a way this property characterises the round sphere S^n entirely. It was shown by S. Gallot in [**62**] that the cone of a complete Riemannian space is either flat or its holonomy is irreducible. Ch. Bär applied the cone construction to a geometric description of complete Riemannian spin spaces admitting real Killing spinors (cf. [**12**]). Interesting from the conformal point of view is the fact that the metric cone construction over an Einstein space is basically equivalent to the *Fefferman-Graham ambient metric* construction (cf. Chapter 2; [**110, 6, 68**]), which we will introduce in Section 0.11. In Chapter 2 we will extend this picture by a *double cone* construction to a wider class of (conformal) geometries (cf. [**68**]). We also note that the *Poincaré-Einstein model* of conformal geometry is related to the Fefferman-Graham ambient model via the cone construction (cf. Section 0.11).

In general, let (A,h) and (M,g) be two semi-Riemannian spaces and let $f : A \to \mathbb{R}$ be a non-vanishing function. Then a metric is given by $h + f^2 g$ on the product space $A \times M$. We call this a *warped-product* metric with warping function f. The canonical projection from $A \times M$ to M is denoted by π. Formulae for the Levi-Civita connection and curvature of a warped product can be found in [**128**] and many other sources.

In particular, let A be a one dimensional manifold. For simplicity let us set $A = \mathbb{R}$ with line element εdt^2, where $\varepsilon = \pm 1$ defines the signature. Then we have, for any semi-Riemannian space (M^n, g) of dimension n and some warping function $f : \mathbb{R} \to \mathbb{R}_+$, the warped-product metric

$$g_f \ := \ \varepsilon dt^2 + f^2 \cdot g$$

defined on $\mathbb{R} \times M$. Let ∇^{g_f} be its Levi-Civita connection and let us denote for any vector fields X, Y on M the pull-backs to $\mathbb{R} \times M$ (which are tangential to the factor M) by X and Y, again. Then the Levi-Civita connection ∇^{g_f} is determined by the formulae

$$\nabla^{g_f}_X Y \ = \ \nabla^g_X Y - \varepsilon g(X,Y) f' f \cdot \frac{\partial}{\partial t} \ ,$$

$$\nabla^{g_f}_X \frac{\partial}{\partial t} = \frac{f'}{f} X \ , \qquad \nabla^{g_f}_{\frac{\partial}{\partial t}} X = \frac{f'}{f} X \qquad \text{and} \qquad \nabla^{g_f}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0 \ ,$$

where $f' = \frac{\partial f}{\partial t}$. The Riemannian curvature operator R^{g_f} is given by

$$R^{g_f}(X,Y)Z \ = \ R^g(X,Y)Z - \varepsilon (f')^2 \cdot \big(\, g(Y,Z)X - g(X,Z)Y \, \big) \ ,$$

$$R^{g_f}(X,Y)\frac{\partial}{\partial t} \ = \ 0 \ , \qquad R^{g_f}(X, \frac{\partial}{\partial t})\frac{\partial}{\partial t} \ = \ \frac{-f''}{f} X \quad \text{and}$$

$$R^{g_f}(X, \frac{\partial}{\partial t})Y \ = \ \varepsilon f'' f \cdot g(X,Y) \frac{\partial}{\partial t} \ ,$$

where X, Y and Z denote vector fields on M (resp. their pull-backs to $\mathbb{R} \times M$). Expressions for the Ricci-tensor and the scalar curvature of g_f follow immediately.

We further specialise and set $A = \mathbb{R}_+$ and $f(t) = t$. In this case we denote $\overline{M} = \mathbb{R}_+ \times M$ and $\hat{g} := \varepsilon dt^2 + t^2 \cdot g$ is the warped-product metric on $\overline{M}$, which we call the *cone metric* (of 1st kind) over the (arbitrary) semi-Riemannian base space (M, g). The set

$\{1\} \times M \subset \overline{M}$ is called the 1-*level* of $\overline{M}$. The 1-level of $\overline{M}$ is as submanifold isometric to the base space (M, g) and

$$\begin{array}{rcccl} \iota & : & M & \hookrightarrow & \overline{M} \ , \\ & & x & \mapsto & (1, x) \ , \end{array}$$

is an isometric embedding. Moreover, we set $\mathbf{E} := t\frac{\partial}{\partial t}$ and call it the Euler vector of the cone $(\overline{M}, \hat{g})$. On the 1-level, we have $\mathbf{E} = \frac{\partial}{\partial t}$. We denote by $\nabla^{\hat{g}}$ the Levi-Civita connection of $\hat{g}$, for which we have the relations

$$\nabla^{\hat{g}}_X Y \ = \ \nabla^g_X Y - \varepsilon g(X,Y)\mathbf{E} \ ,$$

$$\nabla^{\hat{g}}_{\mathbf{E}} X = \nabla^{\hat{g}}_X \mathbf{E} = X \qquad \text{and} \qquad \nabla^{\hat{g}}_{\mathbf{E}} \mathbf{E} = \mathbf{E} \ .$$

The Ricci-tensor of $\hat{g}$ is given by

$$Ric^{\hat{g}}(X,Y) \ = \ Ric^g(X,Y) \ - \ \varepsilon(n-1)g(X,Y) \qquad \text{and} \qquad Ric^{\hat{g}}(\mathbf{E}, \cdot) = 0 \ .$$

The scalar curvature is $scal^{\hat{g}} = \frac{1}{f^2} \cdot (\ scal^g - \varepsilon \cdot n(n-1)\ )$.

Obviously, these formulae show that $\nabla^{\hat{g}}_A \mathbf{E} = A$ for any $A \in T\overline{M}$, which simply says that $\mathbf{E}$ is a homothetic gradient vector field for the cone $(\overline{M}, \hat{g})$, whose causal type is either timelike or spacelike according to the sign ε. Actually, one can show that cone metrics are characterised by the existence of such a homothetic gradient vector field (cf. Lemma 1). Moreover, we see immediately that the cone $(\overline{M}, \hat{g})$ is Ricci-flat if and only if the base space (M, g) is Einstein with non-vanishing scalar curvature $scal^g = \varepsilon \cdot n(n-1)$. If the base (M, g) is a complete Riemannian manifold then it is also known that $(\overline{M}, \hat{g})$ with $\varepsilon = 1$ has either irreducible holonomy or it is flat (cf. [**62**]).

The above observation about Ricci-flat cones is the motivation for another (slightly different) metric cone construction for a semi-Riemannian base space (M, g), whose scalar curvature $scal^g$ is constant and does not vanish. As before, we set $\overline{M} := \mathbb{R}_+ \times M$ with Euler vector $\mathbf{E} := t\frac{\partial}{\partial t}$ and now we define the cone metric $\bar{g}$ (of 2nd kind) on $\overline{M}$ over (M, g) by

$$\bar{g} \ := \ sgn(\lambda^g) \cdot \left(\ \lambda^g t^2 g \ + \ dt^2\ \right) \ ,$$

where λ^g satisfies $scal^g = n(n-1)\lambda^g$ and $sgn(\lambda^g)$ denotes the sign of λ^g. This definition implies that the metric cone $(\overline{M}, \bar{g})$ is Ricci-flat if and only if the base space (M, g) is Einstein with $scal^g \neq 0$. Note that the causal type of $\mathbf{E}$ depends on the sign of $scal^g$. For example, if $scal^g$ is negative we need to attach a timelike direction to the base space (M, g) in order to obtain a Ricci-flat cone. We will use in Chapter 2 the following fact, which follows e.g. from results in [**63**].

Lemma 1. *Let (N, h) be a Ricci-flat semi-Riemannian space of signature $(r+1, s+1)$ admitting a homothetic gradient vector field V, i.e., $\nabla^h_Z V = d \cdot Z$ for all $Z \in TN$ and some constant $d \neq 0$.*

(1) *If V is everywhere spacelike then (N, h) is an open subset of the cone $(\overline{M}, \overline{g})$ defined over some Einstein space (M, g) of positive scalar curvature $scal^g > 0$ with signature $(r, s+1)$.*
(2) *If V is everywhere timelike then (N, h) is an open subset of the cone $(\overline{M}, \overline{g})$ of some Einstein space (M, g) of negative scalar curvature $scal^g < 0$ with signature $(r+1, s)$.*

If the base (M^n, g) is a spin manifold, the metric cone $(\overline{M}, \bar{g})$ is a spin manifold, too. The choice of a spin structure on the base induces a uniquely determined spin structure on the cone $\overline{M}$. We denote the corresponding spinor bundle of the cone by $\bar{\mathcal{S}}$. For n even the restriction of $\bar{\mathcal{S}}$ to the 1-level $\{1\} \times M$ is naturally isomorphic to the spinor bundle $\mathcal{S}$ on the base manifold M^n via a map

$$\Phi \ : \ \mathcal{S} \ \cong \ \bar{\mathcal{S}}|_{\{1\}\times M}$$

with the property $\Phi(X \cdot \varphi) = X \cdot \Phi(\varphi)$ for all $\varphi \in \mathcal{S}$ and $X \in TM$. Similar, if n is odd, there are isomorphisms $\Phi_\pm : \mathcal{S} \cong \bar{\mathcal{S}}^\pm|_{M\times\{1\}}$ for the restricted half spinor bundles such that

$$X \cdot \Phi_\pm(\varphi) = \Phi_\pm(X \cdot \varphi)$$

for all tangent vectors $X \in TM$. With respect to the metric $\bar{g}$ the projection π gives rise to a pull-back $\pi^* : \Gamma(\bar{\mathcal{S}}|_{M\times\{1\}}) \to \Gamma(\bar{\mathcal{S}})$ of spinor fields on the 1-level to the cone space $\overline{M}$. We denote by $\mathcal{K}_\mu(M)$ the space of Killing spinors on (M^n, g) to the Killing number $\mu \in (i\mathbb{R} \cup \mathbb{R}) \setminus \{0\}$. The space $\mathcal{K}_0(M)$ denotes the space of parallel spinors on a manifold M with spin structure (cf. Section 0.8).

Proposition 1. *(cf. [**12**]) Let (M^n, g) be a semi-Riemannian spin manifold of dimension n with $scal^g = 4n(n-1)\mu^2 \neq 0$ and $(\overline{M}, \bar{g})$ its metric cone. The following correspondence exists.*

(1) *If n is even the map*

$$\begin{array}{ccc} \mathcal{K}_\mu(M) \oplus \mathcal{K}_{-\mu}(M) & \cong & \mathcal{K}_0(\overline{M}, \bar{g}) \ , \\ \varphi & \mapsto & \bar{\varphi} := \pi^* \circ \Phi \circ \varphi \end{array}$$

is a natural linear isomorphism.

(2) *If n is odd then (with the choice of appropriate orientations)*

$$\begin{array}{ccc} \mathcal{K}_{\pm\mu}(M) & \cong & \mathcal{K}_0^\pm(\overline{M}) \ , \\ \varphi & \mapsto & \bar{\varphi} := \pi^* \circ \Phi_\pm \circ \varphi \ , \end{array}$$

where $\mathcal{K}_0^\pm(\overline{M})$ denotes the space of parallel $\pm$-Weyl spinors on the cone.

This result (in the Riemannian case) is due to Ch. Bär (cf. [**12**]). It basically says that Killing spinors on the base space correspond to parallel spinors on the cone. Notice that in the presence of a Killing spinor the scalar curvature $scal^g$ is non-zero and constant, which allows the construction of $(\overline{M}, \bar{g})$. The proof of the correspondence established by Proposition 1 is based on the observation that a Killing spinor φ is parallel with respect to the modified spinor derivative $\nabla^\mu_X := \nabla^{\mathcal{S}}_X - \mu X\cdot$, $X \in TM$, coming from an affine connection 1-form on the frame bundle of M, which (in the case of Riemannian geometry with positive scalar curvature) takes values in the subset

$$\mathbb{R}^n \oplus \mathfrak{spin}(n) \ \cong \ \mathfrak{spin}(n+1)$$

of the Clifford algebra $Cl^{\mathbb{C}}_n$. Proposition 1 can be used to establish a classification for complete Riemannian spaces (M, g) admitting real Killing spinors via the holonomy group $Hol(\bar{g})$ of the cone metric (cf. [**12**]). In fact, in this situation the cone $(\overline{M}, \bar{g})$ is a Ricci-flat Riemannian manifold with parallel spinor. Since $(\overline{M}, \bar{g})$ can not be locally symmetric or decomposable unless it is flat, the Berger list states that the holonomy of $(\overline{M}, \bar{g})$ is special or trivial, i.e., $Hol(\bar{g}) = \mathrm{SU}(\frac{n+1}{2})$, $\mathrm{Sp}(\frac{n+1}{4})$, G_2, $\mathrm{Spin}(7)$ or trivial (cf. [**25, 12**]; see also Theorem 4).

A similar result (as Proposition 1) for a geometric description of special Killing p-forms was established by U. Semmelmann in [**143**].

Proposition 2. *(cf. [**143**]) Let (M^n, g) be a semi-Riemannian manifold.*

(1) *There exists a natural 1-to-1-correspondence of special Killing p-forms α to the Killing constant $c = -\varepsilon \cdot (p+1)$ and parallel $(p+1)$-forms on the cone $(\overline{M}, \hat{g})$ (of 1st kind). The correspondence is explicitly given by*

$$\alpha \in \Omega^p(M) \qquad \mapsto \qquad t^p dt \wedge \alpha + \frac{t^{p+1}}{p+1} d\alpha \quad \in \Omega^{p+1}(\overline{M}) \ . \tag{9}$$

(2) *If the base space (M, g) is Einstein with $scal^g \neq 0$ then the Killing constant of any special Killing p-form satisfies $c = -\frac{(p+1)scal^g}{n(n-1)}$ and there exists a natural 1-to-1-correspondence for special Killing p-forms to the Killing constant c and parallel $(p+1)$-forms on the metric cone $(\overline{M}, \bar{g})$ (of 2nd kind). Again, the explicit correspondence is given by the mapping (9).*

Note that not every space admitting a special Killing p-form needs to be an Einstein space. Nevertheless, the first part of Proposition 2 can be generalised to arbitrary Killing numbers $c \neq 0$ (on non-Einstein spaces) by using the cone metric $\bar{g}$ with constant $\lambda^g := \frac{-c}{p+1}$ (which might be unrelated to $scal^g$ in this case) and (9) as explicit cone correspondence.

We want to give a classical example for a geometric structure, which occurs naturally in relation with a cone construction. A semi-Riemannian *Sasaki space* is traditionally defined as a triple (M^n, g, ξ) with $n = 2m+1$ odd, where ξ is a Killing vector field of unit length (i.e. $g(\xi, \xi) = \varepsilon = \pm 1$) such that the linear map $J := -\nabla^g \xi : TM \to TM$ satisfies

$$J^2(X) = -\varepsilon X + g(X, \xi)\xi \qquad \text{and} \qquad (\nabla^g_X J)(Y) = g(X, Y)\xi - g(Y, \xi)X$$

for all $X, Y \in TM$. In particular, the triple (ξ, J, η) with $\eta := g(\xi, \cdot)$ defines a *metric contact structure* on (M, g). In addition, if the metric g is Einstein, then (M^n, g, ξ) is called an *Einstein-Sasaki space*. In this case $scal^g = n(n-1)$.

There is a well known interpretation of Sasaki spaces in terms of the metric cone $\hat{g} = dt^2 + t^2 g$ with $\varepsilon = 1$. In fact, the base space (M, g) admits a Sasaki structure if and only if the cone $(\overline{M}, \hat{g})$ admits a (pseudo)-Kähler structure. To be concrete, if ω is a (pseudo)-Kähler form on $(\overline{M}, \hat{g})$ then the 1-form $\iota_{\mathbf{E}}\omega$ restricted to the 1-level gives rise to a special Killing 1-form on the base (M, g), whose dual vector field ξ defines a Sasaki structure. On the other hand, given a Sasaki structure ξ with dual 1-form $\eta := g(\xi, \cdot)$ the 2-form

$$t dt \wedge \eta \ + \ \frac{t^2}{2} d\eta$$

is a (pseudo)-Kähler structure on $(\overline{M}, \hat{g})$. Moreover, we can see now that a space (M^n, g) is Einstein-Sasaki of signature (r, s) if and only if the cone $(\overline{M}, \hat{g})$ is a Ricci-flat (pseudo)-Kähler space, i.e., the holonomy $Hol(\hat{g})$ of the cone sits in $\mathrm{SU}(\frac{r}{2}, \frac{s+1}{2})$. We remark that every simply connected semi-Riemannian space (M, g) equipped with Einstein-Sasaki structure admits a spin structure. In particular, the cone $(\overline{M}, \hat{g})$ admits parallel spinors, which implies that the base (M, g) always admits Killing spinors.

Einstein-Sasaki spaces can be constructed in the following way. Let (N, h, J) be a $2m$-dimensional Kähler-Einstein space with signature $(2p, 2q)$ and scalar curvature

$scal^h = 4m(m+1)$ and let $U(N)$ denote the complex orthogonal frame bundle over N with structure group $\mathrm{U}(p,q)$. The canonical S^1-bundle over N is given by $M := U(N) \times_{det_{\mathbb{C}}} S^1$ with projection π and the Levi-Civita connection to h on $U(N)$ induces a connection form $\rho : M \to i\mathbb{R}$. We set

$$g \ := \ \pi^* h - \frac{1}{(m+1)^2} \rho \circ \rho \ ,$$

which is an Einstein metric of signature $(2p, 2q+1)$ on the canonical S^1-bundle M. Moreover, one can show that the dual vector field to $\eta := \frac{1}{i(m+1)}\rho$ on (M,g) defines a Sasaki structure with $\varepsilon = 1$.

By our definition of Einstein-Sasaki structures on a space (M^n, g) of odd dimension n the scalar curvature $scal^g$ is always $n(n-1)$ and thus positive. However, in the literature Lorentzian Einstein-Sasaki structures are often assumed to be of negative scalar curvature $-n(n-1)$. In fact, this can be achieved by taking an Einstein-Sasaki metric g (in above sense) of signature $(n-1, 1)$ and then switching to the metric $-g$, which has signature $(1, n-1)$ and scalar curvature $scal^{-g} = -n(n-1)$. We will use the latter convention of negative scalar curvature in the case of Lorentzian geometry. Note also that if we start with a Riemannian Kähler space (N, h, J) of negative scalar curvature $scal^h = -4m(m+1)$ and define the metric $g = \pi^* h + \frac{1}{(m+1)^2}\rho \circ \rho$ on the canonical S^1-bundle M over N, then this gives rise to a Lorentzian Einstein-Sasaki space of negative scalar curvature with timelike Killing vector field ξ.

Finally, we remark that there exist more generalised versions of cone constructions for semi-Riemannian space (cf. [**13, 22**]). We also note that for Ricci-flat spaces one can use the Fefferman-Graham ambient metric $\boldsymbol{h}$ (which is explicitly known in this case; cf. Chapter 2 and e.g. [**6**]) as a replacement for the metric cone $(\overline{M}, \bar{g})$ (of 2nd kind, whose construction is only possible under the assumption of non-zero scalar curvature for the base).

10. CR-Geometry

In this section we introduce partially integrable CR-geometry of hypersurface type on smooth manifolds M^n of dimension $n = 2m+1 \geq 3$. This kind of CR-geometry can be described as a $|2|$-graded parabolic geometry with canonical Cartan connection of type (G, P), where $G = \mathrm{PSU}(p+1, q+1)$ and the parabolic P fixes a complex null line in $\mathbb{C}^{m+2}$ with $m = p+q$. As usual for parabolic geometries, the canonical Cartan geometry and the first order picture correspond to each other uniquely via a prolongation, resp., reduction procedure. Integrable CR-structures are naturally induced on real hypersurfaces of generic type in $\mathbb{C}^{m+1}$. This setting provides an ample source for CR-spaces and a link to the theory of complex analysis in several variables. We will also discuss here basic notions for pseudo-Hermitian structures on integrable CR-spaces, which are related to particular Weyl structures for CR-geometry in the sense of Section 0.5 (cf. [**42**]).

We start our discussion with the structure groups and algebras of *CR-geometry.* The group $\tilde{G} = \mathrm{SU}(p+1, q+1)$, $m = p+q \geq 1$, acts on $\mathbb{C}^{m+2}$ equipped with indefinite Hermitian product $(\cdot,\cdot)_{p+1,q+1}$ via the standard representation. The stabiliser $\tilde{P}$ of a complex null line, which is a parabolic subgroup of $\tilde{G}$, gives rise to a $|2|$-grading

$$\mathfrak{su}(p+1, q+1) \ = \ \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

of the Lie algebra $\mathfrak{g}$ of $\tilde{G}$. Thereby, $\mathfrak{g}_0 \cong \mathfrak{cu}(p,q)$, $\mathfrak{g}_{-1} \cong \mathbb{C}^m$, $\mathfrak{g}_{-2} \cong \mathbb{R}$, $\mathfrak{g}_1 \cong \mathbb{C}^{m*}$ and $\mathfrak{g}_2 \cong \mathbb{R}^*$. Further, let $G = \mathrm{PSU}(p+1,q+1)$ denote the projective special unitary group with Lie algebra $\mathfrak{g}$. This is the quotient of the group $\tilde{G}$ by its centre $\mathbb{Z}_{m+2}$. We denote by P the parabolic subgroup of G, which preserves the corresponding filtration to the $|2|$-grading of $\mathfrak{g}$. The subgroup G_0 which preserves the grading of $\mathfrak{g}$ is isomorphic to the conformal unitary group $\mathrm{CU}(p,q) = \mathrm{U}(p,q) \times \mathbb{R}_+$ and the parabolic P is isomorphic to the semidirect product $G_0 \ltimes exp(\mathfrak{p}_+)$. We call (G,P) the Klein model of CR-geometry.

Now let M^n be a connected smooth manifold of odd dimension $n = 2m+1 \geq 3$. We describe the first order picture of CR-geometry on M. So let H be a subbundle in TM of corank 1 equipped with a pointwise complex structure J on H, i.e., $J_p^2 = -id|_{H_p}$ for all $p \in M$, which is smooth. The Lie bracket of vector fields X, Y on M induces the (tensorial) Levi-bracket

$$\begin{aligned} \mathcal{Z}: \quad H \times H \;&\rightarrow\; Q := TM/_H \ , \\ \mathcal{Z}(X,Y) \;&:=\; \pi_Q[X,Y] \ . \end{aligned}$$

We assume that $\mathcal{Z}$ is non-degenerate and totally real, i.e., $\mathcal{Z}(X,\cdot) \neq 0$ for all $0 \neq X \in H$ and $\mathcal{Z}(X,Y) = \mathcal{Z}(JX,JY)$ for all $X,Y \in H$. In this case the bilinear form $\mathcal{Z}(\cdot, J\cdot)$ on H with values in Q is symmetric and we can see that $\mathcal{Z}$ is pointwise (with respect to an identification of Q with $\mathbb{R}$) the imaginary part of a Hermitian form. We assume from now on that the real line bundle $Q = TM/H$ is equipped with an orientation. Then the Levi-bracket $\mathcal{Z}$ has a uniquely defined signature (p,q) with $p+q = m$, which is inherited from a global Hermitian form on H (with values in $Q \otimes \mathbb{C}$) and a choice of non-zero global section in Q. Under all these assumptions we call (M^n, H, J) a non-degenerate, partially integrable CR-space of hypersurface type with signature (p,q). In particular, the subbundle H of TM is a contact distribution. If, in addition, the Nijenhuis torsion tensor

$$\mathcal{N}_J(X,Y) := [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY]$$

vanishes for all $X,Y \in \Gamma(H)$, we call (M^n,H,J) an *integrable CR-space*. Furthermore, if the Levi-bracket is positive-definite we call (M^n,H,J) a *strictly pseudoconvex* CR-space.

Alternatively, a CR-structure on a smooth space M^n can be defined as a subbundle T_{10} of the complexified tangent bundle $TM^{\mathbb{C}} = TM \otimes \mathbb{C}$, for which $T_{10} \cap \overline{T_{10}} = \{0\}$ and $dim_{\mathbb{C}} T_{10} = m$. We set $T_{01} := \overline{T_{10}}$. The Levi-bracket L on T_{10} is then given by

$$\begin{aligned} L: \ T_{10} \times T_{10} \;&\rightarrow\; E := TM^{\mathbb{C}}/_{T_{10} \oplus T_{01}} \ , \\ (U,V) \;&\mapsto\; i \cdot (pr_E[U,\bar{V}]) \ , \end{aligned}$$

where pr_E denotes the projection onto the quotient E. A (complex defined) CR-structure T_{10} gives rise to a (real defined) CR-structure (H,J) by

$$H := Re(T_{10} \oplus T_{01}) \quad \text{and} \quad J(U + \bar{U}) := i(U - \bar{U}), \quad U \in \Gamma(T_{10}) \ .$$

On the other hand, a (real) CR-structure (H,J) uniquely defines the J-eigenspace T_{10} in $TM^{\mathbb{C}}$ to the eigenvalue i, which is a complex CR-structure. A (complex) CR-structure is non-degenerate if the Levi-form L on T_{10} is non-degenerate and it is partially integrable iff $[\Gamma(T_{10}), \Gamma(T_{10})] \subset \Gamma(T_{10} \oplus T_{01})$, resp., integrable iff $[\Gamma(T_{10}), \Gamma(T_{10})] \subset \Gamma(T_{10})$.

By definition, the distribution H of a partially integrable CR-structure makes the underlying space M into a filtered manifold and the Levi-bracket $\mathcal{Z}$ on the associated graded $\mathrm{gr}(TM)$ is compatible with the almost complex structure J. This makes $\mathrm{gr}(TM)$ into a locally trivial bundle of nilpotent Lie algebras with a reduction of the structure group to $G_0 = \mathrm{CU}(p,q)$, where each fibre is isomorphic to the nilpotent part $\mathfrak{g}_-$ of $\mathfrak{g} = \mathfrak{su}(p+1,q+1)$. In this situation, we can apply the general prolongation procedure for filtered manifolds as mentioned in Section 0.5 (cf. [**41**]), which generates in a unique way a parabolic Cartan geometry $(\mathcal{P}(M), \omega_{nor})$ of CR-type (G,P) on the underlying space M with regular and normal Cartan connection ω_{nor}. We call $(\mathcal{P}(M), \omega_{nor})$ the canonical Cartan geometry of a partially integrable CR-space (M,H,J). It is a matter of fact that the canonical connection ω_{nor} is torsion-free if and only if the underlying CR-structure (H,J) is integrable.

In general, a regular CR-Cartan geometry (of hypersurface type) on a smooth space M^n is given by a pair $(\mathcal{P}(M), \omega)$, where $\mathcal{P}(M)$ is a principal P-bundle on M and ω is a Cartan connection with values in $\mathfrak{g} = \mathfrak{su}(p+1,q+1)$ such that the curvature function κ of ω satisfies the condition $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1}$ for all $i,j \in \{-2,\dots,2\}$. In this situation, the Cartan geometry $(\mathcal{P}(M), \omega)$ induces a filtration $TM \supset T^{-1}M$ on $TM = \mathcal{P} \times_P \mathfrak{g}/\mathfrak{p}$, an almost complex structure J and a J-compatible algebraic bracket on $T^{-1}M$, which makes the associated graded $\mathrm{gr}(TM)$ into a locally trivial bundle with fibre type $\mathfrak{g}_-$ and G_0-reduction of the structure group. The regularity condition for the curvature function κ implies that the algebraic bracket on $\mathrm{gr}(TM)$ is induced by the Lie bracket and M becomes a filtered manifold. Altogether, this means that $(\mathcal{P}(M), \omega)$ induces the partially integrable CR-structure $(T^{-1}M, J)$ (of first order) on M.

A basic example for a regular and normal Cartan geometry is the homogeneous space G/P with Cartan geometry (G, ω_G), where ω_G denotes the Maurer-Cartan form of G. The induced CR-structure on G/P is flat and integrable with signature (p,q). In general, the homogeneous CR-space G/P is naturally identified with a compactification of the Heisenberg group $exp(\mathfrak{g}_-)$, whose Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ induces a left-invariant flat CR-structure. In case $G = \mathrm{PSU}(1, m+1)$ the homogeneous model G/P can be realised as the unit sphere S^{2m+1} in $\mathbb{C}^{m+1}$. Thereby, the flat CR-structure on the hypersurface S^{2m+1} is naturally induced by the complex structure of $\mathbb{C}^{m+1}$ and has definite signature. In general, any generic hypersurface N of $\mathbb{C}^{m+1}$ is naturally equipped with an integrable CR-structure.

In order to introduce CR-tractor calculus we need a $\tilde{P}$-reduction of the canonical CR-Cartan geometry $(\mathcal{P}, \omega_{nor})$ on a space (M,H,J). Such a $\tilde{P}$-reduction $(\tilde{\mathcal{P}}(M), \omega_{nor})$ exists if and only if the complex line bundle $\Lambda^m_{\mathbb{C}} H \otimes Q$ admits a $(m+2)$nd root, i.e., there exists a complex line bundle $\mathcal{E}(1,0) \to M$ such that $\mathcal{E}(1,0)^{\otimes^{m+2}} = \Lambda^m_{\mathbb{C}} H \otimes Q$ (cf. [**33, 39**]). Note that locally such a root bundle $\mathcal{E}(1,0)$ always exists. The standard CR-tractor bundle on M is then defined by

$$\mathcal{T}_{CR}(M) \ := \ \tilde{\mathcal{P}}(M) \times_{\tilde{P}} \mathbb{C}^{m+2} \ ,$$

which is naturally filtered by $\mathcal{T}^{-1}_{CR} \supset \mathcal{T}^0_{CR} \supset \mathcal{T}^1_{CR}$, where $\mathcal{T}^1_{CR}$ is canonically identified with the dual bundle $\mathcal{E}(-1,0)$ of $\mathcal{E}(1,0)$. The standard CR-tractor bundle $\mathcal{T}_{CR}$ on a space (M,H,J) admits a natural complex structure J_{CR} and the canonical connection ω_{nor} induces the tractor connection

$$\nabla^{CR} : \Gamma(\mathcal{T}_{CR}) \to \Omega^1(TM) \otimes \Gamma(\mathcal{T}_{CR}) \ .$$

An automorphism Ψ of a partially integrable CR-space (M, T_{10}) is a diffeomorphism of M such that $\Psi^* T_{10} = T_{10}$. The automorphisms of a CR-space (M, T_{10}) form a Lie group $Aut(M, T_{10})$ acting on M. An infinitesimal automorphism of (M, T_{10}) is a vector field $X \in \mathfrak{X}(M)$ such that the local flow X_t preserves the complex distribution T_{10}. We denote the Lie algebra of infinitesimal automorphisms by $\mathfrak{inf}(M, T_{10})$. The dimension of $\mathfrak{inf}(M, T_{10})$ is bounded for any CR-space by $(m+2)^2 - 1$, which is the dimension of $\mathfrak{su}(p+1, q+1)$. This bound can be reached only if the CR-space (M, T_{10}) is locally flat. We call an element $X \in \mathfrak{inf}(M, H, J)$ a *transverse symmetry* of a CR space M if $X_p \notin H_p$ for all $p \in M$, i.e., X is everywhere on M transverse to H.

We now introduce some basic notions of pseudo-Hermitian geometry in the setting of integrable CR-structures. We will keep this discussion brief, since a more detailed exploration of a generalised (but, completely analogous) version for the partially integrable case will follow in Chapter 5. Let θ be a nowhere vanishing 1-form on a non-degenerate, integrable CR-manifold (M, H, J) of hypersurface type and signature (p, q) with $\theta|_H \equiv 0$. Such a 1-form θ is called a *pseudo-Hermitian form* (resp., pseudo-Hermitian structure) and the data (M, H, J, θ) (resp., (M, T_{10}, θ)) are called a pseudo-Hermitian space. A pseudo-Hermitian form θ is necessarily a contact form on M and gives rise to a Weyl structure for the canonical Cartan geometry $(\mathcal{P}(M), \omega_{nor})$ (cf. Section 0.5). The existence of a θ is guaranteed by the orientability of the quotient bundle Q. Any pseudo-Hermitian form θ uniquely determines a Reeb vector field $T^\theta = T \in \mathfrak{X}(M)$ by the conditions

$$\theta(T) \equiv 1 \qquad \text{and} \qquad d\theta(T, \cdot) \equiv 0 \ .$$

The Hermitian form L_θ defined by

$$\begin{array}{rcl} L_\theta : \ T_{10} \times T_{10} & \to & \mathbb{C} \ , \\ L_\theta(U, V) & := & -id\theta(U, \bar{V}) \end{array}$$

is called the Levi-form of (M, T_{10}, θ). Obviously, $\theta(L(U, V)) = L_\theta(U, V)$. The Levi-form L_θ can be naturally extended to $TM^{\mathbb{C}}$ by

$$L_\theta(U, \bar{V}) := 0 \ , \qquad L_\theta(T, \cdot) = 0 \ ,$$
$$L_\theta(\bar{U}, \bar{V}) := \overline{L_\theta(U, V)} = L_\theta(V, U) \ .$$

The real part of this extension is a symmetric bilinear form on TM, which is non-degenerate on H and has real signature $(2p, 2q)$. We also denote this form by

$$L_\theta : TM \times TM \to \mathbb{R} \ .$$

If the Hermitian form L_θ is positive definite, the CR-space (M, H, J) is strictly pseudoconvex. In general, any two pseudo-Hermitian forms on (M, H, J) differ only by multiplication (= rescaling) with a nowhere vanishing function. If this function is positive the signatures of the two equivalent pseudo-Hermitian structures are the same.

For any choice of pseudo-Hermitian form θ on an integrable (M, H, J) there exists a covariant derivative $\nabla^W : \mathfrak{X}(M) \to \Omega^1(M) \otimes \mathfrak{X}(M)$, which is uniquely determined by the conditions

$$X(L_\theta(Y, Z)) \ = \ L_\theta(\nabla^W_X Y, Z) + L_\theta(Y, \nabla^W_X Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$, i.e., ∇^W is metric with respect to L_θ on $H \subset TM$, and

$$\begin{array}{rcl} Tor^W(X, Y) & = & L_\theta(JX, Y) \cdot T \ , \\ Tor^W(T, X) & = & -\frac{1}{2}\big(\ [T, X] + J[T, JX] \ \big) \end{array}$$

for all $X, Y \in \Gamma(H)$, where the torsion Tor^W is defined in the usual way by

$$Tor^W(X,Y) \ := \ \nabla^W_X Y - \nabla^W_Y X - [X,Y] \ .$$

The connection ∇^W satisfies

$$\nabla^W T = 0 \qquad \text{and} \qquad \nabla^W J = 0 \ .$$

We call ∇^W the (real version of the) Tanaka-Webster connection of the pseudo-Hermitian structure θ. The complex linear extension of ∇^W to $TM^{\mathbb{C}}$ is denoted by ∇^W as well. We remark that the Tanaka-Webster connection ∇^W is *not* induced by the pull-back of the $\mathfrak{g}_0$-part of the canonical Cartan connection ω_{nor} with respect to the Weyl structure $\sigma^\theta : \mathcal{G}_0(M) \to \mathcal{P}(M)$, which is naturally induced by θ.

Finally, we introduce curvature expressions for ∇^W. Thereby, we use the following conventions: indices with letters i, j and k run from 1 to $2m$, whereas indices with Greek letters α, β and γ run from 1 to m. We denote by $\{e_i : \ i = 1, \dots, 2m\}$ an orthonormal basis (resp., local frame) of (H, L_θ) such that

$$J(e_{2\alpha-1}) = e_{2\alpha}, \qquad J(e_{2\alpha}) = -e_{2\alpha-1} \qquad \text{for all } \alpha = 1, \dots, m \ .$$

The complex vectors

$$Z_\alpha := \frac{1}{\sqrt{2}}(e_{2\alpha-1} - iJe_{2\alpha-1}), \quad \alpha = 1, \dots, m \ ,$$

form an orthonormal basis of (T_{10}, L_θ) and the vectors $Z_{\bar\alpha} := \overline{Z_\alpha}$ represent an orthonormal basis of T_{01}. The curvature operator of ∇^W is defined by

$$R^{\nabla^W}(X,Y)Z = \nabla^W_X \nabla^W_Y Z - \nabla^W_Y \nabla^W_X Z - \nabla^W_{[X,Y]} Z \ .$$

The corresponding curvature tensor is given by

$$R^W(X,Y,Z,V) = L_\theta(R^{\nabla^W}(X,Y)Z, \bar V) \ ,$$

where $X, Y, Z, V \in TM^{\mathbb{C}}$ are complex vectors. The *Webster*-Ricci and scalar curvatures are contractions of R^W:

$$Ric^W := \textstyle\sum_{\alpha=1}^m R^W(Z_\alpha, Z_{\bar\alpha}, \cdot, \cdot) \ ,$$

$$scal^W := \textstyle\sum_{\alpha=1}^m Ric^W(Z_\alpha, Z_{\bar\alpha}) \ .$$

The function $scal^W$ on (M, T_{10}, θ) is real. All these definitions of curvature for ∇^W are independent of the choice of orthonormal frame.

In general, the Webster-Ricci curvature Ric^W of a pseudo-Hermitian structure θ is skew-symmetric for real tangent vectors. If Ric^W is a multiple of the 2-form $d\theta$ then the relation

$$Ric^W \ = \ -i\frac{scal^W}{m} \cdot d\theta$$

holds. In this case the pseudo-Hermitian structure θ is called a *pseudo-Einstein structure* for the CR-space (M, H, J) in the sense of J.M. Lee (cf. [**103**]). The (local) existence of a pseudo-Einstein structure is not restrictive to the underlying CR-geometry (cf. [**103, 48**]). Also, the Webster scalar curvature $scal^W$ need not be constant for pseudo-Einstein structures. These properties make the pseudo-Einstein condition quite different from its namesake in (conformal) semi-Riemannian geometry. We will see in Chapter 6 what the natural analog of the conformal Einstein condition is in the framework of CR-geometry.

11. Fefferman-Graham Ambient Metric Construction and Poincaré-Einstein Model

We explain here the Fefferman-Graham ambient metric construction for spaces with conformal structure (cf. [**54**] and e.g. [**87, 59**]). The ambient metric is known to be a powerful tool for studying conformal invariant theory. This is related to the fact that its construction incorporates in a suitable way the higher order jets (up to infinite order) of the underlying conformal structure. We start our discussion with a brief review of the original ambient metric construction due to Ch. Fefferman for pseudoconvex real hypersurfaces N in $\mathbb{C}^{m+1}$, which actually led to the idea of ambient metrics in conformal geometry. We are also interested in the Poincaré-Einstein model with conformal boundary, which is closely related to the ambient metric model via the metric cone construction.

Let $\mathbb{C}^{m+1}$ denote the $(m+1)$-dimensional complex vector space with coordinates $(z^1, \dots, z^{m+1})$ and let us consider a *strictly pseudoconvex domain* Ω in $\mathbb{C}^{m+1}$, i.e., Ω admits a boundary $N = \partial\Omega$, which is locally given as the graph of a twice continuously differentiable real function, whose Levi-form is positive definite. In this case the *boundary* N is called a strictly pseudoconvex real hypersurface in $\mathbb{C}^{m+1}$, which admits a naturally induced integrable CR-structure of definite signature. It is known that a domain Ω is strictly pseudoconvex in $\mathbb{C}^{m+1}$ if and only if Ω is a *domain of holomorphy.* We set $\overline{\Omega} := \Omega \cup \partial\Omega$, where $\overline{\Omega}$ is differentiable up to the boundary of a certain order. Usually, the boundary itself is assumed to be smooth (cf. Section 0.1).

A *defining function* for the boundary N is by definition a (smooth) function $r : \overline{\Omega} \to \mathbb{R}_{\geq 0}$ with $r > 0$ on Ω and $r = 0$, $|dr| \neq 0$ on N. The relevant complex Monge-Ampère problem for the boundary N is to find a function $u > 0$ on Ω, which satisfies $L(u) = 1$ with boundary condition $u|_{\partial\Omega} = 0$, where

$$L(u) = (-1)^{m+1} \det \begin{pmatrix} u & u_{\bar{j}} \\ u_i & u_{i\bar{j}} \end{pmatrix} \qquad \text{and} \qquad u_i = \frac{\partial u}{\partial z^i}, \text{ etc .}$$

Solutions of this Dirichlet boundary value problem are closely related to the Bergman kernel of the domain Ω. There exists an iterative procedure using elementary calculus to produce approximate solutions u of the Monge-Ampère problem such that $L(u) = 1 + o(u^{m+1})$, which means that the coefficients of the Taylor expansion of $L(u) - 1$ (at the boundary with respect to the coordinate u) vanish for all orders $s \leq m+1$. This approximate solution u is unique up to addition of terms in $o(u^{m+2})$ (cf. [**53**]).

The approximate solution u can be used to define an *ambient metric* for the boundary space N with induced CR-structure. This ambient metric lives on $\mathbb{C}^* \times N_\epsilon$, where $N_\epsilon \cong N \times [0, \epsilon)$ is some collar neighbourhood of N in Ω with $\epsilon > 0$, and is defined as the Lorentz-Kähler metric

$$\tilde{g}(u) = \sum_{j,k=0}^{m+1} \frac{\partial^2 f(u)}{\partial z^j \partial \bar{z}^k} dz_j d\bar{z}_k$$

with respect to the potential $f(u) := -|z^0|^2 u(z)$, where $(z^0, z) \in \mathbb{C}^* \times N_\epsilon$. Note that $L(u) = 1 + o(u^{m+1})$ implies $Ric^{\tilde{g}(u)} = o(u^m)$, which shows that the ambient metric $\tilde{g}(u)$ is asymptotically Kähler-Einstein near the boundary $\mathbb{C}^* \times N$ to order m. Replacing the potential $f(u)$ by $-|z^0|^2 u_o(z)$, where u_o is an exact solution of the Monge-Ampère problem, produces a Kähler-Einstein metric on $\mathbb{C}^* \times N_\epsilon$.

We now turn our interest to the *ambient metric* construction in conformal geometry. Let (M^n, c) denote a smooth manifold of dimension $n \geq 3$, where $c = [g]$ is the class of conformally related metrics to some given metric g on M with signature (r, s). The conformal class $[g]$ determines the ray subbundle $\mathcal{Q} \xrightarrow{\pi} M$ in S^2T^*M with $\mathbb{R}_+$-action $\sigma(s)(x, g_x) = (x, s^2 g_x)$ for $x \in M$. Now let $\tilde{M}$ be an $(n+2)$-dimensional manifold endowed with a free $\mathbb{R}_+$-action, which we also denote by σ, and an $\mathbb{R}_+$-equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We call $(\tilde{M}, i)$ an *ambient space* of (M, c) and we write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental vector field generating the $\mathbb{R}_+$-action, i.e., for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$ we have $\boldsymbol{X} f(u) = \frac{d}{dt} f(\sigma(e^t)u)|_{t=0}$. An ambient metric $\boldsymbol{h}$ is a pseudo-Riemannian metric on an ambient space $(\tilde{M}, i)$ of signature $(r+1, s+1)$ such that

(i) the metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$-action σ, i.e., $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$ and $\boldsymbol{X}$ is a homothetic vector field for $\boldsymbol{h}$.
(ii) For $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$.

The second condition implies that the tangent direction along the fibres of $\mathcal{Q}$ is null. To simplify the notations we will usually identify $\mathcal{Q}$ with its image in $\tilde{M}$ and suppress the embedding map i. So far the definition of ambient metrics is rather general. In order to link the geometry of an ambient space $(\tilde{M}, \boldsymbol{h})$ to the underlying conformal structure c on M, further requirements on $\boldsymbol{h}$ are necessary. In [**54**] Ch. Fefferman and R. Graham treat the problem of constructing a formal power series solution along $\mathcal{Q}$ for the (Goursat) problem of finding an ambient metric $\boldsymbol{h}$, which satisfies in addition to (i) and (ii) the condition that it be Ricci-flat, i.e., $Ric(\boldsymbol{h}) = 0$. They observed the following key result.

Theorem 2. *Let (M^n, c) be a space with conformal structure $c = [g]$ of dimension $n \geq 3$ and let $\mathcal{Q} \times (-1, 1)$ be an ambient space with trivially extended $\mathbb{R}_+$-action, where $\mathcal{Q}$ is identified with the level set $\{\rho = 0\}$ of the coordinate ρ in the interval $(-1, 1)$.*

(1) *If n is odd then there exists a formal power series solution $\boldsymbol{h}$, which is unique up to a $\mathbb{R}_+$-equivariant diffeomorphism of $\mathcal{Q} \times (-1, 1)$ fixing $\mathcal{Q}$, satisfying (i) and (ii) and $Ric(\boldsymbol{h}) = o(\rho^\infty)$. If the conformal structure c on M is real analytic, then this formal power series converges so that $\boldsymbol{h}$ actually exists on a neighbourhood of $\mathcal{Q}$ in $\mathcal{Q} \times (-1, 1)$.*
(2) *If n is even the condition $Ric(\boldsymbol{h}) = 0$ for an ambient metric $\boldsymbol{h}$ can not be realised, in general. However, there always exists a formal power series solution $\boldsymbol{h}$ such that $Ric(\boldsymbol{h}) = o(\rho^{n/2-1})$. This solution is uniquely determined up to a $\mathbb{R}_+$-equivariant diffeomorphism fixing $\mathcal{Q}$ and addition of terms in $o(\rho^{n/2})$.*

In the following, we will say that an ambient metric is Ricci-flat to the optimal order if it is a metric with the properties guaranteed by Theorem 2. As a concrete example for a Ricci-flat ambient space, let us consider the pseudo-Euclidean space $\mathbb{R}^{r+1,s+1} \setminus \{0\}$ with origin removed and flat metric $\langle \cdot, \cdot \rangle_{r+1,s+1}$. The conformally flat standard model $S^{r,s} = G/P$ with $G = \mathrm{PO}(r+1, s+1)$ can be realised as the projective light cone of $\mathbb{R}^{r+1,s+1} \setminus \{0\}$, where the conformally flat structure c_o on G/P is naturally induced by the metric $\langle \cdot, \cdot \rangle_{r+1,s+1}$. Conversely, (the upper) half of the light cone can be interpreted as the ray bundle $\mathcal{Q}$ on $S^{r,s}$, which defines an embedding of $\mathcal{Q}$ into $\mathbb{R}^{r+1,s+1} \setminus \{0\}$. The $\mathbb{R}_+$-action σ on $\mathbb{R}^{r+1,s+1} \setminus \{0\}$ is given by multiplication with positive real numbers, which makes the natural embedding of $\mathcal{Q}$ σ-equivariant. By definition of c_o it is clear that $\langle\cdot,\cdot\rangle_{r+1,s+1}$ is a flat ambient metric for $S^{r,s}$ on $\mathbb{R}^{r+1,s+1} \setminus \{0\}$.

Now let us consider, in general, a metric of the form

$$\boldsymbol{h} = 2tdtd\rho + 2\rho dt^2 + t^2\tilde{g}_{ij}(x,\rho)dx^i dx^j \tag{10}$$

defined on a set $\tilde{M} := \mathbb{R}_+ \times \Omega \times I$ with coordinates (t,x,ρ), where the set Ω is open in $\mathbb{R}^n$, the interval I contains zero and the symmetric tensor $\tilde{g}_{ij}(x,\rho)dx^i dx^j$ is the pull-back to $\tilde{M}$ of a 1-parameter family of metrics on Ω. Such a metric $\boldsymbol{h}$ is by definition homogeneous of degree 2 in the coordinate t and the curves $\rho \in I \mapsto (t,x,\rho)$ for fixed t and x are geodesics. The parametrised $\mathbb{R}_+$-orbits $s \in \mathbb{R}_+ \mapsto (st,x,\rho)$ are geodesics as well. Further, we set $\boldsymbol{X} := t\partial_t$. Then for any metric $\boldsymbol{h}$ of the form (10), we have the following: $\boldsymbol{X}$ is homothetic and $\boldsymbol{h}(\boldsymbol{X},\cdot)$ is closed, which implies $Ric(\boldsymbol{h})(\boldsymbol{X},\cdot) = 0$. In particular, $Ric(\boldsymbol{h})_{ti} = Ric(\boldsymbol{h})_{tt} = 0$ for all $i \in \{1,\ldots,n\}$. The other components of $Ric(\boldsymbol{h})$ are given as follows: $Ric(\boldsymbol{h})_{ij}$ is (the pull-back to $\tilde{M}$ of) the tensor

$$\rho\tilde{g}''_{ij} - \rho\tilde{g}^{kl}\tilde{g}'_{ik}\tilde{g}'_{jl} + \frac{1}{2}\rho\tilde{g}^{kl}\tilde{g}'_{kl}\tilde{g}'_{ij} + \frac{2-n}{2}\tilde{g}'_{ij} - \frac{1}{2}\tilde{g}^{kl}\tilde{g}'_{kl}\tilde{g}_{ij} + \mathrm{Ric}(\tilde{g})_{ij}, \tag{11}$$

on M; $\mathrm{Ric}(\boldsymbol{h})_{\rho\rho}$ is

$$-\frac{1}{2}\tilde{g}^{ij}\tilde{g}''_{ij} + \frac{1}{4}\tilde{g}^{ij}\tilde{g}^{kl}\tilde{g}'_{ik}\tilde{g}'_{jl}\ ; \tag{12}$$

and $\mathrm{Ric}(\boldsymbol{h})_{\rho j}$ is

$$\tilde{\nabla}^{(\rho)}_{\ell}(\tilde{g}^{k\ell}\tilde{g}'_{kj}) - \tilde{\nabla}^{(\rho)}_{j}(\tilde{g}^{k\ell}\tilde{g}'_{k\ell}), \tag{13}$$

where the prime denotes differentiation with respect to the coordinate ρ, and for each value of the parameter ρ, $\tilde{\nabla}^{(\rho)}$ is the Levi-Civita covariant derivative on M for $\tilde{g}(x,\rho)$.

One can show that the formal power serious solution $\boldsymbol{h}$ to infinite order for a Ricci-flat ambient metric of a space (M^n,g) of odd dimension n (as guaranteed by Theorem 2) can locally always be brought into the form (10), where $(x^1,\ldots,x^n)$ describes a coordinate patch on M, t is a coordinate in fibre direction of $\mathcal{Q}$ (determined by the corresponding section to g), ρ is the coordinate of a thickening interval I and $g = \sum_{i,j=1}^n \tilde{g}_{ij}(x,0)dx^i dx^j$. Unfortunately, in even dimension it is not correct to assume a priori the form (10) for any ambient metric $\boldsymbol{h}$ which is Ricci-flat to the optimal order. Certainly, we can say that if a metric $\boldsymbol{h}$ of the form (10) is Ricci-flat (to the optimal order), then $\boldsymbol{h}$ is an ambient metric for $c = [\tilde{g}_{ij}(x,0)dx^i dx^j]$ on Ω. An example of a Ricci-flat ambient metric expressed in the form (10) is

$$\boldsymbol{h}_o = 2tdtd\rho + 2\rho dt^2 + t^2(1+\rho/2)^2 g_o\ ,$$

where g_o denotes a conformally flat metric of constant sectional curvature 1 with signature $(r+1,s+1)$. The metric $\boldsymbol{h}_o$ is flat. In fact, $\boldsymbol{h}_o$ is a local coordinate expression for $\langle\cdot,\cdot\rangle_{r,s}$ on the ambient space $\mathbb{R}^{r+1,s+1} \setminus \{0\}$ of $S^{r,s}$ (as discussed above) in a neighbourhood of $\mathcal{Q}$.

If we assume that an ambient metric $\boldsymbol{h}$ for some space (M^n,g) exists in the form (10), then the Taylor expansion (to the optimal order in the coordinate ρ) of the components $\tilde{g}_{ij}(x,\rho)$ can be determined by formally solving Einstein's equation $Ric(\boldsymbol{h}) = 0$. In fact, we obtain the correct coefficients for this expansion by setting (11) - (13) to zero and solving this inductively, beginning with the initial condition $\tilde{g}_{ij}(x,0) = g_{ij}(x)$. For example, setting $\rho = 0$ in (11) and solving for $\tilde{g}'_{ij}$ (such that (11) becomes zero) results in $\tilde{g}'_{ij}(x,0) = -2\mathsf{P}^g_{ij}(x)$. When n is odd this procedure determines the expansion of $\tilde{g}_{ij}(x,\rho)$ up to the ρ^n-term. The vanishing of the other components of $Ric(\boldsymbol{h})$ determines the remainder of the expansion to infinite order. For n even this procedure can be used to determine the Taylor expansion for $\boldsymbol{h}$ up to the $\rho^{n/2}$-term such that $Ric(\boldsymbol{h}) = o(\rho^{n/2-1})$. Then the procedure stops if $Ric(\boldsymbol{h}) \neq 0 \bmod o(\rho^{n/2})$. The $\rho^{n/2}$-term of $Ric(\boldsymbol{h})$ in this (optimal) expansion is called the obstruction tensor denoted by $\mathcal{O}_{ij}$. The ambient obstruction tensor is known to be conformally covariant and trace-free symmetric, involving n derivatives of the metric g. For example, in dimension $n = 4$ the obstruction is given by the Bach tensor B^g from Weyl's relativity (cf. [**11**] and Section 0.4).

We remark that the Ricci-flat ambient metric construction (to the optimal order) is suitable for the construction of conformal scalar invariants and conformally covariant differential operators on the underlying space (M,c). For example, let $P(\boldsymbol{h})$ be a Weyl invariant of the semi-Riemannian metric $\boldsymbol{h}$, then the pull-back of the restriction of $P(\boldsymbol{h})$ to $\mathcal{Q}$ with respect to some metric $g \in c$ produces a conformal scalar invariant on (M,c). The Weyl tensor W^g and the Cotton tensor C^g can be derived this way by pulling back certain parts of the Riemannian curvature tensor of $\boldsymbol{h}$. The conformal Laplacian $P^g_{2,n}$ on $(M,[g])$ is naturally induced by the Laplace operator $\Delta^{\boldsymbol{h}}_{tr}$ on $(\tilde{M},\boldsymbol{h})$. Moreover, conformally covariant differential operators of order $2k$, $k \in \mathbb{N}$, are obtained from the kth power of the ambient Laplacian. In even dimensions $n = 2m$ this works only up to the critical power $k = \frac{1}{2}n$, which gives rise to the celebrated *GJMS-operators* on (M,c) (cf. [**75**]).

Finally, we want to discuss the *Poincaré-Einstein model* for smooth conformal manifolds. For this purpose, let $\underline{M}^{n+1}$ denote a manifold of dimension $n+1$ with boundary a smooth manifold $\partial\underline{M} = M$. We denote the interior of $\underline{M}$ by M^+, which we also call the *bulk* of $\underline{M}$. A *defining function* r for the boundary M is a function on $\underline{M}$ (with a certain regularity up to the boundary), which is positive on M^+ and satisfies $r = 0$, $dr \neq 0$ on the boundary M. A semi-Riemannian metric g^+ on the interior M^+ is said to be *conformally extendible* to the boundary if there exists some defining function r such that the symmetric 2-tensor $\underline{g} = r^2 g^+$ extends (with some specified regularity) to $\underline{M}$ and $\underline{g}$ is non-degenerate up to the boundary. In this case the restriction of $\underline{g}$ to TM in $T\underline{M}|_M$, which we denote by g, determines a conformal structure $c = [g]$ on M. This conformal structure c is independent of the choice of defining function r. Thus we call (M,c) the *conformal infinity* space or *conformal boundary* of (M^+,g^+) (cf. [**101**]). If the space $\underline{M}$ is compact, the metric g^+ is called *conformally compact* in the described situation. In the Riemannian case, any conformally compact metric g^+ is complete on M^+.

Besides the conformal structure c, there exists a second invariant on the boundary of $\underline{M}$. Namely, the function $|dr|^2_{\underline{g}}$ extends to $\underline{M}$ and its restriction to M is independent of the choice of r. In case $|dr|_{\underline{g}} = 1$ along M the sectional curvatures of (M^+,g^+) tend to -1 at the boundary (of infinity). In general, the asymptotic behaviour of the Riemannian curvature tensor R^+ of g^+ is described by

$$R^+_{ijkl} = -(|dr|^2_{\underline{g}})(g^+_{ik}g^+_{jl} - g^+_{il}g^+_{jk}) + o(r^{-4})$$

with respect to local coordinates on $\underline{M}$ at the boundary.

For example, let

$$g^+ \ := \ 4(1-|x|^2)^{-2}\sum_{i=1}^{n+1}(dx^i)^2$$

be the hyperbolic metric (in Riemannian signature) on the unit ball in $\mathbb{R}^{n+1}$. Then $r = \frac{1}{2}(1-|x|^2)$ is a defining function for the boundary sphere S^n and the metric r^2g^+ restricts to the round metric g_o on S^n. This is the flat model, which gives the name *Poincaré metric* to any g^+ on the bulk M^+ of some $\underline{M}$ when the sectional curvatures tend to -1 at infinity. In particular, we note that the condition $|dr|_{\underline{g}} = 1$ along M is satisfied if the metric g^+ on the bulk is an Einstein metric with $\mathrm{Ric}(g^+) = -ng^+$ and negative scalar curvature $-n(n+1)$. In the latter case we call g^+ on the bulk M^+ with boundary M a *Poincaré-Einstein metric*. Notice that we use this terminology even when we do not assume, in general, that $\underline{M}$ is compact. In fact, a collar neighbourhood of the boundary M is enough for our purpose.

In any case, the choice of a defining function r on $\underline{M}$ gives rise to a representative g in the conformal class of the boundary. However, a conformal representative $g \in c$ does not uniquely determine a defining function r on $\underline{M}$ without further conditions. In case g^+ is a Poincaré-Einstein metric, a second condition can be imposed in order to find a distinguished defining function for a boundary metric. Namely, there exists a unique defining function r on some neighbourhood of M such that $\underline{g} = r^2g^+$ induces the chosen $g \in c$ and $|dr|_{\underline{g}} \equiv 1$ on this neighbourhood. We call such a function r *special defining* (cf. [**77, 74**] in the case of Riemannian signature). For example, let us consider again the flat hyperbolic model on the unit ball. A special defining function inducing the round metric g_o on the boundary sphere S^n is given by $r = \frac{1}{2}\cdot\frac{1-|x|}{1+|x|}$. The hyperbolic metric g^+ on the unit ball can then be written (in polar coordinates) as

$$g^+ = r^{-2}\left(\, dr^2 \ + \ (1-r^2)^2 g_o\right)\ .$$

In general, let r be a special defining function for the boundary M of a Poincaré-Einstein space (M^+, g^+). Then g^+ is given on a neighbourhood of M by

$$g^+ = r^{-2}\left(\, dr^2 \ + \ g_r\right)\ , \tag{14}$$

where g_r is a 1-parameter family of metrics with $g_0 = \underline{g}|_M$, the boundary metric.

Similar to the problem of an ambient metric construction for a space with conformal structure, one can ask for the construction of a (not necessarily compact) Poincaré-Einstein space for a given conformal boundary. The following existence result for the bulk is known. Let (M^n, g) be a space of odd dimension $n = 2m+1 \geq 3$ equipped with a real analytic metric g, then there exists (at least locally) some $\epsilon > 0$, a collar neighbourhood $\underline{M}_\epsilon := M \times [0,\epsilon)$ of M and a smooth family g_r of metrics on M with $g = g_0$ such that $g^+ := r^{-2}(\, dr^2 \ + \ g_r)$ is a Poincaré-Einstein metric on $\underline{M}_\epsilon$ with conformal boundary $(M, [g_0])$. If g is a smooth metric then there exists at least a formal power series solution in r to infinite order for the family g_r, which makes $g^+ = r^{-2}(dr^2 \ + \ g_r)$ an Einstein-metric. If the conformal boundary (M^n, c) is of even dimension $n = 2m$ a Poincaré-Einstein metric on a collar neighbourhood does not exist (to all orders), in general. Again, (as for ambient metrics) the obstruction for the existence is the vanishing of the tensor $\mathcal{O} = \mathcal{O}_{ij}$.

There is a close relationship between Ricci-flat ambient metrics and Poincaré-Einstein spaces for $(M^n, [g])$ with conformal structure of signature (r,s). To explain this, let $(\tilde{M}, \boldsymbol{h})$ be a Ricci-flat ambient metric for (M, c) given in the form (10). We define M^+ to be the zero set of the function $\boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X}) + 1$ in $\tilde{M}$. The set M^+ is a smooth hypersurface in $\tilde{M}$ and the ambient metric $\boldsymbol{h}$ restricts to a semi-Riemannian metric on M^+ of signature $(r, s+1)$. With new coordinates $r = \sqrt{-2\rho}$ and $u = rt$ (on the $\rho < 0$ side of $\mathcal{Q}$) in $\tilde{M}$, a direct calculation yields

$$\boldsymbol{h} = u^2 g^+ - du^2 \ ,$$

where

$$g^+ = r^{-2}(\ dr^2 + g(x,r)_{ij} dx^i dx^j)$$

and $g(x,r)_{ij} = \tilde{g}(x, \rho(r))_{ij}$. Thus we observe that the ambient metric $\boldsymbol{h}$ is a metric cone over (M^+, g^+). The *ambient Euler vector field* $\boldsymbol{X}$ is computed to be $\boldsymbol{X} = u\partial_u$ in the new coordinates, and so it has the interpretation of the Euler vector $\mathbf{E}$ for the metric cone over (M^+, g^+). Since $\boldsymbol{h}$ is Ricci-flat, it follows that g^+ is Einstein with $Ric(g^+) = -ng^+$, and since $r(\rho)$ extends smoothly to $\rho = 0$, the space (M^+, g^+) has conformal infinity $(M, [g])$. We will describe in Section 2.4 how the Poincaré-Einstein space belonging to a conformal boundary (M, c) gives rise to a Ricci-flat ambient space (with boundary) for (M, c).

12. The Fefferman Construction

A smooth boundary $\partial\Omega$ of a strictly pseudoconvex domain Ω in $\mathbb{C}^{m+1}$ admits a naturally induced integrable CR-structure. The restriction of an ambient metric $\tilde{g}(u)$ of $\partial\Omega$ to the circle bundle $\partial\Omega \times S^1$ as a subspace of the ambient space $\Omega \times \mathbb{C}^*$ gives rise to a Lorentzian metric. In [**53**] Ch. Fefferman noticed that the conformal class of this Lorentzian metric on $\partial\Omega \times S^1$ is invariant under biholomorphisms of the domain Ω (in fact, the conformal class depends only on the CR-structure of the boundary). He also observed that the knowledge of an explicit formula for the ambient metric $\tilde{g}(u)$ to high orders is not necessary for the construction. Instead, an approximate solution u to 2nd order of the Monge-Ampère problem for the boundary $\partial\Omega$ contains already all necessary information to present a formula for the metric on $\partial\Omega \times S^1$. Such an approximate solution u can be derived by elementary calculus (cf. [**53**]). The induced metric on $\partial\Omega \times S^1$ is called the *Fefferman metric* (corresponding to the solution u) of the strictly pseudoconvex hypersurface $\partial\Omega$ in $\mathbb{C}^{m+1}$. An intrinsic construction of the Fefferman conformal metric for an abstract integrable CR-space was later found by J.M. Lee in [**102**]. The intrinsic construction coincides with Fefferman's original construction for a boundary of a domain in $\mathbb{C}^{m+1}$. We will describe the intrinsic construction with respect to a pseudo-Hermitian form θ later in this section. Our first aim is to formulate a generalised version of the Fefferman construction for parabolic Cartan geometries due to A. Čap (cf. [**34, 39**]; see also [**130, 8, 9**]).

Let $\iota : G \hookrightarrow G'$ be an inclusion of semisimple Lie groups and let P' be a parabolic subgroup of G' such that the G-orbit of eP' in G'/P' is open. Furthermore, let P be a parabolic subgroup of G, which contains the intersection $G \cap P'$. The pairs (G, P) and (G', P') define two parabolic Klein geometries and the corresponding pairs of Lie algebras are denoted by $(\mathfrak{g}, \mathfrak{p})$ and $(\mathfrak{g}', \mathfrak{p}')$. By assumption, we have an inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}'$, which induces a linear isomorphism $\mathfrak{g}/(\mathfrak{g} \cap \mathfrak{p}') \cong \mathfrak{g}'/\mathfrak{p}'$. This isomorphism is equivariant under the action of $G \cap P'$.

Now let $(\mathcal{P},\omega) \xrightarrow{\pi} N$ be a parabolic Cartan geometry of type (G,P) on a smooth manifold N. The space $\mathcal{P}$ is a principal $(G\cap P')$-bundle over $M := \mathcal{P}/(G\cap P')$. Then we can build a principal P'-bundle $\mathcal{P}' \xrightarrow{\pi'} M$ by setting $\mathcal{P}' := \mathcal{P}\times_{G\cap P'} P'$. Moreover, the Cartan connection ω extends by right translation with the action of P' to a Cartan connection ω' on $\mathcal{P}'$. We call this procedure, which uses the initial data $(\mathcal{P},\omega)\to N$, the *generalised Fefferman construction* with *Fefferman space* M and induced parabolic Cartan geometry $(\mathcal{P}',\omega')$ of type (G',P'). Thereby, $\pi : M\to N$ is a natural $(P/(G\cap P'))$-fibre bundle and we call $\dim(\mathfrak{p}/(\mathfrak{g}\cap\mathfrak{p}'))$ the codimension of the Fefferman construction. If G'/P' is connected the Fefferman construction produces for the flat model $G\to G/P$ with Maurer-Cartan form ω_G the model $G'\to G'/P'$ with Maurer-Cartan form $\omega_{G'}$ as Cartan connection, i.e., the resulting Cartan geometry on the Fefferman space G'/P' is flat again.

It is a material programme to classify pairs of parabolic geometries, which allow for a Fefferman construction (with certain codimension). A basic question occurs immediately: *How are the canonical Cartan connections ω_{nor} and ω'_{nor} related in a generalised Fefferman construction?* As an example in the framework of parabolic Cartan geometry, we describe in the following the Fefferman construction which assigns to any partially integrable CR-space (N,H,J) of dimension $n=2m+1$ equipped with a $(m+2)$nd root $\mathcal{E}(1,0)$ of $\Lambda^m_{\mathbb{C}}H\otimes Q$ a Fefferman space M of dimension $2m+2$ with Fefferman conformal structure c. This construction agrees with the classical Fefferman construction for strictly pseudoconvex boundaries due to Ch. Fefferman.

Let $(\tilde G,\tilde P)$, with $\tilde G = \mathrm{SU}(p+1,q+1)$, be the Klein pair which models CR-geometry (with a choice of $(m+2)$nd root), and let $(\tilde G',\tilde P')$ with $\tilde G' = \mathrm{SO}(r+1,s+1)$, $r=2p+1$, $s=2q+1$, be the Klein model of oriented conformal geometry (with a choice of root for densities). The standard representation of $\mathrm{SU}(p+1,q+1)$ on $\mathbb{C}^{m+2}$ is given by restriction of the standard representation of $\mathrm{SO}(r+1,s+1)$ on $\mathbb{R}^{r+1,s+1}\cong\mathbb{C}^{m+2}$. Thereby, the standard representation of $\mathrm{SU}(p+1,q+1)$ acts irreducibly on $\mathbb{C}^{m+2}$ and gives rise to a natural inclusion $\iota:\tilde G\to\tilde G'$. The parabolic $\tilde P'$ in $\mathrm{SO}(r+1,s+1)$ fixes a real null line l in $\mathbb{R}^{r+1,s+1}$ and the intersection $\tilde G\cap\tilde P'$ is contained in the parabolic $\tilde P$, which fixes the complex null line $l_{\mathbb{C}} := \mathbb{C}\cdot l$ in $\mathbb{C}^{p+1,q+1}$. The group $\tilde P$ acts transitively on the set $\mathbb{R}P^1$ of real null lines in $l_{\mathbb{C}}$ with stabiliser $\tilde G\cap\tilde P'$, i.e., $\tilde P/(\tilde G\cap\tilde P')\cong\mathbb{R}P^1$. We conclude that $\tilde G/(\tilde G\cap\tilde P')$ is a circle bundle over the flat homogeneous model $\tilde G/\tilde P$ of CR-geometry, which is diffeomorphic to the homogeneous model $\tilde G'/\tilde P'$ and thus admits in a canonical way a flat $\tilde G'$-invariant conformal geometry. This describes the (classical) Fefferman construction for the flat model of CR-geometry. We remark that the inclusion $\iota:\tilde G\to\tilde G'$ can be lifted to $\hat G=\mathrm{Spin}(r+1,s+1)$, which automatically gives rise to a conformal spin structure on the Fefferman space $\hat G/\hat P\cong\tilde G'/\tilde P'$ (cf. [**39**]).

It is clear now how to extend the construction to curved CR-geometries. Let $(\tilde{\mathcal{P}},\omega_{nor})\to N$ be the principal $\mathrm{SU}(p+1,q+1)$-bundle with canonical Cartan connection ω_{nor} on a partially integrable CR-space (N,H,J) of dimension $n=2m+1$ admitting a $(m+2)$nd root $\mathcal{E}(1,0)$. Then the Fefferman space $\tilde F_c := \tilde{\mathcal{P}}/(\tilde G\cap\tilde P')$ is a circle bundle over N and admits the conformal Cartan geometry $\tilde{\mathcal{P}}' := \tilde{\mathcal{P}}\times_{G\cap\tilde P'}\tilde P'$ with extended connection ω'. In fact, $\tilde F_c$ admits a natural conformal spin structure. However, the Cartan connection ω', which is induced by extension from ω_{nor} on $\mathcal{P}(N)$ is in general not the unique normal Cartan connection of conformal geometry. This is true iff $(\tilde{\mathcal{P}}, \omega_{nor}) \to N$ has no torsion, which means that the underlying CR-geometry is integrable (cf. [**34, 39**])! In the partially integrable case, the difference $\omega' - \omega'_{nor}$ is described by the Nijenhuis torsion tensor $\mathcal{N}_J$ (cf. Chapter 5).

