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Local Existence for Solutions of Fully Non-linear Wave Equations

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Let Ω be a domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}$ be given. We study the initial-boundary value problem for the equation

$$F(t, x, \bar{D}_x^{2m} u(t, x), \bar{D}_x^m \partial_t u(t, x), \partial_t^2 u(t, x)) = f(t, x)$$

with a homogeneous Dirichlet boundary condition; here u is a scalar function, $\bar{D}_x^m u := (\partial_x^\alpha u)_{|\alpha| \leqslant m}$ and certain restrictions are made on F guaranteeing that energy estimates are possible. We prove the existence of a value of $T > 0$ such that a unique classical solution u exists on $[0, T] \times \Omega$. Furthermore, we show that $T \to \infty$ if the data tend to zero.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}$ be given. For every sufficiently smooth function u defined on $[0, T] \times \Omega$ we set

$$\bar{D}_x^m u(t, x) := \left(\partial_x^\alpha u(t, x) = \frac{\partial^{|\alpha|} u(t, x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} : |\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant m \right),$$

$$\mathscr{D}u(t, x) := (\bar{D}_x^{2m} u(t, x), \bar{D}_x^m \partial_t u(t, x), \partial_t^2 u(t, x)).$$

In this paper we study for a scalar function u the mixed problem

$$F(t, x, \mathscr{D}u(t, x)) = f(t, x) \qquad \text{in } [0, T] \times \Omega, \tag{1.1}$$

$$\bar{D}_x^{m-1} u(t, x) = 0 \qquad \text{on } [0, T] \times \partial\Omega, \tag{1.2}$$

$$u(0, x) = u^0(x), \qquad \partial_t u(0, x) = u^1(x) \qquad \text{in } \Omega \tag{1.3}$$

under certain restrictions on F guaranteeing that (1.1) is a non-linear wave equation. For example,

$$F(t, x, \mathscr{D}u(t, x)) = \partial_t^2 u(t, x) - \sum_{|\alpha| \leqslant 2} a_\alpha(t, x, \bar{D}_x^2 u(t, x)) \partial_x^\alpha u(t, x), \tag{1.4}$$

where the coefficient a_α have to obey a special ellipticity condition (compare (2.16)).

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The restrictions made on F can be roughly described as follows. Differentiating (1.1) with respect to t and dividing the result by $\partial F/\partial(\partial_t^2 u)$ we obtain

$$\partial_t^3 u(t,x) + \mathscr{A}_u(t)\partial_t u(t,x) + \mathscr{B}_u(t)\partial_t^2 u(t,x) = g_u(t,x) \tag{1.5}$$

in $[0,T]\times\Omega$; here $\mathscr{A}_u(t)$ and $\mathscr{B}_u(t)$ denote families of spatial operators of order $2m$ and m, respectively, having coefficients depending on $(t, x, \mathscr{D}u(t,x))$, and g_u is suitably defined (compare (2.7)–(2.11)). We suppose that $\mathscr{A}_u(t)$ is uniformly strongly elliptic for every $t\in[0,T]$. Furthermore, we assume that there exists a (not necessarily positive) constant $c\in\mathbb{R}$ such that

$$\operatorname{Re}\langle \mathscr{B}_u(t)\varphi,\varphi\rangle \geqslant c\|\varphi\|^2 \quad \text{for } \varphi\in \mathring{H}^m(\Omega),\quad t\in[0,T], \tag{1.6}$$

where $\langle .,.\rangle$ and $\|.\|$ denote the inner product and the norm in $L_2(\Omega)$, and $\mathring{H}^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the mth Sobolev space $H^m(\Omega)$. We mention that (1.6) holds in the case $m=1$ if only F is supposed to be real valued. For details compare section 2 and especially Assumption 2.1.

We prove the existence of a $T>0$ such that (1.1)–(1.3) has a unique solution

$$u\in\mathscr{C}_T^k := \bigcap_{j=0}^{k} C^j([0,T], H^{(k-j)m}(\Omega)) \tag{1.7}$$

with $k\geqslant[n/2m]+6$ ($[r]:=\max\{k\in\mathbb{N}: k\leqslant r\}$). Then u is a classical solution (compare Lemma 5.1). For functions $u\in\mathscr{C}_T^k$ we understand (1.2) in the sense that

$$u(t) = u(t,.)\in\mathring{H}^m(\Omega) \quad \text{for } t\in[0,T]. \tag{1.8}$$

We require that the solution u of (1.1)–(1.3) satisfies

$$\sup_{[0,T]\times\bar\Omega} \mathscr{D}u(t,x)| \leqslant M \tag{1.9}$$

for a given $M>0$, since the properties of $\mathscr{A}_u$ and $\mathscr{B}_u$ mentioned above will only be supposed if u satisfies (1.9). In order to guarantee (1.9) we have to assume that the data (and F) are suitable.

Definition 1.1. *Let $u^0\in C^{2m}(\bar\Omega)$, $u^1\in C^m(\bar\Omega)$ and $f\in C([0,T]\times\bar\Omega)$. We say that $(u^0,u^1,f)\in\mathscr{I}$ if and only if there exists a $u^2\in C(\bar\Omega)$ such that*

$$F(0,x,\bar D_x^{2m}u^0(x), \bar D_x^m u^1(x), u^2(x)) = f(0,x) \quad \text{in } \Omega, \tag{1.10}$$

$$\sup_{x\in\bar\Omega}(|\bar D_x^{2m}u^0(x)| + |\bar D_x^m u^1(x)| + |u^2(x)|) \leqslant \tfrac{1}{2}M. \tag{1.11}$$

Now we can formulate the main theorem of this paper:

Theorem 1.1. *Let $k\geqslant k_0 := [n/2m]+6$ and $M>0$ be given. Assume that F satisfies Assumption 2.1 and that $u^0\in H^{km}(\Omega)$, $u^1\in H^{(k-1)m}(\Omega)$, $f\in\mathscr{C}_T^{k-2}\cap C^{k-1}([0,T],L_2(\Omega))$ such that $(u^0,u^1,f)\in\mathscr{I}$ satisfies the natural compatibility condition of order k (compare section 3). Then there exists a $T'\in(0,T]$ depending only on*

$$N := \|u^0\|_{k_0 m} + \|u^1\|_{(k_0-1)m} + \int_0^T \left(\sum_{j=1}^{k_0-2}\|\partial_t^j f(t)\|_{(k_0-2-j)m} + \|\partial_t^{k_0-1}f(t)\|\right)dt$$
$$+ \sup_{[0,T]}\left(\sum_{j=0}^{k_0-2}\|\partial_t^j f(t)\|_{(k_0-2-j)m}\right) \tag{1.12}$$

(and F,Ω,M) such that (1.1)–(1.3) *has a unique solution $u\in\mathscr{C}_{T'}^k$, which is a classical*

solution. If the assumptions are satisfied for every $k \geqslant k_0$, then

$$u \in C^\infty([0, T'] \times \bar{\Omega}). \tag{1.13}$$

Remark (1) The regularity conditions made on u^0 and u^1 are necessary conditions for a solution u to be in $\mathscr{C}_T^k$, as will be shown in section 3. Therefore, we can continue the solution u for $t > T'$ by Theorem 1.1 as long as $\sup_{x \in \bar{\Omega}} |\mathscr{D}u(t, x)| \leqslant \frac{1}{2}M$. If Assumption 2.1 is satisfied for every $M > 0$, then we can continue the solution until a blow-up occurs.

Remark (2) In addition to Theorem 1.1 we prove that

$$T' \geqslant O(\log\log N^{-1}) \quad \text{as } N \downarrow 0 \tag{1.14}$$

(compare section 7).

Remark (3) If problem (1.1)–(1.3) is quasilinear as in example 2 at the end of the section 2, it suffices to suppose that $k \geqslant k_0 - 1$ in Theorem 1.1. This can be shown by a modification of the proof of Lemma 6.1.

In the case $m = 1$, $k \geqslant [n/2] + 8$, Theorem 1.1 has already been proved by Shibatah and Tsutsumi [7]. In addition to generalization of their result to higher values of m we give a simpler proof, which makes possible the estimate (1.14).

The proof of Theorem 1.1 is based on the following idea: let $u \in \mathscr{C}_T^{k_0}$ be a solution of (1.1)–(1.3). Differentiating (1.1) with respect to t we obtain (1.5). We set $v := \partial_t u$ and conclude that

$$\left.\begin{array}{ll} \partial_t^2 v(t) + \mathscr{A}_u(t)v(t) + \mathscr{B}_u(t)\partial_t v(t) = g_u(t) & \text{for } t \in [0, T], \\ v(t) \in \mathring{H}^m(\Omega) & \text{for } t \in [0, T], \\ v(0) = u^1, \qquad \partial_t v(0) = u^2, & \end{array}\right\} \tag{1.15}$$

$$u(t) = u^0 + \int_0^t v(\tau)\, d\tau \qquad \text{for } t \in [0, T]. \tag{1.16}$$

Here we omit the variable x and understand the equations in $L_2(\Omega)$. Then (u, v) (sufficiently smooth) is a solution of (1.15) and (1.16) if and only if $v = \partial_t u$ and u solves (1.1)–(1.3).

We define a mapping $\Phi : u \mapsto \Phi[u]$ by

$$\Phi[u](t) := u^0 + \int_0^t v(\tau)\, d\tau \quad \text{for } t \in [0, T], \tag{1.17}$$

where v is the unique solution of (1.15). By an existence theorem for linear equations contained in [6] it follows that Φ is a mapping from

$$\tilde{\mathscr{C}}_T^k := \bigcap_{j=1}^{k} C^j([0, T], H^{(k-j)m}(\Omega)) \tag{1.18}$$

into itself. Note that $C([0, T], H^{km}(\Omega))$ does not contain $\tilde{\mathscr{C}}_T^k$, but $\mathscr{C}_T^k$. By energy estimates it follows that Φ is a contraction. This will give us the existence and uniqueness of a fixed point $u \in \tilde{\mathscr{C}}_{T'}^{k_0}$ of Φ for some $T' > 0$, which is a solution of (1.1)–(1.3). The higher regularity of u can be proved by induction in the same time interval $[0, T']$.

This method could also be applied to hyperbolic systems and to hyperbolic equations with non-linear boundary conditions. In a conference in March 1990 the author learned that H. Koch [5] uses the same idea to study equations of second order ($m = 1$) with non-linear boundary conditions.

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2. Notation and assumptions

By Ω we denote a domain in $\mathbb{R}^n$ having a smooth boundary $\partial\Omega \in C^\infty$ such that $\partial\Omega$ is bounded (or empty) or such that Ω has the form

$$\Omega = \mathbb{R}^{n'} \times \Omega' \tag{2.1}$$

with bounded $\Omega' \subset \mathbb{R}^{n-n'}$. The set $C_0^\infty(\Omega)$ consists of all the infinitely many differentiable functions having compact support in Ω. Futhermore, let

$$C_b^j(\Omega) := \{\varphi \in C^j(\Omega) : \partial_x^\alpha \varphi \text{ is bounded in } \Omega \text{ for } \alpha \in \mathbb{N}_0^n, |\alpha| \leq j\},$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. For every sufficiently smooth function φ we set

$$D_x^j \varphi := (\partial_x^\alpha \varphi : \alpha \in \mathbb{N}_0^n, |\alpha| = j), \qquad \bar{D}_x^j \varphi := (\partial_x^\alpha \varphi : \alpha \in \mathbb{N}_0^n, |\alpha| \leq j) \tag{2.2}$$

($j \in \mathbb{N}$). By $\langle .\,, . \rangle$ and $\|\cdot\|$ we denote the inner product and the norm in $L_2(\Omega)$, and $\|\cdot\|_j$ denotes the norm in the jth Sobolev space $H^j(\Omega)$.

If $u \in \mathscr{C}_T^k$ (compare (1.7)), then

$$|u(t)|_k := \sum_{j=0}^{k} \|\partial_t^j u(t)\|_{(k-j)m} \quad \text{for } t \in [0, T], \tag{2.3}$$

$$|||u|||_{k,T} := \sup_{[0,T]} |u(t)|_k. \tag{2.4}$$

For $u \in \tilde{\mathscr{C}}_T^k$ (compare (1.18)) we set

$$\begin{aligned} |u(t)|_k^\sim &:= \sum_{j=1}^{k} \|\partial_t^j u(t)\|_{(k-j)m} + \|u(t)\|_{(k-1)m} \\ &= |\partial_t u(t)|_{k-1} + \|u(t)\|_{(k-1)m} \quad \text{for } t \in [0, T], \end{aligned} \tag{2.5}$$

$$|||u|||_{k,T}^\sim := \sup_{[0,T]} |u(t)|_k^\sim. \tag{2.6}$$

In order to pose the assumptions to be made on F, we set $\mu := (\mu_0, \mu_1, \mu_2)$ with $\mu_0 = (\mu_0^{(\alpha)} : |\alpha| \leq 2m)$, $\mu_1 = (\mu_1^{(\beta)} : |\beta| \leq m)$ and $\mu_0^{(\alpha)}, \mu_1^{(\beta)}, \mu_2 \in \mathbb{C}$. This notation means that in the case $\mu = \mathscr{D}u(t, x)$ we have $\mu_0^{(\alpha)} = \partial_x^\alpha u(t, x)$, $\mu_1^{(\beta)} = \partial_x^\beta \partial_t u(t, x)$ and $\mu_2 = \partial_t^2 u(t, x)$. The operators $\mathscr{A}_u$ and $\mathscr{B}_u$ used in (1.5) and (1.15) are given by

$$\mathscr{A}_u(t) = \sum_{|\alpha| \leq 2m} a_\alpha(t, ., \mathscr{D}u(t, .))\, \partial_x^\alpha, \tag{2.7}$$

$$\mathscr{B}_u(t) = \sum_{|\beta| \leq m} b_\beta(t, ., \mathscr{D}u(t, .))\, \partial_x^\beta, \tag{2.8}$$

where

$$a_\alpha(t, x, \mu) = \left(\frac{\partial F}{\partial \mu_2}(t, x, \mu)\right)^{-1} \frac{\partial F}{\partial \mu_0^{(\alpha)}}(t, x, \mu), \tag{2.9}$$

$$b_\beta(t, x, \mu) = \left(\frac{\partial F}{\partial \mu_2}(t, x, \mu)\right)^{-1} \frac{\partial F}{\partial \mu_1^{(\beta)}}(t, x, \mu). \tag{2.10}$$

Furthermore, g_u used in (1.5) and (1.15) is given by

$$\left.\begin{aligned} g_u(t, x) &= g(t, x, \mathscr{D}u(t, x)), \\ g(t, x, \mu) &= \left(\frac{\partial F}{\partial \mu_2}(t, x, \mu)\right)^{-1} [\partial_t f(t, x) - (\partial_t F)(t, x, \mu)]. \end{aligned}\right\} \tag{2.11}$$

Note that $\mathscr{D}u(t, x)$ has

$$q(n, m) := \binom{2m+n}{n} + \binom{m+n}{n} + 1 \tag{2.12}$$

components. We impose the following conditions on F.

