Mathematical Methods in the Applied Sciences, Vol. 11, 697–723 (1989)
MOS subject classification: 35 B 40, 35 C 10, 35 E 99

Resonance Phenomena for a Class of Partial Differential Equations of Higher Order in Cylindrical Waveguides

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Communicated by P. Werner

We consider a domain Ω in $\mathbb{R}^n$ of the form $\Omega = \mathbb{R}^l \times \Omega'$ with bounded $\Omega' \subset \mathbb{R}^{n-l}$. In Ω we study the Dirichlet initial and boundary value problem for the equation $\partial_t^2 u + [(-\partial_1^2 - \cdots - \partial_l^2)^m + (-\partial_{l+1}^2 - \cdots - \partial_n^2)^m] u = f e^{-i\omega t}$. We show that resonances can occur if $2m \geqslant l$. In particular, the amplitude of u may increase like t^α (α rational, $0 < \alpha < 1$) or like $\ln t$ as $t \to \infty$. Furthermore, we prove that the limiting amplitude principle holds in the remaining cases.

1. Introduction

Let Ω be an unbounded domain in $\mathbb{R}^n$ with

$$\Omega = \mathbb{R}^l \times \Omega', \tag{1.1}$$

where $\Omega' \subset \mathbb{R}^{n-l}$ is bounded. In the following we study the solution of the initial and boundary value problem

$$\partial_t^2 u + [(-\Delta_x)^m + (-\Delta_y)^m] u = f e^{-i\omega t} \quad \text{in } \Omega \times [0, \infty), \tag{1.2}$$

$$u = \frac{\partial u}{\partial \mathbf{n}} = \ldots = \frac{\partial^{m-1} u}{\partial \mathbf{n}^{m-1}} = 0 \qquad \text{on } \partial\Omega \times [0, \infty), \tag{1.3}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \partial_t u(\mathbf{x}, 0) = u_1(\mathbf{x}) \qquad \text{in } \Omega, \tag{1.4}$$

with $\Delta_x := \partial_1^2 + \cdots + \partial_l^2$ and $\Delta_y := \partial_{l+1}^2 + \cdots + \partial_n^2$, $\mathbf{n}$ denotes the unit normal vector on $\partial\Omega$. For the sake of simplicity, we assume that $f, u_0, u_1 \in C_0^\infty(\Omega)$ and $\partial\Omega \in C^\infty$. If $m = 1$, then (1.2) coincides with the wave equation. We are mainly interested in the asymptotic behaviour of $u(\mathbf{x}, t)$ as $t \to \infty$.

The investigations in this paper are motivated by the results of Ramm and Werner[6] and Werner,[12] who studied problem (1.2)–(1.4) in the case $m = 1$ for the domains $\Omega = \mathbb{R}^l \times (0, 1)$ and $\Omega = \mathbb{R} \times \Omega'$ with bounded $\Omega' \subset \mathbb{R}^{n-1}$, respectively. These configurations allow the following resonances: if $\Omega = \mathbb{R} \times \Omega'$ and if ω^2 is an eigenvalue of

0170–4214/89/060697–27$13.50
Received 7 September 1988

the Dirichlet problem

$$-\Delta_y V - \lambda V = 0 \quad \text{in} \quad \Omega', \qquad V = 0 \quad \text{on} \quad \partial\Omega' \tag{1.5}$$

for the cross-section Ω', then

$$u(\mathbf{x}, t) = t^{1/2} e^{-i\omega t} v(\mathbf{x}) + O(1) \qquad \text{as} \quad t \to \infty \tag{1.6}$$

with a suitable $v \in C(\bar{\Omega})$. If $\Omega = \mathbb{R}^2 \times (0, 1)$ and $\omega = \pi j (j = 0, 1, \ldots)$, then

$$u(\mathbf{x}, t) = \ln t \cdot e^{-i\omega t} v(\mathbf{x}) + O(1) \qquad \text{as } t \to \infty. \tag{1.7}$$

In the case of the remaining frequencies $u(\mathbf{x}, t)$ is bounded as $t \to \infty$ and satisfies the limiting amplitude principle

$$u(\mathbf{x}, t) = U_\omega(\mathbf{x}) e^{-i\omega t} + o(1) \qquad \text{as } t \to \infty, \tag{1.8}$$

where U_ω denotes a solution of the boundary value problem

$$(-\Delta_x - \Delta_y) U_\omega - \omega^2 U_\omega = f \quad \text{in } \Omega, \qquad U_\omega = 0 \quad \text{on } \partial\Omega. \tag{1.9}$$

The estimates (1.6)–(1.8) hold uniformly in every bounded subset of $\bar{\Omega}$. If $\Omega = \mathbb{R}^l \times (0, 1)$ with $l \geq 3$, then the limiting amplitude principle (1.8)–(1.9) holds for every frequency $\omega \geq 0$, so that no resonances occur. Hence, in the special case $m = 1$, problem (1.2)–(1.4) admits resonances only if $l = 1$ or $l = 2$ in (1.1). The resonances are of order $t^{1/2}$ for $l = 1$ and of order $\ln t$ for $l = 2$.

In the following, we study (1.2)–(1.4) also for $m > 1$ and prove that resonances are not possible if $l > 2m$. If $l \leq 2m$, resonances of order t^α with rational $\alpha \in (0, 1)$ or order $\ln t$ can occur for suitable frequencies. In particular, we obtain the following results: if $l > 2m$, then for every given $\omega \geq 0$ there exists a solution U_ω of the boundary value problem

$$\begin{aligned} &[(-\Delta_x)^m + (-\Delta_y)^m] U_\omega - \omega^2 U_\omega = f && \text{in } \Omega, \\ &U_\omega = \frac{\partial U_\omega}{\partial \mathbf{n}} = \cdots = \frac{\partial^{m-1} U_\omega}{\partial \mathbf{n}^{m-1}} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

such that (1.8) holds uniformly in every bounded subset of $\bar{\Omega}$ (limiting amplitude principle). U_ω can be uniquely characterized by a suitable radiation condition, which can be considered as a generalization of a uniqueness condition introduced by Sveshnikov[7] in the case $m = 1$. Furthermore, (1.8) and (1.10) remain valid if $l \leq 2m$ and if ω^2 does not coincide with one of the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the Dirichlet problem

$$\begin{aligned} &(-\Delta_y)^m V - \lambda V = 0 && \text{in } \Omega', \\ &V = \frac{\partial V}{\partial n'} = \cdots = \frac{\partial^{m-1} V}{\partial n'^{m-1}} = 0 && \text{on } \partial\Omega' \end{aligned} \tag{1.11}$$

for the cross-section Ω' ($n' :=$ unit normal vector on $\partial\Omega'$). However, if $l \leq 2m$ and $\omega^2 = \lambda_j$, then the following resonance phenomenon can be observed: if l is odd, then the estimate

$$u(\mathbf{x}, t) = \sum_{s=0}^{m-(l+1)/2} = E_s t^{1-\frac{l+2s}{2m}} e^{-i\omega t} p_s^{(j)}(\mathbf{x}) + U(\mathbf{x}) e^{-i\omega t} + o(1) \text{ as } t \to \infty \tag{1.12}$$

holds; here U is a solution of (1.10), E_s are suitable constants and

$$p_s^{(j)}(\mathbf{x}) = p_s^{(j)}(x, y) := \sum_{k=1}^{\kappa(j)} V_{jk}(y) \int_\Omega f(x', y') V_{jk}(y') |x - x'|^{2s} d(x', y'), \tag{1.13}$$

where $x := (x_1, \ldots, x_l)$, $y := (x_{l+1}, \ldots, x_n)$ and $V_{jk}(k = 1, \ldots \kappa(j))$ denotes an orthonormal basis of the eigenspace E_j belonging to the eigenvalue λ_j of (1.11). For even l (1.12) has to be replaced by

$$u(\mathbf{x}, t) = \sum_{s=0}^{m-1-l/2} E_s t^{1-\frac{l+2s}{2m}} e^{-i\omega t} p_s^{(j)}(\mathbf{x}) + E^* \ln t \cdot e^{-i\omega t} p_{m-l/2}^{(j)}(\mathbf{x})$$
$$+ U(\mathbf{x}) e^{-i\omega t} + o(1) \qquad \text{as } t \to \infty. \tag{1.14}$$

Both estimates hold uniformly in every bounded subset of $\bar{\Omega}$. The precise values of E_s and E^* will be given in Section 6.

The analysis of this paper is based on the spectral theory for unbounded self-adjoint operators. In Section 2, we extend the differential operator $(-\Delta_x)^m + (-\Delta_y)^m$ to a self-adjoint operator A in the Hilbert space $L_2(\Omega)$ with respect to Dirichlet's boundary condition (1.3). Applying the functional calculus for unbounded self-adjoint operators, we obtain a spectral integral representation for the solution u of (1.2)–(1.4). Sections 3 and 4 are deveoted to the study of the resolvent of A and the construction of the spectral family $\{P_\lambda\}$ of A. The explicit form of $\{P_\lambda\}$ is used in Sections 5 and 6 to obtain the above asymptotic for $u(\mathbf{x}, t)$ as $t \to \infty$.

The results of this paper are contained in the author's thesis,[3] to which we refer for a more detailed presentation of some of the proofs.

2. The spectral integral representation of the solution

In order to extend the operator $(-\Delta_x)^m + (-\Delta_y)^m$ to a self-adjoint operator with respect to the Dirichlet condition (1.3), we set

$$\left.\begin{aligned} D(A) &:= \{U \in \mathring{H}_m(\Omega) : [(-\Delta_x)^m + (-\Delta_y)^m] U \in L_2(\Omega)\}, \\ AU &:= [(-\Delta_x)^m + (-\Delta_y)^m] U \quad \text{for} \quad U \in D(A); \end{aligned}\right\} \tag{2.1}$$

here the differential operators $(-\Delta_x)^m$ and $(-\Delta_y)^m$ are interpreted in the sense of distribution theory and $\mathring{H}_m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H_m(\Omega)$ (compare, for example, Reference 8). We prove:

Lemma 2.1. *The operator A defined by* (2.1) *is positive and self-adjoint in the Hilbert space $L_2(\Omega)$.*

Proof. Consider the bilinear form

$$B(U, V) := \begin{cases} (\Delta_x^{m/2} U, \Delta_x^{m/2} V) + (\Delta_y^{m/2} U, \Delta_y^{m/2} V) & (m \text{ even}), \\ (\nabla_x \Delta_x^{(m-1)/2} U, \nabla_x \Delta_x^{(m-1)/2} V + \\ (\nabla_y \Delta_y^{(m-1)/2} U, \nabla_y \Delta_y^{(m-1)/2} V) & (m \text{ odd}), \end{cases} \tag{2.2}$$

for $U, V \in \mathring{H}_m(\Omega)$ with

$$(\nabla_x U, \nabla_x V) := \sum_{i=1}^{l} (\partial_i U, \partial_i V), \qquad (\nabla_y U, \nabla_y V) := \sum_{i=l+1}^{n} (\partial_i U, \partial_i V)$$

($(\,,)$:= inner product in $L_2(\Omega)$). Assume that $U \in D(A)$ and $\varphi \in C_0^\infty(\Omega)$. Then

$$(AU, \varphi) = ([(-\Delta_x)^m + (-\Delta_y)^m]U, \varphi) = [(-\Delta_x)^m U + (-\Delta_y)^m U](\bar{\varphi})$$
$$= \sum_{i=1}^{l} [\partial_i(-\Delta_x)^{m-1}U](\partial_i\bar{\varphi}) + \sum_{i=l+1}^{n} [\partial_i(-\Delta_y)^{m-1}U](\partial_i\bar{\varphi}),$$

where $F(\varphi)$ denotes the application of the distribution F to the test function φ. Repeating this step m times, we obtain $(A, U, \varphi) = B(U, \varphi)$, and hence

$$B(U, V) = (AU, V) \qquad \text{for } U, V \in D(A), \tag{2.3}$$

since $C_0^\infty(\Omega)$ is dense in $D(A)$. This implies that A is symmetric and positive.

Now we show that A is elliptic. Substituting x_i for $\partial_i U$ and $\partial_i V$ in (2.2), we obtain for odd m

$$B(\mathbf{x}, \mathbf{x}) = \sum_{j=1}^{l} \left(x_j \left(\sum_{i=1}^{l} x_i^2 \right)^{(m-1)/2} \cdot x_j \left(\sum_{i=1}^{l} x_i^2 \right)^{(m-1)/2} \right)$$
$$+ \sum_{j=l+1}^{n} \left(x_j \left(\sum_{i=l+1}^{n} x_i^2 \right)^{(m-1)/2} \cdot x_j \left(\sum_{i=l+1}^{n} x_i^2 \right)^{(m-1)/2} \right),$$

and hence

$$B(\mathbf{x}, \mathbf{x}) = \left(\sum_{i=1}^{l} x_i^2 \right)^m + \left(\sum_{i=l+1}^{n} x_i^2 \right)^m. \tag{2.4}$$

By (2.2) this identity holds also for even m. Hölder's inequality implies

$$\left| \sum_{j=1}^{2} a_j \cdot 1 \right| \leqslant \left(\sum_{j=1}^{2} |a_j|^m \right)^{1/m} \left(\sum_{j=1}^{2} 1 \right)^{(m-1)/m},$$

so that

$$B(\mathbf{x}, \mathbf{x}) \geqslant \frac{1}{2^{m-1}} |\mathbf{x}|^{2m} \qquad \text{for } \mathbf{x} \in \mathbb{R}^n. \tag{2.5}$$

Thus A is uniformly strongly elliptic. In particular, Gårdings inequality

$$B(U, U) \geqslant c_1 \|U\|_m^2 - c_2 \|U\|^2 \qquad \text{for} \quad U \in \mathring{H}_m(\Omega) \tag{2.6}$$

holds with suitable $c_1, c_2 > 0$. This implies by a familiar argument that $A + c_2$ is a bijective linear mapping from $D(A)$ onto $L_2(\Omega)$. Hence A is self-adjoint. This concludes the proof of Lemma 2.1.