Since we assume the existence of a $(m+2)$nd root $\mathcal{E}(1,0)$ of the complex line bundle $\Lambda^m_{\mathbb{C}} H \otimes Q$, the standard CR-tractor bundle $\mathcal{T}_{CR}(N) \to N$ with invariant filtration $\mathcal{T}^{-1}_{CR} \supset \mathcal{T}^0_{CR} \supset \mathcal{T}^1_{CR}$ exists, and the Fefferman space $\tilde{F}_c$ over N is canonically identified with the circle bundle of real lines in $\mathcal{T}^1_{CR} \cong \mathcal{E}(1,0)^*$. Furthermore, the lift of $\mathcal{T}_{CR}(N)$ along the Fefferman fibration $\tilde{F}_c \xrightarrow{\pi} N$ is naturally identified with the standard tractor bundle $\mathcal{T}(\tilde{F}_c)$ of conformal (Fefferman) geometry. Since $\mathcal{T}_{CR}(N)$ is equipped with a complex structure J_{CR}, the conformal standard tractor bundle $\mathcal{T}(\tilde{F}_c) \to \tilde{F}_c$ inherits a complex structure (cf. Chapter 5). And, since for integrable CR-geometry the lift of ω_{nor} induces the tractor connection $\nabla^{\mathcal{T}}$ and $\Omega^{\mathcal{T}}(\chi, \cdot) = 0$ for any vertical vector χ in the Fefferman fibration, the tractor holonomy $\mathfrak{hol}(\mathcal{T})$ of $\nabla^{\mathcal{T}}$ on $\mathcal{T}(\tilde{F}_c)$ coincides with the CR-tractor holonomy algebra of ∇^{CR} on $\mathcal{T}_{CR}(N)$. This shows that the conformal holonomy algebra $\mathfrak{hol}(\mathcal{T})$ is reduced for any conformal Fefferman space (in the integrable case) to (a subalgebra of) $\mathfrak{su}(p+1, q+1)$ (cf. Chapter 6 and [**113, 40**]).

Finally, we review the explicit construction of a Fefferman metric on the canonical circle bundle of an integrable, oriented CR-space with respect to a pseudo-Hermitian form θ due to J.M. Lee (cf. [**102**]). The conformal class of the Fefferman metric is a CR-invariant, since it does not depend on the choice of pseudo-Hermitian form, which is shown by direct calculations of the transformation properties (cf. [**102**] and Chapter 5). Note that Lee's intrinsic construction is equivalent to Fefferman's original construction, since the canonical bundle is always trivial for pseudoconvex boundaries. However, Lee's construction differs from the above construction in the context of Cartan geometry, since the canonical bundle is not isomorphic to $\tilde{F}_c$, in general. (In fact, $\tilde{F}_c$ (if it exists) corresponds to a $(m+2)$nd root of the canonical line bundle!) Locally, both constructions of Fefferman conformal classes do always exist and are equivalent. Also, note that there is a direct construction for Fefferman metrics via tractor calculus (cf. [**39**]). We review Lee's construction very briefly here, since the generalised version for partially integrable CR-spaces is completely analogous and will be discussed in more detail in Chapter 5.

Let (N^n, T_{10}) be an integrable, oriented CR-manifold of dimension $n = 2m+1$ and signature (p,q) and let θ be a pseudo-Hermitian structure on N. We denote by

$$\Lambda^{m+1,0}N \ := \ \{ \ \rho \in \Lambda^{m+1}N \otimes \mathbb{C} : \ \iota_X \rho = 0 \ \text{ for all } X \in T_{01} = \overline{T_{10}} \ \}$$

the complex line bundle over N^n, which is the $(m+1)$st exterior power of the annihilator of T_{01}. The bundle $\Lambda^{m+1,0}N$ is called the canonical line bundle of the CR-space (N, T_{10}). The positive real numbers $\mathbb{R}_+$ act by multiplication on $K^* := \Lambda^{m+1,0}N \setminus \{0\}$, which denotes the canonical line bundle with deleted zero section. Then we set $F_c := K^*/_{\mathbb{R}_+}$ and the triple

$$(\ F_c, \pi, N \ )$$

denotes the canonical S^1-principal bundle over (N, T_{10}) whose fibre action is induced by complex multiplication with elements of the unit circle S^1 in $\mathbb{C}$.

The Tanaka-Webster connection ∇^W uniquely gives rise to a connection 1-form on the S^1-principal fibre bundle F_c, which we denote by $A^W : TF_c \to i\mathbb{R}$. Further, we set

$$A_\theta := A^W - \frac{i}{2(m+1)} scal^W \theta \ ,$$

where θ denotes the pull-back to F_c of θ on N. The latter is a connection 1-form on F_c as well, which we call the *Weyl connection* to θ on F_c. The Fefferman metric to θ on F_c is defined by

$$f_\theta \ := \ \pi^* L_\theta - i\frac{4}{m+2}\pi^*\theta \circ A_\theta$$

(in shorter notation, we simply write $f_\theta = L_\theta - i\frac{4}{m+2}\theta \circ A_\theta$). This is, in fact, a symmetric 2-tensor on the real tangent bundle of F_c with signature $(2p+1, 2q+1)$. If the underlying space is strictly pseudoconvex the signature of f_θ is Lorentzian, i.e., $(1, 2m+1)$. The Fefferman conformal class $[f_\theta]$ consists of all smooth metrics $\tilde{f}_\theta$ on F_c, which arise by conformal rescaling of f_θ. The class $[f_\theta]$ depends only on the underlying CR-structure T_{10} of N. In fact, rescaling the pseudo-Hermitian form by $\tilde{\theta} := e^{2l}\theta$ with some real function l on N produces the conformally changed Fefferman metric $f_{\tilde{\theta}} = e^{2l} f_\theta$.

There exists a famous characterisation result for Fefferman metrics of integrable CR-spaces due to G. Sparling in [**146**] and reviewed by C.R. Graham in [**73**].

Theorem 3. *(Sparling's characterisation) Let (M^{n+1}, g) be a pseudo-Riemannian space of dimension $n+1 \geq 4$ and signature $(2p+1, 2q+1)$. Suppose that g admits a Killing vector j (i.e., $\mathcal{L}_j g = 0$) such that*

(1) *$g(j,j) = 0$, i.e., j is lightlike,*
(2) *$\iota_j W^g = 0$ and $\iota_j C^g = 0$,*
(3) *$Ric^g(j,j) > 0$ on M.*

Then g is locally isometric to the Fefferman metric of some integrable CR-space (N, H, J) of hypersurface type with signature (p,q) and dimension n.

On the other hand, any Fefferman metric of an integrable CR-space (N, H, J) of hypersurface type admits a Killing vector field j satisfying (1) to (3).

We remark that it was shown in [**117**] in dimension 4 and for arbitrary even dimension in [**17**] that the Fefferman space of a (strictly pseudoconvex) integrable CR-space admits a certain (pair of) twistor spinor $\phi \in \Gamma(\mathcal{S})$, whose Dirac current V_ϕ is null and twisting. In fact, the Dirac current V_ϕ of this particular twistor spinor satisfies the conditions of Sparling's characterisation (cf. Theorem 1).

13. Summary of the Developments in Part 2

In the final section of this introductory chapter we summarise in brief the investigations and results of Part 2 (Chapter 1 to 6) of the present *Habilitationsschrift*. The work to these investigations was mainly done since the year 2003 and contains an essential part of the author's studies since his PhD Thesis [**106**]. Several parts of the following discussion are published already in articles. Some other parts are new or occur as preprints. In any case we tried to create a unity from the various sources. In particular, notations and conventions should be uniform throughout the chapters and should be in accordance with the current introductory chapter.

Concerning the topics of Part 2 we can certainly say that it is about conformal geometry. In fact, the main motivation of our studies is *conformal invariant theory*. However, we do not touch in the first place the construction of (new) conformal invariants (which is a very interesting research field and which we hope to study increasingly more in the future), but rather try to solve (well known) invariant equations and to describe the (local) conformal geometry of spaces admitting solutions. Thereby, the main character of the studied equations is the *overdeterminateness* and as a consequence the existence of solutions is related to *symmetries* of the underlying geometry. The study of such equations with symmetries occurs in various context and, in particular, is of much interest in geometry, but also in mathematical physics. In fact, our emphasis in many situations is on *conformal Lorentzian geometry*, which provides e.g. the link to *supergravity* as it occurs nowadays in string theories. Also the *Poincaré-Einstein model* in relation with the *holographic principle* of physics and the *AdS/CFT correspondence* are to mention in this direction. To be more concrete, a main motivation for us is for example the *twistor equation* of spin geometry, which can be seen from a certain point of view as the most basic (overdetermined) invariant first order differential equation of conformal geometry. But also *conformal Killing forms* and (ordinary) conformal Killing vectors fields are very much in the centre of interest in the following chapters. Moreover, we will see that these symmetries have to do with a conformal holonomy theory. In fact, *conformal holonomy* will be a leading theme throughout the different chapters (apart from Chapter 4). In detail, the following will happen.

In Chapter 1 we discuss the equation $\nabla^{nor}\alpha = 0$ for $(p+1)$-form tractors on a conformal space (M^n, c) of dimension n. This discussion is based on [**110, 111**] (see also e.g. [**150, 88, 148, 14, 143, 37, 35, 51, 72, 80**]). The projecting part $\alpha_- = \Pi_H(\alpha)$ of a solution α is a conformal Killing p-form of weight $p+1$, which satisfies in addition certain *normalisation conditions*. We compute these conditions with respect to a compatible metric $g \in c$ (which basically results in the computation of the splitting operator $\mathbf{S}$ in this particular case) and also describe the *integrability conditions* for the existence of *normalised* conformal Killing p-forms. On (conformal) Einstein spaces we are able to describe the underlying conformal geometry of spaces with solutions α via the metric cone construction (cf. Theorem 4 and [**12, 143**]). In fact, the conformal Einstein condition itself is described by the existence of a $\nabla^{\mathcal{T}}$-parallel standard tractor (cf. [**28, 93, 64, 92, 6, 7, 65, 69, 122, 123, 3**]).

Obviously, the equation $\nabla^{nor}\alpha = 0$ is closely related to the holonomy theory of the canonical connection ω_{nor} (resp. $\nabla^{\mathcal{T}}$) of conformal geometry (cf. [**6**]). We will show that this has some interesting consequences and applications. First of all, we will prove a conformal analog of the well known *deRham decomposition Theorem* of semi-Riemannian geometry (cf. Theorem 6). Roughly speaking, the essential part of the statement is that the conformal holonomy group decomposes if and only if there exists a product of Einstein metrics in the conformal class c on M. Thereby, the scalar curvatures of the factors satisfy a certain relation (cf. Proposition 6), which will also be of interest in Chapter 2. We want to point out that the *conformal Einstein condition* has a very useful interpretation in terms of conformal holonomy as well. Namely, a space (M, c) is conformally *almost-Einstein* if and only if the conformal holonomy is reduced such that it fixes a parallel standard tractor. This holonomy condition might be very pleasant to test in concrete situations. We will demonstrate this in Chapter 3.

Conformal Killing spinors have the remarkable property that the corresponding twistors (= spinorial tractors) are ∇^{nor}-parallel (cf. [**137, 61, 19, 32**]). Having the spinorial squares at hand it becomes immediately clear that a conformal Killing spinor gives rise to (a number of) normalised conformal Killing forms of certain degree (depending on dimension and signature) on the underlying conformal spin space (M, c). A particularity of (conformal) Lorentzian geometry is the fact that a conformal Killing spinor and its Dirac current (= spinorial square of degree 1) have identical zero sets. The Dirac current is a normalised conformal Killing vector field, and therefore, the corresponding adjoint tractor admits a unique orbit type under the adjoint action of the Möbius group $\mathrm{O}(2, n)$. We will classify the possible orbit types by use of Table 3. The essential outcome of the classification is the existence of exactly four generic types of 2-forms (resp., skew-adjoint endomorphisms) in signature $(2, n)$, which come from a spinor. According to these four generic types we give in Theorem 10 a locally complete geometric description of Lorentzian spin spaces admitting conformal Killing spinors without *singularities*. Among these geometries are *static monopoles, Brinkmann waves*, the *Fefferman spaces*, the *Einstein-Sasaki spaces* and a product class of the form as already mentioned above. This description extends classification results of [**117, 17, 106, 21**].

The metric cone construction for Einstein spaces comes very close to the Fefferman-Graham ambient metric construction for the corresponding conformal class. In fact, as one can almost guess, only a parallel real line needs to be added to the cone in order to obtain the ambient metric. We are able to generalise this idea in Chapter 2, which is based on the author's common work [**68**] with Prof. A.R. Gover from the University of Auckland. In fact, in Theorem 11, we explicitly construct the Fefferman-Graham ambient metric for any space with decomposable conformal holonomy (which implies the existence of a certain product metric in the conformal class as we mentioned above). The interesting point about this construction is that the underlying conformal geometry is not Einstein, in general. We show this point by identifying the conformal holonomy with the holonomy of the ambient metric (cf. Theorem 15). (In general, these two holonomy groups will not have much in common.)

The Fefferman-Graham ambient metric is closely related to the Poincaré-Einstein model with conformal boundary. In fact, the Poincaré-Einstein metric can be realised, in general, on a certain submanifold of the ambient metric space. On the other hand, we show in Section 2.4 that via the cone construction an ambient metric space with boundary can be retrieved from the Poincaré-Einstein model. In this sense both models are shown to be equivalent. The explicit Poincaré-Einstein metric (on a collar) for a boundary space with decomposable conformal holonomy is presented in Theorem 13. We also identify there the Taylor serious expansion with respect to the special defining function of the product boundary (cf. [**74, 87**]). Although our construction works only for a rather special class of conformal structures (as the Taylor expansion indicates, since it terminates already after the 4th order term), to our knowledge it is one of the first explicit constructions with non-Einstein conformal boundary (cf. [**131**] for another explicit construction and [**77, 26, 104, 5, 125**] for pure existence results). In Section 2.6 we describe our Poincaré-Einstein metrics by the existence of certain *special* Killing forms (cf. [**143**]). We want to point out again that the Poincaré-Einstein model (and the ambient metric construction as well) are of much interest in geometry and physics, e.g. in relation with the study of conformal invariants (like *Branson's Q-curvature*), *renormalised volume* and related quantities (cf. [**1, 5, 47, 55, 74, 79, 87**]), or the *AdS/CFT programme* (cf. e.g. [**124, 78**]).

From what we have said so far it becomes clear that the conformal holonomy plays a central role for our investigations. In Chapter 3 we aim to show that a conformal holonomy group can be computed explicitly. For this purpose we develop an invariant calculus for the normal conformal Cartan connection of a group with bi-invariant metric. It is no problem then to apply the classical iterative formula for the holonomy of an invariant connection (cf. e.g. [**91**]). We will do this in the concrete situations of the bi-invariant metrics on SO(3) and SO(4), which are coming from the corresponding Killing forms. Of course, the bi-invariant metric on SO(3) is conformally flat and our invariant calculus shows that the conformal holonomy vanishes in this case. On the other hand, the group SO(4) is locally the product of two round spheres, which is not conformally flat. Indeed, our computation shows that the conformal holonomy algebra $\mathfrak{hol}(\mathfrak{T})$ of the bi-invariant metric on SO(4) is $\mathfrak{so}(7)$. In particular, this result shows that up to constant rescaling the bi-invariant metric on SO(4) is the only Einstein metric in its conformal class. The content of Chapter 3 originated from [**109**]. Our ideas can be extended to much more general circumstances in the framework of homogeneous parabolic geometries as the work [**83**] shows. The latter work also describes nicely how the automorphism group of a homogeneous parabolic geometry can be set into relation with an invariant holonomy theory.

In Chapter 4, which is based on the sources [**105, 108, 115**], our interest in particular solutions of conformally invariant differential equations becomes very obvious. The concrete task is to solve the twistor equation for spinors in (conformal) Lorentzian spin geometry such that the solution admits a *zero*, i.e., the twistor spinor vanishes at a point (cf. [**115**]). We want to briefly motivate why this problem is interesting. In Riemannian geometry one can easily show with classical methods that a twistor spinor with zero is conformally equivalent to a parallel spinor off the zero set (cf. [**19**]). On a compact space a twistor spinor has a zero only if it it is the round sphere, which is conformally flat (cf. [**120**]). On non-compact spaces twistor spinors with zero occur on the conformal completion to infinity of asymptotically flat Riemannian spaces with irreducible (or trivial) holonomy group (cf. [**96, 99**]). As we have mentioned already before, in Lorentzian geometry the Dirac current of a twistor spinor is a conformal Killing vector field with the same zero set as the corresponding spinor. The existence of zeros for the Dirac current of a twistor spinor makes it possible to determine, which normal form the corresponding ∇^{nor}-parallel adjoint tractor can take (cf. Table 3). In fact, this approach makes it possible to show that a twistor spinor with zero is conformally equivalent to a parallel spinor off its zero set (cf. Theorem 19). Thus we obtain the same result as in (conformal) Riemannian geometry, but it seems that the proof of this result in Lorentzian geometry does not work without employing conformal tractor calculus!

Twistor spinors with zeros in Lorentzian geometry are also interesting for the following reason. The corresponding Dirac current is an *essential* conformal Killing vector field on the underlying conformal space (M, c), i.e., the Dirac current is *not* a Killing vector field with respect to any metric $g \in c$ and thus expresses the existence of a *true* conformal symmetry (cf. [**2, 4, 60**]). In Riemannian geometry essential conformal transformation groups are very rare. In fact, a celebrated result says that a complete Riemannian space with essential conformal transformation group is either the round sphere S^n (compact case) or the Euclidean space $\mathbb{R}^n$ (non-compact case), which are both conformally flat spaces (cf. [**134, 116, 2**]). In Lorentzian geometry conformally non-flat spaces with essential transformation group are more or less unknown (cf. [**106, 60**]). The second main achievement of Chapter 4 is the explicit construction of a family of non-compact Lorentzian spaces in dimension 5, which admit twistor spinors with isolated zero, such that the conformal geometry around that zero is not flat (cf. Theorem 20). In particular, this result shows that there exist conformally non-flat Lorentzian spaces with essential conformal transformation group. Our construction is based on the Eguchi-Hanson metric in dimension 4 (which is an asymptotically flat hyperKähler metric in Riemannian geometry) and can be seen as the conformal completion of spaces, which have the asymptotic behaviour of the Minkowski space in dimension 5. However, the metric that we explicitly construct is only continuously differentiable. Nevertheless, it might be possible that the induced conformal class is smooth. This remains unclear here.

So far we have not mentioned CR-geometry in this summary. However, the two final chapters of our work are very much concerned with it. The original motivation for our interest in CR-geometry was again the twistor equation of conformal Lorentzian spin geometry. The discussed topics fit also very well into the theme of conformal holonomy theory. The main point in Chapter 5 and 6 is that via the Fefferman construction CR-geometry is closely related to conformal geometry. In fact, the Fefferman construction produces from CR-structures certain conformal spaces, which admit particular properties concerning its holonomy and the existence of solutions of certain conformally invariant differential equations (cf. [**117, 17, 40**]).

Chapter 5 is based on the source [**112**]. We discuss there a generalised Tanaka-Webster connection and corresponding curvature expressions for *partially integrable* CR-spaces of hypersurface type equipped with a pseudo-Hermitian form. This is done along the lines of [**17**] (cf. also e.g. [**149, 152, 67, 102, 103, 48, 39**]). Thereby, the Tanaka-Webster connection is determined by a certain torsion normalisation, which includes the Nijenhuis tensor (cf. Lemma 17). In Section 5.3 we construct the Fefferman space to a partially integrable CR-structure in the spirit of J.M. Lee by use of a pseudo-Hermitian form and a corresponding Weyl connection form on the canonical circle bundle (cf. [**102**]). This Fefferman construction is shown to be independent from the choice of pseudo-Hermitian form and thus is a CR-invariant construction (cf. Theorem 21). Surprisingly, a very simple consideration shows that the Fefferman construction is still independent from the choice of pseudo-Hermitian form if we replace the corresponding Weyl connection by an arbitrary connection 1-form. We call this construction the *gauged* Fefferman construction of partially integrable CR-geometry (cf. Definition 4). This construction seems to be utterly redundant on the first glance. And at least from the viewpoint of CR-geometry it probably really is. However, we will see in Chapter 6 that it is *good* for something in the realm of conformal geometry. In any case, in the following sections of Chapter 5, we compute the scalar curvature of the gauged Fefferman metric (with respect to the Webster scalar curvature; Theorem 22) and apply the Bochner-Laplacian to the fundamental vector field χ_K, which is vertical in the Fefferman fibration and which is a Killing vector for the gauged Fefferman metric (cf. Proposition 16). Finally, these computations enable us to apply the splitting operator $\mathbf{S}$ to $2\chi_K$ and to identify the result. On this way, we actually compute, in general, the splitting operator $\mathbf{S}$ acting on 1-forms with respect to a compatible metric in a given conformal class. The outcome of the whole computation in the case of a gauged Fefferman space shows that $\mathbf{S}(2\chi_K)$ is an orthogonal complex structure acting on standard tractors, at least under certain conditions on the *gauge* 1-*form* ℓ. The conclusion of Chapter 5 (cf. Theorem 23) is that we have constructed a family of solutions $\mathcal{J}$ of the tractor equation

$$\nabla^{nor}\mathcal{J} = -\Omega^{nor}(\Pi_H(\mathcal{J}), \cdot) \qquad \text{with} \qquad \mathcal{J} \bullet \mathcal{J} = -id|_{\mathcal{T}} .$$

Finally, in Chapter 6 we aim to do three things. First, we consider solutions $\mathcal{J}$ of the above tractor equation on a conformal space (M, c). It turns out that locally there exists a natural construction, which produces a partially integrable CR-structure on a quotient space in one dimension lower (cf. [**39**]). In fact, we can show further that locally the conformal class c on M is given by a Fefferman metric for a certain gauge over some partially integrable CR-space of hypersurface type, i.e., together with results from Chapter 5 we have found a construction principle and a locally complete characterisation of conformal spaces admitting solutions $\mathcal{J}$ of the above equation (cf. Theorem 24). (So much to the justification of our gauged Fefferman construction!)

The second point we make in Chapter 6 is a holonomy characterisation of the classical Fefferman construction for integrable CR-structures. This result was proved in [**113**] (cf. also [**40**]). We can show by use of tractor calculus that a conformal space (M, c), whose conformal tractor holonomy $Hol(\mathcal{T})$ sits in the unitary group $\mathrm{U}(p+1, q+1)$ is locally conformally equivalent to a classical Fefferman metric induced on the canonical circle bundle of an integrable CR-space. Actually, more than this true, since we know that the conformal holonomy algebra $\mathfrak{hol}(\mathcal{T})$ of a classical Fefferman space to an integrable CR-structure is always contained in the special unitary algebra $\mathfrak{su}(p+1, q+1)$ (cf. Theorem 25). In particular, this result implies that there exists no conformal space with tractor holonomy identical to the full $\mathrm{U}(p+1, q+1)$.

The holonomy characterisation of the classical Fefferman construction in combination with the holonomy characterisation of the conformal Einstein condition leads us to make our final stroke. It is well known that a Fefferman metric is never Einstein. But, obviously, if a space has a conformal holonomy group sitting in $\mathrm{U}(p+1, q+1)$, which, in addition, fixes a standard tractor, then this shall be the conformal holonomy of a Fefferman space, which is almost conformally Einstein. Since there is no reason to believe that such a conformal holonomy reduction does not exist (apart from the flat spaces), we shall be able to find conformal Fefferman-Einstein spaces. In fact, based on the work [**114**], we describe in the final sections of Chapter 6 a construction principle of so-called (TSPE)-structures. A (TSPE)-structure is a pseudo-Einstein structure (in the sense of Lee; cf. [**103**]), which induces to the same time a *transverse symmetry* on the underlying integrable CR-space in form of the *Reeb vector* field. Such a (TSPE)-structure can be constructed from any Kähler-Einstein space, and in fact, we show in Theorem 26 that any (TSPE)-structure comes locally by a natural construction from some Kähler-Einstein space. Since a (TSPE)-structure corresponds to a ∇^{nor}-parallel standard CR-tractor, and further, this tractor lifts in the Fefferman construction to a ∇^{nor}-parallel conformal standard tractor, it is clear that a Fefferman metric constructed over an integrable CR-space with (TSPE)-structure must be (almost) conformally Einstein. In fact, we are able to determine the conformal rescaling factor explicitly, which makes the Fefferman metric of a (TSPE)-structure locally to an Einstein metric. The local expressions for the resulting metrics are presented in Theorem 27. Globally, on a circle bundle the Fefferman metrics are never conformally Einstein. This shows that apart from Einstein metrics there might exist other naturally distinguished metrics in a conformal class, resp., there might be occasions when one should consider the almost-Einstein property for the conformal class as the most natural condition.

Part 2

Developments of the Theory

CHAPTER 1

Conformal Killing Forms with Normalisation Condition

Conformal Killing vectors and, more generally, conformal Killing p-forms (as defined in Section 0.7) are solutions of conformally covariant PDE's on spaces equipped with conformal structure. The corresponding PDE's are typically overdetermined, which means that the existence of a solution is subject to integrability conditions in form of (conformal) curvature properties of the underlying space. The solutions itself can be seen as incidences of (higher infinitesimal) symmetries. In particular, conformal Killing p-forms are closely linked to symmetries of the (conformal) Laplacian and other related covariant differential operators (cf. e.g. [**23, 51**]). For this and many other reasons, the investigation of conformal Killing p-forms is of much interest both in geometry and physics (cf. e.g. [**88, 89, 147, 148, 24, 136, 72, 111, 143, 110**]).

We want to discuss in this chapter the equation

$$\nabla^{nor}\mathbf{S}(\alpha) = 0 \ ,$$

where $\alpha \in \Omega^p(M)[p+1]$ is a p-form of conformal weight $p+1$ on a space (M,c) with conformal structure, $\mathbf{S} : \Omega^p(M)[p+1] \to \Omega^{p+1}(\mathcal{T}^*M)$ denotes the splitting operator and ∇^{nor} is the tractor connection as introduced in Section 0.7 coming from the normal conformal Cartan connection ω_{nor} on $\mathcal{P}(M)$. The equation $\nabla^{nor}\mathbf{S}(\alpha) = 0$ is a conformally covariant, overdetermined PDE for conformally weighted differential forms α. It turns out that a solution α of this tractor equation is nothing else, but a conformal Killing p-form on (M,c). However, it is important to observe that such solutions satisfy certain additional *normalisation conditions* (as we will see below) and not every conformal Killing p-form is a solution. Consequently, we call a solution α of $\nabla^{nor}\mathbf{S}(\alpha) = 0$ a *normal conformal Killing p-form* (in short, *nc-Killing p-form*) (cf. [**110, 111**]).

Obviously, since the corresponding $(p+1)$-form tractor $\mathbf{S}(\alpha)$ of a solution α is ∇^{nor}-parallel, the existence of a nc-Killing p-form α is intimately linked to the holonomy theory of ω_{nor} on $\mathcal{P}(M)$, resp., ∇^{nor} on the standard tractor bundle $\mathcal{T}(M)$. For example, if $\mathbf{S}(\alpha)$ is ∇^{nor}-parallel and simple as $(p+1)$-form tractor with $1 < p < n$, then this implies a decomposition of the standard tractor module $\mathbb{R}^{r+1,s+1}$ into a direct sum under the action of the conformal tractor holonomy $Hol(\mathcal{T})$. We will show in this situation with help of α that (at least locally) a certain product metric in the conformal class c on M does exist (cf. Section 1.3). This result should be seen as a conformal analog for the *deRham splitting Theorem* of semi-Riemannian geometry (cf. [**141**]). A special case is the existence of a ∇^{nor}-parallel standard tractor $t \in \Gamma(\mathcal{T})$ on (M,c), which is related to the conformal (almost)-Einstein condition (cf. [**64, 65, 69, 6**]).

Furthermore, the *twistor equation* for spinors is a basic differential equation of conformal geometry and of particular interest in general relativity (cf. [**138**]). In Section 0.8 we did explain that conformal Killing spinors φ correspond via the splitting operator $\mathbf{S}$ to ∇^{nor}-parallel sections $\Psi := \mathbf{S}(\varphi) \in \Gamma(\mathcal{W})$ of the standard spinorial tractor bundle $\mathcal{W}$ on a conformal spin manifold (M,c). It follows immediately that the (non-trivial) spinor squares $\varsigma_p(\Psi) \in \Omega^p(\mathcal{T}^*M)$, $1 \leq p \leq n+1$, are ∇^{nor}-parallel and correspond via the projections Π_H to nc-Killing $(p-1)$-forms on (M,c). All this shows that the existence of conformal Killing spinors on (M,c) is closely linked to the holonomy theory of the tractor connection $\nabla^{\mathcal{T}}$. Again, it becomes clear that the conformal holonomy group $Hol(\mathcal{T})$ and its algebra $\mathfrak{hol}(\mathcal{T})$ are important invariants of conformal geometry, which are suitable to indicate symmetries and solutions of basic conformally covariant differential equations. In Section 1.6 we will pay special attention to the twistor equation of *Lorentzian spin geometry*. In fact, using our discussion of nc-Killing forms we are able to prove a more less complete structure result for solutions of the twistor equation without *singularities* (cf. Theorem 10).

We will proceed as follows. In the first section we describe the action of the canonical connection ∇^{nor} on p-form tractors with respect to a metric g in the conformal class c on M and derive the normalised equations for nc-Killing forms with integrability conditions. This discussion is followed by considerations about nc-Killing forms on (conformally) Einstein spaces (cf. Section 1.2) and in general (cf. Section 1.4). In Section 1.3 we will establish a version of a deRham-like splitting theorem in conformal geometry (cf. Theorem 6). Furthermore, in Section 1.5 we will present a normal form classification of 2-forms in signature $(2,n)$. This classification will enable us to prove in the final section the geometric structure result for solutions of the twistor equation in conformal Lorentzian spin geometry.

1. The Normalised Equations for Conformal Killing p-Forms

Let (M^n,c) be a connected and oriented space with conformal structure of dimension $n \geq 3$ and signature (r,s) and let $(\tilde{P}(M), \omega_{nor})$ denote the corresponding normal conformal Cartan geometry of type $(\tilde{G},\tilde{P})$ with $\tilde{G} = \mathrm{SO}(r+1,s+1)$. The standard tractor bundle of (M^n,c) is given by $\mathcal{T}(M) = \tilde{\mathcal{P}}(M) \times_{\tilde{P}} \mathbb{R}^{r+1,s+1}$. Furthermore, let $g \in c$ denote a compatible metric with local orthonormal frame $\mathcal{B}^s = \{s_1,\dots,s_n\}$. The metric g induces a Weyl structure $\sigma^g : \tilde{\mathcal{G}}_0(M) \to \tilde{\mathcal{P}}(M)$ (cf. Section 0.6). Then, via the natural embedding of $\tilde{\mathcal{P}}(M)$ into the extended $\tilde{G}$-bundle $\tilde{\mathcal{G}}(M)$, the Weyl structure σ^g gives rise to a local frame $\mathcal{B}^{\mathcal{T}} = \{s_-, s_+, s_1, \dots, s_n\}$ of the standard tractor bundle $\mathcal{T}(M) \xrightarrow{\pi} M$, which is an extension of the orthonormal frame $\mathcal{B}^s$, where the vector s_- denotes the unit in $\mathcal{E}[1]$ and s_+ denotes the unit in $\mathcal{E}[-1]$ with respect to $g \in c$.

Now let us consider the $(p+1)$-form tractor bundles $\Lambda^{p+1}\mathcal{T}^*(M)$ on (M,c), which split with respect to $g \in c$ into

$$\Lambda^p(M) \oplus \Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M) \oplus \Lambda^p(M)$$

(cf. Section 0.7). Accordingly, any section α in $\Lambda^{p+1}\mathcal{T}^*(M)$ can locally be written with respect to $g \in c$ and a frame $\mathcal{B}^{\mathcal{T}}$ as

$$s_-^\flat \wedge \alpha_- \ + \ \alpha_0 \ + \ s_-^\flat \wedge s_+^\flat \wedge \alpha_\mp \ - \ s_+^\flat \wedge \alpha_+$$

with uniquely determined differential forms $\alpha_-, \alpha_0, \alpha_\mp$ and α_+. The normal Cartan connection ω_{nor} gives rise to a covariant derivative ∇^{nor} acting on sections of $\Lambda^{p+1}\mathcal{T}^*(M)$. In general, the action of ∇^{nor} on any section γ of some tractor bundle with respect to a compatible metric $g \in c$ is described by

$$\nabla_X^{nor}\gamma \ = \ \nabla_X^g\gamma \ + \ (\ X\ ,\ 0\ ,\ \mathsf{P}^g(X)\ ) \bullet \gamma$$

(cf. Section 0.7). A concrete calculation of $\bullet$ shows that ∇^{nor} acts on sections of the $(p+1)$-form tractor bundles $\Lambda^{p+1}\mathcal{T}^*(M)$ by

$$\nabla^{nor}_X\alpha = \begin{pmatrix} \nabla^g_X & -\mathsf{P}(X)\wedge & -\iota_{\mathsf{P}(X)^\sharp} & 0 \\ -\iota_X & \nabla^g_X & 0 & \iota_{\mathsf{P}(X)^\sharp} \\ -X^\flat\wedge & 0 & \nabla^g_X & -\mathsf{P}(X)\wedge \\ 0 & X^\flat\wedge & -\iota_X & \nabla^g_X \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_\mp \\ \alpha_0 \\ \alpha_- \end{pmatrix} .$$

In the following, we call a section $\alpha \in \Omega^{p+1}(\mathcal{T}^*M)$ a parallel $(p+1)$-form tractor iff $\nabla^{nor}\alpha = 0$. With respect to a metric $g \in c$ this tractor equation is expressed by

$$\nabla^g_X\alpha_- - \iota_X\alpha_0 + X^\flat\wedge\alpha_\mp = 0 \tag{15}$$

$$-\mathsf{P}(X)\wedge\alpha_- + \nabla^g_X\alpha_0 - X^\flat\wedge\alpha_+ = 0 \tag{16}$$

$$\iota_{\mathsf{P}(X)^\sharp}\alpha_- + \nabla^g_X\alpha_\mp - \iota_X\alpha_+ = 0 \tag{17}$$

$$-\iota_{\mathsf{P}(X)^\sharp}\alpha_0 - \mathsf{P}(X)\wedge\alpha_\mp + \nabla^g_X\alpha_+ = 0 . \tag{18}$$

We aim to investigate solutions of the equation $\nabla^{nor}\alpha = 0$ on a space (M,c) (with compatible metric g). So let α be such a parallel $(p+1)$-form tractor with natural projection $\Pi_H(\alpha) \in \Omega^p(M)[p+1]$ and let α_- denote the corresponding p-form with respect to g on M. We can easily recompute the other differential forms $\alpha_0, \alpha_\mp$ and α_+. Remember that

$$d = \sum_{i=1}^n \varepsilon_i s_i^\flat \wedge \nabla^g_{s_i} \qquad \text{and} \qquad d^* = -\sum_{i=1}^n \varepsilon_i \iota_{s_i}\nabla^g_{s_i} .$$

The equations (15) - (17) imply the identities

$$\begin{aligned} d\alpha_- &= (p+1)\alpha_0 , \\ d^*\alpha_- &= (n-p+1)\alpha_\mp , \\ \frac{1}{p+1}d^*d\alpha_- &= (p-n)\alpha_+ - \sum_{i=1}^n \varepsilon_i\iota_{s_i}(\mathsf{P}(s_i)\wedge\alpha_-) , \\ \frac{1}{n-p+1}dd^*\alpha_- &= p\alpha_+ - \sum_{i=1}^n \varepsilon_i s_i^\flat\wedge(\iota_{\mathsf{P}(s_i)^\sharp}\alpha_-) . \end{aligned}$$

For $n \neq 2p$ the sum of the two latter identities gives

$$\alpha_+ = \frac{1}{n-2p}\cdot\Big(\frac{scal^g}{2(n-1)}\alpha_- - \frac{1}{p+1}d^*d\alpha_- - \frac{1}{n-p+1}dd^*\alpha_-\Big),$$

which is

$$\alpha_+ = \frac{1}{n-2p}\Big(\Delta^g_{tr} + \frac{scal^g}{2(n-1)}\Big)\alpha_- ,$$

where $\Delta^g_{tr} = tr^g\nabla^2$ is the Bochner-Laplacian. If $n = 2p$ we have

$$\alpha_+ = \frac{-1}{(n-p)(p+1)}d^*d\alpha_- - \frac{1}{n-p}\sum_{i=1}^n \varepsilon_i\iota_{s_i}(\mathsf{P}(s_i)\wedge\alpha_-) .$$

In any case, we observe that $\alpha_- \equiv 0$ if and only if the $(p+1)$-form tractor α is trivial. In fact, we have simply computed here the splitting operator $\mathbf{S}$ acting on α_- with respect to a metric g (cf. Section 0.7).

With the derived expressions for the components $\alpha_0, \alpha_\mp$ and α_+ of a parallel $(p+1)$-form tractor α inserted into (15) - (18) we obtain the equations

$$0 = \nabla^g_X \alpha_- - \frac{1}{p+1}\iota_X d\alpha_- + \frac{1}{n-p+1} X^\flat \wedge d^*\alpha_- \tag{19}$$

$$0 = -\mathsf{P}(X)\wedge \alpha_- + \frac{1}{p+1}\nabla^g_X d\alpha_- - X^\flat \wedge \Box_p \alpha_- \tag{20}$$

$$0 = \iota_{\mathsf{P}(X)^\sharp}\alpha_- + \frac{1}{n-p+1}\nabla^g_X d^*\alpha_- - \iota_X \Box_p \alpha_- \tag{21}$$

$$0 = \frac{-1}{p+1}\iota_{\mathsf{P}(X)^\sharp} d\alpha_- - \frac{1}{n-p+1}\mathsf{P}(X)\wedge d^*\alpha_- + \nabla^g_X \Box_p \alpha_- \tag{22}$$

for $\alpha_- \in \Omega^p(M)$, where we set

$$\Box_p := \frac{1}{n-2p}\cdot \big(\Delta^g_{tr} + \frac{scal^g}{2(n-1)} id\big) \qquad \text{for } n \neq 2p$$

and

$$\Box_{n/2} := \frac{-1}{n-p}\cdot \Big(\frac{1}{p+1} d^* d \ + \ \sum_{i=1}^n \varepsilon_i \iota_{s_i}(\mathsf{P}(s_i)\wedge \cdot) \Big) \ .$$

Obviously, if only the first of this set of equations is satisfied then the p-form α_- is conformal Killing (cf. (5) in Section 0.6). Hence we call (19) - (22) the normalised conformal Killing form equations and a solution $\alpha_- \in \Omega^p(M)$ is a *normal conformal Killing p-form* (in short, *nc-Killing p-form*). The conformal covariance of the equations implies that if α_- is a nc-Killing p-form to g on M then the rescaled p-form

$$\tilde{\alpha}_- := e^{(p+1)\phi}\cdot \alpha_-$$

is nc-Killing with respect to the conformally changed metric $\tilde{g} = e^{2\phi}\cdot g$.

Equations (19) - (22) are not only conformally covariant. The Hodge star operator $\star$ gives rise to a further symmetry. In order to explain this symmetry we define the tractor Hodge operator $\star_{\mathcal{T}}$ by

$$\alpha \wedge \star_{\mathcal{T}}\alpha = \langle \alpha, \alpha\rangle_{\mathcal{T}} \, dM_{\mathcal{T}} \ ,$$

where we set $dM_{\mathcal{T}} := -s_-^\flat \wedge s_+^\flat \wedge dM$, which is the tractor volume form (given in terms of a frame $\mathcal{B}^{\mathcal{T}}$). The star operator $\star_{\mathcal{T}}$ is parallel: $\nabla^{nor}\star_{\mathcal{T}} = \star_{\mathcal{T}}\nabla^{nor}$. We conclude that if α is a parallel $(p+1)$-form tractor then $\star_{\mathcal{T}}\alpha$ is a parallel $(n-p+1)$-form tractor. The tractor $\star_{\mathcal{T}}\alpha$ has with respect to g the components

$$(\ (-1)^p \star \alpha_- \ , \ \star\alpha_\mp \ , \ -\star\alpha_0 \ , \ (-1)^{p+1}\star \alpha_+ \ ) \ .$$

This proves that if α_- is a nc-Killing p-form then $\star\alpha_-$ is a nc-Killing $(n-p)$-form. Indeed, with

$$\star(\iota_X \beta^p) = (-1)^{p+1} X^\flat \wedge \star\beta \qquad \text{and} \qquad \star\,(X^\flat \wedge \beta^p) = (-1)^p \iota_X \star \beta \ ,$$

$$\star\star|_{\Lambda^p} = (-1)^{p(n-p)+r} \qquad \text{and} \qquad d^* = (-1)^{n(p-1)+r+1}\star d \star \ ,$$

and since $\star\Box_p = -\Box_{n-p}\star$, the normalised equations (19) - (22) are seen to be $\star$-invariant. More generally, it is also true that if α_- is just a conformal Killing p-form then $\star\alpha_-$ is a conformal Killing $(n-p)$-form.

Finally, we discuss here integrability conditions for the existence of nc-Killing p-forms on a semi-Riemannian manifold (M^n, g). Note that the action of the tractor curvature $\Omega^{\mathcal{T}}$ on a $(p+1)$-form tractor $\alpha = (\alpha_-, \alpha_0, \alpha_\mp, \alpha_+)$ is given (in matrix form) by application of

$$\begin{pmatrix} W^g(X,Y)\bullet & -C^g(X,Y)\wedge & -\iota_{C^g(X,Y)^\sharp} & 0 \\ 0 & W^g(X,Y)\bullet & 0 & \iota_{C^g(X,Y)^\sharp} \\ 0 & 0 & W^g(X,Y)\bullet & -C^g(X,Y)\wedge \\ 0 & 0 & 0 & W^g(X,Y)\bullet \end{pmatrix},$$

where $\bullet$ denotes the usual action of $\mathfrak{so}(TM)$ on differential forms. As integrability condition for the existence of a nc-Killing p-form α_- we obtain

$$W^g(X,Y)\bullet\alpha_- = 0 \tag{23}$$

$$W^g(X,Y)\bullet d\alpha_- = (p+1)\cdot C^g(X,Y)\wedge\alpha_- \tag{24}$$

$$W^g(X,Y)\bullet d^*\alpha_- = -(n-p+1)\iota_{C^g(X,Y)^\sharp}\alpha_- \tag{25}$$

$$W^g(X,Y)\bullet\Box_p\alpha_- = \frac{1}{p+1}\iota_{C^g(X,Y)^\sharp}d\alpha_- + \frac{1}{n-p+1}C^g(X,Y)\wedge d^*\alpha_-. \tag{26}$$

Of course, these integrability conditions are conformally covariant and invariant with respect to the Hodge $\star$-operator. Taking the divergence on both sides of (23) - (26) results in

$$\begin{aligned} (n-4)\cdot C^g_X\bullet\alpha_- &= 0 \\ (n-4)\cdot C^g_X\bullet d\alpha_- &= -(p+1)\cdot B^g(X)\wedge\alpha_- \\ (n-4)\cdot C^g_X\bullet d^*\alpha_- &= (n-p+1)\cdot\iota_{B^g(X)^\sharp}\alpha_- \\ (n-4)\cdot C^g_X\bullet\Box_p\alpha_- &= -\frac{1}{p+1}\iota_{B^g(X)^\sharp}\alpha_- - \frac{1}{n-p+1}B^g(X)\wedge\alpha_-\ , \end{aligned}$$

where $C^g_X := C^g(\cdot,\cdot,X)$ for $X \in TM$ and B^g is the Bach tensor (cf. Section 0.4) .

2. Normal Conformal Killing p-Forms on Einstein Manifolds

We call a space (M, g) conformally Einstein if there exists an Einstein metric $\tilde{g}$ in the conformal class $[g]$. We will show that the property for a metric g to be conformally Einstein can be expressed in terms of (dual) standard tractors (cf. [**64, 65, 69, 6**]). Moreover, we will study in this section solutions of the normalised equations (19) - (22) for p-forms on semi-Riemannian Einstein spaces (M, g).

First, let $f_- = \alpha_-$ be a nc-Killing function (= 0-form) without zeros on a space (M, g). We mentioned before that in this case the rescaled function $\tilde{\alpha}_- = \frac{1}{f_-}\alpha_- = 1$ is nc-Killing with respect to the metric $\tilde{g} = \frac{1}{f_-^2} \cdot g$. From the normalised equations (19) - (22) follows immediately

$$\mathsf{P}^{\tilde{g}} \ = \ \frac{-scal^{\tilde{g}}}{2n(n-1)} \cdot \tilde{g} \ ,$$

which means that $\tilde{g}$ is Einstein. On the other hand, every constant function on an Einstein space (M,g) is nc-Killing. Thus we have a criterion, which says that a metric g is conformally Einstein if and only if there exists at least one nc-Killing function without zeros on M. In general, the normalised equations (19) - (22) for a function $f_- = \alpha_-$ on (M,g) reduce to the single second order PDE

$$\textit{trace-free part of } \ (Hess^g(f_-) - f_- \cdot \mathsf{P}^g) \ = \ 0 \ ,$$

which is a classical formulation for the conformal Einstein condition and shows that the kernel of the first differential operator $\mathcal{D}_0$ in the BGG sequence for the standard representation describes the space of parallel sections of $\mathcal{T}(M)$. This is the same behaviour as we observe for the twistor representation (cf. Section 0.8 and 1.6).

Obviously, if a nc-Killing function f_- has zeros the rescaling of the metric g in the above manner is not possible. Examples for nc-Killing functions with zeros on a space (M,c) are well known (cf. e.g. Chapter 4). In general, the set of zeros of a nc-Killing function is singular (i.e. the complement set is dense). In the following we call a space $(M,[g])$, which admits a nc-Killing function (possibly with zeros), an almost (conformally) Einstein space (cf. [**65**]).

Proposition 3. *(cf. [**65**]) A space (M,c) is almost conformally Einstein iff the standard tractor bundle $\mathcal{T}(M)$ (resp., its dual $\mathcal{T}^*(M)$) admits a $\nabla^{\mathcal{T}}$-parallel section.*

One can easily see that if a parallel standard tractor is null, the corresponding Einstein scale g in c is Ricci-flat. Moreover, in the timelike case the Einstein scale has positive scalar curvature $scal^g > 0$, whereas in the spacelike case we have $scal^g < 0$ (up to singularities). The characterisation of Proposition 3 is very useful and becomes even more interesting by the fact that parallel sections in $\mathcal{T}(M)$ can be detected by studying the holonomy of $\nabla^{\mathcal{T}}$.

Proposition 4. *(cf. [**6, 110**]) A space (M,c) is almost conformally Einstein if and only if the holonomy representation of $Hol(\mathcal{T})$ on $\mathbb{R}^{r+1,s+1}$ fixes at least one standard tractor.*

Note in relation with Proposition 4 that in concrete situations it might be more convenient to test whether a standard tractor is fixed by the holonomy representation than to show that a standard tractor is parallel.

For the rest of this section, let us assume that (M^n,g) is a semi-Riemannian Einstein manifold. The 1-form tractor $o := s_-^\flat - \frac{scal^g}{2(n-1)n} s_+^\flat$, which obviously satisfies the equations (15) - (18), is the parallel dual standard tractor that corresponds to the Einstein metric g on $(M,[g])$. The components of o are given with respect to g by $(1,0,0,\frac{scal^g}{2(n-1)n})$, i.e., $o_- = 1$. Now, let

$$\alpha = s_-^\flat \wedge \alpha_- + \alpha_0 + s_-^\flat \wedge s_+^\flat \wedge \alpha_\mp - s_+^\flat \wedge \alpha_+$$

be an arbitrary parallel $(p+1)$-form tractor on (M,g). If $\alpha_0 \equiv 0$ then the $(n-p)$-form $*\alpha_-$ is nc-Killing and coclosed on (M,g). In the other case when $\alpha_0 \not\equiv 0$, it follows immediately that $o\wedge\alpha$ is a parallel $(p+2)$-form tractor, whose components with respect to g are given by

$$(\alpha_0 \ , \ 0 \ , \ \frac{scal^g}{2(n-1)n}\alpha_- - \alpha_+ \ , \ \frac{scal^g}{2(n-1)n}\alpha_0 \ ) \ .$$

This shows that $d\alpha_-$ is a (closed) nc-Killing $(p+1)$-form and again the $(n-p-1)$-form $*d\alpha_-$ is nc-Killing and coclosed. In general, the set of normalised equations (19) - (22) reduces for a coclosed p-form β_- on an Einstein space to

$$\begin{aligned} \nabla^g_X\beta_- &= \frac{1}{p+1}\cdot\iota_X d\beta_- \ , \\ \nabla^g_X d\beta_- &= -\frac{(p+1)\cdot scal^g}{n\cdot(n-1)}\cdot X^\flat\wedge\beta_- \ , \end{aligned}$$

which implies $\Delta_p\beta_- = \frac{(p+1)(n-p)\cdot scal^g}{n\cdot(n-1)}\beta_-$ for the Laplacian $\Delta_p = dd^*+d^*d$. By definition, these two equations determine *special* Killing p-forms to the Killing constant $-\frac{(p+1)\cdot scal^g}{n\cdot(n-1)}$ (not only in the Einstein case; cf. Section 0.4 and [**143**]).

The arguments of the previous paragraph show that any Einstein space, which admits a nc-Killing p-form does also admit a non-trivial special Killing form. Thus we obtain via the cone correspondence and Berger's holonomy list (cf. Section 0.9 and [**12**]) the following structure result in Riemannian geometry.

Theorem 4. *(cf.* [**12, 143**]*) Let (M^n,g) be a simply connected and complete Riemannian Einstein space of positive scalar curvature admitting a nc-Killing p-form. Then M^n is either*

(1) *the round (conformally flat) sphere S^n (up to a constant scale),*
(2) *an Einstein-Sasaki manifold of odd dimension $n\geq 5$ with a special Killing* 1*-form α_-,*
(3) *an Einstein-3-Sasaki space of dimension $n = 4m+3\geq 7$ with a S^2-family of (normed) special Killing* 1*-forms,*
(4) *a nearly Kähler manifold of dimension* 6*, where the Kähler form ω_- is a special Killing* 2*-form or*
(5) *a nearly parallel G_2-manifold in dimension* 7 *with a nice special Killing* 3*-form γ_-.*

The five cases of Theorem 4 correspond to the possible cone holonomies $\{e\}$, $\mathrm{SU}(\frac{n+1}{2})$, $\mathrm{Sp}(\frac{n+1}{4})$, G_2 and Spin(7). Note that the *nice* special Killing 3-form γ_- of case (5) can locally be written with respect to an orthonormal frame $\mathcal{B}^s$ in the form

$$\begin{aligned} \gamma_- \ = \ & s_1^\flat\wedge s_2^\flat\wedge s_7^\flat + s_1^\flat\wedge s_3^\flat\wedge s_5^\flat - s_1^\flat\wedge s_4^\flat\wedge s_6^\flat - s_2^\flat\wedge s_3^\flat\wedge s_6^\flat \\ & -s_2^\flat\wedge s_4^\flat\wedge s_5^\flat + s_3^\flat\wedge s_4^\flat\wedge s_7^\flat + s_5^\flat\wedge s_6^\flat\wedge s_7^\flat \qquad . \end{aligned}$$

Theorem 4 relies strongly on the assumption of completeness in Riemannian geometry. In general when (M^n,g) is pseudo-Riemannian or non-complete, (the connected component of) the holonomy of the cone might be decomposable and non-trivial. Locally, this means that the cone $\overline{M}$ is a product of two spaces (N_i,h_i), $i=1,2$, with dimension $n_i>0$, and the corresponding volume forms $vol(h_i)$, $i=1,2$, (with respect to a local orientation) are parallel on the cone. Via the cone correspondence the volume forms $vol(h_i)$, $i=1,2$, give rise to special Killing forms $\alpha_i = \iota_{\mathbf{E}}vol(h_i)$, $i=1,2$, on the base M^n (which is identified with the 1-level of $\overline{M}$). By construction, the differential forms α_1 and α_2 are simple, and if the base M^n is Einstein with $scal^g \neq 0$, then α_1 and α_2 are nc-Killing.

The (local) existence of α_1, α_2 for decomposable cone holonomy allows a description of the metric g on M by certain (double) warped-products (cf. [**110**]). In fact, let $\mathbf{E}_i := pr_{i*}(\mathbf{E})$, $i = 1, 2$, denote the projections of the Euler vector $\mathbf{E}$ to the (local) factors (N_i, h_i) of the product of the cone $(\overline{M}, \bar{g})$. If we assume that the scalar curvature is normed by $scal^g = \pm n(n-1)$, the following two cases are possible (up to singularities of α_1, α_2, resp., causal singularities of $\mathbf{E}_1, \mathbf{E}_2$):

(1) $\mathbf{E}_1$, $\mathbf{E}_2$ are both spacelike and $scal^g > 0$. Then the metric g is locally isometric to
$$dr^2 + \sin^2(r) \cdot k_1 + \cos^2(r) \cdot k_2 \ , \tag{27}$$
where k_1, k_2 are Einstein metrics with $scal^{k_i} = (n_i - 1)(n_i - 2)$, $i = 1, 2$. The corresponding nc-Killing forms are (locally) given by
$$\alpha_1 = \sin^{n_1}(r) \cdot vol(k_1) \qquad \text{and} \qquad \alpha_2 = \cos^{n_2}(r) \cdot vol(k_2) \ .$$
If $\mathbf{E}_1$ and $\mathbf{E}_2$ are both timelike then the metric $-g$ can locally be expressed in the form (27).

(2) $\mathbf{E}_1$ is spacelike, $\mathbf{E}_2$ is timelike and $scal^g < 0$. Then the metric g is locally isometric to
$$dr^2 + \sinh^2(r) \cdot k_1 + \cosh^2(r) \cdot k_2 \ , \tag{28}$$
where k_1, k_2 are Einstein metrics with $scal^{k_1} = (n_1 - 1)(n_1 - 2)$ and $scal^{k_2} = -(n_2 - 1)(n_2 - 2)$. The special Killing forms are
$$\alpha_1 = \sinh^{n_1}(r) \cdot vol(k_1) \qquad \text{and} \qquad \alpha_2 = \cosh^{n_2}(r) \cdot vol(k_2) \ .$$
If $\mathbf{E}_1$ is timelike and $\mathbf{E}_2$ is spacelike the metric $-g$ is locally of the form (28).

The second case will be of interest in Section 2.6, where we will also give a proof of this statement. The proof of case (1) works the same.

We want to take a closer look to the situation when the cone $(\overline{M}, \overline{g})$ over an Einstein space (M, g) with $scal^g \neq 0$ is decomposable with a parallel covector $\hat{P}$ (of arbitrary causal type). Via the cone correspondence the covector $\hat{P}$ gives rise to a (non-constant) function $f_{\hat{P}} := \iota_{\mathbf{E}} \hat{P}$ on (M^n, g), which satisfies the second order differential equation
$$\nabla^g df_{\hat{P}} = c f_{\hat{P}} \cdot g \qquad \text{with} \ \ c = \frac{-scal^g}{n(n-1)} \ ,$$
i.e., the vector $grad^g(f_{\hat{P}})$ is a *conformal gradient field* and $f_{\hat{P}}$ is a nc-Killing function. (The gradient corresponds to the parallel 2-form tractor $o \wedge \alpha$ in $\Omega^2(\mathcal{T}^*(M))$, where the dual standard tractor α corresponds to $f_{\hat{P}}$.) In general, it is well-known that the existence of a conformal gradient field gives rise to a (local) warped-product structure on the base M (minus a singularity set, which consists of the zeros of the function and the length square of its gradient; cf. [**97**]).

If $\hat{P}$ is not null and $c = -1$ the corresponding warped-product structure for g is either given by
$$dr^2 + \sin^2(r) \cdot k \ , \qquad \text{resp.,} \qquad dr^2 + \cos^2(r) \cdot k$$
or
$$-dr^2 + \sinh^2(r) \cdot k \ , \qquad \text{resp.,} \quad -dr^2 + \cosh^2(r) \cdot k$$

where k is an Einstein metric on a space of dimension $n-1$ with $scal^k = \pm(n-1)(n-2)$. If $c = 1$ the metric $-g$ takes one of the given forms above. All these warped-product structures are special cases of (27) and (28) when one of the metrics k_1 or k_2 does not occur (for dimensional reason). In case when $\hat{P}$ is null and $c = \pm 1$, the metric g is locally given by

$$\mp dt^2 \ + \ \exp^2(t)k \ ,$$

where k is a Ricci-flat metric. For the singularity case, when $grad(f_{\hat{P}})$ has a zero on an Einstein space, it is known that the sectional curvature has to be constant (in a neighbourhood of the critical point), i.e., the base space is conformally flat there (cf. [**95, 97, 98**]).

Conformal gradient fields are known (since a long time) to generate conformal transformations between Einstein spaces (cf. [**28, 93**]). This can be understood in our context as follows. The function $f_{\hat{P}}$ (which generates a conformal gradient) on an Einstein space (M^n, g) is a nc-Killing function and we mentioned already at the beginning of this section that the rescaling of the metric g by a nc-Killing function (without zeros) gives rise to another Einstein metric in the conformal class. More generally, l linearly independent parallel standard tractors gives rise to l *independent* Einstein scales in a conformal class. For example, if $M^n = S^n$ is the round n-sphere, there exists the constant nc-Killing function o_- and there are $n+1$ further, non-constant and *independent* nc-Killing functions, each of them with an isolated zero, which give rise (via stereographic projection) to $n+1$ conformal transformations to Einstein metrics with constant sectional curvature (up to a singular point). This is the conformally flat case, where the number of *independent* nc-Killing functions on the base space is the maximum $n+2$.

We noticed already above that not all nc-Killing p-forms on an Einstein space are special Killing. For example, non-parallel closed differential forms are not special. However, there is an improvement of the cone construction for Einstein spaces (M, g) with $scal^g \neq 0$, which allows a description for any nc-Killing p-form $\alpha_- \in \Omega^p(M)$. For this purpose, we consider the metric

$$\boldsymbol{h} \ := \ \overline{g} - sgn(\lambda^g)ds^2 \ = \ sgn(\lambda^g)(\lambda^g t^2 g + dt^2 - ds^2) \qquad \text{with} \ \ \lambda^g = \frac{scal^g}{n(n-1)}$$

on $\overline{M} \times \mathbb{R}$, which is again Ricci-flat and has signature $(r+1, s+1)$. (cf. Chapter 2 and [**68, 110**]).

Proposition 5. *Let (M^n, g) be a connected semi-Riemannian Einstein space with $scal^g \neq 0$. Then there exists a natural and bijective correspondence between nc-Killing p-forms $\alpha_- \in \Omega^p(M)$ and $\nabla^{\boldsymbol{h}}$-parallel $(p+1)$-forms $\hat{\alpha}$ on $\overline{M} \times \mathbb{R}$.*

This result follows from the fact that the holonomy groups $Hol(\mathcal{T})$ and $Hol(\nabla^{\boldsymbol{h}})$ coincide. We will prove this in Section 2.5 in a more general case (cf. Theorem 15). The correspondence of nc-Killing p-forms on (M, g) and $\nabla^{\boldsymbol{h}}$-parallel $(p+1)$-forms on $\overline{M} \times \mathbb{R}$ in then explicitly given by a natural identification of the tangent space $T(\overline{M} \times \mathbb{R})$ with the standard tractor bundle $\mathcal{T}(M)$ along the level set $M \times \{1, 0\}$ in $\overline{M} \times \mathbb{R}$. The basic reason why all this works is that $\boldsymbol{h}$ is (the explicit form of) a Ricci-flat Fefferman-Graham ambient metric for the conformal space $(M, [g])$ (cf. Chapter 2 and Section 0.11).

Theorem 5. *Let (M, g) be an oriented Einstein space with nc-Killing p-form $\alpha_- \neq 0$.*

(1) *If $scal^g \neq 0$ there exist a special Killing p-form β_1 and a special Killing $(n-p-1)$-form β_2 such that $\alpha_- = \beta_1 + \star d\beta_2$, i.e., α_- is the sum of a coclosed and a closed nc-Killing p-form.*
(2) *If $scal^g = 0$ then either α_-, $d\alpha_-$, $d^*\alpha_-$ or $\Box_p \alpha_-$ is ∇^g-parallel.*

Proof. Let $\alpha = \mathbf{S}(\alpha_-)$ be the corresponding ∇^{nor}-parallel $(p+1)$-form tractor and let $t \in \Gamma(\mathcal{T}(M))$ be the ∇^{nor}-parallel standard tractor that corresponds to the Einstein metric $g \in [g]$. It holds that

$$\iota_t(t^\flat \wedge \alpha) + t^\flat \wedge (\iota_t \alpha) \; = \; \langle t, t \rangle_{\mathcal{T}} \cdot \alpha \; .$$

Thereby, both $(p+1)$-form tractors $\alpha_1 := \iota_t(t^\flat \wedge \alpha)$ and $\alpha_2 := t^\flat \wedge (\iota_t \alpha)$ are ∇^{nor}-parallel. By construction, the corresponding differential forms α_{1-} and $\star\alpha_{2-}$ are both coclosed and, if $scal^g \neq 0$, then $\alpha_- = \frac{1}{\langle t,t \rangle_{\mathcal{T}}}\big(\alpha_{1-} + \alpha_{2-}\big)$.

If $scal^g = 0$ and $\alpha_-, d\alpha_-$ and $d^*\alpha_-$ are not ∇^g-parallel the p-form $\Box_p \alpha_-$ does not vanish and must be parallel by equations (20) - (22). $\Box$

3. DeRham-like Decomposition Theorem

In this section we study conformal spaces (M^n, c), which admit simple ∇^{nor}-parallel $(p+1)$-form tractors

$$\alpha = \alpha_1 \wedge \cdots \wedge \alpha_{p+1} \in \Omega^{p+1}(\mathcal{T}^* M)$$

with $0 < p < n$, i.e., α is (locally) a $(p+1)$-times wedge product of 1-form tractors α_i in $\mathcal{T}^*M$. The existence of such a tractor implies that the conformal holonomy representation of $Hol(\mathcal{T})$ has an invariant (non-trivial, proper) subspace in $\mathbb{R}^{r+1,s+1}$ and thus is not irreducible. We will show that if such an invariant subspace in $\mathbb{R}^{r+1,s+1}$ is non-degenerate with respect to $\langle \cdot, \cdot \rangle_{r+1,s+1}$, then there exists locally a certain product of Einstein metrics in the conformal class c on M. This statement can be understood as a conformal version of the deRham splitting Theorem in semi-Riemannian geometry (cf. [**141**]). (The case $p = 0$ and $p = n+1$ correspond to parallel (dual) standard tractors, which are related to the conformal Einstein condition as we discussed in the previous section.)

First, we observe here the following construction principle for spaces with nc-Killing forms. Let (M_1, g_1) be an Einstein space of dimension n_1 with a coclosed nc-Killing form α_-. Taking the product with another space (M_2, g_2) of dimension n_2 produces the space $(M, g) = (M_1 \times M_2, g_1 + g_2)$. The pull-back of α_- to M still satisfies equation (19), since for every $Y \in TM_2$

$$\nabla^g_Y \alpha_- = \iota_Y d\alpha_- = Y^\flat \wedge d^*\alpha_- = 0 \; ,$$

i.e., the pull-back α_- is a conformal Killing form on (M, g). Now, it is straightforward to see that if we choose g_2 to be an Einstein metric of scalar curvature

$$scal^{g_2} = -\frac{n_2(n_2-1)}{n_1(n_1-1)} \cdot scal^{g_1} \; ,$$

then the normalisation conditions (20) - (22) are also satisfied for the pull-back α_-. Thus we have produced a space (M, g) of dimension $n = n_1 + n_2$ admitting a coclosed nc-Killing form. (If we choose for g_2 a different scalar curvature then α_- does not satisfy the normalised equations. In particular, this shows that not every conformal Killing form is normal, i.e., there is a difference between the space of ∇^{nor}-parallel $(p+1)$-form tractors and the kernel of $\mathcal{D}_0$ in the corresponding BGG sequence.)

On the other hand, we have the following result which says that certain conformal Killing p-forms give rise to a product metric in the conformal class of a space.

Lemma 2. *Let α_- be a conformal Killing p-form with $\|\alpha_-\|^2 \neq 0$ on a space (M^n, g) satisfying the following three properties:*

(1) *α_- is simple, i.e., $\alpha_- = \alpha_1^1 \wedge \ldots \wedge \alpha_p^1$ is a p-times $\wedge$-product of 1-forms,*
(2) *there exists $A \in \mathfrak{X}(M)$ such that $d\alpha_- = A^\flat \wedge \alpha_-$ and*
(3) *there exists $B \in \mathfrak{X}(M)$ such that $d^*\alpha_- = \iota_B \alpha_-$.*

Then a rescaled metric $\tilde{g}$ in the conformal class $[g]$ on M exists such that the rescaled conformal Killing form $\tilde{\alpha}_-$ is parallel. In particular, if $0 < p < n$ then $\tilde{g}$ is (locally) a product metric $g_1 \times g_2$.

Proof. First, observe that all assumptions on α_- are conformally covariant. For example, $d(e^\phi \alpha_-) = \tilde{A}^\flat \wedge e^\phi \alpha_-$ with $\tilde{A}^\flat = d\phi + A^\flat$. Since $\|\alpha_-\|^2 \neq 0$, we can rescale the metric g such that $\tilde{\alpha}_-$ has constant non-zero length. Let us assume that g is already this scale. Then we have

$$\begin{aligned} 0 &= X(g(\alpha_-, \alpha_-)) = 2g(\nabla^g_X \alpha_-, \alpha_-) \\ &= 2g(\frac{1}{p+1}\iota_X d\alpha_- - \frac{1}{n-p+1} X^\flat \wedge d^*\alpha_-, \alpha_-) \end{aligned}$$

for any $X \in TM$. This equation shows that the vector fields A, B have to be zero for the scaling g, i.e., the conformal Killing form α_- is closed and coclosed, hence parallel with respect to g. Moreover, since α_- is simple by assumption, it follows that g is (locally) a product $g_1 \times g_2$ (when the degree of α_- is not 0 or n). $\square$

Lemma 2 generalises the well-known fact that a hypersurface orthogonal, conformal Killing vector field is parallel with respect to some metric in the conformal class. We also remark at this point that, in general, a conformal Killing p-form α_- is conformally related to a parallel p-form for some metric $\tilde{g} = e^{2\phi} \cdot g$ in the conformal class if and only if

$$d\alpha_- = -(p+1) \cdot d\phi \wedge \alpha_- \qquad \text{and} \qquad d^*\alpha_- = (n-p+1) \cdot \iota_{grad(\phi)} \alpha_-$$

(cf. [**143**]). This shows that $A, B \neq 0$ in Lemma 2 are actually (local) gradients (if they exist). For an (anti)-selfdual form $\alpha_- = \pm * \alpha_-$ the two equations above are equivalent.

Now we are prepared to study nc-Killing p-forms α_- on (M, c) such that $\mathbf{S}(\alpha_-)$ is a simple $(p+1)$-form tractor with $0 < p < n$. So let α be such a ∇^{nor}-parallel $(p+1)$-form tractor. The first obvious statement we can make says that the four components α_-, α_0, $\alpha_\mp$ and α_+ of α (with respect to any metric $g \in c$) are simple as well. For example,

$$\alpha_- = \iota_{s_+}(\iota_{s_-}(s_+^\flat \wedge \alpha))$$

and, henceforth, α_- is simple. We can say even more. Let us consider the $(p+1)$-form tractor α at a single point x of (M, g) and let $\mathcal{B}^\mathcal{T} = \{s_-, s_+, s_1, \ldots, s_n\}$ be a fixed frame of $\mathcal{T}_x M$ with respect to g. In any case the $(p+1)$-form tractor α can be written with respect to $\mathcal{B}^{\mathcal{T}}$ either as

$$(s_-^\flat + \beta_1) \wedge (s_+^\flat + \beta_2) \wedge \beta_3 \wedge \ldots \wedge \beta_{p+1}$$

or as

$$(a \cdot s_-^\flat + b \cdot s_+^\flat + \beta_1) \wedge \beta_2 \wedge \ldots \wedge \beta_{p+1} \ ,$$

where the β_i's are linear combinations of the $s_i^\flat$, $i = 1, \ldots n$, and $a, b \in \mathbb{R}$ are some real numbers. If we assume that $\alpha_- = \Pi_H(\alpha) \neq 0$ at $x \in M$ then we can also assume that $d^*\alpha_- \neq 0$ at x (after a possible conformal rescaling of the metric g in a neighbourhood of x in M). This shows that the $(p+1)$-form tractor α takes in a neighbourhood of $x \in M$ with respect to a suitable metric and a local frame $\mathcal{B}^{\mathcal{T}} = \{s_-, s_+, s_1, \ldots, s_n\}$ in any case the normal form

$$(s_-^\flat + \beta_1) \wedge (s_+^\flat + \beta_2) \wedge \beta_3 \wedge \ldots \wedge \beta_{p+1} \ ,$$

for some smooth 1-forms β_j, $j = 1, \ldots, p+1$. From this normal form we finally see that for any parallel $(p+1)$-form tractor α on (M, c) with $\alpha_- \neq 0$ there exist smooth vector fields $A, B \in \mathfrak{X}(M)$ such that

$$d\alpha_- = A^\flat \wedge \alpha_- \qquad \text{and} \qquad d^*\alpha_- = \iota_B \alpha_-$$

with respect to some metric $\tilde{g} \in c$. We are in position to apply Lemma 2.

Lemma 3. *Let α_- be a nc-Killing p-form on (M, g) with $\|\alpha_-\|^2 \neq 0$ such that $\mathbf{S}(\alpha_-)$ is simple. Then there exists a metric $\tilde{g}$ in the conformal class $[g]$ such that the rescaled form $\tilde{\alpha}_-$ is parallel.*

The parallel nc-Killing p-form that is guaranteed by Lemma 3 is simple and, since its degree is different from 0 and n, gives rise (locally by the original deRham Theorem) to a product metric $g_1 \times g_2$ in the conformal class $[g]$. Moreover, the normalising equations (16) and (17) for α show that the factors g_1, g_2 are Einstein with

$$scal^{g_2} = -\frac{(n-p) \cdot (n-p-1)}{p \cdot (p-1)} \cdot scal^{g_1} \ .$$

In this situation the Schouten tensor $\mathsf{P}^{\tilde{g}}$ of the product $\tilde{g} = g_1 \times g_2$ equals the product $\mathsf{P}^{g_1} \times \mathsf{P}^{g_2}$ of the Schouten tensors of the factors. This particular type of semi-Riemannian product spaces will be of interest again in Chapter 2 (cf. [**68**]).

We also want to discuss the case when α_- is a simple and totally lightlike nc-Killing p-form, i.e., a wedge product of orthogonal, lightlike 1-forms only. This condition implies that the vectors $X \in TM$ with $\iota_X \alpha_- = 0$ span a totally lightlike subbundle of TM. There is a version of Lemma 2 for such p-forms.

