Effective equations for macrovariables in randomly coupled quantum systems

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1 Introduction: From the Micro- to the Macro-world

When describing a physical phenomena there is always implicitly or explicitly a statement about the scale such as a length scale, a time scale or an energy scale. The scale on which we talk will more or less set which theory will be used for the description if any is available. There is classical mechanics to describe the non-relativistic macroscopic phenomena just as we have quantum mechanics to describe the microscopic phenomena. At a non-relativistic level quantum mechanics is the most fundamental theory existing. These theories form a basis from which one works upwards in order to describe phenomena. We do not use classical mechanics nor quantum mechanics alone to describe a gas that is in equilibrium but rather thermodynamics and statistical mechanics. One could ask oneself if the postulates of these are derivable in some sense from quantum mechanics. More refined statements about the scale introduce then more approximations which allow then to tackle the problem at hand.

Transport, in large systems, is one of the most complicated phenomena to describe in physics. A straight forward application of the theory, may it be classical or quantum, is often useless. The many degrees of freedom make it hard to solve the equations even for the most simple models. Boltzmann came up with his famous equation to describe on a simple level how the phase space density of a gas would change in time. In order to do this he had to introduce the often criticized assumption of "molecular chaos". It would also result in the H-theorem which states that the entropy of a system will increase in time. These equations were, despite criticism, very useful. The botanist Robert Brown discovered brownian motion, that is , he realized that the movement of a particle suspended in a liquid was eratic and random like. Wiener later on formulated the mathematical description of it which goes now under the name of Wiener process or white noise. It is the prime example of a stochastic process and the most studied one. These equations are without doubt fundamental ones but are not derived from first principles only. A strict derivation of the Boltzmann equation is still lacking. Fourier's law states that the heat current in a metal between two baths will be proportional to the temperature gradient. It works very well but still remains to be proven [4]. The emergence of these laws and equations which typically work very well are still open questions.

Today in order to describe transport in a system the quantum nature of it has to be taken into account. A lot of the non-equilibrium phenomena is analyzed by means of the Kubo formalism or with the help of the Lindblad equations. One of the tools exploited here is the notion of a thermal bath and thermal equilibrium. This means that a much greater part of the system is in thermal equilibrium.

A main characteristic of non-equilibrium is the fact that it returns to equilibrium. This simple and obvious statement generates a serious problem when starting from quantum mechanics because in the microscopic theory there is no notion of equilibrium. As Poincare pointed out, any finite system will return infinitely close to its starting point, hence reversibility.

In order to obtain the irreversible behavior given by stochastic equations there has always been some type of course-graining involved. The ergodic hypothesis states the time scale of a measurements is long compared with the microscopic time scale and so a time average has to be taken. The notion of an ensemble is similar in the sense that we average over a whole set of possible configurations. Many of the approximations used to derive the Lindblad and other master equations are also some type of time coarse graining. All of these "tricks" allow one to make the step from the reversible to the irreversible description. From a mathematical point of view something is taken out or added to the equations and changes fundamentally the nature of the resulting ones.

One of the most simplest ways to try to predict the time evolution of a system is to apply Fermi's golden rule. In a quantum system Fermi's golden rule gives us the transition probabilities of one state of the system to another one. Naively one obtains this rule by expanding the solution of the Schrödinger equation and considering only the first order term. At this point one is assuming that time is small enough for the expansion to the first order to make sense. The transition probability from a state m to a state k is then proportional to $\mid V_{mk}\mid^2 \frac{\sin^2(\omega t/2)}{\omega^2}$ where ω is the energy difference between the two states. Then one assumes that t is large enough so that the transition rate $\frac{1}{t}\mid V_{mk}\mid^2 \frac{\sin^2(\omega t/2)}{\omega^2}$ is proportional to $\mid V_{mk}\mid^2 \frac{\pi}{2}\delta(\omega)$.

If Fermi 's golden rule would be true at all times, then from it an equation could be derived for the probability distribution of the system. One would then jump from a Schrödinger equation to a type of master equation. But Fermi's golden rule is not valid for all system, nor for all times. This has to be checked for the particular model at hand. Partly we can see this by the two possibly contradicting assumptions about the time scale. First it is assumed small but later on large enough. These fuzzy requirements and the success of them are the reasons why it is called a rule.

On the other hand quantum chaos has been an intensive field of research since its discovery. While in Boltzmann's time chaos was a concept emerging from the fact that we have many particles in a system , many degrees of freedom, it became later apparent that even systems with small number of degrees of freedom could exhibit chaotic behavior [3]. Specifically in quantum chaos, signatures of quantum chaos became the new field of study. Although today there is no definition yet of what quantum chaos is, the signatures of quantum chaos are very clear and the main tool for deciding whether or not a system is quantum chaotic is a comparison to the predictions given by random matrix theory (RMT) [20]. In the recent years there have been a number of investigations about the role of quantum chaos in transport properties, [19], [21], [30]. In such investigations deterministic Hamiltonians were taken and their transport properties were analyzed as a function of their "chaoticity". Quantum chaos is said to be exhibited by systems with very few degrees of freedom, so that large compound systems should also exhibit it. And yet there is no theory of transport or dynamics that focusses on its role. We investigate here, in some sense, a marriage between these two ideas, quantum chaos and the derivation of effective equations. And why? If a dynamical law works so well, could it not be that a general statistical characteristic of the system is the one that gives rise to it? If most real systems exhibit signatures of quantum chaos it might be essential that in describing the evolution of a system this inherent part be taken into account. For this sake we take systems which from the start show signatures of quantum chaos. We do this with the help of RMT. Since RMT is the main tool for modelling the statistical behavior of quantum chaotic Hamiltonians, including in our Hamiltonian a random matrix means, in some sense, including quantum chaos, for it would not be surprising to find then that the system is quantum chaotic. Essentially we have a random Hamiltonian and the type of problem to be solved is similar to the Anderson type of model [1]

In this work our line of thoughts goes as follows. There are basically three ensembles in RMT. If different deterministic Hamiltonian show the same statistical behavior of an ensemble would it not be so that they would also show the same dynamical behavior? If we can manage to say what would happen for the ensemble could we not translate this for any deterministic Hamiltonian that seems to belong to this ensemble?

Essentially this dissertation will consist of two parts: First we will calculate the average of the solutions of the Schrödinger equation for a deterministic plus random Hamiltonian. In order to achieve this we will exploit the Feynman diagrammatic expansion of the solution of Schrödinger equation. These diagrams will be classified and analyzed in certain limits such as large size of the Hilbert space and long time-weak coupling limits. Secondly we will calculate the variance of the solution of the Schrödinger equation with respect to the random matrix ensemble. This will tell us if the outcome of an average over the ensemble is a typical result for a member of the ensemble. The relevance of the answers to those questions is as follows. If say a deterministic Hamiltonian, H, exhibits quantum chaos and belongs to a certain ensemble (GOE, GUE), would the evolution calculated by using this Hamiltonian or another one randomly chosen from the ensemble be any different?

2 Expectation value for Macro-variables: Rate equations

2.1 Macro variables and limiting procedures

In this chapter we will derive the average of the time evolution of what we shall call Macro-observables. A Macro-observable here does not only refer to common intuition. Loosely speaking, if the Hilbert space, in which we work, has a certain dimension D, and if our observable is diagonal in some basis, we call an observable macro, when the number of non-zero diagonal matrix elements is a certain fraction of D, where the fraction is independent of D. For example, the probability to be in half of the Hilbert space will be a Macro-observable but the probability to be in a certain state of the Hilbert space will not be a Macro-observable. This means that if we let the size of the Hilbert space grow to ∞ and the spectrum becomes continuous, then our observable will become a distribution function which is non-zero in intervals of length different from zero. We will perform the derivations within certain limits such as large Hilbert space limit and the weak coupling-long time limit (used in [16], [6], [15]). In section 2.2 we introduce the classes of models which will be analyzed. The basic characteristic of these models is that they possess a deterministic part and a random part. The deterministic part is taken to be known and given in its diagonalized form, although we do not fix it. Only some assumptions are made on its spectrum. The random part couples the system and thus generates non trivial dynamics. These are the dynamics which we want to derive. Since our Hamiltonian is partly random we have to compute the average of the dynamics. We derive thus the evolution of certain quantities given by the expectation value of an operator $\hat{O}$.
Our strategy in performing this derivation is similar to those used in [27], [10], [8], [7], [28], in particular [9]. In section 2.3 we will use the Duhamel formula in order to expand our time evolution operator in powers of the interaction to any given order and in section 2.4 we calculate the average over products of random matrices and introduce the graphical representation in order to calculate these averages. With this representation our solution is expanded in Feynman type diagrams. Since we pose the problem in an abstract language of Hilbert space vectors, we view it as Feynman diagrams in the Hilbert space. The different types of Feynman diagrams can be classified according to their contribution in the given limits. In particular there will be three types of diagrams, crossing (section 2.5), nested (section 2.6) and simple (section 2.7). The simple ones will be tractable in the limits considered and can be resummed up afterwards. This will be done in section 2.9. In order to do this we will need a double limit procedure. Initially our systems will be finite dimensional. A parameter N will control so to say the size

of the Hilbert space. It is in the limit $N \to \infty$ that we will derive our equations of motion and so this is the first limit. Secondly we will take the Van Hove limit [16]. In this limit we take $t \to \infty$ and $\lambda \to 0$, where λ is the coupling strength, while maintaining $\lambda^2 t = T$ finite. This is then a weak coupling- long time kind of regime. We call T the macroscopic time. Put mathematically our objective is to derive

$$\lim_{\substack{t\to\infty}}^{\lambda^2 t = T} \lim_{N\to\infty} \mathbb{E}\left[\langle\psi_t|\hat{O}|\psi_t\rangle\right] = O(T) \tag{2.1}$$

We shall see that the system evolves according to Pauli rate equations where the rates are given by Fermi's Golden Rule.
Pauli's rate equations read

$$\boxed{\frac{d}{dt}P(q) = \sum_{q'}\left(M_{q,q'}P(q') - M_{q',q}P(q)\right)} \tag{2.2}$$

where $M_{q,q'}$ is given by Fermi's Golden Rule

$$\boxed{M_{q,q'} = \frac{2\pi}{\hbar}\left|\langle q|\hat{V}|q'\rangle\right|^2 \delta(E_q - E_{q'})} \tag{2.3}$$

In chapter 5 we shall show that the limits need not to be taken independently but that certain scaling relations between time and size would render the same result. Since an average does not say much about what evolution the system follows for an individual realizations of the Hamiltonian we also analyze the variance later on. This will give rise to what we call dynamical typicality in chapter 3. Since we are analyzing classes of systems, the language we use in our derivation is necessarily a bit abstract. We will give some examples in chapter 4.

2.2 Model Class

We choose here a model which has a deterministic Hamiltonian, $\hat{H}_0$, plus a random Hamiltonian, $\hat{V}$. The random part is included as a weak interaction and we calculate the dynamics in the weak coupling - long time limit, the so-called Van Hove limit. $|q,l\rangle$ denotes an eigenstate of $\hat{H}_0$ with energy E_l. q stands for a vector of M quantum numbers, q^a, each allowed to have a finite number, $\aleph_a$, of different values (degeneracy). Our undisturbed Hamiltonian is

$$\begin{aligned}\hat{H}_0 &= \sum_{n=1,q_1=1,\dots q_M=1}^{N,\aleph_1,\dots\aleph_M} E_l|q^1,\dots q^M,l\rangle\langle q^1,\dots q^M,l| \\ &= \sum_{n=1,q=1}^{N,\aleph} E_l|q,l\rangle\langle q,l| \end{aligned} \tag{2.4}$$

In Eq. (2.4) $\sum_{q=1}^{\aleph}$ stands for $\sum_{q_1=1,\dots q_M=1}^{\aleph_1,\dots \aleph_M}$ and $\aleph = \prod_{a=1}^{M} \aleph_a$. We shall treat q as if it were just one quantum number. Since we will analyze the dynamics in the limit $N \to \infty$ we must specify how our density of states behaves, thus how the spectrum of $\hat{H}_0$ behaves when $N \to \infty$. We will take the spectrum to remain bounded and become continues as $N \to \infty$.We specify our discrete density of states as $\nu^N(E)$ such that

$$\sum_{l=1}^{N} \frac{1}{N} F(E_l) = \int dE \nu^N(E) F(E_l)$$

$$= \int dE \sum_{l=1}^{N} \frac{\delta(E - E_l)}{N} F(E) \tag{2.5}$$

$$\lim_{N \to \infty} \int dE \nu^N(E) F(E) = \int dE g(E) F(E) \tag{2.6}$$

Our discrete density of states, $\nu^N(E)$, will tend to a certain function $g(E)$, which we take to be smooth and bounded. The spectrum is taken to remain bounded between 0 and 1 and we take $\frac{dg(E)}{dE}$ to be continues and bounded.
Our interaction will allow transitions between states $|q,l\rangle$ and $|q',l'\rangle$ with some amplitude $W_{q,q'}(l,l')$ weighed by a gaussian distributed independent complex random variable for each entry, $z_{q,q'}(l,l')$. As $N \to \infty$ and the spectrum of $\hat{H}_0$ becomes continues we let our transition amplitudes $W_{q,q'}(l,l')$ go to a continuous bounded function in E_l and $E_{l'}$. We then set

$$W = \left| Max_{q,q',l,l'}(W_{q,q'}(l,l')) \right|^2 \tag{2.7}$$

i.e. W is the square of the maximum transition amplitude. Our interaction term then reads

$$\boxed{\hat{V} = \sum_{l,l'=1,q,q'=1}^{N,\aleph} z_{q,q'}(l,l') W_{q,q'}(l,l') |q,l\rangle\langle q,l'|} \tag{2.8}$$

The distribution over this type of random matrix is

$$P(V) = \frac{1}{Z} \prod_{z_{q,q'}(l,l')} e^{-N\aleph |z_{q,q'}(l,l')|^2}$$

If we define $\tilde{V}$ as the non weighed random matrix

$$\tilde{V} = \sum_{l,l'=1,q,q'=1}^{N,\aleph} z_{q,q'}(l,l') |q,l\rangle\langle q,l'|$$

then

$$P(V) = \frac{1}{Z} e^{-N\aleph\, Tr\left[\tilde{V}^2\right]}$$

$$Z = \left(\frac{\pi}{N\aleph}\right)^{\frac{N^2\aleph(\aleph-1)}{2}}$$

We have the following formula for the average over a pair of random matrix elements:

$$\mathbb{E}[V_{q_1,q_2}(l_1,l_2)V_{q_3,q_4}(l_3,l_4)] = \frac{|W_{q_1,q_2}(l_1,l_2)|^2}{N\aleph}\delta_{q_1,q_4}\delta_{q_2,q_3}\delta_{l_1,l_4}\delta_{l_2,l_3} \tag{2.9}$$

Our total Hamiltonian is then

$$\boxed{\hat{H} = \hat{H}_0 + \lambda\hat{V}} \tag{2.10}$$

We define now Macro-observables:

Definition 2.2.1. *In a Hilbert space,$\mathcal{H}$, of dimension $N\aleph$, an observable $\hat{O}$ diagonal in the basis of eigenfunction of $\hat{H}_0$ has the form*

$$\boxed{\hat{O} = \sum_{q=1}^{\aleph}\sum_{l=1}^{N} O(q,l)|q,l\rangle\langle q,l|} \tag{2.11}$$

It is called a Macro-observable, if the number of values $O(q,l)$ different from zero is a finite fraction of $N\aleph$ in the limit $N\to\infty$.

Since we want to derive the respective Pauli equations we shall analyze the average over the random matrix of the time evolution of the probabilities, $P_t(q,l)$, to be in state $|q,l\rangle$. This will be done by expanding the evolution operator in powers of the interaction and averaging over the random matrices. We will analyze $P_t(q,l)$ as a distribution over a test function and thus the mean values of the expectation values of some operator in the limit $N\to\infty$. Therefore we have to analyze

$$\begin{aligned}\lim_{N\to\infty}\langle\hat{O}\rangle_t &= \lim_{N\to\infty}\langle\psi_t|\hat{O}|\psi_t\rangle \\ &= \lim_{N\to\infty}\sum_{l_0,q_0}\frac{1}{N}O(q_0,E_{l_0})\tilde{P}_t(q_0,l_0) \qquad (2.12)\\ &= \int dE_0 g(E_0)\sum_{q_0}O(q_0,E_0)\tilde{P}_t(q_0,E_0) \qquad (2.13)\end{aligned}$$

where

$$\tilde{P}_t(q_0,l_0) = N|\psi_t(l_0,q_0)|^2 \tag{2.14}$$

remains bounded as $N\to\infty$. We refer to appendix C for more details.

This chapter will be devoted to the proof of the following theorem:

Theorem 2.2.2. *Say* $|\psi_t\rangle$ *is the solution to the Schrödinger equation with the Hamiltonian of Eq. (2.10) and with initial condition* $|\psi_0\rangle$*. If* $\hat{O}$ *is a Macro-observable then the average over the random matrix ensemble of the time evolution of the observable, in the limit of large* N *and in the Van Hove limit,*

$$\lim_{t\to\infty}^{t\lambda^2=T} \lim_{N\to\infty} \mathbb{E}\left[\langle\hat{O}\rangle_t\right] = O(T)$$

is given by

$$O(T) = \sum_q \int d\omega g(\omega) O(q,\omega) P_T(q,\omega) \tag{2.15}$$

were $P_T(q,\omega)$ *solves the following rate equation:*

$$\frac{\partial}{\partial T} P_T(q,\omega) = \sum_{q'} M_{q,q'}(\omega) P_T(q',\omega) \tag{2.16}$$

$$M_{q,q'}(\omega) = \pi g(\omega)\left(-\delta_{q,q'} \sum_{\bar{q}\neq q} \left| W_{q,\bar{q}}(\omega,\omega)\right|^2 + \left| W_{q,q'}(\omega,\omega)\right|^2\right) \tag{2.17}$$

with initial condition

$$P_{T=0}(q,\omega) = \psi_0^*(q,\omega)\psi_0(q,\omega) \tag{2.18}$$

We take $P_{T=0}(q,\omega)$ *to be bounded by some constant.*

2.2.3 Notation

We summarize here some notation which we will use.
In general, when many variables are used, for example the set $\{l_0,\dots l_n\}$, we will write it in short as $\{l_i\}$. Or if there are two of such sets, for example $\{l_0,\dots l_n\}$ and $\{l'_0,\dots l'_n\}$, we will write it in short as $\{l_i,l'_j\}$. In order to refer to the set of these many variables with one variable removed, for example $\{l_1,\dots l_n\}$ out of the set $\{l_0,\dots l_n\}$, we will write $\{l_i\}_0$. If a function, f, depends on such sets of variables we will write it as $f\left(\{l_i,l'_j\}\right)$.
In general there will be many constants appearing, which do not really matter. The only important fact is that they are constant. This is why we use often C to denote a constant but not necessarily the same one. Also C_1 and C_2 is used to denote constants. This is not to be confused with C_π or $C_\pi(n,m)$ which stands for a contraction function of a particular graph. For brevity of notation we will write $\int [dE]$ for $\int dEg(E)$, where $g(E)$ is the density of states of $\hat{H}_0$ in the limit $N\to\infty$.

2.3 The Duhamel expansion

In the following we want to express the time evolution of an observable such as the projector $|q_0,l_0\rangle\langle q_0,l_0|$ in powers of the coupling constant λ. This would give us $P_t(q_0,l_0)$, the probability to be in state $|q_0,l_0\rangle$ at time t. The Duhamel expansion of the time evolution operator is the expansion in terms of the perturbative term of the Hamiltonian. It gives thus the first n-terms of the perturbation series and encodes the rest of the evolution in a remaining term, see eq. (2.25). The following identity for the evolution operator can be verified easily:

$$e^{-iHt} = e^{-iH_0t} - i\lambda \int_0^t e^{-iH(t-s)}\, Ve^{-iH_0 s}\, ds \tag{2.19}$$

By applying successively this identity we can expand the evolution operator in orders of λ. Thus we can rewrite the time evolution operator, and by using this expression the evolution of the wave vector, as follows:

$$e^{-iHt} = \sum_{n=0}^{M-1} (-i\lambda)^n \Gamma_n(t) + \tilde{\Gamma}_M(t) \tag{2.20}$$

$$|\psi_t\rangle = \sum_{n=0}^{M-1} |\psi_t^n\rangle + |\phi_t^M\rangle \tag{2.21}$$

with

$$\begin{aligned}\Gamma_n(t) &= \int_0^t \int_0^{t-s1} \dots \int_0^{t-\Sigma_{j=1}^{n-1} s_j} ds_1 \dots ds_n\, e^{-iH_0(t-\Sigma_{j=1}^{n} s_j)}\, Ve^{-iH_0 s_n}\, V \dots e^{-iH_0 s_1} \\ &= \int_0^t \int_0^t \dots \int_0^t ds_0 \dots ds_n\, e^{-iH_0 s_0}\, Ve^{-iH_0 s_n} \dots e^{-iH_0 s_1} \delta(t - \sum_{j=0}^{n} s_j)\end{aligned} \tag{2.22}$$

and

$$\tilde{\Gamma}_M(t) = \int_0^t \int_0^t \dots \int_0^t ds_0 \dots ds_M\, e^{-iHs_0}\, Ve^{-iH_0 s_n} \dots e^{-iH_0 s_1} \delta(t - \sum_{j=0}^{M} s_j) \tag{2.23}$$

In Eq. (2.21) we have applied $M-1$ times Eq. 2.19 to obtain the perturbation expansion in the interaction of $|\psi_t\rangle$ up to the $M-1^{\text{th}}$ term. $|\phi_t^M\rangle$ encodes the rest of the evolution and fulfills Eq. (2.24).

$$|\phi_t^M\rangle = \int_0^t ds e^{-iH(t-s)} (-i\lambda)\, V |\psi_s^{M-1}\rangle \tag{2.24}$$

We adopt the following notation

$$\int_0^t \dots \int_0^t ds_0 \dots ds_n \delta(t - \sum_{i=0}^{n} s_i) = \int [ds_n]$$

Using the expansion of Eq. (2.20) we find for the probability to be in state $|q_0, l_0\rangle\langle q_0, l_0|$

$$P_t(q_0, l_0) = \sum_{n,m=0}^{M-1} (i\lambda)^m (-i\lambda)^n \langle\psi_0|\Gamma_m^\dagger(t)|q_0, l_0\rangle\langle q_0, l_0|\Gamma_n(t)|\psi_0\rangle + R(M, t, \lambda, q_0, l_0) \tag{2.25}$$

We will focus only on the first term as M goes to ∞. It will be shown in chapter 6 that the contribution of $R(M, t, \lambda, q_0, l_0)$ amounts to zero in the limit $M \to \infty$ and so we may omit in Eq. (2.25) this term. Therefore we will take $P_t(q_0, l_0)$ to be the first term, that is the sum, in Eq. (2.25) and keep in mind that we will finally take the limit $M \to \infty$ in order to have the whole evolution. We replace the initial state of our system by its expansion in terms of our basis vectors, $|q_j, l_j\rangle$.

$$\begin{aligned} P_t^M(q_0, l_0) = \sum_{n,m=0}^{M-1} &\lambda^{m+n} i^m (-i)^n \\ &\sum_{l_n, l'_m, q'_m, q_n} \psi_0^*(q'_m, l'_m)\psi_0(q_n, l_n)\langle q'_m, l'_m|\Gamma_m^\dagger(t)|q_0, l_0\rangle\langle q_0, l_0|\Gamma_n(t)|q_n, l_n\rangle \end{aligned} \tag{2.26}$$

When inserting $\Gamma_n(t)$ into $\langle q_0, l_0|\Gamma_n(t)|q_n, l_n\rangle$ and successive identities, $\sum_{l_j, q_j} |q_j, l_j\rangle\langle q_j, l_j|$ after each interaction term, we obtain:

$$(-i)^n \langle q_0, l_0|\Gamma_n(t)|q_n, l_n\rangle = \prod_{i=1}^{n-1} \sum_{q_i}^{\aleph} \sum_{l_i=1}^{N} K^n(t, \{E_{l_i}\}) L^n(\{q_i\}, \{l_i\}) \tag{2.27}$$

with

$$K^n(t, \{E_{l_i}\}) = (-i)^n \int [ds_n] e^{-iE_{l_0} s_0} e^{-iE_{l_1} s_1} \dots e^{-iE_{l_n} s_n} \tag{2.28}$$

$$L^n(\{q_i\}, \{l_i\}) = \langle q_0, l_0| V|q_1, l_1\rangle\langle q_1, l_1| V|q_2, l_2\rangle \dots \langle q_{n-1}, l_{n-1}| V|q_n, l_n\rangle \tag{2.29}$$

The same can be done with the left hand side expression, $\langle q'_m, l'_m|\Gamma_n(t)|q_0, l_0\rangle$, and the variables of the identities introduced on this side will be denoted by primed variables. Each $K^n(t, \{E_{l_i}\})$ is a set of propagators attached together. It is a "possible history" of the wavevector given by the energy levels it jumps to. This is the quantity we will be analyzing in the Van Hove limit.

The $L^n(\{q_i\}, \{l_i\})$, on the other hand, is the statistical weight given to this history or process by the random interaction. It carries no time dependency and is a random variable.

We thus end up with

$$\begin{aligned} P_t^M(q_0, l_0) = \sum_{n,m=0}^{M-1} \lambda^{n+m} &\sum_{\{l_i, l'_j\}_0, \{q_i, q'_j\}_0} \psi_0^*(q'_m, l'_m)\psi_0(q_n, l_n) K^n(t, \{E_{l_i}\}) \bar{K}^m(t, \{E_{l'_i}\}) \\ &\times L^n(\{q_i\}, \{l_i\}) \bar{L}^m(\{q'_j\}, \{l'_j\}) \end{aligned} \tag{2.30}$$

where $\{l_i, l'_j\}_0$ stands for the set $\{l_i, l'_j\}$ without l_0. The same applies to $\{q_i, q'_j\}_0$. Notice that $q_0 = q'_0$ and $l_0 = l'_0$. This can be seen by inspecting Eq. (2.26). Therefore $E_{l_0} = E_{l'_0}$. Since we want to calculate the average of Eq. (2.30) and the randomness is all encoded in the L^n factors, we will have to calculate $\mathbb{E}[\bar{L}^m(\{q'_j\},\{l'_i\})L^n(\{q_j\},\{l_i\})]$.
There is another useful representation for the K^n terms which we introduce now. Starting from Eq. (2.28) and the following identities

$$\delta\left(t - \sum_{j=0}^{n} s_j\right) = \int_{-\infty}^{\infty} d\alpha e^{-i\alpha(t-\sum_{j=0}^{n} s_j)} e^{\eta(t-\sum_{j=0}^{n} s_j)} \tag{2.31}$$

$$\int_0^{\infty} ds e^{-is(\omega - i\eta)} = \frac{-i}{\omega - i\eta} \tag{2.32}$$

we obtain

$$K^n(t,\{E_{l_i}\}) = i\int_{-\infty}^{\infty} d\alpha e^{-i\alpha t} e^{\eta t} \frac{-1}{E_{l_0} - \alpha - i\eta} \frac{-1}{E_{l_1} - \alpha - i\eta} \cdots \frac{-1}{E_{l_n} - \alpha - i\eta} \tag{2.33}$$

The identity in Eq. (2.31) is valid for any positive η. Later we shall set η equal to t^{-1}. A similar expression as Eq. (2.33) can be found for $\bar{K}^m(t,\{E_{l'_j}\})$. We may thus rewrite the previous equations for the propagators as

$$\bar{K}^m(t,\{E_{l'_i}\}) = -i\int_{-\infty}^{\infty} d\beta e^{i\beta t} e^{\eta t} \prod_{j=0}^{m} \frac{-1}{E_{l'_j} - \beta + i\eta} \tag{2.34}$$

$$K^n(t,\{E_{l_i}\}) = i\int_{-\infty}^{\infty} d\alpha e^{-i\alpha t} e^{\eta t} \prod_{j=0}^{n} \frac{-1}{E_{l_j} - \alpha - i\eta} \tag{2.35}$$

with, once again, $E_{l_0} = E_{l'_0}$.