We consider the following weak formulation of problem (1.2)–(1.4): find $U \in C^2([0, \infty), L_2(\Omega))$ such that

$$U''(t) + AU(t) = f e^{-i\omega t} \qquad \text{for } t \geqslant 0,$$
$$U(0) = u_0, \; U'(0) = u_1; \tag{2.7}$$

here $U'(t)$ denotes the derivative of the mapping U: $[0, \infty) \to L_2(\Omega)$ with respect to the L_2 norm. By using (2.3) it can be shown that U is uniquely determined (compare the proof of Lemma 7.1 in Reference 9). Let $\{P_\lambda\}$ be the (left continuous) spectral family of

A. Note that $P_\lambda = 0$ for $\lambda \leqslant 0$, since A is positive. Set

$$U(t) := \int_0^\infty \cos\sqrt{\lambda}t\, \mathrm{d}(P_\lambda u_0) + \int_0^\infty \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, \mathrm{d}(P_\lambda u_1) + \int_0^\infty \psi(\lambda, t)\mathrm{d}(P_\lambda f), \quad (2.8)$$

where $\psi(\lambda, t)$ is the solution of the initial value problem

$$(\partial_t^2 + \lambda)\psi(\lambda, t) = \mathrm{e}^{-\mathrm{i}\omega t} \quad \text{for } t \geqslant 0,\ \psi(\lambda, 0) = \partial_t \psi(\lambda, 0) = 0, \quad (2.9)$$

given by

$$\psi(\lambda, t) = \begin{cases} \dfrac{1}{\lambda - \omega^2}\left(\mathrm{e}^{-\mathrm{i}\omega t} - \cos\sqrt{\lambda}t + \dfrac{\mathrm{i}\omega}{\sqrt{\lambda}}\sin\sqrt{\lambda}t\right) & \text{if } \lambda \neq \omega^2, \\ \dfrac{\mathrm{i}}{2\omega}\left(t\,\mathrm{e}^{-\mathrm{i}\omega t} - \dfrac{1}{\omega}\sin\omega t\right) & \text{if } \lambda = \omega^2. \end{cases} \quad (2.10)$$

The functional calculus for unbounded self-adjoint operators shows that U is a solution of (2.7). Since $f, u_0, u_1 \in C_0^\infty(\Omega) \subset D(A^r)$ for every $r \in \mathbb{N}$, it follows by the elliptic regularity theory that (1.2)–(1.4) has a classical solution $u(\mathbf{x}, t)$ belonging to $C^\infty(\bar{\Omega} \times [0, \infty))$, which is related to $U(t)$ by

$$U(t)\varphi = \int_\Omega u(\mathbf{x}, t)\varphi(\mathbf{x})\mathrm{d}\mathbf{x} \qquad \text{for every } \varphi \in C_0^\infty(\Omega) \quad (2.11)$$

(compare, for example, the discussion in Reference 10 in a related situation). On the other hand, every solution u of (1.2)–(1.4) having the property

$$u(., t) \in H_m(\Omega) \qquad \text{for every } t \geqslant 0 \quad (2.12)$$

defines a solution u of (2.7) by (2.11). Thus we obtain:

Lemma 2.2. *Problem* (1.2)–(1.4) *has one and only one solution u with the property* (2.12); *u is given by* (2.8) *and* (2.10).

3. The resolvent

In order to estimate the asymptotic behaviour of the solution u given by (2.8) and (2.10), we have to study the behaviour of the resolvent $R_z = (A - zI)^{-1}$ of A near the real axis. The computation of $R_z f$ for $z \in \mathbb{C}\backslash[0, \infty)$ and $f \in C_0^\infty(\Omega)$ leads to the classical problem

$$\begin{aligned} &[(-\Delta_x)^m + (-\Delta_y)^m - z]\,U_z = f && \text{in } \Omega, \\ &U_z = \frac{\partial U_z}{\partial \mathbf{n}} = \cdots = \frac{\partial^{m-1} U_z}{\partial \mathbf{n}^{m-1}} = 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.1)$$

We construct a solution U_z of (3.1) by setting

$$U_z(\mathbf{x}) = U_z(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} u_{jk}(x; z)\, V_{jk}(y), \quad (3.2)$$

where $V_{jk}(k = 1, \ldots, \kappa(j))$ denotes as in (1.13) an orthonormal basis of the eigenspace E_j for the eigenvalue λ_j of (1.11). We assume that the eigenvalues are ordered increasingly: $0 < \lambda_1 < \lambda_2 < \ldots$. Note that $V_{jk} \in C^\infty(\bar{\Omega}')$ by the elliptic regularity theory, since $\partial\Omega' \in C^\infty$. Using $(-\Delta_y)^m V_{jk} = \lambda_j V_{jk}$, we obtain formally

$$[(-\Delta_x)^m + (-\Delta_y)^m - z]\, U_z(\mathbf{x}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} [(-\Delta_x)^m + \lambda_j - z]\, u_{jk}(x;z)\, V_{jk}(y). \tag{3.3}$$

The expansion theorem for the interior boundary value problem (1.11) yields

$$f(\mathbf{x}) = f(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} f_{jk}(x)\, V_{jk}(y) \tag{3.4}$$

with

$$f_{jk}(x) = \int_{\Omega'} f(x, y)\, \bar{V}_{jk}(y)\, \mathrm{d}y. \tag{3.5}$$

These considerations suggest determining the coefficients in (3.2) such that

$$[(-\Delta_x)^m + \lambda_j - z]\, u_{jk} = f_{jk} \qquad \text{in } \mathbb{R}^l. \tag{3.6}$$

In order to compute u_{jk}, we extend $(-\Delta_x)^m$ to a self-adjoint operator $L(m)$ in $L_2(\mathbb{R}^l)$ by setting

$$\left.\begin{aligned} D(L(m)) &:= \{ W \in H_m(\mathbb{R}^l) : (-\Delta_x)^m W \in L_2(\mathbb{R}^l)\}, \\ L(m)\, W &:= (-\Delta_x)^m W \qquad \text{for} \quad W \in D(L(m)). \end{aligned}\right\} \tag{3.7}$$

Since $L(m)$ is uniformly strongly elliptic, the elliptic regularity theory implies $L(m) = L(1)^m$. Hence, the uniquely determined solution u_{jk} of (3.6) belonging to $D(L(m))$ coincides with the solution of

$$[L(1)^m + \lambda_j - z]\, u_{jk} = f_{jk}.$$

Let $\{P_\lambda^{(1)}\}$ denote the spectral family of $L(1)$. Then u_{jk} is given by

$$u_{jk} = \int_0^\infty \frac{1}{\lambda^m + \lambda_j - z}\, \mathrm{d}(P_\lambda^{(1)} f_{jk}).$$

We set $\mu_s := |\lambda_j - z|^{1/m} \mathrm{e}^{\mathrm{i}(\arg(z-\lambda_j) + 2\pi s)/m}$ $(s = 0, \ldots, m-1)$ and obtain

$$u_{jk} = \int_0^\infty \left\{ \sum_{s=0}^{m-1} \frac{1}{m\mu_s^{m-1}(\lambda - \mu_s)} \right\} \mathrm{d}(P_\lambda^{(1)} f_{jk}) =: \sum_{s=0}^{m-1} \frac{1}{m\mu_s^{m-1}}\, w_s.$$

Here w_s is the uniquely determined solution of $(-\Delta_x - \mu_s) w_s = f_{jk}$ in $\mathbb{R}^l$ belonging to $D(L(1))$; w_s has the explicit form

$$w_s(x;z) = \frac{\mathrm{i}|\lambda_j - z|^{\sigma/2m}}{4(2\pi)^\sigma} \mathrm{e}^{\mathrm{i}\sigma\varphi_s(z;j)} \int_{\mathbb{R}^l} \frac{f_{jk}(x')}{|x - x'|^\sigma} H_\sigma^{(1)}(|x - x'|\,|\lambda_j - z|^{1/2m} \mathrm{e}^{\mathrm{i}\varphi_s(z;j)/2})\, \mathrm{d}x', \tag{3.8}$$

where $\sigma := (l/2) - 1$, $H_\sigma^{(1)}$ denotes Hankel's function,

$$H_\sigma^{(1)}(\zeta) = J_\sigma(\zeta) + \mathrm{i}N_\sigma(\zeta) \qquad \text{for } \zeta \in \mathbb{C}, \tag{3.9}$$

and

$$\varphi_s(z;j) := \frac{\arg(z - \lambda_j) + 2\pi s}{m} \qquad (s = 0, \ldots, m-1) \tag{3.10}$$

for $0 < \arg(z - \lambda_j) < 2\pi$. Thus we have

$$u_{jk}(x;z) = \frac{\mathrm{i}|\lambda_j - z|^{\frac{\sigma+2}{2m}-1}}{4m(2\pi)^\sigma} \sum_{s=0}^{m-1} \mathrm{e}^{\mathrm{i}\varphi_s(z;j)[(\sigma/2)+1-m]}$$

$$\int_{\mathbb{R}^l} \frac{f_{jk}(x')}{|x-x'|^\sigma} H_\sigma^{(1)}(|x-x'||\lambda_j - z|^{1/2m} \mathrm{e}^{\mathrm{i}\varphi_s(z;j)/2}) \mathrm{d}x'. \tag{3.11}$$

In order to proof the convergence of the Fourier series (3.2), we start from

$$L(m)u_{jk} + (\lambda_j - z)u_{jk} = f_{jk} \tag{3.12}$$

(compare (3.6) and (3.7)). It follows that

$$(L(m)u_{jk}, u_{jk})_{\mathbb{R}^l} + (\lambda_j - z)\, \| u_{jk} \|^2_{\mathbb{R}^l} = (f_{jk}, u_{jk})_{\mathbb{R}^l} \tag{3.13}$$

($(\cdot,\cdot)_{\mathbb{R}^l}$:= inner product in $L_2(\mathbb{R}^l)$, $\|\cdot\|_{\mathbb{R}^l}$:= norm in $L_2(\mathbb{R}^l)$). Since $L(m)$ is self-adjoint and positive, we obtain

$$\operatorname{Re}(\lambda_j - z)\, \| u_{jk} \|^2_{\mathbb{R}^l} \leqslant \operatorname{Re}(f_{jk}, u_{jk})_{\mathbb{R}^l},$$

$$-\operatorname{Im} z\, \| u_{jk} \|^2_{\mathbb{R}^l} = \operatorname{Im}(f_{jk}, u_{jk})_{\mathbb{R}^l},$$

and by the Cauchy–Schwarz inequality

$$\| u_{jk} \|_{\mathbb{R}^l} \leqslant c_j(z)\, \| f_{jk} \|_{\mathbb{R}^l}, \tag{3.14}$$

where

$$c_j(z) := \begin{cases} \min\left\{ \dfrac{1}{|\operatorname{Im} z|}, \dfrac{1}{\operatorname{Re}(\lambda_j - z)} \right\} & \text{if } \operatorname{Re} z < \lambda_j, \\ \dfrac{1}{|\operatorname{Im} z|} & \text{if } \operatorname{Re} z \geqslant \lambda_j. \end{cases} \tag{3.15}$$

The elliptic regularity theory and (3.12) imply $D^p u_{jk} \in D(L(m))$ and

$$L(m) D^p u_{jk} + (\lambda_j - z) D^p u_{jk} = D^p f_{jk} \tag{3.16}$$

for every multi-index $p \in \mathbb{N}_0^l$, since $f_{jk} \in C_0^\infty(\mathbb{R}^l)$. Hence, the above considerations yield

$$\| u_{jk} \|_{s, \mathbb{R}^l} \leqslant c_j(z)\, \| f_{jk} \|_{s, \mathbb{R}^l} \tag{3.17}$$

for every $s \in \mathbb{N}_0$ ($\|\cdot\|_{s,\mathbb{R}^l}$:= norm in $H_s(\mathbb{R}^l)$). Consider

$$h_z(\mathbf{x}) := \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} u_{jk}(x;z)\, V_{jk}(y), \tag{3.18}$$

with $j_1, j_2 \in \mathbb{N}$, $j_2 > j_1$. Let $d_j > 0$ and $i_j \in \mathbb{N}$ denote suitable constants. Sobolev's inequality implies

$$|h_z(\mathbf{x})|^2 = |h_z(x, y)|^2 \leqslant d_1\, \| h_z(x, .) \|^2_{i_1, \Omega'} \tag{3.19}$$

for every fixed $x \in \mathbb{R}^l$. Since $\Delta_y^{s-1} h_z(x, .) \in L_2(\Omega')$ for every $s \in \mathbb{N}$, it follows from the elliptic regularity theory that

$$\| \Delta_y^{s-1} h_z(x, .) \|^2_{2m, \Omega'} \leqslant d_2 (\| \Delta_y^{s-1} h_z(x, .) \|^2_{\Omega'} + \| \Delta_y^s h_z(x, .) \|^2_{\Omega'}).$$