Lemma 4. *Let α_- be a (non-vanishing) simple and totally lightlike conformal Killing p-form on a space (M, g) with the following two properties:*

(1) *there exists $A \in \mathfrak{X}(M)$ such that $d\alpha_- = A^\flat \wedge \alpha_-$ and*
(2) *there exists $B \in \mathfrak{X}(M)$ such that $d^*\alpha_- = \iota_B \alpha_-$.*

Then there is (locally) a rescaled metric $\tilde{g}$ in the conformal class $[g]$ such that the rescaled nc-Killing form $\tilde{\alpha}_-$ is parallel. In particular, the holonomy of the Levi-Civita connection of $\tilde{g}$ is reducible with a fixed totally lightlike subspace.

Proof. First, we show that we can assume α_- to be a closed form with respect to a suitable metric $\tilde{g} \in [g]$. In fact, the differential form $d\alpha_-$ is simple and closed by assumption. Hence, by Frobenius' Theorem there are (local) coordinates $(x^1, \ldots, x^n)$ such that $d\alpha_- = dx^1 \wedge \ldots \wedge dx^{p+1}$. Moreover, since α_- is simple we can choose these coordinates such that $\alpha_- = f \cdot dx^1 \wedge \ldots \wedge dx^p$, where f is a non-vanishing function depending on $x^1, \ldots, x^{p+1}$. By rescaling the metric we find that $\tilde{\alpha}_- = f^{-1}\alpha_-$ is a closed nc-Killing form.

Now let $\alpha_- = l_1 \wedge \ldots \wedge l_p$ be a totally null and closed conformal Killing form with $d^*\alpha_- = t \cdot l_1 \wedge \ldots \wedge l_{p-1}$, where the l_i's are pairwise orthogonal and lightlike 1-forms and t is some function. Then we can compute in an arbitrary point $x \in M$:

$$\begin{aligned} 0 &= X(g(\iota_{\bar{l}_1} \ldots \iota_{\bar{l}_{p-1}}\alpha_-, \iota_{\bar{l}_1} \ldots \iota_{\bar{l}_{p-1}}\alpha_-)) \\ &= 2 \cdot g(\iota_{\bar{l}_1} \ldots \iota_{\bar{l}_{p-1}} \nabla^g_X \alpha_-, \iota_{\bar{l}_1} \ldots \iota_{\bar{l}_{p-1}}\alpha_-) \\ &= \frac{2 \cdot (-1)^p}{n-p+1} \cdot g(tX^\flat, l_p) \qquad \text{for all } \; X \in T_xM \ , \end{aligned}$$

where we have chosen lightlike 1-forms $\bar{l}_i$ with

$$\nabla^g \bar{l}_i(x) = 0 \ , \qquad g_x(l_i, \bar{l}_i) = 1 \qquad \text{and} \qquad g_x(l_i, \bar{l}_j) = 0 \quad \text{for} \;\; i \neq j \ .$$

This is only possible for all $X \in T_xM$ if $t = 0$, i.e., $d^*\alpha_- = 0$ (at any point $x \in M$). Henceforth, α_- is parallel and totally isotropic. $\square$

Using Lemma 4 and the normal form description for simple $(p+1)$-form tractors with respect to some tractor frame $\mathcal{B}^{\mathcal{T}} = \{s_-, s_+, s_1 \ldots, s_n\}$ leads us to the next result.

Lemma 5. *Let α_- be a simple and totally null nc-Killing p-form on (M, g) such that $\mathbf{S}(\alpha_-)$ is simple. Then there is (at least locally) a metric $\tilde{g}$ in the conformal class such that the rescaled form $\tilde{\alpha}_-$ is parallel.*

Again, we can say more than stated in Lemma 5. With respect to the metric $\tilde{g}$, when $\tilde{\alpha}_-$ is totally null and parallel, the corresponding $(p+1)$-form tractor takes the form

$$\alpha = (s_-^\flat + a \cdot s_+^\flat) \wedge l_1 \wedge \ldots \wedge l_p$$

for some constant $a \in \mathbb{R}$. However, if $a \neq 0$ then $\beta = l_1 \wedge \ldots \wedge l_p$ would be ∇^{nor}-parallel itself, since $s_-^\flat + a \cdot s_+^\flat$ has constant non-zero norm with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{T}}$. This is not possible, since $\Pi_H(\beta) = 0$. Therefore, the constant a must be zero (and α is totally null), which implies that the scalar curvature of $\tilde{g}$ is zero. Furthermore, equations (16) and (17) show that the Ricci tensor of $\tilde{g}$ maps into the totally lightlike subspace of the tangent space consisting of those vectors, which annihilate the nc-Killing form $\tilde{\alpha}_-$. And the metric $\tilde{g}$ has reducible holonomy with an invariant lightlike subspace (which is not dilated under the action). In the generic case we expect that the holonomy is weakly irreducible. We summarise our results so far.

Proposition 6. *Let (M, c) be a conformal space and $Hol(\mathcal{T})$ the conformal tractor holonomy. The standard action of the group $Hol(\mathcal{T})$ fixes a simple $(p+1)$-form on $\mathbb{R}^{r+1,s+1}$ with $1 \leq p \leq n-1$ exactly in the following two situations (up to singular points).*

(1) *There is (locally) a product* $g_1 \times g_2$ *of Einstein metrics in* c *with*

$$scal^{g_2} = -\frac{q(q-1)}{p(p-1)} \cdot scal^{g_1} \ ,$$

where g_1 *is a metric on a space of dimension* p *and* $q := n - p$. *If* $scal^{g_1} \neq 0$ *then* $Hol(\mathcal{T})$ *fixes a non-degenerate subspace (= decomposable case) and if* $scal^{g_1} = 0$ *then the group* $Hol(\mathcal{T})$ *fixes a degenerate subspace of dimension* $p+1$, *which contains an invariant lightlike standard tractor (= indecomposable case).*

(2) *There exists* $g \in c$ *admitting a totally lightlike Ricci tensor and a parallel totally lightlike differential form* α_-. *The group* $Hol(\mathcal{T})$ *fixes a totally isotropic subspace in* $\mathbb{R}^{r+1,s+1}$ *(without dilation) of dimension at least* 2.

The singular points that we exclude in Proposition 6 are those, where the projection $\Pi_H(\alpha)$ of a simple and ∇^{nor}-parallel $(p+1)$-form tractor vanishes or changes the causal type. In the following we always exclude singularities of this type. In both cases of Proposition 6 the tractor holonomy representation on $\mathbb{R}^{r+1,s+1}$ is reducible. A further possibility for a reducible holonomy $Hol(\mathcal{T})$ is the case when a lightlike subspace of dimension s is invariant, but dilated under the action. In this case any s-form on this invariant subspace is not fixed by the action of the holonomy, and therefore, does not give rise to a nc-Killing form, i.e., Proposition 6 does not apply to this situation.

In Chapter 2 we will present an ambient metric construction for conformal spaces (M, c), where $M = M_1 \times M_2$ and c is the conformal equivalence class of a product metric $g_1 \times g_2$ as in Proposition 6(1). If $scal^{g_1} \neq 0$ the ambient metric $\boldsymbol{h}$ of (M, c) is explicitly given as the product

$$\bar{g}_1 \times \bar{g}_2$$

of the metric cones over (M_1, g_1) and (M_2, g_2) (cf. Proposition 9). Moreover, we will show in Section 2.5 that the conformal tractor holonomy $Hol(\mathcal{T})$ coincides with the holonomy group $Hol(\nabla^{\boldsymbol{h}})$ of the Levi-Civita connection of the ambient metric $\boldsymbol{h}$ (cf. Theorem 15). It follows from the construction that the latter holonomy group equals the product $Hol(\bar{g}_1) \times Hol(\bar{g}_2)$ of cone holonomies. Proposition 6 and 9 and Theorem 15 provide the ingredients to prove our deRham-like decomposition theorem (cf. [**6, 110**]).

Theorem 6. *Let* (M^n, c) *be a simply connected conformal* C^∞*-manifold with non-trivial and decomposable tractor holonomy* $Hol(\mathcal{T})$, *i.e., there exists a direct sum decomposition*

$$(\ \mathbb{R}^{r+1,s+1}\ ,\ Hol(\mathcal{T})\ ) \quad = \quad (\mathbb{V}_0, \{e\}) \oplus \left(\bigoplus_{i=1}^{k} (\mathbb{V}_i, G_i) \right) \ ,$$

where $k \in \mathbb{N}$ *is at least* 1 *and all non-trivial factors* $\mathbb{V}_i \subset \mathbb{R}^{r+1,s+1}$, $i = 1, \ldots, k$, *admit (weakly) irreducible representations of some closed subgroup* $G_i \subset Hol(\mathcal{T})$. *Then the conformal class* c *is (at least locally up to singular points) represented by a product metric of the form*

$$g_0 \quad \times \quad \Pi_{i=1}^{k} \ \ (e^{\phi_i} \cdot g_i) \ ,$$

where all g_i*'s,* $i = 1, \ldots, k$, *are Einstein metrics with* $scal^{g_i} = \pm n_i(n_i - 1)$ *defined on certain* M_i *(of dimension* $n_i \geq 4$*), such that the cones* $(\overline{M}_i, \bar{g}_i)$ *are undecomposable. The* *metric g_0 is a flat factor defined on some space M_0 of dimension $n_0 := (k-2)+dim\mathbb{V}_0$, whereas the ϕ_i's are certain real functions depending on coordinates of M_0 only.*

PROOF. The proof works by induction over k. First, let us assume that $k = 1$. Then $dim\mathbb{V}_0 > 0$, since $Hol(\mathcal{T})$ is decomposable. If $dim\mathbb{V}_0 = 1$ the class c is almost conformally Einstein (of non-zero scalar curvature). Hence $c = [e^{\phi_1} \cdot g_1]$ for some Einstein metric with $scal^{g_1} = \pm n_1(n_1 - 1)$ and a constant ϕ_1 on the complement of a singular set in M. If $dim\mathbb{V}_0 \geq 2$ then it follows by Proposition 6 that $c = [\tilde{g}_0 \times g_1]$ with $\tilde{g}_0$, g_1 Einstein and $scal^{g_1} = \pm n_1(n_1 - 1)$. Thereby, the cone $\bar{g}_1$ is (weakly) irreducible and the cone of $\tilde{g}_0$ is flat and has dimension $dim\mathbb{V}_0$, i.e., $\tilde{g}_0$ is conformally flat of dimension $dim\mathbb{V}_0 - 1 = dim\mathbb{V}_0 + k - 2$. After rescaling the metric $\tilde{g}_0$ (up to singular points) we obtain $c = [g_0 \times (e^{\phi_1} g_1)]$, where g_0 is flat on some M_0 and e^{ϕ_1} is the rescaling function defined on M_0.

Now let $k > 1$. Then we have $c = [g_k \times h]$, where g_k is an Einstein metric with $scal^{g_k} = \pm n_k(n_k - 1)$, whose cone $\bar{g}_k$ has (weakly) irreducible holonomy group G_k. The metric h is Einstein as well and its cone $\bar{h}$ has holonomy $H = \Pi_{i=1}^{k-1} G_i$. The ambient metric of the conformal class $[h]$ is given by $\bar{h} \pm dl^2$, where $\pm dl^2$ is a line element with signature depending on the sign of $scal^h$. The tractor holonomy $Hol(\mathcal{T}, [h])$ of $[h]$ equals the holonomy of the ambient metric (cf. Theorem 15), which is again $H = \Pi_{i=1}^{k-1} G_i$. We can see that $Hol(\mathcal{T}, [h])$ splits into $k-1$ (weakly) irreducible factors and a trivial factor of dimension $1 + dim\mathbb{V}_0$.

Obviously, the conformal structure $[h]$ satisfies (locally) all the assumptions of Theorem 6 and, henceforth, by induction we know that $[h]$ is represented by a product metric of the form $g_0 \times \Pi_{i=1}^{k-1}(e^{\phi_i} \cdot g_i)$, where g_0 is a flat metric on a space M_0 of dimension $((k-1)-2) + dim\mathbb{V}_0 + 1 = (k-2) + dim\mathbb{V}_0$ (which might be zero if $k = 2$). In fact, the metric h is given by $e^{-\phi_k}(\ g_0 \times \Pi_{i=1}^{k-1}(e^{\phi_i} \cdot g_i)\ )$, where ϕ_k is some function on M_0. This shows that c is represented (locally up to singular points) by

$$g_0 \times \ \left(\ \Pi_{i=1}^{k-1}(e^{\phi_i} \cdot g_i)\ \right)\ \times (e^{\phi_k} \cdot g_k)$$

as stated in Theorem 6. □

We remark that Theorem 6 does not apply if $k = 1$ and $dim\mathbb{V}_0 = 0$. This is the case of (weakly) irreducible conformal holonomy. For example, Ricci-flat metrics have in the generic case weakly irreducible conformal holonomy $Hol(\mathcal{T})$. A family of conformal spaces with irreducible tractor holonomy are generic Fefferman spaces (cf. Chapter 6). In view of Theorem 6 and the latter remark we might say that every conformal class c on a space M is (locally up to singular points) either built by Einstein metrics (with undecomposable cone) or the conformal holonomy is (weakly) irreducible as in the case of (generalised) Fefferman constructions.

In conformal Riemannian geometry, the Möbius group is the projective Lorentz group $\mathrm{PO}(1, n+1)$ and (the connected component of) the tractor holonomy $Hol(\mathcal{T})$ sits in $\mathrm{SO}_o(1, n+1)$. It is well known that the only subgroup of $\mathrm{SO}_o(1, n+1)$, which acts irreducibly on $\mathbb{R}^{1,n+1}$ is $\mathrm{SO}_o(1, n+1)$ itself (cf. [**142**]). Moreover, one can easily see from (15) - (18) that a $Hol(\mathcal{T})$-invariant lightlike line in $\mathbb{R}^{1,n+1}$ must contain an invariant lightlike vector, which indicates (almost) conformal Ricci-flatness. Combining these facts we see that a conformal space with Riemannian signature is either conformally almost Einstein, a product of Einstein spaces (up to singular points) as in Theorem

6 or its has the full tractor holonomy $\mathfrak{hol}(\mathcal{T}) = \mathfrak{so}(1, n+1)$. In all these cases the holonomy of a Fefferman-Graham ambient metric coincides with the tractor holonomy $Hol(\mathcal{T})$. (For the case $\mathfrak{hol}(\mathcal{T}) = \mathfrak{so}(1, n+1)$ the identity of the holonomies is true for trivial reason, otherwise Theorem 15 applies.)

We finally remark that the conformal tractor holonomy $Hol(\mathcal{T})$ and the holonomy $Hol(\nabla^{\boldsymbol{h}})$ of an ambient metric do not have much in common, in general. For example, let us consider a 4-dimensional Fefferman metric g with non-vanishing Bach tensor (cf. e.g. [**132**] for existence) and let $\boldsymbol{h}$ be an ambient metric for $[g]$. The (reduced) conformal tractor holonomy $Hol^0(\mathcal{T})$ is a subgroup of $\mathrm{SU}(1,2)$. However, $Hol^0(\boldsymbol{h})$ can not be a subgroup of $\mathrm{SU}(1,2)$, since this would imply that $\boldsymbol{h}$ is Ricci-flat, which is not possible, since the obstruction tensor does not vanish. Even if the obstruction vanishes for a conformal space there is no reason to expect that the holonomy groups $Hol(\mathcal{T})$ and $Hol(\nabla^{\boldsymbol{h}})$ coincide, in general!

4. Further Geometric Discussions

In Section 1.2 we have discussed nc-Killing p-forms on Einstein spaces. Using the results of the previous section we want to add here some considerations about the underlying geometries of solutions to the normalised equations (19) - (22), in general. We also want to give some impression of possible geometries admitting nc-Killing p-forms in 3- and 4-dimensional conformal geometry of Riemannian and Lorentzian signature.

So let (M^n, c) be a space of dimension n with conformal structure c of signature (r, s) and let α_- be a nc-Killing p-form. The corresponding ∇^{nor}-parallel $(p+1)$-form tractor $\alpha := \mathbf{S}(\alpha_-)$ gives rise to a $(p+1)$-form $\hat{\alpha}$ on the standard module $\mathbb{R}^{r+1,s+1}$, which is stabilised by the tractor holonomy $Hol(\mathcal{T})$ of (M, c). The orbit type of $\hat{\alpha}$ under the action of $\mathrm{O}(r+1, s+1)$ is uniquely determined. We denote by $Stab_{\mathcal{T}}(\alpha_-) \subset \mathrm{O}(r+1, s+1)$ the stabiliser of the corresponding $\hat{\alpha}$. The tractor holonomy $Hol(\mathcal{T})$ is a subgroup of $Stab_{\mathcal{T}}(\alpha_-)$. The next result is an immediate consequence of Proposition 6.

Proposition 7. *Let (M^n, c) be a conformal space with nc-Killing p-form α_- such that $Stab_{\mathcal{T}}(\alpha_-)$ in $\mathrm{O}(r+1, s+1)$ is isomorphic to a product $G_1 \times G_2$ of groups, which acts decomposable on $\mathbb{R}^{r+s,s+1}$. Then the conformal class c is represented (locally up to singular points) by a product $g_1 \times g_2$ on $M_1 \times M_2 \subset M$ as in Proposition 6(1) with $scal^{g_1} \neq 0$, where g_1 and g_2 admit certain nc-Killing p-forms α_{1-}, resp., α_{2-} such that $\alpha_- = \alpha_{1-} + \alpha_{2-}$ on $M_1 \times M_2$.*

Proposition 7 can be easily generalised to the case when $Stab_{\mathcal{T}}(\alpha_-)$ acts decomposable with more then two factors on the standard module $\mathbb{R}^{r+1,s+1}$. In fact, if a conformal class c on M is represented by a metric

$$g_0 \quad \times \quad \Pi_{i=1}^k \;\; (e^{\phi_i} \cdot g_i) \; ,$$

as discussed in Theorem 6, then any nc-Killing p-form α_- on M can be expressed by a sum $\sum_{i=0}^k e^{f_i} \cdot \alpha_{i-}$ of pull-backs of certain nc-Killing forms α_{i-} on the (M_i, g_i) with appropriate rescaling functions f_i for $i = 0, \ldots, k$.

Next let $\beta \neq 0$ be an arbitrary differential form on a space M. There exists a unique integer $rk(\beta) \geq 0$ such

$$\beta \wedge (d\beta)^{rk(\beta)} \neq 0 \qquad \text{and} \qquad \beta \wedge (d\beta)^{rk(\beta)+1} = 0 \; .$$

We call this integer $rk(\beta)$ the *rank* of the differential form β. Now let us consider an arbitrary nc-Killing p-form α_- on (M,c) with ∇^{nor}-parallel $(p+1)$-form tractor $\alpha = \mathbf{S}(\alpha_-)$. The $(p+1)$-form tractor α is given with respect to a metric $g \in c$ on M and a local tractor frame $\mathcal{B}^{\mathcal{T}} = (s_-, s_+, s_1, \dots, s_n)$ by

$$\alpha = s_-^\flat \wedge \alpha_- + \alpha_0 + s_-^\flat \wedge s_+^\flat \wedge \alpha_\mp - s_+^\flat \wedge \alpha_+ \ .$$

Obviously, for any integer $l \geq 1$ the $(lp+l)$-form tractor α^l is ∇^{nor}-parallel and the canonical projection $\Pi_H(\alpha^l)$ is (with respect to some $g \in c$) given by the differential form

$$\alpha_- \wedge \alpha_0^{l-1} \ = \ (p+1)^{1-l} \cdot \big(\ \alpha_- \wedge (d\alpha_-)^{l-1} \ \big)$$

of degree $l(p+1)-1$ on (M,g).

Proposition 8. *Let (M,c) be a conformal space with nc-Killing p-form α_-. The differential form*

$$\alpha_- \wedge (d\alpha_-)^{l-1}$$

is non-trivial and nc-Killing for any number l with $1 \leq l \leq rk(\alpha_-)+1$.

After the general discussion so far, we state some results about the existence of nc-Killing forms in dimension 3 and 4.

Theorem 7. *Let (M^3,g) be an oriented semi-Riemannian manifold of dimension 3 with arbitrary signature, which admits a non-trivial nc-Killing p-form α_-. Then (M^3,g) is either conformally flat or conformally equivalent (up to singularities) to a Ricci-isotropic Brinkmann wave (cf. Section 0.8).*

Proof. Under the assumptions (using the Hodge star $\star$) there exists in any case either a nc-Killing function or a nc-Killing 1-form on (M^3,g). In the first case g is almost conformally Einstein. Hence the space is conformally flat.

So let us assume that there exists a non-trivial nc-Killing 1-form α_-. The 1-form α_- has either rank 0 or 1. If the rank is 1 then the Hodge dual of $\alpha_- \wedge d\alpha_-$ is a nc-Killing function, which shows that (M,g) is conformally flat. It remains to discuss the situation when $rk(\alpha_-)=0$. In this case the dual V_- of α_- is hypersurface orthogonal by Frobenius' Theorem and there exists (up to singularities) a metric $\tilde{g} \in [g]$ such that V_- is parallel. If V_- is not null the metric $\tilde{g}$ is Ricci-flat. Hence (M,g) is conformally flat. If V_- is null the Ricci-tensor of $\tilde{g}$ maps into the subbundle $\mathbb{R}\cdot V_- \subset TM$, i.e., $\tilde{g}$ is a Ricci-isotropic Brinkmann metric of signature $(1,2)$ or $(2,1)$. $\square$

We also have some geometric description in dimension 4.

Theorem 8. *Let (M^4,g) be an oriented 4-space and let α_- be a non-trivial nc-Killing p-form. a) If g is a Riemannian metric and*

1. *$deg(\alpha_-)=0$ or 4 then (M^4,g) is almost conformally Einstein,*
2. *$deg(\alpha_-)=1$ or 3 then (M^4,g) is conformally flat,*
3. *$deg(\alpha_-)=2$ then (M^4,g) is (up to singularities) conformally related to a Ricci-flat Kähler space.*

b) If g is a Lorentzian metric and $deg(\alpha_-)=1$ then the following cases occur.

1. *$rk(\alpha_-)=1$ and (M^4,g) is a Fefferman space,*
2. *$rk(\alpha_-)=0$ and (M^4,g) is conformally related to a Ricci-isotropic Brinkmann wave (or conformally flat).*

PROOF. a) First, let us consider the Riemannian case with nc-Killing 2-form α_-. We assume without loss of generality that α_- is (anti)-selfdual. Then we set $e^{2\lambda} := \|\alpha\|_g^2$ and one can show (in dimension 4) that

$$d^*\alpha_- = -3 \cdot \iota_{grad(\lambda)}\alpha_-$$

(cf. [**143**]), which implies $d\alpha_- = 3d\lambda \wedge \alpha_-$ as well. It follows that $\|\alpha_-\|_g^{-3} \cdot \alpha_-$ is a parallel Kähler form for the rescaled metric $\tilde{g} = \|\alpha_-\|_g^{-2} \cdot g$ (off the singularity set; cf. Section 1.3). The normalisation conditions (20) and (21) imply that the Ricci-tensor of $\tilde{g}$ vanishes.

If the degree of α_- is 0 or 4 it is clear that (M, g) is conformally almost Einstein. It remains to discuss the case $deg(\alpha_-) = 1$. If $rk(\alpha_-) = 0$ then α_- is conformally related to a parallel 1-form for some metric $\tilde{g} \in [g]$, which has to be Ricci-flat by the normalisation (up to singularities). This implies that $\tilde{g}$ is locally the product of a real line with a Ricci-flat 3-space. Hence, (M, g) is conformally flat. If $rk(\alpha_-) = 1$ the orbit type classification of 2-forms on $\mathbb{R}^{1,5}$ shows that the stabiliser $Stab_{\mathcal{T}}(\alpha_-)$ acts decomposable on $\mathbb{R}^{1,5}$ (cf. Table 3). This fact implies that there exists (locally up to singularities) a product $g_1 \times g_2 \in [g]$ as in Proposition 6(1). In dimension 4 this is only possible if (M, g) is conformally flat.

b) Now we discuss the case of a nc-Killing 1-form on a Lorentzian space (M, g). The case $rk(\alpha_-) = 0$ works similar as above in dimension 3. So let us assume that $rk(\alpha_-) = 1$. Again, the orbit type classification of 2-forms on $\mathbb{R}^{2,4}$ (cf. Table 3) shows that if $\|\alpha\|_g^2 \neq 0$ (up to singularities), then (M^4, g) is locally conformally equivalent to a product of Einstein spaces as in Proposition 6(1), which implies that (M^4, g) is conformally flat. However, if $\|\alpha_-\|_g^2 = 0$ and $rk(\alpha_-) = 1$ then the dual vector V_- of α_- is lightlike and twisting (cf. Section 0.8) and $\alpha = \mathbf{S}(\alpha_-)$ defines an orthogonal complex structure on the standard tractor bundle $\mathcal{T}(M)$. This is the case of Fefferman geometry (cf. Section 0.12 and Chapter 6). □

We remark that if α_- is a nc-Killing 2-form on a Lorentzian space (M^4, g) the notion of selfduality does not apply. It seems that a geometric discussion for Lorentzian spaces (M^4, g) admitting nc-Killing 2-forms needs a further investigation of the $\mathrm{O}(2,4)$-orbit types of 3-form tractors on $\mathbb{R}^{2,4}$. We do not address this problem here. Also the case of neutral signature on 4-dimensional spaces (M^4, g) requires more orbit type investigations for 2- and 3-form tractors (cf. [**140, 85, 80**]). In Table 1 and 2 we give an overview on 4-dimensional Riemannian and Lorentzian geometries admitting nc-Killing p-forms.

5. Normal Form Description for 2-Forms in Signature $(2, n-2)$

In this section we present a complete list (of building blocks) of normal forms for skew-adjoint endomorphisms acting on the pseudo-Euclidean space $\mathbb{R}^{2,n-2}$ of dimension n and signature $(2, n-2)$. The normal form description applies also to 2-forms on $\mathbb{R}^{2,n-2}$, which correspond to skew-adjoint endomorphisms in a natural way. We derived this list from [**27**] (see also [**129, 107**]). Since any nc-Killing 1-form α_- on a conformal space (M, c) of Lorentzian signature $(1, n-1)$ gives rise via $\mathbf{S}(\alpha_-)$ to a uniquely determined 2-form tractor $\hat{\alpha}$ on $\mathbb{R}^{2,n}$ (with respect to a tractor frame), we can assign to any α_- a certain normal form from our list. This application will allow us a geometric

$Hol(\mathfrak{T})$ sitting in	(local conformal) geometry of $[g]$	nc-Killing p-form
$\mathrm{O}(5)$	(almost) Einstein, $scal > 0$	function
$\mathrm{O}(1,4)$	(almost) Einstein, $scal < 0$	function
$Stab(e_-^\flat)$	(almost) Ricci-flat	function
$Stab(e_-^\flat \wedge \omega_o)$	(almost) Ricci-flat, Kähler	Kähler form ω_o
$\{e\}$	conformally flat	maximal
$\mathrm{O}(1,5)$	generic case	non

Table 1. nc-Killing p-forms on Riemannian spaces in dimension 4.

$Hol^0(\mathfrak{T})$ sitting in	(local conformal) geometry of $[g]$	nc-Killing p-form
$\mathrm{O}(1,4)$	(almost) Einstein, $scal > 0$	function
$\mathrm{O}(2,3)$	(almost) Einstein, $scal < 0$	function
$Stab(e_-^\flat)$	(almost) Ricci-flat	function
$Stab(e_-^\flat \wedge l)$, l null, $l \perp e_-^\flat$	pp-wave	1-form without twist, $Ric(V_-, V_-) = 0$
$\mathrm{SU}(1,2)$	Fefferman spaces	1-form with twist, $Ric(V_-, V_-) > 0$
$\{e\}$	conformally flat	maximal
$Stab$(3-form)	?	2-form
$\mathrm{O}(2,4)$	generic case	non

Table 2. nc-Killing p-forms in 4-dimensional Lorentzian geometry.

description of conformal Lorentzian spin manifolds admitting twistor spinors. We will discuss this in the final section of this chapter.

Theorem 9. *(cf. [**27**]) Let β be an arbitrary 2-form on the pseudo-Euclidean space $\mathbb{R}^{2,n-2}$. Then there exist subspaces V_i such that $\mathbb{R}^{2,n-2} = \oplus_i V_i$ is an orthogonal direct sum and the skew-adjoint endomorphism b, which corresponds to β, satisfies $b(V_i) \subset V_i$ for all i. Moreover, there exists a basis $\{e_{i_1}, \ldots, e_{i_{dim(V_i)}}\}$ for every V_i such that the corresponding matrices for (the restriction of) the scalar product $\langle \cdot, \cdot \rangle_{2,n-2}$ and for b give a pair of blocks as presented in the rows of Table 3 below.*

In the following we call a basis of $\mathbb{R}^{2,n-2}$, in which a skew-adjoint operator takes a normal form, an *adapted basis.* There is always an orthogonal decomposition $\mathbb{R}^{2,n-2} = E \oplus P$ to a skew-adjoint operator b such that E is Euclidean and b preserves

signature (p,q)	$A =$ inner product	$B =$ skew-adjoint operator	
$(0,1)$:	(1)	(0)	
$(0,2)$:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$B(\mu) = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}$	$\mu \neq 0$
$(1,0)$:	(-1)	(0)	
$(1,2)$:	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	
$(1,1)$:	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$	$\lambda \neq 0$
$(2,2)$:	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$B_{Ia} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
$(2,1)$:	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$B_{Ib} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	
$(2,4)$:	$\begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_2 & 0 \\ -I_2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ 0 & 0 & 0 \end{pmatrix}$	
$(2,0)$:	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$B_{II}(\nu) = \begin{pmatrix} 0 & -\nu \\ \nu & 0 \end{pmatrix}$	$\nu \neq 0$
$(2,2)$:	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$B_{IIa} = \begin{pmatrix} 0 & -\nu & 1 & 0 \\ \nu & 0 & 0 & 1 \\ 0 & 0 & 0 & -\nu \\ 0 & 0 & \nu & 0 \end{pmatrix}$	$\nu \neq 0$
$(2,4)$:	$\begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_2 & 0 \\ -I_2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & -\nu \\ \nu & 0 \end{smallmatrix} & I_2 & 0 \\ 0 & \begin{smallmatrix} 0 & -\nu \\ \nu & 0 \end{smallmatrix} & I_2 \\ 0 & 0 & \begin{smallmatrix} 0 & -\nu \\ \nu & 0 \end{smallmatrix} \end{pmatrix}$	$\nu \neq 0$
$(2,2)$:	$\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda I_2 & 0 \\ 0 & -\lambda I_2 \end{pmatrix}$	$\lambda \neq 0$
$(2,2)$:	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$	$\lambda \neq 0$
$(2,2)$:	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$B_{IIb} = \begin{pmatrix} \xi & -\nu & 0 & 0 \\ \nu & \xi & 0 & 0 \\ 0 & 0 & -\xi & \nu \\ 0 & 0 & -\nu & -\xi \end{pmatrix}$	$\xi, \nu \neq 0$
$(2,3)$:	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	

TABLE 3. These are the building blocks in the diagonal for the normal forms of skew-adjoint operators in signature $(2, n-2)$. The matrices in the first column (denoted by A) indicate an inner product (of index $s \leq 2$) with respect to some basis. The matrices in the second column (denoted by B) are skew-adjoint endomorphisms with respect to the inner product in column A and the chosen basis.

the decomposition. We call the normal form to b on E an Euclidean block and the normal form to b on P a pseudo-Euclidean block.

Examples.

(1) Let $\omega_o := \sum_{i=1}^{m} e^*_{2i-1} \wedge e^*_{2i}$ be the standard (pseudo)-orthogonal Kähler form on $\mathbb{R}^{2,n-2}$, where $\{e_1, \ldots, e_{2m}\}$ denotes the standard basis. The normal form of the skew-adjoint operator corresponding to a multiple $\omega = \nu \cdot \omega_o$ of the Kähler form with respect to the adapted basis $\{e_1, \ldots, e_{2m}\}$ is built by one block of the form $B_{II}(\nu)$ (pseudo-Euclidean block) and $(m-1)$ blocks of the form $B(\nu)$ (Euclidean blocks) (cf. Table 3).

b) A 2-form $\omega = l_1^\flat \wedge l_2^\flat$ on $\mathbb{R}^{2,n-2}$, where l_1 and l_2 are lightlike vectors, which span a totally lightlike plane, corresponds as skew-adjoint operator with respect to some adapted basis to a composition of a pseudo-Euclidean block of the form B_{Ia} and a trivial Euclidean block of edge length $n-4$.

c) A 2-form $\omega = l_1^\flat \wedge t_1^\flat$ on $\mathbb{R}^{2,n-2}$, where l_1 is lightlike, t_1 is timelike and both vectors are orthogonal, corresponds as skew-adjoint operator with respect to some adapted basis to a composition of a block B_{Ib} and a trivial block of length $n-3$.

Now let $\hat{\varphi} \in \mathbb{\Delta}_{2,n-2}$ be a spinor on the pseudo-Euclidean space $\mathbb{R}^{2,n-2}$ and let $\varsigma_2(\hat{\varphi})$ be the corresponding spinor square in the 2-forms. The following observation will help us to determine the possible normal forms for spinor squares $\varsigma_2(\hat{\varphi})$.

Lemma 6. *Let $\hat{\varphi}$ be a ($\pm$-Weyl) spinor on $\mathbb{R}^{2,n-2}$ and $T \in \mathbb{R}^{2,n-2}$ an arbitrary unit timelike vector. The 1-form $\iota_T \varsigma_2(\hat{\varphi})$ on the Minkowski space $T^\perp \subset \mathbb{R}^{2,n-2}$ (with induced orientation) is the spinor square $\varsigma_1(\varphi)$ of the induced spinor $\varphi \in \mathbb{\Delta}_{1,n-2}$ on $T^\perp$ (by restriction of $\hat{\varphi}$; cf.* [**19**]*).*

Proof. With respect to an orthonormal basis $\mathcal{B} = \{e_0, e_1, \ldots, e_{n-1}\}$ of $\mathbb{R}^{2,n-2}$ such that $e_0 = T$ we have the relation

$$\begin{aligned} \iota_T \varsigma_2(\hat{\varphi}) &= -i \sum_{i<j} \langle e_i e_j \cdot \hat{\varphi}, \hat{\varphi} \rangle_{\mathbb{\Delta}_{2,n-2}} \cdot \iota_T e_i^* \wedge e_j^* \\ &= -i \sum_{j=1}^{n-1} \langle T \cdot e_j \cdot \hat{\varphi}, \hat{\varphi} \rangle_{\mathbb{\Delta}_{2,n-2}} e_j^* = - \sum_{j=1}^{n-1} \langle e_j \varphi, \varphi \rangle_{\mathbb{\Delta}_{1,n-2}} e_j^* = \varsigma_1(\varphi) \end{aligned}$$

holds (cf. Section 0.8 and [**19, 107**]). □

Lemma 6 shows that $\varsigma_2(\hat{\varphi}) \neq 0$ whenever $0 \neq \hat{\varphi} \in \mathbb{\Delta}_{2,n-2}$. This is because $\varsigma_1(\varphi)$ does not vanish (cf. Section 0.8). Moreover, since (the dual of) $\varsigma_1(\varphi)$ is a causal vector, Lemma 6 shows that any 2-form $\varsigma_2(\hat{\varphi})$ induced by a spinor in signature $(2, n-2)$ has the special property that insertion of any timelike vector produces (the dual of) a causal vector. With simple calculations we can figure out, which normal forms built from blocks of our list, are possible for the corresponding skew-adjoint endomorphism to $\varsigma_2(\hat{\varphi})$.

Corollary 1. *Let ω be a 2-form in signature $(2, n-2)$ such that the covector $\iota_T \omega$ is causal for every timelike vector $T \in \mathbb{R}^{2,n-2}$.*

(1) *If there is a timelike T such that $\iota_T\omega$ is lightlike then the normal form corresponding to ω is a composition of a pseudo-Euclidean block of the form B_{Ia} or B_{Ib} with a trivial Euclidean block.*
(2) *If $\iota_T\omega$ is timelike for all timelike T then the normal form of ω is a composition of $B_{II}(\nu)$, $B_{IIa}(\nu)$ or $B_{IIb}(\nu)$ ($\nu^2 \geq \xi^2$) with an Euclidean block consisting of blocks of the form $B(\mu)$ with $\mu < \nu$ and/or a trivial block.*

In the following we will rely on four *generic types* of 2-forms on $\mathbb{R}^{2,n-2}$.

Definition 2. *Let $\mathbb{R}^{2,n-2}$ be the pseudo-Euclidean space of signature $(2,n-2)$ and let $\omega \in \Lambda^2\mathbb{R}^{n*}$ be a non-trivial 2-form. We say that ω is of*

- *Type (I_a) if $\omega = l_1^\flat \wedge l_2^\flat$ for some vectors l_1 and l_2, which span a totally lightlike plane.*
- *Type (I_b) if $\omega = l_1^\flat \wedge t_1^\flat$ for some lightlike vector l_1 and a l_1-orthogonal timelike vector t_1.*
- *Type (II_a) or Kähler Type if ω is a non-trivial multiple of the Kähler form.*
- *Type (II_b) if there exists a non-trivial Euclidean subspace E in $\mathbb{R}^{2,n-2}$ such that ω restricted to E vanishes and ω is of Kähler Type on the orthogonal complement of E in $\mathbb{R}^{2,n-2}$.*

We call these four types *generic*, since their stabilisers in $\mathrm{O}(2,n-2)$ are maximal.

Corollary 2. *A 2-form ω on $\mathbb{R}^{2,n-2}$, which is of Type (I_a), (I_b) , (II_a) or (II_b), is distinguished by the following two properties.*
(1) *The covector $\iota_T\omega$ is causal for every timelike vector $T \in \mathbb{R}^{2,n-2}$ and*
(2) *the stabiliser $Stab(\omega)$ in $\mathrm{O}(2,n-2)$ is maximal, in the sense that there is no non-trivial 2-form satisfying (1), whose stabiliser properly contains $Stab(\omega)$.*

For the proof of Corollary 2 we just note that the stabiliser of a block of the form $B_{IIa}(\nu)$ or $B_{IIb}(\nu)$ is (properly) contained in $\mathrm{U}(1,1)$, which is the stabiliser of $B_{II}(\nu)$.

Finally, we observe that the stabiliser of a 2-form of Type (I_a) or Type (I_b) acts indecomposably on $\mathbb{R}^{2,n-2}$, i.e., there exists no non-trivial, proper and non-degenerate invariant subspaces of $\mathbb{R}^{2,n-2}$ under the action of the stabiliser. The stabiliser of a Kähler (Type) form is $U(1,m-1)$ and acts irreducibly on $\mathbb{R}^{2,2m-2}$. The stabiliser of a 2-form of Type (II_b) acts decomposable on $\mathbb{R}^{2,n-2}$.

6. Application to Twistor Spinors in Lorentzian Spin Geometry

We study here a geometric description for (conformal) Lorentzian spin manifolds admitting solutions of the twistor equation for spinors without singularities (cf. Section 0.8). The discussion is based on the orbit type classification for corresponding spinorial squares in the 2-form tractors deduced from the results of the previous section. Our result extends those of J. Lewandowski in [**117**], H. Baum in [**17**] and Theorem 1. The benefits of tractor and twistor calculus shall become obvious in this section again, since (different to Riemannian spin geometry; cf. [**19**]) a geometric characterisation for solutions of the twistor equation does not seem to work by other means. In Chapter 4 we will apply our methods also to a discussion of conformal Killing spinors with singularities in Lorentzian spin geometry!

So let (M^n, g) be a connected and time-oriented Lorentzian spin manifold of dimension $n \geq 3$ with (complex) spinor bundle $\mathcal{S}$ and indefinite Hermitian product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$.

Recall that a twistor spinor $\varphi \in \Gamma(\mathcal{S})$ is a solution of the overdetermined PDE

$$\nabla^{\mathcal{S}}_X \varphi + \frac{1}{n} X \cdot D^{\mathcal{S}} \varphi = 0 \qquad \text{for all } X \in TM \ .$$

The Dirac current to φ is defined by the relation

$$g(V_\varphi, X) := -\langle X \cdot \varphi, \varphi \rangle_{\mathcal{S}}$$

for any $X \in TM$. If $\varphi \not\equiv 0$ is a conformal Killing spinor then the corresponding Dirac current V_φ is a conformal Killing vector field. Actually, the dual 1-form $\varsigma_1(\varphi)$ to V_φ satisfies the normalised equations (19) - (22), i.e., $\varsigma_1(\varphi)$ is a nc-Killing 1-form on (M, g).

The latter fact becomes clear from the following observation. In general, the corresponding twistor $\Psi := \mathbf{S}(\varphi) \in \Gamma(\mathcal{W})$ to a conformal Killing spinor $\varphi \in \Gamma(\mathcal{S})$ is $\nabla^{\mathcal{W}}$-parallel on $(M, [g])$ (cf. Section 0.8). As usual we can form from Ψ the spinorial squares, which give rise to p-form tractors on $(M, [g])$. These p-form tractors (which might be trivial for certain degrees p) are ∇^{nor}-parallel, by construction. In any case, since (M, g) has Lorentzian signature, the spinorial Möbius group $\mathrm{Spin}(2, n)$ has signature $(2, n)$ and by definition of the Hermitian product $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ on the spin tractor bundle $\mathcal{W}$, it is clear that the spinorial square $\varsigma_2(\Psi)$ in the 2-form tractors is non-trivial if Ψ itself is non-trivial. In fact, the 2-from tractor $\varsigma_2(\Psi)$ is defined by the relation

$$\langle \varsigma_2(\Psi), X^2 \rangle_{\mathcal{T}} := -i \langle X^2 \cdot \Psi, \Psi \rangle_{\mathcal{W}}$$

for all $X^2 \in \Omega^2(\mathcal{T}^* M)$ and its projecting part $\Pi_H(\varsigma_2(\Psi))$ (with respect to the metric g) is nothing else, but the non-trivial nc-Killing 1-form $\varsigma_1(\varphi)$, which is dual to the Dirac current of the twistor spinor $\varphi \not\equiv 0$:

$$\begin{array}{ccc} \begin{array}{c}\varphi \\ \text{twistor spinor}\end{array} & \longleftrightarrow & \begin{array}{c}\Psi \in \Gamma(\mathcal{W}) \\ \nabla^{nor}\text{-parallel twistor}\end{array} \\ \big\downarrow & & \big\downarrow \\ \begin{array}{c}\varsigma_1(\varphi) \\ \text{nc-Killing 1-form}\end{array} & \longleftrightarrow & \begin{array}{c}\varsigma_2(\Psi) \in \Omega^2(\mathcal{T}^* M) \\ \nabla^{nor}\text{parallel 2-form tractor}\end{array} \end{array} .$$

Now we can assign to any conformal Killing spinor φ on a Lorentzian spin manifold (M, g) a 2-form $\hat{\alpha}_\varphi$ on $\mathbb{R}^{2,n}$, which corresponds to the ∇^{nor}-parallel 2-form tractor $\varsigma_2(\Psi)$ (with respect to a tractor frame). The 2-form $\hat{\alpha}_\varphi$ is stabilised by $Hol(\mathcal{T})$ and admits a uniquely defined $\mathrm{O}(2, n)$-orbit type with a certain normal form built from blocks of Table 3. Since $\hat{\alpha}_\varphi$ is a spinorial square, the possible generic types for $\hat{\alpha}_\varphi$ are given in Definition 2.

Note that the normal form of $\hat{\alpha}_\varphi$ is linked to the rank of the nc-Killing 1-form $\varsigma_1(\varphi)$. For example, $rk(\varsigma_1(\varphi)) = 0$ when $\hat{\alpha}_\varphi$ has Type (I_a) or (I_b), whereas the rank of $\varsigma_1(\varphi)$ is $n/2 - 1$ (= maximal) for the Kähler Type (II_a). If $\hat{\alpha}_\varphi$ has Kähler Type on a codimension 1 subspace of $\mathbb{R}^{2,n}$, the rank of $\varsigma_1(\varphi)$ is maximal as well, namely $\frac{n-1}{2}$, which simply reflects that $\varsigma_1(\varphi)$ is a contact form. In the higher codimensional cases of Type (II_b) the rank of $\varsigma_1(\varphi)$ varies between 0 and $\frac{n-2}{2}$. Also note that if $\hat{\alpha}_\varphi$ is not generic, then there exists a nc-Killing 1-form α_- on (M, g) such that the stabiliser of the normal form of $\mathbf{S}(\alpha_-)$ contains the stabiliser of $\hat{\alpha}_\varphi$. This can only happen if $\hat{\alpha}_\varphi$ is built with a block of the form $B_{IIa}(\nu)$ or $B_{IIb}(\nu)$. In fact, α_- can be chosen such that the non-trivial blocks of the normal form of $\mathbf{S}(\alpha_-)$ have the same edge length as the non-trivial block of the normal form of $\hat{\alpha}_\varphi$. This implies that the rank of the 1-forms

$\varsigma_1(\varphi)$ and α_- are the same! The singularity set $sing(\varphi)$ of a twistor spinor φ is defined to be the union of the zeros of φ with the singularities of the length square of its Dirac current V_φ (cf. Chapter 4). The set $\tilde{M} := M \setminus sing(\varphi)$ is dense in M.

Theorem 10. *Let φ be a twistor spinor on a Lorentzian spin manifold (M^n, g) of dimension $n \geq 3$. Then at least one of the following statements is true on $\tilde{M} = M \setminus sing(\varphi)$.*

(1) *It holds that*
$$rk(\varsigma_1(\varphi)) = 0 \qquad and \qquad \|\varsigma_1(\varphi)\|_g^2 = 0$$
and φ is locally conformally equivalent to a parallel spinor with lightlike Dirac current V_φ on a Brinkmann wave.

(2) *It holds that*
$$rk(\varsigma_1(\varphi)) = 0 \qquad and \qquad \|\varsigma_1(\varphi)\|_g^2 < 0$$
and then (locally) $[g] = [-dt^2 + h]$, where h is a Ricci-flat Riemannian metric admitting a parallel spinor (= static monopole case).

(3) *The dimension n is odd and the rank*
$$rk(\varsigma_1(\varphi)) = (n-1)/2$$
is maximal. Then $(\tilde{M}, g)$ is conformally related to a Lorentzian Einstein-Sasaki manifold.

(4) *The dimension n is even and the rank*
$$rk(\varsigma_1(\varphi)) = (n-2)/2$$
is maximal. Then g is locally conformally related to a Fefferman metric on $\tilde{M} = M$.

(5) *It holds that $0 < rk(\varsigma_1(\varphi)) < (n-2)/2$ and there exists (locally) a product metric $g_1 \times g_2 \in [g]$ on $\tilde{M}$, where g_1 is a Lorentzian Einstein-Sasaki metric on a space M_1 of dimension $n_1 := 2 \cdot rk(\varsigma_1(\varphi)) + 1$ admitting a Killing spinor ϕ_1 and g_2 is a Riemannian Einstein metric with Killing spinor ϕ_2 on a space M_2 of positive scalar curvature $scal^{g_2} = -\frac{(n-n_1)(n-n_1-1)}{n_1(n_1-1)} scal^{g_1}$.*

Proof. The conformal Killing spinor φ on (M^n, g) gives rise to a ∇^{nor}-parallel spinorial square $\varsigma_2(\Psi)$ in the 2-form tractors, where $\Psi = \mathbf{S}(\varphi)$. By Corollary 2, it follows that the conformal tractor holonomy $Hol(\mathcal{T})$ of $(M, [g])$ fixes a 2-form $\hat{\alpha}$ of generic type $l_1^\flat \wedge l_2^\flat$, $l_1^\flat \wedge t_1^\flat$, ω_o or $\omega_o|_V$ as described in Definition 2. (Note that the normal form of $\varsigma_2(\Psi)$ might not be exactly one of these four generic types, but in any case there exists a generic type with bigger stabiliser and same edge length as $\varsigma_2(\Psi)$). We can conclude that there exists a ∇^{nor}-parallel 2-form tractor β with normal form $\hat{\alpha}$ and corresponding nc-Killing 1-form α_- on (M, g) such that the tank of $\varsigma_1(\varphi)$ and α_- are the same.

Now let us discuss the four generic cases (one of which has to occur on M). First, let us assume that $\mathbf{S}(\alpha_-)$ has orbit type $l_1^\flat \wedge l_2^\flat$. Then, in fact, $\mathbf{S}(\varsigma_1(\varphi))$ has the same orbit type and, therefore, the rank and the length square of $\varsigma_1(\varphi)$ are zero. This shows that locally on $\tilde{M} \subset M$ (where φ has no zeros) some $\tilde{g} \in [g]$ exists such that the Dirac current V_φ is a $\nabla^{\tilde{g}}$-parallel null vector and the Ricci-tensor of $\tilde{g}$ is totally isotropic.

Moreover, in this scaling the conformally rescaled spinor $\tilde{\varphi}$ to φ has to be $\nabla^{\mathcal{S}}$-parallel (cf. [**106**]).

Next, if n is odd and $rk(\alpha_-) = rk(\varsigma_1(\varphi)) = \frac{n-1}{2}$, the 2-form tractor β has orbit type $\omega_o|_V$ and the $(n+1)$-form tractor $\beta^{\frac{n+1}{2}}$ is non-trivial and ∇^{nor}-parallel. The Hodge dual $\star_{\mathcal{T}}\beta^{\frac{n+1}{2}}$ is a ∇^{nor}-parallel spacelike dual standard tractor, which gives rise to an Einstein scale $\tilde{g} \in [g]$ on $\tilde{M}$. The tractor holonomy $Hol(\mathcal{T})$ sits in $\mathrm{U}(1, \frac{n-1}{2})$ and an explicit ambient metric $\boldsymbol{h}$ for $[\tilde{g}]$ on $\mathbb{R} \times \mathbb{R}_+ \times \tilde{M}$ is known (cf. Theorem 11). The ambient metric $\boldsymbol{h}$ is Ricci-flat and its holonomy $Hol(\nabla^{\boldsymbol{h}})$ equals the conformal holonomy by Theorem 15, i.e., $Hol(\nabla^{\boldsymbol{h}})$ sits in $\mathrm{SU}(1, \frac{n-1}{2})$. This implies that the cone metric of $\tilde{g}$ on $\mathbb{R}_+ \times \tilde{M}$ is Ricci-flat and pseudo-Kähler. It follows that the base metric $\tilde{g}$ on $\tilde{M}$ is a Lorentzian Einstein-Sasaki space of negative scalar curvature (cf. Section 0.9). If n is even and $rk(\varsigma_1(\varphi)) = \frac{n-2}{2}$ then the 2-form tractor β has type ω_o (or $\omega_o|_V$ with V of codimension 2) and the n-form tractor $\beta^{\frac{n}{2}}$ is non-trivial and ∇^{nor}-parallel. This shows that $Hol(\mathcal{T})$ sits in $\mathrm{U}(1, n/2)$, which implies by Corollary 8 that (M, g) is locally conformally equivalent to a Fefferman space. The singularity set is empty in this case (cf. Chapter 6).

In the remaining two cases a product metric will turn up in the conformal class $[g]$ and the factors will admit certain spinors. To prove this, we need the following standard argument. Assume that the representation $\mathbb{R}^{2,n}$ splits into a direct sum of non-degenerate invariant subspaces A of signature $(2, p)$ and B of signature $(0, n-p)$ under the action of $Hol(\mathcal{T})$. Then we denote by Δ_A, resp., Δ_B the corresponding spinor modules. As representations spaces of $\mathrm{Spin}(2, p) \times \mathrm{Spin}(n-p)$ we have

$$\Delta_A \otimes \Delta_B = \Delta_{2,n} \quad \text{for } n \text{ odd} \qquad \text{and} \qquad \Delta_A \otimes \Delta_B = \Delta^{\pm}_{2,n} \quad \text{for } n \text{ even} .$$

In general, if ρ is some representation of a product group $G_1 \times G_2$ and ρ_1, ρ_2 are representations of G_1, resp., G_2 such that

$$\rho \cong \rho_1 \otimes \rho_2 ,$$

then the representation ρ has a fixed vector if and only if ρ_1 and ρ_2 both have fixed vectors. In our situation that means the preimage of a decomposable tractor holonomy group $Hol(\mathcal{T})$ in $\mathrm{Spin}(2, n)$ fixes a spinor in $\Delta_{2,n}$ (resp., $\Delta^{\pm}_{2,n}$) iff the action of both factors of the preimage fix some spinors in Δ_A and Δ_B.

Now let $rk(\varsigma_1(\varphi)) = 0$ and $\|\varsigma_1(\varphi)\| < 0$ on $\tilde{M} \subset M$. Then the Dirac current V_φ is parallel with respect to $\tilde{g} = \|V_\varphi\|_g^{-2} \cdot g \in [g]$ on $\tilde{M}$ and $\tilde{g}$ is Ricci-flat. This shows that $[g]$ contains a static monopole metric $-dt^2 + h$, where h is a Ricci-flat Riemannian metric and $-dt^2 + h$ admits a parallel spinor (cf. Section 0.8). The fact that h admits a parallel spinor as well, follows from the general observation of the previous paragraph.

Finally, if non of the cases so far apply to $\tilde{M}$, then $1 \leq q < \frac{n-2}{2}$ holds for the rank $q := rk(\varsigma_1(\varphi))$, and there exists a ∇^{nor}-parallel 2-form tractor β, whose orbit type is $\omega_o|V$, and β^{q+1} is a ∇^{nor}-parallel volume form of a non-degenerate subbundle $V \subset \mathcal{T}$ of rank $n_1 + 1 = 2q + 2$. This implies that locally on $\tilde{M}$ there exists a product metric $g_1 \times g_2 \in [g]$ as in Proposition 6(1), where g_1 lives on some space M_1 and g_2 on some M_2. In particular, $\boldsymbol{h} = \bar{g}_1 \times \bar{g}_2$ on $(\mathbb{R}_+ \times M_1) \times (\mathbb{R}_+ \times M_2)$ is an explicit ambient metric for $[g]$, whose holonomy is identical to $Hol(\mathcal{T})$ (cf. Theorem 15). Thereby, we can choose g_1 such that the tangent space of its cone $\mathbb{R}_+ \times M_1$ (along the 1-level) is canonically identified with V in $\mathcal{T}$ on $\tilde{M}$, which is furnished with the ∇^{nor}-parallel

2-form tractor β. This shows that the cone metric $\bar{g}_1$, which is Ricci-flat, is equipped with a parallel pseudo-Kähler form, and we can conclude that g_1 is a Lorentzian Einstein-Sasaki metric of dimension n_1 (with scalar curvature $-n_1(n_1-1)$). It remains to be shown that g_1 and g_2 admit Killing spinors. However, this follows again with the general observation from above, since the lift of $Hol(\mathcal{T})$ to $\mathrm{Spin}(2,n)$ fixes a twistor in $\mathbb{\Delta}_{2,n}$ and decomposes the standard module $\mathbb{R}^{2,n}$. Therefore, we have parallel spinors on the metric cones $\bar{g}_1$ and $\bar{g}_2$, which says that there are Killing spinors on the bases (cf. Proposition 1). $\square$

Note that the cases (1) - (4) of Theorem 10 include the standard geometries, which are well known to admit solutions of the twistor equation (cf. [**117, 18**] and Theorem 1). The only new development in the geometric classification of Theorem 10 is the product case (5). Also note that if the Dirac current V_φ of a conformal Killing spinor φ on (M,c) is lightlike then the rank $rk(\varsigma_1(\varphi))$ has to be extremal, i.e., either $rk(\varsigma_1(\varphi)) = 0$ or $\frac{n}{2}-1$ (cf. Theorem 1 and [**40**]).

Finally, we remark that in general, if $g = g_1 \times g_2$ is a product metric such that g_1 and g_2 admit Killing spinors ψ_1 and ψ_2 with Killing numbers $\lambda_1 = \pm i\lambda_2$, then the tensor product $\psi_1 \otimes \psi_2 \in \Gamma(\mathcal{S})$ is a conformal Killing spinor for $g_1 \times g_2$. One can prove this fact by checking the twistor equation directly using the spinor connection

$$\nabla^{\mathcal{S},g} = \nabla^{\mathcal{S},g_1} \otimes \mathbb{1} \ + \ \mathbb{1} \otimes \nabla^{\mathcal{S},g_2} \ .$$

However, for this computation one has to work out carefully a correct identification for the tensor product of the Clifford algebras of the factors with the Clifford algebra of the total space. In any case the product construction is a way to produce Lorentzian spin spaces admitting conformal Killing spinors of the 5th kind of Theorem 10. One simply has to take a (simply connected) Lorentzian Einstein-Sasaki spin space (which always admits an imaginary Killing spinor with Killing constant $\lambda_1 \in i\mathbb{R}$) and a Riemannian spin space with real Killing spinor (and Killing constant $\pm i\lambda_1$), whose Dirac current vanishes! The classification of complete Riemannian spaces with real Killing spinors is well known (cf. [**12**] and Section 0.9). For example, nearly G_2-spaces and nearly Kähler spaces are suitable for the product construction (since the Dirac current vanishes in these two case, whereas the spinor square in the 3-forms, resp., in the 2-forms does not vanish in these situations).

CHAPTER 2

A Sub-Product Construction of Poincaré-Einstein Metrics

Einstein metrics have a distinguished history in geometry and physics. An area of intense recent interest has been the study of conformally compact Einstein metrics and their asymptotically Einstein generalisations. In particular, there have been exciting recent developments relating the topology and scattering theory of these structures with Branson's Q-curvature, renormalised volume and related quantities [**1, 5, 47, 55, 74, 79**]. This programme is intimately linked to the AdS/CFT programme of physics [**124, 78**] which seeks to relate conformal field theory of the boundary conformal manifold to the (pseudo)-Riemannian field theory of the interior *bulk* structure.

Let $\underline{M}$ be a compact smooth manifold with interior M^+ and boundary $M = \partial\underline{M}$. A central question is the existence and uniqueness of a Poincaré-Einstein metric g^+ on the interior M^+ for a given conformal structure on the boundary M (cf. Section 0.11). In [**77**] Graham-Lee showed that each conformal structure on S^n, sufficiently near the standard one, is the conformal infinity of a unique (up to diffeomorphism) asymptotically hyperbolic Poincaré-Einstein metric on the ball near the hyperbolic metric. This idea has been extended to more general circumstances by Biquard [**26**], Lee [**104**] and Anderson, e.g. [**5**]. In another direction, yielding further examples, a connected sum theory for combining Poincaré-Einstein metrics has been developed by Mazzeo and Pacard [**125**]. The problem of obtaining existence and examples is already interesting if we drop the requirement of compactness and simply seek a Poincaré-Einstein collar (and certain extensions thereof) for a given conformal space $(M^n, [g])$. From the physical point of view it is not necessarily expected that Poincaré-Einstein metrics are conformally compact.

In this chapter the main result is a construction of Poincaré-Einstein metrics for a class of boundary conformal structures. More precisely it is this. Given a pair of Einstein manifolds $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$, of signatures resp. (p_1, q_1) and (p_2, q_2), such that their scalar curvatures are related by $m_2(m_2-1)scal^{g_1} = -m_1(m_1-1)scal^{g_2}$, we give a signature (p_1+p_2, q_1+q_2+1) Poincaré-Einstein structure on $M_1 \times M_2 \times I$, where I is a suitable subset of the real line and contains an interval $[0, r_0)$ for some $r_0 > 0$. The conformal infinity is $(M_1 \times M_2, [g_1 \times g_2])$. This is Theorem 13. Using r for the standard coordinate on the interval $[0, r_0)$, explicitly the interior metric is

$$g^+ = r^{-2}(dr^2 + (1-\mu r^2/2)^2 g_1 + (1+\mu r^2/2)^2 g_2),$$

where μ is any constant satisfying $2m_1(m_1-1)\mu = scal^{g_1}$ and $2m_2(m_2-1)\mu = -scal^{g_2}$. We take $m_1 \geq 1$ and $m_2 \geq 0$. In the construction $m_2 = 0$ corresponds to taking $(M_2^{m_2}, g_2)$ as a point (or a collection of isolated points). Thus, as a special case, we recover an explicit Poincaré metric for the case that the boundary metric is Einstein. In the generic situation of our construction, the product metric $g_1 \times g_2$ will sit uniquely as the only metric in its conformal class $[g_1 \times g_2]$ which is a product of Einstein metrics, and so in this setting the Poincaré-metric is determined by the boundary conformal structure.

In Section 2.4 we show (cf. [**77**]) that a certain smooth extension of the metric cone of the interior of a Poincaré-Einstein structure, with conformal infinity $(M,[g])$, yields a Ricci-flat ambient metric for $(M,[g])$ (in the sense of [**54, 38**] except here we obtain the ambient as a manifold with boundary and as a fibred structure over the full Poincaré-Einstein structure). Thus our results above can be rephrased in terms of the ambient metric. In fact our construction proceeds in the other direction. Over a non-Ricci flat Einstein m-manifold (M,g) one may construct the Ricci-flat dimension $m+1$ metric cone $(\overline{M},\overline{g})$ (cf. Section 0.9). If $\overline{M}_1$ and $\overline{M}_2$ are two such cones, over Einstein metrics g_1 and g_2 with an appropriate scalar curvature relation, then we show, in Section 2.2 that the product $(\overline{M}_1 \times \overline{M_2}, \overline{g}_1 \times \overline{g_2})$ is an ambient metric for $(M_1 \times M_2, [g_1 \times g_2])$. The construction generalises in the sense that one may write down the same ambient metric directly and this then extends to the case that g_1 and g_2 are Ricci-flat. This general case is treated first in Theorem 11 of Section 2.1, where we verify explicitly that the metric satisfies the conditions of a Ricci-flat ambient metric as in [**38, 54**]. The Poincaré-Einstein metric arises on the hypersurface of the ambient space given by the zero set of the defining function $\boldsymbol{h}(\boldsymbol{X},\boldsymbol{X})+1$ (cf. Section 0.11). Thus Poincaré-Einstein metrics are equivalent to ambient metrics, at least ambient metrics as manifolds with boundary, in the sense of Section 2.4. (We should point out that starting with a smooth general Poincaré-Einstein metric the ambient metric may fail to be smooth up to the boundary, but will always be differentiable to an order depending on the dimension.)

In Theorem 15 we show that holonomy of the ambient metrics from Theorem 11 agrees with the conformal holonomy. Using this result we show in Theorem 12 that in general the products $g_1 \times g_2$, where $m_2(m_2-1)scal^{g_1} = -m_1(m_1-1)scal^{g_2}$, are not conformally Einstein. In fact we show the stronger result that they are in general not conformally almost-Einstein in the sense of [**65**]. This shows that the general construction of ambient metrics and Poincaré-Einstein metrics here is not a disguised form of the simpler construction for Einstein boundaries. The ambient variant of the latter seems to have been first given in [**110, 76**] (and see also [**6**]). Since the obstruction tensor $\mathcal{O}$ is an obstruction to the type of ambient metrics and Poincaré metrics that we construct [**71, 76**], it also follows that the generic products of this form give a large class of metrics which are not conformally almost-Einstein and yet for which the obstruction tensor $\mathcal{O}$ vanishes, see Corollary 3 and Theorem 12. (The obstruction tensor vanishes on manifolds which are conformally almost-Einstein [**54, 71, 76**].)

In Section 2.6 we show that the Poincaré-Einstein interior metrics, that we obtain, are characterised by the presence of certain special Killing forms (cf. Section 0.4 and 0.9). In the final section we give examples of Poincaré-Einstein metrics where the boundary conformal structure is not conformally Einstein. It is also observed there that one can obviously iterate the construction of Poincaré-Einstein metrics, as described in Theorem 13, to obtain a recursive construction principle for a class of Poincaré-Einstein metrics.

The content of this chapter originates from the paper [**68**], which is a joint work of the author in collaboration with Prof. A. Rod Gover from the University of Auckland in New Zealand.

1. A Family of Explicit Ambient Metrics

For a pair of suitable Einstein metrics g_1 and g_2 we give here an explicit Ricci-flat ambient metric. In the theorem we include the case of just a single Einstein manifold M_1. This is consistent with the general construction by taking the view that the second manifold M_2 is a single point (and so of dimension $m_2 = 0$). It was known to Fefferman and Graham [**54**] that the problem of constructing a formal ambient metric was solvable to all orders whenever the conformal structure underlying manifold M was conformally Einstein. In [**110**] an explicit ambient metric was given for that case. The following theorem may be viewed as an extension and generalisation of those results (cf. Proposition 6(1) of Chapter 1).

Theorem 11. *Suppose that $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$, of signatures resp. (p_1, q_1) and (p_2, q_2), and with $m_1 \geq 1$, $m_2 \geq 0$, are Einstein manifolds such that $m_2(m_2 - 1)scal^{g_1} = -m_1(m_1 - 1)scal^{g_2}$. For each $\mu \in \mathbb{R}$ satisfying $2m_1(m_1 - 1)\mu = scal^{g_1}$ and $2m_2(m_2 - 1)\mu = -scal^{g_2}$, there is a signature $(p_1 + p_2 + 1, q_1 + q_2 + 1)$ Ricci-flat ambient manifold for the conformal manifold $(M_1 \times M_2, [g_1 \times g_2])$, with metric given by the expression (29) below.*

Note that if either of m_1 or m_2 is at least 2 then there is exactly one solution μ to the condition, $2m_1(m_1-1)\mu = scal^{g_1}$ and $2m_2(m_2-1)\mu = -scal^{g_2}$. Otherwise μ is any real number. (In fact if $m_1 = 1$ and $m_2 = 0$ then μ can be taken to be a non-vanishing function. We do not treat this as a special case as the factor may be absorbed as a conformal transformation of g_1.)

Proof. Let us simplify notation by defining $M := M_1 \times M_2$, and $g := g_1 \times g_2$. We write $\pi : \mathcal{Q} \to M$ for the $\mathbb{R}_+$-bundle of metrics conformally related to g. The metric g determines a fibre coordinate t on $\mathcal{Q}$ by writing a general point of $\mathcal{Q}$ in the form $(p, t^2 g(p))$, where $p \in M$ and $t > 0$ (cf. Section 0.11).

The ambient manifold is defined to be $\tilde{M} := \mathcal{Q} \times \tilde{I}$ where

$$\tilde{I} = \begin{cases} \mathbb{R} & \text{if } \mu = 0, \\ \mathbb{R} \setminus \{-\frac{1}{\mu}\} & \text{if } \mu \neq 0 \ \text{ and } \ m_2 = 0, \\ \mathbb{R} \setminus \{\frac{1}{\mu}, -\frac{1}{\mu}\} & \text{otherwise.} \end{cases}$$

There is a projection $\tilde{M} \to \mathcal{Q}$ given by forgetting the $\tilde{I}$ component in the product. Following this with $\pi : \mathcal{Q} \to M$ we have a projection $\tilde{\pi} : \tilde{M} \to M$. It follows that we have the canonical bilinear forms $\tilde{\pi}^* g_1$ and $\tilde{\pi}^* g_2$ on $\tilde{M}$. For notational simplicity let us write, respectively, g_1 and g_2 for these forms on $\tilde{M}$.

We equip $\tilde{M}$ with the metric

$$\boldsymbol{h} := 2tdtd\rho + 2\rho dt^2 + t^2[(1 + \mu\rho)^2 g_1 + (1 - \mu\rho)^2 g_2], \tag{29}$$

where ρ is the standard coordinate on $\mathbb{R}$ viewed as a coordinate on $\tilde{I}$ (and hence on $\tilde{M}$) and $\mathcal{Q}$ is identified with its image $\mathcal{Q} \times \{0\} \subset \tilde{M}$ (cf. Section 0.11). Note that the functions $1 \pm \mu\rho$ are non-vanishing on the set $\tilde{I}$. The metric $\boldsymbol{h}$ on $\tilde{M}$ satisfies by definition the conditions (i) and (ii) of Section 0.11 for an ambient metric. It remains to check that $\boldsymbol{h}$ is Ricci flat. For this purpose we calculate the components of $Ric(\boldsymbol{h})$ as given by the expressions (11) - (13) of Section 0.11. Fixing some choice of local coordinates $x^1, \cdots, x^{m_1}$ on M_1 and $x^{m_1+1}, \cdots, x^{m_1+m_2}$ on M_2, the coordinates (x, t, ρ)

on $\tilde{M}$ are the obvious extension of the coordinates (x,t) on $\mathcal{Q}$. We calculate in these coordinates, and for a function $f(x,t,\rho)$ the notation f' will mean $\partial f/\partial\rho$.

First we calculate (11), (12), and (13) for the metric (29) in the case $m_2 = 0$, that is $\tilde{g} = a^2 g_1$ where $a := 1 + \mu\rho$. For simplicity write $n = m_1$ and $g = g_1$. Then we have $\tilde{g}'_{ij} = 2\mu a g_{ij}$ and hence we have $\tilde{g}^{kl}\tilde{g}'_{kl} = 2\mu a^{-1} n$ and $\tilde{g}''_{ij} = 2\mu^2 g_{ij}$. Substituting these in (11) we obtain

$$-2\rho\mu^2 + 2n\rho\mu^2 + (2-n)a\mu - an\mu + 2n\mu - 2\mu \;=\; \mu(n-1)(2\rho\mu - 2a + 2) \;=\; 0,$$

since $\rho\mu = a - 1$. For (12) we need also $\tilde{g}^{ij}\tilde{g}''_{ij} = 2a^{-2}\mu^2 n$ and substituting this gives $-na^{-2}\mu^2 + na^{-2}\mu^2 = 0$. Finally for (13) observe that $\tilde{g}^{kl}\tilde{g}'_{kl}$ depends only on ρ and so $\tilde{\nabla}^{(\rho)}_j(\tilde{g}^{kl}\tilde{g}'_{kl}) = 0$. On the other hand $\tilde{g}^{kl}\tilde{g}'_{kj} = 2\mu a^{-1}\delta^l_j$, Thus

$$\tilde{\nabla}^{(\rho)}_l(\tilde{g}^{kl}\tilde{g}'_{kj}) = 2\mu a^{-1}\tilde{\nabla}^{(\rho)}_l\delta^l_j = 0 \ ,$$

and so $Ric(\boldsymbol{h})_{\rho j} = 0$.