2.4 Product of multiple random matrices: Graphs

The main purpose of this section will be to introduce graphs as representations of contributions to the ensemble average we want to calculate such that averaging will turn out to be a sum over different graphs. Since $\bar{L}^m(\{l'_i\},\{q'_i\})L^n(\{l_i\},\{q_i\})$ is a product of random variables we can use Wick's formula to rewrite the average as a sum of products of averages of paired matrix elements. It is through this basic formula that the notion of graph comes into play. We recall Wick's theorem here.

Theorem 2.4.1. *Say we have $2n$ random Gaussian variables denoted by X_i, $1 \leq i \leq 2n$, and say we have $Y = X_1 X_2 \ldots X_{2n}$. Denote by $\pi(2n)$ a set of pairings between all the elements of the set s, $s = (1,2\ldots,2n)$. That is $\pi(2n)$ is a list of pairs of elements of s. We then have:*

$$\mathbb{E}[Y] = \sum_{\pi(2n)} \prod_{(i,j)\in\pi(2n)} \mathbb{E}[X_i X_j] \tag{2.36}$$

where (i,j) *refers to a pair of* $\pi(2n)$. $\prod_{(i,j)\in\pi(2n)}$ *is the product of all the pairs of* $\pi(2n)$ *and* $\sum_{\pi(2n)}$ *is a sum over all possible sets of pairings. A pairing* (i,j) *will also be called a contraction and a list of pairings* $\pi(2n)$ *or contractions will also be called a graph.*

We give now a short application of the theorem. If $Y = X_1 X_2 X_3 X_4$ we have $s = \{1,2,3,4\}$. Possible configurations for $\pi(4)$ are then $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$. We have then explicitly

$$\begin{aligned}\mathbb{E}[Y] =& \mathbb{E}[X_1 X_2]\mathbb{E}[X_3 X_4]\\ &+\mathbb{E}[X_1 X_3]\mathbb{E}[X_2 X_4]\\ &+\mathbb{E}[X_1 X_4]\mathbb{E}[X_2 X_3]\end{aligned}$$

The three contributions can be represented as in figure (2.1). By taking the variables X_i to be

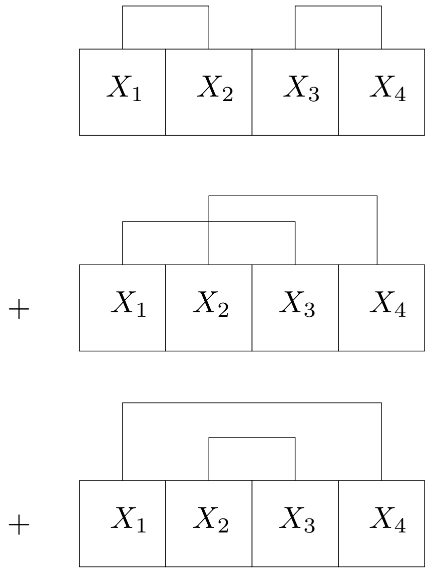

Figure 2.1: Graphical representation of the average of a product of four random variables

the successive matrix elements appearing in $\bar{L}^m L^n$, we can express the average by a sum over the pairings. For this we first introduce the following definitions:

Definition 2.4.2. *We define* $V_{q_1,q_2}(l_1,l_2) = \langle q_1, l_1 | V | q_2, l_2\rangle$ *which allows us to write the product of random variables* $\bar{L}^m(\{l_i'\},\{q_i'\})L^n(\{l_j\},\{q_i\})$ *as*

$$\begin{aligned}\bar{L}^m(\{l_i'\},\{q_i'\})L^n(\{l_j\},\{q_i\}) =& V_{q_m',q_{m-1}'}(l_m',l_{m-1}')\dots V_{q_1',q_0'}(l_1',l_0')\\ &\times V_{q_0,q_1}(l_0,l_1)\dots V_{q_{n-1},q_n}(l_{n-1},l_n) \qquad (2.37)\end{aligned}$$

with $l_0' = l_0$ *and* $q_0' = q_0$. *We associate to each* $\bar{L}^m L^n$ *a set* $\bar{s}$ *having the form*

$$\bar{s} = \{V_{q_m',q_{m-1}'}(l_m',l_{m-1}'),\dots V_{q_1',q_0'}(l_1',l_0'), V_{q_0,q_1}(l_0,l_1),\dots V_{q_{n-1},q_n}(l_{n-1},l_n)\} \qquad (2.38)$$

$$= \{X_1,\dots X_m, X_{m+1},\dots X_{n+m}\} \qquad (2.39)$$

Each element in the set $\bar{s}$ represents a random matrix element. We refer to primed random variables $V_{q'_j,q'_{j-1}}(l'_j,l'_{j-1})$ as left random variables, because they come from the expansion of the left wave function in Eq. 2.26, and to $V_{q_{j-1},q_j}(l_{j-1},l_j)$ as right random variables. We now apply Wick's formula to calculate $\mathbb{E}[(\bar{L}^m(\{l'_i\},\{q'_i\}))L^n(\{l_i\},\{q_i\})]$. $\pi(n,m)$ will now stand for a list of paired up elements of the $n+m$ elements that appear in the product $\bar{L}^m L^n$.

$$\begin{aligned}\mathbb{E}[(\bar{L}^m(\{l'_i\},\{q'_i\}))L^n(\{l_i\},\{q_i\})] &= \sum_{\pi(n,m)} \prod_{(i,j)\in\pi(n,m)} \mathbb{E}[X_i X_j] \\ &= \sum_{\pi(n,m)} C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j,\}) \end{aligned} \tag{2.40}$$

with

$$C_\pi(n,m,\{l_i,l'_i\},\{q_i,q'_j,\}) = \prod_{(i,j)\in\pi(n,m)} \mathbb{E}[X_i X_j] \tag{2.41}$$

We call $C_\pi(n,m,\{l_i,l'_i\},\{q_i,q'_j,\})$ the contraction function and $C_\pi(n,m)$ or C_π is a short hand notation for it. $C_\pi(n,m)$ is thus a function that depends on the list of paired up elements of the $n+m$ random matrix elements where m of them are left random variables and n of them are right random variables. Notice that the product of an odd number of random variables is zero. Thus $n+m$ has to be even. There are two cases for the average of $\mathbb{E}[X_i X_j]$.

$$\mathbb{E}[V_{q'_{i+1},q'_i}(l'_{i+1},l'_i)\,V_{q_j,q_{j+1}}(l_j,l_{j+1})] = \frac{1}{N\aleph}\delta_{l'_{i+1},l_{j+1}}\delta_{l'_i,l_j}\delta_{q'_i,q_j}\delta_{q'_{i+1},q_{j+1}}\left|W_{q_j,q_{j+1}}(l_j,l_{j+1})\right|^2 \tag{2.42}$$

$$\mathbb{E}[V_{q_i,q_{i+1}}(l_i,l_{i+1})\,V_{q_j,q_{j+1}}(l_j,l_{j+1})] = \frac{1}{N\aleph}\delta_{l_i,l_{j+1}}\delta_{l_{i+1},l_j}\delta_{q_i,q_{j+1}}\delta_{q_{i+1},q_j}\left|W_{q_j,q_{j+1}}(l_j,l_{j+1})\right|^2 \tag{2.43}$$

Eq. (2.42) is a pairing of a left and a right random variables. We call this an outer contraction. Eq. (2.43) is a pairing of a right and a right random variable. We call this an inner contraction. We can of course also have inner contractions on the left side. Apart from the delta functions, there are different weights to the average. These depend on $W_{q_j,q_{j+1}}(l_j,l_{j+1})$, N and $\aleph$. These last two relate to the dimension of the Hilbert space. This motivates the following separation of the contraction function:

$$C_\pi(n,m,\{l_i,l'_i\},\{q_i,q'_i\}) = \frac{wC_\pi(n,m,\{l_i,l'_i\},\{q_i,q'_i\})}{(N\aleph)^{\frac{n+m}{2}}} \times \tilde{C}_\pi(n,m,\{l_i,l'_i\},\{q_i,q'_i\}) \tag{2.44}$$

where $\tilde{C}_\pi$ encodes all of the delta relations, and so is either 0 or 1, while wC_π is the weight associated to this set of contractions. wC_π is then a product of transition amplitudes, $W_{q_i q_{i+1}}(l_i,l_{i+1})$. Since these are bounded by Eq. (2.7), we have

$$|wC_\pi(n,m)| \leq W^{\frac{n+m}{2}} \tag{2.45}$$

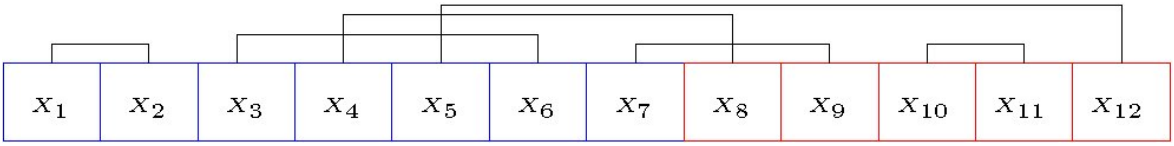

Figure 2.2: Example of a graph contributing to the order $(m,n) = (7,5)$

We build our graphs in the following way. Every X_i is represented by a box and so the product of X_i's is a series of boxes. Every paired up variables are linked to each other by a line which is always above the boxes. An example is shown in Fig. 2.2. In this specific example we have $(m,n) = (7,5)$ and it represents the following expression:

$$\mathbb{E}[X_1X_2]\mathbb{E}[X_3X_6]\mathbb{E}[X_4X_8]\mathbb{E}[X_5X_{12}]\mathbb{E}[X_7X_9]\mathbb{E}[X_{10}X_{11}]$$

It can also be represented as in Fig. 2.3 where we have just explicitly written down the dependency of the random variables. First we note that since the δ functions only relate l-variables

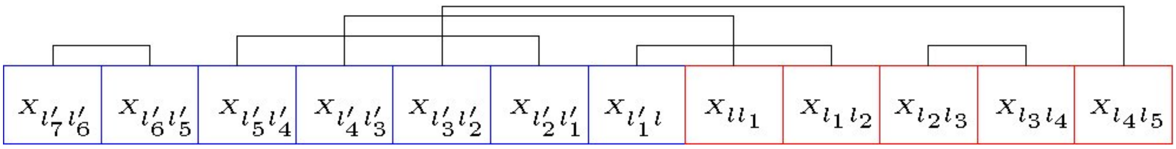

Figure 2.3: Example of a graph contributing to the order $(m,n) = (7,5)$

amongst each other and q-variables amongst each other, there will be a graph on the l-space and one on the q- space. Since one contraction gives the same identities for l- and q-variables these graphs will be the same and the conclusions for graphs on one space will be the same for graphs on the other space. Basically we mean that

$$\tilde{C}_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\}) = \tilde{C}^1_\pi(n,m,\{l_i,l'_j\}) \times \tilde{C}^2_\pi(n,m,\{q_i,q'_j\}) \qquad (2.46)$$

We now want to classify the possible graphs:
In the following definitions (2.4.3 and 2.4.4) we set $\bar{s} = \{X_1 \ldots, X_m, X_{m+1} \ldots X_{n+m}\}$, the set associated to the random variable $\bar{L}^m L^n$.

Definition 2.4.3. *Say we have the set $\bar{s}$ and a graph $\pi(n,m)$ on $\bar{s}$.*

1 *The average of a pair X_iX_j is called an inner contraction, if $i,j \le m$ or when $i,j > m$. It will be called an outer contraction if $i \le m$ and $j > m$ or when $i > m$ and $j \le m$.*

2 *The average of a pair X_iX_j is called a next neighboring (nn) contraction, if $j = i+1$.*

3 *If we have a contraction between X_i and X_j, and a contraction between X_k and X_l and $i < k < j < l$, then we call this a crossing or crossing (c) contraction. If we have a contraction between X_i and X_j and no crossing contraction involving any element X_k with $i < k < j$, the contraction is called non-crossing (nc).*

4 *If we have a contraction, between X_i and X_j, and a contraction between X_k and X_l and, $i < k < l < j \leq m$ or $m < i < k < l < j$ then we call this a nest.*

With these definitions we recognize three classes of graphs:

Definition 2.4.4. *Say we have the set $\bar{s}$ and a graph $\pi(n,m)$ on $\bar{s}$.*

C. G *We say the graph is a Crossing graph (C-graph), if it possesses at least one crossing and we call a graph a Non-Crossing graph (NC-graph) if it possesses no crossings. The set of all C-graphs of order (n,m) is denoted by $\mathcal{G}_2(n,m)$ and the set of all C-graphs is denoted by $\mathcal{G}_2$. An example of a crossing graph is the previous Fig. 2.3.*

N.G *We say the graph is a Nested graph (N-graph) if it is a NC-graph and possesses at least one nest and call a graph a Non-Nested graph (NN-graph) if it possesses none. The set of all N-graphs of order (n,m) is denoted by $\mathcal{G}_1(n,m)$ and the set of all N-graphs is denoted by $\mathcal{G}_1$. An example of a Nested graph is shown in 2.4*

S. G *We say the graph is a Simple graph (S-graph) if it is a NC and NN-graph. The set of all S-graphs of order (n,m) is denoted by $\mathcal{G}_0(n,m)$ and the set of all S-graphs is denoted by $\mathcal{G}_0$. An example of a Simple graph is shown in Fig. (2.5).*

Notice that S-graphs are built only from outer contractions and nn-contractions. The amount of graphs in these classes depend on the order (n,m) and are specified in Eqs. (B.13)-(B.16) From the definitions of Simple, Nested and Crossing graphs, we note that these classes are

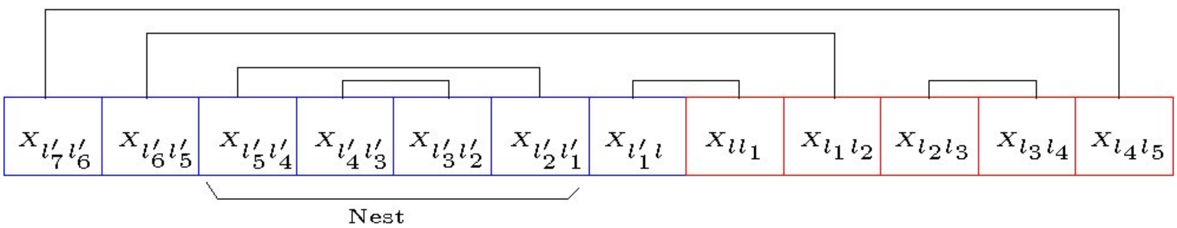

Figure 2.4: Example of a Nested Graph

mutually exclusive and cover the whole set of graphs. The following identity is then valid:

$$\sum_{\pi(n,m)} = \sum_{\pi(n,m)\in\mathcal{G}_0} + \sum_{\pi(n,m)\in\mathcal{G}_1} + \sum_{\pi(n,m)\in\mathcal{G}_2} \tag{2.47}$$

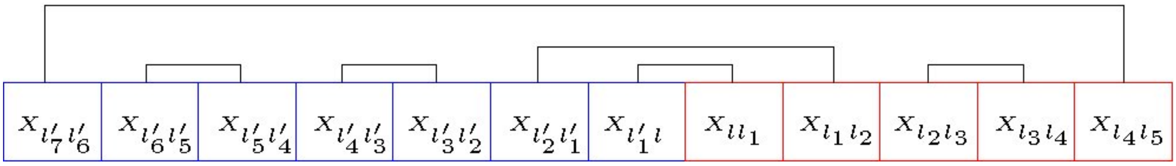

Figure 2.5: Example of a Simple Graph

Thus from Eq. (2.40) we get

$$\mathbb{E}[(\bar{L}^m(\{l'_i\},\{q'_i\}))L^n(\{l_i\},\{q_i\})] = \sum_{a=0}^{2}\sum_{\pi(n,m)\in\mathcal{G}_a} C_\pi(n,m,\{l_i,l'_i\},\{q_i,q'_i\}) \tag{2.48}$$

$$= \sum_{a=0}^{2}\sum_{\pi(n,m)\in\mathcal{G}_a} \frac{wC(n,m,\{l_i,l'_i\},\{q_i,q'_i\})}{(2N\aleph)^{\frac{n+m}{2}}}\tilde{C}^1_\pi(n,m,\{l_i,l'_i\})\tilde{C}^2_\pi(n,m,\{q_i,q'_i\})$$

Of the initial set of variables $\{l_i, l'_j\}$ a graph will impose relations between these variables and so not all of them will remain independent anymore. This motivates the following definitions:

Definition 2.4.5. *For a graph,* $\pi(n,m)$ *we define:*
$\mathcal{A}_\pi$ = *The set of independent variables of the set* $\{l_i, l'_j\}$ *given by the contraction function* $\tilde{C}_\pi$
$\mathcal{A}^q_\pi$ = *The set of independent variables of the* $\{q_i, q'_j\}$ *given by the contraction function* $\tilde{C}_\pi$
$\mathcal{B}_\pi$ = *The set of dependent variables of the set* $\{l_i, l'_j\}$ *given by the contraction function* $\tilde{C}_\pi$
$\mathcal{B}^q_\pi$ = *The set of dependent variables of the set* $\{q_i, q'_j\}$ *given by the contraction function* $\tilde{C}_\pi$
κ_π = *Number of independent variables we have of the set* $\{l_i, l'_j\}$ *given by the contraction function* $\tilde{C}_\pi$.

Note that the dependent and independent variables given by $\tilde{C}_\pi$ are the same as those given by the contraction function C_π since $\tilde{C}_\pi$ encodes the δ-relations of C_π. We will also adopt the following notations regarding dependent and independent variables. The subscript **i** on any variable, as in $\{l_a\}_\mathbf{i}$ for example, will refer to the independent variable of the set $\{l_i, l'_j\}$ to which l_a is equal to. Also if a set has the subscript, as $\{l_2, l_5, l'_1, l'_8\}_\mathbf{i}$, then it refers to the set of independent variables in the set $\{l_i, l'_j\}$ to which these relate to. For example $\{l_i, l'_j\}_\mathbf{i}$ refers to all the independent variables of the set $\{l_i, l'_j\}$ and this would be the set $\mathcal{A}_\pi$.
The number of independent variables we have for a graph will turn out to be very important in the limit $N \to \infty$. We note the following theorems which are proved in the appendix:

Theorem 2.4.6. *A graph* $\pi(n,m)$ *on the set* $\bar{s}$ *(Eq. 2.38) is a Non-Crossing graph if and only if it generates* $\frac{n+m}{2}+1$ *independent variables in the set* $I = \{l_i, l'_j\}$. *That is* $\kappa_\pi = \frac{n+m}{2}+1$ *for NC-graphs.*

This is proved by using lemma B.0.12 and B.0.11. We note that for any graph we have on the set I the following relation:

$$l'_m = l_n \tag{2.49}$$

This is proved in the appendix (B.0.14). Also since the set $I = \{l_i, l'_j\}$ is divided into independent sets, i.e. independent of the rest, we can choose any variable to represent all of them since all the others will be equal to this one. This means the following. For $P_t(q_0, l_0)$ we have that l_0 will belong to a set independent of the rest and all equal among them. Thus we can take l_0 as representing all of the variables in this set. Finally we state the following theorem:

Theorem 2.4.7. *A graph $\pi(n,m)$ on the set $\bar{s}$ (Eq. 2.38) can generate no more than $\frac{n+m}{2}+1$ independent variables in the set $I = \{l_i, l'_j\}$ and so by Theorem 2.4.6 C-graphs have less than $\frac{n+m}{2}+1$ independent variables. That is $\kappa_\pi < \frac{n+m}{2}+1$ for C-graphs.*

This is also a consequence of lemma B.0.12 and B.0.11.

2.5 Bound for crossing graphs

In the previous section we expressed the average over products of random matrices as sums over graphs, which we have classified. In this section we want to introduce this graphical notation into the expressions we want to calculate, such as Eq. (2.30). We use then the properties of these classes to provide bounds on their contribution to the time evolution of the observable. In particular, we bound here the contribution of C-graphs.

When inserting Eq. (2.48) into the average of Eq. (2.30) we obtain:

$$\mathbb{E}\left[P_t^M(q_0, l_0)\right] = \sum_{m,n=0}^{M-1} \lambda^{n+m} \sum_{\{l_i, l'_j\}_0, \{q_i, q'_j\}_0} \psi_0^*(q'_m, l'_m)\psi_0(q_n, l_n) K^n(t, \{E_{l_i}\}) \bar{K}^m(\{t, E_{l'_j}\})$$
$$\times \sum_{a=0}^{2} \sum_{\pi(n,m)\in\mathcal{G}_a} C_\pi(n, m, \{l_i, l'_j\}, \{q_i, q'_j\}) \tag{2.50}$$

Since we want to analyze the limit when $N \to \infty$ and when the energy, E_l, becomes continues we need to renormalize the wave function (see section C). We redefine then our ψ_0 as follows:

$$\psi_0(q_n, l_n) \to \sqrt{N}\psi_0(q_n, l_n) \tag{2.51}$$

That is our amplitude $\psi_0(q_n, l_n)$ in Eq. (2.50) stand now for the original ones times $\sqrt{N}$. We have then

$$\langle \hat{O} \rangle_t = \sum_{q_0}\sum_{l_0} \frac{1}{N} P_t^M(q_0, l_0) O(q_0, l_0) \tag{2.52}$$

which converges to a Riemann integral in the limit $N \to \infty$. We have not explicitly written down the dependency of $\langle \hat{O} \rangle_t$ on M. We keep it in mind and will eventually take the limit $M \to \infty$. In general we will omit these dependencies to avoid an overloaded notation. We can split the evolution into three parts coming from the contributions from the three different type of graphs

$$\mathbb{E}[P_t^M(q_0,l_0)] = P_{0,t}(q_0,l) + P_{1,t}(q_0,l_0) + P_{2,t}(q_0,l_0) \tag{2.53}$$

with

$$P_{a,t}(q_0,l_0) = \sum_{m,n=0}^{M-1} \lambda^{n+m} \sum_{\{l_i,l'_j\}_0,\{q_i,q'_j\}_0} \psi_0^*(q'_m,l'_m)\psi_0(q_n,l_n)K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_j}\})$$
$$\times \sum_{\pi(n,m)\in\mathcal{G}_a} C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\}) \tag{2.54}$$

and a equal to 0, 1 or 2. We therefore have for a diagonal observable

$$\mathbb{E}\left[\langle\hat{O}\rangle_t\right] = \sum_{a=0}^{2}\sum_{q_0}\sum_{l_0}\frac{1}{N}O(q_0,l_0)P_{a,t}(q_0,l_0) \tag{2.55}$$

and define

$$\langle\hat{O}\rangle_t^a = \sum_{q_0}\sum_{l_0}\frac{1}{N}O(q_0,l_0)P_{a,t}(q_0,l_0) \tag{2.56}$$
$$= \sum_{q_0}\int dE_0\nu^N(E_0)O(q_0,E_0)P_{a,t}(q_0,E_0)$$

such that

$$\mathbb{E}\left[\langle\hat{O}\rangle_t\right] = \sum_{a=0}^{2}\langle\hat{O}\rangle_t^a \tag{2.57}$$

We can also use the definitions of independent and dependent sets (2.4.5) to rewrite Eq. (2.54). For this we define the following:

Definition 2.5.1.

$$\mathcal{A}_{\pi 0} = \mathcal{A}_\pi / l_0 \tag{2.58}$$
$$\mathcal{A}_{\pi 0}^q = \mathcal{A}_\pi^q / q_0 \tag{2.59}$$

$\mathcal{A}_{\pi 0}$ is thus the set of independent variables $\mathcal{A}_\pi$ with the independent variable l_0 removed and likewise for $\mathcal{A}_{\pi 0}^q$. We have then

$$\{l_i,l'_j\}_0 = \mathcal{A}_{\pi 0}\bigcup\mathcal{B}_\pi$$
$$\{q_i,q'_j\}_0 = \mathcal{A}_{\pi 0}^q\bigcup\mathcal{B}_\pi^q$$

and thus from Eq. (2.54) we get

$$P_{a,t}(q_0,l_0) = \sum_{m,n=0}^{M-1} \lambda^{n+m} \sum_{\mathcal{A}_{\pi 0}} \sum_{\mathcal{A}^q_{\pi 0}} \sum_{\mathcal{B}_\pi} \sum_{\mathcal{B}^q_\pi} \psi_0^*(q'_m,l'_m)\psi_0(q_n,l_n) K^n(t,\{E_{l_i}\}) \bar{K}^m(t,\{E_{l'_j}\}) \times \sum_{\pi(n,m)\in\mathcal{G}_a} C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\}) \tag{2.60}$$

We now analyze the behavior of the contribution of C-graphs to $\mathbb{E}[P_t^M(q_0,l_0)]$ in the limit $N\to\infty$. In particular, we show that the contribution of C-graphs can be bounded by N^{-1} and so will vanish in the $N\to\infty$ limit. We first prove the following Lemma for C-graphs:

Lemma 2.5.2. *For a bounded Macro-observable and* $\mathbb{E}\left[P_{2,t}(q_0,E_0)\right]$ *given by Eq. (2.54) we have*

$$\begin{aligned}
\lim_{N\to\infty} \langle \hat{O} \rangle_t^2 &= \lim_{N\to\infty} \sum_{q_0} \sum_{l_0} O(q_0,E_{l_0}) P_{2,t}(q_0,E_{l_0}) \\
&= \lim_{N\to\infty} \sum_{q_0} \int dE_0 \nu^N(E_0) O(q_0,E_0) P_{2,t}(q_0,E_0) \\
&= 0
\end{aligned} \tag{2.61}$$