Using this inequality, we obtain by the same argument

$$\|\Delta_y^{s-2} h_z(x,.)\|_{4m,\Omega'}^2 \leqslant d_2 \|\Delta_y^{s-2} h_z(x,.)\|_{\Omega'}^2 + d_2^2(\|\Delta_y^{s-1} h_z(x,.)\|_{\Omega'}^2 + \|\Delta_y^s h_z(x,.)\|_{\Omega'}^2).$$

Continuing in this way, we conclude that

$$\|h_z(x,.)\|_{2sm,\Omega'}^2 \leqslant d_3 \sum_{r=0}^{s} \|\Delta_y^r h_z(x,.)\|_{\Omega'}^2$$

$$= d_3 \sum_{r=0}^{s} \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2r} |u_{jk}(x;z)|^2. \tag{3.20}$$

Since

$$\lambda_j^{2r} = \lambda_1^{2r}\left(\frac{\lambda_j}{\lambda_1}\right)^{2r} \leqslant \lambda_1^{2r}\left(\frac{\lambda_j}{\lambda_1}\right)^{2s} = \frac{1}{\lambda_1^{2s-2r}} \lambda_j^{2s},$$

we have

$$\|h_z(x,.)\|_{2sm,\Omega'} \leqslant d_4 \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2s} |u_{jk}(x;z)|^2. \tag{3.21}$$

Combining (3.21), (3.19), Sobolev's inequality and (3.17), we obtain

$$|h_z(\mathbf{x})|^2 \leqslant d_5 \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_2} |u_{jk}(x;z)|^2$$

$$\leqslant d_6 \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_2} \|u_{jk}(x;z)\|_{i_3,\mathbb{R}^l}^2$$

$$\leqslant d_6 c_{j_1}(z)^2 \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_2} \|f_{jk}\|_{i_3,\mathbb{R}^l}^2, \tag{3.22}$$

since $c_{j_1}(z) \geqslant c_j(z)$ for $j > j_1$. On the other hand, by applying Bessel's inequality to the Fourier expansion (3.4), we conclude that

$$\sum_{j=1}^{N} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_2} |D_x^p f_{jk}(x)|^2 \leqslant \|\Delta_y^{i_2} D_x^p f(x,.)\|_{\Omega'}^2$$

for every $p \in \mathbb{N}_0^l$ and $N \in \mathbb{N}$ $(D_x^p := \partial_1^{p_1}, \ldots, \partial_l^{p_l})$. In particular, we have

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_2} \|f_{jk}\|_{i_3,\mathbb{R}^l}^2 \leqslant \sum_{|p| \leq i_3} \|\Delta_y^{i_2} D_x^p f\|_{\Omega}^2 < \infty. \tag{3.23}$$

Thus (3.18), (3.22) and (3.28) together with (3.15) imply that the Fourier series (3.2) of U_z converges uniformly with respect to $\mathbf{x} \in \bar{\Omega}$ and $z \in K$, if K is an arbitrary compact subset of $\mathbb{C} \setminus [\lambda_1, \infty)$. It follows in the same way that every termwise derivative of (3.2) converges uniformly with respect to $\mathbf{x} \in \bar{\Omega}$ and $z \in K$. Thus U_z is a solution of the differential equation in (3.1). Furthermore, we have

$$\frac{\partial^s U_z}{\partial \mathbf{n}^s}(\mathbf{x}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} \frac{\partial^s}{\partial \mathbf{n}^s}[u_{jk}(x;z) V_{jk}(y)] = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} u_{jk}(x;z) \frac{\partial^s}{\partial n'^s} V_{jk}(y) = 0 \tag{3.24}$$

for every $\mathbf{x} = (x, y) \in \partial\Omega$ and $s = 0, \ldots, m-1$, since $\mathbf{n} = (n_x, n_y)$ with $n_x = (0, \ldots, 0)$ and $n_y = n'$, and since each V_{jk} is an eigenfunction of (1.11). Hence U_z is a solution of (3.1).

The same argument as above yields that for $p \in \mathbb{N}_0^l$, $q \in \mathbb{N}_0^{n-1}$

$$
\begin{aligned}
\|D_x^p D_y^q h_z\|^2 &= \int_{\mathbb{R}^l} \left\{ \int_{\Omega'} |D_x^p D_y^q h_z(x, y)|^2 \, \mathrm{d}y \right\} \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^l} \|D_x^p h_z(x, .)\|_{|q|, \Omega'}^2 \, \mathrm{d}x \\
&\leqslant d_7 \int_{\mathbb{R}^l} \left\{ \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_4} |D_x^p u_{jk}(x; z)|^2 \right\} \mathrm{d}x \\
&= d_7 \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_4} \|D_x^p u_{jk}(.\,; z)\|_{\mathbb{R}^l}^2 \\
&\leqslant d_7 c_{j_1}(z)^2 \sum_{j=j_1}^{j_2} \sum_{k=1}^{\kappa(j)} \lambda_j^{2i_4} \|f_{jk}\|_{|p|, \mathbb{R}^l}^2 \\
&\leqslant d_7 c_1(z)^2 \|\Delta_y^{i_4} f\|_{|p|, \mathbb{R}^l}^2 .
\end{aligned}
$$

This shows that $U_z \in H_s(\Omega)$ for every $s \in \mathbb{N}$. Furthermore, U_z satisfies the boundary conditions (3.24) for $0 \leqslant s \leqslant m-1$. This implies $U_z \in \mathring{H}_m(\Omega)$, and hence $U_z \in D(A)$. Since U_z satisfies (3.1), we obtain $U_z = R_z f$. We collect these results in the following lemma:

Lemma 3.1. *Assume that $f \in C_0^\infty(\Omega)$ and $z \in \mathbb{C} \backslash [\lambda_1, \infty)$. Then the Fourier coefficients u_{jk} of $R_z f$ are given by* (3.11), (3.5) *and* (3.10), *and the Fourier series* (3.2) *of $R_z f$ and its termwise derivatives of arbitrary order converge uniformly with respect to $\mathrm{x} \in \bar{\Omega}$ and $z \in K$ for every compact subset K of* $\mathbb{C} \backslash [\lambda_1, \infty)$.

4. The spectral family

In order to compute the spectral family $\{P_\lambda\}$ of A, we investigate the behaviour of $R_z f$ as $\operatorname{Im} z \to 0$. Consider the Fourier series (3.2) of $R_z f$ and set $z = \rho + \mathrm{i}\tau$ with $\rho, \tau \in \mathbb{R}$. According to (3.12), the Fourier coefficients u_{jk} of $R_{\rho + \mathrm{i}\tau} f$ are solutions of

$$
[L(m) + \lambda_j - \rho - \mathrm{i}\tau] u_{jk}(.\,; \rho + \mathrm{i}\tau) = f_{jk}. \tag{4.1}
$$

Initially we assume $\lambda_j > \rho$. In this case (4.1) has a uniquely determined solution also for $\tau = 0$, since $L(m)$ is positive and self-adjoint. This solution $u_{jk}(.\,; \rho)$ is given by (3.10) and (3.11). We set

$$
S(\mathrm{x}, \rho + \mathrm{i}\tau) := \sum_{j=\iota(\rho)+1}^{\infty} \sum_{k=1}^{\kappa(j)} u_{jk}(x; \rho + \mathrm{i}\tau) V_{jk}(y), \tag{4.2}
$$

where

$$
\iota(\rho) := \begin{cases} \max\{s \in \mathbb{N} : \lambda_s \leqslant \rho\} & \text{if } \rho \geqslant \lambda_1, \\ 0 & \text{if } \rho < \lambda_1. \end{cases} \tag{4.3}
$$

From (3.15), (3.18) and (3.22) it follows that the series (4.2) converges uniformly with respect to $\mathbf{x}\in\bar{\Omega}$ and $\tau\in\mathbb{R}$. It is our first aim to estimate the difference $S(\mathbf{x};\rho+\mathrm{i}\tau)-S(\mathbf{x};\rho)$. We conclude from (4.1) that

$$[L(m)+\lambda_r-\rho][u_{jk}(.;\rho+\mathrm{i}\tau)-u_{jk}(.;\rho)]=\mathrm{i}\tau u_{jk}(.;\rho+\mathrm{i}\tau).$$

The argument leading to (3.17) yields

$$\|u_{jk}(.;\rho+\mathrm{i}\tau)-u_{jk}(.;\rho)\|_{s,\mathbb{R}^l}\leqslant\frac{\tau}{\lambda_j-\rho}\|u_{jk}(.;\rho+\mathrm{i}\tau)\|_{s,\mathbb{R}^l}$$

for every $s\in\mathbb{N}$. Combining this with (3.17), we have

$$\|u_{jk}(.;\rho+\mathrm{i}\tau)-u_{jk}(.;\rho)\|_{s,\mathbb{R}^l}\leqslant\frac{\tau}{(\lambda_j-\rho)^2}\|f_{jk}\|_{s,\mathbb{R}^l}.$$

A consideration analogous to that before (3.22) shows that

$$|D^pS(\mathbf{x};\rho+\mathrm{i}\tau)-D^pS(\mathbf{x};\rho)|^2\leqslant c\frac{\tau}{(\lambda_{\iota(\rho)+1}-\rho)^2}\sum_{j=\iota(\rho)+1}^{\infty}\sum_{k=1}^{\kappa(j)}\lambda_j^r\|f_{jk}\|^2_{s,\mathbb{R}^l}$$

with suitable constants $c>0$ and $r,s\in\mathbb{N}$ (depending on $p\in\mathbb{N}_0^n$). This implies

$$D^pS(\mathbf{x};\rho+\mathrm{i}\tau)-D^pS(\mathbf{x};\rho)=O(\tau)\qquad\text{as }\tau\to0\tag{4.4}$$

for every $p\in\mathbb{N}_0^n$ uniformly with respect to $\mathbf{x}\in\bar{\Omega}$.

The discussion of the Fourier coefficients u_{jk} with $j\leqslant\iota(\rho)$ is based on their representation (3.11). First we study the case $\lambda_j<\rho$. According to the definition (3.10) of φ_s we have

$$\left.\begin{aligned}\varphi_s(\rho+\mathrm{i}0;j)&:=\lim_{\tau\downarrow0}\varphi_s(\rho+\mathrm{i}\tau;j)=\frac{2\pi s}{m},\\ \varphi_s(\rho-\mathrm{i}0;j)&:=\lim_{\tau\uparrow0}\varphi_s(\rho+\mathrm{i}\tau;j)=\frac{2\pi(s+1)}{m}.\end{aligned}\right\}\tag{4.5}$$

From this and (3.11) it follows that $u_{jk}(x;\rho+\mathrm{i}\tau)$ converges uniformly with respect to x in every compact subset M of $\mathbb{R}^l$ as $\tau\downarrow0$ and $\tau\uparrow0$, respectively. The limit functions $u_{jk}(x;\rho+\mathrm{i}0)$ and $u_{jk}(x;\rho-\mathrm{i}0)$ are given by (3.11) with φ_s defined by (4.5). In the same way we conclude from (3.16) that $D^pu_{jk}(x;\rho+\mathrm{i}\tau)$ with arbitrary $p\in\mathbb{N}_0^l$ converges uniformly in M to $D^pu_{jk}(x;\rho+\mathrm{i}0)$ and $D^pu_{jk}(x;\rho-\mathrm{i}0)$ as $\tau\downarrow0$ and $\tau\uparrow0$, respectively. Thus we obtain by (4.1)

$$[(-\Delta_x)^m+\lambda_j-\rho]u_{jk}(x;\rho\pm\mathrm{i}0)=f_{jk}(x)\quad\text{for }x\in\mathbb{R}^l\text{ and }\lambda_j<\rho.\tag{4.6}$$

Now we study the case $\lambda_j=\rho$. In this case we have

$$\varphi_s=\frac{\arg(\mathrm{i}\tau)+2\pi s}{m}=\left(2s+1-\frac{\operatorname{sign}\tau}{2}\right)\frac{\pi}{m}.\tag{4.7}$$

We use the series representations

$$J_\sigma(\zeta)=\zeta^\sigma\sum_{s=0}^{\infty}C_s\zeta^{2s},\tag{4.8}$$

$$N_\sigma(\zeta) = \begin{cases} \zeta^{-\sigma} \sum_{s=0}^{\infty} C'_s \zeta^{2s} & \left(\sigma + \frac{1}{2} \in \mathbb{N}_0\right), \\ \frac{2}{\pi} J_\sigma(\zeta) \ln \frac{\gamma\zeta}{2} + \zeta^\sigma \sum_{s=0}^{\infty} C''_s \zeta^{2s} + \zeta^{-\sigma} \sum_{s=0}^{\sigma-1} C'''_s \zeta^{2s} & (\sigma \in \mathbb{N}_0) \end{cases} \tag{4.9}$$

for $|\arg\zeta| < \pi$, where $\ln\gamma = C_e$ (the Euler–Mascheroni constant) and

$$\left.\begin{aligned} C_s &:= \frac{(-1)^s}{2^{\sigma+2s} s!\, \Gamma(\sigma + s + 1)}, \\ C'_s &:= \frac{(-1)^{\sigma+s+\frac{1}{2}}}{2^{2s-\sigma} s!\, \Gamma(s + 1 - \sigma)}, \\ C''_s &:= \frac{(-1)^{s+1}}{\pi 2^{\sigma+2s} s!(\sigma + s)!} \left(\sum_{r=1}^{s} \frac{1}{r} + \sum_{r=1}^{s+\sigma} \frac{1}{r} \right), \\ C'''_s &:= -\frac{2^{\sigma-s}(\sigma - s - 1)!}{\pi s!} \end{aligned}\right\} \tag{4.10}$$