Assumption 2.1. *Let $k \in \mathbb{N}$ and $M, T > 0$ be given. Then*

(i) $F \in C_b^{(k-2)m+1}([0, T] \times \bar{\Omega} \times \{\mu \in \mathbb{R}^{q(n,m)} : |\mu| \leqslant M\})$.

(ii) $F(t, x, 0) = 0$ *in* $[0, T] \times \bar{\Omega}$.

(iii) $\partial F(t, x, \mu)/\partial \mu_2$ *is real valued and*

$$\frac{\partial F}{\partial \mu_2}(t, x, \mu) \geqslant d_1 > 0 \quad on\ [0, T] \times \bar{\Omega} \times \{\mu : |\mu| \leqslant M\}.$$

(iv) *There exists a $d_2 > 0$ such that*

$$(-1)^m \mathrm{Re} \sum_{|\alpha| = 2m} a_\alpha(t, x, \mu)\xi^\alpha \geqslant d_2 |\xi|^{2m} \quad on\ [0, T] \times \bar{\Omega} \times \{\mu : |\mu| \leqslant M\}$$

for every $\xi \in \mathbb{R}^n$ ($\xi^\alpha := \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$). Furthermore, for every $u \in \mathscr{C}_T^{k_1}$ ($k_1 := [n/2m] + 4$) there exists a $d_3 > 0$, which may depend on $|||u|||_{k_1, T}$, such that

$$\|\mathscr{A}_u(t) - \mathscr{A}_u^*(t))\varphi\| \leqslant d_3 \|\varphi\|_m \ for\ \varphi \in H^{2m}(\Omega) \cap \mathring{H}^m(\Omega),\ t \in [0, T]; \tag{2.13}$$

here $\mathscr{A}_u^*(t)$ denotes the adjoint operator to $\mathscr{A}_u(t)$ with $D(\mathscr{A}_u(t)) = H^{2m}(\Omega) \cap \mathring{H}^m(\Omega)$.

(v) *For every $u \in \mathscr{C}_T^{k_1}$ there exists a $d_4 \in \mathbb{R}$ (not necessarily positive), which may depend on $|||u|||_{k_1, T}$, such that*

$$\mathrm{Re}\langle \mathscr{B}_u(t)\varphi, \varphi\rangle \geqslant d_4 \|\varphi\|^2 \quad for\ \varphi \in \mathring{H}^m(\Omega), \quad t \in [0, T]. \tag{2.14}$$

Condition (2.13) means that the part of $\mathscr{A}_u$ containing the derivatives ∂_x^α with $m + 1 \leqslant |\alpha| \leqslant 2m$ is symmetric. An analogous condition was used in [2].

Condition (2.14) seems to be complicated. However, in the case $m = 1$ this condition is satisfied for every real-valued F, since

$$\begin{aligned} \sum_{|\beta|=1} \langle b_\beta(t, \cdot, \mathscr{D}u(t))\partial_x^\beta \varphi, \varphi\rangle &= -\sum_{|\beta|=1} \langle \varphi, \partial_x^\beta [b_\beta(t, \cdot, \mathscr{D}u(t))\varphi]\rangle \\ &= -\sum_{|\beta|=1} \langle \varphi, b_\beta(t, \cdot, \mathscr{D}u(t))\, \partial_x^\beta \varphi\rangle - \sum_{|\beta|=1} \langle \varphi, [\partial_x^\beta b_\beta(t, \cdot, \mathscr{D}u(t))]\varphi\rangle \end{aligned}$$

and therefore

$$\operatorname{Re}\langle \mathscr{B}_u(t)\varphi, \varphi\rangle = -\frac{1}{2}\sum_{|\beta|=1}\langle \varphi, [\partial_x^\beta b_\beta(t,.,\mathscr{D}u(t))]\varphi\rangle + \langle b_0(t,.,\mathscr{D}u(t))\varphi, \varphi\rangle$$
$$\geqslant c\|\varphi\|^2.$$

In the same way it follows in the case $m = 2$ that (2.14) holds if F is real-valued and satisfies

$$-\sum_{|\beta|=2} b_\beta(t, x, \mu)\xi^\beta \geqslant 0 \quad \text{on } [0, T]\times\bar{\Omega}\times\{\mu : |\mu| \leqslant M\} \tag{2.15}$$

for every $\xi\in\mathbb{R}^n$. If $m\in\mathbb{N}$ and $\mathscr{B}_u(t)$ is elliptic for every $t\in[0, T]$, then (2.14) also holds.

Example 1. Let F be defined by (1.4). Then Assumption 2.1 is satisfied if

$$a_\alpha\in C_b^{(k-2)m+1}([0, T]\times\bar{\Omega}\times\{\mu_0 : |\mu_0| \leqslant M\})$$

is real valued for $|\alpha| = 2$ and

$$-\operatorname{Re}\sum_{|\alpha|=2}\left(a_\alpha(t, x, \mu_0) + \sum_{|\gamma|\leqslant 2}\frac{\partial a_\gamma(t, x, \mu_0)}{\partial\mu_0^{(\alpha)}}\mu_0^{(\gamma)}\right)\xi^\alpha \geqslant d_2|\xi|^{2m} \tag{2.16}$$

on $[0, T]\times\bar{\Omega}\times\{\mu_0 : |\mu_0| \leqslant M\}$ for every $\xi\in\mathbb{R}^n$.

Example 2. For the quasilinear equation let

$$\begin{aligned} F(t, x, \mathscr{D}u(t, x)) :=\ & \partial_t^2 u(t, x) \\ & + \sum_{|\alpha|,|\beta|\leqslant m}(-1)^{|\alpha|}\partial_x^\alpha(a_{\alpha\beta}(t, x, \bar{D}_x^m u(t, x), \partial_t u(t, x))\partial_x^\beta(t, x)) \\ & + h(t, x, \bar{D}_x^m u(t, x), \partial_t u(t, x)) \end{aligned} \tag{2.17}$$

with $a_{\alpha\beta}, h\in C_b^{(k-1)m+1}([0, T]\times\bar{\Omega}\times\{(\mu_1, \mu_2) : |\mu_1| + |\mu_2| \leqslant M\})$, $a_{\alpha\beta} = \bar{a}_{\beta\alpha}$ for $|\alpha| + |\beta| > m$, $h(t, x, 0) = 0$ on $[0, T]\times\bar{\Omega}$ and

$$\operatorname{Re}\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(t, x, \mu_1, \mu_2)\xi^{\alpha+\beta} \geqslant d_2|\xi|^{2m} \tag{2.18}$$

on $[0, T]\times\bar{\Omega}\times\{(\mu_1, \mu_2) : |\mu_1| + |\mu_2| \leqslant M\}$ for every $\xi\in\mathbb{R}^n$, then Assumption 2.1 is satisfied.

3. The compatibility condition

Let $(u^0, u^1, f)\in\mathscr{I}$ be given and let $u\in\mathscr{C}_T^k$ with $k \geqslant [n/2m] + 3$ be a solution of (1.1)–(1.3). Then it holds that $u(t)\in\mathring{H}^m(\Omega)$ and $u\in C^{k-1}([0, T], H^m(\Omega))$. Note that this means that there exist functions $\partial_t^j u\in C([0, T], H^m(\Omega))$ such that

$$\left\|\partial_t^j u(t) - \frac{\partial_t^{j-1}u(t+h) - \partial_t^{j-1}u(t)}{h}\right\|_m \to 0 \quad \text{as } h\to 0 \tag{3.1}$$

for $0 < t < T$, $j = 1, \ldots, k-1$. If $t = 0$ or $t = T$, (3.1) must hold as $h\downarrow 0$ or $h\uparrow 0$, respectively.

Since $\mathring{H}^m(\Omega)$ is closed in $H^m(\Omega)$, we conclude from $u(t)\in\mathring{H}^m(\Omega)$ and (3.1) that $\partial_t^j u(t)\in\mathring{H}^m(\Omega)$ for $t\in[0, T]$, $j = 0, \ldots, k-1$. In particular, we have

$$\partial_t^j u(0)\in\mathring{H}^m(\Omega) \quad \text{for } j = 0, \ldots, k-1. \tag{3.2}$$

In order to formulate the compatibility condition, we note that $\partial_t^j u(0) = u^j$ for $j = 0, 1, 2$, where u^2 is given by $(u^0, u^1, f) \in \mathscr{I}$ and (1.10) and (1.11); the fact that $\partial_t^2 u(0) = u^2$ will be proved by Lemma 3.1 below. We express $\partial_t^j u(0)$ for $j \geqslant 3$ in terms of u^0, u^1, u^2 and f. Differentiating (1.1) with respect to t we obtain

$$\partial_t^3 u(t) = g_u(t) - \mathscr{A}_u(t)\partial_t u(t) - \mathscr{B}_u(t)\partial_t^2 u(t) \quad \text{for } t \in [0, T]$$

(compare (1.5), (2.7)–(2.11)). Further differentiation yields

$$\partial_t^j u(t) = \partial_t^{j-3} g_u(t) - \sum_{\nu=0}^{j-3} \binom{j-3}{\nu} \times \{[\partial_t^\nu \mathscr{A}_u(t)]\partial_t^{j-2-\nu} u(t) + [\partial_t^\nu \mathscr{B}_u(t)]\partial_t^{j-1-\nu} u(t)\} \tag{3.3}$$

$(j \geqslant 3)$, where

$$\partial_t^\nu \mathscr{A}_u(t) := \sum_{|\alpha| \leqslant 2m} [\partial_t^\nu a_\alpha(t, \cdot, \mathscr{D}u(t))]\partial_x^\alpha,$$

and $\partial_t^\nu \mathscr{B}_u(t)$ is defined analogously. We set recursively

$$u^j := \partial_t^{j-3} g_u(0) - \sum_{\nu=0}^{j-3} \binom{j-3}{\nu} \times \{[\partial_t^\nu \mathscr{A}_u(0)] u^{j-2-\nu} + [\partial_t^\nu \mathscr{B}_u(0)] u^{j-1-\nu}\} \tag{3.4}$$

for $j = 3, \ldots, k$, where we have set $\partial_t^\mu u(0) := u^\mu$ for $\mu = 0, \ldots, j-1$ in the coefficients of $\partial_t^\nu \mathscr{A}_u(0)$ and $\partial_t^\nu \mathscr{B}_u(0)$. Note that $u^j = \partial_t^j u(0)$ for $j = 0, \ldots, k$ from our considerations.

Definition 3.1. *Let $(u^0, u^1, f) \in \mathscr{I}$ be given. We say that (u^0, u^1, f) satisfies the compatibility condition of order $k \in \mathbb{N}$, if $u^j \in \mathring{H}^m(\Omega)$ for $j = 0, \ldots, k-1$, where u^2 is given by* (1.10) *and* (1.11), *and u^j is defined by* (3.4) *for $j \geqslant 3$.*
Now we show that the solution u^2 of (1.10) and (1.11) is unique.

Lemma 3.1. *Assume that $u^0 \in C^{2m}(\bar{\Omega})$, $u^1 \in C^m(\bar{\Omega})$, $f(0) \in C(\bar{\Omega})$ and that F satisfies Assumption* 2.1 *for some $k \geqslant 2$. Let $x \in \bar{\Omega}$ be fixed. If $y \in \mathbb{C}$ solves*

$$F(0, x, \bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), y) = f(0, x), \tag{3.5}$$

$$|\bar{D}_x^{2m} u^0(x)| + |\bar{D}_x^m u^1(x)| + |y| \leqslant M, \tag{3.6}$$

then y is unique.

Proof. This proof is the same as a proof in [7]. Let $y_1, y_2 \in \mathbb{C}$ be two solutions of (3.5) and (3.6). Then it holds that

$$|\bar{D}_x^{2m} u^0(x)| + |\bar{D}_x^m u^1(x)| + |y_1 + \Theta(y_2 - y_1)| \leqslant M$$

for $0 \leqslant \Theta \leqslant 1$. Furthermore, we have

$$\begin{aligned} 0 &= F(0, x, \bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), y_1) - F(0, x, \bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), y_2) \\ &= -\int_0^1 \frac{\partial F}{\partial \mu_2}(0, x, \bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), y_1 + \Theta(y_2 - y_1))(y_2 - y_1)\, d\Theta. \end{aligned}$$

Since $\partial F(0, x, \mu)/\partial \mu_2 > 0$, this implies that $y_2 = y_1$.