Proof. Lemma 2.5.2
Inserting Eq. (2.44) and (2.54) in Eq. (2.56) we obtain:

$$\langle \hat{O} \rangle_t^a = \sum_{m,n=0}^{M-1} \lambda^{n+m} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\{l_i,l'_j\}} \sum_{\{q_i,q'_j\}} \psi_0^*(q'_m,l'_m)\psi_0(q_n,l_n) \tag{2.62}$$

$$O(q_0,l_0) K^n(t,\{E_{l_i}\}) \bar{K}^n(t,\{E_{l'_j}\}) \frac{wC(n,m,\{l_i,l'_i\},\{q_i,q'_i\})}{N(N\aleph)^{\frac{n+m}{2}}} \tilde{C}_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\})$$

Using the definitions (2.4.5) of independent and dependent sets under a graph we have

$$\sum_{\{q_i,q'_j\}} \sum_{\{l_i,l'_j\}} = \sum_{\mathcal{A}^q_\pi} \sum_{\mathcal{B}^q_\pi} \sum_{\mathcal{A}_\pi} \sum_{\mathcal{B}_\pi} \tag{2.63}$$

We introduce the following notation:

$$G_\pi^{n,m}(t,\{E_{l_i},E_{l'_j}\}_{\mathbf{i}},\{q_i,q'_j\}_{\mathbf{i}}) = \sum_{\mathcal{B}_\pi} \sum_{\mathcal{B}^q_\pi} K^n(t,\{E_{l_i}\}) \bar{K}^m(t,\{E_{l'_j}\}) \tilde{C}_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\}) \tag{2.64}$$

This means $G_\pi^{n,m}(t,\{E_{l_i},E_{l'_j}\}_{\mathbf{i}},\{q_i,q'_j\}_{\mathbf{i}})$ is just the function $K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_j}\})$, where the restrictions given by the contraction function $C_\pi(n,m)$ have been implemented. The subscript $\mathbf{i}$ in the expression $\{E_{l_i},E_{l'_j}\}_{\mathbf{i}}$ means that out of the sets $\{l_i,l'_j\}$ and $\{q'_i,q_j\}$ it only depends on the independent variables of the sets. We can thus write Eq. (2.62) as follows:

$$
\begin{aligned}
\langle\hat{O}\rangle_t^a =& \sum_{n,m=0}^{M-1}\lambda^{n+m}\sum_{\pi(n,m)\in\mathcal{G}_a}\sum_{\mathcal{A}_\pi^q}\sum_{\mathcal{A}_\pi}\psi_0^*(\{q_m'\}_\mathbf{i},\{l_m'\}_\mathbf{i})\psi_0(\{q_n\}_\mathbf{i},\{l_n\}_\mathbf{i})O(\{q_0\}_\mathbf{i},\{l_0\}_\mathbf{i})\\
&\times\frac{wC_\pi(n,m,\{l_i,l_j'\}_\mathbf{i},\{q_i,q_j'\}_\mathbf{i})}{N^{\frac{n+m}{2}+1}(2\aleph)^{\frac{n+m}{2}}}G_\pi^{n,m}(t,\{E_{l_i},E_{l_j'}\}_\mathbf{i},\{q_i,q_j'\}_\mathbf{i})\\
=&\sum_{n,m=0}^{M-1}\lambda^{n+m}\sum_{\pi(n,m)\in\mathcal{G}_a}\frac{1}{N^{\frac{n+m}{2}+1-\kappa_{C\pi}}}\sum_{\mathcal{A}_{C\pi}^q}\sum_{\mathcal{A}_{C\pi}}\frac{1}{N^{\kappa_{C\pi}}}\\
&\times\psi_0^*(\{q_m'\}_\mathbf{i},\{l_m'\}_\mathbf{i})\psi_0(\{q_n\}_\mathbf{i},\{l_n\}_\mathbf{i})O(\{q_0\}_\mathbf{i},\{l_0\}_\mathbf{i})\\
&\times\frac{wC_\pi(n,m,\{l_i,l_j'\}_\mathbf{i},\{q_i,q_j'\}_\mathbf{i})}{(\aleph)^{\frac{n+m}{2}}}G_\pi^{n,m}(t,\{E_{l_i},E_{l_j'}\}_\mathbf{i},\{q_i,q_j'\}_\mathbf{i})
\end{aligned}
\tag{2.65}
$$

The sum over the independent variables of $\mathcal{A}_\pi$ is now weighted by N^{-1} to the power of the number of independent variables, κ_π, and so in the limit $N\to\infty$ turns into an integral over the independent variables:

$$
\sum_{\{l_i,l_j'\}_\mathbf{i}}\frac{1}{N^{\kappa_\pi}}=\sum_{\mathcal{A}_\pi}\frac{1}{N^{\kappa_\pi}}\to\prod_{j=1}^{\kappa_\pi}\int[d\omega_j]
\tag{2.66}
$$

where ω_j represents an independent variable. By inspecting Eq. (2.28) and Eq. (2.64) we see that $\left|G_\pi^{n,m}\right|$ is bounded by some function of t which we will denote by $G(t)$. By Eq. (2.45) wC_π is also bounded as are the observable and initial wave function. Since the sum over one q_j is a finite sum, the sum over the independent variables $\mathcal{A}_\pi^q$ is bounded by a constant to the power of the number of independent variables for the graph. Thus we have:

$$
\begin{aligned}
&\Big|\lambda^{n+m}\sum_{\mathcal{A}_\pi^q}\sum_{\mathcal{A}_\pi}\frac{1}{N^{\kappa_\pi}}\psi_0^*(\{q_m'\}_\mathbf{i},\{l_m'\}_\mathbf{i})\psi_0(\{q_n\}_\mathbf{i},\{l_n\}_\mathbf{i})O(\{q_0\}_\mathbf{i},\{l_0\}_\mathbf{i})\\
&\times\frac{wC_\pi(n,m,\{l_i,l_j'\}_\mathbf{i},\{q_i,q_j'\}_\mathbf{i})}{\aleph^{\frac{n+m}{2}}}G_\pi^{n,m}(t,\{E_{l_i},E_{l_j'}\}_\mathbf{i},\{q_i,q_j'\}_\mathbf{i})\Big|\\
\le&\lambda^{n+m}\sum_{\mathcal{A}_\pi^q}\sum_{\mathcal{A}_\pi}\frac{1}{N^{\kappa_\pi}\aleph^{\frac{n+m}{2}}}CG(t)W^{\frac{n+m}{2}}\\
\le&C(t,\lambda)\sum_{\mathcal{A}_\pi^q}\frac{1}{\aleph^{\frac{n+m}{2}}}W^{\frac{n+m}{2}}
\end{aligned}
\tag{2.67}
$$

By theorems 2.4.6 and 2.4.7 we have that if $\pi(n,m)\in\mathcal{G}_{0,1}(n,m)$ there are $\frac{n+m}{2}+1$ independent variables and therefore $\kappa_\pi=\frac{n+m}{2}+1$. If $\pi(n,m)\in\mathcal{G}_2(n,m)$ we have $\kappa_\pi<\frac{n+m}{2}+1$. Thus for NC-graphs the factor $\left(\frac{1}{N}\right)^{\frac{n+m}{2}+1-\kappa_\pi}=1$ in Eq. (2.65). For Crossing Graphs $\left(\frac{1}{N}\right)^{\frac{n+m}{2}+1-\kappa_\pi}\le\frac{1}{N}$. For the same reason we have that $\mathcal{A}_\pi^q$ contains less than

$\frac{n+m}{2}+1$ independent variables if $\pi(n,m)\in\mathcal{G}_2$ and so $\sum_{\mathcal{A}^q_\pi}\frac{1}{\aleph^{\frac{n+m}{2}}}\leq 1$. For NC-graphs we have $\sum_{\mathcal{A}^q_\pi}\frac{1}{\aleph^{\frac{n+m}{2}}}\leq\aleph$. Thus by using Eq. (2.67) in Eq. (2.65) we get

$$\begin{aligned}
&\left|\langle\hat{O}\rangle_t^2\right|\leq\frac{1}{N}\sum_{n,m=0}^{M-1}\sum_{\pi(n,m)\in\mathcal{G}_2}C(t,\lambda)\,W^{\frac{n+m}{2}}\\
&\lim_{N\to\infty}\langle\hat{O}\rangle_t^2=0
\end{aligned}\tag{2.68}$$

□

For NC-graphs we get from Eq. (2.65) and the fact that $\kappa_\pi=\frac{n+m}{2}+1$

$$\begin{aligned}
\langle\hat{O}\rangle_t^a=&\sum_{q_0}\sum_{l_0}\frac{1}{N}O(q_0,l_0)P_{a,t}(q_0,E_{l_0})\\
=&\sum_{q_0}\sum_{l_0}\frac{1}{N}O(q_0,l_0)\sum_{m,n=0}^{M-1}\lambda^{n+m}\sum_{\pi(n,m)\in\mathcal{G}_a}\sum_{\mathcal{A}^q_{\pi 0}}\sum_{\mathcal{A}_{\pi 0}}\frac{1}{N^{\frac{n+m}{2}}}\psi_0^*(\{q'_m\}_{\mathbf{i}},\{E_{l'_m}\}_{\mathbf{i}})\\
&\times\psi_0(\{q_n\}_{\mathbf{i}},\{E_{l_n}\}_{\mathbf{i}})\frac{wC_\pi(n,m,\{E_{l_i},E_{l'_j}\}_{\mathbf{i}},\{q_i,q'_j\}_{\mathbf{i}})}{(\aleph)^{\frac{n+m}{2}}}\\
&\times G_\pi^{n,m}(t,\{E_{l_i},E_{l'_j}\}_{\mathbf{i}},\{q_i,q'_j\}_{\mathbf{i}})
\end{aligned}$$

Because of Eq.(2.49) for NC-graphs we have $\{E_{l_n}\}_{\mathbf{i}}=\{E_{l'_m}\}_{\mathbf{i}}$ and so

$$\psi_0^*(\{q'_m\}_{\mathbf{i}},\{E_{l'_m}\}_{\mathbf{i}})\psi_0(\{q_n\}_{\mathbf{i}},\{E_{l_n}\}_{\mathbf{i}})=P_0\left(\{q_n\}_{\mathbf{i}},\{E_{l_n}\}_{\mathbf{i}}\right)$$

The sum over the independent variables of $\mathcal{A}_\pi$ is now weighted by N^{-1} to the power of the number of independent variables, κ_π, and so in the limit $N\to\infty$ turns into an integral over the independent variables:

$$\sum_{\{l_i,l'_j\}_{\mathbf{i}}}\frac{1}{N^{\kappa_\pi}}=\sum_{\mathcal{A}_\pi}\frac{1}{N^{\kappa_\pi}}\to\prod_{E_{l_i},E_{l'_j}\in\mathcal{A}_\pi}\int[dE_{l_i}]\int[dE_{l'_j}]\tag{2.69}$$

We get then

$$\begin{aligned}
\lim_{N\to\infty}\langle\hat{O}\rangle_t^a=&\sum_{q_0}\int[dE_{l_0}]\,O(q_0,E_{l_0})\sum_{m,n}\lambda^{n+m}\sum_{\pi(n,m)\in\mathcal{G}_a}\sum_{\mathcal{A}^q_{\pi 0}}\prod_{E_{l_i},E_{l'_j}\in\mathcal{A}_{\pi 0}}\int[dE_{l_i}]\int[dE_{l'_j}]\\
&\times P_0(\{q_n\}_{\mathbf{i}},\{E_{l_n}\}_{\mathbf{i}})\frac{wC_\pi(n,m,\{E_{l_i},E_{l'_j}\}_{\mathbf{i}},\{q_i,q'_j\}_{\mathbf{i}})}{(\aleph)^{\frac{n+m}{2}}}\\
&\times G_\pi^{n,m}(t,\{E_{l_i},E_{l'_j}\}_{\mathbf{i}},\{q_i,q'_j\}_{\mathbf{i}})
\end{aligned}\tag{2.70}$$

$$=\sum_{q_0}\int dE_{l_0}\,O(q_0,E_{l_0})P_{a,t}(q_0,E_{l_0})\tag{2.71}$$

with

$$P_{a,t}(q_0,E_{l_0}) = \sum_{m,n} \lambda^{n+m} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\mathcal{A}^q_{\pi 0}} \prod_{E_{l_i},E_{l'_j}\in\mathcal{A}_{\pi 0}} \int [dE_{l_i}] \int [dE_{l'_j}] P_0(\{q_n\}_\mathbf{i},\{E_{l_n}\}_\mathbf{i}) \times \frac{wC_\pi(n,m,\{E_{l_i},E_{l'_j}\}_\mathbf{i},\{q_i,q'_j\}_\mathbf{i})}{(\aleph)^{\frac{n+m}{2}}} G^{n,m}_\pi(t,\{E_{l_i},E_{l'_j}\}_\mathbf{i},\{q_i,q'_j\}_\mathbf{i}) \tag{2.72}$$

$P_{a,t}(q_0,E_{l_0})$ is now determined by the initial condition $P_0(\{q_n\}_\mathbf{i},\{E_{l_n}\}_\mathbf{i})$. This means that the contribution for these terms are determined only by the initial data of the probability distribution on states given by the wavefunction and not the initial data of the wave function. Phases are thus irrelevant because two states with different phases, but same initial probability distribution have the same evolution according to Eq. (2.72).
We introduce the following definition:

$$Q_\pi(n,m,\{q_i,q'_j\}_\mathbf{i},E_{l_0},t) = \prod_{E_{l_i},E_{l'_j}\in\mathcal{A}_{\pi 0}} \int [dE_{l_i}] \left[dE_{l'_j}\right] P_0(\{q_n\}_\mathbf{i},\{E_{l_n}\}_\mathbf{i}) \times \frac{wC_\pi(n,m,\{E_{l_i},E_{l'_j}\}_\mathbf{i},\{q_i,q'_j\}_\mathbf{i})}{(\aleph)^{\frac{n+m}{2}}} G^{n,m}_\pi(t,\{E_{l_i},E_{l'_j}\}_\mathbf{i},\{q_i,q'_j\}_\mathbf{i}) \tag{2.73}$$

We substitute $G^{n,m}_\pi$ to have the more explicit form

$$Q_\pi(n,m,\{q_i,q'_j\}_\mathbf{i},E_{l_0},t) = \prod_{E_{l_i},E_{l'_j}\in\mathcal{A}_{\pi 0}} \int [dE_{l_i}] \left[dE_{l'_j}\right] P_0(\{q_n\}_\mathbf{i},\{E_{l_n}\}_\mathbf{i}) \times \sum_{\mathcal{B}_\pi} \sum_{\mathcal{B}^q_\pi} \frac{wC_\pi(n,m,\{q_i,q'_j\},\{E_{l_i},E_{l'_j}\})}{\aleph^{\frac{n+m}{2}}} \tilde{C}_\pi(n,m,\{q_i,q'_j\},\{E_{l_i},E_{l'_j}\}) \times \int_{-\infty}^{\infty} d\alpha d\beta e^{-it(\alpha-\beta)} e^{2\eta t} \prod_{j=0}^{n} \frac{-1}{E_{l_j}-\alpha-i\eta} \prod_{j=0}^{m} \frac{-1}{E_{l'_j}-\beta+i\eta} \tag{2.74}$$

We shall omit sometimes the $\{q_i,q'_j\}_\mathbf{i}$ dependency for brevity of notation. With this we express Eq. (2.72) as:

$$\boxed{P_{a,t}(q_0,E_{l_0}) = \sum_{n,m=0}^{M-1} \lambda^{n+m} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\mathcal{A}^q_{\pi 0}} Q_\pi(n,m,\{q_i,q'_j\}_\mathbf{i},E_{l_0},t)} \tag{2.75}$$

2.6 Bounds for Nested Graphs

We now move on to bounding the contribution coming from Nested graphs. For N-graphs we have the following theorem:

Theorem 2.6.1. *If* $\pi(n,m) \in \mathcal{G}_1(n,m)$ *and*

$$P_{t=0}(q_0, E_0) < C \tag{2.76}$$

we get the following bound:

$$\begin{aligned} I(t) &= \int [dE_0] \, |Q_\pi(n,m,E_0,t)| \\ &\le C \log^{n'+3}(t) t^{\frac{n+m}{2}-1} \end{aligned} \tag{2.77}$$

with C a constant.

Notice that in Eq. 2.75 the Q_π function is weighed by a λ^{n+m} factor and so if theorem 2.6.1 holds then $\lambda^{n+m} I(t)$ would vanish in the Van Hove limit.
In order to prove this we will introduce a new notation which will allow us to exploit the property of nests. In Eq. (2.73) we have the set $\{E_{l_i}, E_{l'_j}\}_\mathbf{i}$ which labels the independent energies variables of the left and right side of the expansion of the evolution of the wave function. These come from the independent variables of the set $\{l_i, l'_j\}$ which in turn come from the graphs.
Depending on the graph structure the variables in $\{E_{l_i}, E_{l'_j}\}_\mathbf{i}$ can appear in many free propagators, i.e. in the denominators of Eq. (2.74). We call this their multiplicity and will relabel these variables according to their multiplicity.
Thus we relabel the set of independent variables $\{E_{l_i}, E_{l'_j}\}_\mathbf{i}$ by $\{\omega_i, \omega'_j\}$ where ω_i denotes the variables having multiplicity higher than 1 and ω'_j those with multiplicity equal to 1.
We will denote by p_j the multiplicity of ω_j on the left hand side and by k_j the multiplicity on the right hand side. This means that for a variable ω_j with left and right multiplicity p_j and k_j respectively we will have a product of free propagators, $\left(\frac{-1}{\omega_j-\alpha-i\eta}\right)^{k_j}\left(\frac{-1}{\omega_j-\beta+i\eta}\right)^{p_j}$ appearing in Eq.(2.74). For variables having multiplicity 1 we will have a term $\left(\frac{-1}{\omega'_j-\alpha-i\eta}\right)$, if it comes from the right and $\left(\frac{-1}{\omega'_j-\beta+i\eta}\right)$, if it comes from the left. We will denote by n' the number of propagators of multiplicity 1. That is n' is equal to the number of ω'_j's. We also designate $\bar{n}+1$ as the number of ω_j and let the index j runs from 0 to $\bar{n}$. Note that since N-graphs are non crossing the number of independent variables $n'+\bar{n}+1$ is equal to $\frac{n+m}{2}+1$ according to theorem 2.4.6. A nest means that we have a contraction between either two left random variables or two right random variables without having a contraction from any variable in between with an outside one. This means that the energy variables in between this contraction have no multiplicity on right or left side. Therefore an variable with multiplicity higher then one that comes from in between a nest will have either k_j or p_j equal to 0. Since for every ω'_j there is just one propagator and ω_i has multiplicity p_i+k_i the total number of propagators, $\sum_{j=0}^{\bar{n}}(p_j+k_j)+n'$, is equal to $n+m+2$. In our new notation we set $E_{l_0}=\omega_0$ and $\{E_{l_n}\}_\mathbf{i}=\omega_{\bar{n}}$. It can be seen that both of these variables always have multiplicity higher then 1.

Depending on whether propagators of multiplicity one will appear on the right or left hand side, they will dependend on α or β.
In short, a specific contraction function $C_\pi(n,m)$ contains the following information: It tells us the value of n' which gives rise to propagators of multiplicity 1. It also tells us whether these propagators are on the left or on the right. It contains the information of the value of $\bar{n}$ and the multiplicity on the left and on the right, p_j and k_j, respectively. We can thus rewrite the information encoded in $C_\pi(n,m)$ as a function of these variables

$$C_\pi(n,m) = C_\pi(n',\{\gamma_i\},\bar{n},\{p_j,k_j\}) \tag{2.78}$$

with the following relations fulfilled:

$$n' + \bar{n} + 1 = \frac{n+m}{2} + 1 \tag{2.79}$$

$$\sum_{j=0}^{\bar{n}} (k_j + p_j - 1) = \frac{n+m}{2} + 1 \tag{2.80}$$

Here $1 \leq i \leq n'$, $0 \leq j \leq \bar{n}$ and the set $\{\gamma_i\}$ is a sequence of α and β. Each α or β stands for a right or left propagator of multiplicity 1. In Eq. (2.74) there is still the weight of a graph to be considered. Since we are taking $W_{q_i,q_j'}(l_i,l_j')$ to tend to a continues and bounded function as $N \to \infty$ in the energy variables, $wC_\pi(n,m,\{q_j,q_j'\},\{l_j,l_j'\})$ will also be continues and bounded and as we are summing in Eq. (2.74) over the dependent variables we define $wC_\pi(n,m,\{q_j,q_j'\}_{\mathbf{i}},\{E_{l_j},E_{l_j'}\}_{\mathbf{i}})$, with the identities imposed and in the new notation:

$$\begin{aligned}
&wC_\pi(n,m,\{q_j,q_j'\}_{\mathbf{i}},\{E_{l_j},E_{l_j'}\}_{\mathbf{i}}) \\
&= \lim_{N\to\infty} \sum_{\mathcal{B}_\pi,\mathcal{B}_\pi^q} wC_\pi(n,m,\{q_j,q_j'\},\{l_j,l_j'\})\,\tilde{C}_\pi(n,m,\{q_j,q_j'\},\{l_j,l_j'\}) \\
&= \mathcal{W}\left(\{\omega_i,\omega_i'\},\{q_i,q_j'\}_{\mathbf{i}}\right)
\end{aligned} \tag{2.81}$$

In this notation we can write $Q_\pi(n,m,\omega_0,t)$ for a specific graph realization as

$$\begin{aligned}
Q_\pi(n',\bar{n},\{k_j,p_j\},\omega_0,t) = &\left(\prod_{j=1}^{\bar{n}} \int [d\omega_j]\right) P_0(q_{\bar{n}},\omega_{\bar{n}}) \int\int d\alpha d\beta e^{-i(\alpha-\beta)t} e^{\eta t} \\
&\times \prod_{l=1}^{n'} \int [d\omega_l'] \frac{-1}{\omega_l' - \gamma_l + i\eta} \\
&\times \prod_{j=0}^{\bar{n}} \left(\frac{-1}{\omega_j - \beta + i\eta}\right)^{p_j} \left(\frac{-1}{\omega_j - \alpha - i\eta}\right)^{k_j} \frac{\mathcal{W}\left(\{\omega_i,\omega_i'\},\{q_i,q_j'\}_{\mathbf{i}}\right)}{\aleph^{\frac{n+m}{2}}}
\end{aligned} \tag{2.82}$$

Proof. Theorem 2.6.1
Remember that η was introduced as an identity and we should have $0 < \eta$. From now on

we set $\eta = t^{-1}$. Since the graph considered is a Nested one there is an integration over a ω_i variable that does not possess any multiplicity on the right or left side. That is either p_j or k_j is equal to 0. We take $p_j = 0$. By use of Eq. (2.82) and (2.76) we have

$$\begin{aligned} I(t) \leq & \frac{C}{\aleph^{\frac{n+m}{2}}} \left(\prod_{j=0, j\neq i}^{\bar{n}} \int [d\omega_j] \right) \int\int d\alpha d\beta e^{\eta t} \prod_{l=1}^{n'} \int [d\omega'_l] \left| \frac{-1}{\omega'_l - \gamma_l + i\eta} \right| \\ & \times \prod_{j=0, j\neq i}^{\bar{n}} \left(\left| \frac{-1}{\omega_j - \beta + i\eta} \right|^{p_j} \left| \frac{-1}{\omega_j - \alpha - i\eta} \right|^{k_j} \right) \\ & \times \left| \int [d\omega_i] \left(\frac{-1}{\omega_i - \alpha - i\eta} \right)^{k_i} \mathcal{W}\left(\{\omega_j, \omega'_j\}, \{q_i, q'_j\}_{\mathbf{i}}\right) \right| \end{aligned} \tag{2.83}$$

The last integral can be done by parts.

$$\begin{aligned} & \left| \int [d\omega_i] \left(\frac{-1}{\omega_i - \alpha - i\eta} \right)^{k_i} \mathcal{W}\left(\{\omega_j\}, \{q_i, q'_j\}_{\mathbf{i}}\right) \right| \\ \leq & \left| g(\omega_i) \mathcal{W}\left(\{\omega_j\}, \{q_i, q'_j\}_{\mathbf{i}}\right) \left(\frac{1}{\omega_i - \alpha - i\eta} \right)^{k_i - 1} \Big|_0^1 \right| \\ + & \left| \int d\omega_i \frac{\partial}{\partial \omega_i} \left(g(\omega_i) \mathcal{W}\left(\{\omega_j\}, \{q_i, q'_j\}_{\mathbf{i}}\right) \right) \left(\frac{-1}{\omega_i - \alpha - i\eta} \right)^{k_i - 1} \right| \\ \leq & C_1 \left| \left(\frac{1}{1 - \alpha - i\eta} \right)^{k_i - 1} \right| + C_1 \left| \left(\frac{1}{\alpha + i\eta} \right)^{k_i - 1} \right| + C_2 \int d\omega_i \left| \frac{-1}{\omega_i - \alpha - i\eta} \right|^{k_i - 1} \end{aligned} \tag{2.84}$$

And so we have for $I(t)$:

$$\begin{aligned} I(t) \leq & \frac{C}{\aleph^{\frac{n+m}{2}}} \left(\prod_{j=0, j\neq i}^{\bar{n}} \int [d\omega_j] \right) \int\int d\alpha d\beta e^{\eta t} \prod_{l=1}^{n'} \int [d\omega'_l] \left| \frac{-1}{\omega'_l - \gamma_l + i\eta} \right| \\ & \times \prod_{j=0, j\neq i}^{\bar{n}} \left(\left| \frac{-1}{\omega_j - \beta + i\eta} \right|^{p_j} \left| \frac{-1}{\omega_j - \alpha - i\eta} \right|^{k_j} \right) \\ & \times \left(C_1 \left| \left(\frac{1}{1 - \alpha - i\eta} \right)^{k_i - 1} \right| + C_1 \left| \left(\frac{1}{\alpha + i\eta} \right)^{k_i - 1} \right| + C_2 \int d\omega_i \left| \frac{-1}{\omega_i - \alpha - i\eta} \right|^{k_i - 1} \right) \end{aligned}$$

By using Eq. (A.4) of the appendix we can derive the following bound

$$I(t) \leq \frac{C}{\aleph^{\frac{n+m}{2}}} \log^{n'}(t) \left(\prod_{j=1, j\neq i}^{\bar{n}-1} t^{p_j+k_j-1} \right) t^{p_0+k_0-2} t^{p_{\bar{n}}+k_{\bar{n}}-2} \int d\alpha d\beta \, [d\omega_0] \, [d\omega_{\bar{n}}] \, \frac{1}{k_i - 1}$$
$$\left(\left| \frac{1}{1-\alpha - i\eta} \right|^{k_i-1} + \left| \frac{1}{\alpha + i\eta} \right|^{k_i-1} + \left| \int d\omega_i \left(\frac{-1}{\omega_i - \alpha - i\eta} \right)^{k_i-1} \right| \right)$$
$$\left| \frac{1}{\omega_0 - \beta + i\eta} \right| \left| \frac{1}{\omega_0 - \alpha - i\eta} \right| \left| \frac{1}{\omega_{\bar{n}} - \beta + i\eta} \right| \left| \frac{1}{\omega_{\bar{n}} - \alpha - i\eta} \right| \tag{2.85}$$

From Eq.(A.9) of the appendix we have

$$\int d\alpha d\beta \, [d\omega_0] \, [d\omega_{\bar{n}}] \left| \frac{1}{x - \alpha - i\eta} \right|^{k_i-1} \left| \frac{1}{\omega_0 - \beta + i\eta} \right| \left| \frac{1}{\omega_0 - \alpha - i\eta} \right|$$
$$\times \left| \frac{1}{\omega_{\bar{n}} - \beta + i\eta} \right| \left| \frac{1}{\omega_{\bar{n}} - \alpha - i\eta} \right| \leq C t^{k_i-1} \log^3(t) \tag{2.86}$$

with x equal to 0, 1 or ω_i. Using Eq. (2.86) in Eq. (2.85) we get

$$\begin{aligned} I(t) \leq & \frac{\tilde{C}}{\aleph^{\frac{n+m}{2}}} \log^{n'+3}(t) \left(\prod_{j=1}^{\bar{n}-1} t^{p_j+k_j-1} \right) t^{p_0+k_0-2} t^{p_{\bar{n}}+k_{\bar{n}}-2} \\ \leq & \frac{\tilde{C}}{\aleph^{\frac{n+m}{2}}} \log^{n'+3}(t) \left(\prod_{j=0}^{\bar{n}} t^{p_j+k_j-1} \right) t^{-2} \\ \leq & \frac{\tilde{C} \log^{n'+3}(t) t^{\frac{m+n}{2}-1}}{\aleph^{\frac{n+m}{2}}} \end{aligned} \tag{2.87}$$

To get the last line we used Eq. (2.80). □

Using Eq. (2.87) in Eq.(2.75) we have

$$\begin{aligned} & \left| \int [d\omega_0] \, O(q_0, \omega_0) P_{1,t}(q_0, \omega_0) \right| \leq O_{max} \int [d\omega_0] \, |P_{1,t}(q_0, \omega_0)| \\ & \leq O_{max} \sum_{n,m} \lambda^{n+m} \sum_{\pi(n,m) \in \mathcal{G}_1} \sum_{\{q_i, q_j'\}_{\mathbf{i}}} \int [d\omega_0] \left| Q_\pi(n, m, \{q_i, q_j'\}_{\mathbf{i}}, \omega_0, t) \right| \\ & \leq O_{max} \sum_{n,m} \lambda^{n+m} \sum_{\pi(n,m) \in \mathcal{G}_1} \sum_{\{q_i, q_j'\}_{\mathbf{i}}} I(t) \\ & \leq C \aleph \sum_{n,m} \sum_{\pi(n,m) \in \mathcal{G}_1} \lambda^{\frac{n+m}{2}} \log^{n'+3}(t) t^{\frac{m+n}{2}-1} \end{aligned} \tag{2.88}$$

In the Van Hove limit, with $\lambda^2 = Tt^{-1}$ we find

$$\lim_{t\to\infty}\left|\int [d\omega_0]\, O(q_0,\omega_0)P_{1,t}(q_0,\omega_0)\right| \leq \lim_{t\to\infty} C\,\aleph\, t^{-1}\sum_{n,m=0}^{M-1}\sum_{\pi(n,m)\in\mathcal{G}_1}\frac{\log^{n'+3}(t)}{T}(T)^{\frac{m+n}{2}}$$
$$\leq 0 \tag{2.89}$$

Thus in the Van Hove limit only S-graphs will contribute.