(compare Reference 5). We insert these expansions into (3.11) and change the order of integration and summation. This leads to terms of the form

$$w_{jk}[h(\zeta)] := \frac{\mathrm{i}|\tau|^{\frac{\sigma+2}{2m}-1}}{4m(2\pi)^\sigma} \sum_{r=0}^{m-1} \mathrm{e}^{\mathrm{i}[(\sigma/2)+1-m]\varphi_r} \int_{\mathbb{R}^l} \frac{f_{jk}(x')}{|x - x'|^\sigma} h(|x - x'||\tau|^{1/2m} \mathrm{e}^{\mathrm{i}\varphi_r/2}) \mathrm{d}x',$$

where $h(\zeta) = \zeta^{2s+\sigma}$, $h(\zeta) = \zeta^{2s-\sigma}$ and $h(\zeta) = \zeta^{2s+\sigma} \ln\zeta$, respectively. Computing the terms $w_{jk}[\zeta^{2s+\sigma}]$, we obtain

$$w_{jk}[\zeta^{2s+\sigma}] = \frac{\mathrm{i}|\tau|^{\frac{\sigma+1+s}{m}-1}}{4m(2\pi)^\sigma} \int_{\mathbb{R}^l} f_{jk}(x')|x - x'|^{2s} \mathrm{d}x' \sum_{r=0}^{m-1} \mathrm{e}^{\mathrm{i}(\sigma+1+s-m)\varphi_r}$$

$$= \begin{cases} 0 & \text{for } \sigma \in \mathbb{N}_0; \quad s = 0, \ldots, m - \sigma - 2, \\ \frac{\mathrm{i}}{4(2\pi)^\sigma} \int_{\mathbb{R}^l} f_{jk}(x')|x - x'|^{2(m-\sigma-1)} \mathrm{d}x' & \text{for } \sigma \in \mathbb{N}_0; \quad s = m - \sigma - 1, \\ \frac{D_s}{|\tau|^{1-(\sigma+s+1)/m}} \int_{\mathbb{R}^l} f_{jk}(x')|x - x'|^{2s} \mathrm{d}x' & \text{for } \sigma + \frac{1}{2} \in \mathbb{N}_0; \quad s = 0, \ldots, m - \sigma - \frac{3}{2}, \end{cases} \tag{4.11}$$

with

$$D_s := \frac{\mathrm{i}}{2m(2\pi)^\sigma} \frac{\mathrm{e}^{\mathrm{i}\pi(\sigma+s+1-m)(2-\operatorname{sign}\tau)/2m}}{1 - \mathrm{e}^{\mathrm{i}2\pi(\sigma+1+s-m)/m}}. \tag{4.12}$$

It follows in the same way that

$$w_{jk}[\zeta^{2s-\sigma}] = \begin{cases} 0 & \text{if} \quad s = 0, \ldots, m - 2, \\ \frac{\mathrm{i}}{4(2\pi)^\sigma} \int_{\mathbb{R}^l} f_{jk}(x')|x - x'|^{2(m-\sigma-1)} \mathrm{d}x' & \text{if} \quad s = m - 1. \end{cases} \tag{4.13}$$

Since

$$\sum_{r=0}^{m-1} \varphi_r e^{i\nu\varphi_r} = \begin{cases} \pi\left(m - \dfrac{\operatorname{sign}\tau}{2}\right) & \text{if } \nu = 0, \\ -\dfrac{2\pi e^{i\nu\pi(2-\operatorname{sign}\tau)/2m}}{1 - e^{i2\nu\pi/m}} & \text{if } \nu = -1, \dots, -(m-1), \end{cases}$$

we obtain for $\sigma \in \mathbb{N}_0$

$$\frac{2}{\pi} w_{jk}[\zeta^{2s+\sigma}\ln\zeta] = \frac{i|\tau|^{\frac{\sigma+1+s}{m}-1}}{m(2\pi)^{\sigma+1}}\left[\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2s}dx' \sum_{r=0}^{m-1} i\frac{\varphi_r}{2} e^{i(\sigma+s+1-m)\varphi_r} + \int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2s}\ln(|x-x'||\tau|^{1/2m})dx' \sum_{r=0}^{m-1} e^{i(\sigma+s+1-m)\varphi_r}\right]$$

$$= \begin{cases} -\dfrac{iD_s}{|\tau|^{1-\frac{\sigma+s+1}{m}}}\displaystyle\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2s}dx' & \text{if } s = 0, \dots, m-\sigma-2, \\ \left(\dfrac{i\ln|\tau|}{2m(2\pi)^{\sigma+1}} - \dfrac{2m-\operatorname{sign}\tau}{8m(2\pi)^{\sigma}}\right)\displaystyle\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2(m-\sigma-1)}dx' & \\ +\dfrac{i}{(2\pi)^{\sigma+1}}\displaystyle\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2(m-\sigma-1)}\ln|x-x'|dx' & \text{if } s = m-\sigma-1. \end{cases} \tag{4.14}$$

Note that $\sigma = l/2 - 1$. Inserting (4.8)–(4.14) in (3.11) we obtain for odd l

$$u_{jk}(x;\lambda_j + i\tau) = \sum_{s=0}^{m-(l+1)/2} \frac{C_s D_s}{|\tau|^{1-(l+2s)/2m}}\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2s}dx' - \frac{C'_{m-1}}{4(2\pi)^{(l/2)-1}}\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2m-l}dx' + O(|\tau|^{1/2m}) \quad \text{as} \quad |\tau| \to 0, \tag{4.15}$$

and for even l

$$u_{jk}(x;\lambda_j + i\tau) = \sum_{s=0}^{m-1-(l/2)} \frac{C_s D_s}{|\tau|^{1-(l+2s)/2m}}\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2s}dx' - \left(\frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}\ln|\tau| + D'(\operatorname{sign}\tau)\right)\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2m-l}dx' - \frac{C_{m-(l/2)}}{(2\pi)^{l/2}}\int_{\mathbb{R}^l} f_{jk}(x')|x-x'|^{2m-l}\ln|x-x'|dx' + O(|\tau|^{1/2m}) \quad \text{as} \quad |\tau| \to 0, \tag{4.16}$$

with

$$D'(\operatorname{sign}\tau) := \frac{1}{4(2\pi)^{(l/2)-1}}\left(C_{m-(l/2)}\left(\frac{2}{\pi}\ln\frac{\gamma}{2} - i\frac{\operatorname{sign}\tau}{2m}\right) + C''_{m-(l/2)} + C'''_{m-1}\right) \tag{4.17}$$

($C_s = C'_s = C''_s = C'''_s := 0$ if $s < 0$, $C'''_s := 0$ if $s > l/2$). Both asymptotic formulae (4.15) and (4.16) hold uniformly in every compact subset of $\mathbb{R}^l$. Hence, u_{jk} may be unbounded as $|\tau| \to 0$. But if $l > 2m$, then (4.15) and (4.16) imply

$$u_{jk}(x; \lambda_j + \mathrm{i}\tau) \to u_{jk}(x; \lambda_j) := \frac{\Gamma(l/2 - m)}{\pi^{l/2} 4^m (m-1)!} \int_{\mathbb{R}^l} \frac{f_{jk}(x')}{|x - x'|^{l-2m}} \mathrm{d}x' \tag{4.18}$$

as $\tau \to 0$. If $l \leqslant 2m$ and

$$\int_{\mathbb{R}^l} f_{jk}(x') |x - x'|^{2s} \mathrm{d}x' = 0 \quad \text{for } x \in \mathbb{R}^l; \quad s = 0, \ldots, [m - l/2] \tag{4.19}$$

($[\alpha] := \max\{i \in \mathbb{N} : i \leqslant \alpha\}$), then the limit $u_{jk}(x; \lambda_j)$ exists also according to (4.15) and (4.16), and we have

$$u_{jk}(x; \lambda_j) := \begin{cases} \dfrac{\Gamma(l/2 - m)}{\pi^{l/2} 4^m (m-1)!} \displaystyle\int_{\mathbb{R}^l} f_{jk}(x') |x - x'|^{2m-l} \mathrm{d}x' & \text{if } l \text{ is odd} \\ \dfrac{(-1)^{m+1-(l/2)}}{\pi^{l/2} 2^{2m-1} (m - l/2)! (m-1)!} \displaystyle\int_{\mathbb{R}^l} f_{jk}(x') |x - x'|^{2m-l} \ln |x - x'| \mathrm{d}x' & \text{if } l \text{ is even.} \end{cases} \tag{4.20}$$

Now we consider the condition (4.19). It is equivalent to

$$0 = \sum_{\alpha + \beta + \gamma = s} \frac{s!}{\alpha! \beta! \gamma!} |x|^{2\alpha} \int_{\mathbb{R}^l} f_{jk}(x') (-2x \cdot x')^\beta |x'|^{2\gamma} \mathrm{d}x' \tag{4.21}$$

for $x \in \mathbb{R}^l$, $s = 0, \ldots, [m - l/2]$. By induction with respect to s we obtain

$$\sum_{\beta + \gamma = s} \frac{(-2)^\beta s!}{\beta! \gamma!} \int_{\mathbb{R}^l} f_{jk}(x') (x \cdot x')^\beta |x'|^{2\gamma} \mathrm{d}x' = 0$$

for $x \in \mathbb{R}^l$, $s = 0, \ldots, [m - l/2]$. Note that for $p \in \mathbb{N}_0^l$

$$\left[D_x^p \int_{\mathbb{R}^l} f_{jk}(x') (x \cdot x')^\beta |x'|^{2\gamma} \mathrm{d}x' \right]_{x=0}$$

$$= \begin{cases} 0 & \text{if } |p| \neq \beta, \\ \beta! \displaystyle\int_{\mathbb{R}^l} f_{jk}(x') x'^p |x'|^{2\gamma} \mathrm{d}x' & \text{if } |p| = \beta, \end{cases}$$

since $D_x^p (x \cdot x')^\beta = \beta \cdot \ldots \cdot (\beta - |p| + 1)(x \cdot x')^{\beta - |p|} x'^p$. Therefore (4.19) implies

$$\int_{\mathbb{R}^l} f^{jk}(x') x'^p |x'|^{2s} \mathrm{d}x' = 0 \quad \text{for } p \in \mathbb{N}_0^l, s \in \mathbb{N}_0 \text{ with } |p| + s \leqslant [m - l/2]. \tag{4.22}$$

By (4.21) also the inverse implication holds, so that (4.19) is equivalent to (4.22).

We summarize our results: the limit of $R_{\rho + \mathrm{i}\tau} f$ as $\tau \downarrow 0$ exists if and only if one of the conditions

$$l > 2m, \tag{4.23}$$

$$l \leqslant 2m \text{ and } \rho \neq \lambda_j \text{ for every } j \in \mathbb{N}, \tag{4.24}$$

$$l \leqslant 2m,\ \rho = \lambda_j \text{ and (4.22) holds for } k = 1, 2, \ldots, \kappa(j) \tag{4.25}$$

is valid. In this case $R_{\rho+i\tau}f$ converges also as $\tau \uparrow 0$. The limits have the Fourier series expansions

$$R_{\rho \pm i0}f(\mathbf{x}) = \sum_{j=1}^{\iota(\rho)} \sum_{k=1}^{\kappa(j)} u_{jk}(x; \rho \pm i0)\, V_{jk}(y) + \sum_{j=\iota(\rho)+1}^{\infty} \sum_{k=1}^{\kappa(j)} u_{jk}(x; \rho)\, V_{jk}(y), \tag{4.26}$$

where u_{jk} is given by (3.11) and (3.10) if $\lambda_j > \rho$, (3.11) and (4.5) if $\lambda_j < \rho$ and (4.18) or (4.20) if $\lambda_j = \rho$, respectively. The series (4.26) converges uniformly with respect to $\mathbf{x} = (x, y)$ in every compact subset M of $\bar{\Omega}$. In the same way we obtain $D^p R_{\rho+i\tau}f(\mathbf{x}) \to D^p R_{\rho+i0}f(\mathbf{x})$ as $\tau \downarrow 0$ and $D^p R_{\rho+i\tau}f(\mathbf{x}) \to D^p R_{\rho-i0}f(\mathbf{x})$ as $\tau \uparrow 0$ uniformly with respect to $\mathbf{x} \in M\,(p \in \mathbb{N}_0^n)$. Together with (3.1) this yields that $R_{\rho+i0}f$ and $R_{\rho-i0}f$ are solutions of the boundary value problem

$$\begin{aligned} &[(-\Delta_x)^m + (-\Delta_y)^m - \rho]\, w = f \qquad \text{in } \Omega, \\ &w = \frac{\partial w}{\partial \mathbf{n}} = \ldots = \frac{\partial^{m-1} w}{\partial \mathbf{n}^{m-1}} = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.27}$$

We conclude these considerations by deriving a condition that characterizes $R_{\rho+i0}f$ uniquely among the solutions of (4.27). Let $w \in C^{2m}(\bar{\Omega})$ be an arbitrary solution of (4.27) and set $w_{jk}(x) := \int_{\Omega'} w(x, y)\, V_{jk}(y)\, dy$. The boundary condition in (4.27) implies that $w(x, .) \in \mathring{H}_m(\Omega') \cap H_{2m}(\Omega')$ for every $x \in \mathbb{R}^l$. The expansion theorem for the interior boundary value problem (1.11) yields

$$w(x, .) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} w_{jk}(x)\, V_{jk},\ (-\Delta_y)^m w(x, .) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} \lambda_j w_{jk}(x)\, V_{jk}.$$