Next we assume $m_2 \geq 1$ and so $\tilde{g} = a^2 g_1 + b^2 g_2$, where $a := (1+\mu\rho)$ and $b := (1-\mu\rho)$. First note that $\tilde{g}'_{ij} = 2\mu a g^1_{ij} - 2\mu b g^2_{ij}$ and hence we have $\tilde{g}^{kl}\tilde{g}'_{kl} = 2\mu(a^{-1}m_1 - b^{-1}m_2)$ and $\tilde{g}''_{ij} = 2\mu^2 g^1_{ij} + 2\mu^2 g^2_{ij}$. Substituting these in (11) and assuming that $1 \leq i \leq m_1$ brings us to

$$-2\rho\mu^2 + 2\rho\mu^2(m_1 - m_2\frac{a}{b}) + (2 - m_1 - m_2)\mu a - \mu a(m_1 - m_2\frac{a}{b}) + \frac{1}{m_1}scal^{g_1}$$

times g^1_{ij}. But now using that $scal^{g_1} = 2m_1(m_1-1)\mu$ and $a - b = 2\mu\rho$ this becomes

$$\begin{aligned}&\frac{\mu}{b}[b(b-a) + (a-b)(bm_1 - am_2)\\ &+ ab(2 - m_1 - m_2) - a(bm_1 - am_2) + 2b(m_1 - 1)] = \mu(b + a - 2)(1 - m_1),\end{aligned}$$

which vanishes identically since $b + a = 2$. A similar calculation for (11) with $m_1 + 1 \leq i \leq m_1 + m_2$ gives $\mu(b + a - 2)(1 - m_2) = 0$ and so $Ric(\boldsymbol{h})_{ij}$ vanishes identically on $\tilde{M}$.

For (12) with $\tilde{g} = a^2 g_1 + b^2 g_2$ we have

$$-\frac{1}{2}(2\mu^2 a^{-2} m_1 + 2\mu^2 b^{-2} m_2) + \mu^2 a^{-2} m_1 + \mu^2 b^{-2} m_2 = 0.$$

Finally the case (13). First observe that $\tilde{g}^{kl}\tilde{g}'_{kl} = 2\mu(a^{-1}m_1 - b^{-1}m_2)$ depends only on ρ and so $\tilde{\nabla}^{(\rho)}_j(\tilde{g}^{kl}\tilde{g}'_{kl}) = 0$. On the other hand $\tilde{g}^{kl}\tilde{g}'_{kj} = 2\mu a^{-1}P_{(1)}{}^l_j - 2\mu b^{-1}P_{(2)}{}^l_j$, where $P_{(1)}$ is the section of $\mathrm{End}(TM)$ projecting onto TM_1 and $P_{(2)}$ is the complementary projection onto TM_2. (Here we view TM as $TM_1 \oplus TM_2$ via the derivative of the product structure $M = M_1 \times M_2$.) Thus

$$\tilde{\nabla}^{(\rho)}_l(\tilde{g}^{kl}\tilde{g}'_{kj}) = 2\mu a^{-1}\tilde{\nabla}^{(\rho)}_l P_{(1)}{}^l_j - 2\mu b^{-1}\tilde{\nabla}^{(\rho)}_l P_{(2)}{}^l_j.$$

But $\tilde{\nabla}^{(\rho)}$ is the Levi-Civita connection for a product metric compatible with the structure $M_1 \times M_2$. Thus $\tilde{\nabla}^{(\rho)}P_{(1)} = 0 = \tilde{\nabla}^{(\rho)}P_{(2)}$ and we conclude that $Ric(\boldsymbol{h})_{\rho j} = 0$. □

2. The Generic Setting and Metric Cones

For (M^m, g) an Einstein manifold of dimension $m > 1$ and scalar curvature $scal^g \neq 0$ we define the metric cone to be $\overline{M} = M \times \mathbb{R}_+$ equipped with the metric

$$\overline{g} = \mathrm{sgn}(\lambda^g)(\lambda^g s^2 g + ds^2) \tag{30}$$

where the non-zero constant λ^g satisfies $scal^g = m(m-1)\lambda^g$ (cf. Section 0.9). This is Ricci-flat. Remember that if g has signature (p,q) then the cone has signature $(p+1,q)$ or $(p,q+1)$ according to whether λ^g is respectively positive or negative.

Now suppose that we have a pair of Einstein manifolds $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$ such that $m_2(m_2-1)scal^{g_1} = -m_1(m_1-1)scal^{g_2}$ as in Theorem 11 (and we allow the case $m_2 = 0$ as explained there.) We will show that the product of the cones over $(M_1 \times M_2, g_1 \times g_2)$ is the ambient manifold from Theorem 11.

With λ satisfying $scal^{g_1} = m_1(m_1-1)\lambda$ there is no essential loss of generality in assuming that $\lambda > 0$. Then the cone metrics are

$$\overline{g}_1 = \lambda s_1^2 g_1 + ds_1^2 \quad \text{and} \quad \overline{g}_2 = \lambda s_2^2 g_2 - ds_2^2 \tag{31}$$

on, respectively, $\overline{M}_1$ and $\overline{M}_2$. A product of Ricci-flat metrics is always Ricci-flat and so in particular this is true for the product metric

$$\boldsymbol{h}_\times := \overline{g}_1 + \overline{g}_2 = ds_1^2 - ds_2^2 + \lambda s_1^2 g_1 + \lambda s_2^2 g_2 \tag{32}$$

on $\overline{M}_1 \times \overline{M}_2$. Now we define functions t and ρ on $\overline{M}_1 \times \overline{M}_2$ by

$$t := \frac{\lambda^{1/2}(s_1+s_2)}{2} \quad \text{and} \quad \rho := \frac{2(s_1-s_2)}{\lambda(s_1+s_2)}, \tag{33}$$

and set $\mu = \lambda/2$. Re-expressing the right-hand-side of (32) in terms of t, ρ, μ, and the pull-back metrics g_1 and g_2, a direct calculation recovers exactly the expression for the ambient metric as given on the right-hand-side of (29). The Jacobian $\partial(t,\rho)/\partial(s_1,s_2)$ is non-vanishing on the positive (s_1,s_2)-quadrant and so, with the pull-back (under the obvious projections) of coordinate sets from M_1 and M_2, the pair t,ρ give coordinates on the entire product $\overline{M}_1 \times \overline{M}_2$. The metric (29) extends this, and since the inverse of the transformation (33) is

$$s_1 = (2\mu)^{-1/2} t(1+\mu\rho) \quad \text{and} \quad s_2 = (2\mu)^{-1/2} t(1-\mu\rho)\ , \tag{34}$$

we see immediately that the points where the ambient metric (29) degenerates (e.g. $\rho = \pm\frac{1}{\mu}$ in the generic case) are points bounding but not in the product $\overline{M}_1 \times \overline{M}_2$. (In stating things this way we are viewing both $\tilde{M}$ and the product $\overline{M}_1 \times \overline{M}_2$ as subspaces, in the obvious way, of the manifold $\mathcal{Q} \times \mathbb{R}$.) In summary we have the following result.

Proposition 9. *Suppose that $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$ are Einstein manifolds such that $m_2(m_2-1)scal^{g_1} = -m_1(m_1-1)scal^{g_2}$. In a neighbourhood of $\mathcal{Q} \subset \tilde{M}$, the ambient metric (29) for $(M_1 \times M_2, [g_1 \times g_2])$ is the product of the cone metrics (31) where $\lambda = 2\mu$ satisfies $scal^{g_1} = m_1(m_1-1)\lambda$ and $scal^{g_2} = -m_2(m_2-1)\lambda$.*

We make some observations in relation to this picture.

Proposition 10. *The ambient metric $\boldsymbol{h}$ given in (29) is independent of constant dilations of the product metric $g_1 \times g_2$ on $M_1^{m_1} \times M_2^{m_2}$.*

Proof. First observe that if $\alpha \in \mathbb{R}_+$ and metrics g and $\widehat{g}$ are related by a constant conformal rescaling according to $\widehat{g} = \alpha g$ then $Ric(\widehat{g}) = Ric(g)$. Thus $scal^{\widehat{g}} = \alpha^{-1} scal^g$ and so, making the compatible transformation of λ^g to $\lambda^{\widehat{g}}$, we have $\lambda^g g = \lambda^{\widehat{g}} \widehat{g}$ and the cone metric (30) for (M,g) is the same as the cone metric for $(M,\widehat{g})$. It follows easily that the product metric $\boldsymbol{h}_\times$ on $\overline{M}_1 \times \overline{M}_2$ depends only on $g_1 \times g_2$ up to dilations. But this extends to $\boldsymbol{h}$ on $\tilde{M}$ since, via (34), there is a formula for $\boldsymbol{h}$ of the form (32) on a dense subspace of $\tilde{M}$. □

Remark. From Proposition 10 it follows that, from the conformal point of view, when $\lambda \neq 0$ with $\max(m_1, m_2) \geq 2$ there is no loss of generality in setting $\mu = 1$.

As a special case of Proposition 9 note that for an Einstein manifold (M^m, g) of scalar curvature $m(m-1)\lambda$, $\lambda > 0$, an ambient metric is given by

$$\lambda s_1^2 g + ds_1^2 - ds_2^2 \tag{35}$$

on $M \times \mathbb{R}_+ \times \mathbb{R}_+$. We may view this as the product of the metric cone with the cone over a point. (In fact we could allow s_2 to range over $\mathbb{R}$ but this extension is not critical for our current discussions.) There is an obvious variant of this for the case $\lambda < 0$.

The observations above lead to Theorem 12 below. First we note that there is an obvious consequence of the ambient construction in Theorem 11. An ambient manifold, as described in Section 0.11, can only be Ricci-flat if the Fefferman-Graham obstruction tensor $\mathcal{O}$ is identically zero [**54, 71**]. Thus we have the following.

Corollary 3. *Suppose that g is a metric conformally related to a product metric $g_1 \times g_2$, where $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$ are Einstein structures such that $m_2(m_2 - 1)scal^{g_1} = -m_1(m_1-1)scal^{g_2} \neq 0$. Then the obstruction tensor $\mathcal{O}^g$ is everywhere vanishing.*

As mentioned earlier, it was already known that the obstruction tensor necessarily vanishes on manifolds that are conformally Einstein (or more generally it vanishes on conformally almost-Einstein manifolds as below). Thus part of the importance of Corollary 3 above is that, according to the next theorem, it gives a more general class of structures for which the obstruction vanishes identically. Recall that an almost-Einstein structure [**65**] on a manifold is a conformal structure with a parallel standard tractor (cf. Section 1.2).

Theorem 12. *Suppose that $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$, are Einstein structures such that $m_2(m_2-1)scal^{g_1} = -m_1(m_1-1)scal^{g_2} \neq 0$. Then the product metric $g_1 \times g_2$ on $M_1 \times M_2$ is conformally almost-Einstein if and only if either $[g_1]$ admits two linearly independent almost-Einstein structures or $[g_2]$ admits two linearly independent almost-Einstein structures.*

Proof. Suppose that $[g_1 \times g_2]$ is conformally almost-Einstein. Then the standard tractor bundle admits a parallel tractor I. From Theorem 15 the holonomy of the standard tractor bundle is canonically the same as the holonomy group for the ($\mathcal{Q}$ connected component of the) ambient metric (29). Thus there is a corresponding parallel vector field $\boldsymbol{I}$ on the ambient space. Since we have a product connection on the ambient space, the projections of $\boldsymbol{I}$, $pr_1(\boldsymbol{I}) \in \mathfrak{X}(\overline{M}_1)$ and $pr_2(\boldsymbol{I}) \in \mathfrak{X}(\overline{M}_2)$ are each parallel. It follows that one of these, without loss of generality $\boldsymbol{I}_1 := pr_1(\boldsymbol{I})$ is not-zero. So $\overline{M}_1$ has the parallel vector field $\boldsymbol{I}_1$. It follows that this is clearly also parallel for the ambient metric

$$\boldsymbol{h}_1 := \lambda s_1^2 g_1 + ds_1^2 - ds_2^2$$

of M_1 (cf. (35)). Note that from the construction of $\boldsymbol{I}_1$ on this ambient space as a trivial extension of a vector field on the cone $\overline{M}_1$, it follows immediately that $ds_2(\boldsymbol{I}_1) = 0$. On the other hand the vector field $V = \partial/\partial s_2$ is also clearly parallel for $\boldsymbol{h}_1$ and linearly independent of $\boldsymbol{I}_1$ (since $ds_2(V) = 1 \neq 0$). Linearly independent parallel vectors on the ambient manifold determine linearly independent parallel standard tractors for the normal tractor connection [**38, 70**] and so $(M_1, [g_1])$ has two almost-Einstein structures.

In the other direction. If $(M_1, [g_1])$ has two linearly independent parallel tractors then, once again using Theorem 15, there are two corresponding, linear independent, parallel vector fields on the ambient space $\tilde{M}$. At least one of these projects to a non-zero parallel vector field on $\overline{M}_1$. Then obviously this parallel field on $\overline{M}_1$ also yields a parallel vector field for the product metric on $\tilde{M} = \overline{M}_1 \times \overline{M}_2$. Thus it determines a parallel tractor on $(M_1 \times M_2, [g_1 \times g_2])$. □

Remark. There exist manifolds that admit exactly one Einstein structure (up to constant dilation of the metric), see Section 2.7 for examples.

Note that it follows from Theorem 12 that if, for example, $[g_1 \times g_2]$ admits an Einstein scale, then on one of the components, say M_1 without loss of generality, on an open dense set the conformal structure $[g_1]$ admits two independent Einstein scales. (Of course we may take one of these to be the Einstein scale on all of M_1 assumed in Theorem 12.) Conversely if we have that M_1 admits two Einstein scales then on an open dense subset of $M_1 \times M_2$ the metric $g_1 \times g_2$ is conformally Einstein.

As a slight digression we point out that when there are multiple (almost-) Einstein scales then these are never isolated. Since almost-Einstein structures are exactly parallel sections of the standard normal conformal tractor bundle [**65**], it follows that if there are two distinct almost-Einstein structures then there is a 2-dimensional family ($\mathbb{R}^2 \setminus \{0\}$) of such structures.

For our later considerations we observe some basic results concerning the Euler vector field for cone products. Given a tensor field on a manifold we use the same notation for the trivial extension of this field to a field on a product of the manifold with another. For a pair of semi-Riemannian manifolds $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, the product metric $\overline{g}_1 \times \overline{g}_2$ is given, using this convention, as a covariant 2-tensor on $\overline{M}_1 \times \overline{M}_2$, by $\overline{g}_1 + \overline{g}_2$. For a constant α we say a vector field V is an α-homothety of a metric g if $\mathcal{L}_V g = \alpha g$ (cf. Section 0.6).

Lemma 7. *Given semi-Riemannian manifolds $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, $\boldsymbol{X}_1$ is an α-homothety of $\overline{g}_1$ and $\boldsymbol{X}_2$ is an α-homothety of $\overline{g}_2$ if and only if $\boldsymbol{X}_1 + \boldsymbol{X}_2$ is an α-homothety of $(\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \times \overline{g}_2)$.*

Proof. From the definition of the Lie derivative it follows immediately that, for any tensor T on $\overline{M}_1$, its trivial extension to $\overline{M}_1 \times \overline{M}_2$ satisfies the condition that $\mathcal{L}_V T = 0$ for any vector field V on $\overline{M}_2$. Of course we can swap the roles of $\overline{M}_1$ and $\overline{M}_2$ in this statement. Using this the result follows from the bilinearity and naturality of the Lie derivative:

$$\begin{aligned}\mathcal{L}_{\boldsymbol{X}_1+\boldsymbol{X}_2}(\overline{g}_1 + \overline{g}_2) &= \mathcal{L}_{\boldsymbol{X}_1}\overline{g}_1 + \mathcal{L}_{\boldsymbol{X}_1}\overline{g}_2 + \mathcal{L}_{\boldsymbol{X}_2}\overline{g}_1 + \mathcal{L}_{\boldsymbol{X}_2}\overline{g}_2 \\ &= \mathcal{L}_{\boldsymbol{X}_1}\overline{g}_1 + \mathcal{L}_{\boldsymbol{X}_2}\overline{g}_2 \ .\end{aligned}$$

Note that (for $i = 1, 2$) $\mathcal{L}_{\boldsymbol{X}_i}\overline{g}_i$ is the trivial extension to the product of a tensor on $\overline{M}_i$. Thus we have $\mathcal{L}_{\boldsymbol{X}_1+\boldsymbol{X}_2}(\overline{g}_1 + \overline{g}_2) = \alpha(\overline{g}_1 + \overline{g}_2)$, if and only if $\mathcal{L}_{\boldsymbol{X}_1}\overline{g}_1 = \alpha\overline{g}_1$ and $\mathcal{L}_{\boldsymbol{X}_2}\overline{g}_2 = \alpha\overline{g}_2$. □

On functions the Lie and exterior derivative agree and so by almost the same argument we have the following result.

Lemma 8. *Given semi-Riemannian manifolds* $(\overline{M}_1, \overline{g}_1)$ *and* $(\overline{M}_2, \overline{g}_2)$, $\boldsymbol{X}_1$ *is a gradient vector field on* $\overline{M}_1$ *and* $\boldsymbol{X}_2$ *is a gradient vector field on* $\overline{M}_2$ *if and only if* $\boldsymbol{X}_1 + \boldsymbol{X}_2$ *is a gradient vector field on* $(\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \times \overline{g}_2)$.

PROOF. Since, for $i = 1, 2$, the vector fields $\boldsymbol{X}_i$ are tangential to the leaf submanifolds, we have

$$(\overline{g}_1 + \overline{g}_2)(\boldsymbol{X}_1 + \boldsymbol{X}_2, \cdot) = \overline{g}_1(\boldsymbol{X}_1, \cdot) + \overline{g}_2(\boldsymbol{X}_2, \cdot).$$

If (for $i = 1, 2$) $\overline{g}_i(\boldsymbol{X}_i, \cdot) = d_i f_i$ then summing shows that the left-hand side is $d(f_1 + f_2)$. (Here d_i denotes the exterior derivative on the factor manifolds which may be identified with the restriction of the exterior derivative d on $\overline{M}_1 \times \overline{M}_2$.) On the other hand if we have $(\overline{g}_1 + \overline{g}_2)(\boldsymbol{X}_1 + \boldsymbol{X}_2, \cdot) = df$, for some function f on $\overline{M}_1 \times \overline{M}_2$ then by restriction we obtain that $\overline{g}_i(\boldsymbol{X}_i, \cdot) = d_i f$ which shows that, on any leaf of the product, $\boldsymbol{X}_i$ is a gradient field. So $\boldsymbol{X}_i$ is a gradient on $\overline{M}_i$. □

Recall, from the proof of Theorem 11, that on the ambient manifold there is a canonical Euler vector field $\boldsymbol{X}$. This is the Euler field from the $\mathbb{R}_+$-action on $\mathcal{Q}$ and the trivial extension of this action via the product $\mathcal{Q} \times I = \tilde{M}$. From the formula (29) for the metric we see that $\boldsymbol{X}$ is a homothetic gradient field. This has an obvious origin in the case of a metric cone product construction, as follows.

Proposition 11. *If the ambient metric is a product of cone metrics, as in Proposition 9, then the canonical ambient Euler vector field* $\boldsymbol{X}$ *is the sum of the Euler fields for the metric cones* $\overline{M}_1$ *and* $\overline{M}_2$.

PROOF. Suppose that the ambient space $(\tilde{M}, \boldsymbol{h})$ is a product $(\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \times \overline{g}_2)$, as in Proposition 9. Each metric cone $(\overline{M}_i, \overline{g}_i)$ $(i = 1, 2)$ has an Euler field $\mathbf{E}_i$ which is a 2-homothetic gradient field (cf. Section 0.9). Thus from the previous lemmata $\mathbf{E}_1 + \mathbf{E}_2$ is a 2-homothetic gradient field on $\tilde{M}$.

In terms of the coordinates used for the ambient metric in expression (29) the ambient Euler field is $t\partial/\partial t$. Using (33) and (34) this is easily re-expressed in terms of the cone coordinates s_1 and s_2:

$$\begin{aligned} t\partial/\partial t &= t\big(\partial s_1 \partial t \partial/\partial s_1 + \partial s_2 \partial t \partial/\partial s_2\big) \\ &= \frac{1}{2}(s_1 + s_2)\big((1 + \mu\rho)\partial/\partial s_1 + (1 - \mu\rho)\partial/\partial s_2\big) \\ &= s_1 \partial/\partial s_1 + s_1 \partial/\partial s_1. \end{aligned}$$

Thus $\boldsymbol{X} = \mathbf{E}_1 + \mathbf{E}_2$. □

Remark. Note that using Lemma 7, Lemma 8 and the formula (32) one can see immediately that the metric $\boldsymbol{h}_\times$ is a Ricci-flat ambient metric without performing coordinate transformations to put it in form of (29): Writing $\mathbf{E}_i$ $(i = 1, 2)$ for the respective cone Euler fields, $\mathbf{E}_1 + \mathbf{E}_2$ is a 2-homothetic gradient field for $\boldsymbol{h}_\times$ (and so property (i) of the ambient metric definition is satisfied). Along the hypersurface $s_1 = s_2$, the metric $\boldsymbol{h}_\times$ obviously restricts to the tautological bilinear form for the conformal structure $\mathcal{Q} \to (M_1 \times M_2)$ (and so property (ii) of the ambient metric definition is satisfied). As mentioned earlier it is a product of Ricci-flat metrics and therefore Ricci-flat.

We should point out that, there is nevertheless considerable value in the normal form (29) for the ambient metric. This form has a very useful geometric interpretation, as outlined in [**54**]. For our purposes here, it enabled an extension of the cone product metric to a larger manifold. It also is valid for the case when the boundary structure is conformal to a product of Ricci-flat metrics (a case for which the metric cones are unavailable). Finally giving the ambient metric in this form yields immediate contact with the previous explicit treatments of the ambient manifold such as [**54, 74**] (where this normalisation of the ambient metric is also used).

3. The Poincaré Metric

Let

$$\boldsymbol{h} := 2tdtd\rho + 2\rho dt^2 + t^2\tilde{g}(x,\rho)_{ij}dx^i dx^j,$$

be a Ricci-flat ambient metric on $\tilde{M} = \mathcal{Q} \times \tilde{I}$ for the conformal class of $g = \tilde{g}(x,0)_{ij}dx^i dx^j$ on the base space M (cf. (10)). We have seen in Section 0.11, in general, that the restriction of $\boldsymbol{h}$ to the hypersurface given by the zero set of the defining function $\boldsymbol{h}(\boldsymbol{X},\boldsymbol{X}) + 1 = 0$ in $\tilde{M}$ gives rise to a Poincaré-Einstein space (M^+, g^+) with boundary $(M,[g])$. In particular, we may apply this to the ambient metric (29) from Theorem 11. To respect that $(M,g) = (M_1 \times M_2, g_1 \times g_2)$, in that case we write $(M^{1,2}, g^{1,2})$ for (M^+, g^+) and have the following result.

Theorem 13. *To each pair of Einstein manifolds $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$, ($m_1 \geq 1$, $m_2 \geq 0$) satisfying $m_2(m_2-1)scal^{g_1} = -m_1(m_1-1)scal^{g_2}$, there is a Poincaré-Einstein manifold $(M^{1,2}, g^{1,2})$, with*

$$Ric(g^{1,2}) = -(m_1+m_2)g^{1,2},$$

and conformal infinity $(M_1 \times M_2, [g_1 \times g_2])$. This is given explicitly by

$$M^{1,2} = M_1 \times M_2 \times I$$

where, with μ satisfying $2m_1(m_1-1)\mu := scal^{g_1}$ and $2m_2(m_2-1)\mu := -scal^{g_2}$, we have

$$I = \begin{cases} [0,\infty) & \textit{if } \mu = 0, \\ [0,\infty) & \textit{if } \mu < 0 \textit{ and } m_2 = 0 \\ [0,\infty) \setminus \{\sqrt{\frac{2}{\mu}}\} & \textit{if } \mu > 0 \;\; \textit{and} \;\; m_2 = 0, \\ [0,\infty) \setminus \{\sqrt{\frac{2}{|\mu|}}\} & \textit{otherwise.} \end{cases}$$

and

$$g^{1,2} = r^{-2}(dr^2 + (1-\mu r^2/2)^2 g_1 + (1+\mu r^2/2)^2 g_2).$$

Note that in the special cases $m_1, m_2 \leq 1$ there is a family of Poincaré metrics parametrised by μ. Obviously, in any case the Poincaré-Einstein metric $g^{1,2}$ of Theorem 13 has the form $r^{-2}(dr^2 + g_r)$, where g_r is given by the power series expansion

$$g_r \;=\; g^{(0)} \;+\; g^{(2)}r^2 \;+\; g^{(4)}r^4$$

with $g^{(0)} = g_1 + g_2$, $g^{(2)} = \mu \cdot (-g_1 + g_2) = \mathsf{P}^{g_1} + \mathsf{P}^{g_2} = \mathsf{P}^{g_1+g_2}$ and $g^{(4)} = \frac{\mu^2}{4} \cdot (g_1 + g_2) = \frac{\|\mathsf{P}\|^2_{g_1+g_2}}{4(m_1+m_2)} \cdot (g_1 + g_2)$. The bilinear form $g^{(4)}$ has no trace-free part, since the Bach tensor B of the product metric $g_1 + g_2$ vanishes. Its trace is determined to $\frac{1}{4}\|\mathsf{P}\|^2_{g_1+g_2}$. No higher order expansion terms for g_r do occur (cf. [**74, 87**]).

4. The Ambient Metric over a Poincaré-Einstein Metric

We digress briefly to observe here that the above construction of the Poincaré-Einstein metric is reversible, and this gives a notion of an ambient metric over any Poincaré-Einstein metric. We recover the ambient metric as a simple extension of the metric cone over the interior (or *bulk*) of the Poincaré metric structure. We will use this result in Section 2.6. For simplicity of exposition we will assume that the Poincaré-Einstein structure is smooth, however the construction extends in an obvious way to metrics with some specified regularity.

Suppose that $(\underline{M}^{n+1}, \underline{g}, r)$ is a Poincaré-Einstein structure. That is $\underline{M}$ is a manifold with boundary a smooth manifold $\partial\underline{M} = M$, r is a non-negative defining function for M, and, off the boundary $g^+ := r^{-2}\underline{g}$ is Einstein with scalar curvature $-n(n+1)$. Then, as mentioned in Section 0.11, the restriction of $\underline{g}$ to TM in $TM^+|_M$ determines a conformal structure $[g]$. We define the ambient manifold over $(\underline{M}^{n+1}, \underline{g}, r)$ to be $\tilde{M} = \underline{M} \times \mathbb{R}_+$. We write $\pi : \tilde{M} \to \underline{M}$ for the projection $\tilde{M} \ni (p, u) \mapsto p \in \underline{M}$ and $\mathcal{Q} := \pi^{-1}(M)$.

The manifold $\tilde{M}$ is equipped with a metric and smooth structure as follows. Off the boundary we use the usual product smooth structure on $M^+ \times \mathbb{R}_+$. We will use u here for the standard coordinate on $\mathbb{R}_+$. The defining function determines, for some $\epsilon > 0$, an identification of $M \times [0, \epsilon)$ with a neighbourhood of M in $\underline{M}$: since $|dr|_{\underline{g}}$ is non-vanishing along the boundary, $(p, y) \in M \times [0, \epsilon)$ is identified with the point obtained by following the flow of the gradient $grad^{\underline{g}} r$, through p, for y units of time. Thus over this we also have an identification of $\tilde{M}$ with $M \times [0, \epsilon) \times \mathbb{R}_+$. Suppose that x^i are local coordinates on $U \subset M$ then on $U \times (0, \epsilon) \times \mathbb{R}_+$ we have coordinates (x^i, r, u). We construct a coordinate patch for $\tilde{M}$ over $U \times [0, \epsilon)$ by taking coordinates (x^i, ρ, t) on $U \times (-\epsilon^2/2, 0] \times \mathbb{R}_+$ and identifying this space with $\pi^{-1}(U \times [0, \epsilon))$ by the coordinate transformation $\rho = -\frac{1}{2}r^2$, $t = u/r$ on $\pi^{-1}(U \times (0, \epsilon))$. This is obviously independent of the coordinates x^i, local on M. Thus, by doing this for all coordinate patches on M, this extends a smooth structure to $\tilde{M}$.

We take the cone metric $h := u^2 g^+ - du^2$ on $\pi^{-1}(M^+)$. This cone metric is Ricci flat and so it remains to verify that it extends to a non-degenerate metric on $\mathcal{Q}$.

The condition $Ric(g^+) = -ng^+$ implies that $|dr|_{\underline{g}} = 1$ on M. However a choice of metric g (from the conformal class) on M determines a unique special defining function r, in a neighbourhood of M, by requiring $|dr|_{\underline{g}} = 1$ and $\underline{g}|_{TM} = g$. The special defining function determines, for some $\epsilon > 0$, an identification of $M \times [0, \epsilon)$ with a neighbourhood of M in $\underline{M}$ and in terms of this the metric g^+ takes the form $g^+ = r^{-2}(g_r + dr^2)$, where g_r is a 1-parameter family of metrics on M (cf. [**77, 74**]). The change from a general defining function to one that satisfies $|dr|_{\underline{g}} = 1$ in a neighbourhood of M is achieved by a smooth rescaling $r \mapsto e^{\omega} r$ (for some smooth function ω) thus assuming that we have such a normalised r does not affect the smooth structure on $\tilde{M}$. Using (14) and the coordinate transformation $\rho = -\frac{1}{2}r^2$, $t = u/r$ on $\pi^{-1}(U\times(0,\epsilon))$ it follows easily that the metric h may be written in the form (10) and so obviously extends as a metric to $\mathcal{Q}$. Note that the coordinate change $\rho = -r^2/2$ means the ambient metric is not smooth at the boundary in general. In fact from the Einstein condition it follows that the Taylor series of g_r involves only even powers of r up to the r^n term, and so the ambient metric is differentiable to any order less than $n/2$ (cf. [**77**]).

Remark. There is an obvious variant of the above construction where one would only assume the Poincaré metric is asymptotically Einstein. In this case the ambient metric will be asymptotically Ricci-flat.

5. Holonomy

Let (M, g) be a semi-Riemannian signature (p, q)-manifold. We write $\mathcal{F}_q$ to denote a frame based at $q \in M$ and $\mathcal{F}_q \cdot A$ for the obvious action of $A \in \mathrm{O}(p, q)$ acting on $\mathcal{F}_q$. If ϕ_q is the trace of a closed curve based at q then we write $\mathcal{F}_q^{\phi_q}$ for the frame obtained from $\mathcal{F}_q$ by parallel translation around ϕ_q. Recall (from Section 0.3 and 0.4) that the holonomy, based at q, of the metric g is by definition the group

$$Hol_q(M, g) = \{A \in \mathrm{O}(p, q) \ : \ \text{for any frame } \mathcal{F}_q \ \exists\ \phi_q \text{ s.t. } \mathcal{F}_q^{\phi_q} = \mathcal{F}_q \cdot A\} \ .$$

Now suppose that on M there we have a homothetic gradient field V. That is a constant c such that $\mathcal{L}_V g = cg$ or, equivalently, $\nabla^g_U V = \frac{c}{2}U$ for all $U \in \mathfrak{X}(M)$. Let us say a hypersurface E in M is *V-transverse* if each maximal integral curve of V meets E in exactly one point. In this setting the holonomy of g is recovered from curves ϕ^E in the V-transverse submanifold E. More precisely we have the following.

Theorem 14. *Let (M, g) be a semi-Riemannian signature (p, q)-manifold with a nowhere-vanishing homothetic gradient field V and a V-transverse hypersurface E. Then for $q \in E$*

$$Hol_q(M, g) = \{A \in \mathrm{O}(p, q) \ : \ \textit{for any frame } \mathcal{F}_q \ \exists\ \phi^E_q \subset E \ \textit{ s.t. } \mathcal{F}_q^{\phi^E_q} = \mathcal{F}_q \cdot A\} \ .$$

We need some preliminary notation and results before we prove Theorem 14. Let us parametrise the integral curves γ^V of V by a smooth function s on M which vanishes on E. For $q \in M$ let us fix attention on a closed smooth path $\phi_q : [0, 1] \to M$, $\phi_q(0) = \phi_q(1) = q$. Over ϕ_q we construct a 2-parameter *path-cone*:

$$\Gamma_q : D \subset [0, 1] \times \mathbb{R} \to M$$

given by

$$\Gamma_q(t, s) := V_s(\phi_q(t)),$$

where D is the open subset of $[0, 1] \times \mathbb{R}$ which gives the maximal range of definition of the flow V_s of V (s runs over an interval that depends on t). By analogy with our treatment of curves we will also use Γ_q to denote the trace (graph) of this function in M since in any instance the meaning should be clear by context. Now for $F_q \in T_qM$ we write $F_q(t)$ for the field (at time t) along the trace of $\phi_q(t)$ given by parallel translation of F_q. We extend this to the path cone by parallel translation along the flow lines of V; we write $F_q(t, s)$ for the vector in $T_{\Gamma_q(t,s)}M$ given by the parallel transport of $F_{\phi_q(t)}$ to $V_s(\phi_q(t))$ along the integral curve of V through $\phi_q(t)$. We need to compare parallel transport in this way with Lie dragging.

Lemma 9.
$$F_q(t,s) = (1 - \frac{c}{2}s)V_{s*}(F_q(t)).$$

PROOF. Since the Levi-Civita connection ∇ is torsion free we have
$$\nabla_V F_q(t,s) = \nabla_{F_q(t,s)}V + \mathcal{L}_V F_q(t,s).$$
This vanishes since by construction $F_q(t,s)$ is parallel along the flow lines of V. On the other hand, since V is homothetic we have $\nabla_{F_q(t,s)}V = \frac{c}{2}F_q(t,s)$ so
$$\mathcal{L}_V F_q(t,s) = -\frac{c}{2}F_q(t,s).$$
□

This sets us up for the key result on the path-cone which is as follows.

Lemma 10. *The field $F_q(t,s)$ is parallel along Γ_q.*

PROOF. Let us write $\dot{\phi}^s_q(t)$ for a tangent field to the curve
$$\Gamma_q(\cdot,s) : [0,1] \to M$$
determined by a fixed value of s. Now it follows from the s derivative of $\Gamma_q(t,s)$ that $\dot{\phi}^s_q(t) = V_{s*}(\dot{\phi}^s_q(t))$. This with the previous lemma implies
$$\nabla_{\dot{\phi}^s_q(t)}F_q(t,s) = (1 - \frac{c}{2}s)\nabla_{V_{s*}(\dot{\phi}^s_q(t))}V_{s*}(F_q(t)),$$
since s is constant. But since V is a homothetic gradient field its flow preserves the Levi-Civita connection. So
$$\nabla_{V_{s*}(\dot{\phi}^s_q(t))}V_{s*}(F_q(t)) = V_{s*}(\nabla_{(\dot{\phi}^s_q(t))}F_q(t)) = 0.$$
So at each point of Γ_q, $F_q(t,s)$ is parallel in the direction $\dot{\phi}^s_q(t)$. But it is also parallel in the direction of V and so the result follows. □

PROOF OF THEOREM 14. First observe that since each integral curve of V meets the V-transverse hypersurface in exactly one point, there is a canonical smooth projection $\pi : M \to E$ which for $p \in M$ finds the point $\pi(p) \in E$ on the flow through p. Thus given an arbitrary closed path ϕ_q (based at $q \in E$) there is a path ϕ^E_q with trace in E given by $\phi^E_q(t) = \pi \circ \phi_q(t)$. This determines a function $s_E(t)$ which gives the value of the parameter s where E meets Γ_q. That is $\phi^E_q(t) = \Gamma_q(t, s_E(t))$. There is a vector field along the trace of ϕ^E:
$$F^E_q(t) := F_q(t, s_E(t)).$$
From the previous lemma this is parallelly transported around ϕ^E:
$$\nabla_{\dot{\phi}^E_q(t)}F^E_q(t) = 0.$$
By construction $F_q(1) = F^E_q(1)$, since $s_E(1) = 0$. Since this holds for all vectors $F_q \in T_qM$ and for all closed paths ϕ_q the proof is complete. □

Theorem 15. *The holonomy group of the $\mathcal{Q}$ connected component of the ambient manifold, with metric (29), is the same as the conformal holonomy of the underlying conformal manifold $(M_1 \times M_2, [g_1 \times g_2])$.*

PROOF. We will retain only the $\mathcal{Q}$ connected component of the ambient manifold and term this the ambient manifold.

First we treat the case that $scal^{g_1} = m_1(m_1-1)\lambda$ for $\lambda \neq 0$. Then $(\tilde{M}, \boldsymbol{h})$ is a product of cones $\overline{M}_1$ and $\overline{M}_2$ with metrics given, respectively, as in (31). The holonomy of the product, the ambient manifold, is the product of the component holonomy groups. On each cone there is a homothetic gradient field in the sense of Theorem 14 above. These are respectively $s_1\partial/\partial s_1$ and $s_2\partial/\partial s_2$. Thus on each cone $\overline{M}_i$ $(i=1,2)$ the holonomy may be computed by considering paths only in the $s_i = 1$ transverse hypersurface. It follows easily that the ambient holonomy is generated by (the transport of full frames for $T(\overline{M}_1 \times \overline{M}_2)$ along) loops in the codimension 2 submanifold $s_1 = 1 = s_2$. But this submanifold is in $\mathcal{Q}$ and is a section over $M_1 \times M_2$ and so this holonomy group is exactly the conformal holonomy, that is holonomy of the normal tractor connection. To see this last claim we use following result of [**38**] (see also [**70**]). Write $T\tilde{M}|_{\mathcal{Q}}$ for the restriction of the ambient tangent bundle to $\mathcal{Q}$ and define an action of $\mathbb{R}_+$ on this space by $s^{-1}\sigma^s_* \cdot \xi$. Here σ is the principal action of $\mathbb{R}_+$ on $\mathcal{Q}$ given by $\sigma^s(g_x) = s^2 g_x$. Then the quotient $(T\tilde{M}|_{\mathcal{Q}})/\mathbb{R}_+$ is a vector bundle over $\mathcal{Q}/\mathbb{R}_+ = M$. This may be identified with the standard conformal tractor bundle and via this identification the ambient parallel transport induces the normal tractor connection. It follows immediately that parallel transport along any fixed section of $\mathcal{Q}$ is sufficient to recover the conformal holonomy.

Now in the case of $\lambda = 0$ suppose the ambient metric is given by (29). Then $\partial/\partial\rho$ is parallel, and hence a homothetic gradient field. This is obviously transverse to $\mathcal{Q}$ and so the ambient holonomy may be calculated by paths in $\mathcal{Q}$. From the result that the $\mathbb{R}_+$-action on $(T\tilde{M}|_{\mathcal{Q}})$ by $\xi \mapsto -1\sigma^s_* \cdot \xi$ agrees with parallel transport [**38, 70**] it follows easily that the ambient holonomy may be calculated via loops in a section of $\mathcal{Q}$ and thus agrees with the conformal holonomy. $\square$

6. Characterisation by Special Killing Forms

We want to characterise sub-product spaces $(M^{1,2}, g^{1,2})$ (as they were constructed in Theorem 13) by the existence of certain special Killing forms (cf. Section 0.4).

To start with, let

$$g^{1,2} = r^{-2}(dr^2 + (1-\mu r^2/2)^2 g_1 + (1+\mu r^2/2)^2 g_2)$$

be a Poincaré-Einstein metric on $M^{1,2} = M_1 \times M_2 \times I$ as in Theorem 13. We assume here that the factors $M_1^{m_1}$ and $M_2^{m_2}$ of the sub-product are oriented spaces of dimensions $m_1 \geq 1$ and $m_2 \geq 0$ and denote by $vol(g_i)$, $i=1,2$, the corresponding volume forms to g_1, resp., g_2. We denote the pull-backs of these volume forms to the sub-product $M^{1,2}$ by $vol(g_i)$, $i=1,2$, as well.

Lemma 11. *Let $\mu > 0$. The m_1-form*

$$\psi := \left(\frac{\mu r}{2} - \frac{1}{r}\right)^{m_1+1} \cdot \ vol(g_1)$$

on $(M^{1,2}, g^{1,2})$ is special Killing (cf. Section 0.4). The function $\tilde{r} := (|\psi|^2_{g^{1,2}})^{-1/2}$ is a defining function for the conformal boundary $\partial M^{1,2} := M_1 \times M_2$ with $|d\tilde{r}|_{\partial M^{1,2}} = 1$.

Proof. We introduce the coordinate $s=\ln(\sqrt{\frac{\mu}{2}}\cdot r)$ and set $h_1:=\sqrt{2\mu}\ \cdot\ \sinh(s)=(\frac{\mu r}{2}-\frac{1}{r})$ and $h_2:=\sqrt{2\mu}\ \cdot\ \cosh(s)$. Then the the metric $g^{1,2}$ takes the form

$$ds^2+2\mu(\sinh^2(s)\cdot g_1+\cosh^2(s)\cdot g_2)\ .$$

Let $\{e_1,\dots,e_{m_1}\}$, resp., $\{f_1,\dots,f_{m_2}\}$ denote local orthonormal frames for (M_1,g_1) and (M_2,g_2). We set $g^1_{ij}:=g_1(e_i,e_j)$ for $i,j\in\{1,\dots,m_1\}$ and $g^2_{ij}:=g_2(f_i,f_j)$ for $i,j\in\{1,\dots,m_2\}$. Moreover, we denote by $e_i^\flat$ and $f_i^\flat$ the dual 1-forms with respect to g_1, resp., g_2. Their pull-backs give rise to a local (orthogonal) coframe $\{ds,e_1^\flat,\dots,e_{m_1}^\flat,f_1^\flat,\dots,f_{m_2}^\flat\}$ on $M^{1,2}$. Locally, we have $\psi=h_1^{m_1+1}\ \cdot\ e_1^\flat\wedge\cdots\wedge e_{m_1}^\flat$. For the covariant derivatives we have (cf. [**128**]; also Section 0.9)

$$\begin{aligned}
&\nabla^{1,2}_{\frac{\partial}{\partial s}}e_i^\flat=-\tfrac{h_1'}{h_1}\cdot e_i^\flat, && \nabla^{1,2}_{\frac{\partial}{\partial s}}f_i^\flat=-\tfrac{h_2'}{h_2}\cdot f_i^\flat, && \nabla^{1,2}_{\frac{\partial}{\partial s}}ds=0,\\
&\nabla^{1,2}_{e_j}e_i^\flat=\nabla^{g_1}_{e_j}e_i^\flat-\tfrac{h_1'}{h_1}g^1_{ji}ds, && \nabla^{1,2}_{e_j}f_i^\flat=0, && \nabla^{1,2}_{e_j}ds=h_1'h_1\cdot e_j^\flat,\\
&\nabla^{1,2}_{f_j}f_i^\flat=\nabla^{g_2}_{e_j}e_i^\flat-\tfrac{h_2'}{h_2}g^2_{ji}ds, && \nabla^{1,2}_{f_j}e_i^\flat=0, && \nabla^{1,2}_{f_j}ds=h_2'h_2\cdot f_j^\flat,
\end{aligned}$$

where $h_i':=\frac{\partial}{\partial s}h_i$ and ∇^{g_i}, $i=1,2$, denote the Levi-Civita connections of g_1, resp., g_2. We obtain

$$\nabla^{1,2}_{e_i}\psi=-g^1_{ii}h_1'h_1^{m_1}\cdot e_1^\flat\wedge\cdots\wedge ds\wedge\cdots\wedge e_{m_1}^\flat=(-1)^i g^1_{ii}\frac{h_1'}{h_1}\cdot ds\wedge(\iota_{e_i}\psi)$$

(where in the middle part of this equation ds replaces e_i at the i-th position of the $\wedge$-product),

$$\nabla^{1,2}_{\frac{\partial}{\partial s}}\psi=\frac{h_1'}{h_1}\psi\qquad\text{and}\qquad\nabla^{1,2}_{f_i}\psi=0\ .$$

This implies $d\psi=(m_1+1)\cdot\frac{h_1'}{h_1}ds\wedge\psi$, which shows that ψ is a Killing form, i.e., $\nabla^{1,2}\psi=\frac{1}{m_1+1}d\psi$. Moreover, we calculate

$$\begin{aligned}
&\nabla^{1,2}_{\frac{\partial}{\partial s}}d\psi=(m_1+1)\cdot\tfrac{h_1''}{h_1}ds\wedge\psi=(m_1+1)\cdot g^{1,2}(\tfrac{\partial}{\partial s},\cdot)\wedge\psi,\\
&\nabla^{1,2}_{f_i}d\psi=(m_1+1)\cdot\tfrac{h_1'h_2'h_2}{h_1}f_j^\flat\wedge\psi=(m_1+1)\cdot g^{1,2}(f_i,\cdot)\wedge\psi\qquad\text{and}\\
&\nabla^{1,2}_{e_i}d\psi=0\ .
\end{aligned}$$

These identities show that the Killing form ψ is special.

For the square length of ψ with respect to $g^{1,2}$ we have $|\psi|^2=h_1^2$ and this shows that $\tilde r=-h_1^{-1}$, which vanishes when r tends to zero. Hence $\tilde r$ is a defining function with $|d\tilde r|=1$ on the conformal boundary $\partial M^{1,2}$. □

We remark that the assumption $\mu>0$ in Lemma 11 is not essential, since we do not make any assumption on the signature of the metric $g^{1,2}$.

In general, we know from Section 2.4 that the metric cone over an (oriented) Poincaré-Einstein space $(\underline{M}^{n+1},\underline{g})$ gives rise to a Ricci-flat ambient metric (with boundary). On the other hand, any Fefferman-Graham ambient metric space contains the corresponding Poincaré-Einstein model as a hypersurface, defined by the zero set of the function $\boldsymbol{h}(\boldsymbol{X},\boldsymbol{X})+1$. Thus we can apply the cone correspondence for special Killing forms (cf. Proposition 2) to prove the following characterisation result for sub-products of the form $g^{1,2}$. Note, in general, that if ϕ is a special Killing form on

$(\underline{M}^{n+1}, \underline{g})$ then the $(n+1-p)$-form $\star d\phi$ is special Killing as well. We call a differential form ϕ non-degenerate and simple if it is at every point of $\underline{M}$ a $\wedge$-product of 1-forms with non-vanishing length.

Theorem 16. *Let $(\underline{M}^{n+1}, \underline{g})$ be a simply connected Poincaré-Einstein space of dimension $n+1$ with $Ric(\underline{g}) = -n\underline{g}$ and conformal boundary $\partial\underline{M} = M^n$.*

(1) *Suppose that there exists a non-degenerate and simple m_1-form ψ, which satisfies the differential equations*

$$\nabla^{\underline{g}}\psi = \gamma \wedge \psi \qquad \text{and} \qquad (\nabla^{\underline{g}}_Y \gamma) \wedge \psi = \underline{g}(Y, \cdot) \wedge \psi$$

for all $Y \in T\underline{M}$ on the bulk of $\underline{M}$, where γ is a 1-form with dual vector $\gamma^\#$ such that

$$\iota_{\gamma^\#}\psi = 0 \qquad \text{and} \qquad \underline{g}(\gamma^\#, \gamma^\#) > 1 \ ,$$

then $(\underline{M}, \underline{g})$ is a sub-product as constructed in Theorem 13 with a metric on the bulk of the form

$$r^{-2}(dr^2 + (1 - \mu r^2/2)^2 g_1 + (1 + \mu r^2/2)^2 g_2) \ .$$

(2) *If $(\underline{M}^{n+1}, \underline{g})$ is a sub-product as described in Theorem 13 then locally on the bulk of $\underline{M}$ there exists a m_1-form ψ, which satisfies the system of differential equations with respect to some 1-form γ as given in (1), and $|\psi|^{-1}$ is a defining function for the boundary.*

Proof. First, let us assume that $\underline{g} := g^{1,2}$ on $\underline{M} = M_1 \times M_2 \times I$ is of the form as described in Theorem 13. Then we know from Lemma 11 that locally (with some choice of orientation) the non-degenerate and simple differential form $\psi = \left(\frac{\mu r}{2} - \frac{1}{r}\right)^{m_1+1} \cdot vol(g_1)$ is special Killing. In particular, $\nabla^{\underline{g}}\psi = \frac{h_1'}{h_1} ds \wedge \psi$. We set $\gamma := \frac{h_1'}{h_1} ds$ and together with the formulae for the covariant derivative of ds from the proof of Lemma 11 we see that the demanded conditions are satisfied for this choice of ψ. In particular, the 1-form γ has length greater than 1. The length function $|\psi|_{\underline{g}}$ tends to infinity at the boundary and its inverse $|\psi|^{-1}_{\underline{g}}$ is locally a defining function (cf. Lemma 11).

On the other hand, let ψ be a differential form on $\underline{M}$ such that

$$\nabla^{\underline{g}}\psi = \gamma \wedge \psi \qquad \text{and} \qquad (\nabla^{\underline{g}}_Y \gamma) \wedge \psi = \underline{g}(Y, \cdot) \wedge \psi$$

for all $Y \in T\underline{M}$, where γ is some smooth 1-form with properties as in (1). The first equation implies immediately that ψ is a Killing form. The second condition implies $\nabla^{\underline{g}}_Y d\psi = (m_1+1) \cdot \underline{g}(Y, \cdot) \wedge \psi$, i.e., ψ is special Killing.

We consider now the ambient metric h on $\tilde{M} = \underline{M} \times \mathbb{R}_+$ (with boundary as in Section 2.4) over the Poincaré-Einstein space $(\underline{M}, \underline{g})$. Over the bulk of $\underline{M}$ the ambient metric h is just the cone metric $u^2 g - du^2$ and the (m_1+1)-form $\tilde{\psi} = u^{m_1} du \wedge \psi + \frac{u^{m_1+1}}{m_1+1} d\psi$ is parallel on this cone. With the assumptions on ψ and γ, it follows that $\tilde{\psi}$ is non-degenerate and simple. Moreover, since $\underline{M}$ is simply connected, the ambient space $\tilde{M}$ itself is simply connected and orientable and we can apply the Hodge star operator to $\tilde{\psi}$ to obtain a non-degenerate and simple parallel differential form $\star\tilde{\psi}$. This shows by the deRham decomposition Theorem (cf. [**141**]) that h is isometric to a product $h_1 \times h_2$ of Ricci-flat metrics on some product space $\overline{M}_1 \times \overline{M}_2$ (which includes $\tilde{M}$ as a submanifold with boundary).

Inserting the Euler vector $X = u\partial/\partial u$ into the parallel differential forms $\tilde{\psi}$ and $\star\tilde{\psi}$ reproduces the special Killing forms ψ, resp., $\star d\psi$ on $\underline{M}$. The latter differential form equals $\frac{1}{m_1+1}\star(\gamma\wedge\psi)$ and has by assumption no zeros on $\underline{M}$. This shows that the projection of the Euler vector X to the factor $\overline{M}_1$ of the product structure on the ambient space has no singularities and, in fact, is everywhere spacelike. The projection of the Euler vector X to $\overline{M}_2$ is everywhere timelike (cf. Section 1.2). Moreover, since these projections of X are homothetic gradient vector fields (cf. Lemma 7 and 8), we can conclude with Lemma 1 of Section 0.9 that h_1 and h_2 are cone metrics over some Einstein spaces (M_1, g_1) and (M_2, g_2). By choosing appropriate scales for the metrics g_1 and g_2 it is straightforward to see that $g_1 \times g_2$ is just a metric in the conformal class of the boundary $M = M_1 \times M_2$ of the initial Poincaré-Einstein space $(\underline{M}, \underline{g})$. In particular, the ambient space $(\tilde{M}, h)$ is a submanifold of the ambient space of $(M_1 \times M_2, g_1 \times g_2)$ that we introduced in Theorem 11. It follows that the initial Poincaré-Einstein space $(\underline{M}, \underline{g})$ is a sub-product space as constructed in Theorem 13. □

7. Examples and Multiple Sub-Products

It should be expected that, for Einstein manifolds, in the generic situation there exists a single Einstein metric in the conformal class (ignoring constant dilations of the metric). In fact, we give here some concrete examples of such Einstein spaces. In particular, these examples demonstrate in relation with Theorem 12 the following statement about Poincaré-Einstein spaces.

Theorem 17. *There exist explicit constructions of Poincaré-Einstein metrics whose boundary structures are not conformally Einstein.*

For example, let us consider the special orthogonal group SO(4) in dimension 4. This is a 6-dimensional compact semisimple Lie group and the Killing form $B_{\mathfrak{so}(4)}$ of the Lie algebra $\mathfrak{so}(4)$ gives rise to a bi-invariant Einstein metric g_B of negative definite signature and negative scalar curvature $-3/2$ on SO(4). In Chapter 3 we will compute the conformal holonomy algebra of the conformal class $[g_B]$ on SO(4). The result is $\mathfrak{hol}(\mathcal{T}) = \mathfrak{so}(7)$ sitting in the structure algebra $\mathfrak{so}(7,1)$. This shows that g_B is (up to constant dilations) the only Einstein metric in $[g_B]$.

Now we define $M := SO(4) \times SO(4)$ with metric $g_{B\times B} := g_B^1 \times g_B^2$, where g_B^1 is $-g_B$ on the first factor, and g_B^2 is g_B on the second factor. This is a product of Einstein metrics, which satisfies the scalar curvature relation of Theorem 11 for the construction of a Ricci-flat ambient metric. By the unique existence (up to multiples) of Einstein scales on the factors and using Theorem 12 we know that M with metric $g_{B\times B}$ is not conformally Einstein. The corresponding Poincaré-Einstein metric is explicitly given on $M \times [0, r_o)$ with $r_o = 4\cdot\sqrt{5}$ by

$$r^{-2}(dr^2 + (1 - r^2/80)^2 g_B^1 + (1 + r^2/80)^2 g_B^2)\ .$$

Another way to make examples is by using 4-manifolds in the sub-product construction. Einstein Riemannian 4-manifolds have only one Einstein scale, unless they are conformally flat. This is easily seen as follows. Suppose we have two linearly independent almost-Einstein structures on a 4-manifold. This exactly means that the manifold admits two linearly independent parallel dual standard tractors I_1 and I_2. The exterior product $I_1 \wedge I_2$ of these is obviously parallel. This (adjoint) tractor $I_1 \wedge I_2$ is a jet prolongation of a conformal gradient field k which annihilates the Weyl curvature W (i.e., $\iota_k W = 0$; cf. [**66**]). Since the parallel tractor $I_1 \wedge I_2$ is a prolongation of k and parallel it follows immediately that k is non-vanishing on an open dense set in the manifold. On the other hand in dimension 4 we have the identity $|W|^2 \delta^a_b = 4W^{acde}W_{bcde}$ (with contractions indicated by Einstein summation convention), and so $|W|^2 = 0$ on an open dense set, and hence everywhere.

Finally, we present a recursive construction principle for multiple sub-products based on Theorem 13 in order to produce Poincaré-Einstein spaces. For this purpose we set up the following initial data. Let $(M_0^{m_0}, g_0)$ be an Einstein space of negative scalar curvature $scal^{g_0} = -m_0(m_0 - 1)$ with $dim(M_0) = m_0$, i.e., $\mu = -1/2$. Further, let $l \geq 1$ be a positive integer and let $(M_i^{m_i}, g_i)$, $i \in \{1, \dots, l\}$, be Einstein spaces with positive scalar curvature $scal^{g_i} = m_i(m_i - 1)$ and $dim(M_i) = m_i$. In the first step, we set $M^{0+} := M_0^{m_0}$ with metric $G^0 := g_0$. And then for $1 \leq s \leq l$ we define recursively

$$G^s := r_s^{-2}(\ dr_s^2 \ + \ (1 + r_s^2/4)^2 \cdot G^{s-1} \ + \ (1 - r_s^2/4)^2 \cdot g_s \ )\ ,$$

which is a metric on the interior of $\underline{M}^s := M^{(s-1)+} \times M_s \times I_s$, where $I_s = [0, 2)$ is an interval of length $\sqrt{2/|\mu|} = 2$ with coordinate r_s. The interior of $\underline{M}^s$ is given by $M^{s+} := M^{(s-1)+} \times M_s \times (0, 2)$. Using Theorem 13 inductively for every step gives rise to a multiple sub-product construction of Poincaré-Einstein metrics, which can be formulated as follows.

Corollary 4. *Let $(\underline{M}^l, G^l)$, $l \geq 1$, be recursively defined as above. Then the metric G^l on the space M^{l+} of dimension $dim(\underline{M}^l) = l + \sum_{i=0}^{l} m_i$ is Poincaré-Einstein with conformal infinity*

$$(M^{(l-1)+} \times M_l, [G^{l-1} \times g_l])\ .$$

An ambient metric of the conformal structure $[G^{l-1} \times g_l]$ is explicitly given by

$$h^l := \bar{g}_0 \times \cdots \times \bar{g}_l$$

on $M_0 \times \cdots \times M_l \times \mathbb{R}_+^{l+1}$, where the $\bar{g}_i$'s denote the cone metrics of the g_i's. By Theorem 12 we can conclude that the conformal structure $[G^{l-1} \times g_l]$ at infinity is not conformally almost-Einstein if every metric g_i for $i \in \{0, \dots, l\}$ of the initial setting admits exactly one almost-Einstein scale.

CHAPTER 3

Conformal Holonomy of Bi-Invariant Metrics

We introduced in Section 0.6 and 0.7 the conformal holonomy groups $Hol(\omega_{nor})$ of the normal conformal Cartan connection and $Hol(\mathcal{T})$ of the tractor connection (which have identical Lie algebras). In Chapter 1 we have already seen that the tractor holonomy $Hol(\mathcal{T})$ is an important and useful invariant for the classification and description of conformal spaces (M^n, c). In particular, via the conformal tractor holonomy it is possible to describe invariant sub-structures, like the conformal Einstein condition, and to detect solutions of certain conformally covariant partial differential equations, e.g. conformal Killing spinors (cf. [**6, 40, 110**]). In this chapter we want to demonstrate that the conformal holonomy can be calculated explicitly, at least in such cases where the underlying conformal geometry has much symmetry. In fact, our aim is to develop an invariant calculus for canonical Cartan connections on conformally homogeneous spaces and to apply this for computations of holonomy. However, in order to simplify the situation we develop our calculus here only for the conformal geometry of (bi)-invariant metrics on (semisimple) Lie groups. Certainly, the invariant calculus works well in much more general circumstances of parabolic geometries as a recent work of M. Hammerl shows (cf. [**83**]).

We will proceed as follows in this chapter. The first section recalls the notion of bi-invariant metrics and, in Section 2, we develop the invariant conformal Cartan calculus adapted to this situation. In particular, we will describe the canonical connection and its curvature by certain linear maps $\gamma_{nor} : \mathfrak{g}_{-1} \to \mathfrak{p}$ and $\kappa : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{p}$. Then we discuss properties of these maps and derive a formula for the conformal holonomy algebra $\mathfrak{hol}(\omega_{nor})$ (cf. Section 3). Finally, we make explicit computations for the bi-invariant metric on SO(4) coming from the Killing form B.

1. Bi-Invariant Metrics

Let N be a connected and compact semisimple Lie group of dimension s and let $\mathfrak{n}$ denote its Lie algebra of left invariant vector fields. Then the Killing form

$$B_{\mathfrak{n}}(X,Y) := \mathrm{tr}\,\mathrm{ad}_X \mathrm{ad}_Y$$

is an $\mathrm{Ad}(N)$-invariant, negative definite scalar product on $\mathfrak{n}$. In particular,

$$B(X,[Y,Z]) = B([X,Y],Z) \qquad \text{for all } X,Y,Z \in \mathfrak{n}\ .$$

The Lie algebra $\mathfrak{n}$ is canonically identified with the tangent space T_eN at the identity element of N. Thus the negative $-B$ of the Killing form gives rise to an inner product on T_eN, which by left translation with L_g, $g \in N$, extends uniquely to a smooth metric $g_{\mathfrak{n}}$ on N. Since $-B$ is $\mathrm{Ad}(N)$-invariant, right translation of $-B$ with R_g, $g \in N$, generates again $g_{\mathfrak{n}}$ and thus the metric $g_{\mathfrak{n}}$ on N is called bi-invariant (cf. e.g. [**91, 128**]).

For the Levi-Civita connection $\nabla^{\mathfrak{n}}$ of the bi-invariant metric $g_{\mathfrak{n}}$ the equation

$$\nabla^{\mathfrak{n}}_X Y = \frac{1}{2}[X,Y] \qquad \text{for all} \;\; X,Y \in \mathfrak{n}$$

holds. The Riemannian curvature tensor is then given by

$$R^{\mathfrak{n}}_{XY} Z = -\frac{1}{4}[[X,Y],Z] \qquad \text{for all} \;\; X,Y,Z \in \mathfrak{n}$$

and the sectional curvature of a plane spanned by orthonormal elements X,Y in $\mathfrak{n}$ is

$$S^{\mathfrak{n}}(X,Y) := \frac{1}{4} g_{\mathfrak{n}}([X,Y],[X,Y]) \ .$$

For the Ricci tensor we have

$$Ric^{\mathfrak{n}} = \frac{1}{4} g_{\mathfrak{n}} \ ,$$

i.e., $g_{\mathfrak{n}}$ is an Einstein metric on N with positive scalar curvature $scal^{\mathfrak{n}} = \frac{s}{4}$. The Schouten tensor is $\mathsf{P}^{\mathfrak{n}} = \frac{-1}{8(s-1)} g_{\mathfrak{n}}$ and the Cotton tensor $C^{\mathfrak{n}}$ vanishes identically. The Weyl tensor is given by

$$W^{\mathfrak{n}} = R^{\mathfrak{n}} + \frac{1}{8(s-1)} g_{\mathfrak{n}} * g_{\mathfrak{n}} \ ,$$

where $*$ denotes the Kulkarni-Nomizu product.

2. Invariant Calculus for Conformal Cartan Geometry

At the end of the last section we computed directly the content of the conformal curvature (Weyl tensor and Cotton tensor) for a bi-invariant metric induced by the Killing form on a compact semisimple Lie group by use of the Lie bracket. Now we establish an invariant conformal Cartan calculus for bi-invariant metrics on Lie groups. In particular, we will describe the normal Cartan connection and its curvature with respect to a certain linear map γ_{nor}. This approach will use a global trivialisation induced by a left invariant frame of second order, which comes from the invariant metric in the conformal class. We note that, although we assume a compact semisimple Lie group with bi-invariant metric coming from the Killing form, our discussion here is actually valid for the more general situation of left invariant metrics on any Lie group. Only the explicit formulae for γ_{nor} and its curvature at the end of this section will depend on the bi-invariance.

So let N be a connected and (compact semisimple) Lie group of dimension s with Lie algebra $\mathfrak{n}$ and (bi)-invariant metric $g_{\mathfrak{n}}$. We fix the conformal structure $c_{\mathfrak{n}} = [g_{\mathfrak{n}}]$ on N. By definition, the Lie group N acts by smooth isometries from the left on $(N, g_{\mathfrak{n}})$, hence by conformal transformations on $(N, c_{\mathfrak{n}})$. This action gives rise to induced left actions of N on the first and second order conformal frame bundles $\mathcal{G}_0(N)$, resp., $\mathcal{P}(N)$. We denote all these actions by L_g for $g \in N$. Since $L_g^* \omega_{nor}$ for any $g \in N$ is a normal Cartan connection on $\mathcal{P}(N)$ inducing via the soldering form the given conformal structure $c_{\mathfrak{n}}$ on N, and since the canonical Cartan connection ω_{nor} on $\mathcal{P}(N)$ is uniquely determined by normality of its curvature, it follows immediately that

$$L_g^* \omega_{nor} = \omega_{nor} \qquad \text{for all } g \in N \ .$$

The space $\mathfrak{g}_{-1} \cong \mathbb{R}^s$ is equipped with the standard inner product $\langle\cdot,\cdot\rangle_s$ and a standard orthonormal basis $\{e_1,\ldots,e_s\}$. Now let

$$\theta : (\mathfrak{n}, -B) \;\cong\; (\mathfrak{g}_{-1}, \langle\;,\;\rangle_s)$$

be an isometry of inner product spaces. We call θ a reference frame. It transfers the Lie bracket $[\;,\;]_{\mathfrak{n}}$ to $\mathfrak{g}_{-1}$ through the expression

$$\rho_{\mathfrak{n},\theta}(a,b) := \theta[\theta^{-1}(a),\theta^{-1}(b)]_{\mathfrak{n}}\;,$$

where $a,b \in \mathfrak{g}_{-1}$. Moreover, the reference frame θ induces the orthonormal frame

$$\left\{E_i := \theta^{-1}(e_i)| \quad i = 1,\ldots,s\;\right\}$$

on $\mathfrak{n}$. The corresponding left-invariant orthonormal frame field on N is a global trivialisation of the first order conformal frame bundle $\mathcal{G}_0(N)$ on $(N, c_{\mathfrak{n}})$:

$$\begin{array}{ccc} \mathcal{G}_0(N) & \cong & N\times \mathrm{CO}(s)\;, \\ \{E_i(p)|\; i=1,\ldots,s\} & \mapsto & (p,e)\;. \end{array}$$

(In fact, this is a trivialisation of the orthonormal frame bundle of $(N, g_{\mathfrak{n}})$.) The bi-invariant metric $g_{\mathfrak{n}}$ induces a $CO(s)$-equivariant lift

$$\sigma^{g_{\mathfrak{n}}} : \mathcal{G}_0(N) \to \mathcal{P}(N)\;,$$

i.e., a Weyl structure (cf. Section 0.6 and [**42**]), which is also equivariant with respect to the induced left actions of N. Thus the composition of the left invariant frame $\{E_i|\; i = 1,\ldots,s\;\}$ (as a section in $\mathcal{G}_0(N)$) with the lift $\sigma^{g_{\mathfrak{n}}}$ gives rise to a trivialisation

$$\iota_{\theta,g_{\mathfrak{n}}} : \mathcal{P}(N) \quad\cong\quad N\times P$$

of the second order frame bundle with parabolic structure group P. As we have chosen the trivialisation $\iota_{\theta,g_{\mathfrak{n}}}$, the induced left action of N on $\mathcal{P}(N)$ is given by

$$\begin{array}{rccc} \iota_{\theta,g_{\mathfrak{n}}}\circ L_g \circ \iota^{-1}_{\theta,g_{\mathfrak{n}}} : & N\times P & \to & N\times P\;, \\ & (n,p) & \to & (g\cdot n, p)\;. \end{array}$$

The fact that the canonical Cartan connection ω_{nor} is left invariant for the action of N and right equivariant for the action of the parabolic structure group P implies that ω_{nor} is uniquely determined by its values at a single point $x_o \in \mathcal{P}(N)$. We can choose x_o as the point (e,e) with respect to the trivialisation $\iota_{\theta,g_{\mathfrak{n}}}$ and then the linear map

$$\omega_{nor}(e,e) : \mathfrak{n}\times\mathfrak{p} \to \mathfrak{g}$$

contains the whole information about ω_{nor} on $\mathcal{P}(N)$. Since the (-1)-part of ω_{nor} corresponds to the soldering form, we have $\omega_{-1}(e,e)(E_i) = e_i$ for $i = 1,\ldots,s$. This leads us to the definition of the map γ_{nor} (which depends on the choice of the global left invariant frame coming from the reference frame θ) as

$$\begin{array}{rccl} \gamma_{nor} & : & \mathfrak{g}_{-1} & \to & \mathfrak{p}\;, \\ & & a & \mapsto & \pi_{\mathfrak{p}}\circ\omega_{nor}(e,e)\circ\pi_{\mathfrak{n}}\circ\omega^{-1}_{nor}(e,e)(a)\;, \end{array}$$

where $\pi_{\mathfrak{p}}$ and $\pi_{\mathfrak{n}}$ denote the obvious projections (the latter with respect to our trivialisation $\iota_{\theta,g_{\mathfrak{n}}}$). This map decomposes to

$$\gamma_{nor} = \gamma_0 + \gamma_1$$

and still contains the whole information about the canonical Cartan connection ω_{nor} on $\mathcal{P}(N)$. In fact, via the relation

$$\omega_{nor}(E_i) = e_i + \gamma_{nor}(e_i)$$

at $x_o = (e,e)$ the canonical Cartan connection ω_{nor} can be recovered from γ_{nor} by right and left translation with P, resp., N.

The curvature Ω inherits the left and right invariance properties from the canonical connection ω_{nor} and thus Ω is again determined by its values at the single point $x_o = (e,e)$ in $P \times N$. With respect to the trivialisation $\iota_{\theta,g_{\mathfrak{n}}}$, we calculate

$$\begin{aligned} E_i(\omega_{nor}(E_j))(e,e) &= \frac{d}{dt}\Big|_{t=0} \omega_{nor}(E_j)(\exp tE_i, e) = \frac{d}{dt}\Big|_{t=0} L^*_{\exp tE_i}\omega_{nor}(E_j)(e,e) \\ &= \frac{d}{dt}\Big|_{t=0} \omega_{nor}(E_j)(e,e) = 0 \end{aligned}$$

for all $i,j \in \{1,\dots,s\}$. This shows the identity

$$\Omega(E_i,E_j) \;=\; -\omega_{nor}(e,e)([E_i,E_j]_{\mathfrak{n}}) \;+\; [e_i + \gamma_{nor}(e_i), e_j + \gamma_{nor}(e_j)]_{\mathfrak{g}} \;.$$

The curvature function κ of the canonical Cartan connection ω_{nor} can then be expressed by

$$\kappa(e_i,e_j) \;=\; -(id + \gamma_{nor}) \circ \rho_{\mathfrak{n},\theta}(e_i,e_j) \;+\; [e_i + \gamma_{nor}(e_i), e_j + \gamma_{nor}(e_j)]_{\mathfrak{g}} \;.$$

Thereby, the (-1)-part of κ is given through

$$\kappa_{-1}(e_i,e_j) \;=\; -\rho_{\mathfrak{n},\theta}(e_i,e_j) \;+\; [e_i,\gamma_0(e_j)]_{\mathfrak{g}} \;+\; [\gamma_0(e_i),e_j]_{\mathfrak{g}} \;.$$

This expression vanishes, since ω_{nor} has no torsion, and we see that the Lie bracket of $\mathfrak{n}$ is given on $\mathfrak{g}_{-1}$ by

$$\rho_{\mathfrak{n},\theta}(e_i,e_j) \;=\; -\gamma_0(e_j)e_i + \gamma_0(e_i)e_j \;. \tag{36}$$

The 0-part of κ is

$$\kappa_0(e_i,e_j) \;=\; -\gamma_0 \circ \rho_{\mathfrak{n},\theta}(e_i,e_j) \;+\; [e_i,\gamma_1(e_j)]_{\mathfrak{g}} \;+\; [\gamma_1(e_i),e_j]_{\mathfrak{g}} \;+\; [\gamma_0(e_i),\gamma_0(e_j)]_{\mathfrak{g}} \;.$$

This part satisfies the trace-free condition

$$\sum_{i=1}^{s} \gamma_0 \circ \rho_{\mathfrak{n},\theta}(e_i,a)(b)(e_i^*) = \left\{ \begin{array}{c} \sum_{i=1}^{s}[e_i,\gamma_1(a)]_{\mathfrak{g}}(b)(e_i^*) + [\gamma_1(e_i),a]_{\mathfrak{g}}(b)(e_i^*) \\ + \sum_{i=1}^{s}[\gamma_0(e_i),\gamma_0(a)]_{\mathfrak{g}}(b)(e_i^*) \end{array} \right\} \tag{37}$$

for all $a,b \in \mathfrak{g}_{-1}$. The 1-part κ_1 of the curvature is

$$\kappa_1(e_i,e_j) \;=\; -\gamma_1 \circ \rho_{\mathfrak{n},\theta}(e_i,e_j) \;+\; [\gamma_0(e_i),\gamma_1(e_j)]_{\mathfrak{g}} \;+\; [\gamma_1(e_i),\gamma_0(e_j)]_{\mathfrak{g}}$$

for all $i,j \in \{1,\dots,s\}$.