2.7 Simple graphs

While introducing the new notation in section 2.6 we have never used the fact that the graphs were nested. This is why we can also use that notation for expressing S-graphs. For S-graphs there are extra relations. For N-graphs we made a distinction between variable of multiplicity 1 and higher. The variables of multiplicity 1 could be associated with a propagator with the α or the β, which we wrote down as a sequence, $\{\gamma_l\}$, that we did not specify. For S-graphs there is an easy way to label this, which we explain now. For S-graphs we can never have k_j or p_j equal to zero, for this would be a nest. In addition, in between two outer contractions there can only be nn-contractions because anything else would be a nest. Thus for S-graphs we have in between two outer contractions, say in between the j^{th} and the $(j+1)^{\text{th}}$, one propagator with multiplicity higher then one and a certain number of propagators of multiplicity one on the left and on the right. This is illustrated in figure 2.6. We can thus label the variable of the propagator with multiplicity higher then one and, between the j^{th} and the $(j+1)^{\text{th}}$ outer contraction, by ω_j. We anticipated this in section 2.6 and already used this notation. Every nn-contraction (definition 2.4.3) in between the j^{th} and the $(j+1)^{\text{th}}$ outer contraction will increase the multiplicity of the propagator that has multiplicity higher then 1 by one. We thus see that if the multiplicity of this propagator on the left is k_j and on the right p_j then there will be k_j-1 propagators of multiplicity 1 on the left and p_j-1 on the right (see figure 2.6). $\bar{n}+1$ is the number of propagators of multiplicity higher then 1 and they are generated by $\bar{n}$ over contractions in the case of S-graphs. Since n' is equal to the number of propagators of multiplicity 1 we find

$$n' = \sum_{j=0}^{\bar{n}}(p_j+k_j-2) \tag{2.90}$$

Since the total number of propagators in $Q_\pi(n,m,\omega_0,t)$ is $n+m+2$, which is equal to $\sum_{j=0}^{\bar{n}}(k_j+p_j)+n'$, Eq. (2.90) implies

$$\frac{n+m}{2}+1 = \sum_{j=0}^{\bar{n}}(k_j+p_j-1) \tag{2.91}$$

Because in between two outer contraction there can only be nn-contraction we have that for a given set of numbers $\bar{n}$, k_j and p_j there exists only one simple graph. $\bar{n}$ gives one the number of outer contractions and k_j and p_j tells us how many nn-contractions there are in between the j^{th} and $(j+1)^{\text{th}}$ outer contraction.

We relabel here the ω'_l, that is the variables with multiplicity 1, by ω^j_l for the right hand side and $\bar{\omega}^j_l$ for the left hand side. The superindex j now refers to the fact that the propagator with this variable lies in between the j^{th} and the $(j+1)^{\text{th}}$ outer contraction. The same notation is used for the $\{q_j, q'_i\}_\mathbf{i}$ variables. The variables $\bar{n}$ and $\{k_j, p_j\}$ completely specify the simple graph $\pi(n,m)$. With this we can rewrite the sum over the q-variables of Eq. (2.82) as

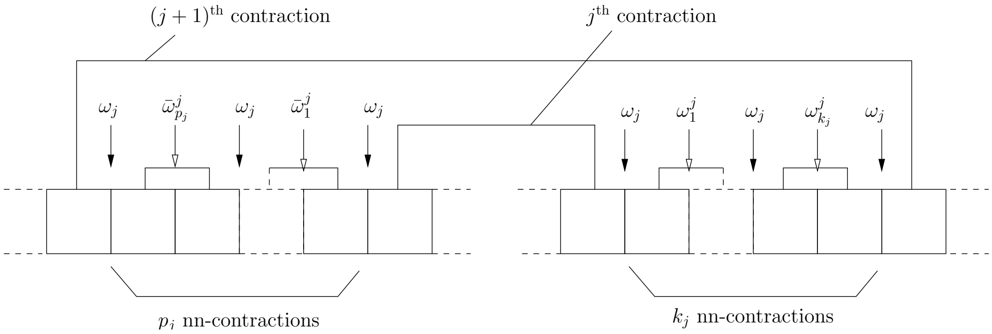

Figure 2.6: Sketch of the graph structure in between two over contractions

$$\sum_{\{q_j, q^j_l, \bar{q}^j_l\}} Q_\pi(\bar{n}, \{k_j, p_j\}, \{q_j, q^j_l, q'^j_l\}, \omega_0, t) =$$

$$\sum_{\{q_j, q^j_l, \bar{q}^j_l\}} \left(\prod_{j=1}^{\bar{n}} \int [d\omega_j]\right) P_0(q_{\bar{n}}, \omega_{\bar{n}}) \int\int d\alpha d\beta e^{-i(\alpha-\beta)t} e^{\eta t}$$

$$\times \prod_{j=0}^{\bar{n}} \left(\left| W_{q_j, q_{j+1}}(\omega_j, \omega_{j+1}) \right|^2 \left(\frac{-1}{\omega_j - \beta + i\eta}\right)^{p_j} \left(\frac{-1}{\omega_j - \alpha - i\eta}\right)^{k_j} \right.$$

$$\times \left(\prod_{l=1}^{k_j-1} \int \left[d\omega^j_l\right] \frac{-\left|W_{q_j, q^j_l}(\omega_j, \omega^j_l)\right|^2}{\omega^j_l - \alpha - i\eta}\right) \left(\prod_{l=1}^{p_j-1} \int \left[d\bar{\omega}^j_l\right] \frac{-\left|W_{q_j, \bar{q}^j_l}(\omega_j, \bar{\omega}^j_l)\right|^2}{\bar{\omega}^j_l - \beta + i\eta}\right) \Bigg) \tag{2.92}$$

were we set $W_{q_{\bar{n}}, q_{\bar{n}+1}}(\omega_{\bar{n}}, \omega_{\bar{n}+1}) = 1$. Notice that the sums over the q^j_l and $\bar{q}^j_l$ variables are summing over W_{q_j, q^j_l} and $W_{q_j, \bar{q}^j_l}$, and can be moved inside of the integral.

We make now a shift of the variables k_j and p_j.

$$p_j \to p_j + 1$$
$$k_j \to k_j + 1$$

We had that neither k_j nor p_j could be equal to zero for S-graphs and so if we were to sum up over different possible k_j's and p_j's we would have to start form 1. With this shift we would start from 0. It is for briefness of notation that we make this shift. This shift implies the following changes in the relationship between the variables n, m and the variables $\bar{n}$, k_j and p_j:

$$\sum_{j=0}^{\bar{n}} (k_j + p_j + 1) = \frac{n+m}{2} + 1 \tag{2.93}$$

And, of course in Eq. (2.92) we have to replace every k_j by $k_j + 1$ and p_j by $p_j + 1$.
By inspecting Eq. (2.92) we see that this quantity will be large whenever the denominators tend to be as small as possible. This will occur when the energies ω_j , ω_l^j and $\bar{\omega}_l^j$ are close to α or β. Our intentions now will be to show that we can replace the integrations over propagators of multiplicity 1, $\frac{-1}{\omega_i^j - \alpha - i\eta}$, by integrations over $\frac{-1}{\omega_i^j - \omega_{\bar{n}} - i\eta}$ by making an error which goes to 0 in the Van Hove limit. This is a kind of resonance condition because it means that Eq. (2.92) will be large when the energies ω_j , ω_l^j and $\bar{\omega}_l^j$ are close to the initial energy $\omega_{\bar{n}}$. We define

$$\Theta(q_j, \omega_j, \alpha, \eta) = \int [d\omega] \frac{-\sum_q \left| W_{q_j,q}(\omega_j, \omega) \right|^2}{\omega - \alpha - i\eta} \tag{2.94}$$

$$\Theta(q_j, \omega_j, \omega_{\bar{n}}) = \lim_{\eta' \to 0} \Theta(q_j, \omega_j, \alpha, \eta') \tag{2.95}$$

$\Theta(q_j, \omega_j, \alpha, \eta)$ is the function appearing multiple times in the last line of Eq. (2.92). Since $W_{q_j,q}(\omega_j, \omega)$ is bounded we have:

$$|\Theta(q_j, \omega_j, \omega_{\bar{n}})| \leq C \tag{2.96}$$

We will show that the difference in the evolution when using $\Theta(q_j, \omega_j, \alpha, \eta)$ or $\Theta(q_j, \omega_j, \omega_{\bar{n}})$ vanishes in the Van Hove limit. In order to analyze the difference in evolution we look at the difference between $\tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t)$ and $Q_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t)$,
where $\tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t)$ is $Q_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t)$ with the $\Theta(q_j, \omega_j, \alpha, \eta)$ are replaced by $\Theta(q_j, \omega_j, \omega_{\bar{n}})$.

$$\begin{aligned} &\Delta Q_\pi(\bar{n}, \{k_j, p_j\}, q_0, \omega_0, t) = \\ &\sum_{\{q_j, q_l^j, \bar{q}_l^j\}} Q_\pi(\bar{n}, \{k_j, p_j\}, \{q_j, q_l^j, \bar{q}_l^j\}, \omega_0, t) - \tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \{q_j, q_l^j, \bar{q}_l^j\}, \omega_0, t) \end{aligned} \tag{2.97}$$

$$\begin{aligned} \Delta Q_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t) &= \sum_{\{q_j\}} \int\int d\alpha d\beta e^{-i(\alpha-\beta)t} e^{\eta t} \prod_{j=0}^{\bar{n}} \int [d\omega_j] \, P_0(q_{\bar{n}}, \omega_{\bar{n}}) \\ &\times \left| W_{q_j,q_{j+1}}(\omega_j, \omega_{j+1}) \right|^2 \left(\frac{1}{\omega_j - \alpha - i\eta} \right)^{k_j+1} \left(\frac{1}{\omega_j - \beta + i\eta} \right)^{p_j+1} \\ &\times \left(\left(\Theta^{k_j}(q_j, \omega_j, \alpha, \eta) \right) \left(\bar{\Theta}^{p_j}(q_j, \omega_j, \beta, \eta) \right) - \left(\Theta^{k_j}(q_j, \omega_j, \omega_{\bar{n}}) \right) \left(\bar{\Theta}^{p_j}(q_j, \omega_j, \omega_{\bar{n}}) \right) \right) \end{aligned} \tag{2.98}$$

We restrict the initial probability distribution to be zero around the borders of the spectrum.

$$P_0(q_{\bar{n}}, \omega_{\bar{n}}) = \chi_\theta(\omega_{\bar{n}}) P_0(q_{\bar{n}}, \omega_{\bar{n}}) \tag{2.99}$$

where $\chi_\theta(\omega)$ is the characteristic function on $[\theta, 1-\theta]$. Thus the range of integration on $\omega_{\bar{n}}$ is between $[\theta, 1-\theta]$. We will now prove the following bound that will allow us to replace $\Theta(q_j, \omega_j, \alpha, \eta)$ by $\Theta(q_j, \omega_j, \omega_{\bar{n}})$.

Theorem 2.7.1.

$$\begin{aligned} \Delta I(\bar{n}, \{k_j, p_j\}, q_0, t) &= \int [d\omega_0] \left| \Delta Q_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t) \right| \\ &\leq C \left(\log(t)\right)^{\frac{n+m}{2}+5} t^{\frac{n+m}{2}-1} \end{aligned} \tag{2.100}$$

Notice that, similar to the case of N-graphs, if theorem 2.7.1 is true then $\lambda^{n+m}\Delta I$ goes to zero in the Van Hove limit and so in this part would vanish in Eq. (2.75).

Proof. Theorem 2.7.1
We have

$$\begin{aligned} |\Delta I| \leq \sum_{\{q_j\}} \int d\alpha d\beta \prod_{j=0}^{\bar{n}} \int [d\omega_j] \left| W_{q_j, q_{j+1}}(\omega_j, \omega_{j+1}) \right|^2 \left| \frac{1}{\omega_j - \alpha - i\eta} \right|^{k_j+1} \left| \frac{1}{\omega_j - \beta + i\eta} \right|^{p_j+1} \\ \times \left| \left(\Theta^{k_j}(q_j, \omega_j, \alpha, \eta) \right) \left(\bar{\Theta}^{p_j}(q_j, \omega_j, \beta, \eta) \right) - \left(\Theta^{k_j}(q_j, \omega_j, \omega_{\bar{n}}) \right) \left(\bar{\Theta}^{p_j}(q_j, \omega_j, \omega_{\bar{n}}) \right) \right| \end{aligned} \tag{2.101}$$

where the integration over $\omega_{\bar{n}}$ is between $[\theta, 1-\theta]$. Notice first that if α or β were not in the range of $[-1, 2]$ then our expression could be bounded by integrals over powers of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. Such integrals can then be bounded by a constant since no poles are present and so in this region ΔI can be bounded by a constant. Thus we need only to consider integration over α and β belonging to a bounded range.
In order to bound $|\Delta I|$ we will first bound
$\left| \Theta^{k_j}(q_j, \omega_j, \alpha, \eta) \bar{\Theta}^{p_j}(q_j, \omega_j, \beta, \eta) - \Theta^{k_j}(q_j, \omega_j, \omega_{\bar{n}}) \bar{\Theta}^{p_j}(q_j, \omega_j, \omega_{\bar{n}}) \right|$.
We have

$$\begin{aligned} &|\Theta(q_j, \omega_j, \alpha, \eta) - \Theta(q_j, \omega_j, \omega_{\bar{n}})| \\ &= \left| \int d\omega g(\omega) \frac{-\sum_q \left| W_{q_j, q}(\omega_j, \omega) \right|^2}{\omega - \alpha - i\eta} - \lim_{\eta' \to 0} \int d\omega g(\omega) \frac{-\sum_q \left| W_{q_j, q}(\omega_j, \omega) \right|^2}{\omega - \omega_{\bar{n}} - i\eta'} \right| \\ &= \lim_{z' \to \omega_{\bar{n}}} \left| \int d\omega G(\omega) \left(\frac{1}{\omega - z} - \frac{1}{\omega - z'} \right) \right| \end{aligned}$$

with

$$\begin{aligned} G(\omega) &= g(\omega) \left| W_{q_j, q}(\omega_j, \omega) \right|^2 \\ z &= \alpha + i\eta \\ z' &= \omega_{\bar{n}} + i\eta' \end{aligned}$$

By integrating by parts we get:

$$
\begin{aligned}
&|\Theta(q_j,\omega_j,\alpha,\eta)-\Theta(q_j,\omega_j,\omega_{\bar n})| \\
&\le \lim_{z'\to\omega_{\bar n}} \left(\left| G(\omega)\left(\log(\omega-z)-\log(\omega-z)\right)\Big|_0^1 \right| + \left| \int d\omega G'(\omega)\left(\log(\omega-z)-\log(\omega-z)\right)\right| \right) \\
&\le \lim_{z'\to\omega_{\bar n}} \left(\left| G(\omega)\left(\log(\omega-z)-\log(\omega-z)\right)\Big|_0^1 \right| + \int_z^{z'} d\,|\chi| \int d\omega \left|\frac{G'(\omega)}{\omega-\chi}\right| \right) \\
&= B_1
\end{aligned} \tag{2.102}
$$

Using the following inequalities

$$
\left|\log(z)-\log(z')\right| \le C\left|z-z'\right|\left(\frac{1}{|z|}+\frac{1}{|z'|}\right) \tag{2.103}
$$

$$
\int_z^{z'} d\,|\chi| \int d\omega \left|\frac{G'(\omega)}{\omega-\chi}\right| \le C\left|z-z'\right|\log(|\eta|) \tag{2.104}
$$

we have

$$
\lim_{\eta'\to 0} B_1 \le C\,|\omega_{\bar n}-\alpha-i\eta|\left(\frac{1}{|1-\alpha-i\eta|}+\frac{1}{|1-\omega_{\bar n}|}+\frac{1}{|\alpha+i\eta|}+\frac{1}{|\omega_{\bar n}|}+\log|\eta|\right) \tag{2.105}
$$

Inequality (2.104) is proved at the end of section A. Using the following decomposition

$$
\begin{aligned}
&\Theta^{k_j}(q_j,\omega_j,\alpha,\eta)\bar\Theta^{p_j}(q_j,\omega_j,\beta,\eta)-\Theta^{k_j}(q_j,\omega_j,\omega_{\bar n})\bar\Theta^{p_j}(q_j,\omega_j,\omega_{\bar n}) = \\
&\bar\Theta^{p_j}(q_j,\omega_j,\beta,\eta)\sum_{l=0}^{k_j-1}\Theta^{k_j-l-1}(q_j,\omega_j,\alpha,\eta)\Theta^{l}(q_j,\omega_j,\omega_{\bar n})\left(\Theta(q_j,\omega_j,\alpha,\eta)-\Theta(q_j,\omega_j,\omega_{\bar n})\right)+ \\
&\Theta^{k_j}(q_j,\omega_j,\omega_{\bar n})\sum_{l=0}^{p_j-1}\bar\Theta^{p_j-l-1}(q_j,\omega_j,\beta,\eta)\bar\Theta^{l}(q_j,\omega_j,\omega_{\bar n})\left(\bar\Theta(q_j,\omega_j,\beta,\eta)-\bar\Theta(q_j,\omega_j,\omega_{\bar n})\right)
\end{aligned} \tag{2.106}
$$

we can bound (2.106) as follows

$$
\left|\Theta^{k_j}(q_j,\omega_j,\alpha,\eta)\bar\Theta^{p_j}(q_j,\omega_j,\beta,\eta)-\Theta^{k_j}(q_j,\omega_j,\omega_{\bar n})\bar\Theta^{p_j}(q_j,\omega_j,\omega_{\bar n})\right| \le A+B \tag{2.107}
$$

$$
\begin{aligned}
A &= \left|\bar\Theta^{p_j}(q_j,\omega_j,\beta,\eta)\right|\sum_{l=0}^{k_j-1}\left|\Theta^{k_j-l-1}(q_j,\omega_j,\alpha,\eta)\Theta^{l}(q_j,\omega_j,\omega_{\bar n})\right|\left|\Theta(q_j,\omega_j,\alpha,\eta)-\Theta(q_j,\omega_j,\omega_{\bar n})\right| \\
&= \sum_{l=0}^{k_j-1} A(l) \\
B &= \left|\Theta^{k_j}(q_j,\omega_j,\omega_{\bar n})\right|\sum_{l=0}^{p_j-1}\left|\bar\Theta^{p_j-l-1}(q_j,\omega_j,\beta,\eta)\bar\Theta^{l}(q_j,\omega_j,\omega_{\bar n})\right|\left|\bar\Theta(q_j,\omega_j,\beta,\eta)-\bar\Theta(q_j,\omega_j,\omega_{\bar n})\right| \\
&= \sum_{l=0}^{p_j-1} B(l)
\end{aligned}
$$

We omitted here all the variable dependencies on A, B, $A(l)$ and $B(l)$ to simplify the notation. The important point here is to see that each factor in the sum is proportional to $(\Theta(q_j,\omega_j,\alpha,\eta)-\Theta(q_j,\omega_j,\omega_{\bar{n}}))$ or $\left(\bar{\Theta}(q_j,\omega_j,\beta,\eta)-\bar{\Theta}(q_j,\omega_j,\omega_{\bar{n}})\right)$. Using Eq. (A.2) and (2.96) to bound $A(l)$ we get

$$A(l) \leq C^l \left|\log(\eta)\right|^{p_j+k_j-l-1} \left|\Theta(q_j,\omega_j,\alpha,\eta)-\Theta(q_j,\omega_j,\omega_{\bar{n}})\right| \tag{2.108}$$

and thus

$$A \leq C \left|\log(\eta)\right|^{p_j+k_j-1} \left|\Theta(q_j,\omega_j,\alpha,\eta)-\Theta(q_j,\omega_j,\omega_{\bar{n}})\right| \tag{2.109}$$

Similarly for B we get

$$B \leq C \left|\log(\eta)\right|^{p_j+k_j-1} \left|\bar{\Theta}(q_j,\omega_j,\beta,\eta)-\bar{\Theta}(q_j,\omega_j,\omega_{\bar{n}})\right| \tag{2.110}$$

When inserting the bound of Eq. (2.107) in Eq. (2.101) we get two parts, one depending on A and the other on B. We shall bound the first part as it is inserted in ΔI and analyze its limit, the second being analogous to the first.
We define

$$\left|\Delta\tilde{I}_A(\bar{n},\{k_j,p_j\},t,l)\right| = \sum_{\{q_j\}} \int d\alpha d\beta \prod_{j=0}^{\bar{n}} \int [d\omega_j] \left|\frac{1}{\omega_j-\alpha-i\eta}\right|^{k_j+1} \left|\frac{1}{\omega_j-\beta+i\eta}\right|^{p_j+1} A(l) \tag{2.111}$$

and

$$\left|\Delta\tilde{I}_B(\bar{n},\{k_j,p_j\},t,l)\right| = \sum_{\{q_j\}} \int d\alpha d\beta \prod_{j=0}^{\bar{n}} \int [d\omega_j] \left|\frac{1}{\omega_j-\alpha-i\eta}\right|^{k_j+1} \left|\frac{1}{\omega_j-\beta+i\eta}\right|^{p_j+1} B(l) \tag{2.112}$$

such that we have

$$\Delta I \leq C \left(\sum_{l=0}^{k_j-1} \left|\Delta\tilde{I}_A(\bar{n},\{k_j,p_j\},t,l)\right| + \sum_{l=0}^{p_j-1} \left|\Delta\tilde{I}_B(\bar{n},\{k_j,p_j\},t,l)\right| \right) \tag{2.113}$$

We bound now Eq. (2.111). By inserting the bound from Eq. (2.109) in Eq. (2.113) and using Eq. (A.4) from the appendix to bound the integrations over all ω_j's except $\omega_{\bar{n}}$ and ω_1 we get for the first sum

$$\sum_{l=0}^{k_j-1} \left|\Delta\tilde{I}_A(\bar{n},\{k_j,p_j\},t,l)\right| \leq C \left(\prod_{j=0}^{\bar{n}} \left(\log^{k_j+p_j-1}(\eta^{-1})\right) \left(\frac{1}{\eta}\right)^{k_j+p_j+1} \right) \eta^2 \int d\alpha d\beta \tag{2.114}$$

$$\times \int [d\omega_{\bar{n}}] [d\omega_1] \left|\frac{1}{\omega_{\bar{n}}-\alpha-i\eta}\right| \left|\frac{1}{\omega_{\bar{n}}-\beta+i\eta}\right| \left|\frac{1}{\omega_1-\alpha-i\eta}\right| \left|\frac{1}{\omega_1-\beta+i\eta}\right|$$

$$\times \left|\Theta(\omega_j,\alpha,\eta)-\Theta(\omega_j,\omega_{\bar{n}})\right|$$

We move to bounding the integrated factor. Inserting the bound from Eq. (2.102) into the integrated factor we get:

$$\begin{aligned}
I_1 &= \int d\alpha d\beta \int [d\omega_{\bar{n}}][d\omega_1] \left|\frac{1}{\omega_{\bar{n}}-\alpha-i\eta}\right| \left|\frac{1}{\omega_{\bar{n}}-\beta+i\eta}\right| \left|\frac{1}{\omega_1-\alpha-i\eta}\right| \left|\frac{1}{\omega_1-\beta+i\eta}\right| \\
&\quad \times |\Theta(\omega_j,\alpha,\eta)-\Theta(\omega_j,\omega_{\bar{n}})| \\
&\leq \int d\alpha d\beta \int [d\omega_{\bar{n}}][d\omega_1] \left|\frac{1}{\omega_{\bar{n}}-\beta+i\eta}\right| \left|\frac{1}{\omega_1-\alpha-i\eta}\right| \left|\frac{1}{\omega_1-\beta+i\eta}\right| \\
&\quad \times \left|\frac{1}{|1-\alpha-i\eta|}+\frac{1}{|1-\omega_{\bar{n}}|}+\frac{1}{|\alpha+i\eta|}+\frac{1}{|\omega_{\bar{n}}|}\right|
\end{aligned} \tag{2.115}$$