Our considerations yield the following lemma.

Lemma 3.2. *Let $k \geqslant [n/2m] + 3$ and $M > 0$ be given. Suppose that F satisfies Assumption 2.1 and that $u^0 \in H^{km}(\Omega)$, $u^1 \in H^{(k-1)m}(\Omega)$, $f \in \mathscr{C}_T^{k-2}$ such that $(u^0, u^1, f) \in \mathscr{I}$. If $u \in \mathscr{C}_T^k$ is a solution of (1.1)–(1.3) with the property (1.9), then it holds that $\partial_t^j u(0) = u^j$, where u^2 is determined by (1.10)–(1.11) and u^j is given by (3.4) for $j = 3, \ldots, k$, Furthermore, (u^0, u^1, f) satisfies the compatibility condition of order k.*

Let $u \in \mathscr{C}_T^k$ $(k \geqslant [n/2m] + 3)$ be a solution of (1.1)–(1.3). According to Lemma 3.2 it holds that

$$|u(0)|_k^{\sim} = \|u^0\|_{(k-1)m} + \sum_{j=1}^{k} \|u^j\|_{(k-j)m}. \tag{3.7}$$

We prove an estimate for $|u(0)|_k^{\sim}$.

Lemma 3.3. *Let $k \geqslant [n/2m] + 4$ and $R > 0$ be given and let all assumptions of Lemma 3.2 be satisfied. Furthermore, suppose that*

$$\|u^0\|_{km} + \|u^1\|_{(k-1)m} + |f(0)|_{k-2} \leqslant R.$$

Then there exists a positive number $c = c(R, k)$ such that

$$\|u^j\|_{(k-j)m} \leqslant c(\|u^0\|_{km} + \|u^1\|_{(k-1)m} + |f(0)|_{k-2}) \tag{3.8}$$

for $j = 2, \ldots, k$.

Proof. At first we prove (3.8) for $j = 2$. We set

$$B := \sup\{|\nabla_\mu F(0, x, \mu)| : x \in \bar{\Omega}, |\mu| \leqslant M\}, \tag{3.9}$$

where $\nabla_\mu F := (\partial F/\partial \mu_0^{(\alpha)}, \partial F/\partial \mu_1^{(\beta)}, \partial F/\partial \mu_2 : |\alpha| \leqslant 2m, |\beta| \leqslant m)$. Furthermore, we set $\mathscr{D}u(0, x) := (\bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), u^2(x))$. Let $x \in \Omega$ be fixed. From $F(0, x, 0) = 0$ and (1.10) we conclude that

$$\begin{aligned} f(0, x) &= F(0, x, \mathscr{D}u(0, x)) - F(0, x, 0) \\ &= \nabla_\mu F(0, x, \Theta \mathscr{D}u(0, x)) \cdot (\bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), u^2(x)) \end{aligned} \tag{3.10}$$

for some $\Theta \in (0, 1)$. Note that $(\partial F/\partial \mu_2)(t, x, \mu) \geqslant d_1 > 0$ by Assumption 2.1. Hence we obtain from (3.9) and (3.10) that

$$|u^2(x)| \leqslant d_1^{-1}[|f(0, x)| + B(|\bar{D}_x^{2m} u^0(x)| + |\bar{D}_x^m u^1(x)|)] \tag{3.11}$$

for $x \in \Omega$. This implies that

$$\|u^2\| \leqslant c(\|f(0)\| + \|u^0\|_{2m} + \|u^1\|_m). \tag{3.12}$$

We differentiate (1.10) with respect to x_i and divide the result by $\partial F/\partial \mu_2$. This yields

$$\begin{aligned} \frac{\partial u^2(x)}{\partial x_i} = &- \mathscr{A}_u(0) \frac{\partial u^0(x)}{\partial x_i} - \mathscr{B}_u(0) \frac{\partial u^1(x)}{\partial x_i} \\ &+ \left(\frac{\partial F}{\partial \mu_2}(0, x, \mathscr{D}u(0, x))\right)^{-1} \left(\frac{\partial f(0, x)}{\partial x_i} - \frac{\partial F}{\partial x_i}(0, x, \mathscr{D}u(0, x))\right) \end{aligned}$$

for $x \in \Omega$, $i = 1, \ldots, n$. It follows by induction that

$$\begin{aligned} \partial_x^\alpha u^2(x) = &\sum_\gamma h_\gamma^{(\alpha)}(0, x, \mathscr{D}u(0, x)) \prod_{j=1}^{|\alpha|} [D_x^j(\bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), f(0, x))]^{\gamma_j} \\ &- \left(\frac{\partial F(0, x, \mathscr{D}u(0, x))}{\partial \mu_2}\right)^{-1} (\partial_x^\alpha F)(0, x, \mathscr{D}u(0, x)) \end{aligned} \tag{3.13}$$

for $x \in \Omega$, $|\alpha| \leqslant (k-2)m$, where the summation extends over all $\gamma = (\gamma_1, \ldots, \gamma_{|\alpha|})$ with

$$1 \leqslant \sum_{j=1}^{|\alpha|} j|\gamma_j| \leqslant |\alpha|$$

and functions $h_\gamma^{(\alpha)}$ consist of derivatives of F. Note that $(k-2)m \geqslant [n/2] + 2$ and that $\partial_x^\alpha F(0, x, 0) = 0$ on $\bar{\Omega}$. With Corollary A.4 and Lemma A.6 we obtain that

$$\|\partial_x^\alpha u^2\| \leqslant c(\|u^0\|_{km} + \|u^1\|_{(k-1)m} + \|f(0)\|_{(k-2)m}) \tag{3.14}$$

for $|\alpha| \leqslant (k-2)m$. This together with (3.12) proves (3.8) for $j = 2$.

In order to prove (3.8) for $j \geqslant 3$ we rewrite (3.3) for $t = 0$ as

$$\begin{aligned}\partial_t^j u(0, x) = \sum_\gamma \tilde{h}_\gamma^{(j)}(0, x, \mathscr{D}u(0, x)) \prod_{i=1}^{j-2} (\bar{D}_x^{2m} u^i(x), \bar{D}_x^m u^{i+1}(x), \partial_t^i f(0, x))^{\gamma_i} \\ - \left(\frac{\partial F(0, x, \mathscr{D}u(0, x))}{\partial \mu_2}\right)^{-1} (\partial_t^{j-2} F)(0, x, \mathscr{D}u(0, x))\end{aligned} \tag{3.15}$$

$(j = 3, \ldots, k)$; here the summation extends over all $\gamma = (\gamma_1, \ldots, \gamma_j)$ with

$$1 \leqslant \sum_{i=1}^{j-2} i|\gamma_i| \leqslant j - 2$$

and the functions $\tilde{h}_\gamma^{(j)}$ consist of derivatives of F. In order to apply Lemma A.6 to the product in (3.15) we have to set

$$w_r := \begin{cases} (\bar{D}_x^{2m} u^{r/m}, \bar{D}_x^m u^{r/m+1}, \ \partial_t^{r/m} f(0)) & \text{for } r = im,\ i = 1, \ldots, j-2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain from (3.15) by Corollary A.4, Lemma A.6 and Lemma A.2 (for functions that are constant in t) that

$$\|u^j\|_{(k-j)m} \leqslant c\left(\sum_{i=1}^{j-1} \|u^i\|_{(k-i)m} + |f(0)|_{k-2}\right). \tag{3.16}$$

From this (3.8) for $j = 3, \ldots, k$ follows by induction.

We conclude this section by proving the existence of data $(u^0, u^1, f) \in \mathscr{I}$ satisfying the compatibility condition of order k.

Lemma 3.4. *Let $k \geqslant [n/2m] + 3$ and $M > 0$ be given. Suppose that F satisfies Assumption 2.1 and that $u^0 \in C_0^{km}(\Omega)$, $u^1 \in C_0^{(k-1)m}(\Omega)$, $f \in C^{(k-2)m}([0, T] \times \Omega)$ such that $f(0) \in C_0^{(k-2)m}(\Omega)$ and*

$$\sup_{x \in \bar{\Omega}} \left[\frac{1}{d_1}|f(0, x)| + \left(\frac{B}{d_1} + 1\right)(|\bar{D}_x^{2m} u^0(x)| + |\bar{D}_x^m u^1(x)|)\right] \leqslant \tfrac{1}{2}M, \tag{3.17}$$

where B is defined by (3.9) and d_1 is given by Assumption 2.1. Then it holds that $(u^0, u^1, f) \in \mathscr{I}$ and $u^j \in C_0^{(k-j)m}(\Omega) \subset \mathring{H}^m(\Omega)$ for $j = 2, \ldots, k-1$. In particular, (u^0, u^1, f) satisfies the compatibility condition of order k.

Proof. Let

$$S := \{x \in \Omega\text{: there exists a } y = y(x) \in \mathbb{C} \text{ such that (3.5) and (3.6) hold}\}. \tag{3.18}$$

Note that $S' := \{x \in \Omega: u^0(x) = u^1(x) = f(0, x) = 0\} \subset S$, since $F(0, x, 0) = 0$ and therefore $y(x) = 0$ on S'.

By the implicit function theorem it follows that S is an open subset of Ω with respect to the usual metric on $\Omega \subset \mathbb{R}^n$. In fact, if $x \in S$, then $(\partial F/\partial \mu_2)(0, x, \mu) \geqslant d_1 > 0$ and the implicit function theorem implies the existence of a neighbourhood U of x and a solution $y \in C(U)$ of (3.5). By (3.11) and (3.17) we obtain that $|\bar{D}_x^{2m} u^0(x)| + |\bar{D}_x^m u^1(x)| + |y(x)| \leqslant \frac{1}{2}M$ on U. Hence we have $U \subset S$. Moreover, we obtain that $y \in C(S)$, since the solution y of (3.5) and (3.6) is unique by Lemma 3.1 and since for every $x \in S$ there exists such a neighbourhood U with $y \in C(U)$.

On the other hand, since $y \in C(S)$, it follows from the continuity assumptions made on F, u^0, u^1 and f that S is a closed subset of Ω. Thus $S = \Omega$ and $y \in C(\Omega)$. Since $S' \subset S$, we have $y \in C_0(\Omega)$. Setting $u^2 := y$ we obtain $(u^0, u^1, f) \in \mathscr{I}$. Using (3.13) and (3.15) we conclude that $u^j \in C_0^{(k-j)m}(\Omega)$ for $j = 2, \ldots, k$. Hence Lemma 3.4 is proved.

4. The existence of the fixed point of Φ

In this section we study the mapping Φ, which is defined by (1.17) and (1.15). First we prove the following Lemma.

Lemma 4.1. *Let $k \geqslant [n/2m] + 5$ and $M > 0$ be given. Suppose that F satisfies Assumption 2.1 and that $u^0 \in H^{km}(\Omega)$, $u^1 \in H^{(k-1)m}(\Omega)$, $f \in \mathscr{C}_T^{k-2} \cap C^{k-1}([0, T], L_2(\Omega))$ such that $(u^0, u^1, f) \in \mathscr{I}$ satisfies the compatibility condition of order k (compare Definitions 1.1 and 3.1). Then for every sufficiently large $R > 0$ there exists a $T_1 > 0$ such that Φ maps*

$$\mathscr{M}(R, T_1) := \{u \in \tilde{\mathscr{C}}_{T_1}^k : |||u|||_{k, T_1}^{\sim} \leqslant R, \partial_t^j u(0) = u^j (j = 0, \ldots, k-1)\} \tag{4.1}$$

into itself, where u^j is given by (1.10), (1.11) and (3.4) for $j \geqslant 2$. Here T_1 depends only on R, k and

$$\tilde{N}_k = |u(0)|_k^{\sim} + \int_0^T |(\partial_t f(\tau)|_{k-3} + \|\partial_t^{k-1} f(\tau)\|) \, d\tau + |||\partial_t f|||_{k-3, T} \tag{4.2}$$

where

$$|u(0)|_k^{\sim} = \sum_{j=1}^{k} \|u^j\|_{(k-j)m} + \|u^0\|_{(k-1)m}.$$

Proof. We suppose that $R, T_1 > 0$ are arbitrary, but fixed. Let $u \in \mathscr{M}(R, T_1)$ be given. In a first step we show that u satisfies (1.9) for sufficiently small T_1.

By Sobolev's lemma we obtain for $(t, x) \in [0, T_1] \times \bar{\Omega}$ that

$$|\mathscr{D}u(t, x) - \mathscr{D}u(0, x)| = |\partial_t \mathscr{D}u(\Theta t, x)| t \leqslant c_1 |||u|||_{k, T_1}^{\sim} t \leqslant c_1 R T_1. \tag{4.3}$$

Since $|\mathscr{D}u(0, x)| \leqslant \frac{1}{2}M$ for $x \in \bar{\Omega}$ by (1.11), it follows that $|\mathscr{D}u(t, x)| \leqslant M$ for $(t, x) \in [0, T_1] \times \bar{\Omega}$ if $T_1 \leqslant M/2c_1 R$.