The linear map $\gamma_{nor} : \mathfrak{g}_{-1} \to \mathfrak{p}$ is uniquely determined by the normalisation conditions (36) and (37) with respect to $\rho_{\mathfrak{n},\theta}$ (which depends on the choice of the reference frame θ). (Otherwise, we would recover from another map γ subject to these properties a further normal connection inducing $c_{\mathfrak{n}}$, which is not possible.) So γ_{nor} depends only on the choice of θ which induces the Lie bracket of $\mathfrak{n}$ on $\mathfrak{g}_{-1}$. We can introduce the following formal notions.

Definition 3. *Let (G, P) with $G = \mathrm{PO}(1, s+1)$ be the flat homogeneous model (of conformal Riemannian geometry) with Lie algebras $(\mathfrak{g}, \mathfrak{p})$ and let*

$$\rho : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$$

be a skew-symmetric map, which satisfies the Jacobi identity and thus defines a Lie algebra bracket (of compact type) on the (-1)-part $\mathfrak{g}_{-1}$ of the grading of $\mathfrak{g}$.

(1) *We call a linear map*

$$\gamma = \gamma_0 + \gamma_1 : \mathfrak{g}_{-1} \to \mathfrak{p}$$

an invariant connection form on $\mathfrak{g}_{-1}$ of type (G, P).

(2) *The curvature*

$$\kappa_{\gamma,\rho} = \kappa_{-1} + \kappa_0 + \kappa_1 : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}$$

of the invariant connection γ with respect to the Lie bracket ρ is defined as

$$\kappa_{\gamma,\rho}(a, b) = -(id + \gamma) \circ \rho(a, b) + [(id + \gamma)(a), (id + \gamma)(b)]_{\mathfrak{g}}$$

for $a, b \in \mathfrak{g}_{-1}$.

(3) *The invariant connection γ is called torsion-free with respect to ρ if $\kappa_{-1} = 0$.*

(4) *The invariant connection γ is called normal with respect to ρ if*

$$\kappa_{-1} = 0 \qquad \text{and} \qquad \mathrm{tr}\, \kappa_0 = 0$$

(cf. (37)).

(5) *There exists a unique normal connection with respect to the bracket ρ. We denote it by $\gamma_\rho : \mathfrak{g}_{-1} \to \mathfrak{p}$ (or γ_{nor} when the bracket is fixed on $\mathfrak{g}_{-1}$) and call it the canonical invariant connection form of type (G, P) corresponding to ρ.*

As we can see from (36), the Lie bracket of $\mathfrak{n}$ is determined on $\mathfrak{g}_{-1}$ by the 0-part γ_0 of the normal connection γ_{nor}, since it has no torsion. In general, a linear map

$$\gamma_0 : \mathfrak{g}_{-1} \to \mathfrak{g}_0$$

with $-\gamma_0(b) \cdot a + \gamma_0(a) \cdot b$, a skew-symmetric expression in $a, b \in \mathfrak{g}_{-1}$, defines a Lie bracket $\rho_{\gamma_0}(a, b)$ on $\mathfrak{g}_{-1}$ if and only if the following sum of even permutations vanishes (Jacobi identity):

$$\sum_{\sigma(i,j,k)} \left[\ \gamma_0[e_i, \gamma_0(e_j)] \ + \ \gamma_0[\gamma_0(e_i), e_j] \ - \ [\gamma_0(e_i), \gamma_0(e_j)]\ ,\ e_k\ \right] \ = \ 0 \tag{38}$$

for all $i, j, k \in \{1, \ldots, s\}$. The map γ_0 can then be extended in an arbitrary manner to a torsion-free connection form γ with respect to ρ_{γ_0} just by adding any linear 1-part γ_1. In such a situation, the curvature function to γ with respect to ρ_{γ_0} is given by

$$\begin{aligned} \kappa_\gamma(a, b) \quad = \quad & -\gamma_0([a, \gamma_0(b)] + [\gamma_0(a), b]) + [a, \gamma_1(b)] + [\gamma_1(a), b] + [\gamma_0(a), \gamma_0(b)] \\ & -\gamma_1([a, \gamma_0(b)] + [\gamma_0(a), b]) + [\gamma_0(a), \gamma_1(b)] + [\gamma_1(a), \gamma_0(b)]\ . \end{aligned}$$

Of course, not every torsion-free map γ_0 can be extended to the normal connection γ_{nor} with respect to ρ_{γ_0}. The condition on γ_0 for being normally extendible is given by the existence of a γ_1 such that

$$\sum_{i=1}^{s} ([e_i, \gamma_1(a)] + [\gamma_1(e_i), a])(b)(e_i^*) = \begin{cases} \sum_{i=1}^{s} \gamma_0([e_i, \gamma_0(a)] + [\gamma_0(e_i), a])(b)(e_i^*) \\ - \sum_{i=1}^{s} [\gamma_0(e_i), \gamma_0(a)](b)(e_i^*) \end{cases} \tag{39}$$

for all $a, b \in \mathfrak{g}_{-1}$.

We note that so far in this section we have only made use of the left invariant nature of a metric on a Lie group, but not of the bi-invariant nature of $g_{\mathfrak{n}}$, which determines the conformal structure $c_{\mathfrak{n}}$. For the explicit computation of γ_{nor} and its curvature we specialise (now really) to the case of a bi-invariant metric (induced by the Killing form) on a compact semisimple Lie group N. The map γ_0 corresponds in this situation to the Levi-Civita connection of the bi-invariant metric $g_{\mathfrak{n}}$ and is given with respect to the reference frame θ by

$$\gamma_0(e_i) = \theta \circ \nabla^{\mathfrak{n}}_{E_i}(\theta^{-1}(\cdot)) = \frac{1}{2}\rho_{\mathfrak{n},\theta}(e_i, \cdot)$$

for all $i = 1, \ldots, s$. Obviously, the so-defined map γ_0, considered as a matrix in $\mathfrak{g}_0 = \mathfrak{co}(s)$ with respect to the basis $\{e_1, \ldots, e_s\}$, satisfies (36), i.e., γ_0 is torsion-free with respect to $\mathfrak{n}$ (and θ). Then we calculate for the traces on the right hand side in (39):

$$\textstyle\sum_{i=1}^{s} \gamma_0([e_i, \gamma_0(a)] + [\gamma_0(e_i), a])(b)(e_i^*) = \frac{1}{2}B_{\mathfrak{n}}(\theta^{-1}(a), \theta^{-1}(b)) \ ,$$

$$\textstyle\sum_{i=1}^{s}[\gamma_0(e_i), \gamma_0(a)](b)(e_i^*) = \frac{1}{4}B_{\mathfrak{n}}(\theta^{-1}(a), \theta^{-1}(b)) \ .$$

(Note that $\{e_1^*, \ldots, e_s^*\}$ denotes the dual basis to $\{e_1, \ldots, e_s\}$ in $\mathbb{R}^s$. This is the dual basis with respect to the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g} \cong \mathfrak{so}(1, s+1)$ only up to a multiple!) We set $\lambda = \frac{-1}{8(s-1)}$ and

$$\gamma_1(a) = \lambda a^*$$

for all $a \in \mathfrak{g}_{-1}$. Calculation of the left hand side in (39) gives

$$\sum_{i=1}^{s}([e_i, \gamma_1(e_k)] + [\gamma_1(e_i), e_k])(e_l)(e_i^*) = 2\lambda(s-1)\delta_{kl}$$

for all $k, l \in \{1, \ldots, s\}$. Comparing both sides of (39) proves that the normal connection form for $\mathfrak{n}$ is determined to

$$\begin{array}{rrcl} \gamma_{nor}: & \mathfrak{g}_{-1} & \longrightarrow & \mathfrak{p} \ , \\ & a & \mapsto & \frac{1}{2}\rho_{\mathfrak{n},\theta}(a, \cdot) - \frac{1}{8(s-1)}a^* \ . \end{array}$$

The curvature functions κ_{-1} and κ_1 vanish identically, since there is no torsion and the Cotton tensor $C^{\mathfrak{n}}$ of $g_{\mathfrak{n}}$ is trivial. The 0-part κ_0 is given by the Weyl tensor $W^{\mathfrak{n}}$ of $g_{\mathfrak{n}}$. In fact,

$$\kappa(a, b) = \kappa_0(a, b) = \theta \circ W^{\mathfrak{n}}(\theta^{-1}(a), \theta^{-1}(b)) \ .$$

3. Invariant Cartan Connections and Holonomy

We derive now a formula, which computes the conformal holonomy algebra for the invariant connection γ_{nor} of the conformal class of a bi-invariant metric. Remember that in general the holonomy group $Hol(\omega_{nor})$ is by definition a subgroup of $G = \mathrm{PO}(1, s+1)$ and $\mathfrak{hol}(\omega_{nor})$ sits in $\mathfrak{so}(1, s+1)$.

Let N be a connected and compact semisimple Lie group with Lie algebra $\mathfrak{n}$ and bi-invariant Riemannian metric $g_{\mathfrak{n}}$ coming from the Killing form and let $\gamma_{nor} : \mathfrak{g}_{-1} \cong \mathfrak{n} \to \mathfrak{p}$ be the invariant connection form corresponding to ω_{nor}. We denote by

$$\Lambda(\mathfrak{g}_{-1}) := span\{(id + \gamma_{nor})(a) |\ a \in \mathfrak{g}_{-1}\} \subset \mathfrak{g}$$

the image of the invariant connection and by

$$\mathfrak{q} := span\{\kappa_{\mathfrak{n}}(a, b) |\ a, b \in \mathfrak{g}_{-1}\} \subset \mathfrak{p}$$

the vector space of curvature values to γ_{nor}. There is a classical formula for the holonomy algebra of an invariant connection with arbitrary structure group G on a homogeneous space (cf. [**91**]). This general result implies directly the following formula for the conformal holonomy algebra of a bi-invariant metric.

Theorem 18. *Let N be a connected and compact semisimple Lie group with conformal structure $[g_{\mathfrak{n}}]$. Then the holonomy algebra of the normal Cartan connection ω_{nor} on $(N,[g_{\mathfrak{n}}])$ is given by the iterative expression*

$$\mathfrak{hol}(\omega_{nor}) = \mathfrak{q} + [\Lambda(\mathfrak{n}),\mathfrak{q}] + [\Lambda(\mathfrak{n}),[\Lambda(\mathfrak{n}),\mathfrak{q}]] + \cdots \ ,$$

which is a subalgebra of $\mathfrak{g} = \mathfrak{so}(1,s+1)$.

4. Examples

We make use of our invariant calculus for the conformal geometry of bi-invariant metrics in order to compute the conformal holonomy algebra in (a trivial and) a non-trivial case.

Example 1. Let $N = \mathrm{SO}(3)$ be the special orthogonal group in dimension 3, which is a 3-dimensional compact and semisimple Lie group. Let $\mathfrak{so}(3)$ denote its Lie algebra. We use for $\mathfrak{so}(3)$ the standard basis $\{E_{ij} |\ 1 \le i < j \le 3\}$ with the E_{ij}'s defined by matrix multiplication as

$$E_{ij} := e_i \cdot e_j^\top - e_j \cdot e_i^\top$$

with respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.

The Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$ and the universal covering of the group $\mathrm{SO}(3)$ is

$$S^3 = \mathrm{Spin}(3) \cong \mathrm{SU}(2)\ .$$

The bi-invariant metric on SO(3) is conformally flat, since the Weyl tensor $W^{\mathfrak{n}}$ always vanishes in dimension 3, and the Cotton $C^{\mathfrak{n}}$ vanishes for any bi-invariant metric. Of course, this is also clear from the fact that the bi-invariant metric on the universal covering group SU(2) is the standard metric on S^3. Therefore, we expect that the calculation of conformal curvature and holonomy produces trivial results here.

In fact, the connection form $\gamma_{nor} = \gamma_0 + \gamma_1$ can be presented in the following form. The chosen basis $\{E_{ij}\}$ in $\mathfrak{so}(3)$ is orthogonal and $-B(E_{ij}, E_{ij}) = 2$ for all its elements. We define the reference frame

$$\theta(\frac{1}{\sqrt{2}}E_{12}) = e_1, \qquad \theta(\frac{1}{\sqrt{2}}E_{13}) = e_2, \qquad \theta(\frac{1}{\sqrt{2}}E_{23}) = e_3\ .$$

Then we have

$$\begin{aligned}
\gamma_0(e_1) &= \tfrac{1}{\sqrt{2}}\nabla^{\mathfrak{n}}_{E_{12}}\cdot\ = \tfrac{1}{2\sqrt{2}}[E_{12},\cdot]_{\mathfrak{n}} = \quad \tfrac{1}{2\sqrt{2}}E_{23}\ ,\\
\gamma_0(e_2) &= \tfrac{1}{\sqrt{2}}\nabla^{\mathfrak{n}}_{E_{13}}\cdot\ = \tfrac{1}{2\sqrt{2}}[E_{13},\cdot]_{\mathfrak{n}} = -\tfrac{1}{2\sqrt{2}}E_{13}\ ,\\
\gamma_0(e_3) &= \tfrac{1}{\sqrt{2}}\nabla^{\mathfrak{n}}_{E_{23}}\cdot\ = \tfrac{1}{2\sqrt{2}}[E_{23},\cdot]_{\mathfrak{n}} = \quad \tfrac{1}{2\sqrt{2}}E_{12}\ .
\end{aligned}$$

(Note that in these formulae the E_{ij}'s form a basis of $\mathfrak{n}$ as well as a basis of the semisimple part of $\mathfrak{g}_0$.) For γ_1 we have

$$\gamma_1(e_i) = -\frac{1}{16}e_i^*, \qquad i = 1,2,3\ .$$

These formulae express the normal Cartan connection on SO(3) in the trivialisation coming from the bi-invariant metric. One can easily that the curvature κ_0 of γ_{nor} vanishes identically. In particular, the conformal holonomy algebra is trivial, which results from the formula in Theorem 18 with $\mathfrak{q} = 0$.

Example 2. We apply our invariant calculus now to the 6-dimensional compact and semisimple Lie group SO(4) with bi-invariant metric induced by the Killing form B. The Lie algebra $\mathfrak{so}(4)$ is isomorphic to

$$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

and as basis we use two copies of the basis $\{E_{ij}\}$ of $\mathfrak{so}(3)$, namely

$$\{E_{ij} |\ 1 \le i < j \le 3\} \cup \{E_{kl} |\ 4 \le k < l \le 6\} \ .$$

This basis is orthogonal with $-B(E_{ij}, E_{ij}) = 2$.

The bi-invariant metric $g_{\mathfrak{so}(4)}$ induced by the Killing form on SO(4) is Einstein with positive scalar curvature. Obviously, it has non-constant sectional curvature $S^{\mathfrak{n}}$ (cf. Section 1), hence it is not conformally flat. For that reason, we expect in our calculation non-trivial curvature terms and holonomy for the conformal structure $c_{\mathfrak{so}(4)}$ on SO(4).

First, we calculate the normal connection form $\gamma_{nor} = \gamma_0 + \gamma_1$. We use the reference frame

$$\theta(\frac{1}{\sqrt{2}}E_{12}) = e_1, \qquad \theta(\frac{1}{\sqrt{2}}E_{13}) = e_2, \qquad \theta(\frac{1}{\sqrt{2}}E_{23}) = e_3 \ ,$$
$$\theta(\frac{1}{\sqrt{2}}E_{45}) = e_4, \qquad \theta(\frac{1}{\sqrt{2}}E_{46}) = e_5, \qquad \theta(\frac{1}{\sqrt{2}}E_{56}) = e_6 \ .$$

From the calculations for the case of $\mathfrak{so}(3)$ we know that

$$\gamma_0(e_1) = \tfrac{1}{2\sqrt{2}}E_{23}, \qquad \gamma_0(e_2) = -\tfrac{1}{2\sqrt{2}}E_{13}, \qquad \gamma_0(e_3) = \tfrac{1}{2\sqrt{2}}E_{12},$$
$$\gamma_0(e_4) = -\tfrac{1}{2\sqrt{2}}E_{56}, \qquad \gamma_0(e_5) = -\tfrac{1}{2\sqrt{2}}E_{46}, \qquad \gamma_0(e_6) = \tfrac{1}{2\sqrt{2}}E_{45} \ .$$

The 1-part γ_1 is given by

$$\gamma_1(e_i) = -\frac{1}{40}e_i^*, \qquad i = 1, \dots, 6 \ .$$

We have only to compute the images of the 0-part κ_0 of the curvature function, which corresponds to the Weyl tensor $W^{\mathfrak{n}}$, to obtain the space $\mathfrak{q}$. It holds that

$$W^{\mathfrak{n}} = R^{\mathfrak{n}} + \frac{1}{8(s-1)} g_{\mathfrak{n}} * g_{\mathfrak{n}} \ .$$

For the Kulkarni-Nomizu product in this expression it is easy to see that

$$\theta(B * B)(\theta^{-1}(e_i), \theta^{-1}(e_j)) = \frac{1}{20}E_{ij} \qquad \text{for all } i < j \in \{1, \dots, 6\} \ .$$

For the Riemannian curvature tensor we find

$$\theta \circ R^{\mathfrak{n}}(\theta^{-1}(e_i), \theta^{-1}(e_j)) = -\frac{1}{8}E_{ij}$$

for all $i < j \in \{1, \dots 3\}$ and all $i < j \in \{4, \dots 6\}$. The remaining curvature expressions for $R^{\mathfrak{n}}$ are zero. This shows that the span of the images of the 0-part κ_0 of the Cartan curvature is equal to $\mathfrak{so}(6)$, which is the semisimple part of $\mathfrak{g}_0$ in the Möbius algebra $\mathfrak{so}(1,7)$. We have

$$\mathfrak{q} = \mathfrak{so}(6) \ \subset \mathfrak{g}_0 \ .$$

Obviously, the span of the normal connection $id + \gamma_{nor}$ is given by

$$\Lambda(\mathfrak{g}_{-1}) = \{e_i + \gamma_0(e_i) - \frac{1}{40}e_i^*|\ i = 1, \ldots, 6\} \ .$$

Then it is a straightforward calculation to see that the space $[\Lambda(\mathfrak{g}_{-1}), \mathfrak{q}]$ of commutators is equal to

$$span\{e_i - \frac{1}{40}e_i^*|\ i = 1, \ldots, 6\} \oplus \mathfrak{so}(6) \ .$$

We denote

$$\mathfrak{l} := span\{x + \gamma_1(x)|\ x \in \mathfrak{g}_{-1}\} \ .$$

The space $\mathfrak{l}$ is stable under the action of $\mathfrak{so}(6)$ sitting in $\mathfrak{g}_0$. This shows that all the spaces

$$[\Lambda(\mathfrak{g}_{-1}), [\cdots, [\Lambda(\mathfrak{g}_{-1}), \mathfrak{q}] \cdots]]$$

of commutators are equal to $\mathfrak{l} \oplus \mathfrak{q}$, which is seen to be isomorphic to the Lie algebra $\mathfrak{so}(7)$ embedded into $\mathfrak{so}(1,7)$. We conclude for the holonomy algebra of SO(4) that

$$\mathfrak{hol}(\omega_{nor}) = \mathfrak{l} \oplus \mathfrak{q} \cong \mathfrak{so}(7) \ .$$

This result has the following interpretation. The bi-invariant metric $g_{\mathfrak{n}}$ in the conformal class $c_{\mathfrak{n}}$ on $N = \mathrm{SO}(4)$ is Einstein with $scal^{\mathfrak{n}} > 0$. Thus this Einstein scale gives rise to a parallel timelike standard tractor, which reduces the conformal holonomy algebra at least to $\mathfrak{so}(7)$. In fact, our calculation shows that the holonomy is not further reduced and we conclude that up to constant rescaling the bi-invariant metric is the only Einstein metric in its conformal class on SO(4). Moreover, we can read off from the holonomy result that SO(4) does not admit twistor spinors nor conformal Killing forms satisfying the normalisation conditions (cf. Chapter 1). Finally, we can say that SO(4) is not (locally) conformally equivalent to a product of Einstein metrics such that the Schouten tensor of the product equals the product of the Schouten tensors of the factors (cf. Proposition 6(1)).

CHAPTER 4

Twistor Spinors with Zeros in Lorentzian Geometry

We gave in Chapter 1 a geometric structure result for Lorentzian spin manifolds admitting conformal Killing spinors without singularities (cf. Theorem 10). In this chapter we want to discuss a *singularity case* in Lorentzian geometry. From the viewpoint of conformal geometry twistor spinors (and also nc-Killing p-forms) with singularities are of particular interest for various reasons (cf. [**120, 82, 95, 96, 99, 65, 105, 106, 115**]).

For example, in Riemannian geometry the length square of a conformal Killing spinor gives rise via rescaling to a Ricci-flat metric in the conformal class on the complement of the zero set, which consists of isolated points. A result by A. Lichnerowicz (cf. [**120**]) states that a compact Riemannian space admitting a twistor spinor with zero is conformally isometric to the round n-sphere S^n and any twistor spinor on S^n admits exactly one isolated zero. A construction by W. Kühnel and H.-B. Rademacher also shows that there exist twistor spinors with zeros on complete, non-compact Riemannian spaces, which are not conformally flat. Such solutions occur typically on the conformal completion to infinity of asymptotically Euclidean spaces with special irreducible holonomy of the Levi-Civita connection (cf. [**96, 99**]). More generally, a space (M, c) admitting a parallel standard tractor t, whose projecting part $\Pi_H(t) \in \Gamma(\mathcal{E}[1])$ has zeros, is called an *almost (conformally) Einstein* space (cf. Chapter 1 and [**65**]). Such spaces are interesting in view of the Poincaré-Einstein construction, which is asymptotically hyperbolic (cf. Section 0.11).

Singularities of twistor spinors and conformal Killing vector fields are also important in the context of *essential* conformal transformation groups (cf. Section 0.6). In fact, an essential conformal Killing vector field needs to have (in the local situation) a zero. In general, an essential conformal transformation group on a space (M, c) is never compact. In Riemannian geometry the reverse statement is true as well, i.e., if a conformal transformation group is non-compact then it has to be essential. On complete Riemannian spaces this situation is very rare, since only the Euclidean space $\mathbb{R}^n$ and the round sphere S^n admit essential conformal transformation groups (cf. [**134, 116, 2**]). Both spaces are conformally flat. A conjecture by A. Lichnerowicz states that compact Lorentzian spaces with essential conformal transformation group are conformally flat as well (cf. [**4, 60**]). In the non-compact case this statement is certainly not true as we will see in Section 2. (cf. also [**106, 115**]).

To be more concrete, in conformal Lorentzian geometry we did describe before already that solutions of the Penrose twistor equation always give rise to conformal Killing vector fields (in form of the Dirac current). The zero set of a twistor spinor and the corresponding Dirac current always coincide (cf. Section 0.8). In fact, we will see here that the Dirac current of a twistor spinor with zero has always the property that its (local) flow consists of essential conformal transformations (cf. Section 1). Then we will show in this chapter the following two main results. First, we prove that a twistor spinor with zero rescales conformally outside the zero set (at least locally) to a parallel spinor. (In Riemannian geometry this result is a straight consequence, since one can simply rescale with the length function of the spinor. However, in Lorentzian geometry this argument does not work and for that reason we employ tractor calculus, which shows in another instance how useful tractor calculus is.) Secondly, we will construct explicitly a family of Lorentzian metrics on (non-compact) open subsets of $\mathbb{R}^5$, which admit twistor spinors with unique isolated zero and essential conformal transformation groups. The conformal geometry of these metrics is not flat in any neighbourhood of the zero. The construction is based on the conformal completion of the *Eguchi-Hanson metric* defined on the complement of a closed ball in $\mathbb{R}^4$ (cf. [**52, 96**]). The constructed family of metrics is of class C^1, i.e., with respect to the standard coordinates on $\mathbb{R}^5$ the coefficients of the metrics are continuously differentiable exactly once. It is not clear yet if the regularity of these metrics improves by rescaling with an appropriate conformal factor. So far solutions with zeros of the twistor equation in Lorentzian spin geometry were only known on conformally flat spaces!

1. Twistor Spinors with Zeros and Their Dirac Currents

We prove in this section a local geometric structure result for Lorentzian spin spaces admitting twistor spinors with *zeros* (on the complement of the zero set). Thereby, we will use results about the shape of the zero set of conformal Killing vectors and spinors from [**105**] (cf. also [**106, 108**]).

Let (M^n, g) be a connected and time-oriented Lorentzian spin manifold of dimension $n \geq 3$ with (complex) spinor bundle $\mathcal{S}$ and indefinite Hermitian product $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ and let us consider a twistor spinor $\varphi \in \Gamma(\mathcal{S})$ with spinor square V_φ, which has a non-empty set $zero(\varphi)$ of zeros on the Lorentzian space (M^n, g). From the definition of $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ and the Dirac current V_φ it is immediately clear that the zero sets $zero(V_\varphi)$ and $zero(\varphi)$ coincide (cf. Section 0.8 and [**106**]). In particular, if φ is a non-trivial twistor spinor with zeros then V_φ is a non-trivial conformal Killing vector field with zeros. Note that if $\varphi(p) = 0$ at some $p \in M$ then $D^{\mathcal{S}}\varphi(p) \neq 0$ (since these are the components with respect to g of the corresponding ∇^{nor}-parallel twistor $\mathbf{S}(\varphi)$; cf. Section 0.8). In [**105**] we discussed the shape of the zero set $zero(\varphi)$ of a twistor spinor.

Proposition 12. *(cf.* [**105, 106**]*) Let φ be a twistor spinor with zero on a Lorentzian space (M^n, g). Then*

(1) *the set $zero(\varphi)$ consists of (a countable number of) isolated points and/or isolated images of maximal null geodesics in (M, g).*
(2) *If $p \in zero(\varphi)$ and $X \cdot D^{\mathcal{S}}\varphi(p) \neq 0$ for all $X \in T_pM$ then the zero p is isolated on M.*
(3) *If γ_p is a geodesic on (M, g) with $p = \gamma_p(0) \in zero(\varphi)$ and $\gamma_p'(0) \cdot D^{\mathcal{S}}\varphi(p) = 0$ then the image of γ_p is contained in $zero(\varphi)$.*

Remember that, in general, the corresponding twistor $\Psi := \mathbf{S}(\varphi) \in \Gamma(\mathcal{W})$ to any non-trivial conformal Killing spinor $\varphi \in \Gamma(\mathcal{S})$ is $\nabla^{\mathcal{W}}$-parallel on $(M, [g])$ and with respect to the metric g the components of the twistor Ψ are given by $(\varphi, D^{\mathcal{S}}\varphi)$. The spinorial square $\varsigma_2(\Psi)$ of Ψ in the 2-form tractors is non-trivial and ∇^{nor}-parallel, and we can assign to it (by use of Table 3) a normal form $\hat{\alpha}_\varphi \in \Lambda^2(\mathbb{R}^{n*})$ (cf. Section 1.6). This normal form is by construction constant on the space (M, g) and the idea for our geometric description of twistor spinors with zero is to discuss, which normal forms $\hat{\alpha}_\varphi$ allow for a zero of the projecting part $V_\varphi = \Pi_H(\varsigma_2(\Psi))$. Off the singularity set the geometric description should then be clear from Theorem 10.

So let us assume that $\varphi \in \Gamma(\mathcal{S})$ is a conformal Killing spinor with zero on (M, g). Since the corresponding Dirac current V_φ is by definition quadratic in the spinor, we can make the following important observation. For any metric $\tilde{g} \in [g]$ the vector field V_φ has a zero at $p \in zero(\varphi)$, and moreover,

$$\nabla^{\tilde{g}} V_\varphi(p) = 0 \qquad \text{and} \qquad div^{\tilde{g}} V_\varphi(p) = 0\ ,$$

which implies $grad^{\tilde{g}}(div^{\tilde{g}} V_\varphi)(p) \neq 0$ (since the 2-jet can not be trivial at a point). This shows that V_φ is an essential conformal Killing vector field on $(M, [g])$ and its local flow consists of essential conformal transformations. In fact, we can use these features of the 2-jet of V_φ for further useful discussion. For this purpose let us consider the adjoint (resp. 2-form) tractor $\varsigma_2(\Psi)$, which is the image of V_φ under the splitting operator $\mathbf{S}$. With respect to any compatible metric $\tilde{g}$ in $[g]$ the adjoint tractor $\varsigma_2(\Psi)$ splits into a set $(\alpha_-, \alpha_0, \alpha_\mp, \alpha_+)$ of differential forms, where $\alpha_- = \varsigma_1(\varphi)$, $\alpha_0 = \frac{1}{2} d\varsigma_1(\varphi)$, $\alpha_\mp = \frac{1}{n} d^* \varsigma_1(\varphi)$ and $\alpha_+ = \frac{1}{n-2}(\Delta_{tr}^{\tilde{g}} + \frac{scal^{\tilde{g}}}{2(n-1)})\varsigma_1(\varphi)$ (cf. Section 1.1). In fact, α_+ is the spinorial 1-form square of $D^{\mathcal{S}}\varphi$, which is in any case a causal 1-form. It follows that at a zero $p \in zero(\varphi)$ the adjoint tractor $\varsigma_2(\Psi)(p)$ is given by $s_+^\flat \wedge \alpha_+$ with respect to any $\tilde{g}$-adapted tractor frame $\{s_-, s_+, s_1, \ldots, s_n\}$ (cf. Section 1.1). This implies that $\varsigma_2(\Psi)$ is a simple 2-form tractor everywhere(!) on M and the corresponding normal form of $\varsigma_2(\Psi)$ is either

$$l_1^\flat \wedge l_2^\flat \qquad \text{or} \qquad l_1^\flat \wedge t_1^\flat\ ,$$

where $l_1^\flat$ and $l_2^\flat$ are lightlike and orthogonal cotractors, resp., $t_1^\flat$ is a timelike cotractor and orthogonal to $l_1^\flat$ (cf. Section 1.5).

The argument so far shows that two types of conformal Killing spinors φ with $zero(\varphi) \neq \emptyset$ on a Lorentzian space (M, g) are possible. According to the normal form of the corresponding 2-form tractor $\varsigma_2(\Psi)$ we define the singularity set of φ as follows. If the normal form to φ is given by a 2-form $l_1^\flat \wedge l_2^\flat$ we set $sing(\varphi) := zero(\varphi)$. In the other case when the normal form is $l_1^\flat \wedge t_1^\flat$ we set $sing(\varphi) := zero(\|V_\varphi\|^2)$.

Proposition 13. *Let $\varphi \in \Gamma(\mathcal{S}[-1/2])$ be a conformal Killing spinor with $zero(\varphi) \neq \emptyset$ on a conformal Lorentzian spin manifold (M, c). Then the set $sing(\varphi)$ is singular on M and there exists for every $p \notin sing(\varphi)$ a neighbourhood $U_p \subset M$ of p and a metric $g \in c|_{U_p}$ such that φ is $\nabla^{\mathcal{S}}$-parallel and V_φ is ∇^g-parallel with respect to g.*

Proof. Let us consider the ∇^{nor}-parallel 2-form tractor $\varsigma_2(\Psi)$. Since $\varsigma_2(\Psi)$ is simple, we have (for both possible normal forms) $\varsigma_2(\Psi) \wedge \varsigma_2(\Psi) = 0$, which implies that the corresponding projecting part $\varsigma_1(\varphi) \wedge d\varsigma_1(\varphi)$ is trivial as well. This means that V_φ is hypersurface orthogonal off the singularity set $sing(\varphi)$.

There are two cases to consider now. First, let us assume that the interior of $zero(\|V_\varphi\|^2)$ is non-empty. In this case Theorem 10 tells us that there exists an open set $U \subset zero(\|V_\varphi\|^2) \setminus zero(\varphi)$, on which φ and V_φ are parallel with respect to some metric $g \in c|_U$. Then $D^{\mathcal{S}}\varphi = 0$ on (U, g) and the scalar curvature $scal^g$ vanishes. We can conclude that with respect to any g-adapted tractor frame $\{s_-, s_+, s_1 \ldots, s_n\}$ the 2-form tractor $\varsigma_2(\Psi)$ is given on (U, g) by $s_-^\flat \wedge \varsigma_1(\varphi)$. In particular, it follows that the normal form of $\varsigma_2(\Psi)$ is $l_1^\flat \wedge l_2^\flat$ and $sing(\varphi) = zero(\varphi)$ is singular on M. In fact, Theorem 10 implies now that for any $p \notin zero(\varphi)$ there exists a neighbourhood U_p such that φ and V_φ a parallel with respect to some $g \in c$ on U_p.

In the other case the set $zero(\|V_\varphi\|^2)$ is singular on M, which means that V_φ is almost everywhere timelike, and there exists a unique metric $g \in c$ on $M \setminus sing(\varphi)$, for which V_φ has constant length square $g(V_\varphi, V_\varphi) = -1$. Since V_φ is hypersurface orthogonal, the vector field V_φ has to be ∇^g-parallel on $M \setminus sing(\varphi)$. In particular, $\varsigma_2(\Psi)$ is given by $s_-^\flat \wedge \varsigma_1(\varphi) + s_+^\flat \wedge \alpha_+$. However, the normal form of $\varsigma_2(\Psi)$ must be $l_1^\flat \wedge t_1^\flat$, which implies $\alpha_+ = 0$ and, therefore, $D^{\mathbb{S}}\varphi = 0$ with respect to g. This shows that φ is $\nabla^{\mathbb{S}}$-parallel on $(M \setminus sing(\varphi), g)$. $\square$

We mentioned already Lorentzian spin spaces with parallel spinors in Section 0.8. There occur exactly two types of geometries, the static monopoles and the Brinkmann waves. As the proof of Proposition 13 already suggest, using the normal form description of $\varsigma_2(\Psi)$ it becomes immediately clear, which of these geometries do occur off the singularity set $sing(\varphi)$ of a twistor spinor φ with zero. Also the shape of the zero and the singularity set of φ is determined by the normal form of $\varsigma_2(\Psi)$.

Theorem 19. *(cf. [**108**]) Let $\varphi \not\equiv 0$ be a twistor spinor on a conformal Lorentzian spin manifold (M, c) with $zero(\varphi) \neq \emptyset$. Then $zero(\varphi)$ consists either of*

(1) *isolated images of lightlike geodesics and, off the zero set, the metric g is locally conformally equivalent to a Brinkmann metric with parallel spinor or*
(2) *isolated points and, off the singularity set $sing(\varphi)$, the metric g is locally conformally equivalent to a static monopole $-ds^2 + h$ admitting a parallel spinor. In a (convex) neighbourhood of $p \in zero(\varphi)$ the singularity set $sing(\varphi)$ is equal to the geodesic light cone $L_{o,p}$, which emerges from $p \in zero(\varphi)$ (cf. Section 0.4).*

PROOF. Let $g \in c$ be a smooth metric on M. First, let us assume that the normal form of $\varsigma_2(\Psi)$ to the twistor spinor φ with zero is given by $l_1^\flat \wedge l_2^\flat$. Then $sing(\varphi) = zero(\varphi)$ and at any zero $p \in zero(\varphi)$ the Dirac current $V_{D^{\mathbb{S}}\varphi}$ is null. This implies that $V_{D^{\mathbb{S}}\varphi} \cdot D^{\mathbb{S}}\varphi(p) = 0$ (cf. [**106**]) and with Proposition 12 we conclude that the (maximal) null geodesic with tangent vector $V_{D^{\mathbb{S}}\varphi}$ running through p lies entirely in $zero(\varphi)$. Moreover, we have seen already in the proof of Proposition 13 that φ and V_φ are locally parallel with respect to some metric $\tilde{g} \in c$ off the set $zero(\varphi)$. Since V_φ is null off the zero set, the metric g is a Brinkmann wave (cf. Theorem 10).

Now, let $l_1^\flat \wedge t_1^\flat$ be the normal form of $\varsigma_2(\Psi)$. Then at any zero $p \in zero(\varphi)$ the Dirac current $V_{D^{\mathbb{S}}\varphi}$ is timelike and therefore $X \cdot D^{\mathbb{S}}\varphi(p) \neq 0$ for all $0 \neq X \in T_pM$ (cf. [**106**]). Proposition 12 implies that $zero(\varphi)$ consists of isolated points on M. Moreover, since V_φ is a conformal Killing vector field and $\nabla^g V_\varphi(p) = 0$ for all $p \in zero(\varphi)$, it follows that V_φ is tangential (or zero) to any lightlike geodesic emerging from any $p \in zero(\varphi)$ (cf. [**105**]). This implies that $L_{o,p}$, $p \in zero(\varphi)$, is contained in $sing(\varphi)$.

In fact, if V_φ is null at $q \in M$ then the integral curve to V_φ through q is a lightlike (pre-)geodesic. If such a point q occurs arbitrary close to some $p \in zero(\varphi)$, but is not contained in $L_{o,p}$, then the corresponding (maximal) lightlike geodesic through q has to intersect the light cone $L_{o,p}$ very close to p. At such an intersection point the vector V_φ would have to be tangential to the null geodesic emerging from p and the null geodesic emerging from q. Since both geodesics are not parallel by assumption, the vector V_φ had to be zero at the intersection. But this is not possible, since p is an isolated zero. This shows that q does not exist in an arbitrary small neighbourhood of $p \in zero(\varphi)$ and we conclude that $sing(\varphi)$ is identical to the geodesic light cone $L_{o,p}$ in a small neighbourhood of any $p \in zero(\varphi)$.

Off the singularity set $sing(\varphi)$ the vector field V_φ is timelike and parallel with respect to the metric $\tilde{g} = \frac{-1}{g(V_\varphi, V_\varphi)} \cdot g$. This shows that $\tilde{g}$ is locally a product of the form $-ds^2 + h$, where h is some Riemannian metric. In fact, the spinor φ is also parallel with respect to $\tilde{g}$, and therefore, φ restricts to a parallel spinor on the (space with) metric h. In particular, it follows that the metrics h and $\tilde{g}$ are Ricci-flat. □

2. Metric with Frame and Spinor

In the previous section we discussed twistor spinors with zeros in Lorentzian geometry without really knowing if such a thing exists at all. Certainly, twistor spinors with zero exist on flat Minkowski space $\mathbb{R}^{1,n}$. However, we would like to know non-trivial conformally curved examples. And, in fact, we describe now the explicit construction of a family of conformally non-flat Lorentzian metrics in dimension 5 admitting twistor spinors with isolated zeros. The construction is based on the Eguchi-Hanson metric in dimension 4 (cf. [**52, 96, 115**]). The proofs for our statements here will be worked out in the following last section of this chapter.

Let us consider the 5-dimensional real vector space $\mathbb{R}^5$ with canonical coordinates $x = (x_0, x_1, x_2, x_3, x_4)$. (In this and the next section we put the indices for coordinates at the bottom.) We set $n = 5$. The Minkowski metric is given by

$$g_0 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \ .$$

This metric is of Lorentzian signature $(1, 4)$ and is flat on $\mathbb{R}^5$. We aim to rewrite the Minkowski metric in cylindrical coordinates. So let E be the 4-dimensional vector subspace in $\mathbb{R}^5$ defined by $x_0 = 0$ and denote by

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

the radial coordinate on E. The space $E \setminus \{0\}$ (with deleted origin) is diffeomorphic to $\mathbb{R}_+ \times S^3$. Thereby, S^3 denotes the 3-sphere, which is given in E by the equation $r = 1$. As the group of elements with unit length in $E \cong \mathbb{H}$ the 3-sphere S^3 is isomorphic to the semisimple Lie group $\mathrm{SU}(2)$. The round metric g_{S^3} on S^3 is $\mathrm{SU}(2)$-invariant and there exist left-invariant 1-forms σ_1, σ_2 and σ_3 on $\mathrm{SU}(2)$ such that

$$g_{S^3} \ = \ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \ .$$

On S^3 in E these left-invariant forms are explicitly given by

$$\begin{aligned}
\sigma_1 &= \tfrac{1}{r^2}(-x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4),\\
\sigma_2 &= \tfrac{1}{r^2}(-x_3 dx_1 + x_4 dx_2 + x_1 dx_3 - x_2 dx_4),\\
\sigma_3 &= \tfrac{1}{r^2}(-x_4 dx_1 - x_3 dx_2 + x_2 dx_3 + x_1 dx_4) \ .
\end{aligned}$$

We denote the dual orthonormal frame on TS^3 by $\{\frac{\partial}{\partial \sigma_1}, \frac{\partial}{\partial \sigma_2}, \frac{\partial}{\partial \sigma_3}\}$. Finally, we see that the Minkowski metric on $\mathbb{R}^5 \setminus \{r = 0\}$ is given in cylindrical coordinates by

$$g_0 \ = \ -dx_0^2 + dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \ .$$

We know that this metric can be smoothly completed to the singular set $\{r = 0\}$ of the cylindrical coordinate system (which is a real line in $\mathbb{R}^5$). The result is the Minkowski metric g_0 on $\mathbb{R}^5$.

Now let us define the cone

$$L \ := \ \{ \ (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 : \ r \le |x_0| \ \}$$

with singular point at the origin of $\mathbb{R}^5$. The boundary set of the cone L in $\mathbb{R}^5$ is

$$L_o \ := \ \{ \ (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 : \ r = |x_0| \ \} \ .$$

Next we define the radial coordinate

$$r_o \ := \ \begin{cases} \quad 0 & \text{on} \ \ L \\ \frac{r^2 - x_0^2}{r} & \text{on} \ \ \mathbb{R}^5 \setminus L \end{cases} \quad .$$

Furthermore, let $a > 0$ be a real parameter. Then we set

$$B_a \ := \ \{ \ (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 : \ 0 < r_o < \frac{1}{a} \ \}$$

and $\tilde{B}_a := B_a \cup L$. Both sets B_a and $\tilde{B}_a$ are open in $\mathbb{R}^5$ for all $a > 0$. The set B_a is a subset of $\mathbb{R}^5 \setminus L$ and $\tilde{B}_a$ is simply connected. We also introduce the set $B_a^{>} := \tilde{B}_a \setminus \{r = 0\}$, where the real line $\{r = 0\}$ is deleted. Figure 1 shows a diagram of the basic domains of definition.

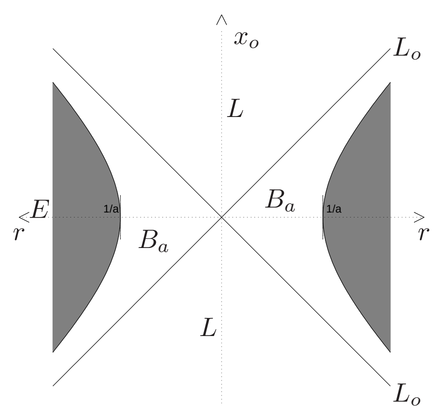

FIGURE 1

On $\tilde{B}_a$ we define a family of pointwise symmetric bilinear forms g_a, $a > 0$, as follows. Let

$$g_a \ := \ \begin{cases} g_0 \ - \ r^2 (a r_o)^4 \cdot \sigma_3^2 \ + \ a^4 (r\beta)^{-2} r_o^2 \cdot \alpha^2 & \text{on} \quad B_a^{>} \\ g_0 & \text{on} \quad \{r = 0\} \end{cases} \quad ,$$

where we set

$$\beta := \sqrt{1 - (a r_o)^4} \qquad \text{and} \qquad \alpha := (r^2 + x_0^2) dr \ - \ 2 x_0 r dx_0 \ .$$

Obviously, the symmetric bilinear form g_a is smoothly defined on $\tilde{B}_a \setminus L_o$ for all $a > 0$, and by definition, g_a restricted to $L \setminus L_o$ is the flat Minkowski metric. The symmetric bilinear form g_a can be rewritten on $B_a^{>}$ as

$$g_a \ = \ -dx_0^2 \ + \ dr^2 \ + \ r^2 (\ \sigma_1^2 + \sigma_2^2 + \beta^2 \sigma_3^2 \ ) \ + \ a^4 (r\beta)^{-2} r_o^2 \cdot \alpha^2 \quad .$$

Proposition 14. *The symmetric bilinear form g_a is a C^1-metric of Lorentzian signature $(1,4)$ on the subset $\tilde{B}_a$ of $\mathbb{R}^5$ for all $a > 0$. The metric g_a is not of class C^2.*

We want to discuss some geometric aspects of the Lorentzian metric g_a. The restriction of g_a to the disk $E \cap B_a$ (with deleted origin) in $E \cong \mathbb{R}^4$ is given by

$$h_a \;:=\; \frac{dr^2}{1-(ar)^4} \;+\; r^2(\;\sigma_1^2+\sigma_2^2+(1-(ar)^4)\sigma_3^2\;)\;.$$

This is a Riemannian metric on $E \cap B_a$, which admits a smooth (even analytic) extension to the origin. Off the origin, the metric g_a is conformally equivalent to the *Eguchi-Hanson metric*

$$g_{EH} \;:=\; \frac{dR^2}{1-(a/R)^4} \;+\; R^2\left(\;\sigma_1^2+\sigma_2^2+(1-(a/R)^4)\sigma_3^2\;\right)$$

defined on the complement of a closed ball in $\mathbb{R}^4$. In fact, with $R := 1/r$ on $E \setminus \{0\}$ we have $g_{EH} = \frac{1}{r^4}\cdot h_a$. We want to point out that the complete Eguchi-Hanson metric is a Ricci-flat Kähler metric defined on the total space of the cotangent bundle of S^2, which is asymptotically locally Euclidean at infinity and hyperKähler with irreducible holonomy group $\mathrm{SU}(2) = \mathrm{Sp}(1)$. The metric g_{EH} on $\mathbb{R}^4 \setminus \{R \le a\}$ which is used throughout this chapter is a simply-connected $\mathbb{Z}_2$ cover of the complete Eguchi-Hanson metric on T^*S^2 with the zero section deleted. The metric g_{EH} is asymptotically Euclidean, Ricci-flat and hyperKähler, and therefore, admits a 2-dimensional space of parallel spinors. The metric g_{EH} is not conformally flat. In fact, g_{EH} is half-conformally flat, i.e., the Weyl curvature tensor $W = W^+ + W^-$ is anti-selfdual ($W^+ = 0$) (cf. [**52, 96**]). We set

$$\tilde{g}_a := \frac{1}{(r^2-x_0^2)^2}\cdot g_a \qquad \text{on } \;\tilde{B}_a \setminus L_o\;.$$

Proposition 15. *a) Let $\tilde{g}_a = (r^2-x_0^2)^{-2}\cdot g_a$, $a > 0$, be a conformally equivalent metric to g_a on $\tilde{B}_a \setminus L_o$. Then*

(1) *the metric $\tilde{g}_a$ is flat for $r < |x_0|$.*
(2) *For $|x_0| < r$ the metric is given by*

$$\tilde{g}_a \;=\; -ds^2 \;+\; g_{EH}\;,$$

where we define the coordinate change Ψ by $s := \frac{-x_0}{r^2-x_0^2}$ and $R := \frac{r}{r^2-x_0^2}$.

In particular, $\tilde{g}_a$ is Ricci-flat on $\tilde{B}_a \setminus L_o$.

b) The Weyl tensor W^{g_a} of the smooth Lorentzian metric g_a on $\tilde{B}_a \setminus L_o$ admits a continuous extension of class C^1 to the singular set L_o. For this extension we have $W^{g_a} \equiv 0$ on L and $W^{g_a} \neq 0$ on B_a, i.e., g_a is not conformally flat.

We note that the Ricci-curvature tensor of the metric g_a on $\tilde{B}_a \setminus L_o$ does not admit a continuous extension to $\tilde{B}_a$. With $\mu := \ln|r^2 - x_0^2|$ it holds that

$$Ric^{g_a} \;=\; -(n-2)\cdot(\;Hess^{\tilde{g}_a}(\mu) \;-\; d\mu^2\;) \;-\; (\;\Delta^{\tilde{g}_a}\mu \;+\; (n-2)\cdot|d\mu|^2\;)\cdot\tilde{g}_a\;.$$

Furthermore, we note that the hypersurface $\{s = 0\}$ is totally geodesic with respect to the metric $-ds^2 + g_{EH}$. This implies that the disk $E \cap \tilde{B}_a$ is a totally umbilic hypersurface in $(\tilde{B}_a, g_a)$. Figure 2 shows the coordinate change Ψ on $\tilde{B}_a$ together with the conformally equivalent metrics g_a and $\tilde{g}_a$.

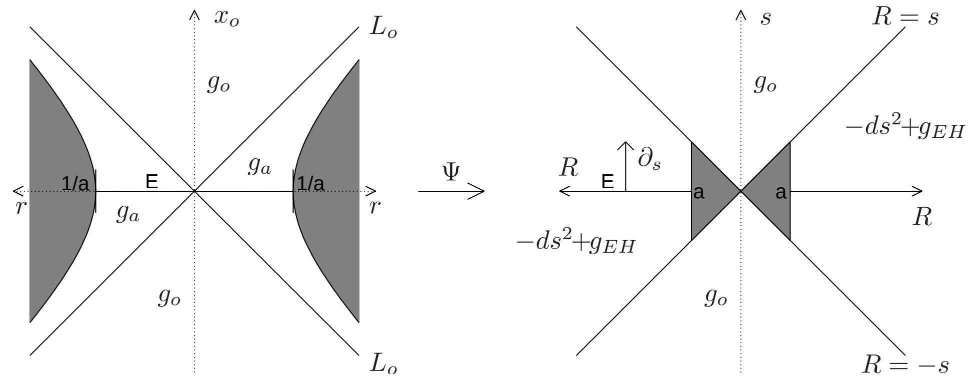

FIGURE 2

Next we define on $B_a^>$ with metric g_a an orthonormal frame $e = \{e_0, e_1, e_2, e_3, e_4\}$ in the following way. Let

$$T := -(r^2 + x_0^2)\frac{\partial}{\partial r} - 2rx_0\frac{\partial}{\partial x_0}$$

be a vector field on $\mathbb{R}^5$. We set

$$\begin{aligned}
e_0 &:= \tfrac{\partial}{\partial x_0} - \tfrac{2x_0}{r} \cdot \tfrac{a^4 r_o^2}{1+\beta} \cdot T \\
e_1 &:= \tfrac{\partial}{\partial r} + \tfrac{r^2+x_0^2}{r^2} \cdot \tfrac{a^4 r_o^2}{1+\beta} \cdot T \\
e_2 &:= r^{-1} \cdot \tfrac{\partial}{\partial \sigma_1} \\
e_3 &:= r^{-1} \cdot \tfrac{\partial}{\partial \sigma_2} \\
e_4 &:= (r\beta)^{-1} \cdot \tfrac{\partial}{\partial \sigma_3} \quad .
\end{aligned}$$

Lemma 12. *The orthonormal frame $e = \{e_0, e_1, e_2, e_3, e_4\}$ on $B_a^>$ is of class C^1.*

Let $\mathrm{Spin}(1,4)$ be the spin group with universal covering map $\lambda : \mathrm{Spin}(1,4) \to \mathrm{SO}(1,4)$ onto the special orthonormal group and let $Cl_{1,4}$ be the Clifford algebra (cf. Section 0.8). The complex spinor module $\Delta_{1,4}$ is isomorphic to $\mathbb{C}^4$ and a realisation of the action of the Clifford algebra $Cl(1,4)$ on $\Delta_{1,4} \cong \mathbb{C}^4$ is given by

$$\gamma_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

where the γ_i's denote the generators of the Clifford algebra with relations $\gamma_0 \cdot \gamma_0 = 1$, $\gamma_i \cdot \gamma_i = -1$ for all $i = 1, \ldots, 4$ and $\gamma_i \cdot \gamma_j = -\gamma_i \cdot \gamma_j$ for all $i \neq j$.

The Lorentzian manifold $(\tilde{B}_a, g_a)$ with C^1-metric is simply connected and oriented. Hence there exists a unique spin structure

$$\pi : Spin(\tilde{B}_a) \to SO(\tilde{B}_a),$$

with corresponding spinor bundle

$$\mathbb{S} := Spin(\tilde{B}_a) \times_{\mathrm{Spin}(1,4)} \mathbb{\Delta}_{1,4} \ .$$

The spinor bundle $\mathbb{S}$ is globally trivial on $\tilde{B}_a$. With respect to a C^1-section of $\pi : Spin(\tilde{B}_a) \to \tilde{B}_a$ (i.e., a global spinor frame of class C^1) the space $C^1(\tilde{B}_a, \mathbb{S})$ of differentiable spinor fields is uniquely identified with the space $C^1(\tilde{B}_a, \mathbb{\Delta}_{1,4})$ of $\mathbb{\Delta}_{1,4}$-valued continuously differentiable functions on $\tilde{B}_a$.

The C^1-frame $e : B_a^> \to SO(\tilde{B}_a)$ (cf. Lemma 12) admits exactly two lifts (of class C^1) to the spinor frame bundle $Spin(\tilde{B}_a)$. We choose one of these lifts and denote it by

$$e_s : B_a^> \to Spin(\tilde{B}_a) \ .$$

Any spinor field ϕ on $B_a^>$ can then be uniquely represented with respect to the spinor frame e_s by a $\mathbb{\Delta}_{1,4}$-valued function, i.e.,

$$\phi \ = \ [\ e_s \ , \ w\ ] \quad \in C^1(B_a^>, \mathbb{S})$$

for some function $w \in C^1(B_a^>, \mathbb{\Delta}_{1,4})$. Now let $w(b,c)$ denote the constant $\mathbb{\Delta}_{1,4}$-valued function $(b, -c, 0, 0)^\top$, where $(b,c) \in \mathbb{C}^2$. We set

$$\psi_{bc}^> := (x_0 e_0 + r e_1) \cdot [\ e_s \ , \ w(b,c)\ ] \qquad \text{on} \quad B_a^> \ .$$

Obviously, the spinor field $\psi_{bc}^>$ is an element of $C^1(B_a^>, \mathbb{S})$ for all (b,c). Calculating the Clifford product results in

$$\psi_{bc}^> \ = \ [\ e_s \ , \ \begin{pmatrix} -x_0 b \\ x_0 c \\ rb \\ rc \end{pmatrix} \ ] \ .$$

We denote by

$$V \ := \ -2x_0 r \frac{\partial}{\partial r} \ - \ (r^2 + x_0^2) \frac{\partial}{\partial x_0}$$

a smooth vector field on $\mathbb{R}^5$. Figure 3 shows the vector field V in a neighbourhood of the origin in $\mathbb{R}^5$ together with integral curves and the flow of the umbilic hypersurface E to some E_+ resp. E_-.

Theorem 20. *Let $(b,c) \in \mathbb{C}^2$ and $\psi_{bc}^>$ be a spinor on $(B_a^>, g_a)$ with $a > 0$.*

(1) *The spinor field $\psi_{bc}^>$ on $B_a^>$ admits a unique extension ψ_{bc} to $(\tilde{B}_a, g_a)$ of class C^1.*
(2) *The unique extension ψ_{bc} is a twistor spinor on $(\tilde{B}_a, g_a)$.*

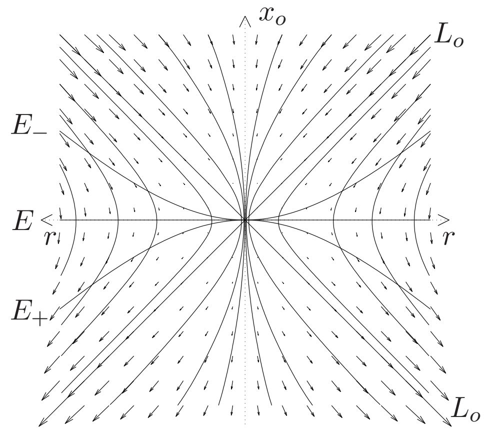

FIGURE 3

(3) *For* $(b,c) \neq 0$ *the twistor spinor* ψ_{bc} *admits exactly one zero at the origin* $\{0\} \in \tilde{B}_a$.

(4) *The zero set of the spinor length square* $u_{bc} := \langle \psi_{bc}, \psi_{bc} \rangle_{\mathbb{S}}$ *is* L_o. *The function* u_{bc} *solves the equation*

$$-u_{bc} \cdot Ric^0 = (n-2) \cdot Hess(u_{bc})^0$$

on $\tilde{B}_a \setminus L_o$, *where* Ric^0 *and* $Hess(u_{bc})^0$ *denote the trace-free parts of the symmetric tensors* Ric^{g_a}, *resp.,* $Hess^{g_a}(u_{bc})$. *In particular, the metric* $\tilde{g}_a = \frac{1}{u_{bc}^2} g_a$ *is Einstein for* $u_{bc} \neq 0$.

(5) *The spinor square* $V_{\psi_{bc}}$ *is a smooth conformal Killing vector field on* $(\tilde{B}_a, g_a)$. *The following equation holds:*

$$V_{\psi_{bc}} = (b^2 + c^2) \cdot V \ .$$

(6) *The vector* $V_{\psi_{bc}}$ *is timelike on* $\tilde{B}_a \setminus L_o$, *lightlike on* $L_o \setminus \{0\}$ *and zero only in the origin* $\{0\} \in \tilde{B}_a$.

In short, Theorem 20 says that there exists a 2-dimensional set of twistor spinors on $(\tilde{B}_a, g_a)$ for all $a > 0$, which admit an isolated zero at the origin. There exist no further twistor spinors on $(\tilde{B}_a, g_a)$, since the Eguchi-Hanson metric g_{EH} admits exactly two linearly independent (parallel) twistor spinors for $a > 0$. Together with Proposition 15 we obtain from Theorem 20 the following important observation to our construction.

Corollary 5. *There exists a family of Lorentzian* C^1*-metrics* g_a, $a > 0$, *in dimension* 5, *which admit twistor spinors and a smooth causal conformal Killing vector field, all with isolated zero at some point* $\{p\}$ *such that* g_a *is non-conformally flat around the zero at* $\{p\}$.

We remark that the vector field V is complete on $\tilde{B}_a$, i.e., the flow of V to the time t generates a one-parameter group of conformal transformations on $\tilde{B}_a$. All these conformal transformations are essential (cf. Section 1 and Section 0.6).

We want to add some further comments concerning our construction. The metric g_a can be considered as a completion of the metric $-ds^2 + g_{EH}$, which is Ricci-flat and *asymptotically Minkowskian*, to the set L with infinity L_o. The twistor spinors extend to L as well with a zero at some point of infinity. In general, it is known that a Lorentzian metric with differentiable Weyl tensor has to be conformally flat in the causal past and future of a zero of a twistor spinor (cf. Section 3). Therefore, it is reasonable in our construction to do the conformal completion to L by using the flat Minkowski metric g_0 on the *other side* of the infinity set L_o. There exists no extension (conformal completion) with differentiable Weyl tensor of g_a on B_a to a neighbourhood of the origin, which is not conformally flat on L, but preserves the existence of a twistor spinor. This fact implies that our completion of g_a can not be analytic.

We also want to point out again that our construction is even not smooth. However, the question remains whether there is a conformally equivalent metric to g_a on $\tilde{B}_a$, whose regularity is better than of class C^1. The existence of the C^1-extension of the Weyl tensor of g_a to the infinity set L_o certainly does not pose an obstruction to this question. By the way, the reason that we work here with C^1-spinors is due to the fact that our metrics g_a are only of class C^1. It might well be that there exists a reasonable differentiable structure of better regularity on the spinor bundle $\mathbb{S}$ over $(\tilde{B}_a, g_a)$ than we use here. In fact, the spinors ψ_{bc} of Theorem 20 on $(\tilde{B}_a, g_a)$ might be more regular in a reasonable sense. However, since we do not know, we decided to work (as a precaution) only in class C^1.

3. Proof of Statements

We prove here the statements which we made in the previous section. We start with a discussion of differentiability of certain functions on $\tilde{B}_a \subset \mathbb{R}^5$. For some arbitrary p-tuple $I_p = (i_1, \ldots, i_p) \in \{0, \ldots, 4\}^p$ let us denote by

$$\partial_{I_p} := \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}}$$

a partial derivative of order p. Moreover, for any 5-tuple $l = (l_r, l_0, \ldots, l_4)$ with $l_r, l_0, \ldots, l_4 \in \mathbb{N} \cup \{0\}$ we set $s_l := -l_r + \sum_{i=0}^{4} l_i$ and define the smooth function

$$f_l = f(l_r, l_0, \cdots, l_4) := r^{-l_r} \cdot x_0^{l_0} \cdot \ldots \cdot x_4^{l_4} \qquad \text{on } \; B_a \; .$$

We say that the rational function f_l is of order s_l. Remember that we defined the radial function r_o to be $(r^2 - x_0^2)/r$ on B_a and identically zero on L (cf. Section 2). For any function f on B_a we understand the product $r_o \cdot f$ in a unique way as a function on $\tilde{B}_a = B_a \cup L$, which is identically zero on L. For $t > 0$ a real number we denote

$$B_a^t := B_a \cap \{x \in \mathbb{R}^5 : \; r \leq t\} \; .$$

Notice that if a function f is continuous on B_a and its absolute value $|f|$ is bounded on B_a^t for all $t > 0$ then $r_o \cdot f$ is continuous on $\tilde{B}_a$. In fact, for this conclusion it is sufficient for $|f|$ to be bounded on $B_{\tilde{a}}^t$ for all $t > 0$ with some $\tilde{a} > a$.

Lemma 13. *Any function on $\tilde{B}_a$ of the form $r_o^m \cdot f_l$ with $m > 0$ is of class C^{k-1}, but not of class C^k, where $k := \min\{m, m + s_l\}$.*

Proof. First, we note that $|x_i| < r$ on B_a for all $i = 0, \ldots, 4$, and we see that the absolute value $|f_l|$ of any function of the form f_l with $s_l \geq 0$ is bounded on B_a^t by t^{s_l} for all $t > 0$. More generally, the absolute value of the partial derivative $\partial_{I_p} f_l$ is bounded on B_a^t for all $t > 0$ if $s_l - p \geq 0$. In particular, the absolute value of x_0/r is bounded on B_a. Moreover, x_0/r is continuous on $(B_a \cup L_o) \setminus \{0\}$. This shows that the extension of the function $r - x_0 \cdot x_0/r$ by zero to the origin in $\mathbb{R}^5$ is a continuous function on $B_a \cup L_o$

and is identically zero on L_o. And this implies that the coordinate r_o is continuously defined on $\tilde{B}_a$. The function r_o is not continuously differentiable. However, we have

$$dr_o = \frac{-2x_0}{r}dx_0 \ + \ \sum_{i=1}^{4}\left(\frac{x_i x_0^2}{r^3} + \frac{x_i}{r}\right)dx_i \ ,$$

and we see that the coefficients of dr_o are bounded on B_a. (This implies that $d(r_o^2) = 2r_o dr_o$ is continuous on $\tilde{B}_a$, i.e., r_o^2 is of class C^1.)

Since a function of the form $r_o^m \cdot f_l$ admits terms of lowest order $m + s_l$, such a function is at most of class C^{m+s_l-1}. However, a pth order derivative of $r_o^m \cdot f_l$ on B_a can be extended continuously to the zero function on L only if $p < m$. In fact, any application of a derivative ∂_{I_m} of order m admits a non-trivial term of the form $m! \cdot f_l \cdot \Pi_{k=1}^m dr_o(\frac{\partial}{\partial x_{i_k}})$, which can not be extended continuously by zero to L. On the other hand, $\partial_{I_{m-1}}(r_o^m \cdot f_l) = r_o \cdot h$, where $|h|$ is bounded on B_a^t for all $t > 0$ if $s_l \geq 0$. This shows that $r_o^m \cdot f_l$ is of class C^{k-1} with $k := \min\{m, m + s_l\}$, but it is not k-times continuously differentiable. □

Now we set

$$\omega_a := (ar_o)^4 \cdot (r\sigma_3)^2 \qquad \text{and} \qquad \rho_a = \frac{a^4 r_o^2}{1-(ar_o)^4} \cdot \left(\frac{\alpha}{r}\right)^2 \ .$$

With these notations we have $g_a = g_0 - \omega_a + \rho_a$ on $\tilde{B}_a$, where g_0 is the flat Minkowski metric on $\tilde{B}_a$.

Proof of Proposition 14. The metric g_0 is smooth on $\tilde{B}_a$. We have to discuss the differentiability of ω_a and ρ_a. The coefficients of the 1-forms $r \cdot \sigma_3$ and α / r are of order $s_l = 0$, resp., $s_l = 1$. With application of Lemma 13 we conclude that ω_a is of class C^3 and ρ_a is of class C^1. The symmetric 2-form ρ_a is not of class C^2. This implies that the symmetric bilinear form g_a on $\tilde{B}_a$ is of class C^1 for all $a > 0$, but it is not of class C^2.

We postpone the proof that g_a is a metric of Lorentzian signature until the proof of Proposition 15. The proof of Lemma 12 about the existence of the orthonormal frame e will show the Lorentzian nature of g_a as well. □

For the proof of Proposition 15 we use the coordinate change

$$\begin{array}{cccc} \Psi: & \mathbb{R}^5 \setminus L_o & \longrightarrow & \mathbb{R}^5 \setminus L_o \ , \\ & (x_0, r, \varphi_i) & \mapsto & (s, R, \varphi_i) = \left(\frac{-x_0}{r^2 - x_0^2} \ , \ \frac{r}{r^2 - x_0^2} \ , \ \varphi_i \right) \ , \end{array}$$

where the φ_i's are some (local) coordinates on S^3 which remain unchanged. The coordinate transformation Ψ is smooth on $\mathbb{R}^5 \setminus L_o$ and we have

$$\begin{aligned} dx_0 &= -\tfrac{s^2+R^2}{(R^2-s^2)^2}ds + \tfrac{2sR}{(R^2-s^2)^2}dR, \\ dr &= \tfrac{2sR}{(R^2-s^2)^2}ds - \tfrac{s^2+R^2}{(R^2-s^2)^2}dR, \\ \tfrac{\partial}{\partial r} &= -2sR\tfrac{\partial}{\partial s} - (s^2+R^2)\tfrac{\partial}{\partial R}, \\ \tfrac{\partial}{\partial x_0} &= -(s^2+R^2)\tfrac{\partial}{\partial s} - 2sR\tfrac{\partial}{\partial R} \ . \end{aligned}$$

This shows also $T = \frac{\partial}{\partial R}$ and $V = \frac{\partial}{\partial s}$.

Proof of Proposition 15. First, we calculate the symmetric bilinear form g_a on B_a with respect to the coordinate transformation Ψ. Remember that $\alpha = (x_0^2 + r^2)dr - 2x_0 r dx_0$. Then

$$\begin{array}{rcl} \alpha & = & \frac{-dR}{(R^2-s^2)^2} \ , \\ -dx_0^2 + dr^2 & = & \frac{-ds^2+dR^2}{(R^2-s^2)^2} \ . \end{array}$$

With $R^2 = r_o^{-2}$ on B_a and $r^2 - x_0^2 = (R^2 - s^2)^{-1}$ we obtain

$$\begin{aligned} g_a \ = \ & \frac{1}{(R^2-s^2)^2}\big(-ds^2 + dR^2 + R^2(\sigma_1^2 + \sigma_2^2 + (1-(a/R)^4)\sigma_3^2) \ \big) \\ & - \frac{R^2 \cdot dR^2}{r^2(1-(R/a)^4)\cdot (R^2-s^2)^4} \\ = \ & \frac{1}{(R^2-s^2)^2}\Big(-ds^2 + dR^2 + R^2(\sigma_1^2 + \sigma_2^2 + (1-(a/R)^4)\sigma_3^2) \\ & \qquad - \frac{1}{1-(R/a)^4} dR^2 \ \Big) \\ = \ & \frac{1}{(R^2-s^2)^2}\Big(\ -ds^2 + \frac{dR^2}{1-(a/R)^4} \ + \ R^2(\sigma_1^2 + \sigma_2^2 + (1-(a/R)^4)\sigma_3^2 \ \Big) \end{aligned}$$

and we can conclude that

$$\tilde{g}_a \ = \ \frac{1}{(r^2-x_0^2)^2} \cdot g_a \ = \ -ds^2 \ + \ g_{EH}$$

on B_a. The corresponding (even simpler) calculation on $L \setminus L_o$, where $r_o \equiv 0$, shows that

$$\tilde{g}_a \ = \ \frac{1}{(r^2-x_0^2)^2} g_a \ = \ -ds^2 \ + \ dR^2 \ + \ R^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \ ,$$

i.e., $\tilde{g}_a$ is the flat metric for $r < |x_0|$. In particular, since $\tilde{g}_a$ on $\tilde{B}_a \setminus L_o$ is a metric of Lorentzian signature, we have shown that the conformally equivalent symmetric bilinear form g_a of class C^1 on $\tilde{B}_a$ is a metric and admits Lorentzian signature as well, which completes the proof of Proposition 14.