On the other hand, we have

$$(-\Delta_x)^m w(x, .) = \sum_{j=1}^{\infty} \sum_{k=1}^{\kappa(j)} w_{jk}^*(x)\, V_{jk}$$

with $w_{jk}^*(x) = \int_{\Omega'} (-\Delta_x)^m w(x, y)\, V_{jk}(y)\, dy = (-\Delta_x)^m w_{jk}(x)$. Inserting this and the Fourier expansion (3.4) of f into (4.27), we obtain

$$[(-\Delta_x)^m + \lambda_j - \rho]\, w_{jk} = f_{jk} \qquad \text{in } \mathbb{R}^l \tag{4.28}$$

for $j = 1, 2, \ldots$ and $k = 1, 2, \ldots \kappa(j)$. The Fourier coefficients u_{jk} of $R_{\rho+i0}f$ can be uniquely characterized in the following way: if $\lambda_j > \rho$, then u_{jk} is uniquely determined by (4.28) and the property $u_{jk} \in D(L(m))$, since $L(m)$ is self-adjoint and positive. If $\lambda_j < \rho$, it can be shown that u_{jk} is the only solution of (4.28) satisfying

$$\left.\begin{aligned} &\Delta_x^s u_{jk}(x; \rho + i0) = O\left(\frac{1}{r^{(l-1)/2}}\right) && \text{as } r = |x| \to \infty, \\ &\left(\frac{\partial}{\partial r} - i|\lambda_j - \rho|^{1/2m}\right) \Delta_x^s u_{jk}(x; \rho + i0) = o\left(\frac{1}{r^{(l-1)/2}}\right) && \text{as } r = |x| \to \infty \end{aligned}\right\} \tag{4.29}$$

for $s = 0, \ldots, m-1$. Formula (4.29) can be considered as a generalization of Sommerfeld's radiation condition. If $\lambda_j = \rho$ and (4.22) holds, then u_{jk} may be unbounded as $|x| \to \infty$. It will be shown in Reference 4 that u_{jk} is the only solution of

(4.28) with

$$\int_{|x-x_0|=R} u_{jk}(x;\rho)\,\mathrm{d}S_x = o(R^{l-1}) \qquad \text{as } R\to\infty \text{ for every } x_0\in\mathbb{R}^l. \tag{4.30}$$

Hence, $R_{\rho+\mathrm{i}0}f$ is uniquely determined by (4.27) and the properties of its Fourier coefficients u_{jk} collected above. Thus we have proved:

Lemma 4.1. *Assume that $f\in C_0^\infty(\Omega)$ and $\rho\in\mathbb{R}$. The resolvent $R_{\rho+\mathrm{i}\tau}f$ of the operator A converges as $\tau\downarrow 0$ if and only if one of the conditions (4.23)–(4.25) holds. The limit function $R_{\rho+\mathrm{i}0}f$ is the uniquely determined solution of (4.27) with the property that the Fourier coefficients u_{jk} belong to $D(L(m))$ if $\lambda_j>\rho$ and satisfy the condition (4.29) for $s=0,\dots,m-1$ if $\lambda_j<\rho$ or (4.30) if $\lambda_j=\rho$, respectively.*

Our next aim is to construct of the spectral family $\{P_\lambda\}$ of A. First we summarize some properties of $R_z f$ obtained above:

(i) We have

$$R_{\rho+\mathrm{i}\tau}f(\mathbf{x}) - R_{\rho-\mathrm{i}\tau}f(\mathbf{x}) \to R_{\rho+\mathrm{i}0}f(\mathbf{x}) - R_{\rho-\mathrm{i}0}f(\mathbf{x}) \qquad \text{as } \tau\downarrow 0 \tag{4.31}$$

uniformly with respect to $(\mathbf{x},\rho)\in M\times K$, where M and K are arbitrary compact subsets of $\bar\Omega$ and $\mathbb{R}\backslash\{\lambda_j : j\in\mathbb{N}\}$, respectively.

(ii) From (3.11), (3.9), (4.8) and (4.9) we obtain

$$u_{jk}(x;z) = O\left(\frac{\ln|z-\lambda_j|}{|z-\lambda_j|^{1-(1/2m)}}\right) \qquad \text{as } z\to\lambda_j$$

and by (4.2) and (4.4)

$$R_z f(\mathbf{x}) - R_{\bar z}f(\mathbf{x}) = O\left(\frac{\ln|z-\lambda_j|}{|z-\lambda_j|^{1-(1/2m)}}\right) \qquad \text{as } z\to\lambda_j \tag{4.32}$$

uniformly in every compact subset M of $\bar\Omega$.

We conclude from (i) and (ii) by the same argument used in the verification of formula (2.36) in Reference 12 that

$$(P_\lambda f)(\mathbf{x}) = \frac{1}{2\pi\mathrm{i}}\int_0^\lambda [R_{\rho+\mathrm{i}0}f(\mathbf{x}) - R_{\rho-\mathrm{i}0}f(\mathbf{x})]\,\mathrm{d}\rho \tag{4.33}$$

and that P_λ is continuous with respect to λ. In particular, A has no eigenvalues.

In order to compute the integrand of (4.33), its convenient to set

$$\left.\begin{aligned} u^{(j)}(\mathbf{x};z) &:= \sum_{k=1}^{\kappa(j)} u_{jk}(x)\,V_{jk}(y),\\ f^{(j)}(\mathbf{x};z) &:= \sum_{k=1}^{\kappa(j)} f_{jk}(x)\,V_{jk}(y). \end{aligned}\right\} \tag{4.34}$$

Relation (4.26) implies

$$R_{\rho+\mathrm{i}0}f(\mathbf{x}) - R_{\rho-\mathrm{i}0}f(\mathbf{x}) = \sum_{j=1}^{\iota(\rho)} [u^{(j)}(\mathbf{x};\rho+\mathrm{i}0) - u^{(j)}(\mathbf{x};\rho-\mathrm{i}0)]. \tag{4.35}$$

Since

$$\varphi_{r+1}(\rho+\mathrm{i}0;j) = \varphi_r(\rho-\mathrm{i}0;j) \qquad \text{for } r=0,\dots,m-2,$$

$$\varphi_0(\rho+\mathrm{i}0;j) = 0, \qquad \varphi_{m-1}(\rho-\mathrm{i}0;j) = 2\pi$$

(compare (4.5)), it follows from the representation (3.11) of u_{jk} that

$$\begin{aligned} &u^{(j)}(\mathbf{x};\rho+\mathrm{i}0)-u^{(j)}(\mathbf{x};\rho-\mathrm{i}0) \\ &=\frac{\mathrm{i}|\rho-\lambda_j|^{\frac{\sigma+2}{2m}-1}}{4m(2\pi)^\sigma}\int_{\mathbf{R}^l}\frac{f^{(j)}(x',y)}{|x-x'|^\sigma}\{H_\sigma^{(1)}(|x-x'||\rho-\lambda_j|^{1/2m} \\ &\quad-\mathrm{e}^{\mathrm{i}\sigma\pi}H_\sigma^{(1)}(-|x-x'||\rho-\lambda_j|^{1/2m})\}\,\mathrm{d}x' \end{aligned}$$

if $\lambda_j<\rho$. Using

$$J_\sigma(-\zeta)=\mathrm{e}^{\mathrm{i}\sigma\pi}J_\sigma(\zeta),\qquad N_\sigma(-\zeta)=\mathrm{e}^{-\mathrm{i}\sigma\pi}N_\sigma(\zeta)+2\mathrm{i}\cos(\sigma\pi)J_\sigma(\zeta)$$

(compare Reference 5), we obtain

$$\begin{aligned} u^{(j)}(\mathbf{x};\rho+\mathrm{i}0)-u^{(j)}(\mathbf{x};\rho-\mathrm{i}0)&=\frac{\mathrm{i}|\rho-\lambda_j|^{\frac{\sigma+2}{2m}-1}}{2m(2\pi)^\sigma}\int_{\mathbf{R}^l}\frac{f^{(j)}(x',y)}{|x-x'|^\sigma} \\ &\quad\times J_\sigma(|x-x'||\rho-\lambda_j|^{1/2m})\,\mathrm{d}x' \end{aligned}\tag{4.36}$$

for $\lambda_j<\rho$. Thus the spectral family of A is given by the improper integral

$$P_\lambda f(\mathbf{x})=\int_0^\lambda\left\{\sum_{j=1}^{\iota(\rho)}\frac{|\rho-\lambda_j|^{\frac{\sigma+2}{2m}-1}}{2m(2\pi)^{\sigma+1}}\int_{\mathbf{R}^l}\frac{f^{(j)}(x',y)}{|x-x'|^\sigma}J_\sigma(|x-x'||\rho-\lambda_j|^{1/2m})\,\mathrm{d}x'\right\}\mathrm{d}\rho.\tag{4.37}$$

This formula shows that $P_\lambda=0$ for $\lambda\leqslant\lambda_1$. Furthermore, it can be shown in the same way as in Reference 12, page 186, that the spectrum $\sigma(A)$ of A consists of the interval $[\lambda_1,\infty)$. If $\lambda\neq\lambda_1,\lambda_2,\ldots$, then we have by (4.37)

$$\frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda}=\frac{1}{2m(2\pi)^{\sigma+1}}\sum_{j=1}^{\iota(\lambda)}|\lambda-\lambda_j|^{\frac{\sigma+2}{2m}-1}\int_{\mathbf{R}^l}\frac{f^{(j)}(x',y)}{|x-x'|^\sigma}J_\sigma(|x-x'||\lambda-\lambda_j|^{1/2m})\,\mathrm{d}x'.\tag{4.38}$$

If $l\leqslant 2m$ or, equivalently, $\sigma+1\leqslant m$, then $P_\lambda f$ is not differentiable at $\lambda=\lambda_k$ ($k\in\mathbb{N}$). Inserting (4.8) into (4.38), we obtain

$$\frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda}=\begin{cases} g_k(\mathbf{x})+O(|\lambda-\lambda_k|) & \text{as } \lambda\uparrow\lambda_k, \\ \dfrac{1}{2m(2\pi)^{\sigma+1}}\displaystyle\sum_{s=0}^{[m-\sigma-1]}\frac{C_s}{|\lambda-\lambda_k|^{1-(\sigma+1+s)/m}}\int_{\mathbf{R}^l}f^{(k)}(x',y)|x-x'|^{2s}\mathrm{d}x' & \\ \quad+g_k(\mathbf{x})+O(|\lambda-\lambda_k|^{1/2m}) & \text{as } \lambda\downarrow\lambda_k \end{cases}\tag{4.39}$$

uniformly in every compact subset M of $\bar{\Omega}$, where

$$g_k(\mathbf{x})=\frac{1}{2m(2\pi)^{\sigma+1}}\sum_{j=1}^{k-1}|\lambda_k-\lambda_j|^{\frac{\sigma+2}{2m}-1}\int_{\mathbf{R}^l}\frac{f^{(j)}(x',y)}{|x-x'|^\sigma}J_\sigma(|x-x'||\lambda_k-\lambda_j|^{1/2m})\,\mathrm{d}x'.\tag{4.40}$$

If $l>2m$, then $P_\lambda f$ is continuously differentiable with respect to $\lambda\in\mathbb{R}$. We collect our results in the following lemma:

Lemma 4.2.

(i) *The spectral family P_λ of A is continuous on $\mathbb{R}$ and vanishes for $\lambda\leqslant\lambda_1$; $P_\lambda f$ is given by* (4.37) *if $f\in C_0^\infty(\Omega)$.*

(ii) *The operator A has no eigenvalues. The spectrum $\sigma(A)$ of A consists of the interval $[\lambda_1, \infty)$.*

(iii) *The derivative $(\mathrm{d}/\mathrm{d}\lambda)(P_\lambda f(\mathbf{x}))$ is given by (4.38). It is continuous on $\mathbb{R}$ if $l > 2m$. If $l \leqslant 2m$, then $P_\lambda f$, in general, is not differentiable with respect to λ at $\lambda = \lambda_k (k \in \mathbb{N})$.*

5. The principle of limiting amplitude

We want to investigate the asymptotic behaviour of

$$u(\mathbf{x}, t) = \left[\int_{\lambda_1}^{\infty} \cos\sqrt{\lambda} t \,\mathrm{d}(P_\lambda u_0)\right](\mathbf{x}) + \left[\int_{\lambda_1}^{\infty} \frac{\sin\sqrt{\lambda} t}{\sqrt{\lambda}} \mathrm{d}(P_\lambda u_1)\right](\mathbf{x})$$
$$+ \left[\int_{\lambda_1}^{\infty} \psi(\lambda, t)\mathrm{d}(P_\lambda f)\right](\mathbf{x}) \tag{5.1}$$

as $t \to \infty$. Since $P_\lambda = 0$ for $\lambda \leqslant \lambda_1$, u coincides with the uniquely determined solution of (1.2)–(1.4) satisfying (2.12). We use the following lemma:

Lemma 5.1. *Let M be an arbitrary compact subset of $\bar{\Omega}$ and s the smallest integer with $2(s+1) > n/2$. Assume that $g \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$ such that*

$$\left|\left[\int_\alpha^\beta \varphi(\lambda)\mathrm{d}(P_\lambda g)\right](\mathbf{x})\right| \leqslant c\,\|(P_\beta - P_\alpha)A^s g\| \sup_{\alpha<\lambda<\beta} |\varphi(\lambda)| \tag{5.2}$$

(with $P_\infty := I$) for every $\mathbf{x} \in M$, every interval (α, β) with $\lambda_1 \leqslant \alpha < \beta \leqslant \infty$ and every bounded $\varphi \in C(\alpha, \beta)$.

The proof of Lemma 5.1 can be obtained by a modification of the proof of Lemma 3.1 in Reference 12.