In the next step we prove $\Phi[u] \in \tilde{\mathscr{C}}_{T_1}^k$. Consider (1.15). Formally differentiating this equation $j - 2$ times with respect to t and comparing the result with (3.4) we conclude that (formally)

$$\partial_t^j v(0) = u^{j+1} \in \mathring{H}^m(\Omega), \quad j = 0, \ldots, k-2. \tag{4.4}$$

Hence (u^1, u^2, g_u) satisfies the natural compatibility condition for problem (1.15) of order $k - 1$. From (2.9) and Lemma A.1 we obtain

$$a_\alpha(.,.,\mathscr{D}u) - a_\alpha(.,.,0) \in \tilde{\mathscr{C}}_{T_1}^{k-2}.$$

The same holds for b_β. From (2.11) we obtain by Lemma A.1 and Corollary A.5 that

$$g_u \in \tilde{\mathscr{C}}_{T_1}^{k-2} \subset \mathscr{C}_{T_1}^{k-3} \cap \mathscr{C}^{k-2}([0, T_1], L_2(\Omega)).$$

From the linear existence theorem ((1.1) in [6]) it follows that (1.15) has a unique solution $v \in \mathscr{C}_{T_1}^{k-1}$. Therefore (4.4) holds. Furthermore we conclude from (1.17) and $v \in \mathscr{C}_{T_1}^{k-1}$ that $\Phi[u] \in C([0, T_1], H^{(k-1)m}(\Omega))$, $\partial_t \Phi[u] = v \in \mathscr{C}_{T_1}^{k-1}$. This proves that $\Phi[u] \in \tilde{\mathscr{C}}_{T_1}^{k}$. Note that we have

$$\partial_t^j \Phi[u](0) = u^j \quad \text{for } j = 0, \ldots, k-1 \tag{4.5}$$

by (1.17) and (4.4).

In the next step we prove $\Phi[u] \in \mathscr{M}(R, T_1)$. We suppose that $T_1 \leqslant 1$, $T_1 \leqslant 1/R$. By d_j $(j = 1, 2, \ldots)$ we denote positive constants depending only on k and $\tilde{N}_k$. It holds that

$$|u(t)|_{k-1} \leqslant |u(0)|_{k-1} + \int_0^t |\partial_t u(\tau)|_{k-1} \, d\tau \leqslant \tilde{N}_k + \int_0^{T_1} R \, d\tau \leqslant d_1 \tag{4.6}$$

for $t \in [0, T_1]$. With Lemma 8.1 we obtain that

$$|a_\alpha(t, . , \mathscr{D}u(t)) - a_\alpha(t, . , 0)|_{k-3} \leqslant d_2, \tag{4.7}$$

$$|a_\alpha(t, . , \mathscr{D}u(t)) - a_\alpha(t, . , 0)|\tilde{}_{k-2} \leqslant d_3 R \tag{4.8}$$

for $t \in [0, T_1]$. The same estimates hold for b_β. We apply Theorem 1.1 of [6] to (1.15). This yields

$$\begin{aligned} |v(t)|_{k-1} \leqslant d_4 \exp(c_2 t) \left[|v(0)|_{k-1} + c_2 \int_0^t (\|\partial_t^{k-2} g_u(\tau)\| + |g_u(\tau)|_{k-3}) \, d\tau \right] \\ + d_5 |g_u(t)|_{k-3} \end{aligned} \tag{4.9}$$

for $t \in [0, T_1]$, where $c_2 = c_2(R)$. Note that

$$|v(0)|_{k-1} = |\partial_t u(0)|_{k-1} \leqslant \tilde{N}_k$$

and that by Lemma A.1 and Corollary A.5

$$|g_u(t)|_{k-3} \leqslant d_6,$$

$$\int_0^t (\|\partial_t^{k-2} g_u(\tau)\| + |g_u(\tau)|_{k-3}) \, d\tau \leqslant d_7 T_1 + d_8 R T_1$$

for $t \in [0, T_1]$. Hence it follows from (4.9) that

$$|v(t)|_{k-1} \leqslant d_9 + d_{10} \exp(c_2 t)(1 + c_2 T_1 (1 + R))$$

for $t \in [0, T_1]$. With (1.15) we conclude that

$$\begin{aligned} |\Phi[u](t)|\tilde{}_k &\leqslant \|u^0\|_{(k-1)m} + \int_0^t \|v(\tau)\|_{(k-1)m} \, d\tau + |v(t)|_{k-1} \\ &\leqslant d_{11} + d_{10} \exp(c_2 t) \left(1 + \frac{1}{c_2}\right)(1 + c_2 T_1 (1 + R)) \end{aligned} \tag{4.10}$$

for $t \in [0, T_1]$. Note that c_2 in (4.9) can be chosen so that $c_2 \geqslant 1$. Furthermore, note that d_{10} and d_{11} depend only on k and $\tilde{N}_k$. We suppose that

$$R \geqslant d_{11} + 4 d_{10} e, \tag{4.11}$$

$$T_1 := \min\left\{T, 1, \frac{1}{R}, \frac{M}{2c_1 R}, \frac{1}{c_2(1+R)}\right\}. \tag{4.12}$$

Then (4.10) implies that $|||\Phi[u]|||^{\sim}_{k,T_1} \leqslant R$. This together with (4.5) proves that $\Phi[u] \in \mathcal{M}(R, T_1)$.

The next lemma contains the contraction property of Φ.

Lemma 4.2. *Let all assumptions of Lemma* 4.1 *be satisfied. Furthermore let* $R > 0$ *be given. Then there exists a* $T_2 > 0$ *such that*

$$|||\Phi[u_1] - \Phi[u_2]|||^{\sim}_{k-1,T_2} \leqslant \tfrac{1}{2}|||u_1 - u_2|||^{\frown}_{k-1,T_2} \tag{4.13}$$

for all $u_1, u_2 \in \mathcal{M}(R, T_2)$ *with* $\Phi[u_1], \Phi[u_2] \in \mathcal{M}(R, T_2)$. *Here* T_2 *depends only on* R, k *and* $|||\partial_t f|||^{\sim}_{k-3,T}$.

Remark. If $u \in \mathcal{M}(R, T_2)$, then $\Phi[u] \in \mathcal{M}(R, T_2)$ if and only if $|||\Phi[u]|||^{\sim}_{k,T_2} \leqslant R$.

Proof of Lemma 4.2. We suppose that $T_2 > 0$ is arbitrary, but fixed. We set $v_j := \partial_t \Phi[u_j]$ $(j = 1, 2)$. Then we have

$$|v_j(t)|_{k-1} \leqslant |\Phi[u_j](t)|^{\sim}_k \leqslant R \tag{4.14}$$

for $t \in [0, T_2]$; $j = 1, 2$. Let $w := v_1 - v_2$. Since v_j satisfies (1.15) with $u = u_j$ $(j = 1, 2)$, it holds that

$$\begin{aligned}
&\partial_t^2 w(t) + \mathcal{A}_{u_1}(t)w(t) + \mathcal{B}_{u_1}(t)\partial_t w(t) \\
&\quad = g_{u_1}(t) - g_{u_2}(t) + [\mathcal{A}_{u_2}(t) - \mathcal{A}_{u_1}(t)]v_2(t) \\
&\qquad + [\mathcal{B}_{u_2}(t) - \mathcal{B}_{u_1}(t)]\,\partial_t v_2(t) \quad \text{for } t \in [0, T_2],
\end{aligned} \tag{4.15}$$

$$w(t) \in \mathring{H}^m(\Omega) \qquad \text{for } t \in [0, T_2], \tag{4.16}$$

$$\partial_t^j w(t) = 0 \qquad \text{for } j = 0, \ldots, k. \tag{4.17}$$

We estimate the right-hand side of (4.15). From Lemma A.1 and Corollary A.3 we conclude that

$$|g_{u_1}(t) - g_{u_2}(t)|^{\frown}_{k-3} \leqslant d_1 |u_1(t) - u_2(t)|^{\sim}_{k-1} \leqslant d_1 |||u_1 - u_2|||^{\sim}_{k-1,T_2},$$

$$|g_{u_1}(t) - g_{u_2}(t)|_{k-4} \leqslant d_2 |u_1(t) - u_2(t)|^{\frown}_{k-2} \leqslant d_2 T_2 |||u_1 - u_2|||^{\frown}_{k-1,T_2},$$

for $t \in [0, T_2]$, since

$$\begin{aligned}
\|\partial_t^j u_1(t) - \partial_t^j u_2(t)\|_{(k-2-j)m} &\leqslant \int_0^t \|\partial_t^{j+1}[u_1(\tau) - u_2(\tau)]\|_{(k-2-j)m}\, d\tau \\
&\leqslant T_2 |||u_1 - u_2|||^{\sim}_{k-1,T_2}
\end{aligned} \tag{4.18}$$

for $j = 0, \ldots, k-2$. Here and in the following d_j $(j = 1, 2, \ldots)$ denote positive constants depending only on R, k and $|||\partial_t f|||^{\sim}_{k-3,T}$.

Let $t \in [0, T_2]$. From Lemma A.1 and Corollary A.3 we obtain that

$$\begin{aligned}
&|[a_\alpha(t, \cdot, \mathcal{D}u_2(t)) - a_\alpha(t, \cdot, \mathcal{D}u_1(t))]\partial_x^\alpha v_2(t)|^{\sim}_{k-3} \\
&\leqslant a_\alpha(t, \cdot, \mathcal{D}u_2(t) - a_\alpha(t, \cdot, \mathcal{D}u_1(t))|^{\sim}_{k-3} |v_2(t)|_{k-1} \\
&\leqslant d_4 |u_1(t) - u_2(t)|^{\frown}_{k-1}
\end{aligned}$$

for $|\alpha| \leq 2m$. This implies that

$$|[\mathscr{A}_{u_2}(t) - \mathscr{A}_{u_1}(t)]v_2(t)|\tilde{}_{k-3} \leq d_5 |||u_1 - u_2|||\tilde{}_{k-1,T_2}.$$

In the same way it can be shown with (4.18) that

$$|[\mathscr{A}_{u_2}(t) - \mathscr{A}_{u_1}(t)]v_2(t)|_{k-4} \leq d_6 T_2 |||u_1 - u_2|||\tilde{}_{k-1,T_2}.$$

Analogous estimates hold for $[\mathscr{B}_{u_2}(t) - \mathscr{B}_{u_1}(t)]v_2(t)$. Note that for $\varphi \in \tilde{\mathscr{C}}_{T_2}^{k-3}$

$$\|\partial_t^{k-3}\varphi(t)\| + |\varphi(t)|_{k-4} \leq |\varphi(t)|\tilde{}_{k-3} \quad \text{for } t \in [0, T_2].$$

We apply Theorem 1.1 of [6] to (4.15)–(4.17) and obtain

$$\begin{aligned} |v_1(t) - v_2(t)|_{k-2} &\leq d_7 \exp(d_8 t) \int_0^t |||u_1 - u_2|||\tilde{}_{k-1,T_2} \, d\tau \\ &\quad + d_9 T_2 |||u_1 - u_2|||\tilde{}_{k-1,T_2} \\ &\leq (d_7 + d_9) T_2 \exp(d_8 T_2) |||u_1 - u_2|||\tilde{}_{k-1,T_2}. \end{aligned}$$

Using (1.17) we conclude that

$$\begin{aligned} |\Phi[u_1](t) - \Phi[u_2](t)|\tilde{}_{k-1} &\leq \int_0^t \|v_1(\tau) - v_2(\tau)\|_{(k-2)m} \, d\tau + |v_1(t) - v_2(t)|_{k-2} \\ &\leq (d_7 + d_9)(T_2^2 + T_2) \exp(d_8 T_2) |||u_1 - u_2|||\tilde{}_{k-1,T_2} \end{aligned} \tag{4.19}$$

for $t \in [0, T_2]$. We choose $T_2 > 0$ so small that

$$(d_7 + d_9)(T_2^2 + T_2) \exp(d_8 T_2) \leq \tfrac{1}{2}.$$

Then (4.19) implies (4.13) and Lemma 4.2 is proved.

By Lemma 4.1 and Lemma 4.2 we obtain the existence of a fixed point of Φ in the following way: suppose that all the assumptions of Lemma 4.1 are satisfied. Let $u_0 \in \tilde{\mathscr{C}}_T^k$ be a function with $\partial_t^j(0) = u^j$ for $j = 0, \dots, k-1$ (compare, for example, Lemma A.8). We choose

$$R := \max\{|||u_0|||\tilde{}_{k,T}, d_{12} + 2d_{11}e\}, \tag{4.20}$$

where d_{11} and d_{12} are the constants of (4.11). Note that R depends only on k and $\tilde{N}_k$. Furthermore, we set

$$T' := \min\{T_1, T_2\}, \tag{4.21}$$

with T_1, T_2 being the numbers of Lemma 4.1 and Lemma 4.2. Then the following statements hold:

(i) $\mathscr{M}(R, T') \neq \varnothing$ (since $u_0 \in \mathscr{M}(R, T')$).

(ii) Φ maps $\mathscr{M}(R, T')$ into itself.

(iii) For $u_1, u_2 \in \mathscr{M}(R, T')$ (4.13) holds with T_2 being replaced by T'.

The proof of Banach's fixed-point theorem yields the existence of a fixed-point $u \in \tilde{\mathscr{C}}_{T'}^{k-1}$ of Φ.

In order to prove the uniqueness of a fixed point of Φ in $\tilde{\mathscr{C}}_{T'}^k$, we suppose that all the assumptions of Lemma 4.1 are satisfied. Let $u_1, u_2 \in \tilde{\mathscr{C}}_T^k$ be two fixed points of Φ. We choose

$$R := \max\{|||u_1|||^{\sim}_{k,T'}, |||u_2|||^{\sim}_{k,T'}\}.$$

By $\Phi[u_j] = u_j$ ($j = 1, 2$) and Lemma 4.2 it follows that $u_1(t) = u_2(t)$ in some interval $[0, T_2]$. Note that $T_2 > 0$ depends only R. Hence a further application of Lemma 4.2 yields $u_1(t) = u_2(t)$ in $[0, 2T_2]$. After a finite number of steps we obtain $u_1(t) = u_2(t)$ in $[0, T']$. We have proved the following theorem.

Lemma 4.3. *Let all the assumptions of Lemma* 4.1 *be satisfied. Then there exists a* $T' > 0$ *depending only on k and* $\tilde{N}_k$ *such that* Φ *has a fixed point* $u \in \tilde{\mathscr{C}}^{k-1}_{T'}$. *If* $k \geqslant [n/2m] + 6$, *then this fixed point is unique in* $\mathscr{C}^{k-1}_{T'}$.