Notice that $\left|\frac{1}{\omega_{\bar{n}}-\alpha-i\eta}\right|$ was canceled out by the bound on $|\Theta(\omega_j,\alpha,\eta)-\Theta(\omega_j,\omega_{\bar{n}})|$.
Because $\omega_{\bar{n}}$ is in $[\theta, 1-\theta]$ we have $\frac{1}{|\omega_{\bar{n}}|}$ and $\frac{1}{|1-\omega_{\bar{n}}|}$ bounded. From bounds (A.11) in the appendix we obtain

$$I_1 \leq \log^4(\eta^{-1})$$

We then have for $\Delta\tilde{I}_A$

$$\sum_{l=0}^{k_j-1} \left|\Delta\tilde{I}_A(\bar{n},k_j,p_j,t,l)\right| \leq C \log^{\sum_{j=0}^{\bar{n}}(k_j+p_j-1)+4}(\eta^{-1}) \left(\frac{1}{\eta}\right)^{\sum_{j=0}^{\bar{n}}(k_j+p_j+1)-2} \tag{2.116}$$

If we apply the same strategy to bound the $\Delta\tilde{I}_B$ factor and use Eq. (2.93), we get:

$$\Delta I \leq C \log^{\frac{n+m}{2}+5-2(\bar{n}+1)}(t)\, t^{\frac{n+m}{2}-1} \tag{2.117}$$

□

This result, as we will show now, allows to say that in the Van Hove limit the difference will tend to zero, once again because the power of t is not strong enough.
Since we changed variables from n,m to $\{p_j,k_j\}$ and $\bar{n}$ we have to express the sum Eq. (2.75) in terms of the new variables introduced. We have then

$$\begin{aligned}
(2.75) = &\sum_{\bar{n},k_j,p_j=0}^{2\sum_{j=0}^{\bar{n}}(k_j+p_j+1)\leq 2M} \left(\lambda^2\right)^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)} \\
&\times \sum_{\pi(\bar{n},\{k_j,p_j\})\in\mathcal{G}_0} \sum_{q_j,q_l^j,q_l'^j} Q_\pi(\bar{n},\{k_j,p_j\},\{q_j,q_j^l,q_j'^l\},\omega_0,t)
\end{aligned} \tag{2.118}$$

The upper limit of the sum is a complicated condition which comes from the fact that we are summing n and m from 0 until $M-1$. That is, the upper limit of the sum over k_j and p_j depends on $\bar{n}$ and vice versa. Since each S-graph $\pi(\bar{n},\{k_j,p_j\})$ is uniquely specified by $\bar{n}$, k_j

and p_j the sum, $\sum_{\pi(\bar{n},\{k_j,p_j\})\in\mathcal{G}_0}$, in Eq. (2.118) is actually just a sum over one element and so can be taken out. We set

$$\tilde{P}_{0,t}(q_0,\omega_0) = \sum_{\bar{n},k_j,p_j=0}^{2\sum_{j=0}^{\bar{n}}(k_j+p_j+1)\le 2M} (\lambda^2)^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)} \times \sum_{q_j,q_l^j,q_l'^j} \tilde{Q}_\pi(\bar{n},\{k_j,p_j\},\{q_j,q_l^j,q_l'^j\},\omega_0,t) \tag{2.119}$$

such that the difference between the time evolution of the probability distribution , $P_{0,t}(q_0,\omega_0)$ and $\tilde{P}_{0,t}(q_0,\omega_0)$, evaluated on an observable can be bounded as follows:

$$\begin{aligned} &\left|\int d\omega_0\, O(q_0,\omega_0)\left(P_{0,t}(q_0,\omega_0)-\tilde{P}_{0,t}(q_0,\omega_0)\right)\right| \\ &=\left|\int d\omega_0\, O(q_0,\omega_0)\sum_{\bar{n},k_j,p_j}(\lambda^2)^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)}\sum_{\pi(\bar{n},\{k_j,p_j\})\in\mathcal{G}_0}\Delta Q_\pi(\bar{n},\{k_j,p_j\},q_0,\omega_0,t)\right| \\ &\le O_{max}\sum_{\bar{n},k_j,p_j=0}^{2\sum_{j=0}^{\bar{n}}(k_j+p_j+1)\le 2M}(\lambda^2)^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)}\Delta I(\bar{n},\{k_j,p_j\},q_0,t) \end{aligned} \tag{2.120}$$

Inserting the bound of Eq. (2.116) in Eq. (2.120) we get in the Van Hove limit ($\lambda^2 = Tt^{-1}$):

$$\begin{aligned} &\lim_{t\to\infty}\left|\int d\omega_0\, O(q_0,\omega_0)\left(P_{0,t}(q_0,\omega_0)-\tilde{P}_{0,t}(q_0,\omega_0)\right)\right| \\ &\le C\lim_{t\to\infty}\sum_{\bar{n},k_j,p_j=0}^{2\sum_{j=0}^{\bar{n}}(k_j+p_j+1)\le 2M}(\lambda^2)^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)}\, t^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)-1}\log^{\sum_{j=0}^{\bar{n}}(k_j+p_j-1)+4}(t) \\ &\le C\lim_{t\to\infty}t^{-1}\sum_{\bar{n},k_j,p_j=0}^{2\sum_{j=0}^{\bar{n}}(k_j+p_j+1)\le 2M} T^{\bar{n}+\sum_{j=0}^{\bar{n}}(k_j+p_j)-1}\log^{\sum_{j=0}^{\bar{n}}(k_j+p_j-1)+4}(t) \end{aligned}$$

Thus

$$\lim_{t\to\infty}\left|\int d\omega_0\, O(q_0,\omega_0)\left(P_{0,t}(q_0,\omega_0)-\tilde{P}_{0,t}(q_0,\omega_0)\right)\right| \le 0 \tag{2.121}$$

This means that in the Van Hove limit $\tilde{P}_{0,t}(q_0,\omega_0)$ captures properly the evolution of the probability density.

2.8 Comments on the interpretation

Before proceeding to derive the rate equations in the respective limits some comments and remarks should be in order about the non contributing parts of the evolution. We refer here to an analogy with a free quantum particle. We can write down the evolution of the probability to find a quantum particle at point y, when it starts out in position x as

$$P(y,t) = \sum_{\text{path a,path b}} A^*_{\text{path a}(y|x)}(t) A_{\text{path b}(y|x)}(t) \tag{2.122}$$

where $A_{\text{path a}(y|x)}(t)$ is the probability amplitude for a path that starts from the point x and goes to the point y. Figure 2.7 shows two different paths the quantum particle may take with the respective amplitudes, $A_1(t)$ and $A_2(t)$. When calculating the probability to be at point y, these two paths can interfere.

If we sum up over the paths that are nearly the same there would practically be no interference.

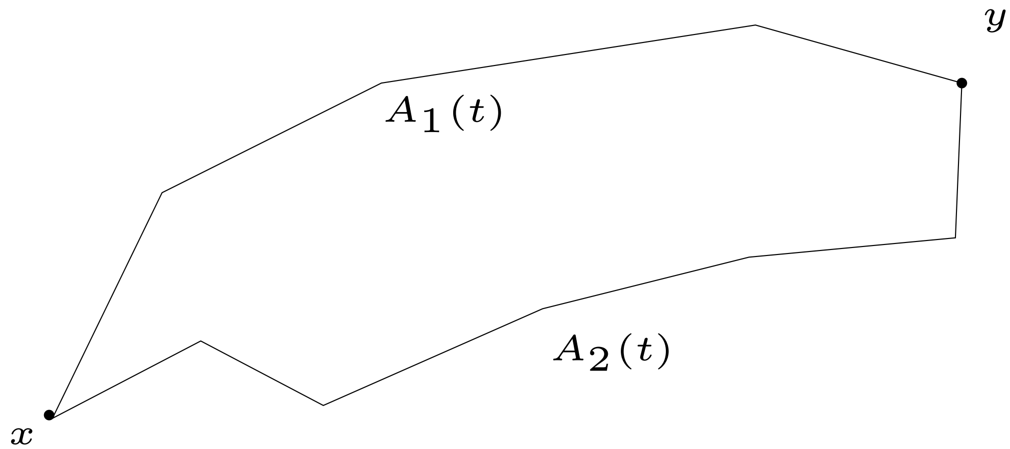

Figure 2.7: Two different paths the quatum particle can take

The sum over the product of amplitudes of paths becomes the probability of that path being taken. This is sketched in figure 2.8. Thus if the paths were identical and only those were contributing, our particle would behave stochastically, and thus classical, it would look very similar except that instead of having sums over amplitudes of paths we would have sums over probabilities of paths:

$$P(y,t) = \sum_{\text{path a}} P_{\text{path a}(y|x)}(t) \tag{2.123}$$

If all quantum effects were negligible we could pass from Eq. (2.122) to Eq.(2.123).

In our case we have paths in the Hilbert space. This is not so different then for the case of the free particle. If we represent as dots our Hilbert state vectors and as lines the transitions from one vector to another, we would have the following representation for a pair of paths coming from the expansion of the evolution operators (see figure 2.9). The weight of such a pair of trajectories in Hilbert space turns out to be zero, because the transition amplitudes do not correlate at all. An example of a Crossing path is depicted in figure 2.10. We only show

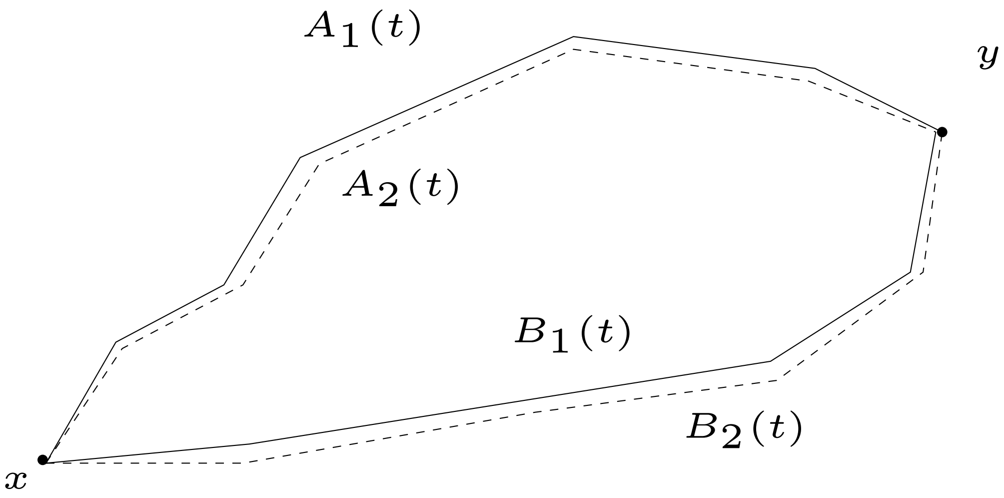

Figure 2.8: Sketch of the contribution if only similar paths are taken

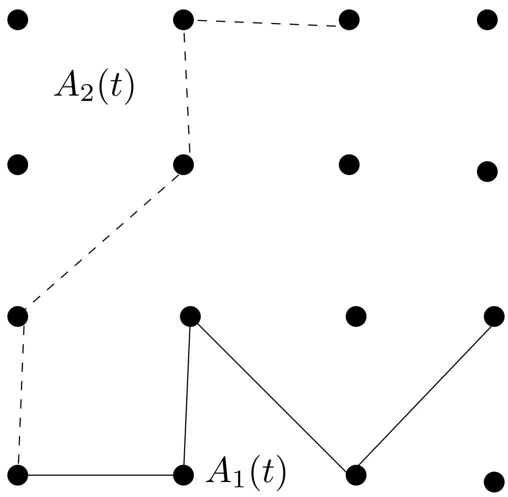

Figure 2.9: Representation of pairs of paths in the Hilbert space

three dots for simplicity. The time variables are ordered here and symbolize the time it takes to make the jump from one state to the other. The arrows indicate the time direction and an oriented arrow can only correlate with another one that is identical, including the direction. For a crossing graph the arrow of path t_2 correlates with the one of t_3' and the one of t_3 correlates with the one of t_2'. In some sense this means that if t_2 and t_3' denote the present of the two paths, then the future of the first path t_3 correlates with the past,t_2' , of the second part. This is very odd and quantum like.

An example of a Nested path is shown in figure 2.11. The two paths follow each other but they are allowed to have a completely different evolution at some point, as long as they return

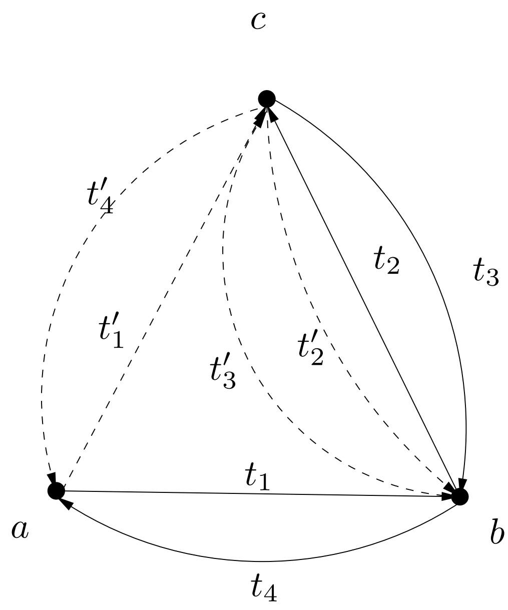

Figure 2.10: Example of a Crossing path

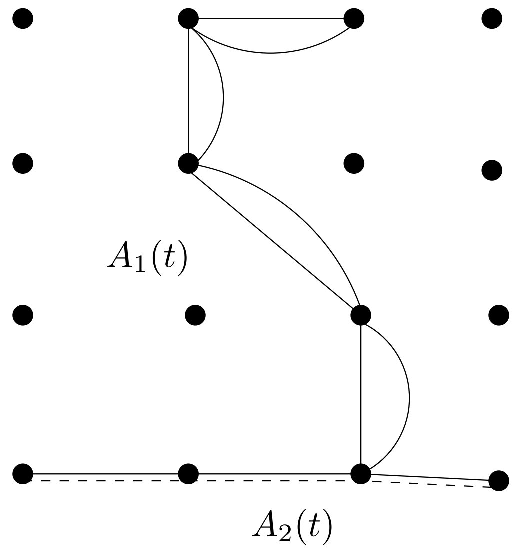

Figure 2.11: Example of a Nested path

to the point, where they started to differ and then continue their way together.

For a Simple path we depict an example in figure 2.12. Here once again the paths can differ

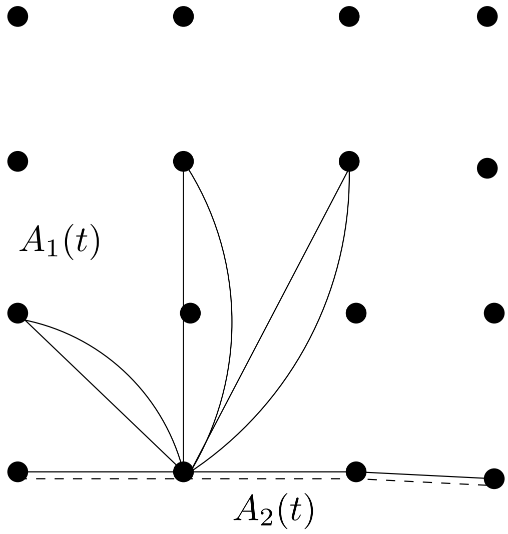

Figure 2.12: Simple paths

but have to return after each jump. The more classical like contributions are, of course, the simple paths.

2.9 The rate equations

As we have seen in section 2.7, we can use $\tilde{P}_{0,t}(q_0, \omega_0)$ to calculate the contribution from S-graphs to the time evolution of the probability distribution in the Van Hove limit. Since we already showed that C-graphs and N-graphs do not contribute in the $N \to \infty$ limit and the Van Hove limit, $\tilde{P}_{0,t}(q_0, \omega_0)$ describes the evolution of the probability distribution in these limits. We now proceed to showing that the average of the probability distribution $P_t(q_0, \omega_0)$ is a solution of the rate equations in the limits considered. We will resum up the expression we have for $\tilde{P}_{0,t}(q_0, \omega_0)$ over the S-graphs. Our Duhamel expansion is truncated at the $M-1$ term and in the limit $M \to \infty$ the rest would go to zero (see chapter 6). Thus we calculate $\tilde{P}_{0,t}(q_0, \omega_0)$ for $M \to \infty$. This means that instead of the complicated condition we had in the sum of Eq. (2.118) we have to sum $\bar{n}$, k_j and p_j up to ∞. We have for $\tilde{Q}_\pi$ the following expression:

$$\sum_{q_j \neq q_0, q_l^j, q_l'^j} \tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \{q_j, q_l^j, q_l'^j\}, \omega_0, t) = \sum_{\{q_j\}_0} \left(\prod_{j=1}^{\bar{n}} \int [d\omega_j] \right) P_0(q_{\bar{n}}, \omega_{\bar{n}})$$
$$\times \int d\alpha d\beta e^{-i(\alpha-\beta)t} e^{\eta t} \prod_{j=0}^{\bar{n}} \left| W_{q_j, q_{j+1}}(\omega_j, \omega_{j+1}) \right|^2 \left(\frac{-1}{\omega_j - \beta + i\eta} \right)^{p_j+1} \left(\frac{-1}{\omega_j - \alpha - i\eta} \right)^{k_j+1}$$
$$\times \Theta^{k_j}(q_j, \omega_j, \omega_{\bar{n}}) \bar{\Theta}^{p_j}(q_j, \omega_j, \omega_{\bar{n}}) \tag{2.124}$$

For the probability density we have:

$$\begin{aligned} \mathbb{E}[P_T(q_0, \omega_0)] &= \lim_{M\to\infty} \lim_{t\to\infty}^{\lambda^2 t = T} \mathbb{E}\left[P_t^M(q_0, \omega_0) \right] \\ &= \lim_{M\to\infty} \lim_{t\to\infty}^{\lambda^2 t = T} \tilde{P}_{0,t}(q_0, \omega_0) \\ &= \lim_{t\to\infty}^{\lambda^2 t = T} \sum_{\bar{n}=0}^{\infty} \lambda^{2\bar{n}} \sum_{k_i, p_i = 0}^{\infty} (\lambda^2)^{\sum_{j=0}^{\bar{n}}(k_j + p_j)} \sum_{q_j \neq q_0, q_l^j, q_l'^j} \tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t) \end{aligned} \tag{2.125}$$

We absorb the factor $(\lambda^2)^{k_j+p_j}$ by redefining our Θ function as

$$\Theta(q_j, \omega_j, \omega_{\bar{n}}) \to \lambda^2 \Theta(q_j, \omega_j, \omega_{\bar{n}}) \tag{2.126}$$

By inspecting Eq.(2.124) this entails

$$(\lambda^2)^{k_j+p_j} \tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t) \to \tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t)$$

We then find

$$\mathbb{E}[P_T(q_0, \omega_0)] = \lim_{t\to\infty}^{\lambda^2 t = T} \sum_{\bar{n}=0}^{\infty} \lambda^{2\bar{n}} \sum_{k_j, p_j = 0}^{\infty} \sum_{q_j \neq q_0, q_l^j, q_l'^j} \tilde{Q}_\pi(\bar{n}, \{k_j, p_j\}, \omega_0, t) \tag{2.127}$$

In section 2.3 we have replaced the delta function over the time integrals $\delta\left(t - \sum_j s_j\right)$ by the integrations over the α and β variables. We now apply to some extent the opposite procedure, starting from Eq. (2.124). Inserting the following equality in Eq. (2.124)

$$\left(\frac{-1}{\omega_j - \alpha - i\eta} \right)^{k_j+1} = -i \int_0^\infty ds_j \, e^{-is_j(\omega_j - \alpha - i\eta)} \frac{(-is_j)^{k_j}}{k_j!} \tag{2.128}$$

and integrating over α and β we have

$$\tilde{Q}_\pi(\bar{n},\{k_i,p_i\},q_0,\omega_0,t) = \left(\prod_{j=1}^{\bar{n}}\int[d\omega_j]\right)P_0(q_{\bar{n}},\omega_{\bar{n}})\sum_{\{q_j\}_0}\prod_{j=0}^{\bar{n}}\left|W_{q_j,q_{j+1}}(\omega_j,\omega_{j+1})\right|^2$$
$$\times\left(\prod_{j=0}^{\bar{n}}\int_0^\infty ds_j\delta(t-\sum_{j=0}^{\bar{n}}s_j)e^{-is_j\omega_j}\frac{(-is_j)^{k_j}}{k_j!}\Theta^{k_j}(q_j,\omega_j,\omega_{\bar{n}})\right)$$
$$\times\left(\prod_{j=0}^{\bar{n}}\int_0^\infty d\tau_j\delta(t-\sum_{j=0}^{\bar{n}}\tau_j)e^{i\tau_j\omega_j}\frac{(i\tau_j)^{p_j}}{p_j!}\bar{\Theta}^{p_j}(q_j,\omega_j,\omega_{\bar{n}})\right) \quad (2.129)$$

From this expression we can sum up over k_j and p_j to evaluate Eq. (2.127). Since

$$\sum_{k_j=0}^{\infty}\frac{(-is_j)^{k_j}}{k_j!}\Theta^{k_j}(q_j,\omega_j,\omega_{\bar{n}}) = e^{-is_j\Theta(q_j,\omega_j,\omega_{\bar{n}})} \quad (2.130)$$

we have

$$\mathbb{E}[P_T(q_0,\omega_0)] = \lim_{\substack{\lambda^2 t=T\\ t\to\infty}}\sum_{\bar{n}=0}^{\infty}\lambda^{2\bar{n}}\left(\prod_{j=1}^{\bar{n}}\int[d\omega_j]\right)P_0(q_{\bar{n}},\omega_{\bar{n}})\sum_{\{q_j\}}\prod_{j=0}^{\bar{n}}\left|W_{q_j,q_{j+1}}(\omega_j,\omega_{j+1})\right|^2$$
$$\times\left(\prod_{j=0}^{\bar{n}}\int_0^\infty ds_j\delta(t-\sum_{j=0}^{\bar{n}}s_j)e^{-is_j\left(\omega_j+\Theta(q_j,\omega_j,\omega_{\bar{n}})\right)}\right)$$
$$\times\left(\prod_{j=0}^{\bar{n}}\int_0^\infty d\tau_j\delta(t-\sum_{j=0}^{\bar{n}}\tau_j)e^{i\tau_j\left(\omega_j+\bar{\Theta}(q_j,\omega_j,\omega_{\bar{n}})\right)}\right) \quad (2.131)$$

We now set out to calculate the following part of Eq. (2.131) in the Van Hove limit:

$$\mathcal{P} = \lambda^{2\bar{n}}\left(\prod_{j=0}^{\bar{n}}\int_0^\infty ds_j\delta(t-\sum_j s_j)e^{-is_j\left(\omega_j+\Theta(q_j,\omega_j,\omega_{\bar{n}})\right)}\right)$$
$$\left(\prod_{j=0}^{\bar{n}}\int_0^\infty d\tau_j\delta(t-\sum_j\tau_j)e^{i\tau_j\left(\omega_j+\bar{\Theta}(q_j,\omega_j,\omega_{\bar{n}})\right)}\right) \quad (2.132)$$

By performing the change of variables

$$a_j = \frac{s_j+\tau_j}{2} \quad (2.133)$$
$$b_j = \frac{s_j-\tau_j}{2} \quad (2.134)$$

we obtain

$$\mathcal{P} =(2\lambda^2)^{\bar{n}} \prod_{j=0}^{\bar{n}} \int_0^t da_j \delta\left(t-\sum_{j=0}^{\bar{n}} a_j\right) e^{2a_j Im\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]}$$
$$\times \prod_{j=0}^{\bar{n}} \int_{-a_j}^{+a_j} db_j \delta\left(\sum_{j=0}^{\bar{n}} b_j\right) e^{-i2\sum_j b_j\left(\omega_j+Re\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]\right)} \tag{2.135}$$

We can rewrite the integrals over b_j as follows:

$$\int_{-a_j}^{+a_j} db_j \delta\left(\sum_{b_j} b_j\right) e^{i2\sum_{j=0}^{\bar{n}} b_j\left(\omega_j+Re\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]\right)}$$
$$=\int_{-\infty}^{\infty} db_j \chi_a(b) e^{2i\sum_{j=0}^{\bar{n}-1} b_j\left(\omega_j-\omega_{\bar{n}}+Re\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]-Re\left[\Theta(q_j,\omega_{\bar{n}},\omega_{\bar{n}})\right]\right)} \tag{2.136}$$

Here $\chi_a(b)$ is a product of characteristic functions,

$$\chi_a(b) =\chi(-a_0 \leq \sum_{j=0}^{\bar{n}} b_j \leq a_0) \prod_{j=0}^{\bar{n}-1} \chi(-a_j \leq b_j \leq a_j)$$

We now perform the Van Hove limit, $t\to\infty$ with $\lambda^2 t = T$. This means we have for Eq. (2.132):

$$\mathcal{P} = \left(2\lambda^2\right)^{\bar{n}} \prod_{j=0}^{\bar{n}} \int_0^{T/\lambda^2} da_j \delta\left(t-\sum_{j=0}^{\bar{n}} a_j\right) e^{2a_j Im\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]}$$
$$\times \prod_{j=0}^{\bar{n}} \int_{-\infty}^{\infty} db_j \chi_a(b) e^{-i\sum_{j=0}^{\bar{n}-1} 2b_j\left(\omega_j-\omega_{\bar{n}}+Re\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]-Re\left[\Theta(q_j,\omega_{\bar{n}},\omega_{\bar{n}})\right]\right)} \tag{2.137}$$

Remember that in Eq. (2.126) we have included in our Θ function the λ^2 factor for shorter notation such that we have now

$$\Theta(q_j,\omega_j,\omega_{\bar{n}}) = \lambda^2 \lim_{\eta\to 0} \sum_q \int d\omega g(\omega) \frac{-\left|W_{q_j,q}(\omega_j,\omega)\right|^2}{\omega-\omega_{\bar{n}}-i\eta}$$
$$Im\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right] = -\lambda^2 \pi g(\omega_{\bar{n}}) \sum_{q\neq q_j} \left|W_{q_j,q}(\omega_j,\omega_{\bar{n}})\right|^2 \tag{2.138}$$

We perform the change of variables,

$$\alpha_j = 2\lambda^2 a_j$$

to obtain

$$\mathcal{P} = \prod_{j=0}^{\bar{n}} \int_0^T d\alpha_j \delta\left(T-\sum_{j=0}^{\bar{n}} \alpha_j\right) e^{-\frac{\alpha_j}{\lambda^2} Im\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]}$$
$$\times \prod_{j=0}^{\bar{n}} \int_{-\infty}^{\infty} db_j \chi_{\alpha/(2\lambda^2)}(b) e^{-i\sum_{j=0}^{\bar{n}-1} 2b_j\left(\omega_j-\omega_{\bar{n}}+Re\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]-Re\left[\Theta(q_j,\omega_{\bar{n}},\omega_{\bar{n}})\right]\right)} \tag{2.139}$$

For the integrals over b_j we obtain in the limit

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} db_j \chi_{\alpha/(2\lambda^2)}(b)\, e^{i\sum_{j=0}^{\bar{n}-1} 2b_j\left(\omega_j-\omega_{\bar{n}}+Re\left[\Theta(q_j,\omega_j,\omega_{\bar{n}})\right]-Re\left[\Theta(q_j,\omega_{\bar{n}},\omega_{\bar{n}})\right]\right)} = \prod_{j=0}^{\bar{n}-1}\pi\delta(\omega_j-\omega_{\bar{n}}) \tag{2.140}$$

By using Eqs. (2.140) and (2.138) in $\mathcal{P}$ and Eq. (2.131) we can now take the Van Hove limit of the expectation value of an operator,

$$\begin{aligned}
&\int [d\omega_0]\, \bar{O}(q_0,\omega_0) P_T(q_0,\omega_0) \\
&= \sum_{\bar{n}=0}^{\infty} \int [d\omega_0]\, \bar{O}(q_0,\omega_0) \sum_{\{q_j\}} \int [d\omega_j] \prod_{j=0}^{\bar{n}} \int_0^T d\alpha_j \delta\left(T-\sum_{j=0}^{\bar{n}}\alpha_j\right) e^{-\alpha_j \pi g(\omega_{\bar{n}})\sum_{q\neq q_j}\left|W_{q_j,q}(\omega_j,\omega_{\bar{n}})\right|^2} \\
&\times \prod_{j=0}^{\bar{n}-1} \pi\delta(\omega_j-\omega_{\bar{n}}) \left|W_{q_j,q_{j+1}}(\omega_j,\omega_{j+1})\right|^2 \\
&= \sum_{\bar{n}=0}^{\infty} \int [d\omega_0]\, \bar{O}(q_0,\omega_0) \sum_{\{q_j\}} \prod_{j=0}^{\bar{n}} \int_0^T d\alpha_j \delta\left(T-\sum_{j=0}^{\bar{n}}\alpha_j\right) e^{-\alpha_j \pi g(\omega_{\bar{n}})\sum_{q\neq q_j}\left|W_{q_j,q}(\omega_{\bar{n}},\omega_{\bar{n}})\right|^2} \\
&\times \prod_{j=0}^{\bar{n}-1} \pi g(\omega_{\bar{n}}) \left|W_{q_j,q_{j+1}}(\omega_{\bar{n}},\omega_{\bar{n}})\right|^2 P_0(q_{\bar{n}},\omega_{\bar{n}})
\end{aligned}$$

This corresponds to the expansion of the solution to the rate equations. Thus the expectation value of a diagonal operator is

$$O(T) = \sum_{q_0} \int [d\omega_0]\, O(q_0,\omega_0) P_T(q_0,\omega_0) \tag{2.141}$$

with $P_T(q_0,\omega_0)$ satisfying

$$\frac{\partial}{\partial T} P_T(q,\omega) = \sum_{q'} M_{q,q'}(\omega) P_T(q',\omega) \tag{2.142}$$

$$M_{q,q'}(\omega) = \pi g(\omega)\left(-\delta_{q,q'}\sum_{\bar{q}\neq q}\left|W_{q,\bar{q}}(\omega,\omega)\right|^2 + \left|W_{q,q'}(\omega,\omega)\right|^2\right) \tag{2.143}$$

which is the desired Pauli master equation with the rates given by Fermi's golden rule. Eqs. (2.142) and (2.143) are our central result.