Assume that $g \in C_0^\infty(\Omega)$ and that φ is bounded and continuous in (α, β) with $\lambda_1 \leqslant \alpha < \beta \leqslant \infty$. Lemma 5.1 implies

$$\left[\int_\alpha^\beta \varphi(\lambda)\mathrm{d}(P_\lambda g)\right](\mathbf{x}) = \int_\alpha^\beta \varphi(\lambda)\mathrm{d}(P_\lambda g(\mathbf{x})). \tag{5.3}$$

Since $\{P_\lambda\}$ is continuous for $\lambda \in \mathbb{R}$ and continuously differentiable with respect to $\lambda \in \mathbb{R}\backslash\{\lambda_1, \lambda_2, \ldots\}$, it follows from (4.39) that

$$\int_\alpha^\beta \varphi(\lambda)\mathrm{d}(P_\lambda g(\mathbf{x})) = \int_\alpha^\beta \varphi(\lambda)\frac{\mathrm{d}(P_\lambda g(\mathbf{x}))}{\mathrm{d}\lambda}\mathrm{d}\lambda. \tag{5.4}$$

Consider the first integral in (5.1),

$$I_1(\mathbf{x}, t) := \int_{\lambda_1}^{\infty} \cos\sqrt{\lambda} t \,\mathrm{d}(P_\lambda u_0(\mathbf{x})). \tag{5.5}$$

In the remaining part of this section we consider a fixed compact subset M of $\bar{\Omega}$ and a fixed $\varepsilon > 0$. According to Lemma 5.1, there exists an $a > 0$ such that

$$I_1(\mathbf{x}, t) = \int_{\lambda_1}^{a} \cos\sqrt{\lambda} t \,\mathrm{d}(P_\lambda u_0(\mathbf{x})) + w_1(\mathbf{x}, t),$$

where $|w_1(\mathbf{x}, t)| < \varepsilon$ for $\mathbf{x} \in M$ and $t \geqslant 0$. The interval $[\lambda_1, a]$ contains only a finite number of eigenvalues $\lambda_1, \ldots \lambda_s$ of (1.11). We set $U_\delta := \bigcap_{j=1}^{s}(\lambda_j - \delta, \lambda_j + \delta) \cap [\lambda_1, a]$.

Since $\{P_\lambda\}$ is continuous, Lemma 5.1 implies

$$\left|\int_{U_\delta} \cos\sqrt{\lambda}t\,\mathrm{d}(P_\lambda u_0(\mathbf{x}))\right| < \varepsilon \qquad \text{for } \mathbf{x}\in M, \tag{5.6}$$

for sufficiently small $\delta > 0$. Note that the functions $(\mathrm{d}/\mathrm{d}\lambda)(P_\lambda u_0)$ and $(\mathrm{d}^2/\mathrm{d}\lambda^2)(P_\lambda u_0)$ are continuous and bounded for $\lambda\in[\lambda_1, a]\backslash U_\delta$ and $\mathbf{x}\in M$. Integrating by parts, we obtain

$$\int_{[\lambda_1,a]\backslash U_\delta} \cos\sqrt{\lambda}t\frac{\mathrm{d}(P_\lambda u_0(\mathbf{x}))}{\mathrm{d}\lambda}\mathrm{d}\lambda = O\left(\frac{1}{t}\right) \qquad \text{as } t\to\infty$$

uniformly with respect to $\mathbf{x}\in M$. This, together with (5.4) and (5.6), yields $I_1(\mathbf{x}, t) = o(1)$ as $t\to\infty$ uniformly in M.

The following lemma can be shown by analogous estimates:

Lemma 5.2. *Let M be a compact subset of $\bar{\Omega}$ and assume that $g\in C_0^\infty(\Omega)$. Then we have for every interval (α, β) with $\lambda_1 \leqslant \alpha < \beta \leqslant \infty$ and every bounded $\varphi\in C(\alpha, \beta)$*

$$\int_\alpha^\beta \varphi(\lambda)\cos\sqrt{\lambda}t\,\mathrm{d}(P_\lambda g(x)) = o(1) \qquad \textit{as } t\to\infty, \tag{5.7}$$

$$\int_\alpha^\beta \varphi(\lambda)\sin\sqrt{\lambda}t\,\mathrm{d}(P_\lambda g(x)) = o(1) \qquad \textit{as } t\to\infty, \tag{5.8}$$

uniformly with respect to $\mathbf{x}\in M$.

We apply Lemma 5.2 to (5.1). Since

$$\psi(\lambda, t) = \frac{\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}t}}{\lambda-\omega^2} - \frac{\mathrm{i}\sin\sqrt{\lambda}t}{\sqrt{\lambda}(\sqrt{\lambda}+\omega)}$$

(compare (2.10), we obtain

$$u(\mathbf{x}, t) = \mathrm{e}^{-\mathrm{i}\omega t}\int_{\lambda_1}^\infty \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\mathrm{d}(P_\lambda f(\mathbf{x})) + o(1) \qquad \text{as } t\to\infty \tag{5.9}$$

uniformly in M. For $\delta > 0$ we set

$$I_2(\mathbf{x}, t; \delta) := \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\mathrm{d}(P_\lambda f(\mathbf{x})), \tag{5.10}$$

$$I_3(\mathbf{x}; \delta) := \int_{|\lambda-\omega^2|\geqslant\delta} \frac{1}{\lambda-\omega^2}\mathrm{d}(P_\lambda f(\mathbf{x})), \tag{5.11}$$

$$W_1(\mathbf{x}, t; \delta) := -\int_{|\lambda-\omega^2|\geqslant\delta} \frac{\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\mathrm{d}(P_\lambda f(\mathbf{x})). \tag{5.12}$$

We denote by $W_i (i\in\mathbb{N})$ terms that will be shown later to be of order $o(1)$ as $t\to\infty$. By (5.9)–(5.12) we have

$$u(\mathbf{x}, t) = \mathrm{e}^{-\mathrm{i}\omega t}[I_2(\mathbf{x}, t; \delta) + I_3(\mathbf{x}; \delta) + W_1(\mathbf{x}, t; \delta)] + o(1) \qquad \text{as } t\to\infty, \tag{5.13}$$

uniformly with respect to $\mathbf{x}\in M$ and $\delta > 0$.

Consider I_3 and note that

$$\int_{\lambda_1}^{\infty} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} \mathrm{d}(P_\lambda f(\mathbf{x})) = R_{\omega^2+\mathrm{i}\tau} f(\mathbf{x}), \tag{5.14}$$

by the functional calculus for self-adjoint operators. Since

$$\left|\frac{1}{\lambda-\omega^2} - \frac{1}{\lambda-\omega^2-\mathrm{i}\tau}\right| = \frac{|\tau|}{|\lambda-\omega^2|\,|\lambda-\omega^2-\mathrm{i}\tau|} \leqslant \frac{|\tau|}{\delta^2}$$

for $|\lambda-\omega^2| > \delta$, we obtain by Lemma 5.1

$$I_3(\mathbf{x};\delta) = \lim_{\tau\downarrow 0}\left[R_{\omega^2+\mathrm{i}\tau} f(\mathbf{x}) - \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau}\mathrm{d}(P_\lambda f(\mathbf{x}))\right]. \tag{5.15}$$

In the following we suppose that one of the conditions (4.23)–(4.25) holds with $\rho=\omega^2$. Then $R_{\omega^2+\mathrm{i}0} f(\mathbf{x}) := \lim_{\tau\downarrow 0} R_{\omega^2+\mathrm{i}\tau} f(\mathbf{x})$ exists. Furthermore, (4.38) and (4.39) imply that in this case $\lim_{\lambda\to\omega^2} (\mathrm{d}/\mathrm{d}\lambda)(P_\lambda f(\mathbf{x}))$ exists and that

$$\left|\frac{\mathrm{d}(P_\lambda f(\mathbf{x}))}{\mathrm{d}\lambda} - \left[\frac{\mathrm{d}(P_\lambda f(\mathbf{x}))}{\mathrm{d}\lambda}\right]_{\lambda=\omega^2}\right| = O(|\lambda-\omega^2|^{1/2m}) \qquad \text{as } \lambda\to\omega^2$$

uniformly in M. Since

$$\lim_{\tau\downarrow 0}\int_{\omega^2-\delta}^{\omega^2+\delta}\frac{1}{\lambda-\omega^2-\mathrm{i}\tau}\mathrm{d}\lambda = \mathrm{i}\pi, \tag{5.16}$$

we conclude from (5.15) that

$$I_3(\mathbf{x};\delta) = R_{\omega^2+\mathrm{i}0} f(\mathbf{x}) - \mathrm{i}\pi\left[\frac{\mathrm{d}(P_\lambda f(\mathbf{x}))}{\mathrm{d}\lambda}\right]_{\lambda=\omega^2} + w_2(\mathbf{x};\delta), \tag{5.17}$$

where $w_2(\mathbf{x};\delta)\to 0$ as $\delta\downarrow 0$ uniformly in M.

Now we turn to the discussion of I_2. As above we obtain

$$I_2(\mathbf{x},t;\delta) = \int_{\omega^2-\delta}^{\omega^2+\delta}\frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\left[\frac{\mathrm{d}(P_\lambda f(\mathbf{x}))}{\mathrm{d}\lambda}\right]_{\lambda=\omega^2}\mathrm{d}\lambda + w_3(\mathbf{x},t;\delta), \tag{5.18}$$

where $w_3(\mathbf{x},t;\delta)\to 0$ as $\delta\downarrow 0$ uniformly in $M\times[0,\infty)$. It suffices to compute the integral in (5.18) for $\omega^2 \geqslant \lambda_1$, since $[(\mathrm{d}/\mathrm{d}\lambda)(P_\lambda f(\mathbf{x}))]_{\lambda=\omega^2} = 0$ for $\omega^2 < \lambda_1$. We suppose that $0<\delta<\lambda_1$ and set $\mu := \sqrt{\lambda}$, $\xi := \mu-\omega$. Then

$$\begin{aligned}
\int_{\omega^2-\delta}^{\omega^2+\delta}\frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\mathrm{d}\lambda &= 2\int_{\sqrt{(\omega^2-\delta)}}^{\sqrt{(\omega^2+\delta)}}\frac{1-\mathrm{e}^{-\mathrm{i}(\mu-\omega)t}}{(\mu-\omega)(\mu+\omega)}\mu\,\mathrm{d}\mu \\
&= \int_{\sqrt{(\omega^2-\delta)}}^{\sqrt{(\omega^2+\delta)}}\frac{1-\mathrm{e}^{-\mathrm{i}(\mu-\omega)t}}{(\mu+\omega)}\mathrm{d}\mu + \int_{\sqrt{(\omega^2-\delta)}-\omega}^{\sqrt{(\omega^2+\delta)}-\omega}\frac{1-\mathrm{e}^{-\mathrm{i}\xi t}}{\xi}\mathrm{d}\xi \\
&= \mathrm{i}\int_{\sqrt{(\omega^2-\delta)}-\omega}^{\sqrt{(\omega^2+\delta)}-\omega}\frac{\sin\xi t}{\xi}\mathrm{d}\xi + w_4(t;\delta),
\end{aligned}$$

where

$$w_4(t;\xi) = \int_{\sqrt{(\omega^2-\delta)}}^{\sqrt{(\omega^2+\delta)}} \frac{1-e^{-i(\mu-\omega)t}}{(\mu+\omega)} d\mu + \int_{\sqrt{(\omega^2-\delta)}-\omega}^{\sqrt{(\omega^2+\delta)}-\omega} \frac{1-\cos\xi t}{\xi} d\xi.$$

Note that $(1-\cos\xi t)/\xi$ is an odd function with respect to ξ. Therefore we have

$$\left|\int_{\sqrt{(\omega^2-\delta)}-\omega}^{\sqrt{(\omega^2+\delta)}-\omega} \frac{1-\cos\xi t}{\xi} d\xi\right| = \left|\int_{\omega-\sqrt{(\omega^2-\delta)}}^{\sqrt{(\omega^2+\delta)}-\omega} \frac{1-\cos\xi t}{\xi} d\xi\right|$$

$$\leqslant 2\left\{\ln\left[\sqrt{\left(1+\frac{\delta}{\omega^2}\right)}-1\right]-\ln\left[1-\sqrt{\left(1-\frac{\delta}{\omega^2}\right)}\right]\right\} = O(\delta) \qquad \text{as } \delta\downarrow 0.$$

This implies $w_4(t;\delta)\to 0$ as $\delta\downarrow 0$ uniformly in $[0,\infty)$. Setting $\zeta := \xi t$, we obtain

$$\int_{\sqrt{(\omega^2-\delta)}-\omega}^{\sqrt{(\omega^2+\delta)}-\omega} \frac{\sin\xi t}{\xi} d\xi = \int_{[\sqrt{(\omega^2-\delta)}-\omega]t}^{[\sqrt{(\omega^2+\delta)}-\omega]t} \frac{\sin\zeta}{\zeta} d\zeta = \pi + W_2(t;\delta) \tag{5.19}$$

with

$$W_2(t;\delta) := -\int_{[\sqrt{(\omega^2+\delta)}-\omega]t}^{\infty} \frac{\sin\zeta}{\zeta} d\zeta - \int_{-\infty}^{[\sqrt{(\omega^2-\delta)}-\omega]t} \frac{\sin\zeta}{\zeta} d\zeta, \tag{5.20}$$

and hence

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-e^{-i(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} d\lambda = i\pi + w_4(t;\delta) + i W_2(t;\delta). \tag{5.21}$$

We conclude from (5.13), (5.17), (5.18) and (5.21) that

$$u(\mathbf{x},t) = e^{-i\omega t} R_{\omega^2+i0} f(\mathbf{x}) + W_3(\mathbf{x},t;\delta) + w_5(\mathbf{x},t;\delta) + o(1) \qquad \text{as } t\to\infty \tag{5.22}$$

uniformly with respect to $\mathbf{x}\in M$, where

$$W_3(\mathbf{x},t;\delta) := e^{-i\omega t}\left(W_1(\mathbf{x},t;\delta) + i W_2(t;\delta)\left[\frac{d(P_\lambda f(\mathbf{x}))}{d\lambda}\right]_{\lambda=\omega^2}\right)$$

and $w_5(\mathbf{x},t;\delta)\to 0$ as $\delta\downarrow 0$ uniformly with respect to $(\mathbf{x},t)\in M\times[0,\infty)$. In particular, there exists a $\delta_3>0$ with $|w_5(\mathbf{x},t;\delta_3)|<\varepsilon$ for $\mathbf{x}\in M$ and $t\geqslant 0$. From (5.20), (5.12) and Lemma 5.2 we obtain $W_3(\mathbf{x},t;\delta_3)=o(1)$ as $t\to\infty$ uniformly in M. Altogether we have

$$u(\mathbf{x},t) = e^{-i\omega t} R_{\omega^2+i0} f(\mathbf{x}) + o(1) \qquad \text{as } t\to\infty \tag{5.23}$$

uniformly in every bounded subset of $\bar{\Omega}$. This shows that the principle of limiting amplitude (1.8) holds if one of the conditions (4.23)–(4.25) is satisfied.