5. The fixed point of Φ and the solution

Let $u \in \tilde{\mathscr{C}}^k_T$ with $k \geqslant [n/2m] + 4$ be a fixed point of Φ. Note that

$$\bar{D}^{jm}_x u \in C^{2-j}_b([0, T] \times \bar{\Omega}) \quad j = 0, 1, 2 \tag{5.1}$$

by the following lemma:

Lemma 5.1. *Let* $k \geqslant k_2 := [n/2m] + 1$ *and let* $u \in \mathscr{C}^k_T$. *Then it holds that*

$$\bar{D}^{jm}_x u \in C^{k-k_2-j}_b([0, T] \times \bar{\Omega}) \quad j = 0, \ldots, k - k_2. \tag{5.2}$$

We shall give the proof of Lemma 5.1 at the end of this section. Consider the fixed point $u \in \tilde{\mathscr{C}}^k_T$ of Φ. It follows from the definition of Φ that $(u, v) = (u, \partial_t u)$ is a solution of (1.15) and (1.16). By the definition of $\mathscr{A}_u$, $\mathscr{B}_u$ and g_u (compare (2.7)–(2.11)) we obtain from $v = \partial_t u$ and (1.15) that

$$\begin{aligned} 0 &= \int_0^t [\partial_t^3 u(\tau, x) + \mathscr{A}_u(\tau)\partial_t u(\tau, x) + \mathscr{B}_u(\tau)\partial_t^2 u(\tau, x) \\ &\quad - g_u(\tau, x)] \frac{\partial F}{\partial \mu_2}(\tau, x, \mathscr{D}u(\tau, x))\, d\tau \\ &= \int_0^t \left(\frac{\partial}{\partial \tau}[F(\tau, x, \mathscr{D}u(\tau, x)) - f(\tau, x)]\right) d\tau \\ &= F(t, x, \mathscr{D}u(t, x)) - f(t, x) - F(0, x, \mathscr{D}u(0, x)) + f(0, x) \end{aligned}$$

for $(t, x) \in [0, T] \times \Omega$. Taking into account the fact that $F(0, x, \mathscr{D}u(0, x)) = f(0, x)$ by (1.10), we conclude that (1.1) holds

Note that $\partial_t u(t) = v(t) \in \mathring{H}^m(\Omega) \cap C^m_b(\bar{\Omega})$ for $t \in [0, T]$. Therefore, the following relation holds:

$$\bar{D}^{m-1}_x v(t, x) = 0 \quad \text{on } [0, T] \times \partial\Omega.$$

Since the same holds for $u^0(x)$, we obtain from

$$u(t, x) = u^0(x) + \int_0^t v(\tau, x)\, d\tau$$

that u satisfies (1.2). Furthermore, (1.3) holds. Hence u is a classical solution of (1.1)–(1.3). In addition to this it follows from Lemma A.7 that $u \in \mathscr{C}^k_T$.

On the other hand, every solution $u \in \mathscr{C}_T^k$ of (1.1)–(1.3) is a fixed point of Φ, which follows from the construction of problem (1.15) and (1.16). Together with Lemma 4.3 we have proved the following lemma.

Lemma 5.2. *Let all assumptions of* 4.1 *be satisfied. Then there exists a* $T' > 0$ *depending only on* k *and* $\tilde{N}_k$ *such that* (1.1)–(1.3) *has a classical solution,* $u \in \mathscr{C}_{T'}^{k-1}$. *Furthermore, it holds*

$$\bar{D}_x^{jm} u \in C_b^{k-k_2-1-j}([0, T'] \times \bar{\Omega}) \quad \text{for } j = 0, \ldots, k - k_2 - 1 \tag{5.3}$$

$(k_2 = [n/2m] + 1)$. *If* $k \geqslant [n/2m] + 6$, *then* u *is unique in* $\mathscr{C}_{T'}^{k-1}$.

Remark. Lemma 5.2 differs from Theorem 1.1 in two points. First, if all the assumptions of Theorem 1.1 are satisfied, then Lemma 5.2 gives a solution $u \in \mathscr{C}_{T'}^{k-1}$ instead of $u \in \mathscr{C}_{T'}^k$. The second difference is that T' in Lemma 5.2 depends on k. Both problems will be solved in the next section.

Proof of Lemma 5.1. We prove that for $v \in C([0, T], H^{k_2 m}(\Omega))$ that the following holds:

$$v \in C_b([0, T] \times \bar{\Omega}). \tag{5.4}$$

Note that $k_2 m \geqslant [n/2] + 1$. Hence Sobolev's lemma yields

$$v(t) \in H^{k_2 m}(\Omega) \subset C_b(\bar{\Omega}) \quad \text{for } t \in [0, T]. \tag{5.5}$$

Moreover, we obtain that

$$|v(t_1, x) - v(t_2, x)| \leqslant c \| v(t_1) - v(t_2) \|_{k_2 m} \to 0 \quad \text{as } t_1 \to t_2 \tag{5.6}$$

for $x \in \bar{\Omega}$. Since (5.6) is uniform with respect to $x \in \bar{\Omega}$, we conclude from (5.5) and (5.6) that (5.4) holds.

Let $k \geqslant k_2$, $u \in \mathscr{C}_T^k$. Then it holds that $\partial_x^\alpha \bar{D}_x^{jm} \partial_t^i u \in C([0, T], H^{k_2 m}(\Omega))$ for $|\alpha| + i \leqslant k - k_2 - j$. Thus we obtain by (5.4) that (5.2) is valid.

6. Higher regularity and the proof of Theorem 1.1

The gap between Lemma 5.2 and Theorem 1.1 will be bridged by the following lemma.

Lemma 6.1. *Let* $k \geqslant k_0 = [n/2m] + 6$ *and let all the assumptions of Lemma* 4.1 *be satisfied. If* $u \in \mathscr{C}_T^{k-1}$ *is a solution of* (1.1)–(1.3), *then* $u \in \mathscr{C}_T^k$.

Before proving Lemma 6.1 we consider Theorem 1.1. Let $k \geqslant k_0$ and let all the assumptions of Theorem 1.1 be satisfied. Lemma 5.2 gives us the existence of a $T' > 0$ such that (1.1)–(1.3) has a solution $u \in \mathscr{C}_{T'}^{k_0-1}$. This T' depends only on $\tilde{N}_{k_0}$. Note that $\tilde{N}_{k_0} \leqslant cN$ by Lemma 3.3. Hence T' depends only on N. From Lemma 6.1 we obtain that $u \in \mathscr{C}_{T'}^{k_0}$. Further application of Lemma 6.1 yields successively $u \in \mathscr{C}_{T'}^{k_0+1}, \ldots, \mathscr{C}_{T'}^k$. Finally it follows from Lemma 5.1 that u has the regularity property (5.2), so that (1.13) follows if the assumptions of Theorem 1.1 are satisfied for every $k \geqslant k_0$. This proves Theorem 1.1.

Proof of Lemma 6.1. This proof is a modification of a proof contained in [7]. Let $u \in \mathscr{C}_T^{k-1}$ be a solution of (1.1)–(1.3). Then (1.15) and (1.16) hold with $v = \partial_t u$. We differentiate (1.15) twice with respect to t:

$$\begin{aligned}
&\partial_t^5 u(t) + \mathscr{A}_u(t)\partial_t^3 u(t) + \mathscr{B}_u(t)\partial_t^4 u(t) \\
&\quad = \partial_t^2 g_u(t) - 2[\partial_t \mathscr{A}_u(t)]\partial_t^2 u(t) - 2[\partial_t \mathscr{B}_u(t)]\partial_t^3 u(t) \\
&\qquad - [\partial_t^2 \mathscr{A}_u(t)]\partial_t u(t) - [\partial_t^2 \mathscr{B}_u(t)]\partial_t^2 u(t) \quad \text{for } t \in [0, T].
\end{aligned} \tag{6.1}$$

It holds that (compare (2.11))

$$\begin{aligned}
\partial_t^2 g_u(t) &= \partial_t^2 [g(t, ., \mathscr{D}u(t))] = \nabla_\mu g(t, ., \mathscr{D}u(t)) \cdot \mathscr{D}\partial_t^2 u(t) + \tilde{g}(t) \\
&= \nabla_{\mu_0} g(t, ., \mathscr{D}u(t)) \cdot \bar{D}_x^{2m} \partial_t^2 u(t) + \nabla_{\mu_1} g(t, ., \mathscr{D}u(t)) \cdot \bar{D}_x^m \partial_t^3 u(t) \\
&\quad + \frac{\partial g}{\partial \mu_2}(t, ., \mathscr{D}u(t))\partial_t^4 u(t) + \tilde{g}(t),
\end{aligned} \tag{6.2}$$

where $\nabla_\mu g := (\partial g/\partial \mu_0^{(\alpha)}, \partial g/\partial \mu_1^{(\beta)}, \partial g/\partial \mu_2 : |\alpha| \leqslant 2m, |\beta| \leqslant m)$ and

$$\begin{aligned}
\tilde{g}(t) &:= (\partial_t^2 g)(t, ., \mathscr{D}u(t)) + 2\nabla_\mu (\partial_t g)(t, ., \mathscr{D}u(t)) \cdot \mathscr{D}\partial_t u(t) \\
&\quad + \sum_{|\gamma|=2} \nabla_\mu^\gamma g(t, ., \mathscr{D}u(t)) \cdot (\mathscr{D}\partial_t u(t))^\gamma.
\end{aligned} \tag{6.3}$$

In the same way we obtain

$$\partial_t^2 \mathscr{A}_u(t) = \sum_{|\alpha| \leqslant 2m} [\nabla_\mu a_\alpha(t, ., \mathscr{D}u(t)) \cdot \mathscr{D}\partial_t^2 u(t) + \tilde{a}_\alpha(t)]\partial_x^\alpha$$

$$\partial_t^2 \mathscr{B}_u(t) = \sum_{|\beta| \leqslant m} [\nabla_\mu b_\beta(t, ., \mathscr{D}u(t)) \cdot \mathscr{D}\partial_t^2 u(t) + \tilde{b}_\beta(t)]\partial_x^\beta$$

with $\tilde{a}_\alpha$ and $\tilde{b}_\beta$ being defined analogously to $\tilde{g}$. We set $w(t) := \partial_t^2 u(t)$ and rewrite (6.1) as

$$\begin{aligned}
&\partial_t^3 w(t) + \mathscr{A}'(t)\partial_t w(t) + \mathscr{B}'(t)\partial_t^2 w(t) \\
&\quad = h(t) - \mathscr{G}(t)w(t) - 2[\partial_t \mathscr{B}_u(t)]\partial_t w(t)
\end{aligned} \tag{6.4}$$

with

$$\begin{aligned}
\mathscr{A}'(t) &= \sum_{|\gamma| \leqslant 2m} a'_\gamma(t)\partial_x^\gamma \\
&:= \mathscr{A}_u(t) - \nabla_{\mu_1} g(t, ., \mathscr{D}u(t)) \cdot \bar{D}_x^m \\
&\quad + \left(\sum_{|\alpha| \leqslant 2m} [\partial_x^\alpha \partial_t u(t)] \nabla_{\mu_1} a_\alpha((t, ., \mathscr{D}u(t)) \right) \cdot \bar{D}_x^m \\
&\quad + \left(\sum_{|\beta| \leqslant m} [\partial_x^\beta \partial_t^2 u(t)] \nabla_{\mu_1} b_\beta((t, ., \mathscr{D}u(t)) \right) \cdot \bar{D}_x^m,
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\mathscr{B}'(t) &= \sum_{|\delta| \leqslant m} b'_\delta(t)\partial_x^\delta \\
&:= \mathscr{B}_u(t) - \frac{\partial g}{\partial \mu_2}(t, ., \mathscr{D}u(t)) + \sum_{|\alpha| \leqslant 2m} [\partial_x^\alpha \partial_t u(t)] \frac{\partial a_\alpha}{\partial \mu_2}(t, ., \mathscr{D}u(t)) \\
&\quad + \sum_{|\beta| \leqslant m} [\partial_x^\beta \partial_t^2 u(t)] \frac{\partial b_\beta}{\partial \mu_2}(t, ., \mathscr{D}u(t)),
\end{aligned} \tag{6.6}$$

$$h(t) := \tilde{g}(t) - \sum_{|\alpha| \leqslant 2m} \tilde{a}_\alpha(t)\partial_x^\alpha \partial_t u(t) - \sum_{|\beta| \leqslant m} \tilde{b}_\beta(t)\partial_x^\beta \partial_t^2 u(t) \tag{6.7}$$

and

$$\begin{aligned}\mathscr{G}(t) &= \sum_{|\gamma| \leqslant 2m} g_\gamma(t)\partial_x^\gamma \\ &:= 2[\partial_t \mathscr{A}_u(t)] - \nabla_{\mu_0} g(t,.., \mathscr{D}u(t)) \cdot \bar{D}_x^{2m} \\ &\quad + \left(\sum_{|\alpha| \leqslant 2m} [\partial_x^\alpha \partial_t u(t)] \nabla_{\mu_0} a_\alpha(t,.., \mathscr{D}u(t)) \right) \cdot \bar{D}_x^{2m} \\ &\quad + \left(\sum_{|\beta| \leqslant m} [\partial_x^\beta \partial_t^2 u(t)] \nabla_{\mu_0} b_\beta(t,.., \mathscr{D}u(t)) \right) \cdot \bar{D}_x^{2m}. \end{aligned} \tag{6.8}$$

Note that $a'_\alpha(t) = a_\alpha(t,.., \mathscr{D}u(t))$ for $|\alpha| = 2m$, so that $\mathscr{A}'(t)$ is a family of uniformly elliptic operators. Furthermore, $b'_\beta(t) = b_\beta(t,.., \mathscr{D}u(t))$ for $|\beta| \geqslant 1$. Hence (2.14) holds with $\mathscr{B}_u(t)$ being replaced by $\mathscr{B}'(t)$. From Lemma A.1 and Corollary A.3 we obtain that $h \in \mathscr{C}_T^{k-4}$ and that the coefficients $a'_\alpha(t)$ have a representation

$$a'_\alpha(t, x) = p_\alpha(t, x) + q_\alpha(t, x) \tag{6.9}$$

($(t, x) \in [0, T] \times \Omega$, $|\alpha| \leqslant 2m$) such that

$$\left.\begin{aligned} &p_\alpha \in C_b^{(k-2)m}([0, T] \times \bar{\Omega}), \\ &q_\alpha \in \mathscr{C}_T^{k-4}. \end{aligned}\right\} \tag{6.10}$$

The same holds for $b'_\beta(t)$, $g_\gamma(t)$, $\partial_t b_\beta(t,.., \mathscr{D}u(t))$.