We finally prove two last bounds. One is on the contribution of S-graphs and the other is a bound for the norm of $|\psi_t^n\rangle$. We do not need these results in this section but prove it here since it has to do with S-graphs.

Theorem 2.9.1. *If* $\pi(n,m) \in \mathcal{G}_0(n,m)$ *we have the following bound:*

$$\begin{aligned} I(t) =& \left| \int [d\omega_0]\, \lambda^{n+m} \sum_{\{q_i,q_j'\}_i} \tilde{Q}_\pi(n,m,\{q_i,q_j'\}_i,\omega_0,t) \right| \\ \leq& \frac{(C\lambda^2 t)^{\frac{n+m}{2}}}{\frac{n+m}{2}!} \end{aligned} \tag{2.144}$$

where $\tilde{Q}_\pi(n,m,\omega_0,t)$ *is given by Eq. (2.124)*

Proof. Theorem 2.9.1
From Eq. (2.129) we get the contribution of a graph and so

$$\begin{aligned} I(t) \leq& \lambda^{2\bar{n}} \Bigg| \left(\prod_{j=0}^{\bar{n}} \int [d\omega_j] \right) P_0(q_{\bar{n}},\omega_{\bar{n}}) \sum_{\{q_j\}} \prod_{j=0}^{\bar{n}} \left| W_{q_j,q_{j+1}}(\omega_j,\omega_{j+1}) \right|^2 \\ &\times \left(\prod_{j=0}^{\bar{n}} \int_0^\infty ds_j \delta(t - \sum_{j=0}^{\bar{n}} s_j) e^{-is_j\omega_j} \frac{(-is_j)^{k_j}}{k_j!} \Theta^{k_j}(q_j,\omega_j,\omega_{\bar{n}}) \right) \\ &\times \left(\prod_{j=0}^{\bar{n}} \int_0^\infty d\tau_j \delta(t - \sum_{j=0}^{\bar{n}} \tau_j) e^{i\tau_j\omega_j} \frac{(i\tau_j)^{p_j}}{p_j!} \bar{\Theta}^{p_j}(q_j,\omega_j,\omega_{\bar{n}}) \right) \Bigg| \end{aligned} \tag{2.145}$$

Since $\left| W_{q_j,q_{j+1}}(\omega_j,\omega_{j+1}) \right|^2$ is positive and bounded we can bound the product by $C_1^{\bar{n}}$. We apply the same change of variable as in Eq. (2.133) and (2.134).

$$\begin{aligned} I(t) \leq& C_1^{\bar{n}} \lambda^{2\bar{n}} \prod_{j=0}^{\bar{n}} \int_0^\infty da_j \delta(t - \sum_{j=0}^{\bar{n}} a_j) \\ \times& \left| \prod_{j=0}^{\bar{n}} \int [d\omega_j] \int_{-a_j}^{a_j} db_j \delta\left(\sum_{j=0}^{\bar{n}} b_j \right) e^{-ib_j\omega_j} \frac{s_j^{k_j}}{k_j!} \frac{\tau_j^{p_j}}{p_j!} \Theta^{k_j}(q_j,\omega_j,\omega_{\bar{n}}) \bar{\Theta}^{p_j}(q_j,\omega_j,\omega_{\bar{n}}) \right| \end{aligned} \tag{2.146}$$

Here, s_j and τ_j are now of course functions of the new variables and bounded by a_j. The $\Theta(q_j,\omega_j,\omega_{\bar{n}})$ functions are proportional to λ^2 but also bounded and smooth. We can thus use the bound of Eq. (A.21) on the double integrations of b_j and ω_j to obtain

$$\begin{aligned}
I(t) \leq & C_1^{\bar{n}} \lambda^{2\bar{n}} \prod_{j=0}^{\bar{n}} \int_0^\infty da_j \delta(t - \sum_{j=0}^{\bar{n}} a_j) \prod_{j=0}^{\bar{n}} \frac{\left(C\lambda^2 a_j\right)^{k_j}}{k_j!} \frac{\left(C\lambda^2 a_j\right)^{p_j}}{p_j!} \\
\leq & C_1^{\bar{n}} \lambda^{2\bar{n}} \frac{\left(C\lambda^2 t\right)^{\sum_{j=0}^{\bar{n}}(p_j+k_j)}}{\prod_{j=0}^{\bar{n}} k_j! p_j!} \prod_{j=0}^{\bar{n}} \int_0^\infty da_j \delta(t - \sum_{j=0}^{\bar{n}} a_j) \\
\leq & \frac{\left(C\lambda^2 t\right)^{\sum_{j=0}^{\bar{n}}(p_j+k_j)}}{\prod_{j=0}^{\bar{n}} (k_j! p_j!)} \frac{\left(C_1 \lambda^2 t\right)^{\bar{n}}}{\bar{n}!}
\end{aligned} \tag{2.147}$$

Since from Eq. (2.93) we have $\bar{n} + \sum_{j=0}^{\bar{n}} (p_j + k_j) = \frac{n+m}{2}$ we get the following inequality for some constant C:

$$\frac{1}{\bar{n}! \prod_{j=0}^{\bar{n}} (k_j! p_j!)} \leq \frac{C^{\frac{n+m}{2}}}{\frac{n+m}{2}!} \tag{2.148}$$

Inserting this in Eq. (2.147) and selecting the largest out of all constants present we find

$$I(t) \leq \frac{\left(C\lambda^2 t\right)^{\frac{n+m}{2}}}{\frac{n+m}{2}!} \tag{2.149}$$

□

We now want to explicitly have a bound on the norm of $|\psi_t^n\rangle$.

Theorem 2.9.2.

$$\mathbb{E}[\langle\psi_t^n|\psi_t^n\rangle] \leq \frac{(C'T)^n}{n!} \tag{2.150}$$

Proof. Theorem 2.9.2 Similarly as how we found the expression for Eq. (2.50) we find

$$\begin{aligned}
\mathbb{E}[\langle\psi_t^n|\psi_t^n\rangle] = \lambda^{2n} & \sum_{\{l_i, l'_j\}, \{q_i, q'_j\}} \psi_0^*(q'_n, l'_n) \psi_0(q_n, l_n) K^n(t, \{E_{l_i}\}) \bar{K}^n(\{t, E_{l'_j}\}) \\
& \times \sum_{a=0}^{2} \sum_{\pi(n,m) \in \mathcal{G}_a} C_\pi(n, m, \{l_i, l'_j\}, \{q_i, q'_j\})
\end{aligned} \tag{2.151}$$

Once again we have written our expectation value as a function of the different classes of graphs and for the same reasons as before, that is Eq. (2.67) and theorem 2.6.1, the C-graphs and N-graphs will vanish in the limit $N \to \infty$ and $t \to \infty$ with $\lambda^2 t = T$ finite. Therefore we have

$$\lim_{t\to\infty}\lim_{N\to\infty}\mathbb{E}[\langle\psi_t^n|\psi_t^n\rangle] = \lim_{t\to\infty}\lim_{N\to\infty}\lambda^{2n}\sum_{\{l_i,l_j'\},\{q_i,q_j'\}}\psi_0^*(q_n',l_n')\psi_0(q_n,l_n)K^n(t,\{E_{l_i}\})\bar{K}^n(\{t,E_{l_j'}\})$$
$$\times\sum_{\pi(n,n)\in\mathcal{G}_0}C_\pi(n,m,\{l_i,l_j'\},\{q_i,q_j'\})$$
$$= \lim_{t\to\infty}\lambda^{2n}\sum_{\pi(n,n)\in\mathcal{G}_0}\sum_{\mathcal{A}_\pi^q}\int d\omega_0\,Q_\pi(n,n,\{q_i,q_j'\}_{\mathbf{i}},\omega_0,t) \quad (2.152)$$

where $Q_\pi(n,n,\{q_i,q_j'\}_{\mathbf{i}},\omega_0,t)$ is defined through Eq. (2.75). We can rewrite $Q_\pi(n,n,\{q_i,q_j'\}_{\mathbf{i}},\omega_0,t)$ as a function of dependent and independent variables which would give $Q_\pi(\bar{n},\{k_j,p_j\},\omega_0,t)$ and by theorem 2.7.1 we have the in the Van Hove limit we can replace $Q_\pi(\bar{n},\{k_j,p_j\},\omega_0,t)$ by $\tilde{Q}_\pi(\bar{n},\{k_j,p_j\},\omega_0,t)$. Thus we have

$$\lim_{t\to\infty}\lim_{N\to\infty}\mathbb{E}[\langle\psi_t^n|\psi_t^n\rangle] \leq \lim_{t\to\infty}\lambda^{2n}\sum_{\pi(n,n)\in\mathcal{G}_0}\left|\sum_{\mathcal{A}_\pi^q}\int d\omega_0\,\tilde{Q}_\pi(n,n,\{q_i,q_j'\}_{\mathbf{i}},\omega_0,t)\right| \quad (2.153)$$

Finally by use of theorem 2.9.1 and the fact that the number of S-graphs of order (n,m) can be bounded by $\tilde{C}^{n+m}$ with $\tilde{C}$ being a constant we get

$$\lim_{t\to\infty}\lim_{N\to\infty}\mathbb{E}[\langle\psi_t^n|\psi_t^n\rangle] \leq \sum_{\pi(n,n)\in\mathcal{G}_0}\frac{(CT)^n}{n!}$$
$$\leq \frac{(C'T)^n}{n!}$$

□

3 Dynamical Typicality

When something is characterized as " typical " (for an ensemble) one refers to the fact that out of most cases (ensemble members) the same result (property) comes out. In physics there appears to be a limited number of qualitatively different behaviors described by effective equations. Nevertheless, there are probably infinitely many realizations of systems. For example, the empirical law of heat transfer, Fourier's law, states that the rate of heat flow through a homogeneous solid is directly proportional to the area of the section at right angle to the direction of heat flow, and to the temperature difference along the path of heat flow. This is a statement typically valid for homogeneous solids. Thus we say that Fourier's law is a typical behavior for solids.

Over the last years there have been studies of typical behavior starting from quantum mechanical principles ([12], [18], [24], [23], [13], [22], [11]), [25] in order to understand better the properties of matter, may they be large or small. The main claim in [18] is about canonical typicality. The basic statement is that for large Hilbert spaces, most state vectors are such that the reduced density matrix we obtain from it, for a small system, will be canonical.

What we mean by "dynamical typicality " is quite similar. We state that no matter what the realization will be of the random interaction, the dynamics of the observable will be the same. Mathematically this means that the variance of the observable, where the variance refers to the variance with respect to the probability distribution of the random matrix, goes to zero as N goes to ∞:

$$\mathrm{Var}_{N,t}\left[\hat{O}\right] = \left(\mathbb{E}\left[\langle\psi_t|\hat{O}|\psi_t\rangle^2\right] - \mathbb{E}\left[\langle\psi_t|\hat{O}|\psi_t\rangle\right]^2\right) \tag{3.1}$$

$$\boxed{\lim_{N\to\infty} \mathrm{Var}_{N,t}\left[\hat{O}\right] = 0} \tag{3.2}$$

Notice that no statement is made about λ and t, although implicitly one takes these to be finite. In order to prove Eq. (3.2) we will use in a crucial way the perturbation expansion and the assumption that t and λ stay finite. This is why this result cannot be extended to the limits of our previous chapter.

3.1 Bounding the error on the approximate solution

In this section we want to find a bound for the difference between the average of the observable using the solution of the Schrödinger equation $|\psi_t\rangle$ and the average of the observable using the approximate solution. From the expansion in Eq. (2.21) we have

$$\begin{aligned}|\psi_t\rangle &= \sum_{n=0}^{M-1} |\psi_t^n\rangle + |\phi_t^M\rangle \\ &= |\Psi_t^M\rangle + |\phi_t^M\rangle \end{aligned} \tag{3.3}$$

and so we intend to show that the error made by using $|\Psi_t^M\rangle$ instead of $|\psi_t\rangle$ becomes zero, when $N \to \infty$ and $M \to \infty$. Thus we want to find a proper bound for the following function:

$$\begin{aligned}\mathrm{Error}(t,\lambda,M,N) &= \mathbb{E}\left[\langle\psi_t|\hat{O}|\psi_t\rangle\right] - \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right] \\ &= -\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle\right] + \mathbb{E}\left[\langle\psi_t|\hat{O}|\phi_t^M\rangle\right] + \mathbb{E}\left[\langle\phi_t^M|\hat{O}|\psi_t\rangle\right]\end{aligned} \tag{3.4}$$

We prove the following theorem:

Theorem 3.1.1. *For any $t_0 < \infty$, $\lambda < \infty$ and $\varepsilon > 0$ there exist an N_0 and an M such that for $t \le t_0$ and $N \ge N_0$*

$$\mathit{Error}(t,\lambda,M,N) \le \varepsilon \tag{3.5}$$

In particular

$$\lim_{M\to\infty} \mathit{Error}(t,\lambda,M,N) = 0 \tag{3.6}$$

To prove this we first prove the following theorem:

Theorem 3.1.2. *For any $t_0 < \infty$, $\lambda < \infty$ and $\varepsilon > 0$ there exist an N_0 and an M such that for all $N \ge N_0$ and $t \le t_0$*

$$\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle\right] \le \varepsilon \tag{3.7}$$

In particular

$$\lim_{M\to\infty} \mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle\right] = 0 \tag{3.8}$$

Proof. Theorem 3.1.2
We take M odd such that later on at least one left random variable will have to contract with a

right one. By using Eq. (2.24) for $|\phi_t^M\rangle$ and the Cauchy-Schwartz inequality we get

$$\begin{aligned}
\langle\phi_t^M|\hat{O}|\phi_t^M\rangle &\le O_{max}\int_0^t dsd\tau\sqrt{\langle\psi_{M-1}(\tau)|V^2|\psi_{M-1}(\tau)\rangle\langle\psi_{M-1}(s)|V^2|\psi_{M-1}(s)\rangle} \\
&\le O_{max}t\int_0^t ds\langle\psi_{M-1}(s)|V^2|\psi_{M-1}(s)\rangle \\
&= O_{max}t\lambda^{2M}\int_0^t [ds_M]\int_0^t [d\tau_M] \\
&\times\langle\psi_0|\prod_{j=0}^{M-1} e^{-is_jH_0}V^2\prod_{j=0}^{M-1} e^{-is_jH_0}|\psi_0\rangle \\
&= O_{max}t\lambda^{2M}\int_0^t [ds_M]\int_0^t [d\tau_M]\,O(M,\tau_i,s_j) \qquad (3.9)
\end{aligned}$$

with

$$\begin{aligned}
O(M,\tau_i,s_j) = &\sum_{\{l'_j,l_i\}}\sum_{\{q'_j,q_i\}}\psi_0^*(l'_M,q'_M)\psi_0(l_M,q_M)\left(\prod_{j=0}^{M-1} e^{iE_{l'_j}\tau_j}\right)\left(\prod_{j=0}^{M-1} e^{-iE_{l_j}s_j}\right) \\
&\times\langle l'_M,q'_M|V|l'_{M-1},q'_{M-1}\rangle\ldots\langle l'_1,q'_1|V|l'_0,q'_0\rangle \\
&\times\langle l'_0,q'_0|l_0,q_0\rangle\langle l_0,q_0|V|l_1,q_1\rangle\ldots\langle l_{M-1},q_{M-1}|V|l_M,q_M\rangle \qquad (3.10)
\end{aligned}$$

The average will then be

$$\begin{aligned}
|\mathbb{E}[O(M,\tau_i,s_j)]| \le &\sum_{\{l'_j,l_i\}}\sum_{\{q'_j,q_i\}}\left|\psi_0^*(l'_M,q'_M)\psi_0(l_M,q_M)\right| \\
&\times\mathbb{E}[\langle l'_M,q'_M|V|l'_{M-1},q'_{M-1}\rangle\ldots\langle l'_1,q'_1|V|l_0,q_0\rangle \\
&\times\langle l_0,q_0|V|l_1,q_1\rangle\ldots\langle l_{M-1},q_{M-1}|V|l_M,q_M\rangle] \qquad (3.11)
\end{aligned}$$

and so by Eq. (2.40) and (C.5)

$$\begin{aligned}
|\mathbb{E}[O(M,\tau_i,s_j)]| \le &C\sum_{\{l'_j,l_i\}}\sum_{\{q'_j,q_i\}}\frac{1}{N} \\
&\times\sum_{\pi(M,M)}\left|C_\pi(M,M,\{l'_j,l_i\},\{q'_j,q_i\})\right| \qquad (3.12)
\end{aligned}$$

By using Eq. (2.44) and (2.45) in Eq.(3.12) and dividing our sum over the graphs between C and NC-graphs we get

$$
\begin{aligned}
|\mathbb{E}[O(M,\tau_i,s_j)]| &\leq C \sum_{\pi(M,M)} \sum_{\{l'_j,l_i\}} \sum_{\{q'_j,q_i\}} \left| C_\pi(M,M,\{l'_j,l_i\},\{q'_j,q_i\}) \right| \frac{1}{N} \\
&\leq \sum_{\pi(M,M)} \sum_{\{l'_j,l_i\}_\mathrm{i}} \sum_{\{q'_j,q_i\}_\mathrm{i}} \left(\frac{W}{N\aleph} \right)^M \frac{1}{N} \\
&\leq \sum_{\pi(M,M)\in\mathcal{G}_{0,1}(M,M)} \sum_{\{l'_j,l_i\}_\mathrm{i}} \sum_{\{q'_j,q_i\}_\mathrm{i}} \frac{W^M}{(N\aleph)^M} \frac{1}{N} \\
&+ \sum_{\pi(M,M)\in\mathcal{G}_2(M,M)} \sum_{\{l'_j,l_i\}_\mathrm{i}} \sum_{\{q'_j,q_i\}_\mathrm{i}} \frac{W^M}{(N\aleph)^M} \frac{1}{N}
\end{aligned}
\tag{3.13}
$$

where $\sum_{\{l'_j,l_i\}_\mathrm{i}}$ refers to a sum over the independent variables of the set $\{l'_j,l_i\}$ generated by the contraction function C_π and the same holds for the set of q variables. According to theorem 2.4.6 and 2.4.7, the number of independent variables of the set $\{l'_j,l_i\}$ is equal to $M+1$ for graphs in $\mathcal{G}_0(M,M)$ or $\mathcal{G}_1(M,M)$ and less then $M+1$ for graphs in $\mathcal{G}_2(M,M)$. Each of these sums will render a factor of N. Thus

$$
\begin{aligned}
|\mathbb{E}[O(M,\tau_i,s_j)]| &\leq \sum_{\pi(M,M)\in\mathcal{G}_{0,1}(M,M)} \sum_{\{q'_j,q_i\}_\mathrm{i}} \frac{W^M}{\aleph^M} \\
&+ \frac{1}{N} \sum_{\pi(M,M)\in\mathcal{G}_2(M,M)} \sum_{\{q'_j,q_i\}_\mathrm{i}} \frac{W^M}{\aleph^M}
\end{aligned}
$$

Applying once again theorems 2.4.6 and 2.4.7 on the set $\{q'_j,q_i\}$ we find there are $M+1$ independent variables for graphs in $\mathcal{G}_1$ and less than $M+1$ for graphs in $\mathcal{G}_2$, and so

$$
\begin{aligned}
|\mathbb{E}[O(M,\tau_i,s_j)]| &\leq W^M \aleph \sum_{\pi(M,M)\in\mathcal{G}_{0,1}(M,M)} 1 \\
&+ \frac{W^M}{N} \sum_{\pi(M,M)\in\mathcal{G}_2(M,M)} 1 \\
|\mathbb{E}[O(M,\tau_i,s_j)]| &\leq W^M \left(\aleph \frac{(2M)!}{M!(M+1)!} + \frac{1}{N} G_2(M) \right) \\
G_2(M) &= \sum_{\pi(M,M)\in\mathcal{G}_2(M,M)} 1
\end{aligned}
\tag{3.14}
$$

Observing that

$$
\int_0^t [ds_M] = \frac{t^M}{M!} \tag{3.15}
$$

and inserting Eq. (3.14) and (3.15) in Eq. (3.9) we have

$$\begin{aligned}\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle\right] &\leq C\lambda^{2M} t \frac{t^{2M}}{M!M!}\left(\aleph \frac{(2M)!}{M!(M+1)!} + \frac{1}{N} G_2(M)\right) \\ &\leq C\left(B_1(M,\aleph,t\lambda) + B_2(M,N,t\lambda)\right) \end{aligned} \tag{3.16}$$

with

$$B_1(M,\aleph,t\lambda) = \frac{t\,(\lambda t)^{2M}}{M!M!}\aleph\frac{(2M)!}{M!(M+1)!} \tag{3.17}$$

$$B_2(M,N,t\lambda) = \frac{t\,(\lambda t)^{2M}}{M!M!}\frac{1}{N}G_2(M) \tag{3.18}$$

And according to Eq. (B.13) we have

$$G_2(M) \leq \frac{(2M)!}{2^M M!} \tag{3.19}$$

We have absorbed the W factor in the λ factor for brevity. $B_2(M,N,t\lambda)$ is bounded by $\frac{1}{N}$ and so in the limit $N \to \infty$ it goes to zero. As for $B_1(M,\aleph,t\lambda)$ it goes to zero when $M \to \infty$. Thus for fixed t_0, λ and ε there exists always an M such that $B_1(M,\aleph,t\lambda) \leq \varepsilon$ when $t \leq t_0$ For fixed t_0, λ and ε and for any M there is always an N_0 such that $B_2(M,N,\lambda t) \leq \varepsilon$ for $t \leq t_0$ and $N \geq N_0$. □

We have shown that the first term of Eq. (3.4) is bounded by any ε provided N and M are large enough. In fact just M needs to be large. We now prove the same for the other two terms, which proves theorem 3.1.1.