Next we review curvature properties of the Eguchi-Hanson metric g_{EH}. This discussion will provide us with all the information that we need to prove our claims about the curvature properties of the Lorentzian metrics $\tilde{g}_a$ and g_a. Let us fix the orthonormal frame

$$\{f_1, f_2, f_3, f_4\} := \left\{ \ -\beta\frac{\partial}{\partial R} \ , \ R^{-1}\frac{\partial}{\sigma_1} \ , \ R^{-1}\frac{\partial}{\sigma_2} \ , \ (R\beta)^{-1}\frac{\partial}{\sigma_3} \ \right\} \ ,$$

where $\beta := \sqrt{1-(a/R)^4}$. We denote by $\{f^i : i = 1, \ldots, 4\}$ the dual frame. The connection 1-form ω and the curvature 2-form Ω of the Levi-Civita connection ∇^{EH} are determined by the structure equations

$$df^i = -\sum_{k=1}^{4} \omega_k^i \wedge f^k \qquad \text{and} \qquad \Omega_j^i = d\omega_j^i + \sum_{k=1}^{4} \omega_k^i \wedge \omega_j^k \ .$$

It holds that

$$\omega^i_j = -g_{EH}(\nabla^{EH} f_i, f_j) \qquad \text{and} \qquad \Omega^i_j = -g_{EH}(R(e_i,e_j)\cdot,\cdot) \ ,$$

where

$$R^{EH}(e_i,e_j) = \nabla^{EH}_{e_i}\nabla^{EH}_{e_j} - \nabla^{EH}_{e_j}\nabla^{EH}_{e_i} - \nabla^{EH}_{[e_i,e_j]} \ .$$

The components are explicitly calculated to

$$\begin{array}{rcccccc} \omega^1_2 & = & \omega^3_4 & = & \beta R^{-1}\cdot f^2 & = & \beta\cdot\sigma^1, \\ \omega^1_3 & = & -\omega^2_4 & = & \beta R^{-1}\cdot f^3 & = & \beta\cdot\sigma^2, \\ \omega^1_4 & = & \omega^2_3 & = & \gamma\cdot f^4 & = & \gamma R\beta\cdot\sigma^3, \end{array}$$

where $\gamma = \beta R^{-1} + \beta'$ and $\beta' = \frac{\partial\beta}{\partial R} = 2(a/R)^4(R\beta)^{-1}$, and

$$\begin{array}{rl} \Omega^1_2 = \ \ \Omega^3_4 = & \frac{2a^4}{R^6}\lambda^1_- \ , \\ \Omega^1_3 = -\Omega^2_4 = & \frac{2a^4}{R^6}\lambda^2_- \ , \\ \Omega^1_4 = \ \ \Omega^2_3 = & -\frac{4a^4}{R^6}\lambda^3_- \ , \end{array}$$

where the λ^i_-'s build a basis of the anti-selfdual 2-forms for g_{EH} and are defined as

$$\begin{aligned} \lambda^1_- &= f^1\wedge f^2 - f^3\wedge f^4 \ , \\ \lambda^2_- &= f^1\wedge f^3 - f^4\wedge f^2 \ , \\ \lambda^3_- &= f^1\wedge f^4 - f^2\wedge f^3 \ . \end{aligned}$$

It follows that the Riemannian curvature tensor R^{EH} of g_{EH} is anti-selfdual. This implies that g_{EH} is Ricci-flat and R^{EH} equals the Weyl tensor W^{EH}, i.e., we have

$$R^{EH} = W^{EH} = W^- \neq 0 \ .$$

In particular, since the Weyl tensor is a complete obstruction to conformal flatness in dimension 4, we can see that g_{EH} is nowhere conformally flat on its domain of definition (which is $B_a\cap E$, resp., $\Psi(B_a\cap E)$).

The metric $\tilde g_a = -ds^2 + g_{EH}$ is an ordinary semi-Riemannian product. Hence the curvature components of $\tilde g_a$ in the direction of the coordinate $\frac{\partial}{\partial s}$ vanish, i.e., the curvature tensor of $-ds^2+g_{EH}$ is entirely determined by the components of the Riemannian curvature tensor R^{EH}. In particular, we see that the metric $-ds^2+g_{EH}$ is Ricci-flat and the components of the Weyl tensor $W^{\tilde g_a}$ of $\tilde g_a$ in the direction of the coordinate $\frac{\partial}{\partial s}$ do vanish as well. Since, by construction, the metric $\tilde g_a = -ds^2 + g_{EH}$ is conformally equivalent to g_a on B_a, we know the Weyl tensor W^{g_a} of g_a on B_a as well. It is simply a rescaling of $W^{\tilde g_a}$. Obviously, the metric g_a is not conformally flat on B_a. On $L\setminus L_o$ the metric g_a is flat and therefore conformally flat, i.e., $W^{g_a}\neq 0$ on B_a and $W^{g_a}\equiv 0$ on $L\setminus L_o$.

Finally, on the light cone L_o the Weyl tensor of g_a is not defined in the usual way, because g_a is only of class C^1 at L_o. We aim to show that the Weyl tensor of g_a on $\tilde B_a\setminus L_o$ admits a continuous extension to L_o. For this purpose we note that the Weyl tensor rescales explicitly by $W^{g_a} = r_o^4 r^4\cdot W^{\tilde g_a}$. Then calculating the components of

W^{g_a} with respect to the coordinate system $u := \{\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_4}\}$ using our formulae for W^{EH} from above results in expressions of the form

$$W^{g_a}\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\Big) = \begin{cases} A \cdot r_o^2 \ + \ B \cdot r_o^6/r^4 & \text{on} \ \ B_a \\ \qquad 0 & \text{on} \ \ L \smallsetminus L_o \end{cases}$$

for all $i, j, k, l \in \{0, \ldots, 4\}$, where A, B are sums of functions of the form f_l, $\beta \cdot f_l$ and $\beta^{-1} \cdot f_l$ with order $s_l = 4$, i.e., the extensions of all components to L_o by zero are C^1-functions on $\tilde{B}_a$. We conclude that the Weyl tensor W^{g_a} has a continuous extension of class C^1 on $\tilde{B}_a$. □

Next we consider the frame $e = \{e_0, \ldots, e_4\}$, which we have defined in Section 2 and which was claimed there to be orthonormal for g_a on $B_a^>$ and of class C^1.

Proof of Lemma 12. First, we show that the frame e is orthonormal at every point of $(B_a^>, g_a)$. Obviously, this is true on $L \smallsetminus \{r = 0\}$, since g_a is the flat Minkowski metric thereon. It is also obvious that the vectors e_2, e_3 and e_4 are orthonormal for g_a on B_a and that they are orthogonal to the remaining basis vectors e_0 and e_1. For the latter we find with $\frac{a^4 r_o^2}{1+\beta} = R^2(1-\beta)$ and $T = \frac{\partial}{\partial R}$ the expressions

$$\begin{aligned} e_0 \ &= \ -(s^2+R^2)\tfrac{\partial}{\partial s} \ \ - \ \ 2sR\beta\tfrac{\partial}{\partial R} \qquad \text{and} \\ e_1 \ &= \ -2sR\tfrac{\partial}{\partial s} \qquad\quad\ \ - \ \ (s^2+R^2)\beta\tfrac{\partial}{\partial R} \ , \end{aligned}$$

from which we see that e_0 and e_1 are orthonormal with respect to $g_a = (R^2 - s^2)^{-2}(-ds^2 + g_{EH})$ on B_a as well. We conclude that the frame e is a pointwise orthonormal basis on B_a.

It remains to discuss the differentiability of the coefficients of the vectors $\{e_0, \ldots, e_4\}$. For this we notice that the function $\frac{a^4 r_o^2}{1+\beta}$ is only of class C^1 on $B_a^>$. The function β^{-1} is of class C^3 and all other functions, which are involved in the coefficients are smooth on $B_a^>$. □

Let us introduce the vectors

$$\begin{aligned} \tilde{e}_0 &:= \tfrac{-1}{R^2-s^2}\big(\ (s^2+R^2)\tfrac{\partial}{\partial s} \ + \ 2sR\beta\tfrac{\partial}{\partial R} \ \big) \qquad \text{and} \\ \tilde{e}_1 &:= \tfrac{-1}{R^2-s^2}\big(\ 2sR\tfrac{\partial}{\partial s} \ + \ (s^2+R^2)\beta\tfrac{\partial}{\partial R} \ \big) \end{aligned}$$

with respect to the Ψ-transformed coordinates, and let us denote

$$\tilde{e} := \Big\{ \ \tilde{e}_0 \ , \ \tilde{e}_1 \ , \ R^{-1} \cdot \frac{\partial}{\partial \sigma_1} \ , \ R^{-1} \cdot \frac{\partial}{\partial \sigma_2} \ , \ (R\beta)^{-1} \cdot \frac{\partial}{\partial \sigma_3} \ \Big\} \ ,$$

which is an orthonormal frame with respect to $\tilde{g}_a = \frac{1}{(r^2 - x_0^2)^2} \cdot g_a$ on $B_a^> \smallsetminus L_o$. As we know from the proof of Lemma 12, $\Psi_*(e_i) = (R^2 - s^2) \cdot \tilde{e}_i$, $i = 0, \ldots, 4$. Moreover, we set

$$f := \Big\{ \ -\frac{\partial}{\partial s} \ , \ -\beta\frac{\partial}{\partial R} \ , \ R^{-1} \cdot \frac{\partial}{\partial \sigma_1} \ , \ R^{-1} \cdot \frac{\partial}{\partial \sigma_2} \ , \ (R\beta)^{-1} \cdot \frac{\partial}{\partial \sigma_3} \ \Big\}$$

on $B_a^> \setminus L_o$. On B_a this is just the extension by f_0 of the frame $\{f_1, f_2, f_3, f_4\}$ that we introduced already for the Eguchi-Hanson metric g_{EH}. The matrix

$$\kappa = \frac{1}{R^2 - s^2} \begin{pmatrix} s^2 + R^2 & 2sR & 0 & 0 & 0 \\ 2sR & s^2 + R^2 & 0 & 0 & 0 \\ 0 & 0 & R^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & R^2 - s^2 & 0 \\ 0 & 0 & 0 & 0 & R^2 - s^2 \end{pmatrix}$$

gives the transformation $\tilde{e} = f \cdot \kappa$. With $t := \ln \frac{R-s}{R+s}$ and

$$E_{01} := \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have $\kappa = \exp(tE_{01})$. The elements in the preimage of κ by the group covering $\lambda : \mathrm{Spin}(1,4) \to \mathrm{SO}(1,4)$ are given by $\pm \exp(\frac{t}{2}\gamma_0\gamma_1)$, where we use the γ-matrices introduced in Section 2. We choose in the following $\tilde{\kappa} := \exp(\frac{t}{2}\gamma_0\gamma_1)$, which is given by

$$\tilde{\kappa} = \frac{1}{\sqrt{R^2 - s^2}} \begin{pmatrix} R & 0 & -s & 0 \\ 0 & R & 0 & s \\ -s & 0 & R & 0 \\ 0 & s & 0 & R \end{pmatrix}.$$

Let us remind to the conformal covariance property of a twistor spinor with respect to conformally related metrics. In general, a twistor spinor φ on a spin space (M, c) with conformal structure c is an element of $\Gamma(\mathcal{S}[-1/2])$ with conformal weight $-1/2$. With respect to some metric $g \in c$ and a spinor frame v_s the twistor spinor φ is given by $[v_s, w]$, where w is some function with values in the spinor module Δ. If $\tilde{g} = e^{2\sigma} g$ is some conformally related metric to g in c with corresponding rescaled spinor frame $\tilde{v}_s$, we denote $\tilde{\varphi} := e^{\sigma/2} \cdot [\tilde{v}_s, w]$, which is the conformally rescaled twistor spinor with respect to $\tilde{g}$. (Of course, in abuse of notations, φ and $\tilde{\varphi}$ represent the same spinor field in $\Gamma(\mathcal{S}[-1/2])$ (cf. Section 0.8).

Proof of Theorem 20. The verification of the first two statements of Theorem 20 is the main work of the proof. We will show this in some few steps. First, we prove that $\psi_{bc}^>$ is a twistor spinor on B_a and also on $L \setminus (L_o \cup \{r = 0\})$, which already implies that $\psi_{bc}^>$ is a twistor spinor on $B_a^>$. Thereby, we will not directly check the twistor equation for $\psi_{bc}^>$, but first use the conformal transformation from g_a to the Ricci-flat metric $\tilde{g}_a$. In the next step we show that $\psi_{bc}^>$ extends to a C^1-spinor on $\tilde{B}_a \setminus \{0\}$. This spinor will still solve the twistor equation. Finally, we show that the latter spinor can also be extended to the origin by a zero. The resulting spinor ψ_{bc} is a unique continuous extension of $\psi_{bc}^>$, which is of class C^1 and solves the twistor equation everywhere on $\tilde{B}_a$.

To start with, let us consider $\psi_{bc}^>$ on $B_a^> \setminus L_o$. The spinor $\psi_{bc}^>$ is given with respect to the spinor frame e_s by $[\ e_s\ ,\ (-x_0 b, x_0 c, rb, rc)^\top\ ]$. We have $e = \tilde{e} \cdot ((R^2 - s^2) id)$, where $\tilde{e}$ is orthonormal with respect to $\tilde{g}_a = \frac{1}{(r^2 - x_0^2)^2} g_a$. Let $\tilde{e}_s$ be the corresponding lift of the rescaled frame $\tilde{e}$. Then the (conformally rescaled) spinor $\tilde{\psi}^{>}_{bc}$ (cf. Section 0.8) is given with respect to $\tilde{e}_s$ by

$$[\ \tilde{e}_s\ ,\ \sqrt{R^2-s^2}\cdot(-x_0 b, x_0 c, rb, rc)^\top\ ]\ .$$

Further, we have

$$\tilde{\psi}^{>}_{bc} = \sqrt{R^2-s^2}\cdot[\ f_s\ ,\ \tilde{\kappa}\cdot(-x_0 b, x_0 c, rb, rc)^\top\ ]\ ,$$

where f_s denotes the lift of the frame f, which corresponds to the lift $\tilde{e}_s$. With

$$\sqrt{R^2-s^2}\cdot\tilde{\kappa}\begin{pmatrix}-x_0 b\\ x_0 c\\ rb\\ rc\end{pmatrix} = \begin{pmatrix}R & 0 & -s & 0\\ 0 & R & 0 & s\\ -s & 0 & R & 0\\ 0 & s & 0 & R\end{pmatrix}\begin{pmatrix}-x_0 b\\ x_0 c\\ rb\\ rc\end{pmatrix} = \begin{pmatrix}0\\ 0\\ b\\ c\end{pmatrix}$$

we find that

$$\tilde{\psi}^{>}_{bc} = [\ f_s\ ,\ (0,0,b,c)^\top\ ]\ .$$

The spinor derivative of $\tilde{\psi}^{>}_{bc}$ with respect to $\tilde{g}_a$ is given by

$$\tilde{\nabla}^{S}\tilde{\psi}^{>}_{bc} = [\ f_s\ ,\ \frac{-1}{2}\cdot\sum_{0\le i<j\le 4}\omega^i_j\otimes\gamma_i\gamma_j\cdot(0,0,b,c)^\top\ ]\ ,$$

where the ω^i_j's are the components of the Levi-Civita connection of $\tilde{g}_a$. On B_a we have $\omega^0_j = 0$ and the other ω^i_j's are just the components that we calculated in the proof of Proposition 15 for the Eguchi-Hanson metric g_{EH}. Notice also that on $L\smallsetminus(L_o\cup\{r=0\})$ the components ω^i_j admit the same expressions (with $\beta\equiv 1$) as on B_a with respect to the frame f. The relations for the ω^i_j's immediately prove that $\tilde{\psi}^{>}_{bc}$ is a parallel spinor with respect to $\tilde{g}_a$ on $B^{>}_a\smallsetminus L_o$ for any $(b,c)\in\mathbb{C}^2\smallsetminus 0$. (In fact, the spinors of the form $\tilde{\psi}^{>}_{bc}$ restricted to the Eguchi-Hanson metric g_{EH}, which is a hyperKähler metric for any $a>0$, form the space of all parallel spinors thereon.) Any parallel spinor is a twistor spinor. In particular, $\tilde{\psi}^{>}_{bc}$ is a twistor spinor. Hence, by conformal covariance and the fact that $\psi^{>}_{bc}$ is of class C^1 on $B^{>}_a$, the spinor $\psi^{>}_{bc}$ is a twistor spinor on $(B^{>}_a, g_a)$.

Now let

$$G = r^{-1}\cdot\begin{pmatrix}r & 0 & 0 & 0 & 0\\ 0 & x_1 & x_2 & x_3 & x_4\\ 0 & -x_2 & x_1 & -x_4 & x_3\\ 0 & -x_3 & x_4 & x_1 & -x_2\\ 0 & -x_4 & -x_3 & x_2 & x_1\end{pmatrix}$$

be a matrix valued function on $B^{>}_a$. We have $e\cdot G = \{\frac{\partial}{\partial x_0},\ldots,\frac{\partial}{\partial x_4}\}$ on $L\smallsetminus(L_o\cup\{r=0\})$. The standard frame u is orthonormal on $L\smallsetminus(L_o\cup\{r=0\})$ and admits a smooth extension to $L\smallsetminus L_o$. (Of course, the matrix G is singular for $r=0$.) A transformation matrix for corresponding spinor frames is given by

$$\tilde{G} = r^{-1}\cdot\begin{pmatrix}r & 0 & 0 & 0\\ 0 & r & 0 & 0\\ 0 & 0 & x_1+ix_2 & x_3+ix_4\\ 0 & 0 & -x_3+ix_4 & x_1-ix_2\end{pmatrix}\ .$$

This form of the matrix is due to the fact that $\mathrm{Spin}(4)$ is isomorphic to $\mathrm{SU}(2)\times\mathrm{SU}(2)$. The spinor $\psi^{>}_{bc} = [e_s, (-x_0 b, x_0 c, rb, rc)^\top)]$ is presented with respect to the spinor frame u_s on $L \setminus (L_o \cup \{r=0\})$ by

$$\psi^{>}_{bc} = [\ u_s\ ,\ \tilde{G}^{-1}(-x_0 b, x_0 c, rb, rc)^\top)\ ]\ .$$

Obviously, the vector valued function

$$\tilde{G}^{-1}\begin{pmatrix} -x_0 b \\ x_0 c \\ rb \\ rc \end{pmatrix} = \begin{pmatrix} -x_0 b \\ x_0 c \\ (x_1 - ix_2)b - (x_3 + ix_4)c \\ (x_3 - ix_4)b + (x_1 + ix_2)c \end{pmatrix}$$

is non-singular and smooth on $L \setminus L_o$. Hence the spinor $\psi^{>}_{bc}$ on $B^{>}_a$ admits a C^1-extension to $\tilde{B}_a \setminus \{0\}$. We denote this extension by ψ^{o}_{bc}, which is by continuity reasons a twistor spinor on $\tilde{B}_a \setminus \{0\}$.

We still have to show that ψ^{o}_{bc} extends further to a C^1-spinor ψ_{bc} on $\tilde{B}_a$. For this purpose, we improve our change of frame from above and introduce a non-singular C^1-frame around the origin. Then we show that the components of ψ^{o}_{bc} with respect to a corresponding non-singular spinor frame are of class C^1. So let

$$Q = \begin{pmatrix} k & q & 0 & 0 & 0 \\ q & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$k \ := \ \sqrt{\tfrac{1+r_o^2\rho}{1-4x_0^2\rho}} \ \cdot \ \Big(1 \ - \ \tfrac{(r^2+x_0^2)\beta^2}{r^2(1+\beta)} \cdot \rho \Big) \qquad \text{and}$$

$$q \ := \ \sqrt{\tfrac{1+r_o^2\rho}{1-4x_0^2\rho}} \ \cdot \ \tfrac{2x_0(r^2+x_0^2)\beta^2}{r(1+\beta)} \ \cdot \ \rho,$$

where $\rho = a^4\beta^{-2}r_o^2$. We have $Q \equiv \mathbb{1}$ on L and $4x_0^2\rho < 1$ on an open neighbourhood of L. For $4x_0^2\rho < 1$ the function k is well defined and of class C^1 (cf. Lemma 13). It follows that on a certain open neighbourhood of L in $\tilde{B}_a$ the function k is positive. We denote this set by C_a. In fact, $k^2 - q^2 \equiv 1$ on C_a, i.e., the transformation matrix Q takes values in $\mathrm{SO}_o(1,4)$ and is of class C^1 on C_a. The matrix Q is useful, because the transformed frame $\{h_0, \ldots, h_4\} := e \cdot Q$ is given by

$$\begin{aligned} h_0 &= (1 - 4x_0^2\rho)^{-1/2} \cdot \tfrac{\partial}{\partial x_0}\ , \\ h_1 &= (1+\rho_1)^{-1/2} \cdot \tfrac{\partial}{\partial r} \ + \ \rho_2(1+\rho_1)^{-1/2} \cdot (1 - 4x_0^2\rho)^{-1} \cdot \tfrac{\partial}{\partial x_0}\ , \\ h_i &= e_i \qquad \text{for} \ \ i = 2,3,4 \end{aligned}$$

on $B^{>}_a \cap C_a$ with $\rho_1 := r^{-2}(r^2 + x_0^2)^2\rho$ and $\rho_2 := -4x_0 r^{-1}(r^2+x_0^2)\rho$, i.e., the first basis vector h_0 admits now a continuous extension to $\{r=0\}$. The remaining basis vectors are still singular at $\{r=0\}$. However, a straightforward calculation shows that the frame $\tilde{h} = \{\tilde{h}_0, \ldots, \tilde{h}_4\} := e \cdot (QG)$ admits a C^1-extension to $\{r=0\}$, i.e., $\tilde{h}$ is a non-singular C^1-frame on C_a, which is an open neighbourhood of the origin.

A corresponding transformation matrix to Q for spinor frames is given by

$$\tilde{Q} = \begin{pmatrix} \sqrt{\frac{k+1}{2}} & 0 & \frac{-q}{\sqrt{2(k+1)}} & 0 \\ 0 & \sqrt{\frac{k+1}{2}} & 0 & \frac{q}{\sqrt{2(k+1)}} \\ \frac{-q}{\sqrt{2(k+1)}} & 0 & \sqrt{\frac{k+1}{2}} & 0 \\ 0 & \frac{q}{\sqrt{2(k+1)}} & 0 & \sqrt{\frac{k+1}{2}} \end{pmatrix} .$$

This matrix is again of class C^1 on C_a. In particular, it is non-singular and equal to the identity on L. In fact, the matrix $\tilde{Q}$ can be written as $\tilde{Q} = \mathbb{1} + r_o^2 \cdot \hat{Q}$, where $\hat{Q}$ is some matrix valued function on C_a whose components are sums of functions of the form f_l with $s_l \geq 0$. The spinor ψ^o_{bc} is expressed with respect to the corresponding spinor frame $\tilde{h}_s$ by

$$\psi^o_{bc} = [\ \tilde{h}_s\ ,\ \tilde{G}^{-1} \cdot \tilde{Q}^{-1}(-x_0 b, x_0 c, rb, rc)^\top\ ]\ .$$

We have

$$\Phi := \tilde{G}^{-1} \cdot \tilde{Q}^{-1} \begin{pmatrix} -x_0 b \\ x_0 c \\ rb \\ rc \end{pmatrix} = \tilde{G}^{-1} \begin{pmatrix} -\sqrt{\frac{k+1}{2}} \cdot x_0 b & - & \frac{qrb}{\sqrt{2(k+1)}} \\ \sqrt{\frac{k+1}{2}} \cdot x_0 c & + & \frac{qrc}{\sqrt{2(k+1)}} \\ \frac{q x_0 b}{\sqrt{2(k+1)}} & + & \sqrt{\frac{k+1}{2}} \cdot rb \\ \frac{q x_0 c}{\sqrt{2(k+1)}} & + & \sqrt{\frac{k+1}{2}} \cdot rc \end{pmatrix} .$$

From Lemma 13 we know that the function $\frac{q x_0}{r}$ is of class C^1, since it behaves like $r_o^2 \cdot \frac{x_0}{r}$, where $\frac{x_0}{r}$ has order zero. This observation is sufficient to conclude that the vector valued function Φ extends to a C^1-function on C_a. Obviously, the extended C^1-function Φ is zero at the origin. We can conclude that ψ^o_{bc} extends to a C^1-spinor ψ_{bc} on $\tilde{B}_a$ with zero at the origin. If $(b, c) \neq 0$, the origin is the only zero of ψ_{bc}. Moreover, since ψ^o_{bc} is a twistor spinor and ψ_{bc} is C^1, the expression $\nabla^{\mathbb{S}}_X \psi_{bc} + \frac{1}{n} X \cdot D^{\mathbb{S}} \psi_{bc}$ is continuous on $\tilde{B}_a$ and zero on $\tilde{B}_a \setminus \{0\}$ for all differentiable vector fields X. This shows that ψ_{bc} satisfies the twistor equation in the origin. Altogether we have proven yet the first three statements of Theorem 20.

The length square u_{bc} of $\psi^>_{bc} = [e_s, (-x_0 b, x_0 c, rb, rc)^\top]$ is by definition (cf. [**15**]) equal to

$$(\ \gamma_0 \cdot (-x_0 b, x_0 c, rb, rc)^\top\ ,\ (-x_0 b, x_0 c, rb, rc)^\top\ )_{\mathbb{C}^4}\ =\ (r^2 - x_0^2) \cdot (b^2 + c^2)\ .$$

Obviously, the function u_{bc} is smooth on $\tilde{B}_a$ and its zero set is L_o. We know already from Proposition 15 that $u_{bc}^{-2} \cdot g_a$ is a Ricci-flat metric on $\tilde{B}_a \setminus L_o$, i.e., the function u_{bc} provides a rescaling to an Einstein metric in the conformal class. In general, such a rescaling function satisfies the partial differential equation

$$-u_{bc} \cdot Ric^0 = (n-2) \cdot Hess(u_{bc})^0\ .$$

It is interesting to note that the function u_{bc} has a non-trivial zero set (cf. [**65**]).

Furthermore, using the definition $g_a(V_{\psi_{bc}}, X) = \langle \psi_{bc}, X \cdot \psi_{bc} \rangle_{\mathbb{S}}$ and calculating the products $(\gamma_0 \cdot (-x_0 b, x_0 c, rb, rc)^\top, \gamma_i \cdot (-x_0 b, x_0 c, rb, rc)^\top)$ for $i = 0, \ldots, 4$ shows that the spinor square of the twistor spinor ψ_{bc} is equal to

$$V_{\psi_{bc}} = (b^2 + c^2) \cdot \big(- (x_0^2 + r^2) e_0 - 2 x_0 r e_1 \big) = (b^2 + c^2) \cdot V \ .$$

For $(b, c) \neq 0$ the vector field $V_{\psi_{bc}}$ is smooth with unique zero at the origin. Finally, since $\alpha(V) = 0$, we obtain $g_a(V, V) = -(r^2 - x_0^2)^2$. This shows that $V_{\psi_{bc}}$ for $(b, c) \neq 0$ is everywhere timelike except on L_o, where the spinor square is lightlike, resp., zero only at the origin. This behaviour fits to our observations in Theorem 19. $\square$

Corollary 5 is a simple conclusion using Theorem 20 and Proposition 15. We add some remarks about the vector field V, which is smooth! Since ψ_{bc} is a twistor spinor we immediately know that V is a conformal Killing vector field for g_a on $\tilde{B}_a$. However, we simply reprove this statement here directly. Namely,

$$\begin{array}{rcl}
L_V g_0 & = & -4x_0 \cdot g_0 \\
L_V r^m & = & -2m x_0 \cdot r^m \\
L_V r_o^2 & = & 0 \\
L_V \sigma_3^2 & = & 0 \\
L_V(-r^2 (a r_o)^4 \cdot \sigma_3^2) & = & -4x_0 \cdot (-r^2 (a r_o)^4 \cdot \sigma_3^2) \\
L_V \alpha & = & -4x_0 \cdot \alpha \\
L_V(a^4 (r\beta)^{-2} r_o^2 \cdot \alpha^2) & = & (4x_0 - 2 \cdot 4 \cdot x_0)(a^4 (r\beta)^{-2} r_o^2 \cdot \alpha^2) \\
 & = & -4x_0 \cdot (a^4 (r\beta)^{-2} r_o^2 \cdot \alpha^2) \ .
\end{array}$$

This proves that $L_V g_a = -4x_0 \cdot g_a$ on $\tilde{B}_a$, i.e., V is a conformal Killing vector with $div^{g_a}(V) = -10 \cdot x_0$.

Finally, we want to state a reason why an extension of the metric g_a on B_a to L with differentiable Weyl tensor has to be conformally flat in order to preserve twistor spinors and the conformal Killing vector V. One observes the following facts. All integral curves of V on L converge in one flow direction to the origin, i.e., the origin is in the closure of any integral curve on L. The length square $|W^{2,2}|^2$ of the Weyl $(2,2)$-tensor is constant along integral curves of V. Moreover, with our assumptions we know that at the origin W^{g_a} has to vanish (cf. [**17, 106**]), i.e., $|W^{2,2}|^2$ is identically zero on the closure L of $L \setminus L_o$. Then, since V inserted into the Weyl tensor W^{g_a} produces zero (cf. [**17, 106**]) and V is timelike on $L \setminus L_o$, it follows that the length square of W^{g_a} is non-negative on $L \setminus L_o$ and it is zero if and only if the Weyl tensor vanishes. With the argument from before we can conclude that the Weyl tensor of the extension has to vanish on L.

CHAPTER 5

Partially Integrable CR-Spaces and the Gauged Fefferman Construction

The original Fefferman construction was used to define in an invariant way a conformal structure on a circle bundle over the boundary of a strictly pseudoconvex domain in $\mathbb{C}^{m+1}$ (cf. [**53**]). This construction was made intrinsic for the case of integrable CR-spaces (M, T_{10}) by J.M. Lee (cf. [**102**]). In Section 0.12 we explained already a further generalised version of Fefferman constructions for parabolic geometries via Cartan geometry due to A. Cap (cf. [**34**]). This approach coincides in the special case with the classical Fefferman construction for integrable CR-geometry. Immediately, it becomes clear that the generalised version applies also to the case of partially integrable CR-geometry. As in the integrable case the corresponding Fefferman spaces admit a conformal structure, which is determined by the underlying CR-geometry only.

In this chapter we aim to work out this generalised Fefferman construction for *partially integrable CR-spaces* of *hypersurface type*. Our approach will go along the lines of Lee's intrinsic construction, i.e., we will make the construction on the level of pseudo-Hermitian geometry and the resulting conformal classes will be given by a generalised version of Fefferman metrics. For this purpose we introduce *(generalised) Tanaka-Webster connections* for pseudo-Hermitian structures on partially integrable CR-spaces. We are neither going to describe the Fefferman construction on the level of Cartan geometry nor do we discuss the relation of CR-tractor calculus and conformal tractor calculus. This approach is worked out [**39**].

However, once we have constructed the Fefferman space for partially integrable CR-geometry we will discuss some features of its conformal geometry in terms of conformal tractor calculus. In fact, we will show that there exists a certain complex structure acting on the conformal standard tractor bundle of a Fefferman space. This complex structure is a section of the adjoint tractor bundle and corresponds via the splitting operator $\mathbf{S}$ to the fundamental vector field in the fibre of the Fefferman construction. In Chapter 6 we will investigate orthogonal complex structures on the conformal standard tractor bundle, which correspond to conformal Killing vectors fields, in general. In view of the needs of the discussion there, we will actually introduce in this chapter a slightly more general version of the Fefferman construction for partially integrable CR-geometry as we announced so far. We call this generalised version the *gauged Fefferman construction*. As before the (gauged) Fefferman space is equipped with a conformal structure. However, in this construction the generalised Tanaka-Webster connection (resp., the corresponding Weyl connection) is basically replaced by an arbitrary connection on the canonical line bundle.

We start our investigations by introducing a generalised version of Tanaka-Webster connections for the case of pseudo-Hermitian geometry on partially integrable CR-spaces in Section 1 (cf. [**102, 126**]). Using these connections (resp., the corresponding

Weyl connections) we construct the Fefferman metrics on the canonical S^1-bundle such that the conformal class does not depend on the choice of pseudo-Hermitian form, but only on the underlying partially integrable CR-geometry (cf. Section 3 and [**112**]). It turns out that we can add a gauge form ℓ to the Weyl connection form and still obtain a uniquely defined conformal structure on the Fefferman space, which does not depend on the choice of pseudo-Hermitian form. An important part of our study is the computation of the relation between Webster scalar curvature and Riemannian scalar curvature of the ℓ-gauged Fefferman space in Section 5 (cf. Theorem 22). Moreover, we are able to calculate explicitly the application of the Laplacian to the fundamental vector field in the Fefferman construction (cf. Proposition 16). The result identifies the explicit form of the adjoint tractor that belongs via the splitting operator to the vertical Killing vector field in the fibre of the gauged Fefferman construction. In a certain situation of the gauged Fefferman construction this adjoint tractor acts as complex structure on standard tractors (cf. Theorem 23).

1. The Tanaka-Webster Connection

Let (M, H, J) (or (M, T_{10})) denote a partially integrable CR-manifold equipped with a pseudo-Hermitian form θ (cf. Section 0.10 and e.g. [**102, 41**]). For the sake of simplified notations we will assume throughout this chapter that the corresponding Levi-form L_θ is positive definite on H (and then we call the CR-structure *strictly pseudoconvex*). This restriction is not essential at all for our investigations! The purpose of this section is to introduce a certain covariant derivative, which naturally belongs to the given pseudo-Hermitian structure. We call this connection the (generalised) Tanaka-Webster connection to θ. In the integrable case our definition of this connection coincides with the original one (cf. [**102, 126**]). We will also introduce curvature expressions for the generalised Tanaka-Webster connection.

The following Lemmata 14 to 17 are known facts, certainly for the case of integrable CR-structures, where its statements and proofs can be found in [**16**] and [**17**]. We discuss here the modified statements for the weaker condition of partial integrability. Thereby, we mainly explain the refinements of the formulae that have to be taken into consideration due to partial integrability. We usually omit those parts of the proofs of the statements, which do not depend on the Nijenhuis tensor $\mathcal{N}_J$ (cf. also Section 0.10).

Lemma 14. *(cf. [**17**]) Let $L_\theta : TM^{\mathbb{C}} \times TM^{\mathbb{C}} \to \mathbb{C}$ be the Levi-form to θ on a partially integrable CR-manifold (M, T_{10}) (resp. (M, H, J)) and let T be the Reeb vector field that belongs to θ. Then*

$$
\begin{aligned}
&[T, Z] \in \Gamma(H^{\mathbb{C}}) \qquad \textit{for all } Z \in \Gamma(H^{\mathbb{C}}) \ , \\
&L_\theta([T, U], V) + L_\theta(U, [T, V]) = T(L_\theta(U, V)) \ , \\
&L_\theta([T, \bar{U}], V) = L_\theta([T, \bar{V}], U) \ , \\
&L_\theta([T, \mathrm{U}], \bar{V}) = L_\theta([T, V], \bar{U})
\end{aligned}
$$

for all $U, V \in \Gamma(T_{10})$*, and*

$$L_\theta(X,Y) = d\theta(X,JY)\ ,$$
$$L_\theta(JX,JY) = L_\theta(X,Y)\ , \qquad L_\theta(JX,Y) + L_\theta(X,JY) = 0\ ,$$
$$L_\theta([T,X],Y) - L_\theta([T,Y],X) = L_\theta([T,JX],JY) - L_\theta([T,JY],JX)$$

for all $X, Y \in \Gamma(H)$.

There belongs a certain connection to any pseudo-Hermitian structure θ on a partially integrable CR-manifold (M, T_{10}) (cf. [**102, 17, 126**]).

Lemma 15. *(cf. [**17**]) Let* (M, T_{10}, θ) *be a pseudo-Hermitian manifold and let* T *be the Reeb vector field to* θ*. Then there exists a uniquely determined covariant derivative*

$$\nabla^W : \Gamma(T_{10}) \to \Gamma(T^*M^{\mathbb{C}} \otimes T_{10})$$

such that

(1)
$$\nabla^W_T U = pr_{10}[T,U]\ , \qquad \nabla^W_{\bar V} U = pr_{10}[\bar V, U]\ ,$$

(2)
$$X(L_\theta(U,V)) = L_\theta(\nabla^W_X U, V) + L_\theta(U, \nabla^W_{\bar X} V)$$

for all $U, V \in \Gamma(T_{10})$ *and* $X \in TM^{\mathbb{C}}$*, where* pr_{10} *denotes the projection onto* T_{10}*. The connection* ∇^W *satisfies*

$$\nabla^W_U V - \nabla^W_V U = pr_{10}[U,V]\ .$$

We call the connection ∇^W, which is guaranteed by Lemma 15, the (generalised) Tanaka-Webster connection corresponding to θ. The connection ∇^W is uniquely determined by relation (1) of Lemma 15 and by $L_\theta(\nabla^W_Z U, V) = Z(L_\theta(U,V)) - L_\theta(U, [\bar Z, V])$, which follows from the metric condition (2). The proof that these defining relations give rise to a connection uses mainly Lemma 14 (cf. [**16**]). Note that in case of an integrable CR-structure T_{10} the more special relation

$$\nabla^W_U V - \nabla^W_V U = [U,V]$$

holds. We extend now the Tanaka-Webster connection ∇^W to the complex tangent bundle $TM^{\mathbb{C}}$ by

$$\nabla^W T := 0 \qquad \text{and} \qquad \nabla^W_X \bar U := \overline{\nabla^W_{\bar X} U} \qquad \text{for all } X \in TM^{\mathbb{C}},\ U \in \Gamma(T_{10})\ .$$

The torsion Tor^W of this connection is defined in the usual manner as

$$Tor^W(X,Y) := \nabla^W_X Y - \nabla^W_Y X - [X,Y]$$

for $X, Y \in \Gamma(TM^{\mathbb{C}})$.

Lemma 16. *(cf. [**17**]) The torsion* Tor^W *of the Tanaka-Webster connection*

$$\nabla^W\ :\ \Gamma(TM^{\mathbb{C}}) \to \Gamma(T^*M^{\mathbb{C}} \otimes TM^{\mathbb{C}})$$

satisfies

$$Tor^W(U, \bar V) = iL_\theta(U,V)\cdot T\ ,$$
$$Tor^W(U,V) = -pr_{01}[U,V]\ , \qquad Tor^W(\bar U, \bar V) = -pr_{10}[\bar U, \bar V]\ ,$$
$$Tor^W(T,U) = -pr_{01}[T,U]\ , \qquad Tor^W(T, \bar U) = -pr_{10}[T, \bar U]$$

for all $U, V \in \Gamma(T_{10})$*, where* pr_{01} *denotes the projection onto* T_{01}.

In the integrable case the formulae for the torsion simplify to

$$Tor^W(U,V) = Tor^W(\bar{U},\bar{V}) = 0$$

when $U, V \in T_{10}$. However, due to the refinement in Lemma 15, for the partially integrable case the relation $\nabla^W_U V - \nabla^W_V U - [U,V] = -pr_{01}[U,V]$ holds. This implies immediately the torsion formulae of Lemma 16. As usual, we can restrict the Tanaka-Webster connection to the real part and obtain a linear connection ∇^W on the (real) tangent bundle TM.

Lemma 17. *(cf. [**17**]) Let θ be a pseudo-Hermitian structure on a partially integrable CR-manifold (M,H,J). The (real) Webster connection*

$$\nabla^W : \mathfrak{X}(M) \to \Gamma(T^*M \otimes TM)$$

is uniquely determined by the following properties

(1)
$$X(L_\theta(Y,Z)) = L_\theta(\nabla^W_X Y, Z) + L_\theta(Y, \nabla^W_X Z)$$
for all $X, Y, Z \in \mathfrak{X}(M)$, i.e., ∇^W is metric with respect to L_θ on H,

(2)
$$Tor^W(X,Y) = L_\theta(JX,Y)\cdot T - \frac{1}{4}\mathcal{N}_J(X,Y) \qquad \textit{and}$$

(3)
$$Tor^W(T,X) = -\frac{1}{2}\big(\ [T,X] + J[T,JX]\ \big)$$

for all $X, Y \in \Gamma(H)$. In addition, the connection ∇^W satisfies

$$\nabla^W T = 0 \qquad \textit{and} \qquad \nabla^W J = 0\ .$$

Proof. The same proof as in [**16**] (p.16) applies. However, we have to take into consideration the refined formula $Tor^W(U,V) = -pr_{01}[U,V]$ and the relation

$$-\frac{1}{4}\mathcal{N}_J(X,Y) = Tor^W(U,V) + Tor^W(\bar{U},\bar{V})$$

for all $U, V \in T_{10}$, where $X = U + \bar{U}$ and $Y = V + \bar{V}$. Together with $Tor^W(U,\bar{V}) = iL_\theta(U,V)\cdot T$ this shows that

$$\begin{aligned} Tor^W(X,Y) &= Tor^W(\bar{U},V) + Tor^W(U,\bar{V}) + Tor^W(U,V) + Tor^W(\bar{U},\bar{V}) \\ &= L_\theta(JX,Y)\cdot T - \frac{1}{4}\mathcal{N}_J(X,Y) \end{aligned}$$

for all $X, Y \in H$. Since

$$\begin{aligned} \nabla^W_Y JX &= \nabla^W_Y(i(U-\bar{U})) = i(\nabla^W_Y U - \overline{\nabla^W_Y U}) \\ &= J(\nabla^W_Y U\ +\ \overline{\nabla^W_Y U}) = J\nabla^W_Y X \end{aligned}$$

for all $X \in \Gamma(H)$ and $Y \in TM$, it follows that J is parallel with respect to ∇^W. □

The important point in Lemma 17 is the fact that the Nijenhuis tensor is part of the torsion of the Tanaka-Webster connection due to partial integrability. The way in which the Nijenhuis tensor $\mathcal{N}_J$ occurs in the formula for the torsion Tor^W is essentially implicated by condition (1) of Lemma 15. (The metric condition (2) of Lemma 15 is a rather inevitable choice.) There are other suitable connections, which occur in the framework of partially integrable CR-geometry and pseudo-Hermitian geometry. These sorts of connections differ from our definition of the Tanaka-Webster connection ∇^W e.g. by the torsion normalisation (cf. e.g. [**126**]).

Next we define curvature expressions for ∇^W. Thereby, we use the conventions for indices and orthonormal bases as introduced in Section 0.10. The curvature operator of ∇^W is defined by

$$R^{\nabla^W}(X,Y)Z = \nabla^W_X\nabla^W_Y Z - \nabla^W_Y\nabla^W_X Z - \nabla^W_{[X,Y]}Z \ ,$$

where $X,Y,Z \in \Gamma(TM^{\mathbb{C}})$ are complex vectors, in general. The operator R^{∇^W} is tensorial in the entries X,Y and Z. The corresponding curvature tensor is given by

$$R^W(X,Y,Z,V) = L_\theta(R^{\nabla^W}(X,Y)Z,\bar{V})$$

for $X,Y,Z,V \in TM^{\mathbb{C}}$. A straightforward calculation proves the following symmetry properties,

$$\begin{aligned} R^W(X,Y,Z,V) &= -R^W(Y,X,Z,V) = -R^W(X,Y,V,Z), \\ R^W(A,\overline{B},C,\overline{D}) &= R^W(C,\overline{B},A,\overline{D}) - L_\theta(Tor^W(\overline{B},Tor^W(C,A)),D), \\ R^{\nabla^W}(A,B)\overline{C} &= (\nabla_{\overline{C}}Tor^W)(A,B) \end{aligned}$$

for all vectors X,Y,Z,V in $TM^{\mathbb{C}}$ and A,B,C,D in T_{10}. If the Nijenhuis tensor $\mathcal{N}_J$ vanishes the two latter identities simplify to

$$R^W(A,\overline{B},C,\overline{D}) = R^W(C,\overline{B},A,\overline{D}) \qquad \text{and} \qquad R^{\nabla^W}(A,B)\overline{C} = 0 \ .$$

The Webster-Ricci and scalar curvatures are defined (as before in the integrable case; cf. Section 0.10 and e.g. [**103**]) by contractions of R^W through

$$\begin{aligned} Ric^W &:= \textstyle\sum_{\alpha=1}^m R^W(Z_\alpha,Z_{\bar{\alpha}},\cdot,\cdot) \ , \\ scal^W &:= \textstyle\sum_{\alpha=1}^m Ric^W(Z_\alpha,Z_{\bar{\alpha}}) \ . \end{aligned}$$

These definitions are independent of the choice of orthonormal basis. Note that the tensor $Ric^W(\cdot,J\cdot)$ is symmetric by definition. With respect to an adapted real basis $\{e_1,\ldots,e_{2m}\}$ we have

$$\begin{aligned} Ric^W(X,Y) &= i\cdot \textstyle\sum_{\alpha=1}^m R^W(e_{2\alpha-1},Je_{2\alpha-1},X,Y) \ , \\ scal^W &= i\cdot \textstyle\sum_{\alpha=1}^m Ric^W(e_{2\alpha-1},Je_{2\alpha-1}) \ . \end{aligned}$$

Obviously, the function $scal^W$ on (M,T_{10},θ) is real. We set

$$\omega_\alpha^\beta := L_\theta(\nabla^W Z_\alpha, Z_\beta) \ .$$

With the components ω_α^β of the Tanaka-Webster connection the Webster scalar curvature can be expressed by

$$\begin{aligned} scal^W &= \sum_{\alpha=1}^m Ric^W(Z_\alpha, Z_{\bar\alpha}) \\ &= \sum_{\alpha,\gamma=1}^m L_\theta([\nabla^W_{Z_\gamma}, \nabla^W_{Z_{\bar\gamma}}]Z_\alpha - \nabla^W_{[Z_\gamma, Z_{\bar\gamma}]}Z_\alpha, Z_\alpha) \\ &= \sum_{\gamma=1}^m \left(\sum_{\alpha=1}^m d\omega_\alpha^\alpha - \sum_{\alpha,\beta=1}^m \omega_\alpha^\beta \wedge \omega_\beta^\alpha \right)(Z_\gamma, Z_{\bar\gamma}) \\ &= \sum_{\alpha,\gamma=1}^m d\omega_\alpha^\alpha(Z_\gamma, Z_{\bar\gamma}) \ . \end{aligned}$$

2. Rescaling of a Pseudo-Hermitian Structure

We discuss here the transformation rules for the Tanaka-Webster connection and its scalar curvature under rescaling of a given pseudo-Hermitian structure. We will use these transformation rules later to prove the independence of the Fefferman construction from the choice of a pseudo-Hermitian form. We note that calculations and resulting expressions in this section do not differ formally from those in the classical integrable case, since the Nijenhuis torsion does not play a role in the transformation (cf. [**102**]).

Let (M, T_{10}, θ) be a partially integrable CR-space of hypersurface type with pseudo-Hermitian structure θ of positive definite signature and corresponding Tanaka-Webster connection ∇^W. As before, we use for our calculations local J-adapted frames $\{e_i : i = 1, \dots, 2m\}$. Then $Z_\alpha = \frac{1}{\sqrt{2}}(e_{2\alpha-1} - iJe_{2\alpha-1})$. Moreover, we set

$$\theta^\alpha = L_\theta(\cdot, Z_\alpha) \ , \qquad \theta^{\bar\alpha} = L_\theta(\cdot, Z_{\bar\alpha})$$

and

$$\delta_\alpha^\beta = L_\theta(Z_\alpha, Z_\beta) \ , \qquad \omega_\alpha^\beta := L_\theta(\nabla^W Z_\alpha, Z_\beta) \ .$$

We have $\nabla^W Z_\alpha = \sum_{\beta=1}^m \omega_\alpha^\beta \otimes Z_\beta$. For a real smooth function $f \in C^\infty(M)$ we define

$$f_\alpha := Z_\alpha(f) \ , \qquad f_{\bar\alpha} := Z_{\bar\alpha}(f) \ , \qquad f_o = T(f),$$
$$f_{\alpha\bar\beta} := (\nabla^W_{Z_{\bar\beta}} df)(Z_\alpha) \qquad \text{and} \qquad f_{\bar\alpha\beta} := (\nabla^W_{Z_\beta} df)(Z_{\bar\alpha}) \ .$$

We also set

$$\delta f := \sum_{\alpha=1}^m f_{\bar\alpha} Z_\alpha \qquad \text{and} \qquad \Delta_b f := -\sum_{\alpha=1}^m \left(f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha} \right) \ .$$

The latter definitions for the differential operators δ and Δ_b are independent of the choice of orthonormal frame. We have $\delta f(f) = \sum_{\alpha=1}^m f_\alpha \cdot f_{\bar\alpha}$ and Δ_b is called the *sub-Laplacian* (cf. e.g [**102**]).

Now let $f \in C^\infty(M)$ be an arbitrary function and let $\tilde\theta = e^{2f}\theta$. The Hermitian form $L_{\tilde\theta}$ is again positive definite and $\tilde\theta$ is another pseudo-Hermitian structure on (M, T_{10}).

We want to examine the transformation rules for the Tanaka-Webster connections and the corresponding scalar curvatures under such a rescaling. First of all, we notice that

$$\tilde{\theta}^\alpha = e^f(\theta^\alpha + 2if_{\bar{\alpha}}\theta) \qquad \text{for all } \ \alpha = 1, \dots, m,$$

where $\tilde{\theta}^\alpha$ is dual to $\tilde{Z}_\alpha = e^{-f} \cdot Z_\alpha$ with respect to $L_{\tilde{\theta}}$, and

$$\tilde{T} = e^{-2f} \cdot (T - 2i\delta f + 2i\overline{\delta f}) \ .$$

Lemma 18. *(cf. [**102**]) Let $\tilde{\theta} = e^{2f}\theta$ be a rescaled pseudo-Hermitian structure on a partially integrable CR-space (M, T_{10}). Then the relation*

(1)
$$\begin{aligned} \tilde{\nabla}^W_X U \quad = \quad & \nabla^W_X U + 2df(U) \cdot pr_{10}X + 2df(pr_{10}X) \cdot U - 2L_\theta(U, \bar{X}) \cdot \delta f \\ + \quad & i\theta(X) \cdot \big(4df(U) \cdot \delta f + 4U \cdot \delta f(f) + 2 \cdot \nabla^W_U \delta f\big) \end{aligned}$$

holds for any $U \in \Gamma(T_{10})$ and $X \in TM^{\mathbb{C}}$.

(2) *For the connection components we have the transformation rule*

$$\begin{aligned} \tilde{\omega}^\beta_\alpha \ = \ \omega^\beta_\alpha \ + \ 2(f_\alpha\theta^\beta - f_{\bar{\beta}}\theta^{\bar{\alpha}}) + \delta^\beta_\alpha \cdot \left(\sum_{\gamma=1}^m f_\gamma\theta^\gamma - f_{\bar{\gamma}}\theta^{\bar{\gamma}}\right) \\ + \ i \ \theta(\cdot) \left(f_{\bar{\beta}\alpha} + f_{\alpha\bar{\beta}} + 4f_\alpha f_{\bar{\beta}} + 4\delta^\beta_\alpha \cdot \sum_{\gamma=1}^m f_\gamma f_{\bar{\gamma}}\right) \ , \end{aligned}$$

where $\alpha, \beta = 1, \dots, m$.

(3) *The Webster scalar curvature rescales by*

$$\widetilde{scal}^W = e^{-2f} \cdot \big(\ scal^W \ + \ 2(m+1)\Delta_b f \ - \ 4m(m+1)\delta f(f) \ \big) \quad .$$

Proof. (1) We take the expression for $\tilde{\nabla}^W$ in Lemma 18 as definition for a connection and verify that it satisfies the determining properties for the Tanaka-Webster connection of Lemma 15. First, we have

$$\tilde{\nabla}^W_{\bar{V}} U = \nabla^W_{\bar{V}} U - 2L_\theta(U, V)\delta f \ ,$$

and on the other hand,

$$\tilde{pr}_{10}[\bar{V}, U] = pr_{10}[\bar{V}, U] - 2L_\theta(U, V)\delta f$$

for all $U, V \in \Gamma(T_{10})$. This shows that $\tilde{\nabla}^W_{\bar{V}} U = \tilde{pr}_{10}[\bar{V}, U]$. Next we see that

$$\begin{aligned} & L_{\tilde{\theta}}(\tilde{\nabla}^W_X U, V) + L_{\tilde{\theta}}(U, \tilde{\nabla}^W_{\bar{X}} V) \\ = \quad & e^{2f} \cdot L_\theta(\nabla^W_X U + 2df(U)pr_{10}X + 2df(pr_{10}X)U - 2L_\theta(U, \bar{X})\delta f, V) \\ & + e^{2f} \cdot L_\theta(U, \nabla^W_{\bar{X}} V + 2df(V)pr_{10}\bar{X} + 2df(pr_{10}\bar{X})V - 2L_\theta(V, X)\delta f) \\ = \quad & e^{2f} \cdot X(L_\theta(U, V)) + 2e^{2f}df(X) \cdot L_\theta(U, V) \\ & + e^{2f} \cdot \big(2df(U) \cdot L_\theta(X, V) - 2df(\bar{V}) \cdot L_\theta(U, \bar{X})\big) \\ & + e^{2f} \cdot \Big(2df(\bar{V}) \cdot L_\theta(U, \bar{X}) - 2df(U) \cdot \overline{L_\theta(V, X)}\Big) \\ = \quad & X(L_{\tilde{\theta}}(U, V)) \end{aligned}$$

for all $U, V \in \Gamma(T_{10})$. It remains to show the relation $\tilde{\nabla}^W_{\tilde{T}} U = \tilde{pr}_{10}[\tilde{T}, U]$. For this purpose we calculate

$$\begin{aligned} e^{2f} \cdot \tilde{pr}_{10}[\tilde{T}, U] &= pr_{10}[T, U] + 2ie^{2f} pr_{10}[e^{-2f}(-\delta f + \overline{\delta f}), U] \\ &= pr_{10}[T, U] - 4i df(U)\delta f - 2i \cdot \nabla^W_{\delta f} U + 2i \cdot \nabla^W_{\overline{\delta f}} U \\ &\quad + 2i \cdot \sum_{\alpha=1}^m U(f_{\bar{\alpha}}) \cdot Z_\alpha + 2i \cdot \sum_{\alpha=1}^m f_{\bar{\alpha}} \cdot \nabla^W_U Z_\alpha \ , \end{aligned}$$

and on the other hand,

$$\begin{aligned} e^{2f} \cdot \tilde{\nabla}^W_{\tilde{T}} U &= \nabla^W_T U - 2i \nabla^W_{\delta f} U + 2i \nabla^W_{\overline{\delta f}} U - 8i \cdot df(U)\delta f - 4i \sum_{\alpha=1}^m f_\alpha f_{\bar{\alpha}} \cdot U \\ &\quad + i(4df(U)\delta f + 4\delta f(f) \cdot U + 2 \cdot \nabla^W_U \delta f) \ , \end{aligned}$$

which obviously equals the previous expression.

(2) To calculate the connection components $\tilde{\omega}^\beta_\alpha$, we use the following identity

$$\begin{aligned} & i f_o \delta^\beta_\alpha + 2 \cdot L_\theta(\nabla^W_{Z_\alpha} \delta f, Z_\beta) \\ = \ & d\theta(Z_\alpha, Z_{\bar{\beta}}) T(f) + 2 \cdot Z_\alpha(f_{\bar{\beta}}) - 2 \cdot \nabla^W_{Z_\alpha} Z_{\bar{\beta}}(f) \\ = \ & 2 \cdot Z_\alpha(f_{\bar{\beta}}) + [Z_{\bar{\beta}}, Z_\alpha](f) - pr_{10}[Z_{\bar{\beta}}, Z_\alpha](f) - pr_{01}[Z_\alpha, Z_{\bar{\beta}}](f) \\ = \ & Z_\alpha(f_{\bar{\beta}}) + Z_{\bar{\beta}}(f_\alpha) - \nabla^W_{Z_\alpha} Z_{\bar{\beta}}(f) - \nabla^W_{Z_{\bar{\beta}}} Z_\alpha(f) \\ = \ & f_{\bar{\beta}\alpha} + f_{\alpha\bar{\beta}} \ . \end{aligned}$$

We have

$$\begin{aligned} \tilde{\omega}^\beta_\alpha &= L_{\tilde{\theta}}(\tilde{\nabla}^W \tilde{Z}_\alpha, \tilde{Z}_\beta) = L_\theta(\tilde{\nabla}^W Z_\alpha, Z_\beta) - \delta^\beta_\alpha \cdot df \\ &= \omega^\beta_\alpha - \delta^\beta_\alpha \cdot df + 2 \cdot f_\alpha \theta^\beta - 2 f_{\bar{\beta}} \theta^{\bar{\alpha}} + 2\delta^\beta_\alpha \cdot \sum_{\gamma=1}^m f_\gamma \theta^\gamma \\ &\quad + i \cdot \left(4 f_\alpha f_{\bar{\beta}} + 2 \cdot L_\theta(\nabla^W_{Z_\alpha} \delta f, Z_\beta) + 4\delta^\beta_\alpha \cdot \sum_{\gamma=1}^m f_\gamma f_{\bar{\gamma}} \right) \theta \\ &= \omega^\beta_\alpha + 2(f_\alpha \theta^\beta - f_{\bar{\beta}} \theta^{\bar{\alpha}}) + \delta^\beta_\alpha \cdot \sum_{\gamma=1}^m (f_\gamma \theta^\gamma - f_{\bar{\gamma}} \theta^{\bar{\gamma}}) \\ &\quad + i \cdot \left(4 f_\alpha f_{\bar{\beta}} + i f_o \delta^\beta_\alpha + 2 \cdot L_\theta(\nabla^W_{Z_\alpha} \delta f, Z_\beta) + 4\delta^\beta_\alpha \cdot \sum_{\gamma=1}^m f_\gamma f_{\bar{\gamma}} \right) \theta \\ &= \omega^\beta_\alpha + 2(f_\alpha \theta^\beta - f_{\bar{\beta}} \theta^{\bar{\alpha}}) + \delta^\beta_\alpha \cdot \sum_{\gamma=1}^m (f_\gamma \theta^\gamma - f_{\bar{\gamma}} \theta^{\bar{\gamma}}) \\ &\quad + i \cdot \left(4 f_\alpha f_{\bar{\beta}} + f_{\bar{\beta}\alpha} + f_{\alpha\bar{\beta}} + 4\delta^\beta_\alpha \cdot \sum_{\gamma=1}^m f_\gamma f_{\bar{\gamma}} \right) \theta \quad . \end{aligned}$$

(3) We use the latter formula for the connection components to calculate the Webster scalar curvature. We have

$$\begin{aligned}\sum_{\alpha=1}^m \tilde{\omega}_\alpha^\alpha \quad = \quad & \sum_{\alpha=1}^m \omega_\alpha^\alpha \quad + \quad (m+2)\cdot\sum_{\alpha=1}^m (f_\alpha\theta^\alpha - f_{\bar\alpha}\theta^{\bar\alpha}) \\ & + \quad i\cdot\sum_{\alpha=1}^m \big(\ f_{\bar\alpha\alpha} + f_{\alpha\bar\alpha} \ + \ 4(m+1)f_\alpha f_{\bar\alpha}\big)\,\theta \quad .\end{aligned}$$

The trace of the exterior differential of this expression is equal to the Webster scalar curvature. With

$$\begin{aligned} d(\sum_{\alpha=1}^m f_\alpha\theta^\alpha)(Z_\gamma, Z_{\bar\beta}) \ &= \ \sum_{\alpha=1}^m (df_\alpha\wedge\theta^\alpha + f_\alpha d\theta^\alpha)(Z_\gamma, Z_{\bar\beta}) \\ &= \ -Z_{\bar\beta}(f_\gamma) + \sum_{\alpha=1}^m f_\alpha\theta^\alpha([Z_{\bar\beta}, Z_\gamma]) \\ &= \ -f_{\gamma\bar\beta}\end{aligned}$$

we obtain

$$\begin{aligned}\sum_{\alpha,\beta=1}^m d\tilde{\omega}_\alpha^\alpha(Z_\beta, Z_{\bar\beta}) \quad = \quad & \sum_{\alpha,\beta=1}^m d\omega_\alpha^\alpha(Z_\beta, Z_{\bar\beta}) \quad - \quad (m+2)\cdot\sum_{\beta=1}^m (f_{\beta\bar\beta} + f_{\bar\beta\beta}) \\ & + \quad i\sum_{\alpha,\beta=1}^m \big(\ f_{\bar\alpha\alpha} + f_{\alpha\bar\alpha} \ + \ 4(m+1)\cdot f_\alpha f_{\bar\alpha}\ \big)\cdot d\theta(Z_\beta, Z_{\bar\beta}) \quad .\end{aligned}$$

As result we find

$$\widetilde{scal}^W = e^{-2f}\cdot\big(\ scal^W \ + \ 2(m+1)\Delta_b f \ - \ 4m(m+1)\delta f(f)\ \big) \quad .$$

□

3. The Fefferman Metric

We construct the Fefferman metric to a pseudo-Hermitian structure θ on the total space of the canonical S^1-principal bundle over a partially integrable, strictly pseudoconvex CR-manifold (M, T_{10}) and show that the Fefferman conformal class is independent of the choice of pseudo-Hermitian form. This makes the Fefferman construction a natural and important device for the study of partially integrable CR-geometry. The construction that we describe coincides with the classical Fefferman construction in the case of integrable CR-geometry (cf. Section 0.12). In fact, the Fefferman metric and calculations for the independence of the the conformal class look formally the same as in the integrable case (cf. [**102, 17**]). However, at the end of the section we aim to introduce a slightly more generalised class of metrics on the canonical S^1-principal bundle, which we call the *gauged* Fefferman metrics. A gauged Fefferman metric depends on a gauge 1-form ℓ, but again it does not depend on the choice of pseudo-Hermitian structure. The reason for the invention of this extended construction will become clear in Chapter 6 when we investigate complex structures in conformal tractor calculus.

So let (M^n, T_{10}) be a partially integrable, strictly pseudoconvex CR-manifold of dimension $n = 2m+1$ and let θ be a pseudo-Hermitian structure on this CR-space.

We denote by

$$\Lambda^{m+1,0}M \;:=\; \{\; \rho \in \Lambda^{m+1}M \otimes \mathbb{C} : \;\iota_X \rho = 0 \;\text{ for }\; \text{all } X \in T_{01} = \overline{T_{10}} \;\}$$

the complex line bundle over M^n, which consists of all those complex $(m+1)$-forms that vanish by insertion of any element of T_{01}. The bundle $\Lambda^{m+1,0}M$ is called the canonical line bundle of the CR-space (M, T_{10}). The positive real numbers $\mathbb{R}_+$ act by multiplication on $K^* := \Lambda^{m+1,0}M \setminus \{0\}$, which denotes the canonical line bundle without zero section. We set $F_c := K^*/_{\mathbb{R}^+}$ and the triple

$$(\; F_c \;,\; \pi \;,\; M \;)$$

denotes the canonical S^1-principal bundle of (M, T_{10}) whose fibre action is induced by complex multiplication with the elements of the unit circle S^1 in $\mathbb{C}$.

Let $\{Z_\alpha : \; \alpha = 1, \ldots, m\}$ be some local orthonormal frame of (T_{10}, L_θ) and let θ^α, $\alpha = 1, \ldots, m$, denote the corresponding dual 1-forms. The $(m+1,0)$-form

$$\tau := \theta \wedge \theta^1 \wedge \ldots \wedge \theta^m$$

is a local section of $\Lambda^{m+1,0}M$. We denote by $[\tau]$ the corresponding local section in $F_c = K^*/_{\mathbb{R}^+}$. With the help of the projection π every 1-form ρ on M can be lifted to F_c. The result is a 1-form $\pi^*\rho$ on F_c. For convenience, we shall usually denote the lifted 1-form simply by ρ again. On the other hand, by use of the section $[\tau]$ we are also able to pull back 1-forms σ on F_c to the underlying space M. This pullback is usually denoted by $[\tau]^*\sigma$ or just σ again.

The Tanaka-Webster connection ∇^W naturally extends to a covariant derivative acting on sections of the complex line bundle $\Lambda^{m+1,0}M$. In fact, the covariant derivative ∇^W on $\Lambda^{m+1,0}M$ is induced by a uniquely determined connection 1-form on the S^1-principal fibre bundle F_c, which we denote by

$$A^W : TF_c \;\longrightarrow\; i\mathbb{R} \;\; .$$

We have $[\tau]^*A^W = -\sum_{\alpha=1}^m \omega_\alpha^\alpha$ with respect to local frame forms $\theta^\alpha, \alpha = 1, \ldots, m$. Further, we set

$$A_\theta := A^W - \frac{i}{2(m+1)} scal^W \theta \;.$$

This expression is a connection 1-form on F_c as well. In fact, this is the induced Weyl connection on the canonical S^1-bundle of (M, T_{10}), which belongs to the given pseudo-Hermitian structure θ (cf. Section 0.5). The curvature of A^W is the 2-form $\Omega^W = dA^W$. We have

$$\Omega^W = -\sum_{\alpha=1}^m d\omega_\alpha^\alpha \;.$$

Moreover, we denote the curvature of A_θ by $\Omega_\theta = dA_\theta$. Then

$$\Omega_\theta = \Omega^W - \frac{i}{2(m+1)} scal^W d\theta - \frac{i}{2(m+1)} d(scal^W) \cdot \theta \;.$$

We define now the Fefferman metric to θ on F_c by

$$f_\theta \;:=\; \pi^* L_\theta - i\frac{4}{m+2}\pi^*\theta \circ A_\theta \;.$$

This is, in fact, a symmetric 2-tensor on the real tangent bundle of F_c. To shorten the notation, we simply use the expression $f_\theta = L_\theta - i\frac{4}{m+2}\theta \circ A_\theta$. Since we assume

L_θ to be positive definite, the signature of f_θ is Lorentzian (i.e. $sig = (1, 2m+1)$). The Fefferman conformal class $[f_\theta]$ consists of all smooth metrics $\tilde{f}_\theta$ on F_c which arise by conformal rescaling of f_θ, i.e., $\tilde{f}_\theta = e^{2l} f_\theta$ for some smooth function l on F_c. We want to prove now the independence of the conformal class $[f_\theta]$ from the choice of pseudo-Hermitian form. For this purpose we need to find the transformation rule for the Tanaka-Webster connection form A^W, resp., for the Weyl connection A_θ on F_c under rescaling of θ. Thereby, we use the results from the previous section and we will see that the transformation rule for the Weyl connection A_θ is particular easy.

To start with, let $\tilde{\theta} = e^{2\phi} \cdot \theta$ be a rescaled pseudo-Hermitian form on (M, T_{10}), and accordingly, let $\tilde{Z}_\alpha = e^{-\phi} \cdot Z_\alpha$, $\alpha = 1, \ldots, m$, be rescaled basis vectors for T_{10}. We have

$$\tilde{\theta}^\alpha = e^\phi(\theta^\alpha + 2i\phi_{\bar{\alpha}}\theta) \qquad \text{and} \qquad \tilde{\tau} = e^{(m+2)\phi}\tau \ ,$$

i.e., $[\tilde{\tau}]$ and $[\tau]$ are identical as local sections of F_c. Moreover, we have

$$[\tau]^* A^W = -\textstyle\sum_{\alpha=1}^m \omega_\alpha^\alpha \quad \text{and}$$

$$[\tau]^* \tilde{A}^W = -\textstyle\sum_{\alpha=1}^m \tilde{\omega}_\alpha^\alpha \ .$$

By using Lemma 18, we obtain

$$\begin{aligned} [\tau]^*(\tilde{A}^W - A^W) \ &= \ -\,(m+2) \cdot \sum_{\alpha=1}^m (\phi_\alpha \theta^\alpha - \phi_{\bar{\alpha}} \theta^{\bar{\alpha}}) \\ &\qquad + i(\ \Delta_b \phi \ - \ 4(m+1) \cdot \sum_{\alpha=1}^m \phi_\alpha \phi_{\bar{\alpha}}\ ) \cdot \theta, \end{aligned}$$

and further,

$$\begin{aligned} [\tau]^*(A_{\tilde{\theta}} - A_\theta) \ &= \ -\,(m+2) \cdot \sum_{\alpha=1}^m (\ \phi_\alpha \theta^\alpha - \phi_{\bar{\alpha}} \theta^{\bar{\alpha}}\ ) \\ &\quad - i(\ -\Delta_b \phi \ + \ 4m(m+1) \cdot \sum_{\alpha=1}^m \phi_\alpha \phi_{\bar{\alpha}}\ ) \cdot \theta \\ &\quad - \frac{i}{2(m+1)} \left(scal^W + 2(m+1)\Delta_b \phi - 4(m+1) \cdot \sum_{\alpha=1}^m \phi_\alpha \phi_{\bar{\alpha}} \right) \cdot \theta \\ &\quad + \frac{i}{2(m+1)} \cdot scal^W \theta \\ &= \ -\,(m+2) \cdot \sum_{\alpha=1}^m (\ \phi_\alpha \theta^\alpha - \phi_{\bar{\alpha}} \theta^{\bar{\alpha}}\ ) \\ &\quad - \left(2i(m+2) \cdot \sum_{\alpha=1}^m \phi_\alpha \phi_{\bar{\alpha}} \right) \cdot \theta \qquad . \end{aligned}$$

We conclude that

$$A_{\tilde{\theta}} = A_\theta \ - \ (m+2) \sum_{\alpha=1}^m (\ \phi_\alpha \theta^\alpha - \phi_{\bar{\alpha}} \theta^{\bar{\alpha}}\ ) \ - \ 2i(m+2) \sum_{\alpha=1}^m \phi_\alpha \phi_{\bar{\alpha}} \theta \ .$$

Now we can consider the transformation rule for the Fefferman metric f_θ under rescaling of θ. We have

$$\begin{aligned} f_{\tilde\theta} &= 2\cdot\sum_{\alpha=1}^m \tilde\theta^\alpha\circ\tilde\theta^{\bar\alpha} - i\frac{4}{m+2}\tilde\theta\circ A_{\tilde\theta} \\ &= e^{2\phi}\cdot\Big(\sum_{\alpha=1}^m 2\cdot\big(\theta^\alpha\circ\theta^{\bar\alpha}+2i\phi_{\bar\alpha}\theta\circ\theta^{\bar\alpha}-2i\phi_\alpha\theta^\alpha\circ\theta+4\phi_{\bar\alpha}\phi_\alpha\cdot\theta\circ\theta\big) \\ &\quad - i\frac{4}{m+2}\theta\circ A_\theta+\sum_{\alpha=1}^m\big(4i\phi_\alpha\theta\circ\theta^\alpha-4i\phi_{\bar\alpha}\theta\circ\theta^{\bar\alpha}-8\phi_\alpha\phi_{\bar\alpha}\cdot\theta\circ\theta\big)\Big) \\ &= e^{2\phi}\cdot f_\theta\ . \end{aligned}$$

Theorem 21. *(cf. [**102**]) Let (M^n,T_{10}) be a partially integrable CR-space with a pseudo-Hermitian structure θ and Fefferman metric f_θ on F_c. Let $\tilde\theta=e^{2\phi}\theta$ be a rescaled pseudo-Hermitian structure. Then the corresponding Fefferman metric rescales by $f_{\tilde\theta}=e^{2\phi}\cdot f_\theta$.*

The independence of the conformal class $[f_\theta]$ of the Fefferman metric from the choice of the pseudo-Hermitian structure θ relies basically on the use of the Weyl connection A_θ for the construction. The Weyl connection has simply the correct transformation law, which makes the construction CR-invariant. However, we can easily see now that on this basis we can add an arbitrary fixed 1-form to the connection A_θ (for any θ) and define a more general family of metrics on F_c, whose conformal classes are still independent from the pseudo-Hermitian structure, i.e., the new metrics are invariants of the underlying CR-structure and the additional datum of a gauge. To be concrete, let $\ell\in\Omega^1(M;i\mathbb{R})$ be a 1-form on the base space M. We set $A_{\theta,\ell}:=A_\theta+\ell$. Actually, this is the general form of an arbitrary connection on F_c.