From (1.15) and (6.4) we conclude that $w = \partial_t^2 u \in \mathscr{C}_T^{k-3}$ is a solution of the following linear problem

$$\partial_t^3 w(t) + \mathscr{A}'(t)\partial_t w(t) + \mathscr{B}'(t)\partial_t^2 w(t)$$

$$= h(t) - \mathscr{G}(t)w(t) - 2[\partial_t \mathscr{B}_u(t)]\partial_t w(t) \quad \text{for } t \in [0, T], \tag{6.11a}$$

$$w(t) \in \mathring{H}^m(\Omega) \quad \text{for } t \in [0, T], \tag{6.11b}$$

$$w(0) = u^2, \quad \partial_t w(0) = u^3, \quad \partial_t^2 w(0) = u^4. \tag{6.11c}$$

We prove the existence of a solution $w \in \tilde{\mathscr{C}}_T^{k-2}$ of (6.11) by the method of successive approximation (compare [3]). Let $w_0 \in \tilde{\mathscr{C}}_T^{k-2}$ with $\partial_t^j w_0(0) = u^{j+2}$ for $j = 0, \ldots, k-3$ (compare Lemma A.8). We define w_j for $j \geqslant 1$ successively by

$$\partial_t^2 v_{j+1}(t) + \mathscr{A}'(t)v_{j+1}(t) + \mathscr{B}'(t)\partial_t v_{j+1}(t)$$

$$= h(t) - \mathscr{G}(t)w_j(t) - 2[\partial_t \mathscr{B}_u(t)]\partial_t w_j(t) \quad \text{for } t \in [0, T], \tag{6.12a}$$

$$v_{j+1}(t) \in \mathring{H}^m(\Omega) \quad \text{for } t \in [0, T], \tag{6.12b}$$

$$v_{j+1}(0) = u^3, \quad \partial_t v_{j+1}(0) = u^4, \tag{6.12c}$$

$$w_{j+1}(t) := u^2 + \int_0^t v_{j+1}(\tau)\, d\tau \quad \text{for } t \in [0, T]. \tag{6.13}$$

We prove $w_j \in \tilde{\mathscr{C}}_T^{k-2}$ by induction. Let $w_j \in \tilde{\mathscr{C}}_T^{k-2}$, then

$$h - \mathscr{G}(\cdot)w_j - 2[\partial_t \mathscr{B}_u(\cdot)]\partial_t w_j \in \tilde{\mathscr{C}}_T^{k-4}.$$

We apply Theorem 1.1 of [6] to problem (6.12) and obtain $v_{j+1} \in \mathscr{C}_T^{k-3}$. Hence (6.13) implies $w_{j+1} \in \tilde{\mathscr{C}}_T^{k-2}$.

Note that $\partial_t^\nu w_j(0) = u^{\nu+2}$ ($\nu = 0, \ldots, k-2$) for every $j \in \mathbb{N}$. We consider the differences $v_{j+1} - v_j$, $w_{j+1} - w_j$. The following relations hold

$$[\partial_t^2 + \mathscr{A}'(t) + \mathscr{B}'(t)\partial_t]\,[v_{j+1}(t) - v_j(t)]$$

$$= \{ - \mathscr{G}(t) - 2[\partial_t \mathscr{B}_u(t)]\partial_t\}\,[w_j(t) - w_{j-1}(t)] \quad \text{for } t \in [0, T], \tag{6.14a}$$

$$v_{j+1}(t) - v_j(t) \in \mathring{H}^m(\Omega) \quad \text{for } t \in [0, T], \tag{6.14b}$$

$$\partial_t^\nu v_{j+1}(0) - \partial_t^\nu v_j(0) = 0 \quad \text{for } \nu = 0, \ldots, k-2 \tag{6.14c}$$

and

$$w_{j+1}(t) - w_j(t) = \int_0^t [v_{j+1}(\tau) - v_j(\tau)]\,d\tau \quad \text{for } t \in [0, T]. \tag{6.15}$$

With Corollary A.3 we conclude that

$$|\{\mathscr{G}(t) + 2[\partial_t \mathscr{B}_u(t)]\partial_t\}\,(w_j(t) - w_{j-1}(t))|^{\sim}_{k-4} \leqslant c_1 |w_j(t) - w_{j-1}(t)|^{\sim}_{k-2},$$

$$|\{\mathscr{G}(t) + 2[\partial_t \mathscr{B}_u(t)]\partial_t\}\,(w_j(t) - w_{j-1}(t))|_{k-5} \leqslant c_2 |w_j(t) - w_{j-1}(t)|_{k-3}$$

$$\leqslant c_2 \int_0^t |w_j(\tau) - w_{j-1}(\tau)|^{\sim}_{k-2}\,d\tau.$$

Applying the energy estimate of Theorem 1.1 in [6] to (6.14) we obtain that

$$|v_{j+1}(t) - v_j(t)|_{k-3} \leqslant c_3 \int_0^t |w_j(\tau) - w_{j-1}(\tau)|^{\sim}_{k-2}\,d\tau$$

with $c_3 > 0$ depending on T. With (6.15) it follows that

$$|w_{j+1}(t) - w_j(t)|^{\sim}_{k-2} \leqslant c_4 \int_0^t |w_j(\tau) - w_{j-1}(\tau)|^{\sim}_{k-2}\,d\tau, \tag{6.16}$$

with $c_4 := c_3 T$. By induction we obtain from (6.16) that

$$|w_{j+1}(t) - w_j(t)|^{\sim}_{k-2} \leqslant \frac{(c_4 t)^j}{j!}\,|||w_1 - w_0|||^{\sim}_{k-2,T} \tag{6.17}$$

for $t \in [0, T]$. This implies that $\{w_j\}$ converges in $\mathscr{C}_T^{k-2}$. We denote the limit by w. Since $v_j = \partial_t w_j$, the sequence $\{v_j\}$ converges in $\mathscr{C}_T^{k-3}$ to $\partial_t w$. Hence it follows from (6.12) and (6.13) that $w \in \tilde{\mathscr{C}}_T^{k-2}$ is a solution of (6.11).

It remains to be shown that $w = \partial_t^2 u$. Let $w^{(1)}, w^{(2)} \in \mathscr{C}_T^{k-3}$ be two solutions of (6.11). Then (6.14) and (6.15) hold with $w_j - w_{j-1}$, $v_{j+1} - v_j$ and $w_{j+1} - w_j$ being replaced by $w^{(1)} - w^{(2)}$, $\partial_t w^{(1)} - \partial_t w^{(2)}$ and $w^{(1)} - w^{(2)}$, respectively. The same argument that leads to (6.17) gives us

$$|w^{(1)}(t) - w^{(2)}(t)|^{\sim}_{k-3} \leqslant \left(\frac{c_4 t}{j!}\right)^j |||w^{(2)} - w^{(1)}|||^{\sim}_{k-3,T} \quad \text{for } t \in [0, T]$$

for every $j \in \mathbb{N}$. It follows that $w^{(1)} = w^{(2)}$. This proves $\partial_t^2 u = w \in \tilde{\mathscr{C}}_T^{k-2}$.

We rewrite (1.15) as

$$\mathscr{A}_u(t)\partial_t u(t) = g_u(t) - \partial_t^3 u(t) - \mathscr{B}_u(t)\partial_t^2 u(t). \tag{6.18}$$

Differentiation with respect to t yields

$$\mathscr{A}_u(t)\partial_t^2 u(t) = \partial_t g_u(t) - \partial_t^4 u(t) - \mathscr{B}_u(t)\partial_t^3 u(t)$$

$$- [\partial_t \mathscr{A}_u(t)]\partial_t u(t) - [\mathscr{B}_u(t)]\partial_t^2 u(t). \tag{6.19}$$

Taking into account that $u \in \mathscr{C}_T^{k-1}$, $\partial_t^2 u \in \tilde{\mathscr{C}}_T^{k-2}$ we conclude that the right-hand side of (6.19) is an element of $C([0, T], H^{(k-4)m}(\Omega))$. The elliptic regularity theory (compare [6]) yields $\partial_t^2 u \in C([0, T], H^{(k-2)m}(\Omega))$. Hence we have $\partial_t^2 u \in \mathscr{C}_T^{k-2}$.

In the same way we conclude from (6.18) that $\partial_t u \in \mathscr{C}_T^{k-1}$ and therefore $u \in \tilde{\mathscr{C}}_T^k$. Finally, it follows by Lemma A.7 that $u \in \mathscr{C}_T^k$.

7. The existence interval for small data

In this section we study solutions $u \in \mathscr{C}_T^k$ $(k \geqslant k_0)$ of (1.1)–(1.3) with $|||u|||_{k_0, T}^{\sim} \leqslant R_0$, where $R_0 > 0$ is chosen so that

$$|\mathscr{D}u(t, x)| \leqslant c\,\|\mathscr{D}u(t)\|_{(k_0-3)m} \leqslant cR_0 \leqslant \tfrac{1}{2}M \quad \text{for } (t, x) \in [0, T] \times \bar{\Omega} \tag{7.1}$$

by Sobolev's lemma. We suppose that all the assumptions of Theorem 1.1 are satisfied for every $T > 0$ and that

$$\tilde{N} := \sum_{j=0}^{k_0} \|u^j\|_{(k_0-j)m} + \int_0^\infty (|\partial_t f(t)|_{k_0-3} + \|\partial_t^{k_0-1} f(t)\|)\,dt + \sup_{[0,\infty)} |\partial_t f(t)|_{k_0-3} \leqslant \tfrac{1}{2}R_0. \tag{7.2}$$

According to Theorem 1.1 there exists a $T > 0$ and a solution $u \in \mathscr{C}_T^k$ of (1.1)–(1.3). Since $|u(0)|_{k_0}^{\sim} \leqslant \tilde{N} \leqslant \frac{1}{2}R_0$ by (7.2), we can choose T so small that $|u(t)|_{k_0}^{\sim} \leqslant R_0$ for $t \in [0, T]$. We use $\Phi[u] = u$ to prove an a priori estimate for $|u(t)|_{k_0}^{\sim}$.

By (4.9) and $v = \partial_t \Phi[u]$ we have

$$|\partial_t \Phi[u](t)|_{k_0-1} \leqslant d_1 \exp(d_2 t)\left[|\partial_t \Phi[u](0)|_{k_0-1} + \int_0^t (\|\partial_t^{k_0-2} g_u(\tau)\| + |g_u(\tau)|_{k_0-3})\,d\tau\right] + d_3 |g_u(t)|_{k_0-3} \quad \text{for } t \in [0, T]. \tag{7.3}$$

Here and in the following we denote by d_j $(j = 1, 2, \ldots)$ positive constants that can be chosen independently of all $u \in \mathscr{C}_T^k$ with $|||u|||_{k_0, T}^{\sim} \leqslant R_0$. From (2.11) and $\partial_t F(t, x, 0) = 0$ on $[0, T] \times \bar{\Omega}$ we conclude by Lemma A.1 and Corollary A.4 that

$$|g_u(t)|_{k_0-3} \leqslant d_4(|\partial_t f(t)|_{k_0-3} + |u(t)|_{k_0-1}),$$
$$\|\partial_t^{k_0-2} g_u(t)\| \leqslant d_5(|\partial_t f(t)|_{k_0-3} + \|\partial_t^{k_0-1} f(t)\| + |u(t)|_{k_0}^{\sim})$$

and therefore

$$\int_0^t (\|\partial_t^{k_0-2} g_u(\tau)\| + |g_u(\tau)|_{k_0-3})\,d\tau \leqslant (d_4 + d_5)\left(\tilde{N} + \int_0^t |u(\tau)|_{k_0}^{\sim}\,d\tau\right), \tag{7.4}$$

$$|g_u(t)|_{k_0-3} \leqslant d_4\left(\tilde{N} + |u(0)|_{k_0-1} + \int_0^t |u(\tau)|_{k_0}^{\sim}\,d\tau\right) \leqslant d_6 \tilde{N} + d_4 \int_0^t |u(\tau)|_{k_0}^{\sim}\,d\tau \tag{7.5}$$

for $t \in [0, T]$. With $|\partial_t \Phi[u](0)|_{k_0-1} \leqslant \tilde{N}$ we obtain from (7.3) that

$$|\partial_t \Phi[u](t)|_{k_0-1} \leqslant \exp(d_2 t)\left(d_8 \tilde{N} + d_9 \int_0^t |u(\tau)|_{k_0}^{\sim}\,d\tau\right) \text{ for } t \in [0, T],$$

where we can choose $d_2 \geqslant 1$. It follows that

$$\begin{aligned}|\Phi[u](t)|_{k_0}^{\sim} &\leqslant \|\Phi[u](0)\|_{(k_0-1)m} + \int_0^t \|\partial_t \Phi[u](\tau)\|_{(k_0-1)m}\,d\tau + |\partial_t \Phi[u](t)|_{k_0-1} \\ &\leqslant \tilde{N} + \exp(d_2 t)\left(d_8 \tilde{N} + d_9 \int_0^t |u(\tau)|_{k_0}^{\sim}\,d\tau\right)\left(1 + \frac{1}{d_2}\right) \\ &\leqslant \exp(d_2 t)\left(d_{10}\tilde{N} + d_{11}\int_0^t |u(\tau)|_{k_0}^{\sim}\,d\tau\right) \quad \text{for } t \in [0, T]\end{aligned}$$