Proof. Theorem 3.1.1
By Cauchy-Schwartz inequality for vector spaces we have

$$\begin{aligned}\left|\langle\psi_t|\hat{O}|\phi_t^M\rangle\right| &\leq \sqrt{\langle\psi_t|\hat{O}|\psi_t\rangle}\sqrt{\langle\phi_t^M|\phi_t^M\rangle} \\ &\leq \sqrt{O_{max}}\sqrt{\langle\phi_t^M|\phi_t^M\rangle}\end{aligned}$$

$\langle\phi_t^M|\phi_t^M\rangle$ is a sum of products of random variables and consequently, a function of random variables in the corresponding probability space. In a probability space we have by Hölder's inequality:

$$\mathbb{E}\left[|X|\right] \leq \mathbb{E}\left[|X|^2\right]^{\frac{1}{2}} \tag{3.20}$$

Applying this to our case, we would have

$$\mathbb{E}\left[\sqrt{\langle\phi_t^M|\phi_t^M\rangle}\right] \leq \mathbb{E}\left[\langle\phi_t^M|\phi_t^M\rangle\right]^{\frac{1}{2}} \tag{3.21}$$

which gives us

$$\mathbb{E}\left[\left|\langle\psi_t|\hat{O}|\phi_t^M\rangle\right|\right] \leq \sqrt{O_{max}}\mathbb{E}\left[\left|\langle\phi_t^M|\phi_t^M\rangle\right|^{\frac{1}{2}}\right] \leq \sqrt{O_{max}}\mathbb{E}\left[\left|\langle\phi_t^M|\phi_t^M\rangle\right|\right]^{\frac{1}{2}} \tag{3.22}$$

Using theorem (3.1.2) and Eq. (3.22) in Eq. (3.4) we see that for any t_0, λ and small ε' there exists an N_0 and M such that for $N \geq N_0$ and $t \leq t_0$

$$\boxed{|\text{Error}(t,\lambda,N,M)| \leq C\left(\varepsilon' + 2\varepsilon'^{\frac{1}{2}}\right)} \tag{3.23}$$

where C depends on O_{max} . By choosing $C\left(\varepsilon' + 2\varepsilon'^{\frac{1}{2}}\right) \leq \varepsilon$ we have proved theorem 3.1.1. □

3.2 Variance

In this section we will turn to the variance of observables and prove the following theorem:

Theorem 3.2.1. *For any fixed t_0, λ, M and $\varepsilon > 0$ and Macro-Observable $\hat{O}$ there exists an N_0 such that if $N \geq N_0$ and $t \leq t_0$ we have*

$$Var_N^M\left[O(t,\lambda)\right] = \left(\mathbb{E}\left[\left(\langle\hat{O}\rangle_t^M\right)^2\right] - \mathbb{E}\left[\langle\hat{O}\rangle_t^M\right]^2\right) \leq \varepsilon \tag{3.24}$$

with

$$\langle\hat{O}\rangle_t^M = \langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle \tag{3.25}$$

Thus

$$\lim_{N\to\infty} Var_N^M\left[O(t,\lambda)\right] = 0 \tag{3.26}$$

When calculating $\mathbb{E}\left[\left(\langle O\rangle_t^M\right)^2\right]$, graphs will arise again, but in a broader sense, since we can have random variable correlations between the two $\langle O\rangle_t^M$'s. These graphs do not appear when calculating $\mathbb{E}\left[\langle O\rangle_t^M\right]^2$, thus the idea is to show that the contribution of the extra graphs appearing in $\mathbb{E}\left[\left(\langle O\rangle_t^M\right)^2\right]$ is bounded by a constant times N^{-1}.

Proof. Theorem 3.2.1
For a diagonal observable we have

$$\mathbb{E}\left[\left(\langle O\rangle_t^M\right)^2\right]=\mathbb{E}\left[\sum_{l_0,q_0}O(l_0,q_0)P_t(q_0,l_0)\sum_{\bar{l}_0,\bar{q}_0}O(\bar{l}_0,\bar{q}_0)P_t(\bar{q}_0,\bar{l}_0)\right] \tag{3.27}$$

and from Eq. (2.30)

$$\begin{aligned}P_t(q_0,l_0)P_t(\bar{q}_0,\bar{l}_0)=&\sum_{n,m=0}^{M}\lambda^{n+m}\sum_{\{l_i,l'_j\}_0}\sum_{\{q_i,q'_j\}_0}\psi_0^*(l'_m,q'_m)\psi_0(l_n,q_n)K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_i}\})\\&\times L^n(\{l_i\},\{q_i\})\bar{L}^m(\{l'_i\},\{q'_i\})\\&\sum_{\bar{n},\bar{m}=0}^{M}\lambda^{\bar{n}+\bar{m}}\sum_{\{\bar{l}_i,\bar{l}'_j\}_0}\sum_{\{\bar{q}_i,\bar{q}'_j\}_0}\psi_0^*(\bar{l}'_{\bar{m}},\bar{q}'_{\bar{m}})\psi_0(\bar{l}_{\bar{n}},\bar{q}_{\bar{n}})K^{\bar{n}}(t,\{E_{\bar{l}_i}\})\bar{K}^{\bar{m}}(t,\{E_{\bar{l}'_i}\})\\&\times L^{\bar{n}}(\{\bar{l}_i\},\{\bar{q}_i\})\bar{L}^{\bar{m}}(\{\bar{l}'_i\},\{\bar{q}'_i\})\end{aligned} \tag{3.28}$$

where the subscript 0 in $\{l_i,l'_j\}_0$ refers to the fact that l_0 is not included in this sum. The same goes for $\{\bar{l}_i,\bar{l}'_j\}_0$, $\{q_i,q'_j\}_0$ and $\{\bar{q}_i,\bar{q}'_j\}_0$, which do not include $\bar{l}_0$, q_0, and $\bar{q}_0$, respectively. We set

$$\mathbb{E}\left[P_t(q_0,l_0)P_t(\bar{q}_0,\bar{l}_0)\right]=\mathbf{O}^{l_0,\bar{l}_0}(q_0,\bar{q}_0) \tag{3.29}$$

such that

$$\mathbb{E}\left[\left(\langle\hat{O}\rangle_t^M\right)^2\right]=\sum_{l_0,q_0}\sum_{l_0,q_0}O(l_0,q_0)O(\bar{l}_0,\bar{q}_0)\mathbf{O}^{l_0,\bar{l}_0}(q_0,\bar{q}_0) \tag{3.30}$$

and so

$$\begin{aligned}\mathbf{O}^{l_0,\bar{l}_0}(q_0,\bar{q}_0)=&\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\lambda^{n+m+\bar{n}+\bar{m}}\sum_{\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\}_0}\sum_{\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\}_0}\psi_0^*(l'_m,q'_m)\psi_0(l_n,q_n)\\&\times\psi_0^*(\bar{l}'_{\bar{m}},\bar{q}'_{\bar{m}})\psi_0(\bar{l}_{\bar{n}},\bar{q}_{\bar{n}})K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_i}\})K^{\bar{n}}(t,\{E_{\bar{l}_i}\})\bar{K}^{\bar{m}}(t,\{E_{\bar{l}'_i}\})\\&\times\mathbb{E}\left[L^n(\{l_i\},\{q_i\})\bar{L}^m(\{l'_i\},\{q'_i\})L^{\bar{n}}(\{\bar{l}_i\},\{\bar{q}_i\})\bar{L}^{\bar{m}}(\{\bar{l}'_i\},\bar{q}'_i)\right]\end{aligned} \tag{3.31}$$

The average to be performed in Eq. (3.31) is similar to the one in Eq. (2.40). Using Wick's theorem we have graphs on a set $s=\{V_{l'_m,l'_{m-1}}(q'_m,q'_{m-1})\dots V_{l_{n-1},l_n}(q_{n-1},q_n)V_{\bar{l}'_{\bar{m}},\bar{l}'_{\bar{m}-1}}(\bar{q}'_{\bar{m}},\bar{q}'_{\bar{m}-1})\dots V_{\bar{l}_{\bar{n}-1},\bar{l}_{\bar{n}}}(\bar{q}_{\bar{n}-1},\bar{q}_{\bar{n}})\}$ that induce δ-relations between the quantum numbers $\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\}$ and $\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\}$.

$$\begin{aligned}&\mathbb{E}\left[L^n(\{l_i\},\{q_i\})\bar{L}^m(\{l'_i\},\{q'_i\})L^{\bar{n}}(\{\bar{l}_i\},\{\bar{q}_i\})\bar{L}^{\bar{m}}(\{\bar{l}'_i\},\{\bar{q}'_i\})\right]\\=&\mathbb{E}\left[V_{l'_m,l'_{m-1}}(q'_m,q'_{m-1})\dots V_{l_{n-1},l_n}(q_{n-1},q_n)V_{\bar{l}'_{\bar{m}},\bar{l}'_{\bar{m}-1}}(\bar{q}'_{\bar{m}},\bar{q}'_{\bar{m}-1})\dots V_{\bar{l}_{\bar{n}-1},\bar{l}_{\bar{n}}}(\bar{q}_{\bar{n}-1},\bar{q}_{\bar{n}})\right]\\=&\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}(n,m,\bar{n},\bar{m})}\mathcal{C}_\pi(n,m,\bar{n},\bar{m},\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\},\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\})\end{aligned} \tag{3.32}$$

We conclude that $n + m + \bar{n} + \bar{m}$ has to be even and from now on we consider this always to be the case in our calculations. Thus sums over n, m, $\bar{n}$ and $\bar{m}$ have to be restricted to sums, where this relation is maintained. We can classify the graphs again as NC-graphs and C-graphs, but before that we introduce the following additional classification of these graphs:

Definition 3.2.2. *Separable and Non-Separable Graphs*
If for a graph no contractions occur between the $V_{l'_m,l'_{m-1}}(q'_m, q'_{m-1})\ldots V_{l_{n-1},l_n}(q_{n-1}, q_n)$ variables and the $V_{\bar{l}'_{\bar{m}},\bar{l}'_{\bar{m}-1}}(\bar{q}'_{\bar{m}}, \bar{q}'_{\bar{m}-1})\ldots V_{\bar{l}_{\bar{n}-1},\bar{l}_{\bar{n}}}(\bar{q}_{\bar{n}-1}, \bar{q}_{\bar{n}})$ variables, we call this a separable graph (Sep-graph) and non separable (NSep-graph), if there is at least one contraction present. We denote by $\mathcal{G}_s(n,m,\bar{n},\bar{m})$ the set of all separable graphs of order $(n,m,\bar{n},\bar{m})$ and by $\mathcal{G}_s$ the set of all separable graphs. We denote by $\mathcal{G}_{non-s}(n,m,\bar{n},\bar{m})$ the set of all non separable graphs of order $(n,m,\bar{n},\bar{m})$ and by $\mathcal{G}_{non-s}$ the set of all non separable graphs.

Figures 3.1 and 3.2 are examples of a Non-Separable graph and a Separable graph respectively.

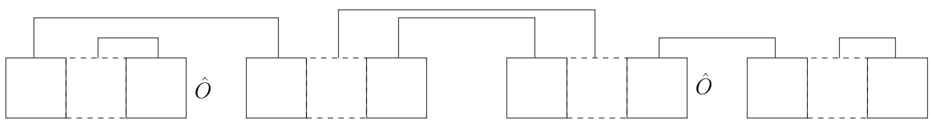

Figure 3.1: Example of a Non-Separable graph

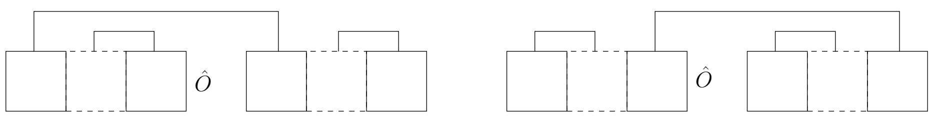

Figure 3.2: Example of a Separable graph

A similar definition is used in [5] in order to analyze higher moments of the norm of the wave function. According to this classification, for Sep-graphs randomness coming from the first $\langle O \rangle_t^M$ correlates with randomness from that same expression and never with randomness of the second $\langle O \rangle_t^M$. The same goes of course for the second $\langle O \rangle_t^M$. This is then equivalent to just averaging over $\langle O\rangle_t^M$ and squaring it:

$$\mathbf{O}^{l_0,\bar{l}_0}(q_0,\bar{q}_0)=\sum_{n,m,\bar{n},\bar{m}=0}^{M}\lambda^{n+m+\bar{n}+\bar{m}}\sum_{\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\}_0}\sum_{\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\}_0}\psi_0^*(l'_m,q'_m)\psi_0(l_n,q_n)$$
$$\times\psi_0^*(\bar{l}'_{\bar{m}},\bar{q}'_{\bar{m}})\psi_0(\bar{l}_{\bar{n}},\bar{q}_{\bar{n}})K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_i}\})K^{\bar{n}}(t,\{E_{\bar{l}_i}\})\bar{K}^{\bar{m}}(t,\{E_{\bar{l}'_i}\})$$
$$\times\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}(n,m,\bar{n},\bar{m})}C_\pi(n,m,\bar{n},\bar{m},\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\},\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\}) \tag{3.33}$$

With definition 3.2.2 we see that

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}(n,m,\bar{n},\bar{m})}C_\pi(n,m,\bar{n},\bar{m},\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\},\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\})$$
$$=\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_s(n,m,\bar{n},\bar{m})}C_\pi(n,m,\bar{n},\bar{m},\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\},\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\})$$
$$+\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_{non-s}(n,m,\bar{n},\bar{m})}C_\pi(n,m,\bar{n},\bar{m},\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\},\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\}) \tag{3.34}$$

We can define a contribution coming from Sep-graphs and one from NSep-graphs as

$$\mathbf{O}^{l_0,\bar{l}_0}(q_0,\bar{q}_0)=\mathbf{O}_s^{l_0,\bar{l}_0}(q_0,\bar{q}_0)+\mathbf{O}_{ns}^{l_0,\bar{l}_0}(q_0,\bar{q}_0) \tag{3.35}$$

Since for graphs in $\mathcal{G}_s$ the contractions only occur among the $V_{l'_m,l'_{m-1}}(q'_m,q'_{m-1}),\dots V_{l_{n-1},l_n}(q_{n-1},q_n)$ variables and among the $V_{\bar{l}'_{\bar{m}},\bar{l}'_{\bar{m}-1}}(\bar{q}'_{\bar{m}},\bar{q}'_{\bar{m}-1}),\dots V_{\bar{l}_{\bar{n}-1},\bar{l}_{\bar{n}}}(\bar{q}_{\bar{n}-1},\bar{q}_{\bar{n}})$, each graph in $\mathcal{G}_s(n,m,\bar{n},\bar{m})$ is equal to a graph in $\mathcal{G}(n,m)$ times a graph in$\mathcal{G}(\bar{n},\bar{m})$. Thus

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_s(n,m,\bar{n},\bar{m})}C_\pi(n,m,\bar{n},\bar{m},\{l_i,l'_j,\bar{l}_i,\bar{l}'_j\},\{q_i,q'_j,\bar{q}_i,\bar{q}'_j\})$$
$$=\sum_{\pi(n,m)\in\mathcal{G}(n,m)}C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\})\sum_{\pi(\bar{n},\bar{m})\in\mathcal{G}(\bar{n},\bar{m})}C_\pi(\bar{n},\bar{m},\{\bar{l}_i,\bar{l}'_j\},\{\bar{q}_i,\bar{q}'_j\}) \tag{3.36}$$

If we take only the contribution of Sep-graphs by inserting Eq. (3.36) in Eq. (3.33), we obtain

$$\mathbf{O}_s^{l_0,\bar{l}_0}(q_0,\bar{q}_0)$$
$$=\sum_{n,m}\lambda^{n+m}\sum_{\{l_i,l'_j\}_0}\sum_{\{q_i,q'_j\}_0}\psi_0^*(l'_m,q'_m)\psi_0(l_n,q_n)K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_i}\})$$
$$\times\sum_{\pi(n,m)\in\mathcal{G}(n,m)}C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\})$$
$$\times\sum_{\bar{n},\bar{m}}\lambda^{\bar{n}+\bar{m}}\sum_{\{\bar{l}_i,\bar{l}'_j\}_0}\sum_{\{\bar{q}_i,\bar{q}'_j\}_0}\psi_0^*(\bar{l}'_{\bar{m}},\bar{q}'_{\bar{m}})\psi_0(\bar{l}_{\bar{n}},\bar{q}_{\bar{n}})K^{\bar{n}}(t,\{E_{\bar{l}_i}\})\bar{K}^{\bar{m}}(t,\{E_{\bar{l}'_i}\})$$
$$\times\sum_{\pi(\bar{n},\bar{m})\in\mathcal{G}(\bar{n},\bar{m})}C_\pi(\bar{n},\bar{m},\{\bar{l}_i,\bar{l}'_j\},\{\bar{q}_i,\bar{q}'_j\})$$
$$=\mathbb{E}\left[P_t\left(l_0,q_0\right)\right]\mathbb{E}\left[P_t\left(\bar{l}_0,\bar{q}_0\right)\right] \tag{3.37}$$

Combining Eq. (3.37) and (3.35) we get

$$\mathbf{O}^{l_0,\bar{l}_0}(q_0,\bar{q}_0) = \mathbb{E}\left[P_t\left(l_0,q_0\right)\right]\mathbb{E}\left[P_t\left(\bar{l}_0,\bar{q}_0\right)\right] + \mathbf{O}_{ns}^{l_0,\bar{l}_0}(q_0,\bar{q}_0) \tag{3.38}$$

Inserting Eq.(3.38) in Eq.(3.30), will thus lead to

$$\mathbb{E}\left[\left(\langle\hat{O}\rangle_t^M\right)^2\right] = \mathbb{E}\left[\langle\hat{O}\rangle_t^M\right]\mathbb{E}\left[\langle\hat{O}\rangle_t^M\right] + \sum_{l_0,q_0,\bar{l}_0,\bar{q}_0} O(l_0,q_0)O(\bar{l}_0,\bar{q}_0)\mathbf{O}_{ns}^{l_0,\bar{l}_0}(q_0,\bar{q}_0) \tag{3.39}$$

Therefore we have

$$\begin{aligned}\mathrm{Var}_N^M\left[O(t,\lambda)\right] = &\sum_{n,m,\bar{n},\bar{m}=0}^{M-1} \lambda^{n+m+\bar{n}+\bar{m}} \sum_{\{l_i,l_j',\bar{l}_i,\bar{l}_j'\}}\sum_{\{q_i,q_j',\bar{q}_i,\bar{q}_j'\}} \psi_0^*(l_m',q_m')\psi_0(l_n,q_n)\\ &\times\psi_0^*(\bar{l}_{\bar{m}}',\bar{q}_{\bar{m}}')\psi_0(\bar{l}_{\bar{n}},\bar{q}_{\bar{n}})K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l_i'}\})K^{\bar{n}}(t,\{E_{\bar{l}_i}\})\bar{K}^{\bar{m}}(t,\{E_{\bar{l}_i'}\})\\ &\times O(l_0,q_0)O(\bar{l}_0,\bar{q}_0)\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_{non-s}(n,m,\bar{n},\bar{m})} \mathcal{C}_\pi(n,m,\bar{n},\bar{m},\{l_i,l_j',\bar{l}_i,\bar{l}_j'\},\{q_i,q_j',\bar{q}_i,\bar{q}_j'\})\end{aligned} \tag{3.40}$$

Using the fact that the observable is bounded and Eq. (C.5), we have

$$\begin{aligned}\left|\mathrm{Var}_N^M\left[O(t,\lambda)\right]\right| \leq &\sum_{n,m,\bar{n},\bar{m}} \lambda^{n+m+\bar{n}+\bar{m}} \sum_{\{l_i,l_j',\bar{l}_i,\bar{l}_j'\}}\sum_{\{q_i,q_j',\bar{q}_i,\bar{q}_j'\}} \kappa(t)\frac{C}{N^2}\\ &\times\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_{non-s}(n,m,\bar{n},\bar{m})} \left|\mathcal{C}_\pi(n,m,\bar{n},\bar{m},\{l_i,l_j',\bar{l}_i,\bar{l}_j'\},\{q_i,q_j',\bar{q}_i,\bar{q}_j'\})\right|\end{aligned} \tag{3.41}$$

with C a constant depending on the observable and

$$\begin{aligned}\kappa(t) &= \left|K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l_i'}\})K^{\bar{n}}(t,\{E_{\bar{l}_i}\})\bar{K}^{\bar{m}}(t,\{E_{\bar{l}_i'}\})\right|\\ &\leq \frac{t^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!}\end{aligned} \tag{3.42}$$

There are $\frac{n+m+\bar{n}+\bar{m}}{2}$ contractions to be performed and thus a weighing factor smaller than $\left(\frac{W}{N\aleph}\right)^{\frac{n+m+\bar{n}+\bar{m}}{2}}$, where W is the maximum value of the transition amplitudes squared (see Eq. (2.7)). We have then

$$\begin{aligned}\left|\mathrm{Var}_N^M\left[O(t,\lambda)\right]\right| \leq &\sum_{n,m,\bar{n},\bar{m}} \lambda^{n+m+\bar{n}+\bar{m}}\kappa(t)C\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_{non-s}(n,m,\bar{n},\bar{m})}\\ &\times\sum_{\{l_i,l_j',\bar{l}_i,\bar{l}_j'\}_\mathrm{i}}\sum_{\{q_i,q_j',\bar{q}_i,\bar{q}_j'\}_\mathrm{i}} \left(\frac{W}{N\aleph}\right)^{\frac{n+m+\bar{n}+\bar{m}}{2}}\frac{1}{N^2}\end{aligned} \tag{3.43}$$

According to lemma B.0.15 and B.0.18 Non-separable graphs will produce on the set $I = L_0 \cup L_1$, with $L_0 = \{l_i, l'_j\}$ and $L_1 = \{\bar{l}_i, \bar{l}'_j\}$, a number of independent variables M_I, with $M_I \leq \frac{n+m+\bar{n}+\bar{m}}{2} + 1$. In addition the equality sign will only hold for NC-graphs. The same holds for $L_0 = \{q'_m, \dots q_n\}$ and $L_1 = \{\bar{q}'_m, \dots \bar{q}_n\}$. Then for NC-graphs

$$\sum_{\{l_i, l'_j, \bar{l}_i, \bar{l}'_j\}_{\mathrm{i}}} 1 = N^{\frac{n+m+\bar{n}+\bar{m}}{2}+1}$$
$$\sum_{\{q_i, q'_j, \bar{q}_i, \bar{q}'_j\}_{\mathrm{i}}} 1 = \aleph^{\frac{n+m+\bar{n}+\bar{m}}{2}+1}$$

and for C-graphs

$$\sum_{\{l_i, l'_j, \bar{l}_i, \bar{l}'_j\}_{\mathrm{i}}} 1 \leq N^{\frac{n+m+\bar{n}+\bar{m}}{2}}$$
$$\sum_{\{q_i, q'_j, \bar{q}_i, \bar{q}'_j\}_{\mathrm{i}}} 1 \leq \aleph^{\frac{n+m+\bar{n}+\bar{m}}{2}}$$

We denote by $\mathcal{G}^{0,1}_{non-s}$ the set of non-separable graphs that are also NC-graphs and by $\mathcal{G}^{2}_{non-s}$ the set of non-separable graphs that are also C-graphs. Inserting this in Eq. (3.43) we get:

$$\begin{aligned} \mathrm{Var}^M_N\left[O(t,\lambda)\right] \leq & \frac{C\,\aleph}{N} \sum_{n,m,\bar{n},\bar{m}=0}^{M-1} \frac{(t\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!2^{\frac{n+m+\bar{n}+\bar{m}}{2}+2}} \sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{0,1}_{non-s}(n,m,\bar{n},\bar{m})} 1 \\ & + \frac{C}{N^2} \sum_{n,m,\bar{n},\bar{m}=0}^{M-1} \frac{(t\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!2^{\frac{n+m+\bar{n}+\bar{m}}{2}+2}} \sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{2}_{non-s}(n,m,\bar{n},\bar{m})} 1 \end{aligned} \quad (3.44)$$

Since the sum over the graphs is bounded by some function of M we conclude that for any fixed t_0, λ, M and small ε there exists a N_0 such that for $N \geq N_0$ and $t \leq t_0$ we have $Var^M_N\left[O(t,\lambda)\right] \leq \varepsilon$. □

So far we have proved that the average of the approximate solution approximates well the average of the real solution up to some time and we have also proved that the average of the approximate solution is a typical outcome for any realization. Now we move on to proving that the average of the solution is a typical outcome for any realization.