Definition 4. *Let (M^n,T_{10}) be a partially integrable CR-space, θ a pseudo-Hermitian structure and $\ell\in\Omega^1(M;i\mathbb{R})$ an arbitrary 1-form on M.*

(1) *We call the metric*
$$f_{\theta,\ell}=L_\theta-i\frac{4}{m+2}\theta\circ A_{\theta,\ell}$$
the ℓ-gauged Fefferman metric with respect to θ on (M^n,T_{10}).

(2) *Let $\tilde\theta=e^{2\phi}\theta$ be a rescaled pseudo-Hermitian structure. Then $f_{\tilde\theta,\ell}=e^{2\phi}\cdot f_{\theta,\ell}$, which shows that the conformal class $[f_{\theta,\ell}]$ is independent of the choice of pseudo-Hermitian form θ. We denote $c_\ell:=[f_{\theta,\ell}]$ and call it the conformal ℓ-gauged Fefferman class of the CR-space (M,T_{10}).*

Obviously, for $\ell=0$ the metric $f_{\theta,\ell}=f_\theta$ is the usual Fefferman metric to θ. Actually, if ℓ is a closed form on M then the metrics f_θ and $f_{\theta,\ell}$ are locally isometric. The local isometry is given by a gauge transformation on F_c, which transforms the connection form $A_{\theta,\ell}$ into A_θ. The fibres are preserved under this (local) isometry. On the other hand, a local isometry between f_θ and $f_{\theta,\ell}$, which preserves the fibres, can only exist if there is a gauge transformation, i.e., the difference $\ell=A_{\theta,\ell}-A_\theta$ has to be closed. In any case, the metrics $f_{\theta,\ell}$ and $f_{\tilde\theta,\tilde\ell}$ are locally isometric only if the rescaling function ϕ on M is constant zero, i.e., when the underlying pseudo-Hermitian geometries θ and $\tilde\theta$

are identical. Altogether, we conclude that generalised Fefferman metrics $f_{\theta,\ell}$ and $f_{\tilde{\theta},\tilde{\ell}}$ are locally isometric with preserved fibre if and only if $\tilde{\theta} = \theta$ and $\tilde{\ell} - \ell$ is a closed form on M.

In particular, the ℓ-gauged Fefferman class $c_\ell = [f_{\theta,\ell}]$ is locally conformally equivalent to any $c_{\tilde{\ell}} = [f_{\theta,\tilde{\ell}}]$ with $\tilde{\ell} \in [\ell]$, where $[\ell]$ denotes the class of 1-forms on M with purely imaginary values, which differ from ℓ only by a closed form (i.e. by a local gauge transformation). We will sometimes use the notation $c_{[\ell]}$ for this conformal class, which is uniquely given by a CR-structure T_{10} and a local gauge class $[\ell]$.

4. The Torsion Tensor of Partial Integrability

We examine here properties of the torsion tensor Tor^W with respect to the Tanaka-Webster connection on pseudo-Hermitian spaces. The torsion consists essentially of the Nijenhuis tensor $\mathcal{N}_J$ and $Tor^W(T,\cdot)$, where the latter part is the deviation of the Reeb vector T from being a transverse symmetry (cf. Lemma 17). The Nijenhuis tensor is a CR-invariant.

Let (M^n, H, J) be a strictly pseudoconvex, partially integrable CR-space with dimension $n = 2m+1$ and let θ denote a pseudo-Hermitian structure on M. As usual let $\{e_i : \ i = 1, \dots, 2m\}$ be a local J-adapted orthonormal frame of L_θ on H. We use the following conventions. If $A \in T^{(r,2)}M^{\mathbb{C}}$ is a $(r,2)$-tensor, then we denote its trace (or contraction) with respect to θ by

$$tr^\theta A := \sum_{i=1}^{2m} A(e_i, e_i) \ .$$

More generally, we use the notation $tr^\theta_{k,l} A$ for the trace of a (r,s)-tensor A, where the contraction takes place in the kth and lth entry of A. If A is a skew-symmetric 2-tensor on M then we set

$$L_A(\cdot,\cdot) := iA(\cdot, J\cdot)$$

and we have

$$tr^\theta L_A = i \cdot tr^\theta A(\cdot, J\cdot) = 2 \cdot \sum_{\alpha=1}^{m} A(Z_\alpha, Z_{\bar{\alpha}}) \ .$$

If A is a symmetric $(0,2)$-tensor then

$$tr^\theta A = 2 \cdot \sum_{\alpha=1}^{m} A(Z_\alpha, Z_{\bar{\alpha}}) \ .$$

The Nijenhuis tensor $\mathcal{N}$ on (M, H, J) is defined by

$$\mathcal{N}(X,Y) := [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY] \ ,$$

where $X, Y \in \Gamma(H)$. Since $Tor^W(X,Y) = L_\theta(JX,Y)\cdot T - \frac{1}{4}\mathcal{N}(X,Y)$, we can consider the Nijenhuis tensor as the essential part of the torsion restricted to the contact distribution H. We have

$$\begin{aligned} J\mathcal{N}(X,Y) &= -\mathcal{N}(JX,Y) = -\mathcal{N}(X,JY) \qquad \text{and} \\ tr^\theta L_\theta(\mathcal{N}(X,\cdot),\cdot) &= 0 \end{aligned}$$

for all $X, Y, Z \in H$. We form from $\mathcal{N}$ the $\mathcal{B}_\theta$-tensor by

$$\mathcal{B}_\theta(X,Y,Z) := \frac{1}{8}\Big(\ L_\theta(\mathcal{N}(X,Y),Z) + L_\theta(\mathcal{N}(Z,Y),X) + L_\theta(\mathcal{N}(Z,X),Y)\ \Big)\ .$$

Moreover, let

$$\mathcal{B}(X,Y) := \sum_{i=1}^{2m} \mathcal{B}_\theta(X,Y,e_i)e_i$$

denote the corresponding $(1,2)$-tensor. The definition of the tensor $\mathcal{B}$ does not depend on the chosen θ and the orthonormal frame $\{e_i\}$. We have

$$\mathcal{B}(X,Y) - \mathcal{B}(Y,X) = \frac{1}{4}\mathcal{N}(X,Y)$$

and $\mathcal{B}$ vanishes identically if and only if $\mathcal{N}$ vanishes identically. In other words, the tensor $\mathcal{B}$ contains the same information as $\mathcal{N}$. Moreover, the relations

$$\begin{aligned}
\mathcal{B}_\theta(X,Y,Z) &= -\mathcal{B}_\theta(X,Z,Y)\ ,\\
\mathcal{B}_\theta(X,Y,Z) &= -\mathcal{B}_\theta(JX,JY,Z) = -\mathcal{B}_\theta(JX,Y,JZ) = -\mathcal{B}_\theta(X,JY,JZ)\ ,\\
J\mathcal{B}(X,Y) &= -\mathcal{B}(JX,Y) = -\mathcal{B}(X,JY)\ ,\\
tr^\theta\mathcal{B} &= \textstyle\sum_{i=1}^{2m}\mathcal{B}(e_i,e_i) = 0\ ,\\
tr^\theta_{1,3}B_\theta(X) &= \textstyle\sum_{i=1}^{2m}\mathcal{B}_\theta(e_i,X,e_i) = 0\ ,\\
tr^\theta_{2,3}B_\theta(X) &= \textstyle\sum_{i=1}^{2m}\mathcal{B}_\theta(X,e_i,e_i) = 0
\end{aligned}$$

hold. Straightforward calculations using essentially the condition of partial integrability also show the following identities:

$$\begin{aligned}
tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),Y),\cdot) &= tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),\cdot),Y) - tr^\theta L_\theta(\mathcal{N}(X,\cdot),\mathcal{N}(Y,\cdot))\\
tr^\theta\mathcal{B}_\theta(\mathcal{N}(X,\cdot),\cdot,Y) &= \frac{1}{4}tr^\theta L_\theta(\mathcal{N}(X,\cdot),\mathcal{N}(Y,\cdot))\\
tr^\theta L_\theta(\mathcal{B}(X,\cdot),\mathcal{B}(Y,\cdot)) &= \frac{1}{8}tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),\cdot),Y)\\
tr^\theta L_\theta(\mathcal{B}(X,\cdot),\mathcal{B}(\cdot,Y)) &= \frac{1}{16}tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),\cdot),Y)\\
\sum_{i,j=1}^{2m} L_\theta(\mathcal{N}(\mathcal{N}(e_i,e_j),e_j),e_i) &= \frac{1}{2}\cdot\sum_{i,j=1}^{2m} L_\theta(\mathcal{N}(e_i,e_j),\mathcal{N}(e_i,e_j))\ .
\end{aligned}$$

The third identity above shows that the tensor $tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),\cdot),Y)$ is symmetric in X and Y.

The other part of the torsion is

$$Tor^W(T,X) = -\frac{1}{2}(\ [T,X] + J[T,JX]\ )\ ,$$

where $X \in H$. We define the tensor

$$\mathcal{R}_\theta(X,Y) := -2\cdot L_\theta(Tor^W(T,X),Y) \qquad \text{for}\ \ X,Y \in H\ .$$

By Lemma 14, it is clear that $\mathcal{R}_\theta$ is a symmetric tensor. Moreover, we have

$$\begin{aligned} Tor^W(T,JX) &= -J(Tor^W(T,X)), \\ \mathcal{R}_\theta(X,JY) &= \mathcal{R}_\theta(JX,Y) \qquad\qquad \text{and} \\ tr^\theta\mathcal{R}_\theta &= tr^\theta\mathcal{R}_\theta(\cdot,J\cdot) \;=\; 0 \;. \end{aligned}$$

If X, Y are (local) sections of H such that $L_\theta(X,Y)$ and $L_\theta(X,JY)$ are constant then the equation

$$\mathcal{R}_\theta(X,Y) = \mathcal{R}_\theta(Y,X) = L_\theta([T,X],Y) + L_\theta([T,Y],X)$$

holds.

5. The Scalar Curvature of Gauged Fefferman Metrics

We compute in this section certain parts of the Ricci-curvature tensor and the scalar curvature for ℓ-gauged Fefferman metrics. The formulae that we obtain generalise the results of [**102**]. Due to partial integrability the Nijenhuis tensor $\mathcal{N}$ and also the gauge ℓ will enter the curvature expressions. This makes the calculations more laborious.

Let (M^n, H, J) be a strictly pseudoconvex, partially integrable CR-space with dimension $n = 2m+1$, let θ denote a pseudo-Hermitian structure on M and let

$$f_{\theta,\ell} = L_\theta - i\frac{4}{m+2}\theta \circ A_{\theta,\ell} \;,$$

be the gauged Fefferman metric on the canonical S^1-bundle F_c to θ on M and some gauge ℓ. We will sometimes denote the ℓ-gauged Fefferman metric simply by f.

The S^1-action on the fibres of F_c induces a (vertical) fundamental vector field for each element in the Lie algebra $i\mathbb{R}$ of S^1. We denote by χ_K the fundamental field which is determined by

$$A_{\theta,\ell}(\chi_K) = i\frac{m+2}{2} \quad .$$

The field χ_K is lightlike on $(F_c, f_{\theta,\ell})$. Moreover, let T be the Reeb vector field to θ on M and let X^* denote the horizontal lift to F_c of any vector X on M with respect to the connection $A_{\theta,\ell}$. We have $f_{\theta,\ell}(\chi_K, T^*) = 1$ and

$$\{e_1^*, \ldots, e_n^*, T^*, \chi_K\}$$

is a local frame on $(F_c, f_{\theta,\ell})$. Throughout this section we use (local) vector fields X, Y, Z and V on M which have constant coefficients with respect to the chosen J-adapted frame $\{e_i : \; i = 1, \ldots, 2m\}$. This implies that scalar products of such vector fields (and their images by J) are constant with respect to $f_{\theta,\ell}$. To start with our calculations, we note that

$$\begin{aligned} [X^*, \chi_K] &= 0 \\ [X^*, Y^*]_{Vert} &= i\tfrac{2}{m+2}\Omega_{\theta,\ell}(X,Y)\cdot\chi_K \\ [X^*, Y^*]_{Horiz} &= [X,Y]^* \\ [T^*, X^*] &= [T,X]^* + i\tfrac{2}{m+2}\Omega_{\theta,\ell}(T,X)\cdot\chi_K \\ [X^*, Y^*] &= pr_H[X,Y]^* - d\theta(X,Y)\cdot T^* + i\tfrac{2}{m+2}\Omega_{\theta,\ell}(X,Y)\cdot\chi_K \;, \end{aligned}$$

where $\Omega_{\theta,\ell} = dA_{\theta,\ell}$ is the curvature of the connection form $A_{\theta,\ell}$ on F_c.

Lemma 19. *(cf. [**17**]) For the ℓ-gauged Fefferman metric f on F_c the Levi-Civita connection ∇^f satisfies*

$$
\begin{aligned}
f(\nabla^f_{X^*}Y^*, Z^*) &= L_\theta(\nabla^W_X Y, Z) + \mathcal{B}_\theta(X,Y,Z) \\
f(\nabla^f_{\chi_K}Y^*, Z^*) &= \tfrac{1}{2}L_\theta(JY, Z) \\
f(\nabla^f_{X^*}Y^*, \chi_K) &= -\tfrac{1}{2}L_\theta(JX, Y) \\
f(\nabla^f_{T^*}Y^*, Z^*) &= \tfrac{1}{2}\big(\, L_\theta([T,Y],Z) - L_\theta([T,Z],Y) - i\tfrac{2}{m+2}\Omega_{\theta,\ell}(Y,Z)\,\big) \\
f(\nabla^f_{X^*}Y^*, T^*) &= \tfrac{1}{2}\big(\, L_\theta([T,X],Y) + L_\theta([T,Y],X) + i\tfrac{2}{m+2}\Omega_{\theta,\ell}(X,Y)\,\big) \\
f(\nabla^f_{T^*}T^*, Z^*) &= -\, i\tfrac{2}{m+2}\Omega_{\theta,\ell}(T,Z) \\
f(\nabla^f\chi_K, \chi_K) &= f(\nabla^f\chi_K, T^*) = f(\nabla^f T^*, T^*) = 0 \\
f(\nabla^f_{\chi_K}\chi_K, Z^*) &= f(\nabla^f_{\chi_K}T^*, Z^*) = f(\nabla^f_{T^*}\chi_K, Z^*) = 0
\end{aligned}
$$

for all $X,Y,Z \in \Gamma(H)$ (which have pairwise constant scalar products with respect to L_θ).

Proof. We apply the Koszul formula for the Levi-Civita connection ∇^f, namely

$$f(\nabla^f_D B, C) = \frac{1}{2}\big(\, f([D,B],C) + f([C,B],D) + f([C,D],B)\,\big)$$

for all vector fields B, C, D on F_c, which have constant length and pairwise constant scalar products. In fact, the formulae of Lemma 19 result immediately from the Koszul formula and above expressions for commutators of vector fields on F_c after projection to M and replacing the scalar products with respect to f by those with respect to L_θ. For example, for the first formula in Lemma 19 we calculate

$$
\begin{aligned}
2f(\nabla^f_{X^*}Y^*, Z^*) &= L_\theta([X,Y],Z) + L_\theta([Z,Y],X) + L_\theta([Z,X],Y) \\
&= 2L_\theta(\nabla^W_X Y, Z) + \frac{1}{4}L_\theta(\mathcal{N}_J(X,Y),Z) + \frac{1}{4}L_\theta(\mathcal{N}_J(Z,Y),X) \\
&\qquad + \frac{1}{4}L_\theta(\mathcal{N}_J(Z,X),Y) \\
&= 2L_\theta(\nabla^W_X Y, Z) + 2\mathcal{B}_\theta(X,Y,Z)
\end{aligned}
$$

for all sections X, Y, Z in H with pairwise constant scalar products. The other formulae follow in a similar way. □

Note that the Nijenhuis torsion occurs only in the H-part of the connection components in Lemma 19. The 1-form ℓ influences the curvature expression that appear.

Lemma 20. *(cf. [**102**])*

$$Ric^{f_{\theta,\ell}}(\chi_K\ ,T^*) \quad = \quad \frac{1}{2(m+1)} scal^W \ - \ \frac{i}{2(m+2)} tr^\theta d\ell(\cdot, J\cdot) \qquad \textit{and}$$

$$\begin{aligned} Ric^{f_{\theta,\ell}}(X^*,V^*) \quad = \quad & \frac{scal^W}{(m+1)(m+2)} \cdot L_\theta(X,V) \\ & - i\frac{m}{2(m+2)}(\ \Omega^W(X,JV) + \Omega^W(V,JX)\ ) \\ & - \frac{m}{4}(\ \mathcal{R}_\theta(X,JV) + \mathcal{R}_\theta(V,JX)\ ) \\ & + tr^\theta_{1,4}(\nabla^W \mathcal{B}_\theta)(X,V) + tr^\theta_{1,4}(\nabla^W\mathcal{B}_\theta)(V,X) \\ & - \frac{1}{8} tr^\theta L_\theta(\mathcal{N}(X,\cdot),\mathcal{N}(V,\cdot)) \ + \ \frac{1}{4} tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),\cdot),V) \\ & + \frac{i}{m+2}(\ d\ell(X,JV) + d\ell(V,JX)\ ) \end{aligned}$$

for all vectors X^*, V^* in the horizontal lift of H to TF_c.

Proof. We use the connection components of Lemma 19 in order to obtain second covariant derivatives of vector fields on F_c and certain components of the Riemannian curvature tensor $R^{f_{\theta,\ell}}$. We set

$$G(X,V) := \mathcal{R}_\theta(X,V) + i\frac{2}{m+2}\Omega_{\theta,\ell}(X,V)$$

for all X, V in H. First, we have

$$\begin{aligned} f(\nabla^f_{X^*}\nabla^f_{Y^*}Z^*,V^*) \quad &= \quad X^*(f(\nabla^f_{Y^*}Z^*,V^*)) - f(\nabla^f_{Y^*}Z^*,\nabla^f_{X^*}V^*) \\ &= \quad L_\theta(\nabla^W_X\nabla^W_Y Z,V) \\ & \qquad + \frac{1}{4}L_\theta(JY,Z)\cdot G(X,V) + \frac{1}{4}L_\theta(JX,V)\cdot G(Y,Z) \\ & \qquad - L_\theta(\nabla^W_Y Z,\mathcal{B}(X,V)) + L_\theta(\nabla^W_X(\mathcal{B}(Y,Z)),V) \\ & \qquad - L_\theta(\mathcal{B}(Y,Z),\mathcal{B}(X,V))\ , \end{aligned}$$

$$\begin{aligned} f(\nabla^f_{[X^*,Y^*]}Z^*,V^*) \quad = \quad & L_\theta(\nabla^W_{pr_H[X,Y]}Z,V) + \frac{i}{m+2}\Omega_{\theta,\ell}(X,Y)\cdot L_\theta(JZ,V) \\ & - \frac{1}{2}L_\theta(JX,Y)\cdot \\ & \Big(\ L_\theta([T,Z],V) - L_\theta([T,V],Z) - i\frac{2}{m+2}\Omega_{\theta,\ell}(Z,V)\ \Big) \\ & + \mathcal{B}_\theta(pr_H[X,Y],Z,V) \quad , \end{aligned}$$

which results in the curvature component

$$\begin{aligned}
R^f(X^*,Y^*,Z^*,V^*) \;=\;& R^W(X,Y,Z,V) \\
& - \frac{i}{m+2}L_\theta(JZ,V)\cdot\Omega_{\theta,\ell}(X,Y) - \frac{1}{2}L_\theta(JX,Y)\cdot G(Z,V) \\
& - L_\theta(JX,Y)\cdot L_\theta(Tor^W(T,Z),V) \\
& + \frac{1}{4}L_\theta(JY,Z)\cdot G(X,V) + \frac{1}{4}L_\theta(JX,V)\cdot G(Y,Z) \\
& - \frac{1}{4}L_\theta(JX,Z)\cdot G(Y,V) - \frac{1}{4}L_\theta(JY,V)\cdot G(X,Z) \\
& - (\nabla^W_Y \mathcal{B}_\theta)(X,Z,V) + (\nabla^W_X \mathcal{B}_\theta)(Y,Z,V) \\
& - \frac{1}{4}\mathcal{B}_\theta(\mathcal{N}(X,Y),Z,V) \\
& - L_\theta(\mathcal{B}(Y,Z),\mathcal{B}(X,V)) + L_\theta(\mathcal{B}(X,Z),\mathcal{B}(Y,V))
\end{aligned}$$

for X,Y,Z and V in H. Moreover, we have

$$\begin{aligned}
f(\nabla^f_{T^*}\nabla^f_{X^*}\chi_K,V^*) &= -f(\nabla^f_{X^*}\chi_K,\nabla^f_{T^*}V^*) \\
&= -\frac{1}{4}(L_\theta([T,V],JX) - L_\theta([T,JX],V) - \frac{2i}{m+2}\Omega_{\theta,\ell}(V,JX)),
\end{aligned}$$

$$f(\nabla^f_{[X^*,T^*]}\chi_K,V^*) = \frac{1}{2}L_\theta([T,X],JV)$$

and we obtain

$$\begin{aligned}
R^f(X^*,T^*,\chi_K,V^*) + R^f(X^*,\chi_K,T^*,V^*) =& -\frac{i}{2(m+2)}\big(\Omega_{\theta,\ell}(V,JX) + \Omega_{\theta,\ell}(X,JV)\big) \\
& -\frac{1}{4}\left(\; \mathcal{R}_\theta(X,JV) + \mathcal{R}_\theta(V,JX)\;\right) \;.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=1}^{2m} R^{f_{\theta,\ell}}(X^*,e_i^*,e_i^*,V^*) \;=\;& \sum_{i=1}^{2m} R^W(X,e_i,e_i,V) \\
& + \frac{3i}{2(m+2)}\left(\; \Omega_{\theta,\ell}(X,JV) - \Omega_{\theta,\ell}(JX,V)\;\right) \\
& - \sum_{i=1}^{2m}\Big(\; (\nabla^W_{e_i}\mathcal{B}_\theta)(X,e_i,V) + \frac{1}{4}\mathcal{B}_\theta(\mathcal{N}(X,e_i),e_i,V)\;\Big) \\
& + \sum_{i=1}^{2m} L_\theta(\mathcal{B}(X,e_i),\mathcal{B}(e_i,V)) \;,
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{2m} R^{f_{\theta,\ell}}(X^*, e_i^*, Je_i^*, JV^*) \quad = \quad & \sum_{i=1}^{2m} R^W(X, e_i, e_i, V) \\
& - \frac{1}{2(m+1)} scal^W L_\theta(X,V) + i\frac{m+1}{m+2}\Omega_{\theta,\ell}(X,JV) \\
& - \frac{m-1}{2} L_\theta(Tor^W(T,X), JV) \\
& - \frac{m-1}{2} L_\theta(Tor^W(T,V), JX) \\
& + \sum_{i=1}^{2m} \Big((\nabla^W_{e_i}\mathcal{B}_\theta)(X, e_i, V) + \frac{1}{4}\mathcal{B}_\theta(\mathcal{N}(X,e_i), e_i, V) \Big) \\
& - \sum_{i=1}^{2m} L_\theta(\mathcal{B}(X,e_i), \mathcal{B}(e_i,V))
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{2m} R^{f_{\theta,\ell}}(X^*, JV^*, e_i^*, Je_i^*) \quad = \quad & \sum_{i=1}^{2m} R^W(X, JV, e_i, Je_i) \\
& + \frac{scal^W}{m+1} L_\theta(X,V) \;-\; i\frac{2m+2}{m+2}\Omega_{\theta,\ell}(X,JV) \\
& - 2\cdot \sum_{i=1}^{2m} L_\theta(\mathcal{B}(X,e_i), \mathcal{B}(V,e_i)) \quad .
\end{aligned}$$

By using

$$\begin{aligned}
\sum_{i=1}^{2m} R^{f_{\theta,\ell}}(X^*, e_i^*, Je_i^*, JV^*) \;=\; & \sum_{\alpha=1}^{m} R^{f_{\theta,\ell}}(X^*, e_{2\alpha-1}^*, Je_{2\alpha-1}^*, JV^*) \\
& - \sum_{\alpha=1}^{m} R^{f_{\theta,\ell}}(X^*, Je_{2\alpha-1}^*, e_{2\alpha-1}^*, JV^*) \\
=& - \sum_{\alpha=1}^{m} R^{f_{\theta,\ell}}(X^*, JV^*, e_{2\alpha-1}^*, Je_{2\alpha-1}^*)
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{i=1}^{2m} R^{f_{\theta,\ell}}(X^*, e_i^*, e_i^*, V^*) = \quad & - \; i\Omega^W(X, JV) \\
& + \; \frac{3i}{2(m+2)}\big(\Omega_{\theta,\ell}(X,JV) - \Omega_{\theta,\ell}(JX,V) \big) \\
& + \; \frac{m-1}{2}(L_\theta(Tor^W(T,X), JV) + L_\theta(Tor^W(T,V), JX)) \\
& - \; \sum_{i=1}^{2m} \Big(2(\nabla^W_{e_i}\mathcal{B}_\theta)(X, e_i, V) + \frac{1}{2}\mathcal{B}_\theta(\mathcal{N}(X,e_i), e_i, V) \Big)
\end{aligned}$$

$$+ \quad \sum_{i=1}^{2m} \big(2L_\theta(\mathcal{B}(X,e_i),\mathcal{B}(e_i,V)) + L_\theta(\mathcal{B}(X,e_i),\mathcal{B}(V,e_i)) \big).$$

Adding the expression for $R^{f_\theta}(X^*,T^*,\chi_K,V^*) + R^{f_\theta}(X^*,\chi_K,T^*,V^*)$ and using the identities for torsion terms of Section 4 we obtain

$$\begin{aligned} Ric^{f_{\theta,\ell}}(X^*,V^*) = \quad & - \ i\Omega^W(X,JV) \ + \ 2 \cdot tr^\theta_{1,4}(\nabla^W \mathcal{B}_\theta)(X,V) \\ & + \ \frac{i}{m+2}\big(\ \Omega_{\theta,\ell}(X,JV) + \Omega_{\theta,\ell}(V,JX) \ \big) \\ & - \ \frac{m}{4}\big(\ \mathcal{R}_\theta(X,JV) + \mathcal{R}_\theta(V,JX) \ \big) \\ & - \ \frac{1}{8} tr^\theta L_\theta(\mathcal{N}(X,\cdot),\mathcal{N}(V,\cdot)) \ + \ \frac{1}{4} tr^\theta L_\theta(\mathcal{N}(\mathcal{N}(X,\cdot),\cdot),V) \ . \end{aligned}$$

After symmetrisation in X and V of (the first line of) the right hand side of the latter equation, we obtain the component $Ric^{f_{\theta,\ell}}(X^*,V^*)$ as stated in the lemma. Furthermore, we have

$$\begin{aligned} Ric^{f_{\theta,\ell}}(\chi_K,T^*) = \ & \sum_{i=1}^{2m} \Big(\ f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{[T^*,e_i^*]}\chi_K, e_i^*) - f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{T^*}\nabla^{f_{\theta,\ell}}_{e_i^*}\chi_K, e_i^*) \ \Big) \\ = \ & \frac{i}{2(m+2)} \sum_{i=1}^{2m} \Big(\ \Omega_{\theta,\ell}(Je_i,e_i) - L_\theta([T,e_i],Je_i) + L_\theta([T,Je_i],e_i) \ \Big) \\ = \ & \frac{1}{2(m+1)} scal^W \ - \ \frac{i}{2(m+2)} tr^\theta d\ell(\cdot,J\cdot) \qquad . \end{aligned}$$

□

Theorem 22. *(cf. [**102**]) The scalar curvature of the ℓ-gauged Fefferman metric $f_{\theta,\ell}$ on F_c is given by*

$$scal^{f_{\theta,\ell}} \ = \ \frac{2m+1}{m+1}\cdot scal^W \ + \ \frac{1}{m+2}\cdot tr^\theta L_{d\ell} \qquad .$$

Proof. We compute

$$\begin{aligned} scal^{f_{\theta,\ell}} \quad = \quad & 2\cdot Ric^{f_{\theta,\ell}}(T^*,\chi_K) + \sum_{i=1}^{2m} Ric^{f_{\theta,\ell}}(e_i^*,e_i^*) \\ = \quad & \left(\frac{1}{m+1} + \frac{2m}{(m+1)(m+2)} + \frac{2m}{m+2} \right)\cdot scal^W \\ & - \frac{1}{8}\cdot \sum_{i,j=1}^{2m} L_\theta(\mathcal{N}(\mathcal{N}(e_i,e_j),e_j),e_i) \\ & + \frac{1}{4}\cdot \sum_{i,j=1}^{2m} L_\theta(\mathcal{N}(e_i,e_j),\mathcal{N}(e_i,e_j)) \end{aligned}$$

$$+\ \frac{2i}{m+2} tr^\theta d\ell(\cdot, J\cdot) \quad - \frac{i}{m+2} tr^\theta d\ell(\cdot, J\cdot)$$

$$= \quad \frac{2m+1}{m+1} \cdot scal^W \ + \ \frac{i}{m+2} tr^\theta d\ell(\cdot, J\cdot) \quad .$$

Thereby, we use again the torsion identities from Section 4, which hold under the condition of partial integrability. □

Theorem 22 shows that the scalar curvature of the Fefferman metric f_θ and the Webster scalar curvature are proportional. For arbitrary $\ell \in \Omega^1(M; i\mathbb{R})$ this is not true.

6. The Laplacian of the Fundamental Vector χ_K

We discuss now properties of the fundamental vector field χ_K which is vertical along the S^1-fibre bundle F_c in the Fefferman construction. It is easy to see that χ_K is a Killing vector with respect to any ℓ-gauged Fefferman metric $f_{\theta,\ell}$. However, the main goal of this section is to calculate the Laplacian of χ_K. We are able to give an explicit expression for it. The result is the first step in direction of a tractor calculus description for ℓ-gauged Fefferman metrics, which come from partially integrable CR-spaces (cf. also Chapter 6 and [**112**]).

The fundamental vector field χ_K is uniquely determined by $A_{\theta,\ell}(\chi_K) = i\frac{m+2}{2}$ and it is lightlike with respect to the ℓ-gauged Fefferman metric $f_{\theta,\ell}$, where $\ell \in \Omega^1(M; i\mathbb{R})$. Lemma 19 shows that

$$f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_C \chi_K, B) = -f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_B \chi_K, C)$$

for all vectors $B, C \in TF_c$, i.e., χ_K is a Killing vector for any ℓ-gauged Fefferman metric $f_{\theta,\ell}$. Equivalently, for the dual 1-form θ to χ_K on $(F_c, f_{\theta,\ell})$ the equation

$$\nabla^{f_{\theta,\ell}} \theta = 1/2 \cdot d\theta$$

holds. The Bochner-Laplacian $\Delta^{f_{\theta,\ell}}_{tr} = tr^{f_{\theta,\ell}} \nabla^2$ acts on arbitrary tensor fields ρ through

$$tr^{f_{\theta,\ell}} \nabla^2 \rho \ = \ \nabla^{f_{\theta,\ell}}_{\chi_K} \nabla^{f_{\theta,\ell}}_{T^*} \rho + \nabla^{f_{\theta,\ell}}_{T^*} \nabla^{f_{\theta,\ell}}_{\chi_K} \rho \ + \ \sum_{i=1}^{2m} \Big(\nabla^{f_{\theta,\ell}}_{e_i^*} \nabla^{f_{\theta,\ell}}_{e_i^*} \rho - \nabla^{f_{\theta,\ell}}_{\nabla^{f_{\theta,\ell}}_{e_i^*} e_i^*} \rho \Big)$$

(with respect to our special choice of frame on F_c; cf. Section 0.4). In general, for any Killing 1-form θ the equations

$$d^*\theta = 0 \qquad \text{and} \qquad \Delta^{f_{\theta,\ell}}_{tr} \theta = -1/2 \cdot d^* d\theta = -1/2 \cdot \Delta^{f_{\theta,\ell}}_1 \theta$$

hold, where $\Delta^{f_{\theta,\ell}}_1$ denotes the Laplace-Beltrami operator. By using the formulae of Lemma 19 we find a simple expression for the Laplacian $\Delta^{f_{\theta,\ell}}_1$ applied to θ on $(F_c, f_{\theta,\ell})$.

Proposition 16. *(cf. [**112**]) Let $(F_c, f_{\theta,\ell})$ be the ℓ-gauged Fefferman space of a partially integrable CR-space (M^n, T_{10}) of dimension $n = 2m+1$ with pseudo-Hermitian structure θ and $\ell \in \Omega^1(M; i\mathbb{R})$. Then*

(1) *the fundamental vector χ_K is Killing and the lift θ to F_c of the pseudo-Hermitian form is the dual Killing 1-form. In particular,*

$$\nabla^{f_{\theta,\ell}} \theta = 1/2 \cdot d\theta \ .$$

(2) *For the Laplace-Beltrami operator applied to* θ *on* $(F_c, f_{\theta,\ell})$ *the equation*

$$\Delta_1^{f_{\theta,\ell}}\theta = -2i\frac{(n-1)}{n+3}\cdot A_{\theta,\ell} + \left(\frac{scal^{f_{\theta,\ell}}}{n} - \frac{2(n+1)}{n\cdot(n+3)}tr^\theta L_{d\ell}\right)\cdot\theta\ ,$$

resp.,

$$\Box_1^{f_{\theta,\ell}}\theta = \frac{i}{n+3}\cdot A_{\theta,\ell} + \frac{(n+1)\cdot tr^\theta L_{d\ell}}{n(n-1)\cdot(n+3)}\cdot\theta$$

holds, where $\Box_1^{f_{\theta,\ell}} = \frac{1}{n-1}\left(\ \Delta_{tr}^{f_{\theta,\ell}} + \frac{scal^{f_{\theta,\ell}}}{2n}\ \right)$ *(cf. Section 1.1).*

Proof. Let X, Y, Z denote sections in H on M such that their coefficients with respect to a local J-adapted frame $\{e_i :\ i = 1,\dots,2m\}$ are constant. We have

$$\begin{aligned} f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{X^*}\nabla^{f_{\theta,\ell}}_{Y^*}\chi_K,\chi_K) &= -f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{Y^*}\chi_K,\nabla^{f_{\theta,\ell}}_{X^*}\chi_K) = -\frac{1}{4}f_{\theta,\ell}(\,(JY)^*,(JX)^*\,) \\ &= -\frac{1}{4}f_{\theta,\ell}(X^*,Y^*)\ , \end{aligned}$$

$$\begin{aligned} f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{X^*}\nabla^{f_{\theta,\ell}}_{Y^*}\chi_K,T^*) &= -f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{Y^*}\chi_K,\nabla^{f_{\theta,\ell}}_{X^*}T^*) \\ &= -\frac{1}{4}\big(\ L_\theta([X,T],JY) + L_\theta([JY,T],X)\ \big) \\ &\quad -i\frac{1}{2(m+2)}\Omega_{\theta,\ell}(JY,X), \end{aligned}$$

$$\begin{aligned} f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{X^*}\nabla^{f_{\theta,\ell}}_{Y^*}\chi_K,Z^*) &= -f_{\theta,\ell}(\nabla^{f_{\theta,\ell}}_{Y^*}\chi_K,\nabla^{f_{\theta,\ell}}_{X^*}Z^*) \\ &= \frac{1}{2}\big(\ L_\theta(\nabla^W_X JZ,Y) + \mathcal{B}_\theta(X,JZ,Y)\ \big). \end{aligned}$$

These formulae show that

$$\begin{aligned} f_{\theta,\ell}(tr^{f_{\theta,\ell}}\nabla^2\chi_K,\chi_K) &= -\tfrac{m}{2} \\ f_{\theta,\ell}(tr^{f_{\theta,\ell}}\nabla^2\chi_K,Z^*) &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2m}\big(\ L_\theta(\nabla^W_{e_i}JZ,e_i) + L_\theta(\nabla^W_{e_i}e_i,JZ)\ \big) \\ &\quad +\tfrac{1}{2}\textstyle\sum_{i=1}^{2m}\mathcal{B}_\theta(e_i,JZ,e_i) \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2m} e_i\big(\ L_\theta(e_i,JZ)\ \big)\ =\ 0 \end{aligned}$$

$$\begin{aligned} f_{\theta,\ell}(tr^{f_{\theta,\ell}}\nabla^2\chi_K,T^*) &= -i\tfrac{1}{2(m+2)}\textstyle\sum_{i=1}^{2m}\Omega_{\theta,\ell}(Je_i,e_i) \\ &= -\tfrac{scal^W}{2(m+1)} + \tfrac{i}{2(m+2)}tr^\theta d\ell(\cdot,J\cdot) \\ &= -\tfrac{1}{2n}scal^{f_{\theta,\ell}} + \tfrac{n+1}{n(n+3)}tr^\theta L_{d\ell}\ \ , \end{aligned}$$

where we use the relation

$$scal^{f_{\theta,\ell}} = \frac{2m+1}{m+1}scal^W + \frac{1}{m+2}tr^\theta L_{d\ell}\ .$$

With the identity $f_{\theta,\ell}(tr^{f_{\theta,\ell}}\nabla^2\chi_K,\cdot) = tr^{f_{\theta,\ell}}\nabla^2\theta(\cdot) = -\frac{1}{2}\Delta_1^{f_{\theta,\ell}}\theta(\cdot)$ we obtain the stated formula for the Laplace-Beltrami operator applied to θ. $\square$

We obtain the following simple relation for $\square_1\theta$ and the ℓ-gauged Weyl connection $A_{\theta,\ell}$.

Corollary 6. *Let $(F_c, f_{\theta,\ell})$ be the ℓ-gauged Fefferman space of (M,T_{10}) with respect to θ,ℓ on M. Then*

$$\square_1^{f_{\theta,\ell}}\theta = \frac{i}{n+3}A_{\theta,\ell} \qquad \textit{if and only if} \qquad tr^\theta L_{d\ell} = 0\ .$$

Finally, we define the vector space

$$H^1_{tr}(M,T_{10}) := \{\ell \in \Omega^1(M;i\mathbb{R}):\ tr^\theta d\ell(\cdot,J\cdot) = 0\}/\{\ell\in\Omega^1(M;i\mathbb{R}):\ d\ell = 0\}\ ,$$

for any partially integrable CR-space (M,T_{10}). This space is the affine model for the space of connections, which admit the same scalar curvature as the Weyl connection on F_c belonging to some pseudo-Hermitian form θ, modulo gauge transformations. The map

$$\Psi:\quad [\ell]\in H^1_{tr}(M,T_{10}) \qquad\mapsto\qquad c_{[\ell]} = [f_{\theta,\ell}]$$

is onto the space of (local) $[\ell]$-gauged Fefferman conformal structures on F_c with the property

$$\square_1^{f_{\theta,\tilde\ell}}\theta = \frac{i}{n+3}A_{\theta,\tilde\ell}$$

for any pseudo-Hermitian form θ and $\tilde\ell \in [\ell]$.

7. Conformal Tractor Calculus for Gauged Fefferman Spaces

Since the canonical principal S^1-bundle F_c over a partially integrable CR-space (M,T_{10}) is equipped with the ℓ-gauged Fefferman conformal classes, we can employ tractor calculus for the investigation of the conformal geometry and the underlying CR-geometry in a Fefferman construction. Our first aim here is to calculate the splitting operator $\mathbf{S}:\mathfrak{X}(F)\to\Gamma(\mathcal{A}(F))$ with respect to a metric $g\in c$ on a space (F,c) with conformal structure, in general (cf. Section 0.7). We will see that for any conformal Killing vector field τ on (F,g) the component of the adjoint tractor $\mathbf{S}(\tau)$ in $\mathcal{A}^1(F,c)$ is just given by $\square_1^g\tau$ (cf. Section 1.1). With the help of Proposition 16 we can then see that the corresponding adjoint tractor to the fundamental vector χ_K on a Fefferman space F_c is related to a certain complex structure, which acts on the standard tractors $\mathcal{T}(F,c)$. In fact, in Chapter 6 we will show that the existence of such a complex structure characterises ℓ-gauged Fefferman spaces.

So let (F^{n+1},c) be a smooth manifold with conformal structure c of dimension $n+1$ (and arbitrary signature) and let $g\in c$ denote a (fixed) metric in the conformal class. The standard and adjoint tractor bundles of (F^{n+1},c) are denoted by $\mathcal{T}$, resp. $\mathcal{A}$ (cf. Section 0.7). With respect to the metric g the adjoint tractor bundle decomposes into $TF\oplus\mathfrak{co}(TF)\oplus T^*F$ and an adjoint tractor $A\in\mathcal{A}_p$, $p\in F$, is given with respect to g by a triple $(\tau,\psi,\eta)\in\mathcal{A}_p$. Now let $\{e_i:\ i=1,\ldots,n+1\}$ be an arbitrary orthonormal basis of T_pF with respect to g. We set $\varepsilon_i := g(e_i,e_i)$. The induced algebraic bracket

$\{\cdot,\cdot\}$ on $\mathcal{A}_p$ has the following expressions with respect to (τ,ψ,η) and the basis $\{e_i\}$ (cf. Section 0.7):

$$\begin{array}{rcl}
\{e_i,\psi\} &=& -\,\psi(e_i)\\
\{e_i^*,\psi\} &=& e_i^*\circ\psi\\
\{e_i,e_j^*\} &=& e_i\otimes e_j^* \;-\; \varepsilon_i\varepsilon_j e_j\otimes e_i^* \qquad \text{for } \; i\neq j\\
\{e_i,e_i^*\} &=& id|_{TF}\\
\{e_i,\eta\} &=& e_i\otimes\eta \;-\; \sum_{j=1}^{n+1}\varepsilon_i\varepsilon_j\eta(e_j)e_j\otimes e_i^* \;+\; \eta(e_i)id\\
\{\tau,e_j^*\} &=& \tau\otimes e_j^* \;-\; \varepsilon_j e_j\otimes g(\tau,\cdot) \;+\; e_j^*(\tau)id\;.
\end{array}$$

We aim to calculate the splitting operator $\mathbf{S}:\mathfrak{X}(F)\to\Gamma(\mathcal{A})$, which is uniquely determined by the conditions

$$\Pi_H(\mathbf{S}(\tau))=\tau \qquad \text{and} \qquad \partial^*(d^{nor}\mathbf{S}(\tau))=0\;,$$

where $d^{nor}=\nabla^{nor}$ is the covariant exterior derivative $d^{nor}=\nabla^{nor}$ induced by the canonical connection ω_{nor}. The connection ∇^{nor} acts on a section (τ,ψ,η) in $\mathcal{A}$ by

$$\begin{array}{rcl}
d^{nor}(\tau,\psi,\eta) &=& (\;\nabla^g_{\cdot}\tau\;,\;\nabla^g_{\cdot}\psi\;,\;\nabla^g_{\cdot}\eta\;)+\{\;(\cdot,0,\mathsf{P}^g(\cdot))\;,\;(\tau,\psi,\eta)\;\}\\
&=& (\;\nabla^g_{\cdot}\tau\;,\;\nabla^g_{\cdot}\psi\;,\;\nabla^g_{\cdot}\eta\;)+(\{\cdot,\psi\}\;,\;\{\cdot,\eta\}+\{\mathsf{P}^g(\cdot),\tau\}\;,\;\{\mathsf{P}^g(\cdot),\psi\}),
\end{array}$$

where ∇^g denotes the Levi-Civita connection of g and

$$\begin{array}{rrcl}
\mathsf{P}^g: & TF &\to& T^*F\;,\\
& X &\mapsto& \frac{1}{n-1}\left(\;\frac{scal^g}{2n}g(X,\cdot)-Ric^g(X,\cdot)\;\right)
\end{array}$$

is the Schouten tensor with respect to g (cf. Section 0.7). Applying the codifferential ∂^* results in

$$\partial^* d(\tau,\psi,\eta)=\begin{pmatrix} 0\\ \sum_{i=1}^{n+1}(\;\{e_i^*,\nabla^g_{e_i}\tau\}+\{e_i^*,\{e_i,\psi\}\}\;)\\ \sum_{i=1}^{n+1}(\;\{e_i^*,\nabla^g_{e_i}\psi\}\;+\;\{e_i^*,\{e_i,\eta\}\}\;+\;\{e_i^*,\{\mathsf{P}^g(e_i),\tau\}\}\;)\end{pmatrix}.$$

With above formulae for the bracket $\{\cdot,\cdot\}$ we obtain

$$\begin{array}{rcl}
\sum_{i=1}^{n+1}\{e_i^*,\{e_i,\psi\}\} &=& \psi\;+\;tr^g\psi\cdot id\;-\;\sum_{i=1}^{n+1}\varepsilon_i g(\psi(e_i),\cdot)e_i\\
\sum_{i=1}^{n+1}\{e_i^*,\nabla^g_{e_i}\tau\} &=& -\,\nabla^g_{\cdot}\tau\;-\;tr^g\nabla^g\tau\;+\;\sum_{i=1}^{n+1}\varepsilon_i g(\nabla^g_{e_i}\tau,\cdot)e_i\\
\sum_{i=1}^{n+1}\{e_i^*,\{e_i,\eta\}\} &=& (n+1)\cdot\eta\\
\sum_{i=1}^{n+1}\{e_i^*,\nabla^g_{e_i}\psi\} &=& \sum_{i=1}^{n+1}e_i^*\circ\nabla^g_{e_i}\psi\\
\sum_{i=1}^{n+1}\{e_i^*,\{\mathsf{P}^g(e_i),\tau\}\} &=& -\,2\mathsf{P}^g(\tau)\;+\;tr^g\mathsf{P}^g\cdot g(\tau,\cdot),
\end{array}$$

where $tr^g\mathsf{P}^g=-\frac{scal^g}{2n(n+1)}$. One can immediately see that the equation $\partial^* d(\tau,\psi,\eta)=0$ is solved for an arbitrary vector field τ on F by setting

$$\begin{array}{rcl}
\psi &=& asym(\nabla^g\tau)+tr^g(\nabla^g\tau)\cdot id \qquad\qquad \text{and}\\
\eta &=& -\,\frac{1}{n+1}\left(\;(\sum_{i=1}^{n+1}e_i^*\circ\nabla^g_{e_i}\psi)-2\mathsf{P}^g(\tau)+tr^g\mathsf{P}^g\cdot g(\tau,\cdot)\;\right)\;,
\end{array}$$

where $asym$ denotes the skew-symmetric part of an endomorphism on TF. By insertion of the first equation into the second one we obtain

$$\eta = \frac{-1}{n+1}\left(\big(\sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i}(asym\nabla^g\tau)(\cdot)\big) + d(div^g\tau) - 2\mathsf{P}^g(\tau) + tr^g\mathsf{P}^g \cdot g(\tau,\cdot) \right) .$$

We reformulate the latter expression for η in case when τ is a conformal Killing vector field. In this situation, we have $\psi = \nabla^g\tau$,

$$\sum_{i=1}^{n+1} e_i^* \circ \nabla^g_{e_i}(\nabla^g\tau) = -g(tr^g\nabla^2\tau,\cdot) + \frac{2}{n+1}d(div(\tau))$$

and together with (4) we obtain

$$\eta = \frac{1}{n-1}\cdot g(tr^g\nabla^2\tau,\cdot) \ + \ \frac{scal^g}{2n\cdot(n-1)}g(\tau,\cdot) .$$

We conclude that the splitting operator $\mathbf{S}$ applied to a conformal Killing vector τ on (F,c) is given with respect to $g \in c$ by

$$\mathbf{S}(\tau) = (\ \tau\ ,\ \nabla^g_{\cdot}\tau\ ,\ \Box^g_1(g(\tau,\cdot))\ )\ ,$$

where the differential operator $\Box^g_1$ is defined as in Proposition 16 by

$$\Box^g_1 := \frac{1}{n-1}\left(\ \Delta^g_{tr} + \frac{scal^g}{2n}\ \right) .$$

Now we specialise the situation to the realm of Fefferman spaces and study their conformal tractor calculus. So let $(F_c^{n+1}, f_{\theta,\ell})$ be a ℓ-gauged Fefferman space over a strictly pseudoconvex, partially integrable CR-space (M^n, T_{10}) with pseudo-Hermitian structure θ and $\ell \in \Omega^1(M; i\mathbb{R})$. (We note once again that the restriction to the strictly pseudoconvex case is not essential for the considerations in this chapter.) Let $\mathcal{T}(F_c)$ and $\mathcal{A}(F_c)$ be the standard, resp., adjoint tractor bundle to the conformal structure $[f_{\theta,\ell}]$ on F_c. We set $j := 2\chi_K$, where χ_K is the fundamental vector field in F_c with $A_{\theta,\ell}(\chi_K) = i\frac{m+2}{2}$. The adjoint tractor bundle $\mathcal{A}(F_c)$ decomposes with respect to the ℓ-gauged Fefferman metric $f_{\theta,\ell}$ into $TF_c \oplus \mathfrak{co}(TF_c) \oplus TF_c^*$ and we define

$$\mathcal{J}\ :=\ (\ j\ ,\ J_{\theta,\ell}\ ,\ \frac{2i}{n+3}A_{\theta,\ell}\ )\ ,$$

where $J_{\theta,\ell}$ denotes the horizontal lift of the almost complex structure J on H to TF_c (with respect to $A_{\theta,\ell}$). The triple $\mathcal{J}$ can be considered via the metric $f_{\theta,\ell}$ as a section in the adjoint tractor bundle $\mathcal{A}(F_c)$, which acts by $\bullet$ as an endomorphism on the standard tractor bundle $\mathcal{T}(F_c)$ (cf. Section 0.7).

Proposition 17. *(1) The definition of the adjoint tractor*

$$\mathcal{J} = (j, J_{\theta,\ell}, \frac{2i}{n+3}A_{\theta,\ell})\ \in\ \Gamma(\mathcal{A})$$

as a triple with respect to the Fefferman metric $f_{\theta,\ell}$ does not depend on the choice of θ, i.e., $\mathcal{J}$ is a CR-invariant of (M, T_{10}).

(2) The adjoint tractor $\mathcal{J}$ acts on the standard tractor bundle $\mathcal{T}(F_c)$ as complex structure, i.e., the relation $\mathcal{J}^2 := \mathcal{J}\bullet\mathcal{J} = -id|_{\mathcal{T}(F_c)}$ holds.

Proof. We only proof here the second statement that $\mathcal{J}$ acts as complex structure on $\mathcal{T}(F_c)$. The proof of independence of the definition of $\mathcal{J}$ from the choice of pseudo-Hermitian form θ will be postponed until Theorem 23. In fact, we will see there that $\mathcal{J}$ is even a conformally invariant object on F_c.

The standard tractor bundle $\mathcal{T}(F_c)$ splits with respect to the ℓ-gauged Fefferman metric $f_{\theta,\ell}$ into $\mathbb{R}\oplus TF_c\oplus\mathbb{R}$ and the action of $\mathcal{J}$ on a standard tractor $t=(a,\xi,b)$ with respect to this splitting is given by an application of the matrix

$$\mathcal{J}=\begin{pmatrix} 0 & \frac{2i}{n+3}A_{\theta,\ell} & 0 \\ j & J_{\theta,\ell} & \frac{-2i}{n+3}A^{\sharp}_{\theta,\ell} \\ 0 & -f_{\theta,\ell}(j,\cdot) & 0 \end{pmatrix} \quad ,$$

(cf. Section 0.7). We have

$$\mathcal{J}\bullet t=\begin{pmatrix} \frac{2i}{n+3}A_{\theta,\ell}(\xi) \\ b\cdot j+J_{\theta,\ell}(\xi)-\frac{2ia}{n+3}A^{\sharp}_{\theta,\ell} \\ -f_{\theta,\ell}(j,\xi) \end{pmatrix} \quad ,$$

where $J_{\theta,\ell}$ acts trivially on the orthogonal complement of the horizontally lifted distribution H. Applying $\mathcal{J}$ again results in

$$\mathcal{J}\bullet\mathcal{J}\bullet t=\begin{pmatrix} \frac{2ib}{n+3}A_{\theta,\ell}(j) \\ \frac{2i}{n+3}A_{\theta,\ell}(\xi)\cdot j-pr_H(\xi)+\frac{2i}{n+3}f_{\theta,\ell}(j,\xi)A^{\sharp}_{\theta,\ell} \\ \frac{2ia}{n+3}A_{\theta,\ell}(j) \end{pmatrix} \;=\; -t \quad ,$$

where we use $\frac{2i}{n+3}A_{\theta,\ell}(j)=-1$ and pr_H denotes the orthogonal projection of TF_c to the horizontal lift of H. □

We know that the projection $\Pi_H(\mathcal{J})=j$ is a Killing vector field on $(F_c,f_{\theta,\ell})$. Next we aim to compute $\mathbf{S}(j)\in\Gamma(\mathcal{A})$, where $\mathbf{S}$ is the splitting operator. We set

$$U:=\frac{2(n+1)}{n\cdot(n-1)(n+3)}(0,0,tr^{\theta}L_{d\ell}\cdot\theta)\in\mathcal{A}(F_c)$$

with respect to $f_{\theta,\ell}$. Note that the 1-form $tr^{\theta}L_{d\ell}\cdot\theta$ does not depend on the pseudo-Hermitian structure θ, i.e., $tr^{\theta}L_{d\ell}\cdot\theta$ is a uniquely defined section in T^*F_c. Since T^*F_c is canonically included as subbundle in the adjoint tractor bundle (cf. Section 0.7), we can conclude that U is a uniquely defined section in $\mathcal{A}(F_c)$ and does not depend on the choice of $f_{\theta,\ell}$.

Theorem 23. *Let $(F_c^{n+1},[f_{\theta,\ell}])$ be the gauged Fefferman space of a strictly pseudo-convex, partially integrable CR-space (M^n,T_{10}) with pseudo-Hermitian form θ, gauge $\ell\in\Omega^1(M;i\mathbb{R})$ and fundamental vector field $j=2\chi_K$. Then*

(1) *the relation*

$$\mathbf{S}(j)\;=\;\mathcal{J}\;+\;U$$

holds. In particular, $\mathcal{J}$ is a conformal invariant on F_c.

(2)

$$\mathbf{S}(j)=\mathcal{J}\qquad\textit{if and only if}\qquad[\ell]\in H^1_{tr}(M,T_{10})\ .$$

In this case the equation

$$\nabla^{nor}_{\cdot}\mathcal{J} = -\Omega^{nor}(j,\ \cdot\ )$$

is satisfied.

PROOF. From Proposition 16 and Lemma 19 we know that j is a Killing vector with $\nabla^{f_{\theta,\ell}}_X j = J_{\theta,\ell}X$ for all $X \in TF_c$ and

$$\Box_1 f_{\theta,\ell}(j,\cdot) = \frac{2i}{n+3}\cdot A_{\theta,\ell} \ + \ \frac{2(n+1)\cdot tr^\theta L_{d\ell}}{n(n-1)\cdot(n+3)}\cdot\theta\ .$$

This shows

$$\mathbf{S}(j) \ = \ (\ j\ ,\ J_{\theta,\ell}\ ,\ \frac{2i}{n+3}A_{\theta,\ell} + U\ ) \ = \ \mathcal{J} \ + \ U$$

with respect to $f_{\theta,\ell}$. Since the adjoint tractors $\mathbf{S}(j)$ and U are conformally invariant by definition, the adjoint tractor $\mathcal{J}$ is a conformally invariant object as well. In particular, we see that $\mathcal{J}$ does not depend on the pseudo-Hermitian form θ, which completes the proof of Proposition 17. Moreover, Corollary 6 shows that $\mathbf{S}(j) = \mathcal{J}$ if and only if $tr^\theta L_{d\ell} = 0$. In general, the equation $\nabla^{nor}_{\cdot}\mathbf{S}(\tau) = -\Omega^{nor}(\tau,\cdot)$ holds for any conformal Killing vector τ (cf. Section 0.7). Replacing $\mathbf{S}(\tau)$ by $\mathcal{J}$ and τ by j in this formula yields the claimed equation for $\mathcal{J}$. $\Box$

We remark that the curvature expression $\Omega^{nor}(j,\cdot)$ is entirely determined by the Nijenhuis tensor $\mathcal{N}$ and the gauge 1-form ℓ. If $\mathcal{N} = 0$ and $\ell = 0$, i.e., in the situation of the classical intrinsic Fefferman construction, we have $\nabla^{nor}\mathbf{S}(j) = 0$ and $\Omega^{nor}(j,\cdot) = 0$. This statement follows with Sparling's characterisation of classical Fefferman metrics (cf. Theorem 3 and Chapter 6).

Finally, we remark that the complex structure $\mathcal{J}$ on a ℓ-gauged Fefferman space admits a natural explanation via the Fefferman construction using Cartan geometry (cf. Section 0.12). In fact, remember that if $\Lambda^m_{\mathbb{C}}H\otimes Q$ admits a $(m+2)$-nd root $\mathcal{E}(1,0)$ then there exists a $\tilde{P}$-reduction $\tilde{\mathcal{P}}(M)$ of the normal CR-Cartan geometry $(\mathcal{P}(M),\omega_{nor}) \to (M,T_{10})$ of type $(\mathrm{PSU}(1,m+1),P)$, where $\tilde{P}$ denotes the corresponding parabolic in $\mathrm{SU}(1,m+1)$. The standard CR-tractor bundle is then given by $\mathcal{T}_{CR}(M) = \tilde{\mathcal{P}}(M)\times_{\tilde{P}} \mathbb{C}^{m+2}$, which is equipped with a natural complex structure J_{CR} (cf. Section 0.10). The Fefferman space is given by $\tilde{F}_c = \tilde{\mathcal{P}}(M)/(\mathrm{SU}(1,m+1)\cap\tilde{P}')$, where $\tilde{P}'$ is the parabolic in $\mathrm{SO}(2,2m+2)$ of conformal geometry. It is a matter of fact that the lift of $\mathcal{T}_{CR}(M)$ along the S^1-fibering $\tilde{F}_c \to M$ gives rise to the conformal standard tractor bundle $\mathcal{T}(\tilde{F}_c)$ on the Fefferman space (cf. Section 0.12). One can show now that the lift of the natural complex structure J_{CR} on $\mathcal{T}_{CR}(M)$ to $\mathcal{T}(\tilde{F}_c)$ produces $\mathcal{J}$ (cf. [**39**]).

CHAPTER 6

Unitary Conformal Holonomy and Einstein Reductions

This chapter is dedicated to the geometric study of the (tractor) equation

$$\nabla^{nor}_{\cdot}\mathbf{S}(j) = -\Omega^{nor}(j,\cdot) \qquad \text{with} \qquad \mathbf{S}(j)^2 = -id|_{\mathcal{T}(F)} \tag{40}$$

for a vector field $j \in \mathfrak{X}(F)$ on a conformal space (F,c) (where $\mathbf{S}$ denotes the splitting operator as introduced in Section 0.7), i.e., we are asking here for the existence of a conformal Killing vector field j on F, whose 2-jet prolongation $\mathcal{J} := \mathbf{S}(j)$ in the adjoint tractors acts as complex structure on the standard tractor bundle $\mathcal{T}(F)$. A special case of this equation is

$$\nabla^{nor}_{\cdot}\mathcal{J} = 0 \qquad \text{with} \qquad \mathcal{J}^2 = -id|_{\mathcal{T}(F)} \ .$$

Obviously, the existence of such a ∇^{nor}-parallel complex structure $\mathcal{J} \in \Gamma(\mathcal{A})$ is equivalent to the reduction of the conformal tractor holonomy $Hol(\mathcal{T})$ to (a subgroup of) the unitary group $\mathrm{U}(p+1,q+1)$.

To be more concrete, we aim to discuss in this chapter three features of complex structures in conformal tractor calculus and unitary holonomy reductions. First, we have seen already in the previous chapter that certain ℓ-gauged Fefferman spaces of partially integrable CR-spaces admit a solution $j \in \mathfrak{X}(F)$, resp., $\mathcal{J} \in \Gamma(\mathcal{A})$ of (40). We will show in the first part of this chapter that if a solution of (40) exists then the underlying conformal structure is locally given by some ℓ-gauged Fefferman class on the canonical S^1-bundle of a uniquely determined partially integrable CR-space (with arbitrary signature; cf. Chapter 5). The reconstruction of the underlying partially integrable CR-structure from a complex structure $\mathcal{J}$ will be discussed in detail in Section 1. In particular, we will see that if the complex structure $\mathcal{J} \in \Gamma(\mathcal{A})$ is ∇^{nor}-parallel then the underlying CR-geometry is integrable and the gauge ℓ is trivial (resp., closed), i.e., we are in the situation of the classical Fefferman construction. In fact, the equation $\nabla^{nor}\mathcal{J} = 0$ implies $\Omega^{nor}(j,\cdot) = 0$ and one can immediately see that this curvature condition allows the application of Sparling's characterisation of classical Fefferman geometry (cf. Section 0.7 and Theorem 3).

The result about the reconstruction of an integrable CR-structure in case of $\nabla^{nor}\mathcal{J} = 0$ also leads us to a discussion of the unitary conformal holonomy reduction with a remarkable outcome. Due to a result of A. Čap it is known that the lift of the canonical Cartan connection of CR-geometry gives rise to the normal conformal Cartan connection of the conformal structure on a Fefferman space if and only if the CR-geometry is integrable (cf. [**34, 39**]). Since the structure group of CR-geometry is $\mathrm{PSU}(p+1,q+1)$, the latter fact can be used to conclude that at least the conformal holonomy algebra $\mathfrak{hol}(\mathcal{T})$ is reduced to the special unitary algebra $\mathfrak{su}(p+1,q+1)$ if a ∇^{nor}-parallel complex structure $\mathcal{J}$ does exist. This further holonomy reduction might be surprising on the first glance given that the stabiliser of an orthogonal complex structure is the unitary group $\mathrm{U}(p+1,q+1)$. In the second part of this chapter we will explain this automatic $\mathfrak{su}(p+1,q+1)$-reduction of $\mathfrak{hol}(\mathcal{T})$ by direct computations using only conformal tractor calculus (cf. Section 2). The proof relies essentially on the condition of normality of the canonical Cartan connection of conformal geometry. In particular, the result says that the conformal tractor holonomy $Hol(\mathcal{T})$ of a conformal space is never the full unitary group $\mathrm{U}(p+1,q+1)$. Our results about unitary conformal holonomy reductions are confirmed by [**40**] presenting more general computations and further conclusions.

In the third and final part of this chapter (Section 3 to 7) we discuss Einstein conditions for integrable CR-geometries and their corresponding Fefferman spaces. This topic is related to the study of pseudo-Einstein Hermitian structures (in the sense of Lee; cf. [**103**]) and Einstein reductions of conformal holonomy (cf. [**6, 7, 110**]). In the literature, it is well known that a classical Fefferman metric constructed over an integrable CR-space equipped with a pseudo-Hermitian structure is never Einstein (cf. [**102**]). However, the approach of conformal holonomy theory and its relation to the conformal Einstein condition via parallel standard tractors (cf. Chapter 1) suggests that a Fefferman conformal class could possibly be Einstein without being conformally flat, in general. To be more concrete again, the equivalence of the canonical Cartan connections of integrable CR-geometry and conformal Fefferman geometry implies that $\nabla^{\mathcal{T}}$-parallel standard CR-tractors and conformal tractors correspond naturally and bijectively to each other. It is also known that a $\nabla^{\mathcal{T}}$-parallel standard CR-tractor gives rise to a pseudo-Einstein Hermitian structure θ on the underlying CR-space (up to singularities), whose Reeb vector T^θ is in addition a transverse symmetry (cf. [**39**]). In the following we will call such a pseudo-Hermitian structure θ a TS-pseudo-Einstein or simply (TSPE)-structure. Our discussion will show that (TSPE)-structures are closely related to Kähler-Einstein spaces (via factoring through the transverse symmetry). In fact, with our approach we are able to state a local construction and characterisation principle for (TSPE)-structures on integrable CR-spaces (cf. Theorem 26). In a second step, we will prove that a Fefferman conformal class, which is induced by a (TSPE)-space admits locally always an Einstein scale and we are able to present an explicit form of the corresponding *Fefferman-Einstein metric* (cf. Theorem 27). The Einstein scales that occur on these spaces admit globally always singularities. This instance shows that in a conformal Einstein class there might exist certain naturally distinguished metrics apart from the Einstein metrics.

1. The Reconstruction of CR-Structures

In Chapter 5 we have proven the existence of a complex structure $\mathcal{J}$ on the standard tractor bundle of certain ℓ-gauged Fefferman spaces (cf. Theorem 23). We want to argue now in the reversed direction and explain the reconstruction of CR-structures from such $\mathcal{J}$. Our discussion will be of local nature, since we do not want to assume a S^1-fibration on our initial space.

We will use the following general observation about orthogonal complex structures on $\mathbb{R}^{2p+2,2q+2}$.

Lemma 21. *(cf.* [**112**]*) Let*

$$\beta \;=\; \begin{pmatrix} -a & l & 0 \\ m & A & -\mathbb{J}l^\top \\ 0 & -m^\top\mathbb{J} & a \end{pmatrix}$$

be a matrix in $\mathfrak{g} = \mathfrak{so}(2p+2, 2q+2)$. *Then the property* $\beta^2 = -id$ *is equivalent to the following conditions on* $m, -\mathbb{J}l^\top \in \mathfrak{g}_{-1}$ *and* $A \in \mathfrak{so}(\mathbb{J}_{2p+2,2q+2})$*:*

(1) m *and* $-\mathbb{J}l^\top$ *are non-zero lightlike eigenvectors of* A *to the eigenvalue* a*,*
(2) *the scalar product of* m *with* $-\mathbb{J}l^\top$ *equals* $1+a^2$ *and*
(3) A^2 *restricted to* $(span\{m, -\mathbb{J}l^\top\})^\perp$ *in* $\mathfrak{g}_{-1}$ *is equal to* $-id$*.*

PROOF. We have

$$\beta^2 = \begin{pmatrix} a^2 + lm & -al + lA & -l\mathbb{J}\ {}^tl \\ -am + Am & ml + A^2 + \mathbb{J}\ {}^tl\,{}^tm\mathbb{J} & -A\mathbb{J}\ {}^tl - a\mathbb{J}\ {}^tl \\ -{}^tm\mathbb{J}m & -{}^tm\mathbb{J}A - a\,{}^tm\mathbb{J} & {}^tm\,{}^tl + a^2 \end{pmatrix} .$$

From this matrix square the statement of Lemma 21 is obvious. □

Now let (F^{n+1}, c) be a smooth manifold of dimension $n+1$ with conformal structure c of signature $(2p+1, 2q+1)$ and let $\mathcal{J} \in \Gamma(\mathcal{A}(F))$ be a complex structure acting on the standard tractors $\mathcal{T}(F)$ subject to the tractor equation

$$\nabla^{nor}_{\cdot}\mathcal{J} = -\Omega^{nor}(j, \cdot)$$

with projection $\Pi_H(\mathcal{J}) = j$ (which is a conformal Killing vector field under these conditions). The splitting operator $\mathbf{S}$ applied to j reproduces the adjoint tractor $\mathcal{J}$. Furthermore, let $f \in c$ be some metric on F. Then the adjoint tractor $\mathcal{J}$ is given by the triple (j , $\nabla^f j$, $\Box^f_1(f(j,\cdot))$), resp., a matrix

$$\mathcal{J} \;=\; \begin{pmatrix} -J_c & J_\eta & 0 \\ j & J_s & -J^\sharp_\eta \\ 0 & -g(j,\cdot) & J_c \end{pmatrix} ,$$

where J_s is the skew-symmetric part of $\nabla^f j$, $J_c = \frac{2}{n+1} div^f(j)$ and $J_\eta = \Box^f_1(f(j,\cdot))$. From Lemma 21 we know that j and $-J^\sharp_\eta$ are lightlike J_c-eigenvectors of J_s with $-f(j, J^\sharp_\eta) = 1 + J_c^2$. The skew-symmetric endomorphism J_s restricts to an orthogonal complex structure on the subbundle $W(\mathcal{J}, f)$, which is defined to be the f-orthogonal complement of the span of j and $-J^\sharp_\eta$ in TF.

Now we denote by θ^f the dual of $\frac{1}{2}j$ with respect to f in c. The subbundles $\mathbb{R}j$ and $E := ker\theta^f$ give rise to a flag in TF. The quotient $Q := E/\mathbb{R}j$ is via f identified with the subbundle $W(\mathcal{J}, f)$ and inherits a metric denote by f_Q. The image of the tensorial map

$$\nabla^f_{\cdot} j : E \to TF$$

lies again in E and j is mapped to $\frac{2\cdot div^f(j)}{n+1} \cdot j$. Hence $\nabla^f j$ can be interpreted as an endomorphism of the quotient bundle Q. The skew-symmetric part $J : Q \to Q$ of this endomorphism acts as f_Q-orthogonal complex structure, i.e., $f_Q(JX, JY) = f_Q(X, Y)$ for all vectors X, Y in Q. Moreover, the 2-form $f_Q(J\cdot, \cdot)$ on Q equals the 2-form that is induced by $d\theta^f$ on the quotient Q. Thereby, note that $d\theta^f(j, \cdot) = 0$ on E.