From this and $|u(t)|_{k_0}^{\sim} \leqslant R_0$ we conclude by induction that

$$|\Phi^j[u](t)|_{k_0}^{\sim} \leqslant \tilde{N}\, d_{10} \sum_{v=0}^{j-1} \frac{1}{v!}\left(\frac{d_{11}}{d_2}\right)^v \exp[d_2(v+1)t] + tR_0 \frac{d_{11}^j \exp(jd_2 t)}{(j-1)!\, d_2^{j-1}} \tag{7.6}$$

($t \in [0, T]; j = 0, 1, \ldots$). Note that $\Phi^j[u] = u$. Letting $j \to \infty$ we obtain

$$\begin{aligned}|u(t)|_{k_0}^{\sim} &\leqslant \tilde{N}\, d_{10} \exp\left(d_2 t + \frac{d_{11}}{d_2}\exp(d_2 t)\right) \\ &\leqslant d_{10}\exp[(d_{11}+1)\exp(d_2 t)]\,\tilde{N}\end{aligned} \tag{7.7}$$

for $t \in [0, T]$. We set

$$T_3 := \frac{1}{d_2}\log\left(\frac{1}{d_{11}+1}\log\frac{R_0}{\tilde{N}}\right) \tag{7.8}$$

for sufficiently small $\tilde{N}$. Then it holds that $|u(t)|_{k_0}^{\sim} \leqslant R_0$ for $t \in [0, T_3]$.

We denote by $[0, T_4]$ the largest interval on which the solution $u \in \mathscr{C}_T^k$ of (1.1)–(1.3) exists. It holds that $T_4 > T_3$. In fact, if we assume $T_4 \leqslant T_3$, then it follows with $|u(T_4)|_{k_0}^{\sim} \leqslant R_0$ and (7.1) that all the assumptions of Theorem 1.1 are satisfied in $[T_4, \infty)$. Hence there exists a $\Delta T > 0$ and a solution

$$\tilde{u} \in \bigcap_{j=0}^{k} C^j([T_4, T_4 + \Delta T], H^{(k-j)m}(\Omega)).$$

Note that $\partial_t^j \tilde{u}(T_4) = \partial_t^j u(T_4)$ for $j = 0, \ldots, k$ by the considerations of section 3. Therefore we can extend the solution $u \in \mathscr{C}_{T_4}^k$ to a solution $u \in \mathscr{C}_{T_4+\Delta T}^k$ in contradiction to the definition of T_4. Thus we have proved $T_4 > T_3 = \mathcal{O}(\log\log(1/\tilde{N}))$ as $\tilde{N} \downarrow 0$. Using Lemma 3.3 we obtain (1.14).

Appendix

In the first part of this section we study $G(t, x, \mathscr{D}u(t, x))$, where

$$G \in C_b^{(k-2)m+1}([0, T] \times \bar{\Omega} \times \{\mu \in \mathbb{R}^{q(n,m)}\}) \tag{A.1}$$

and $u \in \mathscr{C}_T^k$ for some $k \geqslant k_2 := [n/2m] + 4$.

Lemma A.1. *Let $k \geqslant k_2 = [n/2m] + 4$ and G satisfying (A.1) be given.*

(i) *We set $\tilde{G}(t, x) := G(t, x, \mathscr{D}u(t, x)) - G(t, x, 0)$. If $u \in \tilde{\mathscr{C}}_T^k \cap \mathscr{C}_T^{k_2}$, then $\tilde{G} \in \tilde{\mathscr{C}}_T^{k-2}$*

and

$$|\tilde{G}(t)|_{k-2}^{\sim} \leqslant c_1(1 + |u(t)|_{k-1})^{(k-2)m}|u(t)|_k^{\sim}, \tag{A.2}$$

$$|\tilde{G}(t)|_{k-3} \leqslant c_2(1 + |u(t)|_{k-1})^{(k-3)m}|u(t)|_{k-1} \tag{A.3}$$

for $t \in [0, T]$ *with* $c_1, c_2 > 0$ *depending only on* k, G *and* Ω.

(ii) *If* $u_1, u_2 \in \tilde{\mathscr{C}}_T^k \cap \mathscr{C}_T^{k_2}$, *then*

$$|G(t, . , \mathscr{D}u_1(t)) - G(t, . , \mathscr{D}u_2(t))|_{k-2}^{\sim} \leqslant c_3|u_1(t) - u_2(t)|_k^{\sim}, \tag{A.4}$$

$$|G(t, . , \mathscr{D}u_1(t)) - G(t, . , \mathscr{D}u_2(t))|_{k-3} \leqslant c_4|u_1(t) - u_2(t)|_{k-1}, \tag{A.5}$$

for $t \in [0, T]$, *where* $c_3, c_4 > 0$ *depend only on* k, G, Ω *and* $|||u_1|||_{k,T}^{\sim}$, $|||u_2|||_{k,T}^{\sim}$.

Before we can prove Lemma A.1 we need two lemmata given in the following.

Lemma A.2. *Suppose that* $i, j \in \mathbb{N}_0$ *such that* $r := \min\{i, j, i + j - [n/2] - 1\} \geqslant 0$. *Then for* $f \in C([0, T], H^i(\Omega))$, $g \in C([0, T], H^j(\Omega))$ *it holds that* $fg \in C([0, T], H^r(\Omega))$ *and*

$$\| f(t)g(t) \|_r \leqslant c \| f(t) \|_i \| g(t) \|_j \quad \text{for } t \in [0, T] \tag{A.6}$$

with $c > 0$ *depending only on* Ω, i *and* j; *here* fg *denotes the pointwise multiplication of* f *and* g.

Proof. Let $t \in [0, T]$ be fixed and suppose that $r = 0$. We prove that

$$\| f(t)g(t) \| \leqslant c \| f(t) \|_i \| g(t) \|_j \quad \text{for } t \in [0, T]. \tag{A.7}$$

If $i = [n/2] + 1$, then $f(t) \in C_b(\bar{\Omega})$ and $|f(t, x)| \leqslant c \| f(t) \|_i$ for $(t, x) \in [0, T] \times \bar{\Omega}$ by the lemma of Sobolev. Hence (A.7) follows immediately. The same holds if $j = [n/2] + 1$.

We suppose that $i, j \leqslant [n/2]$ and $i + j = [n/2] + 1$. Note that $i, j \geqslant 1$. We set

$$q := \frac{2n}{n - 2i}, \qquad p := \frac{n}{i}.$$

Then we have $2/q + 2/p = 1$, $q \geqslant 2$ and

$$2 \leqslant p = \frac{n}{[n/2] + 1 - j} \leqslant \frac{2n}{n - 2j}.$$

From Sobolev's lemma (compare [1]) it follows that $f(t) \in L_q(\Omega)$, $g(t) \in L_p(\Omega)$ and

$$\| f(t) \|_{L_q(\Omega)} \leqslant c \| f(t) \|_i, \qquad \| g(t) \|_{L_p(\Omega)} \leqslant c \| g(t) \|_j$$

for $t \in [0, T]$. Hölders inequality yields (A.7).

From (A.7) we conclude that

$$\begin{aligned} \| f(t_1)g(t_1) - f(t_2)g(t_2) \| \leqslant{}& \| f(t_1) \|_i \| g(t_1) - g(t_2) \|_j \\ &+ \| f(t_1) - f(t_2) \|_i \| g(t_2) \|_j \end{aligned}$$

for $t_1, t_2 \in [0, T]$ and hence $fg \in C([0, T], L_2(\Omega))$. This proves Lemma A.2 in the case $r = 0$. For $r > 0$, Lemma A.2 follows by induction.

Corollary A.3 (*i*) *Let* $u \in \mathscr{C}_T^k$, $v \in \mathscr{C}_T^l$ *with* $r := \min[k, l, k + l - [n/2m] - 1\} \geqslant 0$. *Then* $uv \in \mathscr{C}_T^r$ *and*

$$|u(t)v(t)|_r \leqslant c|u(t)|_k |v(t)|_l \quad \text{for } t \in [0, T], \tag{A.8}$$

where $c > 0$ depends only on Ω, k, l and r.
(ii) Let $u \in \tilde{\mathscr{C}}^k_T$, $v \in \tilde{\mathscr{C}}^l_T$ with $r := \min\{k, l, k + l - [n/2m] - 2\} \geqslant 1$. Then $uv \in \tilde{\mathscr{C}}^r_T$ and

$$|u(t)v(t)|_r^{\sim} \leqslant c|u(t)|_k^{\sim}|v(t)|_l^{\sim} \quad \text{for } t \in [0, T], \tag{A.9}$$

with $c > 0$ depending on Ω, k, l and r.

Proof. Note that $(k + l)m \geqslant [n/2] + 1$ in case (i) and $(k + l - 2)m \geqslant [n/2] + 1$ in case (ii), respectively. Corollary A.3 follows from Lemma A.2 by the Leibnitz rule and the definition of $\mathscr{C}^k_T$ and $\tilde{\mathscr{C}}^k_T$ (compare (1.7), (1.18), (2.3) and (2.5)).

Proof of Lemma A.1. At first we prove by induction that

$$\left.\begin{aligned} &\tilde{G} \in C([0, T], H^l(\Omega)),\\ &\|\tilde{G}(t)\|_l \leqslant c_1(1 + |u(t)|_{k-1})^l |u(t)|_{k-1} \quad \text{for } t \in [0, T] \end{aligned}\right\} \tag{A.10}$$

for $l = 0, \ldots, (k - 3)m$.

It holds that

$$\|\tilde{G}(t)\| \leqslant c_2|u(t)|_2 \quad \text{for } t \in [0, T], \tag{A.11}$$

since

$$|G(t, x, \mu) - G(t, x, 0)| \leqslant c_2|\mu| \quad \text{for } (t, x, \mu) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^{q(n, m)}.$$

In the same way we obtain from

$$\begin{aligned} \partial_t \tilde{G}(t, x) = {} & (\partial_t G)(t, x, \mathscr{D}u(t, x)) - (\partial_t G)(t, x, 0)\\ & + \nabla_\mu G(t, x, \mathscr{D}u(t, x)) \cdot \mathscr{D}\partial_t u(t, x) \end{aligned} \tag{A.12}$$

that $\|\partial_t \tilde{G}(t)\| \leqslant c_3|u(t)|_3$ for $t \in [0, T]$. This implies $\tilde{G} \in C([0, T], L_2(\Omega)) = \mathscr{C}^0_T$ and (A.10) is proved for $l = 0$. In the case $l = 1$ we obtain from

$$\begin{aligned} \frac{\partial \tilde{G}(t, x)}{\partial x_i} = {} & \frac{\partial G}{\partial x_i}(t, x, \mathscr{D}u(t, x)) - \frac{\partial G}{\partial x_i}(t, x, 0)\\ & + \nabla_\mu G(t, x, \mathscr{D}u(t, x)) \cdot \mathscr{D}\frac{\partial u(t, x)}{\partial x_i} \end{aligned} \tag{A.13}$$

and the above argument that $\|(\partial \tilde{G}/\partial x_i)(t)\| \leqslant c_4|u(t)|_3$ $(i = 1, \ldots, n)$. Furthermore it follows that $\partial \tilde{G}/\partial x_i \in C([0, T], L_2(\Omega))$, since

$$\begin{aligned} & \left\| [G(t_1, ., \mathscr{D}u(t_1)) - G(t_1, ., \mathscr{D}u(t_2))]\, \mathscr{D}\frac{\partial u(t_1)}{\partial x_i} \right\|\\ & \leqslant c_5 \sup_{x \in \Omega} |\mathscr{D}u(t_1, x) - \mathscr{D}u(t_2, x)| \left\| \mathscr{D}\frac{\partial u(t_1)}{\partial x_i} \right\|\\ & \leqslant c_6 \|u(t_1) - u(t_2)\|_{(k-1)m} \left\| \mathscr{D}\frac{\partial u(t_1)}{\partial x_i} \right\| \end{aligned}$$

for $t_1, t_2 \in [0, T]$.

Let (A.10) be proved for $l = 0, \ldots, L - 1$ with $2 \leqslant L \leqslant (k - 3)m$ and consider (A.13). We obtain from the induction hypothesis that

$$\left.\begin{aligned} &\frac{\partial G}{\partial x_i}(., ., \mathscr{D}u) - \frac{\partial G}{\partial x_i}(., ., 0) \in C([0, T], H^{L-1}(\Omega)),\\ &\nabla_\mu G(., ., \mathscr{D}u) - \nabla_\mu G(., ., 0) \in C([0, T], H^{L-1}(\Omega)). \end{aligned}\right\} \tag{A.14}$$

Obviously it holds that

$$\left.\begin{aligned}&\nabla_\mu G(.,.,0)\cdot\mathscr{D}\frac{\partial u}{\partial x_i}\in\mathscr{C}_T^{k-4},\\&\left|\nabla_\mu G(t,.,0)\cdot\mathscr{D}\frac{\partial u(t)}{\partial x_i}\right|_{k-4}\leqslant c|u(t)|_{k-1}\quad\text{for } t\in[0,T].\end{aligned}\right\}\tag{A.15}$$

With Lemma A.2 it follows from (A.13)–(A.15) that $\partial G/\partial x_i\in C([0,T],H^{L-1}(\Omega))$ for $i=1,\ldots,n$. This proves the first part of (A.10) for $l=L$. By an analogous argument we obtain the estimate in (A.10) for $l=L$. Hence (A.10) is proved for $l=0,\ldots,(k-3)m$.