6. Resonances

In this section we investigate the solution u under the assumptions $l\leqslant 2m$ and $\omega^2=\lambda_k$. As above, we denote by M a fixed compact subset of $\bar{\Omega}$.

We start our considerations from the asymptotic relation (5.13). First we consider the integral I_2 introduced in (5.10). Together with (5.4), (4.39), (4.40) and $\sigma = l/2-1$, we obtain

$$I_2(\mathbf{x}, t; \delta) = \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{[m-l/2]} C_s I^*_{\beta_s}(t;\delta) p_s^{(k)}(\mathbf{x})$$

$$+ \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-e^{-i(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} d\lambda\, g_k(\mathbf{x}) + w_6(\mathbf{x}, t; \delta), \tag{6.1}$$

where

$$\beta_s := 1 - \frac{\sigma+1+s}{m} = 1 - \frac{l+2s}{2m},$$

$$I^*_\beta(t;\delta) := \int_{\omega^2}^{\omega^2+\delta} \frac{1-e^{-i(\sqrt{\lambda}-\omega)t}}{(\lambda-\omega^2)^{1+\beta}} d\lambda, \tag{6.2}$$

$$p_s^{(k)}(\mathbf{x}) := \int_{\mathbb{R}^l} f^{(k)}(x', y)|x-x'|^{2s} dx' \tag{6.3}$$

and $w_6(\mathbf{x}, t; \delta) \to 0$ as $\delta \downarrow 0$ uniformly in $M \times [0, \infty)$. Using (5.21), we conclude that

$$I_2(\mathbf{x}, t; \delta) = \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{[m-l/2]} C_s I^*_{\beta_s}(t;\delta) p_s^{(k)}(\mathbf{x}) + i\pi g_k(\mathbf{x})$$

$$+ i W_2(t;\delta) g_k(\mathbf{x}) + w_7(\mathbf{x}, t; \delta), \tag{6.4}$$

where $w_7(\mathbf{x}, t; \delta) \to 0$ as $\delta \downarrow 0$ uniformly in $M \times [0, \infty)$.

Consider I^*_β. The substitution $\mu := \sqrt{\lambda}$ yields

$$I^*_\beta(t;\delta) = 2\int_\omega^{\sqrt{(\omega^2+\delta)}} \frac{1-e^{-i(\mu-\omega)t}}{(\mu-\omega)^{1+\beta}} h_\beta(\mu) d\mu,$$

with $h_\beta(\mu) := \mu/(\mu+\omega)^{1+\beta}$. Note that

$$|h_\beta(\mu) - h_\beta(\omega)| \leqslant |\mu-\omega| \left|\frac{dh_\beta}{d\mu}(\omega)\right|$$

for $\mu \in [\omega, \omega/\beta]$. This implies

$$I^*_\beta(t;\delta) = \frac{1}{(2\omega)^\beta} \int_\omega^{\sqrt{(\omega^2+\delta)}} \frac{1-e^{-i(\mu-\omega)t}}{(\mu-\omega)^{1+\beta}} d\mu + w_8(t;\delta;\beta), \tag{6.5}$$

where $w_8(t;\delta;\beta) \to 0$ as $\delta \downarrow 0$ uniformly with respect to $t \geqslant 0$ and $\beta \in [0, 1-1/2m]$. Setting $\xi := (\mu-\omega)t$, we obtain

$$I^*_\beta(t;\delta) = \frac{t^\beta}{(2\omega)^\beta} \int_0^{[\sqrt{(\omega^2+\delta)}-\omega]t} \frac{1-e^{-i\xi}}{\xi^{1+\beta}} d\xi + w_8(t;\delta;\beta). \tag{6.6}$$

Assume that $0<\beta<1-1/2m$. By (6.6) we have

$$\begin{aligned}I_\beta^*(t;\delta)&=\frac{t^\beta}{(2\omega)^\beta}\left[\int_0^\infty\frac{1-\mathrm{e}^{-\mathrm{i}\xi}}{\xi^{1+\beta}}\mathrm{d}\xi-\int_{[\sqrt{(\omega^2+\delta)}-\omega]t}^\infty\frac{\mathrm{d}\xi}{\xi^{1+\beta}}\right.\\&\quad\left.+\int_{[\sqrt{(\omega^2+\delta)}-\omega]t}^\infty\frac{\mathrm{e}^{-\mathrm{i}\xi}}{\xi^{1+\beta}}\mathrm{d}\xi\right]+w_8(t;\delta;\beta)\\&=\frac{1}{(2\omega)^\beta}\left[\frac{\pi\mathrm{e}^{\mathrm{i}\beta\pi/2}}{\beta\Gamma(\beta)\sin(\beta\pi)}t^\beta-\frac{1}{\beta[\sqrt{(\omega^2+\delta)}-\omega]^\beta}\right]\\&\quad+w_8(t;\delta;\beta)+W_4(t;\delta;\beta),\end{aligned}\tag{6.7}$$

with

$$W_4(t;\delta;\beta):=\frac{t^\beta}{(2\omega)^\beta}\int_{[\sqrt{(\omega^2+\delta)}-\omega]t}^\infty\frac{\mathrm{e}^{-\mathrm{i}\xi}}{\xi^{1+\beta}}\mathrm{d}\xi$$

(compare the integrals (11c) and (12b) in Reference 1). Note that

$$|W_4(t;\delta;\beta)|\leqslant\frac{2}{(2\omega)^\beta[\sqrt{(\omega^2+\delta)}-\omega]^{1+\beta}}\frac{1}{t},\tag{6.8}$$

as an integration by parts shows.

Now let $\beta=0$. In this case (6.6) can be rewritten as

$$\begin{aligned}I_0^*(t;\delta)&=\int_1^{[\sqrt{(\omega^2+\delta)}-\omega]t}\frac{\mathrm{d}\xi}{\xi}+\int_0^1\frac{1-\mathrm{e}^{-\mathrm{i}\xi}}{\xi}\mathrm{d}\xi-\int_1^\infty\frac{\mathrm{e}^{-\mathrm{i}\xi}}{\xi}\mathrm{d}\xi\\&\quad+\int_{[\sqrt{(\omega^2+\delta)}-\omega]t}^\infty\frac{\mathrm{e}^{-\mathrm{i}\xi}}{\xi}\mathrm{d}\xi+w_8(t;\delta;0)\\&=\ln t+\ln[\sqrt{(\omega^2+\delta)}-\omega]+C_e+\mathrm{i}\frac{\pi}{2}+w_8(t;\delta;0)+W_4(t;\delta;0)\end{aligned}\tag{6.9}$$

($C_e:=$ the Euler–Mascheroni constant; compare (3.67) in Reference 11).

We insert (6.7) and (6.9) in (6.4) and use the abbreviations

$$\left.\begin{aligned}E_s&:=\frac{C_s}{2m(2\pi)^{l/2}}\frac{\pi\mathrm{e}^{\mathrm{i}\beta_s\pi/2}}{(2\omega)^{\beta_s}\beta_s\Gamma(\beta_s)\sin(\beta_s\pi)}\quad\left(s=0,\ldots,\left[m-\frac{l}{2}\right]\right),\\E^*&:=\frac{1}{2m(2\pi)^{l/2}}C_{m-(l/2)}\end{aligned}\right\}\tag{6.10}$$

with $\beta_s=1-(l+2s)/2m$ and C_s defined by (4.10). Then we obtain for odd l

$$\begin{aligned}I_2(\mathbf{x},t;\delta)&=\sum_{s=0}^{m-(l+1)/2}E_st^{\beta_s}p_s^{(k)}(\mathbf{x})+\mathrm{i}\pi g_k(\mathbf{x})\\&\quad-\frac{1}{2m(2\pi)^{l/2}}\sum_{s=0}^{m-(l+1)/2}\frac{C_s}{(2\omega)^{\beta_s}\beta_s[\sqrt{(\omega^2+\delta)}-\omega]^{\beta_s}}p_s^{(k)}(\mathbf{x})\\&\quad+w_9(\mathbf{x},t;\delta)+W_5(\mathbf{x},t;\delta),\end{aligned}\tag{6.11}$$

and for even l

$$I_2(\mathbf{x}, t; \delta) = \sum_{s=0}^{m-1-l/2} E_s t^{\beta_s} p_s^{(k)}(\mathbf{x}) + E^* \ln t \cdot p_{m-(l/2)}^{(k)}(\mathbf{x}) + \mathrm{i}\pi g_k(\mathbf{x})$$
$$- \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{m-1-l/2} \frac{C_s}{(2\omega)^{\beta_s} \beta_s [\sqrt{(\omega^2+\delta)} - \omega]^{\beta_s}} p_s^{(k)}(\mathbf{x})$$
$$+ \frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}} \left\{ \ln [\sqrt{(\omega^2+\delta)} - \omega] + C_e + \frac{\mathrm{i}\pi}{2} \right\} p_{m-(l/2)}^{(k)}(\mathbf{x})$$
$$+ w_9(\mathbf{x}, t; \delta) + W_5(\mathbf{x}, t; \delta), \tag{6.12}$$

where

$$W_5(\mathbf{x}, t; \delta) := \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{[m-l/2]} C_s W_4(t; \delta; \beta_s) p_s^{(k)}(\mathbf{x}) + \mathrm{i} W_2(t; \delta) g_k(\mathbf{x}) \tag{6.13}$$

and $w_9(\mathbf{x}, t; \delta) \to 0$ as $\delta \downarrow 0$ uniformly in $M \times [0, \infty)$.

Now we turn to the discussion of I_3, using the respresentation (5.15). We conclude from (4.39) that

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} \frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} \mathrm{d}\lambda = \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{[m-l/2]} C_s J_{\beta_s}^*(\delta; \tau) p_s^{(k)}(\mathbf{x})$$
$$+ \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} \mathrm{d}\lambda\, g_k(\mathbf{x}) + w_{10}(\mathbf{x}; \delta; \tau), \tag{6.14}$$

where

$$J_\beta^*(\delta; \tau) := \int_{\omega^2}^{\omega^2+\delta} \frac{1}{(\lambda - \omega^2 - \mathrm{i}\tau)(\lambda - \omega^2)^\beta} \mathrm{d}\lambda \tag{6.15}$$

and $w_{10}(\mathbf{x}; \delta; \tau) \to 0$ as $\delta \downarrow 0$ uniformly with respect to $\mathbf{x} \in M$ and $\tau \geqslant 0$.