So far we have constructed from $\mathcal{J}$ with help of the some metric f in c the data

$$\mathbb{R}j, \quad E, \quad Q := E/\mathbb{R}j \quad \text{and} \quad J : Q \to Q .$$

It is clear from the definition of the projection $\Pi_H : \mathcal{A}(F) \to TF = H^1_{\mathfrak{g}}F$ (cf. Section 0.7) that $\mathbb{R}j$ is independently given from the choice of f in c. This implies also that

$E := ker\theta^f$ and the quotient space Q are independent from $f \in c$. We also can show that J does not depend on the choice of f in c. For this purpose, let $\tilde{f} = e^{2\phi} f$ be an arbitrary rescaled metric in the conformal class c. The transformation law for the Levi-Civita connection under conformal change implies that

$$\nabla^{\tilde{f}}_X j = \nabla^f_X j + d\phi(X)j + d\phi(j)X \qquad \text{for all } X \in E \ .$$

This relation shows that the skew-symmetric part of $\nabla^{\tilde{f}} j$ induces on the quotient space Q the same complex structure J as $\nabla^f j$ does. Altogether, this reasoning shows that $\mathbb{R}j$, E and the pair (Q, J) are uniquely determined by $\mathfrak{J}$ on the conformal space (F, c)

In general, there exists a naturally defined Lie derivative $\mathfrak{L}$ acting on sections of any tractor bundle in the direction of conformal Killing vector fields. This derivative is given by differentiation of a given tractor along the flow of the vector field, which consists of (local) conformal diffeomorphisms. If $A \in \Gamma(\mathcal{A}(F))$ is an adjoint tractor, $\mathfrak{L}_V$ in the direction of a conformal Killing vector V on a space (F, c) is given by

$$\mathfrak{L}_V A \ = \ -\{\mathbf{S}(V), A\} \ + \ \Omega^{nor}(V, \Pi_H(A))$$

(cf. [**37**]). In particular, $\mathfrak{L}_V \mathbf{S}(V) = 0$ for any conformal Killing vector V. Note that in case V is Killing with respect to some metric $f \in c$ above Lie derivative $\mathfrak{L}_V$ coincides with the usual Lie derivative in the direction of V applied to the components of A in the tensorial decomposition $TF \oplus \mathfrak{co}(TF) \oplus T^*F$ with respect to f.

Applied to our situation here we can see that $\mathfrak{L}_j \mathfrak{J} = 0$ for the complex structure $\mathfrak{J}$ with $j = \Pi_H(\mathfrak{J})$ on (F, c), i.e., the complex structure $\mathfrak{J}$ is preserved under the (local) flow j_t of the conformal Killing vector field j. With this remark we can conclude that $\mathbb{R}j, E$ and the complex structure J on Q are preserved by the (local) flow j_t on (F, c) as well. Alternatively, we can see this point without tractor calculus as follows. Since the conformal Killing vector field j has no zero, we can find locally a metric f in c such that j is Killing, i.e., j is an infinitesimal isometry of f, which implies $\mathfrak{L}_j f = 0$ and $[\mathfrak{L}_j, \nabla^f] = 0$ (also $[\mathfrak{L}_j, \Box^f_1] = 0$). Then we have $[j, j] = 0$ and obtain $\mathfrak{L}_j \theta^f = 0$, which proves that $j_t^*(\mathbb{R}j) = \mathbb{R}j$ and $j_t^*(E) = E$ to any time t. Moreover, $\mathfrak{L}_j(\nabla^f j) = 0$, which shows that J on Q is invariant under the flow j_t as well.

The Lie bracket of vector fields on F gives rise to a well defined map

$$\begin{aligned} L_Q : \ \ Q \times Q \ \ &\rightarrow \ \ K := TF/E \ , \\ (X, Y) \ \ &\mapsto \ \ pr_K[X, Y] \ , \end{aligned}$$

for which the relation $\theta^f \circ L_Q(\cdot, \cdot) = -f_Q(J\cdot, \cdot)$ holds for any $f \in c$ and $L_Q(JX, JY) = L_Q(X, Y)$ for all X, Y in Q. This shows that L_Q is an algebraic, non-degenerate and totally real bracket on Q. Moreover, the map L_Q is by definition invariant under the flow j_t along the integral curves of j.

Let us consider once more the conformal Killing vector j with its integral curves γ^j on F. Since j admits no zeros on F, the integral curves γ^j_p through any point $p \in F$ are locally diffeomorphic to $\mathbb{R}$. That means we can factorise the manifold F locally around every point through these integral curves γ^j and obtain a smooth projection onto a C^∞-manifold M_U of dimension n:

$$\pi_U : \ U \subset F \ \rightarrow \ M_U := U/Im(\gamma^j) \ .$$

Thereby, we can assume that the open submanifold U of F is diffeomorphic to $M_U \times \mathbb{R}$ and π_U is the natural projection onto the first factor. Since the subbundle E of TF

is invariant under the flow j_t of the vector field j, its projection by π_U gives rise to a subset H of TM_U. The subbundle $\mathbb{R}j$ of TF is the kernel of the projection π_{U*}. Hence the subset H is a smooth distribution of corank one in TM_U and the quotient bundle Q projects naturally onto the distribution H. Since the algebraic bracket L_Q is non-degenerate and also j_t-invariant on F, the distribution H has to be contact on M_U. The corresponding algebraic bracket $\mathcal{Z}$ is just the projection of L_Q (cf. Section 0.10). Moreover, the j_t-invariant complex structure J on Q projects to a complex structure on the contact distribution H on M_U, which we again denote by J. In particular, it follows that the bracket $\mathcal{Z}$ is totally real with respect to this J. At this point we have shown that the adjoint tractor $\mathcal{J}$ on (F, c) generates locally a uniquely determined CR-manifold (M_U, H, J) of hypersurface type and dimension n, which is partially integrable.

Finally, we want to construct pseudo-Hermitian structures on (M_U, H, J). For this purpose, let us consider (locally) any metric f in c on $U \subset F$ with $\mathcal{L}_j f = 0$. Then $\mathcal{L}_j \theta^f = 0$ and θ^f projects uniquely to a 1-form θ on M_U such that $\theta|_H = 0$, i.e., any (local) metric f with $\mathcal{L}_j f = 0$ gives rise to a pseudo-Hermitian form θ on (M_U, H, J). From such a θ we obtain the Levi-form L_θ on the CR-space (M_U, H, J). The signature of L_θ is (p, q). In summary, we have the following result.

Proposition 18. *Let (F, c) be a space with conformal structure c admitting a conformal Killing vector $j \in \mathfrak{X}(F)$ such that $\mathbf{S}(j) \in \Gamma(\mathcal{A}(F))$ acts as complex structure on $\mathcal{T}(F)$. Then*

(1) *the local factorisation of F through the integral curves γ^j of j generates in a natural way a smooth space M_U admitting a uniquely determined partially integrable CR-structure (H, J).*
(2) *Any (local) metric f in c with $\mathcal{L}_j f = 0$ generates in a natural way a pseudo-Hermitian form θ on that CR-space.*

So far we have not used for our discussion all the information that is encoded in the existence of the complex structure $\mathcal{J}$. And, in fact, there is still an open issue. It is the question how the conformal structure c on $U \subset F$ is related to the induced CR-space (M_U, H, J). To give an answer to this question, we consider the 1-form $A^f := \Box_1^f(f(j, \cdot))$ for a (local) metric f with $\mathcal{L}_j f = 0$. Then we have $A^f(j) = -1$ and $\mathcal{L}_j A^f = 0$. These properties show that we can understand A^f as a *local connection form* on the fibration $\pi_U : U \subset F \to M_U$. From our construction so far, it is clear that f on U takes the form

$$\pi_U^* L_\theta - 4 \cdot \theta^f \circ A^f \ ,$$

where L_θ is the Levi-form on (M_U, H, J) belonging to θ, which in turn was induced by f on $U \subset F$.

If we choose U suitably small we can find a diffeomorphism Ψ_U, which identifies U with an open subset of the canonical S^1-bundle F_c over (M_U, H, J) such that $\theta^f = \Psi_U^* \theta$ and $\Psi_{U*}(j) = 2\chi_K$, where χ_K is the fundamental field introduced in Section 5.5. (Such an identification Ψ_U is unique up to gauge transformations of the canonical S^1-bundle.) We can compare now f on U with the Fefferman construction to θ on the canonical S^1-bundle of (M_U, H, J). Let A_θ be the Weyl connection to (M_U, H, J) (cf. Section 5.3). We denote by $\ell := \Psi_U^{-1*} A^f \, - \, A_\theta$ the difference between the two given connections (via Ψ_U^{-1}). Eventually, we see from this discussion that f equals the ℓ-gauged Fefferman metric over (M_U, H, J) via Ψ_U, i.e., $f = \Psi_U^* f_{\theta,\ell}$. However, the difference ℓ between the two connections can not be arbitrary. In fact, by construction the adjoint tractor $\mathbf{S}(2\chi_K)$ is via Ψ_U identified with the complex structure $\mathcal{J}$. Yet we know from Theorem 23 that the vertical Killing vector $2\chi_k$ in the ℓ-gauged Fefferman construction gives rise to a complex structure via the splitting operator $\mathbf{S}$ if and only if the contraction $tr^\theta d\ell(\cdot, J\cdot)$ vanishes on M_U, i.e., the gauge class $[\ell]$ is an element of $H^1_{tr}(M_U, H, J)$. This completes our discussion of reconstruction.

Theorem 24. *Let (F, c) be a space with conformal structure of arbitrary signature c admitting a conformal Killing vector $j \in \mathfrak{X}(F)$ such that $\mathbf{S}(j) \in \Gamma(\mathcal{A}(F))$ acts as complex structure on $\mathcal{T}(F)$ and let (M_U, H, J) be the (locally) induced CR-space (cf. Proposition 18). Then c on F is (locally) conformally equivalent to some $[\ell]$-gauged Fefferman metric, which is constructed on (M_U, H, J) with suitable $[\ell] \in H^1_{tr}(M_U, H, J)$.*

Note that Theorem 24 is a statement about conformal spaces of arbitrary signature $(2p+1, 2q+1)$, whereas in Chapter 5 we introduced the ℓ-gauged Fefferman construction only in the strictly pseudoconvex case (i.e. positive definite case). However, the restriction to the positive definite case in Chapter 5 is not essential and the theory extends straightforwardly to the arbitrary signature case.

A special case of Theorem 24 is when $\mathcal{J} = \mathbf{S}(j)$ is a ∇^{nor}-parallel complex structure on $\mathcal{T}(F)$. The Sparling's characterisation tells us what happens in this situation (cf. Theorem 3).

Corollary 7. *(cf.* [**146, 73**]*) Let (F^{n+1}, c) be a manifold with conformal structure c admitting a conformal Killing vector j such that $\mathcal{J} = \mathbf{S}(j)$ is a ∇^{nor}-parallel complex structure on $\mathcal{T}(F)$. Then c is locally conformally equivalent to a (classical) Fefferman metric f_θ constructed over an integrable CR-space.*

Proof. The vector field j admits no zeros on F. This implies that the partial differential equation

$$j(\phi) \;=\; \frac{1}{n} \cdot div^f j$$

admits locally a solution ϕ for any $f \in c$. We choose such a local solution ϕ for some fixed f. Then the vector field j is Killing with respect to $g = e^{2\phi} f$. We explained in Section 0.7 that with $\nabla^{nor}\mathcal{J} = 0$ the curvature condition $\iota_j \Omega^{nor} = 0$ follows, which is equivalent to $\iota_j W^g = 0$ and $\iota_j C^g = 0$ with respect to the metric g. Moreover,

$$div^g j = 0 \quad \text{and} \quad \Box_1^g(g(j,\cdot)) \;=\; \frac{1}{n-1}\left(-Ric^g(j,\cdot) \;+\; \frac{scal^g}{2n} \cdot g(j,\cdot) \right) \;,$$

and with $g(j,j) = 0$ and $g(j, \Box_1^g j) = -1 - (div^g j)^2$ we obtain

$$Ric^g(j,j) \;=\; n-1 \;>\; 0 \;.$$

These curvature properties for g and j show that we can apply Sparling's characterisation (cf. Theorem 3). The conclusion is that g is (locally) isometric to the (classical) Fefferman metric of some integrable CR-space of hypersurface type. $\square$

We remark that the underlying CR-space, which is predicted by Sparling's characterisation and actually is explicitly constructed in [**73**], is indeed (locally) equivalent to the CR-space (M_U, H, J), whose reconstruction was established in Proposition 18.

Corollary 7 (resp., Sparling's characterisation) also tells us that in the situation of Theorem 24 we have

$$\Omega^{nor}(j,\cdot)=0 \qquad \text{if and only if} \qquad \mathcal{N}_J=0 \ \text{ and } \ [\ell]=0$$

for the reconstructed CR-space (M_U,H,J). Finally, we want to remark that the statement of Corollary 7 can be rephrased as a holonomy characterisation.

Corollary 8. *(cf.* [**113, 40**]*) Let (F,c) be a manifold with conformal structure c of signature $(2p+1,2q+1)$ and conformal tractor holonomy $Hol(\mathcal{T})$ contained in* $\mathrm{U}(p+1,q+1)$. *Then the conformal class c is locally the (classical) Fefferman conformal class of some integrable CR-space.*

2. The Holonomy Reduction to $\mathrm{SU}(p+1,q+1)$

We have just seen in Corollary 8 that a conformal space whose tractor holonomy $Hol(\mathcal{T})$ is contained in $\mathrm{U}(p+1,q+1)$ is locally equivalent to a (classical) Fefferman space constructed over an integrable CR-space. However, since the structure group of CR-geometry is $\mathrm{PSU}(p+1,q+1)$, the lift of the canonical CR-Cartan connection along the Fefferman construction gives rise to the normal conformal Cartan connection and since $\Omega^{nor}(\chi,\cdot)=0$ for any vertical χ (cf. Section 0.12 and [**34**]), we understand that at least the holonomy algebra $\mathfrak{hol}(\mathcal{T})$ of a classical Fefferman space is actually further reduced to $\mathfrak{su}(p+1,q+1)$. This observation simply says that

$$Hol(\mathcal{T}) \ \neq \ \mathrm{U}(p+1,q+1)$$

for any conformal space! We want to explain this phenomena here using conformal tractor calculus (without making the detour of reconstructing the underlying CR-space and using Sparling's characterisation; see also [**40**]).

So let $\mathcal{J}$ be an adjoint tractor on a space (M^n,c) of dimension $n:=2m+2$ with conformal structure c such that

$$\mathcal{J}^2:=\mathcal{J}\bullet\mathcal{J}=-id|_{\mathcal{T}(M)}\in\Gamma(End(\mathcal{T}(M))) \qquad \text{and} \qquad \nabla^{nor}\mathcal{J}=0\ .$$

The adjoint tractor $\mathcal{J}$ is given with respect to a metric $g\in c$ by

$$\mathcal{J} \ = \ \begin{pmatrix} -J_c & J_\eta & 0 \\ j & J_s & -J_\eta^\sharp \\ 0 & -g(j,\cdot) & J_c \end{pmatrix}$$

(cf. Section 1). As before, let $W(\mathcal{J},g)$ be the g-orthogonal complement of $span\{j,J_\eta^\sharp\}$ in TM, which can be understood as a subbundle of $\mathcal{T}(M)$ with respect to g. Note that

$$\langle\cdot,\cdot\rangle_{\mathcal{T}}|_{W(\mathcal{J},g)} \ = \ g|_{W(\mathcal{J},g)} \qquad \text{and} \qquad \mathcal{J}|_{W(\mathcal{J},g)} \ = \ J_s|_{W(\mathcal{J},g)}\ .$$

If u_- and u_+ are generating elements of $\mathcal{E}[1]\cong_g\mathbb{R}$, resp., $\mathcal{E}[-1]\cong_g\mathbb{R}$ with $\langle u_-,u_+\rangle_{\mathcal{T}}=1$ then

$$W(\mathcal{J},g) \ := \ span\{j,J_\eta^\sharp\}^{\perp_g} \ = \ span\{u_-,\mathcal{J}u_-,u_+,\mathcal{J}u_+\}^{\perp(\mathcal{T},\langle\cdot,\cdot\rangle)}\ .$$

In this situation we can choose a local complex frame $\{e_\alpha:\ \alpha=1,\ldots,m\}$ of $W(\mathcal{J},g)$ such that $\{e_\alpha,\mathcal{J}e_\alpha:\ \alpha=1,\ldots,m\}$ is an orthogonal frame of $W(\mathcal{J},g)$ and

$$\{\ u_-\ ,\ \mathcal{J}u_-\ ,\ u_+\ ,\ \mathcal{J}u_+\ ,\ e_1\ ,\ \mathcal{J}e_1\ ,\ \ldots\ ,\ e_m\ ,\ \mathcal{J}e_m\ \}$$

is a local frame of the standard tractor bundle $\mathcal{T}(M)$. We call the corresponding complex frame

$$\mathcal{B}(\mathcal{J},g) \ := \ \{ \ u_- \ , \ u_+ \ , \ e_1 \ , \ \dots \ , \ e_m \ \}$$

a $(\mathcal{J},g)$-adapted frame of $\mathcal{T}(M)$.

Next let us denote by

$$\mathcal{T}^{\mathbb{C}}(M) = \mathcal{T}(M) \otimes \mathbb{C}$$

the complexified standard tractor bundle on (M,c). We extend the complex structure $\mathcal{J}$ on $\mathcal{T}(M)$ to a $\mathbb{C}$-linear complex structure on the complexification $\mathcal{T}^{\mathbb{C}}(M)$, which we denote again by $\mathcal{J}$. The bundle $\mathcal{T}^{\mathbb{C}}(M)$ decomposes into the direct sum $\mathcal{T}_{10} \oplus \mathcal{T}_{01}$, where $\mathcal{T}_{10}$ denotes the i-eigenspace of $\mathcal{J}$ and $\mathcal{T}_{01}$ is the complex conjugate. The determinant bundle

$$\mathcal{C} := \Lambda^{m+2}\mathcal{T}^*_{10}$$

is a complex line bundle on M. We call $\mathcal{C}$ the canonical complex line tractor bundle of $(\mathcal{T}(M),\mathcal{J})$. (If we denote by $\tilde{\mathcal{U}}(M)$ the $\mathrm{U}(p+1,q+1)$-reduction of the $\mathrm{O}(2p+2,2q+2)$-principal fibre bundle $\tilde{\mathcal{G}}(M)$ induced by $\mathcal{J}$ then the canonical complex line tractor bundle is given by $\mathcal{C} = \tilde{\mathcal{U}}(M) \times_{det^{-1}_{\mathbb{C}}} \mathbb{C}$.) The connection ω_{nor} induces on $\mathcal{C}$ a covariant derivative ∇^{nor} and we denote by $\Omega^{\mathcal{C}}$ the conformal curvature for the complex line tractor bundle $\mathcal{C}$.

Lemma 22. *Let $\mathcal{C}$ be the canonical complex line tractor bundle to $\mathcal{T}(M)$ with ∇^{nor}-parallel complex structure $\mathcal{J}$. Then the curvature $\Omega^{\mathcal{C}}$ vanishes identically on $\mathcal{C}$.*

Proof. We aim to compute the curvature $\Omega^{\mathcal{C}}$ on $\mathcal{C}$. First of all, we remark that by assumption we have

$$\Omega^{nor}(j, \ \cdot \ ) \ = \ 0$$

with $j := \Pi_H(\mathcal{J})$. With respect to any metric $g \in c$ we can conclude that $\iota_j W^g = 0$ and $\iota_j C^g = 0$.

Now let $\{E_\alpha : \ \alpha = 1, \dots, m+2\}$ be a local complex frame of $(\mathcal{T}(M), \mathcal{J})$ such that $\{E_\alpha, \mathcal{J}E_\alpha : \ \alpha = 1, \dots, m+2\}$ is a local orthonormal frame of $(\mathcal{T}(M), \langle\cdot,\cdot\rangle)$. Then we denote

$$\varrho_\alpha \ := \ \frac{1}{\sqrt{2}} \langle \ \cdot \ , \ E_\alpha + i\mathcal{J}E_\alpha \ \rangle_{\mathcal{T}}$$

and the $(m+2)$-form

$$\varrho \ := \ \varrho_1 \wedge \dots \wedge \varrho_{m+2}$$

is a local complex tractor volume form on (M,c), i.e., a local section in $\mathcal{C}$. We have

$$\Omega^{\mathcal{C}}(X,Y) \circ \varrho \ = \ -i \sum_{\alpha=1}^{m+2} \langle E_\alpha, E_\alpha \rangle_{\mathcal{T}} \cdot \langle \Omega(X,Y) \bullet E_\alpha, \mathcal{J}E_\alpha \rangle_{\mathcal{T}} \cdot \varrho$$

for all $X, Y \in TM$. In particular, this expression proves that

$$\Omega^{\mathcal{C}}(j, \ \cdot \ ) \ = \ 0 \ .$$

Next we reformulate above expression for $\Omega^{\mathcal{C}}$ with respect to an arbitrary metric $g \in c$ on M and a local $(\mathcal{J},g)$-adapted frame $\mathcal{B}(\mathcal{J},g) = \{u_-, u_+, e_1, \dots, e_m\}$ with $\varepsilon_\alpha := g(e_\alpha, e_\alpha)$. We have

$$\Omega^{\mathcal{C}}(X,Y) \ = \ -2i \cdot \langle \Omega(X,Y) \bullet u_-, \mathcal{J}u_+ \rangle_{\mathcal{T}} \ - \ i \sum_{\alpha=1}^{m} \varepsilon_\alpha \cdot \langle \Omega(X,Y) \bullet e_\alpha, \mathcal{J}e_\alpha \rangle_{\mathcal{T}},$$

which is a purely imaginary number for $X,Y\in TM$. We remember that the curvature Ω^{nor} has with respect to g the matrix form

$$\begin{pmatrix} 0 & C^g & 0 \\ 0 & W^g & -C^{g\sharp} \\ 0 & 0 & 0 \end{pmatrix} .$$

Then we obtain

$$\begin{aligned}
\Omega^{\mathcal{C}}(X,Y) &= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot\langle\Omega^{nor}(X,Y)\bullet e_\alpha,\mathcal{J}e_\alpha\rangle_{\mathcal{T}}\\
&= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot\langle\Omega_0(X,Y)\bullet e_\alpha,\mathcal{J}e_\alpha\rangle_{\mathcal{T}}\\
&= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot W^g(X,Y,e_\alpha,J_se_\alpha)\\
&= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot\big(\ W^g(X,J_se_\alpha,e_\alpha,Y)\ -\ W^g(X,e_\alpha,J_se_\alpha,Y)\ \big)\\
&= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot\big(\langle\Omega_0(X,J_se_\alpha)\bullet e_\alpha,Y\rangle_{\mathcal{T}}-\langle\Omega_0(X,e_\alpha)\bullet\mathcal{J}e_\alpha,Y\rangle_{\mathcal{T}}\big)
\end{aligned}$$

for all $X,Y\in TM$.

Let us assume now that Y is an element of $W(\mathcal{J},g)$ and $X\in TM$ is arbitrary. We set $u_{2\alpha-1}:=e_\alpha$ and $u_{2\alpha}:=J_se_\alpha$ for $\alpha=1,\dots,m$. With $\iota_jW^g=0$ and $\iota_jC^g=0$ we obtain

$$\begin{aligned}
\Omega^{\mathcal{C}}(X,Y) &= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot\big(\langle\Omega_0(X,J_se_\alpha)\bullet\mathcal{J}e_\alpha,\mathcal{J}Y\rangle_{\mathcal{T}}+\langle\Omega_0(X,e_\alpha)\bullet e_\alpha,\mathcal{J}Y\rangle_{\mathcal{T}}\big)\\
&= -i\sum_{\alpha=1}^{m}\varepsilon_\alpha\cdot\big(W^g(X,J_se_\alpha,J_se_\alpha,J_sY)+W^g(X,e_\alpha,e_\alpha,J_sY)\big)\\
&= -\ i\sum_{i=1}^{n-2}\varepsilon_\alpha\cdot W^g(X,u_i,u_i,J_sY)\\
&\qquad +\ \frac{i}{1+J_c^2}\cdot\big(\ W^g(X,j,J_\eta^\sharp,J_sY)\ +\ W^g(X,J_\eta^\sharp,j,J_sY)\ \big)\\
&= -i\cdot tr_g^{23}W^g(X,\ \cdot\ ,\ \cdot\ ,J_sY)\\
&= 0\ .
\end{aligned}$$

This shows that $\iota_Y\Omega^{\mathcal{C}}=0$ for all $Y\in W(\mathcal{J},g)$. Since the orthogonal complement of $W(\mathcal{J},g)$ in TM has rank 2 and spans together with $W(\mathcal{J},g)$ the tangent space TM, we can conclude that the only possible non-vanishing component of the curvature on $\mathcal{C}$ is $\Omega^{\mathcal{C}}(j,J_\eta^\sharp)$. However, we know already that $\iota_j\Omega^{\mathcal{C}}=0$, i.e., the latter component of $\Omega^{\mathcal{C}}$ vanishes as well. $\square$

The proof of Lemma 22 uses strongly the symmetries of the Weyl curvature W^g, which are a direct consequence of the normalisation condition $\partial^* \circ \kappa = 0$ for ω_{nor} and the generalised Bianchi identity (cf. Section 0.6). We note further that Lemma 22 shows that the local complex tractor volume form ϱ is parallel with respect to ∇^{nor}. This property also implies the local existence of a conformal Killing spinor on (M, c) (cf. Theorem 1 and 10; see also e.g. [**17**]). Lemma 22 is the main ingredient for the proof of our reduction claim on the conformal holonomy.

Theorem 25. *Let (M, c) be a space of dimension $n = 2m + 2 \geq 4$ with conformal structure c of signature $(2p+1, 2q+1)$ such that the conformal holonomy group $Hol(\mathcal{T})$ is contained in the unitary group $\mathrm{U}(p+1, q+1)$. Then*

(1) *the holonomy algebra $\mathfrak{hol}(\mathcal{T})$ is a subalgebra of the special unitary algebra $\mathfrak{su}(p+1, q+1)$.*
(2) *If, in addition, the space M is simply connected then the holonomy group $Hol(\mathcal{T})$ is contained in the special unitary group $\mathrm{SU}(p+1, q+1)$.*

Proof. (1) The assumption of Theorem 25 about the holonomy group $Hol(\mathcal{T})$ implies the existence of a ∇^{nor}-parallel complex structure $\mathcal{J} \in \Gamma(\mathcal{A}(M))$ on the standard tractor bundle $\mathcal{T}(M)$ of (M, c). Lemma 22 shows that the values $\Omega^{nor}(X, Y) \in \mathcal{A}(M)$ of the tractor curvature have vanishing complex trace for all $X, Y \in TM$. It follows that the curvature form Ω on $\tilde{\mathcal{G}}(M)$ takes values only in $\mathfrak{su}(p+1, q+1) \subset \mathfrak{g}$. Since the special unitary algebra $\mathfrak{su}(p+1, q+1)$ is an ideal in $\mathfrak{u}(p+1, q+1)$, the Ambrose-Singer holonomy Theorem (cf. Section 0.3) shows that the holonomy algebra $\mathfrak{hol}(\mathcal{T})$ is contained in $\mathfrak{su}(p+1, q+1)$.

(2) In general, when M is simply connected the holonomy group $Hol(\mathcal{C}, \nabla^{nor})$ of the canonical complex line tractor bundle is a connected Lie subgroup of $\mathrm{U}(1)$. Here, since $\mathcal{C}$ is locally flat by Lemma 22, $Hol(\mathcal{C}, \nabla^{nor})$ is also a discrete subgroup of $\mathrm{U}(1)$. It follows that the holonomy group $Hol(\mathcal{C}, \nabla^{nor})$ is trivial and the complex line bundle $\mathcal{C}$ is globally flat admitting a parallel complex tractor volume form on M. This proves that $Hol(\mathcal{T})$ is contained in $\mathrm{SU}(p+1, q+1)$. □

We remark that A. Čap and A.R. Gover can also prove Theorem 25. More generally, their proof shows that for any ∇^{nor}-parallel adjoint tractor A the Killing form $B_{\mathfrak{so}(r+1,s+1)}(\Omega^{nor}(\cdot,\cdot), A)$ vanishes identically on a space (M, c) (in the framework of conformal geometry)! This shows again for our case $A = \mathcal{J}$ that Ω^{nor} has no complex trace (cf. [**40**]).

3. Transverse Symmetry

In the remaining sections of this chapter we want to discuss Einstein reductions (in the sense of holonomy) for integrable CR-spaces and their corresponding Fefferman spaces. The idea for this construction comes from the observation that the conformal tractor holonomy $Hol(\mathcal{T})$ of some space might be contained in $\mathrm{U}(p+1, q+1)$ (resp. $\mathrm{SU}(p+1, q+1)$) and to the same time fixes a standard tractor without being the trivial holonomy. This property does imply for the underlying integrable CR-space the existence of a pseudo-Einstein Hermitian structure θ (up to singularities), whose Reeb vector is simultaneously a transverse symmetry.

Let (M,H,J) be an integrable CR-manifold of hypersurface type and dimension $n=2m+1$ with arbitrary signature (p,q). In the following, we will call a pseudo-Hermitian form θ on (M,H,J) a transverse symmetry if the corresponding Reeb vector $T^\theta\in\mathfrak{X}(M)$ satisfies

$$[T,X]+J[T,JX]=0 \qquad \text{for all } X\in\Gamma(H)\ ,$$

i.e., the (local) Reeb flow consists of CR-automorphisms (cf. Section 0.10). In short, we then say that θ is a TS-pseudo-Hermitian structure or just (TSPH)-structure on (M,H,J). Obviously, a pseudo-Hermitian form θ is a transverse symmetry if and only if the torsion part $Tor^W(T^\theta,X)$ of the Tanaka-Webster connection ∇^W to θ vanishes for all vectors $X\in H$ (cf. Chapter 5). Equivalently, these conditions also mean that θ is a Killing 1-form for the metric

$$g_\theta\ :=\ L_\theta\ +\ \theta\circ\theta\ ,$$

resp., T^θ is a Killing vector, i.e., $\mathfrak{L}_T g_\theta=0$. The latter fact uses the properties $\mathfrak{L}_T J=0$ and $\mathfrak{L}_T\theta=0$ (for the case of transverse symmetry). Furthermore, we extend our notation and say that θ is a TS-pseudo-Einstein structure (in short: (TSPE)-structure) on (M,H,J) if and only if θ is a transverse symmetry and simultaneously the Webster-Ricci tensor Ric^W is a constant multiple of $d\theta$, i.e.,

$$Ric^W=-i\frac{scal^W}{m}\cdot d\theta \qquad \text{and} \qquad Tor^W(T,X)=0$$

for all $X\in H$. In this case we call (M,H,J,θ) a TS-pseudo-Einstein space (cf. Section 0.10 and e.g. [**103, 48**]).

For later use we aim to compare the Tanaka-Webster connection and corresponding curvature tensors for a given (TSPH)-structure θ with the Levi-Civita connection and metric curvature tensors of the induced metric g_θ. So let θ be a fixed (TSPH)-structure on an integrable CR-space (M,H,J). First, we determine the endomorphism

$$D^\theta:=\nabla^W-\nabla^{g_\theta}\ .$$

A straightforward calculation shows that the covariant derivative

$$\nabla^W-\frac{1}{2}d\theta\cdot T+\frac{1}{2}(\theta\otimes J+J\otimes\theta)$$

is metric and has no torsion with respect to g_θ. We conclude that it is the Levi-Civita connection of g_θ and we obtain as comparison tensor

$$D^\theta:=\nabla^W-\nabla^{g_\theta}=\frac{1}{2}\left(d\theta\cdot T-(\theta\otimes J+J\otimes\theta)\right)\ .$$

Another straightforward calculation shows that for any $X,Y,Z\in\mathfrak{X}(M)$ we have

$$\begin{aligned} R^{\nabla^W}(X,Y)Z \ =\ & R^{g_\theta}(X,Y)Z-\frac{1}{2}\left(\nabla^{g_\theta}_Z d\theta(X,Y)\right)\cdot T-\frac{1}{2}d\theta(X,Y)\cdot J(Z)\\ &+\frac{1}{4}d\theta(Y,Z)\cdot J(X)-\frac{1}{4}d\theta(X,Z)\cdot J(Y)\\ &+\frac{1}{4}\theta(Z)\cdot\theta(X)\cdot Y-\frac{1}{4}\theta(Z)\cdot\theta(Y)\cdot X\ . \end{aligned}$$

This is the comparison of the curvature tensors. The formula immediately proves that (in the transverse symmetric case!) the Webster curvature operator R^{∇^W} (resp. tensor R^W) satisfies formally the first Bianchi identity of a Riemannian curvature tensor.

Lemma 23. *Let θ be a (TSPH)-structure on (M,H,J). Then the Webster curvature tensor R^W satisfies*

$$R^W(X,Y,Z,V)+R^W(Y,Z,X,V)+R^W(Z,X,Y,V)=0$$

for all $X,Y,Z,V\in TM$. In particular, we have the symmetries

$$\begin{aligned} R^W(X,Y,Z,V) &= R^W(Z,V,X,Y) \qquad \textit{and} \\ R^W(X,JY,JZ,V) &= R^W(JX,Y,Z,JV)\ . \end{aligned}$$

Also note that under the condition of transverse symmetry we have

$$\Omega^W \;=\; dA^W \;=\; -\pi^* Ric^W$$

(cf. Section 5.3).

Using the symmetry properties for the Webster curvature tensor stated in Lemma 23 in case of transverse symmetry, we compute the comparison between the Ricci tensor of g_θ and the Webster-Ricci tensor. For this purpose, let $\{e_i : i=1,\dots,2m\}$ denote a local J-adapted frame of H in TM (cf. Section 0.10). We have

$$Ric^{g_\theta}(X,Y)=R^{g_\theta}(X,T,T,Y)+\sum_{i=1}^{2m}\varepsilon_i R^{g_\theta}(X,e_i,e_i,Y)$$

and

$$\begin{aligned} Ric^W(X,Y) &= \quad i\sum_\alpha \varepsilon_{2\alpha-1}R^W(e_{2\alpha-1},Je_{2\alpha-1},X,Y) \\ &= \quad i\sum_\alpha \varepsilon_{2\alpha-1}R^W(X,Je_{2\alpha-1},e_{2\alpha-1},Y) \\ &\quad\; - i\sum_\alpha \varepsilon_{2\alpha-1}R^W(X,e_{2\alpha-1},Je_{2\alpha-1},Y) \\ &= \quad i\sum_\alpha \varepsilon_{2\alpha-1}R^W(X,Je_{2\alpha-1},Je_{2\alpha-1},JY) \\ &\quad\; + i\sum_\alpha \varepsilon_{2\alpha-1}R^W(X,e_{2\alpha-1},e_{2\alpha-1},JY) \\ &= \quad i\sum_i \varepsilon_i R^W(X,e_i,e_i,JY) \end{aligned}$$

for all $X,Y\in TM$. With the comparison formula for the curvature tensors R^{g_θ} and R^W we obtain $R^{g_\theta}(X,T)T=\frac{1}{4}X$ and

$$\sum_i \varepsilon_i R^{\nabla^W}(X,e_i)e_i = Ric^{g_\theta}(X)-R^{g_\theta}(X,T)T+\frac{3}{4}X$$

for all $X \in H$. These formulae combined with the fact that T inserted into R^W is zero result in

$$Ric^{g_\theta}(X,Y) = iRic^W(X,JY) - \tfrac{1}{2}g_\theta(X,Y),$$
$$Ric^W(T,X) = 0, \qquad Ric^W(T,T) = 0$$

and

$$Ric^{g_\theta}(T,X) = 0 \ , \qquad Ric^{g_\theta}(T,T) = \frac{m}{2}g_\theta(T,T) \ ,$$

where $X, Y \in H$.

4. The (Local) Submersion to a (TSPH)-Structure

We assume here that θ is a (TSPH)-structure on the integrable CR-manifold (M,H,J) of dimension $n = 2m+1$. This implies that the Reeb vector T to θ is Killing for the induced metric g_θ. At least locally, we can factorise through the integral curves of T on M and obtain a semi-Riemannian metric h on a quotient space, which has dimension $2m$. We describe this process in detail. In particular, we calculate the relation for the Ricci curvatures of the induced metric g_θ and the metric h on the quotient.

Let θ be a (TSPH)-structure on (M,H,J) of signature (p,q). To every point in $x \in M$ exists a neighbourhood (e.g. some small ball) $U \subset M$ and a map ϕ_U such that ϕ_U is a diffeomorphism between U and $\mathbb{R}^n$ and $d\phi_U(T) = \frac{\partial}{\partial x_1}$, which is the first standard coordinate vector in $\mathbb{R}^n$. This implies that there exists a smooth submersion

$$\pi_U : \ U \subset M \ \rightarrow \ N \subset \mathbb{R}^{2m}$$

such that for all $v \in N$ the preimage $\pi_U^{-1}(v)$ consists of an integral curve of T through some point in U parametrised by an interval in $\mathbb{R}$.

The distribution H in TU is orthogonal to T with respect to g_θ and the projection π_{U*} restricted to H has no kernel. For any vector X in TN we denote by X^* the unique lift of X to TU, which is tangential to H. Then we define

$$h(X,Y) := g_\theta(X^*,Y^*) = L_\theta(X^*,Y^*)$$

for arbitrary tangent vectors X, Y at some point of N. This defines in a unique way a smooth metric tensor of signature $(2p,2q)$ on N and the projection π_U becomes a Riemannian submersion with respect to g_θ and h. In particular, the distribution H in TU is horizontal for this Riemannian submersion. The construction itself is naturally derived from θ only (and some chosen neighbourhood U).

For simplicity, we assume now that

$$\pi : (M, g_\theta) \rightarrow (N,h)$$

is globally a smooth Riemannian submersion, where the preimages are the integral curves of the Reeb vector T to a (TSPH)-structure θ on M with CR-structure (H,J). Since the complex structure J acts on H and T is an infinitesimal automorphism of J, the complex structure can be uniquely projected to a smooth endomorphism on N, which we also denote by J and which satisfies $J^2 = -id|_{TN}$. Since J is integrable on H, the endomorphism J is integrable on N as well, i.e., J is a complex structure on N. In fact, J is a Kähler structure on (N,h), i.e.,

$$\nabla^h J = 0 \ .$$

The latter fact can be seen with the comparison tensor D^θ. We have

$$\begin{aligned}(\nabla^{g_\theta}_{X^*}J)(Y^*) &= \nabla^{g_\theta}_{X^*}(JY^*) - J\nabla^{g_\theta}_{X^*}Y^* \\ &= \nabla^{W}_{X^*}(JY^*) - (J\nabla^{W}_{X^*}Y^*) - \frac{1}{2}d\theta(X^*, J(Y^*))\cdot T \\ &= -\frac{1}{2}g_\theta(X^*,Y^*)\cdot T\end{aligned}$$

and

$$Vert_\pi \nabla^{g_\theta}_{X^*}(J(Y^*)) = -\frac{1}{2}g_\theta(Y^*,X^*)\cdot T\ .$$

Together with $\nabla^h \circ \pi_* = \pi_* \circ \nabla^{g_\theta}$ this implies $\nabla^h J = 0$ on N.

Altogether, we know yet that a (TSPH)-space (M,H,J,θ) gives rise (locally) in a natural manner to a $(2m)$-dimensional Kähler space (N,h,J). We use now the standard formulae for the Ricci tensor of a Riemannian submersion to calculate Ric^h (cf. [**127**]). We obtain

$$Ric^h(X,Y) = Ric^{g_\theta}(X^*,Y^*) + \frac{1}{2}g_\theta(X^*,Y^*)$$

for all $X,Y \in TN$. Using our formula for the Ricci tensor of g_θ with respect to the Webster-Ricci curvature, we find

$$Ric^h(X,Y) = iRic^W(X^*,JY^*)$$

for all $X,Y \in TN$. Basically, this result says that the Webster-Ricci curvature of a (TSPH)-structure is the Ricci curvature of the base space of the natural submersion.

5. Description and Construction of TS-Pseudo-Einstein Spaces

We explain here an explicit construction of TS-pseudo-Einstein spaces with arbitrary Webster scalar curvature. We also show that locally this construction principle generates all TS-pseudo-Einstein structures, i.e., we obtain a locally complete picture.

Let (M,H,J,θ) be a TS-pseudo-Einstein space with arbitrary signature (p,q), i.e.,

$$Ric^W = -i\frac{scal^W}{m}\cdot d\theta \qquad \text{and} \qquad Tor^W(T,X) = 0$$

for all $X \in H$. Moreover, we assume for simplicity that θ generates globally a smooth Riemannian submersion

$$\pi : (M, g_\theta) \to (N,h)\ .$$

With the relation for the Ricci tensors from the end of the last section we have

$$\pi^* Ric^h = \frac{scal^W}{m}d\theta(\cdot, J\cdot) = \frac{scal^W}{m}\pi^* h\ .$$

This shows that the base space of the natural submersion to the (TSPH)-structure θ is a Kähler-Einstein space of scalar curvature

$$scal^h = 2\cdot scal^W\ .$$

We conclude that a TS-pseudo-Einstein space (M,H,J,θ) of dimension $n = 2m+1$ determines uniquely (at least locally) a Kähler-Einstein manifold (N,h,J) of dimension $2m$ and signature $(2p,2q)$.

We want to show now that there is an inverse construction, which assigns to any Kähler-Einstein metric (with signature $(2p, 2q)$) a uniquely determined pseudo-Hermitian structure (which is then TS-pseudo-Einstein). The construction itself is natural and unique up to a gauge. In fact, it is straightforward to check that the resulting pseudo-Hermitian structures for different gauges are isomorphic.

To start with, let (N^{2m}, h, J) be a Kähler-Einstein space of dimension $2m$ with $scal^h \neq 0$ and let $P(N)$ be the $\mathrm{U}(p,q)$-reduction of the orthonormal frame bundle on (N, h). Then

$$\mathcal{O}_{ac}(N) := P(N) \times_{det_{\mathbb{C}}} S^1$$

with projection $\pi_{\mathcal{O}_{ac}(N)}$ is the principal S^1-fibre bundle over N, which is associated to the anti-canonical complex line bundle $\mathcal{O}(-1)$ of the Kähler manifold (N, h, J). The Levi-Civita connection to h induces a connection form ρ_{ac} on the anti-canonical S^1-bundle $\mathcal{O}_{ac}(N)$ with values in $i\mathbb{R}$. For its curvature we have

$$\Omega^{\rho_{ac}}(X^*, Y^*) = iRic^h(X, JY) ,$$

where $X, Y \in TN$ and X^*, Y^* are their horizontal lifts with respect to ρ_{ac}.

At first, we see from the latter formula that the horizontal spaces of $(\mathcal{O}_{ac}(N), \rho_{ac})$ generate a contact distribution H of corank 1 in $T\mathcal{O}_{ac}(N)$ and the horizontal lift of the complex structure J to H produces a non-degenerate CR-structure (H, J) on $\mathcal{O}_{ac}(N)$. This CR-structure is integrable as can be seen from the relation

$$\Omega^{\rho_{ac}}(X^*, JY^*) + \Omega^{\rho_{ac}}(JX^*, Y^*) = 0$$

for all $X, Y \in TN$ and the fact that the Nijenhuis tensor $\mathcal{N}(X^*, Y^*)$ is the horizontal lift of

$$J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] = 0 .$$

Secondly, we see that

$$\theta := i\frac{2m}{scal^h}\rho_{ac}$$

is a pseudo-Hermitian structure on $M := \mathcal{O}_{ac}(N)$ furnished with the CR-structure (H, J). The Reeb vector T on the pseudo-Hermitian space

$$(\mathcal{O}_{ac}(N), H, J, \theta)$$

is vertical along the fibres (in fact, it is a fundamental vector field generated by the S^1-action on the fibres) and by construction of (H, J) a transverse symmetry. Since $d\theta = \pi^*_{\mathcal{O}_{ac}(N)} h(J\cdot, \cdot)$ on H, the base space of the corresponding submersion is again the Kähler-Einstein space (N, h, J) that we started with. For that reason, we know that the Webster-Ricci curvature of θ must be given by

$$iRic^W(X^*, JY^*) = Ric^h(X, Y), \qquad X, Y \in H .$$

Since h is Einstein, we can conclude that the pseudo-Hermitian space $(\mathcal{O}_{ac}(N), H, J, \theta)$ is TS-pseudo-Einstein with Webster-Ricci curvature

$$Ric^W = -i\frac{scal^h}{2m} \cdot d\theta .$$

As mentioned before, for the inverse construction on the Kähler-Einstein space (N, h, J), the choice of $\theta = i\frac{2m}{scal^h}\rho_{ac}$ as pseudo-Hermitian 1-form is not unique. One might replace θ by $\hat{\theta} := \theta + df$ for some smooth function f on $\mathcal{O}_{ac}(N)$ with $T(f) \neq -\theta(T)$, where T is vertical. The latter condition ensures that $\hat{\theta}$ is *transverse* with respect to the fibration, which makes it possible to lift the complex structure to the kernel of $\hat{\theta}$. We obtain again a TS-pseudo-Einstein structure on $\mathcal{O}_{ac}(N)$ with induced CR-structure. It is straightforward to see that there exits a diffeomorphism (gauge transformation) on $\mathcal{O}_{ac}(N)$, which transforms $\theta+df$ into θ, i.e., there is an isomorphism of pseudo-Hermitian structures. Since (locally) $\theta + df$ is the most general choice of a *transverse* 1-form whose exterior differential is the lift of $h(J\cdot,\cdot)$ on N, we know that our gauged construction exhausts locally all (isomorphism classes of) TS-pseudo-Einstein structures with non-zero Webster scalar curvature.

As we have seen above a Webster-Ricci flat TS-pseudo-Einstein space (M, H, J, θ) gives rise to a Ricci flat Kähler space. Again we aim to find a reconstruction. So let (N, h, J) be a Ricci flat Kähler space furnished with a 1-form γ such that $d\gamma = h(\cdot, J\cdot)$, i.e., $\omega := d\gamma$ is the Kähler form. The S^1-principal fibre bundle $\mathcal{O}_{ac}(N)$ has a Levi-Civita connection form ρ_{ac} with values in $i\mathbb{R}$ which is flat, i.e., $d\rho_{ac} = 0$. We set

$$\theta := i\rho_{ac} - \pi^*\gamma$$

on $\mathcal{O}_{ac}(N)$. Obviously, it holds that

$$d\theta = -\pi^*\omega \ ,$$

i.e., θ is a contact form on $\mathcal{O}_{ac}(N)$ and the distribution H in $T\mathcal{O}_{ac}(N)$, which is defined to be the kernel of θ, is contact as well. By definition, the distribution H is transverse to the vertical direction of the fibre. For that reason we can lift J to H. Again, the CR-structure (H, J) on $M := \mathcal{O}_{ac}(N)$ is integrable. Moreover, θ is a pseudo-Hermitian structure on $(\mathcal{O}_{ac}(N), H, J)$. As the construction is done, it is clear that locally around every point of $(\mathcal{O}_{ac}(N), g_\theta)$ the base of the natural Riemannian submersion is naturally identified with a subset of the Ricci flat space (N, h, J). We conclude that

$$iRic^W(X, Y) = Ric^h(X, Y) = 0$$

for all $X, Y \in H$, i.e., the pseudo-Hermitian space

$$(\mathcal{O}_{ac}(N), H, J, \theta)$$

over a Ricci flat Kähler space (N, h, J) with Kähler form $d\gamma$, where $\theta = i\rho_{ac} - \pi^*\gamma$, is TS-pseudo-Einstein with $scal^W = 0$.

In the Webster-Ricci flat construction, the pseudo-Hermitian form θ can be replaced by $\hat{\theta} = i\rho_{ac} - \pi^*\gamma + df$, where f is some smooth function on $\mathcal{O}_{ac}(N)$ with $T(f) \neq -i\rho_{ac}(T)$ for any vertical vector T. This is the most general *transverse* 1-form on $\mathcal{O}_{ac}(N)$ with $d\hat{\theta} = -\pi^*\omega$. However, again one can see that θ and $\hat{\theta} = \theta + df$ are gauge equivalent on $\mathcal{O}_{ac}(N)$, i.e., they are isomorphic as pseudo-Hermitian structures. We conclude that with our construction and the help of a particular gauge we found (locally) the most general form of a Webster-Ricci flat TS-pseudo-Einstein space. We summarise our results.

Theorem 26. *Let (N, h, J) be a Kähler-Einstein space of dimension $2m$ and signature $(2p, 2q)$ with scalar curvature $scal^h$.*

(1) *If $scal^h \neq 0$ then the anti-canonical S^1-principal bundle*

$$\mathcal{O}_{ac}(N) = P(N) \times_{det_{\mathbb{C}}} S^1$$

with connection 1*-form*

$$\theta := i\frac{2m}{scal^h}\rho_{ac} \ ,$$

where ρ_{ac} *is the Levi-Civita connection to* h*, and with induced horizontal CR-structure* (H,J) *is a TS-pseudo-Einstein space with*

$$scal^W = \frac{1}{2}scal^h \neq 0 \ .$$

(2) *If* $scal^h = 0$ *and the Kähler form is* $\omega = d\gamma$ *for some* 1*-form* γ *on* N *then* $(\mathcal{O}_{ac}(N), H, J)$ *with (TSPH)-structure* $\theta = i\rho_{ac} - \pi^*\gamma$ *is Webster-Ricci flat.*

Locally, any TS-pseudo-Einstein space (M,H,J,θ) *is isomorphic to one of these two models depending on the Webster scalar curvature* $scal^W$*.*

We remark that for the case $scal^h \neq 0$ we could have chosen (locally) the gauge $\theta = i\rho_{ac} + \pi^*\eta - \pi^*\gamma$, where $d\gamma$ is the Kähler form and $d\eta$ the Ricci form. This would enable us to treat the two cases of Theorem 26 simultaneously as a single case. However, for the following discussion of the corresponding Fefferman spaces we consider the chosen gauge of Theorem 26(1) as more convenient.

6. The Fefferman Construction under Transverse Symmetry

We specialise now the classical Fefferman construction due to J.M. Lee (cf. Section 0.12 and e.g. [**102**]) to our case of TS-pseudo-Einstein spaces. This construction comes (locally) as torus bundle over a Kähler-Einstein space.

In general, for a pseudo-Hermitian space (M,H,J,θ) of dimension $n = 2m+1$ the Fefferman metric is given on the canonical S^1-principal fibre bundle (F_c, π_M, M) by

$$\pi_M^* L_\theta \ - \ i\frac{4}{m+2}\pi_M^*\theta \circ A_\theta \ .$$

Now let (M,H,J,θ) be a TS-pseudo-Einstein space given as the anti-canonical S^1-principal fibre bundle $M = \mathcal{O}_{ac}(N)$ over a Kähler-Einstein space (N,h,J) equipped with the naturally induced CR-structure (H,J) and pseudo-Hermitian form θ as described in Theorem 26:

$$\pi_N^{ac} : (\mathcal{O}_{ac}(N), H, J, \theta) \ \to \ (N,h,J) \ .$$

We denote by

$$\mathcal{O}_c(N) := P(N) \times_{det_{\mathbb{C}}^{-1}} S^1$$

the canonical S^1-principal fibre bundle over (N,h,J), which is equipped with the Levi-Civita connection denoted by ρ_c. Further, let F_c be the total space of the canonical S^1-fibre bundle over the CR-manifold $(\mathcal{O}_{ac}(N), H, J)$. We denote by π the projection of F_c to N:

$$\pi : F_c \to N \ .$$

Obviously, the pull-back of $\mathcal{O}_c(N)$ along the anti-canonical projection π_N^{ac} is isomorphic to F_c. The concrete isomorphism is given by the choice of the gauge θ. This shows that we can understand the total space F_c as a torus bundle over (N,h,J):

$$F_c = P(N) \times_{(det_{\mathbb{C}}, det_{\mathbb{C}}^{-1})} S^1 \times S^1 \ .$$

The Tanaka-Webster and Weyl connection forms are given on the torus bundle F_c over the Kähler-Einstein space (N, h, J) by

$$A^W = \pi_c^*\rho_c \qquad \text{and} \qquad A_\theta = \pi_c^*\rho_c - \frac{i \cdot scal^W}{2(m+1)}\pi_{ac}^*\theta \ ,$$

where $\pi_c : F_c \to \mathcal{O}_c(N)$ and $\pi_{ac} : F_c \to \mathcal{O}_{ac}(N)$ are the natural projections. The Fefferman metric f_θ to the TS-pseudo-Einstein space $(\mathcal{O}_{ac}(N), H, J, \theta)$ on the torus bundle F_c (over N) is then given by

$$f_\theta = \pi^* h - i\frac{4}{m+2}\pi_{ac}^*\theta \circ \left(\pi_c^*\rho_c - i\frac{scal^W}{2(m+1)}\pi_{ac}^*\theta\right) \ .$$

Notice that the metric f_θ is uniquely derived from (N, h, J) if we assume θ to be given in the gauge of Theorem 26. We will omit in the following the subscripts for the various projections. It will be clear from the context which projection is meant.

Definition 5. *Let (N, h, J) be a Kähler-Einstein space and let*

$$F_c = P(N) \times_{(det_{\mathbb{C}}, det_{\mathbb{C}}^{-1})} S^1 \times S^1$$

be the canonical-anti-canonical torus bundle over N. Then we denote by $f_h := f_\theta$ (where θ is the gauge of Theorem 26 depending on whether $scal^h = 0$ or $\neq 0$) the Fefferman metric on F_c, which belongs to the TS-pseudo-Einstein space $(\mathcal{O}_{ac}(N), H, J, \theta)$. We call f_h the Fefferman metric of the Kähler-Einstein space (N, h, J).

In general, since θ is a transverse symmetry, we have

$$d(\pi^*\rho_c) + d(\pi^*\rho_{ac}) = \pi^* Ric^W - \pi^* Ric^W = 0 \ ,$$

i.e., the 1-form $\pi^*\rho_c + \pi^*\rho_{ac}$ is closed on F_c. In fact, we will see in the next section that this 1-form is parallel in case of a TS-pseudo-Einstein structure with $scal^W \neq 0$. A f_h-orthogonal 1-form to $\pi^*\rho_c + \pi^*\rho_{ac}$ is given by $\pi^*\rho_c - \frac{1}{m+1}\pi^*\rho_{ac}$ and we can rewrite the Fefferman metric from the previous expression as

$$f_h = \pi^* h + \frac{4m(m+1)}{(m+2)^2 \cdot scal^h} \cdot \left(\ (\pi^*\rho_c + \pi^*\rho_{ac})^2 - (\pi^*\rho_c - \frac{1}{m+1}\pi^*\rho_{ac})^2 \ \right) \ .$$

If $scal^h = 0$ the Fefferman metric of (N, h, J) is computed to be

$$f_h = \pi^* h - i\frac{4}{m+2}(i\pi^*\rho_{ac} - \pi^*\gamma) \circ \pi^*\rho_c \ ,$$

where $d\gamma = \omega$ is the Kähler form and the 1-form $\pi^*\rho_c$ is closed (in fact, parallel).

7. The Einstein Metric in the Fefferman Conformal Class

Finally, we find an explicit local scale for an Einstein metric in the conformal class of any Fefferman metric, which comes from a TS-pseudo-Einstein space. For this purpose, we compute a convenient formula for the Ricci tensor of f_h. The result shows how to choose the conformal rescaling factor.

Let (M, H, J, θ) be a TS-pseudo-Einstein space. As before, we assume that $M = \mathcal{O}_{ac}(N)$ is the anti-canonical S^1-bundle over a Kähler-Einstein space (N, h, J) and θ is a gauge as in Theorem 26. The Fefferman metric $f_h = f_\theta$ is defined on the torus bundle $F_c = P(N) \times_{(det_{\mathbb{C}}, det_{\mathbb{C}}^{-1})} S^1 \times S^1$. We denote by χ_K the fundamental vector field along the Fefferman fibration determined by $A_\theta(\chi_K) = \frac{m+2}{2} i$ and by T^* the horizontal lift of the Reeb vector $T \in \mathfrak{X}(M)$ to F_c with respect to A_θ. Furthermore, let

$$\{e_i : i = 1, \dots, 2m\}$$

denote some local orthonormal frame on (N, h) and let the e_i^*'s be the horizontal lifts to F_c with respect to θ and then A_θ. We obtain a local frame on F_c of the form

$$\{e_i^*, T^*, \chi_K\} \ .$$

With our definitions, we have

$$\begin{aligned}
&f_h(\chi_K, T^*) = 1 \ , \\
&\pi^*\rho_c(\chi_K) = \tfrac{m+2}{2} i \ , \quad \pi^*\rho_{ac}(\chi_K) = 0 \ , \\
&\pi^*\theta(T^*) = 1 \ , \qquad \pi^*\rho_c(T^*) = 0 \ , \\
&\pi_*(e_i^*) = e_i \ , \qquad \pi^*\rho_{ac}(e_i^*) = \pi^*\rho_c(e_i^*) = 0 \qquad \text{and} \\
&[T^*, e_i^*] = [\chi_K, e_i^*] = [\chi_K, T^*] = 0 \qquad \text{for all } \ i \in 1, \dots, 2m \ .
\end{aligned}$$

In the following, local computations are made with respect to a frame of the form $\{e_i^*, T^*, \chi_K\}$.

From the formulae of Lemma 19 for covariant derivatives (with respect to the Levi-Civita connection of $f_h = f_\theta$) combined with the fact that

$$dA_\theta = \Omega_\theta = -\pi^* Ric^W - i \frac{scal^W}{2(m+1)} \pi^* d\theta = i \frac{(m+2) \cdot scal^h}{4m(m+1)} \cdot \pi^* d\theta$$

in the TS-pseudo-Einstein case we obtain

$$\begin{aligned}
&\nabla^{f_h}_{e_i^*} e_j^* = (\nabla^W_{e_i} e_j)^* - \tfrac{1}{2}\pi^* d\theta(e_i^*, e_j^*) T^* - \tfrac{1}{2} S^W \pi^* d\theta(e_i^*, e_j^*) \chi_K \ , \\
&\nabla^{f_h}_{T^*} e_i^* = \nabla^{f_h}_{e_i^*} T^* = \tfrac{1}{2} S^W (J e_i)^* \ , \\
&\nabla^{f_h}_{\chi_K} e_i^* = \nabla^{f_h}_{e_i^*} \chi_K = \tfrac{1}{2} (J e_i)^* \ , \\
&\nabla^{f_h}_{\chi_K} T^* = \nabla^{f_h}_{T^*} \chi_K = \nabla^{f_h}_{T^*} T^* = \nabla^{f_h}_{\chi_K} \chi_K = 0 \ ,
\end{aligned}$$

where we set

$$S^W := \frac{scal^h}{2m(m+1)} \ .$$

It follows immediately that

$$f_h(\nabla^{f_h}_A T^*, B) = -f_h(\nabla^{f_h}_B T^*, A)$$

for all $A, B \in \mathfrak{X}(F_c)$, i.e., T^* is a Killing vector field on (F_c, f_h). (In general, for any pseudo-Hermitian space the horizontal lift of the Reeb vector T is a Killing vector on the Fefferman space if and only if T is a transverse symmetry and $\Omega^{A_\theta}(T^*, \cdot) = 0$.) Moreover, we see that the vertical vector field

$$T^* - S^W \chi_K$$

is parallel. The dual of this vector field with respect to f_h is a parallel 1-form, which is equal to $-i\frac{2}{m+2} \cdot (\pi^*\rho_c + \pi^*\rho_{ac})$ for $scal^h \neq 0$, resp., $-i\frac{2}{m+2} \cdot \pi^*\rho_c$ for $scal^h = 0$ (cf. Section 6).

For the Riemannian curvature tensor of f_h we find the formulae

$$\begin{aligned}
&R^{f_h}(e_i^*,e_j^*)e_j^* = (R^{\nabla^W}(e_i,e_j)e_j)^* + \tfrac{3}{2}S^W d\theta(e_i,e_j)(Je_j)^* \ ,\\
&R^{f_h}(e_i^*,\chi_K)T^* = \tfrac{1}{4}S^W\cdot e_i^* \ ,\\
&R^{f_h}(T^*,e_j^*)e_j^* = \tfrac{1}{4}S^W\cdot (T^*+S^W\cdot\chi_K) \ ,\\
&R^{f_h}(\chi_K,e_j^*)e_j^* = \tfrac{1}{4}(T^*+S^W\cdot\chi_K) \ ,\\
&R^{f_h}(\chi_K,T^*) = 0 \ .
\end{aligned}$$

Then we obtain for the Ricci tensor

$$\begin{aligned}
&Ric^{f_h}(e_i^*,e_j^*) = iRic^W(e_i,Je_j) - S^W f_h(e_i^*,e_j^*) \ ,\\
&Ric^{f_h}(T,e_i^*) = Ric^{f_h}(\chi_K,e_i^*) = 0 \ ,\\
&Ric^{f_h}(T^*,T^*) = \tfrac{m}{2}(S^W)^2 \ ,\\
&Ric^{f_h}(T^*,\chi_K) = \tfrac{m}{2}S^W \ ,\\
&Ric^{f_h}(\chi_K,\chi_K) = \tfrac{m}{2} \ ,
\end{aligned}$$

i.e., the Ricci tensor of f_h takes the form

$$\begin{aligned}
Ric^{f_h} \quad &= \quad i\pi^* Ric^W(\cdot,J\cdot) - S^W\pi^* f_h\\
&+ \ \frac{m}{2}\left((S^W)^2\pi^*\theta\circ\pi^*\theta - \frac{4}{(m+2)^2}A_\theta\circ A_\theta - i\frac{4}{m+2}S^W A_\theta\circ\pi^*\theta\right)\\
&= \quad \frac{scal^h}{2(m+1)}f_h - \frac{2m}{(m+2)^2}\left(A_\theta - \frac{i(m+2)\cdot scal^h}{4m(m+1)}\pi^*\theta\right)^2 \ .
\end{aligned}$$

In particular, if $scal^h = 0$ on N we have

$$Ric^{f_h} = -\frac{2m}{(m+2)^2}(\pi^*\rho_c)^2$$

and if $scal^h \neq 0$ then

$$Ric^{f_h} = \frac{scal^h}{2(m+1)}f_h - \frac{2m}{(m+2)^2}(\pi^*\rho_c + \pi^*\rho_{ac})^2 \ .$$

The computations show that the Fefferman metric f_h to a Kähler-Einstein space (N,h,J) is never Einstein. (In fact, any Fefferman metric is not Einstein (cf. [**102**]).) For example, a Webster-Ricci flat TS-pseudo-Einstein space gives rise to a Fefferman metric f_h with totally isotropic Ricci tensor. However, the Einstein condition should not be expected for the Fefferman metric itself. Instead, we will show now that the Fefferman metric to any TS-pseudo-Einstein space (resp. Kähler-Einstein space) is (locally) conformally Einstein, i.e., there exists locally around every point of F_c a conformally rescaled metric $\tilde{f}_h$ to f_h, which is Einstein. For the computation of the explicit Einstein scale we introduce the coordinate function t on the torus fibre bundle F_c by

$$\begin{aligned}
&dt = i\pi^*\rho_c &&\text{when} \quad scal^h = 0 \quad \text{and}\\
&dt = i\pi^*\rho_c + i\pi^*\rho_{ac} &&\text{when} \quad scal^h \neq 0 \ .
\end{aligned}$$

First, we consider the (Webster)-Ricci flat case. So let $\tilde{f}_h = e^{2\phi} f_h$ be a conformally rescaled metric of f_h, where ϕ is a real function on F_c. For the Ricci tensor of $\tilde{f}_h$ we find by using standard transformation formulae

$$Ric^{\tilde{f}_h} - Ric^{f_h} = -2m(Hess^{f_h}(\phi) - d\phi \circ d\phi) + (-\Delta^{f_h}\phi - 2m\|d\phi\|^2_{f_h}) f_h$$

(cf. e.g. [**93**]). We denote this difference by C_ϕ, which is a symmetric $(0,2)$-tensor. Now let $\phi(t)$ be a function on F_c, which depends only on the coordinate t in the direction of the canonical S^1-fibration. Then the only non-trivial component of C_ϕ is

$$C_\phi(\chi_K, \chi_K) = -2m\left(\ \chi_K\chi_K(\phi) - \chi_K(\phi)^2\ \right) \ .$$

This shows that for any function $\phi(t)$ on F_c, which solves the ODE

$$\partial_t\partial_t\phi - (\partial_t\phi)^2 = \frac{1}{(m+2)^2} \ ,$$

the Ricci tensor $Ric^{\tilde{f}_h}$ of $\tilde{f}_h = e^{2\phi} f_h$ vanishes identically. The most general solution of this ODE is

$$\phi = c_1 - \ln\left(\cos\left(\frac{t}{m+2} + c_2\right)\right),$$

where c_1, c_2 are constants. We choose here $\phi = -\ln(\cos(\frac{t}{m+2}))$, which gives as conformal rescaling factor

$$e^{2\phi} = \cos^{-2}\left(\frac{t}{m+2}\right) \ .$$

Then the conformally changed Fefferman metric

$$\tilde{f}_h = \cos^{-2}(t/(m+2)) \cdot \left(\pi^*h - i\frac{4}{m+2}(i\pi^*\rho_{ac} - \pi^*\gamma) \circ \pi^*\rho_c\right)$$

is Ricci flat on an open subset of F_c around the hypersurface given by $\{t = 0\}$. Obviously, a global conformal Einstein scale for f_h on F_c does not exist. Locally, it does exist everywhere.

Now we assume that (N, h, J) is Kähler-Einstein with $scal^h \neq 0$. Then we find with respect to a conformal scaling function $\phi(t)$, which depends only on the coordinate t with $dt = i(\pi^*\rho_c + \pi^*\rho_{ac})$,

$$\begin{aligned} Ric^{\tilde{f}_h} - Ric^{f_h} \ = \ C_\phi \ = \ & \quad 2m(\partial_t\partial_t\phi - (\partial_t\phi)^2)(d\rho_c + d\rho_{ac})^2 \\ & + \ \frac{(m+2)^2 \cdot scal^h}{4m(m+1)}(\partial_t\partial_t\phi - (\partial_t\phi)^2) f_h \ . \end{aligned}$$

Again, if we choose $\phi = -\ln(\cos(\frac{t}{m+2}))$, the metric $\tilde{f}_h = e^{2\phi} f_h$ is Einstein. In fact, with this choice we obtain

$$Ric^{\tilde{f}_h} = \frac{(2m+1) \cdot scal^h}{4m(m+1)} f_h$$

for the Ricci tensor of the rescaled metric and $scal^{\tilde{f}_h} = \frac{2m+1}{2m} \cdot scal^h$ for the scalar curvature.

Theorem 27. *Let (N, h, J) be a Kähler-Einstein space of dimension $2m$ and signature $(2p, 2q)$ with scalar curvature $scal^h$.*

(1) *If* $scal^h = 0$ *and the Kähler form is* $\omega = d\gamma$ *for some* 1*-form* γ *on* N *then the metric*

$$\tilde{f}_h = \cos^{-2}(t) \cdot (\ \pi^* h + 4dt \circ (\pi^*\gamma + ds)\ )$$

on $N \times \{\ (s,t)\ :\ -\frac{\pi}{2} < t < \frac{\pi}{2}\ \} \subset N \times \mathbb{R}^2$ *(with natural projection* π *onto* N*) is conformally Fefferman and Ricci flat with signature* $(2p+1, 2q+1)$.

(2) *If* $scal^h \neq 0$ *then the metric*

$$\tilde{f}_h = \cos^{-2}(t) \cdot \left(\ \pi^* h - \frac{4m(m+1)}{scal^h} \cdot (\ dt^2 + \frac{\rho_{ac}^2}{(m+1)^2})\right)$$

on $\mathcal{O}_{ac}(N) \times (-\frac{\pi}{2}, \frac{\pi}{2})$*, where* $(\mathcal{O}_{ac}(N), \pi, N)$ *is the anti-canonical* S^1*-bundle over* N *with Levi-Civita connection* $\rho_{ac} : T\mathcal{O}_{ac}(N) \to i\mathbb{R}$*, is conformally Fefferman and Einstein with* $scal^{\tilde{f}_h} = \frac{2m+1}{2m} \cdot scal^h$ *and signature* $(2p+1, 2q+1)$.

On the other hand, if a Fefferman metric f_θ *to an integrable CR-space is locally conformally Einstein, then any Einstein metric* $\tilde{f} \in [f_\theta]$ *can be brought into the form (1) or (2).*

In Theorem 27 we simplified the expressions for the Fefferman metrics. In fact, in the Ricci flat case both the Levi-Civita connections ρ_c and ρ_{ac} are flat, i.e., the torus bundle is globally a product and so we parametrised the vertical directions by the coordinates t, s, where the coordinate t is rescaled (compared with our notation from before) by a factor $(m+1)$. We remark that $\tilde{f}_h$ has the interesting property that there exists (locally) a conformal Killing spinor $\varphi \in \Gamma(\tilde{\mathcal{S}})$ such that $\tilde{D}^{\mathcal{S}}\varphi$ is a non-trivial parallel spinor.

For the case when $scal^h \neq 0$ we replaced the 1-form $\pi^*\rho_c - \frac{1}{m+1}\pi^*\rho_{ac}$ on F_c by $-\frac{m+2}{m+1} \cdot \rho_{ac}$ on $\mathcal{O}_{ac}(N) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. This is possible, since locally the canonical bundle $\mathcal{O}_c(N)$ and the anti-canonical bundle $\mathcal{O}_{ac}(N)$ can be identified such that the Levi-Civita connection ρ_c becomes $-\rho_{ac}$. It is useful to note here that the Fefferman metric $f_h = \cos^2(t) \cdot \tilde{f}_h$ as presented in Theorem 27 is the product of a real line with the metric

$$\pi^* h - \frac{4m(m+1)}{(m+1)^2 \cdot scal^h} \cdot \rho_{ac}^2\ .$$

The latter is the well-known Einstein-Sasaki metric, which is constructed over the Kähler-Einstein space (N, h) (cf. Section 0.9).

The proof of the fact that any Einstein metric in a Fefferman conformal class can be brought into the form (1) or (2) of Theorem 27 uses the unique and natural correspondence of parallel standard tractors of CR and conformal geometry in the classical Fefferman construction for integrable CR-structures (cf. [**39**]). This is the most easy and elegant way to do this and, actually, reflects the initial idea to our construction. It is also possible to prove the (TSPE)-condition for the underlying CR-space directly without this argument of tractor calculus. However, we omit this here.

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