In the next step we prove

$$\left.\begin{aligned}&\tilde G\in\mathscr{C}_T^l,\\&|\tilde G(t)|_l<c(1+|u(t)|_{k-1})^{lm}|u(t)|_{k-1}\quad\text{for } t\in[0,T]\end{aligned}\right\}\tag{A.16}$$

for $l=0,\ldots,k-3$ by induction. In the case $l=0$, (A.16) is already proved by (A.10). For $l=1$, (A.16) follows from (A.12) by the same argument that proves (A.10) in the case $l=1$. Let (A.16) be proved for $l=0,\ldots,L-1$ with $2\leqslant L\leqslant k-3$ and consider (A.12). The induction hypothesis yields

$$(\partial_t G)(.,.,\mathscr{D}u)-(\partial_t G)(.,.,0)\in\mathscr{C}_T^{L-1},$$
$$\nabla_\mu G(.,.,\mathscr{D}u)-\nabla_\mu G(.,.,0)\in\mathscr{C}_T^{L-1}.$$

Together with

$$\nabla_\mu G(.,.,0)\cdot\mathscr{D}\frac{\partial u}{\partial t}\in\mathscr{C}_T^{k-3},\tag{A.17a}$$

$$\left|\nabla_\mu G(.,.,0)\cdot\mathscr{D}\frac{\partial u}{\partial t}\right|_{k-3}\leqslant c|u(t)|_k^{\sim}\quad\text{for } t\in[0,T],\tag{A.17b}$$

$$\left|\nabla_\mu G(.,.,0)\cdot\mathscr{D}\frac{\partial u}{\partial t}\right|_{k-4}\leqslant c|u(t)|_{k-1}\quad\text{for } t\in[0,T],\tag{A.17c}$$

and Corollary A.3 it follows that

$$\partial\tilde G/\partial t\in\mathscr{C}_T^{L-1}.\tag{A.18}$$

Using (A.10) we conclude that $\tilde G\in\mathscr{C}_T^L$. The estimate in (A.16) with $l=L$ can be proved in the same way. Hence (A.16) holds for $l=0,\ldots,k-3$, which proves (A.3). In order to prove $\tilde G\in\mathscr{C}_T^{k-2}$ and (A.2), we start from (A.16) with $l=k-3$. The argument leading to (A.18) yields $\partial\tilde G/\partial t\in\mathscr{C}_T^{k-3}$. This implies together with $\tilde G\in\mathscr{C}_T^{k-3}$ that $\tilde G\in\tilde{\mathscr{C}}_T^{k-2}$. The estimate (A.2) follows analogously.

In order to prove the second part of Lemma A.1 we note that

$$|G(t,x,\mathscr{D}u_1(t,x))-G(t,x,\mathscr{D}u_2(t,x))|\leqslant c|\mathscr{D}u_1(t,x)-\mathscr{D}u_2(t,x)|$$

($(t,x)\in[0,T]\times\bar\Omega$) and therefore

$$\|G(t,.,\mathscr{D}u_1(t))-G(t,.,\mathscr{D}u_2(t)\|\leqslant c|u_1(t)-u_2(t)|_{k-1}.$$

From this estimate the assertion follows by the same argument as above.

Corollary A.4. *Suppose that $k \geqslant [n/2m] + 4$, $R > 0$ and $u^j \in H^{(k-j)m}(\Omega)$ for $j = 0, 1, 2$ with $\|u^0\|_{km} + \|u^1\|_{(k-1)m} + \|u^2\|_{(k-2)m} \leqslant R$. Let G satisfy (A.1). We set*

$$G^*(x) := G(0, x, \bar{D}_x^{2m} u^0(x), \bar{D}_x^m u^1(x), u^2(x)) - G(0, x, 0).$$

Then $G^* \in H^{(k-2)m}(\Omega)$ and

$$\|G^*\|_{(k-2)m} \leqslant c(\|u^0\|_{km} + \|u^1\|_{(k-1)m} + \|u^2\|_{(k-2)m}), \tag{A.19}$$

where $c > 0$ depends on k, R, G and Ω.

The proof of Corollary A.4 is the same as the proof of (A.10).

Corollary A.5. *Let $k \geqslant k_2 = [n/2m] + 4$ and G satisfying (A.1) be given. Furthermore suppose that $u \in \tilde{\mathscr{C}}_T^k \cap \mathscr{C}_T^{k_2}$ and $v \in \tilde{\mathscr{C}}_T^{k-2}$. Then $G(.,., \mathscr{D}u)v \in \tilde{\mathscr{C}}_T^{k-2}$ and*

$$|G(t,., \mathscr{D}u(t))\, v(t)|_{k-3} \leqslant c_1(1 + |u(t)|_{k-1})^{(k-3)m} |u(t)|_{k-1} |v(t)|_{k-3}, \tag{A.20}$$

$$\|\partial_t^{k-2}[G(t,., \mathscr{D}u(t))v(t)]\| \leqslant c_2(1 + |u(t)|_{k-1})^{(k-2)m} |u(t)|_k^{\sim} |v(t)|_{k-3} + c_3 \|\partial_t^{k-2} v(t)\| \tag{A.21}$$

for $t \in [0, T]$, where $c_1, c_2, c_3 > 0$ depend only on k, G and Ω.

Proof. The assertion (A.20) can be shown by Lemma A.1, Corollary A.3 and

$$G(t,., \mathscr{D}u(t))v(t) = [G(t,., \mathscr{D}u(t)) - G(t,.,0)]v(t) + G(t,.,0)v(t)$$

($t \in [0, T]$). Equation (A.21) follows analogously from the Leibnitz rule and Lemma A.2.

Lemma A.6. *Let $k \geqslant [n/2] + 1$ and $w_j = (w_j^{(1)}, \ldots, w_j^{(s_j)}) \in C([0, T], H^{k-j}(\Omega))$ for $1 \leqslant j \leqslant k$ be given. If $\gamma_j \in \mathbb{N}_0^{s_j}$ such that*

$$1 \leqslant \sum_{j=1}^{k} j\,|\gamma_j| \leqslant l \tag{A.22}$$

for some $l \leqslant k$, then

$$\prod_{j=1}^{l} w_j^{\gamma_j} \in C([0, T], H^{k-l}(\Omega)), \tag{A.23}$$

$$\left\| \prod_{j=1}^{l} w_j^{\gamma_j}(t) \right\|_{k-l} \leqslant \prod_{j=1}^{l} \|w_j(t)\|_{k-j}^{|\gamma_j|} \quad \text{for } t \in [0, T], \tag{A.24}$$

where

$$w_j^{\gamma_j} := \prod_{i=1}^{s_j} (w_j^{(i)})^{\gamma_j^{(i)}}, \qquad \|w_j(t)\|_{k-j} := \sum_{i=0}^{s_j} \|w_j^{(i)}(t)\|_{k-j}$$

and the constant $c > 0$ depends only on k, l and Ω.

Proof. We perform induction with respect to $l \in \mathbb{N}$. If $l = 1$, we have $\gamma_1 = (0, \ldots, 0, 1, 0, \ldots, 0)$ and (A.23), (A.24) are obvious.

Let (A.23) and (A.24) be proved for $1 \leqslant l \leqslant L - 1$ with $L \leqslant k$. We assume that $\gamma_j \in \mathbb{N}_0^{s_j}$ ($j = 1, \ldots, L$) are given such that (A.22) holds with $l = L$. We have the following two cases:

(1) $|\gamma_L| = 1$. Then $|\gamma_1| = \cdots = |\gamma_{L-1}| = 0$ and (A.23), (A.24) are obvious.

(2) $|\gamma_L| = 0$. We assume that $\Sigma_{j=1}^{L-1} j|\gamma_j| = L$. Otherwise the assertion follows directly from the induction hypothesis. Let $J := \max\{j \in \mathbb{N} : |\gamma_j| \neq 0\}$. We set

$$v := \left(\sum_{j=1}^{J-1} w_j^{\gamma_j}\right) w_J^{\tilde{\gamma}_J}, \tag{A.25}$$

where $\tilde{\gamma}_J \in \mathbb{N}_0^{s_J}$ is chosen so that $|\gamma_J - \tilde{\gamma}_J| = 1$, $\tilde{\gamma}_J \leqslant \gamma_J$. Note that

$$\sum_{j=1}^{J=1} j|\gamma_j| + J|\tilde{\gamma}_j| = L - J.$$

This implies that $|\tilde{\gamma}_J| = 0$ if $J > L - J$ and $|\gamma_j| = 0$ if $L - J < j < J - 1$. Hence it follows from $w = vw_J^{\gamma_J - \tilde{\gamma}_J}$ together with the induction hypothesis and Lemma A.2 that (A.23) and (A.24) hold for $l = L$. This concludes the proof of Lemma A.6.

Lemma A.7. *Let $k \geqslant [n/2m] + 5$ and suppose that Assumption 2.1 holds and that $u \in \tilde{\mathscr{C}}_T^k$ is a solution of (1.1)–(1.3), where $f \in \mathscr{C}_T^{k-2}$. Then $u \in \mathscr{C}_T^k$.*

Proof. Let $\psi \in C^\infty(\Omega)$ with $\overline{\text{supp}\ \psi} \subset \Omega$. We multiply (1.1) with ψ and differentiate the result with respect to x_i ($i = 1, \ldots, n$). This yields

$$\mathscr{A}_u(t)\left(\psi(x)\frac{\partial u(t,x)}{\partial x_i}\right) = h(t,x) \quad \text{for } (t,x) \in [0,T] \times \Omega, \tag{A.26}$$

where

$$\begin{aligned} h(t,x) := & \sum_{|\alpha| \leqslant 2m} a_\alpha(t,x,\mathscr{D}u(t,x))\left[\partial_x^\alpha\left(\psi(x)\frac{\partial u(t,x)}{\partial x_i}\right) - \psi(x)\partial_x^\alpha \frac{\partial u(t,x)}{\partial x_i}\right] \\ & - \psi(x)\mathscr{B}_u(t)\left(\frac{\partial}{\partial x_i}\partial_t u(t,x)\right) - \psi(x)\frac{\partial}{\partial x_i}\partial_t^2 u(t,x) \\ & + \left(\frac{\partial F(t,x,\mathscr{D}u(t,x))}{\partial \mu_2}\right)^{-1}\left(\frac{\partial}{\partial x_i}(\psi(x)f(t,x))\right. \\ & \left. - \psi(x)\frac{\partial F}{\partial x_i}(t,x,\mathscr{D}u(t,x)) - \frac{\partial \psi(x)}{\partial x_i}F(t,x,\mathscr{D}u(t,x))\right). \end{aligned} \tag{A.27}$$

Note that $h \in C([0,T], H^{(k-3)m}(\Omega))$. From elliptic regularity theory (compare [6]) we obtain $\psi \partial u/\partial x_i \in C([0,T], H^{(k-1)m}(\Omega))$. Similar considerations can be made at the boundary of Ω, so that it follows that $u \in C([0,T], H^{(k-1)m+1}(\Omega))$ (if Ω has the form (2.1), compare the considerations in [6]). From this and (A.27) we obtain $h \in C([0,T], H^{(k-3)m+1}(\Omega))$. Repeating this step $m-1$ times we conclude that $u \in C([0,T], H^{km}(\Omega))$. Since $u \in \tilde{\mathscr{C}}_T^k$, this implies $u \in \mathscr{C}_T^k$.

Lemma A.8. *Let $k \geqslant 2$ and suppose that $u^j \in H^{(k-j)m}(\Omega)$ for $j = 0, \ldots, k-1$. Then there exists a function $u \in \tilde{\mathscr{C}}_T^k$ with $\partial_t^j u(0) = u^j$ for $j = 0, \ldots, k-1$.*

Proof. At first we assume that k is even. Let $\tilde{u}^j \in H^{(k-j)m}(\mathbb{R}^n)$ for $j = 0, \ldots, k-1$ be extensions of u^j onto $\mathbb{R}^n$ (compare [1]). We define an operator A by

$$\left.\begin{aligned} D(A) &:= \{v \in L_2(\mathbb{R}^n) : (-\Delta)^m v \in L_2(\mathbb{R}^n)\}, \\ Av &:= (-\Delta)^m v \quad \text{for } v \in D(A). \end{aligned}\right\} \tag{A.28}$$

Then A is a positive and self-adjoint in $L_2(\mathbb{R}^n)$. Let v be the solution of

$$\left.\begin{aligned} \left(\prod_{j=1}^{k/2} (\partial_t^2 + jA)\right) v = 0 \quad & \text{for } t \in [0, T]' \\ \partial_t^j v(0) = \tilde{u}^j \quad & \text{for } j = 0, \ldots, k-1. \end{aligned}\right\} \tag{A.29}$$

It follows by spectral theory and by elliptic regularity theory that (A.29) has a solution

$$v \in \bigcap_{j=0}^{k} C^j([0, T], H^{(k-j)m}(\mathbb{R}^n)).$$

The restriction u of v on Ω is the desired function. Note that $u \in \mathscr{C}_T^k$.

If k is odd, then we choose $w \in \mathscr{C}_T^{k-1}$ such that $\partial_t^j w(0) = u^{j+1}$ for $j = 0, \ldots, k-2$. Then

$$u(t) := u^0 + \int_0^t w(\tau)\, d\tau$$

is the desired function.

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