Consider J_β^*. Setting $\mu := (\lambda - \omega^2)/\tau$, we obtain in the case $\beta > 0$

$$J_\beta^*(\delta; \tau) = \frac{1}{\tau^\beta} \int_0^{\delta/\tau} \frac{\mathrm{d}\mu}{(\mu - \mathrm{i})\mu^\beta}$$
$$= \frac{1}{\tau^\beta} \left\{ \int_0^\infty \frac{\mathrm{d}\mu}{(\mu - \mathrm{i})\mu^\beta} - \int_{\delta/\tau}^\infty \left(\frac{1}{\mu^{\beta+1}} + \frac{\mathrm{i}}{(\mu - \mathrm{i})\mu^{\beta+1}} \right) \mathrm{d}\mu \right\},$$
$$= \frac{1}{\tau^\beta} C^*(\beta) - \frac{1}{\beta \delta^\beta} + O(\tau) \qquad \text{as } \tau \downarrow 0, \tag{6.16}$$

with

$$C^*(\beta) := \int_0^\infty \frac{\mathrm{d}\mu}{(\mu - \mathrm{i})\mu^\beta} = \int_0^\infty \frac{\mathrm{d}\mu}{(\mu^2+1)\mu^{\beta-1}} + \mathrm{i} \int_0^\infty \frac{\mathrm{d}\mu}{(\mu^2+1)\mu^\beta}$$
$$= \frac{\pi}{2} \left(\frac{1}{\sin \beta\pi/2} + \frac{\mathrm{i}}{\sin(\beta+1)\pi/2} \right) \tag{6.17}$$

(compare the integral 42 on page 70 in Reference 1). In the case $\beta = 0$ we have

$$J_0^*(\delta; \tau) = \ln \delta - \ln \tau + \frac{\mathrm{i}\pi}{2} + o(1) \qquad \text{as } \tau \downarrow 0. \tag{6.18}$$

Together with (5.16) we obtain from (6.14)

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau}\mathrm{d}(P_\lambda f(\mathbf{x})) = \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{m-(l+1)/2} C_s\left(\frac{C^*(\beta_s)}{\tau^{\beta_s}} - \frac{1}{\beta_s\delta^{\beta_s}}\right)p_s^{(k)}(\mathbf{x}) + \mathrm{i}\pi g_k(\mathbf{x}) + w_{10}(\mathbf{x}, t;\delta) + o(1) \quad \text{as } \tau\downarrow 0 \tag{6.19}$$

if l is odd, and

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau}\mathrm{d}(P_\lambda f(\mathbf{x})) = \frac{1}{2m(2\pi)^{l/2}} \sum_{s=0}^{m-1-l/2} C_s\left(\frac{C^*(\beta_s)}{\tau^{\beta_s}} - \frac{1\cdot}{\beta_s\delta^{\beta_s}}\right)p_s^{(k)}(\mathbf{x}) + \frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}\left(\ln\delta - \ln\tau + \frac{\mathrm{i}\pi}{2}\right)p_{m-(l/2)}^{(k)}(\mathbf{x}) + \mathrm{i}\pi g_k(\mathbf{x}) + w_{10}(\mathbf{x};\delta;\tau) + o(1) \quad \text{as } \tau\downarrow 0 \tag{6.20}$$

if l is even.

Now we investigate the first term in (5.15). By Lemma 3.1 and (4.34) we have

$$R_{\omega^2+\mathrm{i}\tau}f(\mathbf{x}) = \sum_{j=1}^{\infty} u^{(j)}(\mathbf{x};\omega^2+\mathrm{i}\tau). \tag{6.21}$$

The discussion in Sections 3 and 4 yields

$$\sum_{\substack{j=1\\ j\neq k}}^{\infty} u^{(j)}(\mathbf{x};\omega^2+\mathrm{i}\tau) = \sum_{\substack{j=1\\ j\neq k}}^{\infty} u^{(j)}(\mathbf{x};\omega^2+\mathrm{i}0) + o(1) \qquad \text{as } \tau\downarrow 0, \tag{6.22}$$

uniformly in M. From (4.15), (4.16) and (6.3) we obtain

$$u^{(k)}(\mathbf{x};\omega^2+\mathrm{i}\tau) = v^{(k)}(\mathbf{x}) + \sum_{s=0}^{m-(l+1)/2}\frac{C_sD_s}{\tau^{\beta_s}}p_s^{(k)}(\mathbf{x}) + O(\tau^{1/2m}) \qquad \text{as } \tau\downarrow 0 \tag{6.23}$$

for odd l, and

$$u^{(k)}(\mathbf{x};\omega^2+\mathrm{i}\tau) = v^{(k)}(\mathbf{x}) - D'(1)p_{m-(l/2)}^{(k)}(\mathbf{x}) - \frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}\ln\tau\cdot p_{m-(l/2)}^{(k)}(\mathbf{x}) + \sum_{s=0}^{m-1-(l/2)}\frac{C_sD_s}{\tau^{\beta_s}}p_s^{(k)}(\mathbf{x}) + O(\tau^{1/2m}) \qquad \text{as } \tau\downarrow 0 \tag{6.24}$$

for even l, where

$$v^{(k)}(\mathbf{x}) := \begin{cases} \dfrac{\Gamma((l/2)-m)}{\pi^{l/2}4^m(m-1)!} \\ \quad\times\displaystyle\int_{\mathbf{R}^l} f^{(k)}(x',y)|x-x'|^{2m-l}\,\mathrm{d}x' & \text{if } l \text{ is odd,} \\ \dfrac{(-1)^{m+1-(l/2)}}{\pi^{l/2}2^{2m-1}(m-(l/2))!(m-1)!} \\ \quad\times\displaystyle\int_{\mathbf{R}^l} f^{(k}(x',y)|x-x'|^{2m-l}\ln|x-x'|\,\mathrm{d}x' & \text{if } l \text{ is even.}\end{cases} \tag{6.25}$$

We set

$$V_\omega(\mathbf{x}) := \begin{cases} \sum_{\substack{j=1\\ j\neq k}}^{\infty} u^{(j)}(\mathbf{x};\omega^2+\mathrm{i}0)+v^{(k)}(\mathbf{x}) & \text{for odd } l\\ \left(\frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}[C_e-\ln(2\omega)]-D'(1)\right)p^{(k)}_{m-(l/2)}(\mathbf{x}) & \\ +\sum_{\substack{j=1\\ j\neq k}}^{\infty} u^{(j)}(\mathbf{x};\omega^2+\mathrm{i}0)+v^{(k)}(\mathbf{x}) & \text{for even } l. \end{cases} \tag{6.26}$$

Note that $[(-\Delta_x)^m+(-\Delta_y)^m]\,p^{(k)}_{m-(l/2)}(\mathbf{x})-\omega^2 p^{(k)}_{m-(l/2)}(\mathbf{x})=0$ by (6.3) and (4.34) and that therefore

$$\begin{aligned} &[(-\Delta_x)^m+(-\Delta_y)^m]\,V_\omega-\omega^2 V_\omega=f \qquad \text{in } \Omega,\\ &V_\omega=\frac{\partial V_\omega}{\partial \mathbf{n}}=\ldots=\frac{\partial^{m-1}V_\omega}{\partial \mathbf{n}^{m-1}} \qquad \text{on } \partial\Omega. \end{aligned} \tag{6.27}$$

Now we assume that l is odd. Recall that $\beta_s=1-(l+2s)/2m$ and $\sigma=(l/2)-1$. Note that by (4.12) and (6.17)

$$C_sD_s=\frac{C_sC^*(\beta_s)}{2m(2\pi)^{l/2}} \qquad \left(s=0,1,\ldots,m-\frac{l+1}{2}\right)$$

if $\tau>0$. Therefore (5.15), (6.19) and (6.21)–(6.26) yield

$$I_3(\mathbf{x};\delta)=V_\omega(\mathbf{x})+\frac{1}{2m(2\pi)^{l/2}}\sum_{s=0}^{m-(l+1)/2}\frac{C_s}{\beta_s\delta^{\beta_s}}p^{(k)}_s(\mathbf{x})-\mathrm{i}\pi g_k(\mathbf{x})-w_{10}(\mathbf{x};\delta;0). \tag{6.28}$$

Combining this with (6.11), we obtain

$$\begin{aligned} I_2(\mathbf{x},t;\delta)+I_3(\mathbf{x};\delta)=&\sum_{s=0}^{m-(l+1)/2}E_st^{\beta_s}p^{(k)}_s(\mathbf{x})+V_\omega(\mathbf{x})\\ &+\sum_{s=0}^{m-(l+1)/2}\frac{1}{2m(2\pi)^{l/2}}C_s\\ &\times\left(\frac{1}{\beta_s\delta^{\beta_s}}-\frac{1}{(2\omega)^{\beta_s}\beta_s[\sqrt{(\omega^2+\delta)}-\omega)-\omega]^{\beta_s}}\right)p^{(k)}_s(\mathbf{x})\\ &+w_{11}(\mathbf{x},t;\delta)+W_5(\mathbf{x},t;\delta), \end{aligned} \tag{6.29}$$

where $w_{11}(\mathbf{x},t;\delta)\to 0$ as $\delta\downarrow 0$ uniformly in $M\times[0,\infty)$. Note that

$$\begin{aligned} \lim_{\delta\downarrow 0}\left[\frac{1}{\beta\delta^\beta}-\frac{1}{(2\omega)^\beta\beta[\sqrt{(\omega^2+\delta)}-\omega]^\beta}\right]&=\frac{1}{\beta}\lim_{\delta\downarrow 0}\frac{(2\omega)^\beta-[\sqrt{(\omega^2+\delta)}+\omega]^\beta}{(2\omega)^\beta\delta^\beta}\\ &=-\frac{1}{2\beta(2\omega)^\beta}\lim_{\delta\downarrow 0}\frac{[\sqrt{(\omega^2+\delta)}+\omega]^{\beta-1}}{\delta^{\beta-1}\sqrt{(\omega^2+\delta)}}\\ &=0 \end{aligned}$$

for $0<\beta<1$. Hence, for any given $\varepsilon>0$ there exists a $\delta_4>0$ such that

$$\left|\frac{1}{2m(2\pi)^{l/2}}\sum_{s=0}^{m-(l+1)/2} C_s\left(\frac{1}{\beta_s\delta_4^{\beta_s}}-\frac{1}{(2\omega)^{\beta_s}\beta_s[\sqrt{(\omega^2+\delta_4)}-\omega]^{\beta_s}}\right)p_s^{(k)}(\mathbf{x})\right|<\varepsilon$$

and $|w_{11}(\mathbf{x},t;\delta_4)|<\varepsilon$ for $\mathbf{x}\in M$. We insert (6.29) into (5.13). Note that by (5.12), Lemma 5.2, (6.13), (5.20) and (6.8)

$$W_5(\mathbf{x},t;\delta_4)+W_1(\mathbf{x},t;\delta_4)=o(1) \qquad \text{as } t\to\infty$$

uniformly with respect to $\mathbf{x}\in M$. Thus we obtain

$$u(\mathbf{x},t)=\sum_{s=0}^{m-(l+1)/2} E_s\mathrm{e}^{-\mathrm{i}\omega t}t^{\beta_s}p_s^{(k)}(\mathbf{x})+\mathrm{e}^{-\mathrm{i}\omega t}V_\omega(\mathbf{x})+o(1) \qquad \text{as } t\to\infty \tag{6.30}$$

uniformly with respect to $\mathbf{x}\in M$ if l is odd.

Now we assume that l is even. As above we conclude from (5.15), (6.20) and (6.21)–(6.26)

$$\begin{aligned}I_3(\mathbf{x};\delta)&=V_\omega(\mathbf{x})+\frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}(\ln 2\omega-C_e)p_{m-(l/2)}^{(k)}(\mathbf{x})-\mathrm{i}\pi g_k(\mathbf{x})\\&\quad+\frac{1}{2m(2\pi)^{l/2}}\sum_{s=0}^{m-1-(l/2)}\frac{C_s}{\beta_s\delta^{\beta_s}}p_s^{(k)}(\mathbf{x})\\&\quad-\frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}\left(\ln\delta+\frac{\mathrm{i}\pi}{2}\right)p_{m-(l/2)}^{(k)}(\mathbf{x})-w_{10}(\mathbf{x};\delta;0)\end{aligned} \tag{6.31}$$

and with (6.12)

$$\begin{aligned}&I_2(\mathbf{x},t;\delta)+I_3(\mathbf{x};\delta)\\&\quad=\sum_{s=0}^{m-1-(l/2)}E_st^{\beta_s}p_s^{(k)}(\mathbf{x})+E^*\ln t\cdot p_{m-(l/2)}^{(k)}(\mathbf{x})+V_\omega(\mathbf{x})\\&\qquad+\frac{C_{m-(l/2)}}{2m(2\pi)^{l/2}}\{\ln 2\omega+\ln[\sqrt{(\omega^2+\delta)}-\omega]-\ln\delta\}\,p_{m-(l/2)}^{(k)}(\mathbf{x})\\&\qquad+w_{12}(\mathbf{x},t;\delta)+W_5(\mathbf{x},t;\delta),\end{aligned} \tag{6.32}$$

where $w_{12}(\mathbf{x},t;\delta)\to 0$ as $\delta\downarrow 0$ uniformly with respect to $\mathbf{x}\in M$ and $t\geqslant 0$. Since

$$\lim_{\delta\downarrow 0}\{\ln[\sqrt{(\omega^2+\delta)}-\omega]-\ln\delta\}=-\ln 2\omega.$$

we obtain together with (5.13)

$$\begin{aligned}u(\mathbf{x},t)&=\sum_{s=0}^{m-1-(l/2)}E_s\mathrm{e}^{-\mathrm{i}\omega t}t^{\beta_s}p_s^{(k)}(\mathbf{x})+E^*\mathrm{e}^{-\mathrm{i}\omega t}\ln t\cdot p_{m-(l/2)}^{(k)}(\mathbf{x})\\&\quad+\mathrm{e}^{-\mathrm{i}\omega t}V_\omega(\mathbf{x})+o(1) \qquad \text{as } t+\infty\end{aligned} \tag{6.33}$$

uniformly in M. Thus we have proved:

Theorem 6.1. *Let u be the uniquely determined solution of* (1.2)–(1.4) *satisfying* (2.12) *and assume that* $f, u_1, u_2\in C_0^\infty(\Omega)$, $\partial\Omega\in C^\infty$ *and* $\Omega=\Omega'\times\mathbb{R}^l$. *The principle of limiting amplitude* ((1.8) *and* (1.10)) *holds if and only if one of the conditions* (4.23)–(4.25) *is* *satisfied. The limit amplitude* U_ω *coincides with the limit* $R_{\omega^2+i0}\,f$ *of the resolvent of the operator A introduced in* (2.1) *and is uniquely characterized by the conditions stated in Lemma* 4.1.

In all other cases resonances of order t^α $(0<\alpha<1)$ *or* $\ln t$ *occur. The asymptotic behaviour of u in the resonance case is given by* (6.30) *if l is odd and by* (6.33) *if l is even (with* $\beta_s = 1-(l+2s)/2m$).

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