Theorem 3.2.3. *For any t_0, λ and $0 < \varepsilon$ and a Macro-Observable $\hat{O}$ there exists an N_0 and an M such that if $N \geq N_0$ and $t \leq t_0$ we have*

$$Var_N\left[O(t,\lambda)\right] \leq \varepsilon \quad (3.45)$$

with

$$Var_N\left[O(t,\lambda)\right] = \mathbb{E}\left[\langle\psi_t|O|\psi_t\rangle^2\right] - \mathbb{E}\left[\langle\psi_t|O|\psi_t\rangle\right]^2 \quad (3.46)$$

We can express this variance as a function of $|\phi_t^M\rangle$ and $|\Psi_t^M\rangle$. Through theorem 3.1.2, and by taking $\hat{O}$ as the identity, we see that the norm of $|\phi_t^M\rangle$ tends to zero. Thus the norm of $|\Psi_t^M\rangle$ tends to 1 and every average, where the vector $|\phi_t^M\rangle$ is involved, should go to zero. Our strategy will then be to first bound the expressions involving $|\phi_t^M\rangle$ by averages over the norm of this vector or powers of it.
For this we introduce here the following short hand notation:

$$\begin{aligned} X &= \langle\psi_t|\hat{O}|\psi_t\rangle \\ Y &= \langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle \\ Z &= \langle\phi_t^M|\hat{O}|\phi_t^M\rangle \\ R &= \langle\Psi_t^M|\hat{O}|\phi_t^M\rangle \end{aligned}$$

In this notation our previous results are

$$\begin{aligned} \mathbb{E}[Z] &\xrightarrow{M\to\infty} 0 \\ \mathbb{E}[X-Y] &\xrightarrow{M\to\infty} 0 \\ Var_N[Y] &\xrightarrow{N\to\infty} 0 \end{aligned}$$

Proof. Theorem 3.2.3
We want to calculate $Var[X]_N$, which, can be written as:

$$\begin{aligned} Var[X]_N =& \mathbb{E}\left[\left(Y+Z+R+\bar{R}\right)^2\right] - \mathbb{E}\left[\left(Y+Z+R+\bar{R}\right)\right]^2 \\ =& \mathbb{E}\left[Y^2+Z^2+R^2+\bar{R}^2+2\left(YZ+YR+Y\bar{R}+ZR+Z\bar{R}+R\bar{R}\right)\right] \\ &- \left(\mathbb{E}[Y]^2+\mathbb{E}[Z]^2+\mathbb{E}[R]^2+\mathbb{E}\left[\bar{R}\right]^2\right) \\ -2\Big(\mathbb{E}[Y]\mathbb{E}[Z]&+\mathbb{E}[Y]\mathbb{E}[R]+\mathbb{E}[Y]\mathbb{E}\left[\bar{R}\right]+\mathbb{E}[Z]\mathbb{E}[R]+\mathbb{E}[Z]\mathbb{E}\left[\bar{R}\right]+\mathbb{E}[R]\mathbb{E}\left[\bar{R}\right]\Big) \end{aligned} \tag{3.47}$$

The only term that does not involve $|\phi_t^M\rangle$ is $\mathbb{E}\left[Y^2\right]-\mathbb{E}[Y]^2$. We can pair up the terms as follows:

$$\begin{aligned} Var_N[X] =& Var_N[Y]+Var_N[Z]+Var_N[R]+Var_N\left[\bar{R}\right] \\ &+2\left(\mathbb{E}[(Y-\mathbb{E}[Y])Z]+\mathbb{E}[(Y-\mathbb{E}[Y])R]+\mathbb{E}\left[(Y-\mathbb{E}[Y])\bar{R}\right]\right) \\ &+2\left(\mathbb{E}[(Z-\mathbb{E}[Z])R]+\mathbb{E}\left[(Z-\mathbb{E}[Z])\bar{R}\right]+\mathbb{E}\left[(R-\mathbb{E}[R])\bar{R}\right]\right) \end{aligned} \tag{3.48}$$

Hölder's inequality states for probability spaces:

$$\mathbb{E}[|AB|] \leq \mathbb{E}\left[|A|^2\right]^{\frac{1}{2}}\mathbb{E}\left[|B|^2\right]^{\frac{1}{2}} \tag{3.49}$$

Applying this here we have:

$$
\begin{aligned}
|Var_N[X]| \leq & Var_N[Y] + Var_N[Z] + Var_N[|R|] + Var_N\left[\left|\bar{R}\right|\right] \\
& +2\left(Var_N[Y]^{\frac{1}{2}}\mathbb{E}\left[Z^2\right]^{\frac{1}{2}} + Var_N[Y]^{\frac{1}{2}}\mathbb{E}\left[|R|^2\right]^{\frac{1}{2}} + Var_N[Y]^{\frac{1}{2}}\mathbb{E}\left[\left|\bar{R}\right|^2\right]^{\frac{1}{2}}\right) \\
& +2\left(Var_N[Z]^{\frac{1}{2}}\mathbb{E}\left[|R|^2\right]^{\frac{1}{2}} + Var_N[Z]^{\frac{1}{2}}\mathbb{E}\left[\left|\bar{R}\right|^2\right]^{\frac{1}{2}} + Var_N[|R|]^{\frac{1}{2}}\mathbb{E}\left[\left|\bar{R}\right|^2\right]^{\frac{1}{2}}\right)
\end{aligned}
\tag{3.50}
$$

Now Z is a positive variable by definition. In addition we showed that its average goes to zero. If a positive random variable has average zero then it is likely that the square of such a variable will also be zero, unless we consider some pathological case. This would mean of course that the variance also goes to zero. For now we will suppose this to be true. It will be proved in section 3.3. All the terms proportional to $Var_N[Z]$ or $\mathbb{E}\left[Z^2\right]$ will thus go to zero. The only terms remaining are then those involving only R and $\bar{R}$.
We will now show that $\mathbb{E}\left[|R|^2\right]$ is bounded by $\mathbb{E}\left[Z^2\right]^{\frac{1}{2}}$ and that $\mathbb{E}[|R|]$ is bounded by $\mathbb{E}[Z]^{\frac{1}{2}}$. These would tend to zero. The same conclusions can be applied, of course, if we replace R by $\bar{R}$. For $\mathbb{E}\left[|R|^2\right]$ we have:

$$
\begin{aligned}
\mathbb{E}\left[\left|R^2\right|\right] =& \mathbb{E}\left[\left|\langle\Psi_t^M|\hat{O}|\phi_t^M\rangle\right|^2\right] \\
\leq& \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\langle\phi_t^M|\phi_t^M\rangle\right] \\
\leq& \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle^2\right]^{\frac{1}{2}}\mathbb{E}\left[\langle\phi_t^M|\phi_t^M\rangle^2\right]^{\frac{1}{2}} \\
\leq& C\mathbb{E}\left[Y^2\right]^{\frac{1}{2}}\mathbb{E}\left[Z^2\right]^{\frac{1}{2}}
\end{aligned}
\tag{3.51}
$$

We applied the Hölder inequality to pass from the second to the third line and C is some constant. Analoguesly we have for $\mathbb{E}[|R|]$:

$$
\begin{aligned}
\mathbb{E}[|R|]^2 =& \mathbb{E}\left[\left|\langle\Psi_t^M|\hat{O}|\phi_t^M\rangle\right|\right]^2 \\
\leq& \mathbb{E}\left[\sqrt{\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\langle\phi_t^M|\phi_t^M\rangle}\right]^2 \\
\leq& \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right]\mathbb{E}\left[\langle\phi_t^M|\phi_t^M\rangle\right] \\
\leq& C\mathbb{E}[Y]\mathbb{E}[Z]
\end{aligned}
\tag{3.52}
$$

Using inequalities (3.51) and (3.52) in Eq.(3.50) we have :

$$
\begin{aligned}
|Var_N[X]| \leq\ & Var_N[Y] + Var_N[Z] + C\left(\mathbb{E}[Y]\mathbb{E}[Z] + \mathbb{E}\left[Y^2\right]^{\frac{1}{2}}\mathbb{E}\left[Z^2\right]^{\frac{1}{2}}\right) \\
& +2\left(Var_N[Y]^{\frac{1}{2}}\mathbb{E}\left[Z^2\right]^{\frac{1}{2}} + Var_N[Y]^{\frac{1}{2}}\left(\mathbb{E}\left[Y^2\right]^{\frac{1}{4}}\mathbb{E}\left[Z^2\right]^{\frac{1}{4}}\right)\right) \\
& +2\,Var_N[Z]^{\frac{1}{2}}\left(\mathbb{E}\left[Y^2\right]^{\frac{1}{4}}\mathbb{E}\left[Z^2\right]^{\frac{1}{4}}\right) \\
& +2\left(\mathbb{E}[Y]^{\frac{1}{2}}\mathbb{E}[Z]^{\frac{1}{2}} + \mathbb{E}\left[Y^2\right]^{\frac{1}{4}}\mathbb{E}\left[Z^2\right]^{\frac{1}{4}}\right)\mathbb{E}\left[Y^2\right]^{\frac{1}{4}}\mathbb{E}\left[Z^2\right]^{\frac{1}{4}}
\end{aligned}
\tag{3.53}
$$

According to theorems 3.2.1, 3.1.2 and 3.3.1, there exists for any t_0, λ and $0 < \varepsilon$ and a Macro-Observable $\hat{O}$ there exists an N_0 and an M such that if $N \geq N_0$ and $t \leq t_0$

$$
\begin{aligned}
Var_N[Y] &\leq \varepsilon \\
\mathbb{E}[Z] &\leq \varepsilon \\
\mathbb{E}\left[Z^2\right] &\leq \varepsilon
\end{aligned}
$$

Therefore for any t_0, λ and $0 < \varepsilon$ and a Macro-Observable $\hat{O}$ there exists an N_0 and an M such that if $N \geq N_0$ and $t \leq t_0$

$$|Var_N[X]| \leq C_1\varepsilon + C_2\varepsilon^{\frac{3}{4}} + C_3\varepsilon^{\frac{1}{2}} + C_4\varepsilon^{\frac{1}{4}}$$

Thus

$$\boxed{\lim_{M\to\infty}\lim_{N\to\infty}|Var_N[X]| = 0} \tag{3.54}$$

□

3.3 Bound for $\mathbb{E}\left[Z^2\right]$

In this section we will show that $\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle^2\right]$ also goes to zero as N and M go to ∞. Our theorem is the following:

Theorem 3.3.1. *For any t_0, λ and $0 < \varepsilon$ and a Macro-Observable $\hat{O}$ there exists an N_0 and an M such that if $N \geq N_0$ and $t \leq t_0$ we have*

$$\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle^2\right] < \varepsilon \tag{3.55}$$

Proof. Theorem 3.3.1 For this we can use Eq. (3.9) and (3.10). We have then:

$$\langle\phi_t^M|\hat{O}|\phi_t^M\rangle^2 \leq O_{max}^2 t^2\lambda^{4M}\int_0^t [ds_M]\int_0^t [d\tau_M]\,O(M,\tau_i,s_j)\int_0^t [d\bar{s}_M]\int_0^t [d\bar{\tau}_M]\,O(M,\bar{\tau}_i,\bar{s}_j) \tag{3.56}$$

with

$$\begin{aligned}
&\mathbb{E}\left[O(M,\tau_i,s_j)O(M,\bar{\tau}_i,\bar{s}_j)\right] = \sum_{\{l_j',l_i\}}\sum_{\{q_j',q_i\}}\sum_{\{\bar{l}_j',\bar{l}_i\}}\sum_{\{\bar{q}_j',\bar{q}_i\}}\\
&\times\psi_0^*(l_M',q_M')\psi_0(l_M,q_M)\psi_0^*(\bar{l}_M',\bar{q}_M')\psi_0(\bar{l}_M,\bar{q}_M)\\
&\times\left(\prod_{j=0}^{M-1} e^{iE_{l_j'}\tau_j}\right)\left(\prod_{j=0}^{M-1} e^{-iE_{l_j}s_j}\right)\left(\prod_{j=0}^{M-1} e^{iE_{\bar{l}_j'}\bar{\tau}_j}\right)\left(\prod_{j=0}^{M-1} e^{-iE_{\bar{l}_j}\bar{s}_j}\right)\\
&\times\mathbb{E}\Big[\langle l_M',q_M'|V|l_{M-1}',q_{M-1}'\rangle\ldots\langle l_1',q_1'|V|l_0',q_0'\rangle\\
&\times\langle l_0',q_0'|l_0,q_0\rangle\langle l_0,q_0|V|l_1,q_1\rangle\ldots\langle l_{M-1},q_{M-1}|V|l_M,q_M\rangle\\
&\times\langle\bar{l}_M',\bar{q}_M'|V|\bar{l}_{M-1}',\bar{q}_{M-1}'\rangle\ldots\langle\bar{l}_1',\bar{q}_1'|V|\bar{l}_0',\bar{q}_0'\rangle\\
&\times\langle\bar{l}_0',\bar{q}_0'|\bar{l}_0,\bar{q}_0\rangle\langle\bar{l}_0,\bar{q}_0|V|\bar{l}_1,\bar{q}_1\rangle\ldots\langle\bar{l}_{M-1},\bar{q}_{M-1}|V|\bar{l}_M,\bar{q}_M\rangle\Big]
\end{aligned} \tag{3.57}$$

and so by Eq. (C.5)

$$\begin{aligned}
&|\mathbb{E}\left[O(M,\tau_i,s_j)O(M,\bar{\tau}_i,\bar{s}_j)\right]| \leq \sum_{\{l_j',l_i\}}\sum_{\{q_j',q_i\}}\sum_{\{\bar{l}_j',\bar{l}_i\}}\sum_{\{\bar{q}_j',\bar{q}_i\}}\frac{1}{N^2}\\
&\times\Big|\mathbb{E}\Big[\langle l_M',q_M'|V|l_{M-1}',q_{M-1}'\rangle\ldots\langle l_1',q_1'|V|l_0',q_0'\rangle\\
&\times\langle l_0',q_0'|l_0,q_0\rangle\langle l_0,q_0|V|l_1,q_1\rangle\ldots\langle l_{M-1},q_{M-1}|V|l_M,q_M\rangle\\
&\times\langle\bar{l}_M',\bar{q}_M'|V|\bar{l}_{M-1}',\bar{q}_{M-1}'\rangle\ldots\langle\bar{l}_1',\bar{q}_1'|V|\bar{l}_0',\bar{q}_0'\rangle\\
&\times\langle\bar{l}_0',\bar{q}_0'|\bar{l}_0,\bar{q}_0\rangle\langle\bar{l}_0,\bar{q}_0|V|\bar{l}_1,\bar{q}_1\rangle\ldots\langle\bar{l}_{M-1},\bar{q}_{M-1}|V|\bar{l}_M,\bar{q}_M\rangle\Big]\Big|
\end{aligned} \tag{3.58}$$

Once again this average can be split into C-graphs and NC-graphs. The NC-graphs have $2M+2$ independent variables while the C-graphs have necessarily less then $2M+2$ independent variables. Since there are $4M$ random variables, there are $2M$ pairings and so the weight is inversely proportional to $(N\aleph)^{2M}$. Since the transition elements are bounded the total weight of any graphs is bounded by $\left(\frac{W}{N\aleph}\right)^{2M}$. Thus we have:

$$\begin{aligned}
&\mathbb{E}\left[\bar{L}^M(\{l_j'\},\{q_j'\})L^M(\{l_j\},\{q_j\})\bar{L}^M(\{\bar{l}_j'\},\{\bar{q}_j'\})L^M(\{\bar{l}_j\},\{\bar{q}_j\})\right]\\
&=\sum_{\pi(M,M,M,M)\in\mathcal{G}_{0,1}(M,M,M,M)} C_\pi(M,M,M,M,\{l_j',l_i,\bar{l}_j',\bar{l}_i\},\{q_j',q_i,\bar{q}_j',\bar{q}_i\})\\
&+\sum_{\pi(M,M,M,M)\in\mathcal{G}_2(M,M,M,M)} C_\pi(M,M,M,M,\{l_j',l_i,\bar{l}_j',\bar{l}_i\},\{q_j',q_i,\bar{q}_j',\bar{q}_i\})
\end{aligned} \tag{3.59}$$

If $\pi(M,M,M,M)$ is a NC-graph and thus belongs to $\mathcal{G}_{0,1}(M,M,M,M)$, we have:

$$\sum_{\{l_j',l_i\}}\sum_{\{q_j',q_i\}}\sum_{\{\bar{l}_j',\bar{l}_i\}}\sum_{\{\bar{q}_j',\bar{q}_i\}} \mathcal{C}_\pi(M,M,M,M,\{l_j',l_i,\bar{l}_j',\bar{l}_i\},\{q_j',q_i,\bar{q}_j',\bar{q}_i\})$$
$$\leq W^{2M}N^2\aleph^3 \tag{3.60}$$

But if $\pi(M,M,M,M)$ is a C-graph and thus belongs to $\mathcal{G}_2(M,M,M,M)$, we have:

$$\sum_{\{l_j',l_i\}}\sum_{\{q_j',q_i\}}\sum_{\{\bar{l}_j',\bar{l}_i\}}\sum_{\{\bar{q}_j',\bar{q}_i\}} \mathcal{C}_\pi(M,M,M,M,\{l_j',l_i,\bar{l}_j',\bar{l}_i\},\{q_j',q_i,\bar{q}_j',\bar{q}_i\})$$
$$\leq W^{2M}N\aleph^3 \tag{3.61}$$

Inserting Eq. (3.59) in Eq.(3.58) and using the last two identities we arrive at

$$|\mathbb{E}[O(M,\tau_i,s_j)O(M,\bar{\tau}_i,\bar{s}_j)]| \leq W^{2M}\aleph^3 \sum_{\pi(M,M,M,M)\in\mathcal{G}_{0,1}(M,M,M,M)} 1$$
$$+\frac{W^{2M}\aleph^3}{N}\sum_{\pi(M,M,M,M)\in\mathcal{G}_2(M,M,M,M)} 1 \tag{3.62}$$

Inserting this in Eq.(3.56) we have

$$\left|\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle^2\right]\right| \leq O_{max}^2 t^2\lambda^{4M}\int_0^t[ds_M]\int_0^t[d\tau_M]\int_0^t[d\bar{s}_M]\int_0^t[d\bar{\tau}_M]$$
$$\left(W^{2M}\aleph^3\sum_{\pi(M,M,M,M)\in\mathcal{G}_{0,1}(M,M,M,M)}1+\frac{W^{2M}\aleph^3}{N}\sum_{\pi(M,M,M,M)\in\mathcal{G}_2(M,M,M,M)}1\right)$$
$$\leq O_{max}^2 t^2\lambda^{4M}\frac{t^{4M}}{M!^4}W^{2M}\aleph^3\left(\frac{(4M)!}{(2M)!(2M+1)!}+\frac{1}{N}G_2(4M)\right) \tag{3.63}$$

with

$$\sum_{\pi(M,M,M,M)\in\mathcal{G}_2(M,M,M,M)} 1 = G_2(4M) \tag{3.64}$$

Thus for any t_0, λ and ε there exists an M and N_0 such when $t<t_0$ and $N>N_0$ we have $\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle^2\right]<\varepsilon$.
Thus

$$\boxed{\lim_{M\to\infty}\lim_{N\to\infty}\left|\mathbb{E}\left[\langle\phi_t^M|\hat{O}|\phi_t^M\rangle^2\right]\right| = 0} \tag{3.65}$$

□

4 Examples

4.1 An illustration of theorem 3.2.3

Basically theorem 3.2.3 tells us that for any time, t_0, there exists a system size, N_0, for which the behavior is typical up to that time. The larger t_0 is the larger N_0 will have to be. However, it does not give any information about what the relationship might be between N_0 and t_0. In this section we want to show that the time t_0 up to which the variance is small increases at least logarithmicly with the size of the Hilbert space.
If, according to theorem 3.1.1, we have a system with Hilbert space size larger than N_0 the average of the time evolution of an observable, $\mathbb{E}\left[\langle\psi_t|\hat{O}|\psi_t\rangle\right]$, can be well approximated by the average using our approximate solution, $\mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right]$, until time t_0. Now, according to theorem 3.2.1 the average using the approximated solution faithfully represents the dynamics of any realization until up to a time t_0'. That is $\mathrm{Var}_{N_0}^M\left[\hat{O}\right] < \varepsilon$. Thus until $t = \min\{t_0, t_0'\}$ we can say the average faithfully represents the dynamics. We now examine how t_0 and t_0' behave by varying N_0 by using some very rough bounds just to illustrate their meaning. We demand that $B_1(M, \aleph, t_0\lambda)$ and $B_2(M, N_0, t_0\lambda)$ be less then ε, with ε small. We have then from Eqs. (3.17) and (3.18)

$$2M\log(\lambda t_0) + \log(t_0) \leq \log\left(\frac{\varepsilon M!^3(M+1)!}{\aleph(2M)!}\right) \tag{4.1}$$

$$2M\log(\lambda t_0) + \log(t_0) \leq \log\left(\frac{\varepsilon M!^3 2^M N_0}{(2M)!}\right) \tag{4.2}$$

We suppose M to be very large so that we can apply Stirling's approximations. In doing so we obtain

$$\log(\lambda t_0) + \log(t_0) \leq \frac{\log\varepsilon - \log\aleph}{2M} - \log 2 - 2 + \log M \tag{4.3}$$

$$\log(\lambda t_0) + \log(t_0) \leq \frac{\log\varepsilon}{2M} + \frac{\log N_0}{2M} + \frac{\log M}{2} \tag{4.4}$$

We now look at the variance. We demand

$$\frac{C\aleph}{N_0}\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\frac{(t_0'\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!2^{\frac{n+m+\bar{n}+\bar{m}}{2}+2}}\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{0,1}_{non-s}(n,m,\bar{n},\bar{m})} 1 \leq \varepsilon \tag{4.5}$$

$$\frac{C}{N_0^2}\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\frac{(t_0'\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!2^{\frac{n+m+\bar{n}+\bar{m}}{2}+2}}\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{2}_{non-s}(n,m,\bar{n},\bar{m})} 1 \leq \varepsilon \tag{4.6}$$

Since the number of non-separable graphs that are NC-graphs is less then the number of NC-graphs we have by Eq. (B.16)

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{0,1}_{non-s}(n,m,\bar{n},\bar{m})} 1 \leq \frac{(n+m+\bar{n}+\bar{m})!}{\left(\frac{n+m+\bar{n}+\bar{m}}{2}\right)!\left(\frac{n+m+\bar{n}+\bar{m}}{2}+1\right)!} \tag{4.7}$$

and since the number of non-separable graphs that are also C-graphs is less then the number of graphs we have by Eq. (B.15)

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{2}_{non-s}(n,m,\bar{n},\bar{m})} 1 \leq \frac{(n+m+\bar{n}+\bar{m})!}{\left(\frac{n+m+\bar{n}+\bar{m}}{2}\right)!2^{\frac{n+m+\bar{n}+\bar{m}}{2}}} \tag{4.8}$$

Using the following inequality for factorials, where C is a constant,

$$(a+b)! \leq C^{a+b}a!b! \tag{4.9}$$

we get

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{0,1}_{non-s}(n,m,\bar{n},\bar{m})} 1 \leq C^{n+m+\bar{n}+\bar{m}} \tag{4.10}$$

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}^{2}_{non-s}(n,m,\bar{n},\bar{m})} 1 \leq C^{n+m+\bar{n}+\bar{m}}\left(\frac{n+m+\bar{n}+\bar{m}}{2}\right)! \tag{4.11}$$

By using the bound (4.10) and (4.11) in Eq. (4.5) and (4.6) we obtain

$$\frac{C\aleph}{N_0}\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\frac{(Ct_0'\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!2^{\frac{n+m+\bar{n}+\bar{m}}{2}+2}} \leq \varepsilon \tag{4.12}$$

$$\frac{C}{N_0^2}\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\frac{(Ct_0'\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!}\left(\frac{n+m+\bar{n}+\bar{m}}{2}\right)! \leq \varepsilon \tag{4.13}$$

By once again using bounds on the factorial in Eq. (4.13) we get

$$\frac{C\aleph}{N_0}\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\frac{(Ct_0'\lambda)^{n+m+\bar{n}+\bar{m}}}{n!m!\bar{n}!\bar{m}!2^{\frac{n+m+\bar{n}+\bar{m}}{2}+2}} \leq \frac{C\aleph}{N_0}e^{4Ct_0'\lambda} \leq \varepsilon \tag{4.14}$$

$$\frac{C}{N_0^2}\sum_{n,m,\bar{n},\bar{m}=0}^{M-1}\frac{(Ct_0'\lambda)^{n+m+\bar{n}+\bar{m}}}{\frac{n}{2}!\frac{m}{2}!\frac{\bar{n}}{2}!\frac{\bar{m}}{2}!} \leq \frac{C}{N_0^2}e^{4C(t_0'\lambda)^2} \leq \varepsilon \tag{4.15}$$

and so finally we get that if the following conditions are fulfilled

$$t_0'\lambda \leq C\left(\log(\varepsilon)+\log(N_0)\right) \tag{4.16}$$

$$t_0'\lambda \leq C\left(\log(\varepsilon)+2\log(N_0)\right)^{\frac{1}{2}} \tag{4.17}$$

then the variance will be smaller then ε. The bounds made here were very rough and that is why N_0 has to be extremely large compared to t_0'.

4.2 Example of a system exhibiting typicality

Our example is a bipartite system similar to the one used in [17], where one part is considered to be large, as an environment for example. We have then

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \mathrm{Dim}\,[\mathcal{H}_1] &= D_1 \\ \mathrm{Dim}\,[\mathcal{H}_2] &= D_2 \end{aligned}$$

with the index 2 referring to the large system and thus large Hilbert space.

$$\begin{aligned} H &= H_1 \otimes I_2 + I_1 \otimes H_2 \\ |n,m\rangle &= |E_n^1\rangle \otimes |E_m^2\rangle \\ H|n,m\rangle &= \left(E_n^1 + E_m^2\right)|n,m\rangle \end{aligned} \tag{4.18}$$

We couple these two systems randomly

$$\hat{V} = \sum_{n,m,n',m'} V_{(n,m),(n',m')}|n,m\rangle\langle n',m'| \tag{4.19}$$

which can be written down as

$$\hat{V} = \sum_{\bar{n},\bar{m}=1}^{D_1 \times D_2} V_{\bar{n},\bar{m}}|\bar{n}\rangle\langle\bar{m}| \tag{4.20}$$

We set $D_1 \times D_2 = D$. If system 2 is a reservoir then D_2 tends to be extremely large and so is D. D_2 plays the role of the N we used in previous sections and D_1 of $\aleph$. If we are interested in the dynamics of systems 1 such as the probability for certain levels to be populated this would correspond to the following observable

$$\hat{O} = \sum_{n=1}^{D_1} \sum_{m=1}^{D_2} E_n|n,m\rangle\langle n,m| \tag{4.21}$$

this corresponds to a Macro-observable and thus for finite time and coupling the system behaves typically. It is with this example that we best understand that the statement of typicality is not restricted to what one usually understands as Macro, for system 1 could be very small and just possess a couple of states. The fact that the environment is large makes the observable macro with respect to the total Hilbert space.

4.3 Relaxation of a 2-site model

Here we discuss a discrete one particle model where the particle can move on 2 sites (similar to [2]). Each site possesses N energy levels and the transition amplitudes are complex gaussian random variables. We have then the following Hamiltonian:

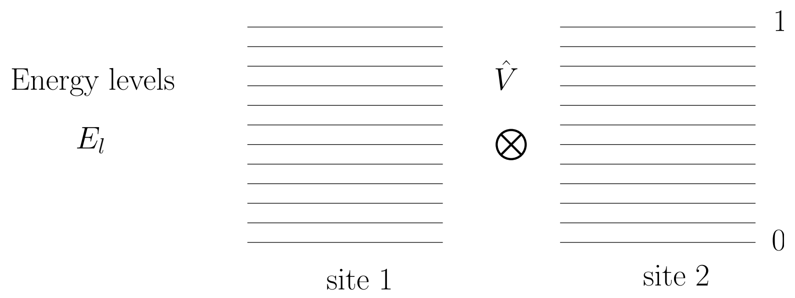

Figure 4.1: Sketch of the two site model

$$\hat{H} = \sum_{x=1}^{2}\sum_{l=1}^{N} E_l |x,l\rangle\langle x,l| + \lambda \sum_{x_1,x_2=1}^{2}\sum_{l_1,l_2=1}^{N} z_{x_1,x_2}(l_1,l_2)\, W_{x_1,x_2}(l_1,l_2)\, |x_1,l_1\rangle\langle x_2,l_2| \tag{4.22}$$

with

$$W_{x_1,x_2}(l_1,l_2) = (1-\delta_{x_1,x_2}) \tag{4.23}$$

If we take the spectrum of H_0 to be equidistant and bounded between 0 and 1, we have $E_l = l/N$. The density of states then becomes constant in the limit $N \to \infty$. In the limit $N \to \infty$ and in the Van Hove limit we get, according to (2.142) and (2.143), the following rate equations for the probability to be on site 1 and 2.

$$P_T(i) = \int_0^1 d\omega P_T(i,\omega) \tag{4.24}$$

$$\frac{d}{dT} P_T(1) = \pi\,(P_T(2) - P_T(1)) \tag{4.25}$$

$$\frac{d}{dT} P_T(2) = \pi\,(P_T(1) - P_T(2)) \tag{4.26}$$

4.4 Diffusion in a randomly coupled chain

The next model is an extension of our previous example. Instead of having just two sites we have a now a chain. Nevertheless, the interaction between each site is the same. This is a tight binding type of model with random coupling similar to the one used in [31]. Our Hamiltonian is then

$$\hat{H} = \sum_{x=-L}^{L} \sum_{l=1}^{N} E_l |x,l\rangle\langle x,l| + \lambda \sum_{x,y=-L}^{L} \sum_{l_1,l_2=1}^{N} z_{x,y}(l_1,l_2)\, W_{x,y}(l_1,l_2)\, |x,l_1\rangle\langle y,l_2| \tag{4.27}$$

Since we are taking only neighboring sites to be coupled we have

$$W_{x,y}(l_1,l_2) = \delta_{x,y-1} + \delta_{x,y+1} \tag{4.28}$$

We can then calculate the effective equation by use of Eq. (2.142) and (2.143). We have then

$$\frac{\partial}{\partial T} P_T(x,\omega) = \sum_y M_{x,y}(\omega) P_T(y,\omega) \tag{4.29}$$

By Eq. (4.28) and (2.143) we have

$$\begin{aligned} M_{x,y}(\omega) =& \pi g(\omega) \left(-\delta_{x,y} \sum_{z \neq x} |W_{x,z}(\omega,\omega)|^2 + |W_{x,y}(\omega,\omega)|^2 \right) \\ =& \pi g(\omega) \left(-2\delta_{x,y} + \delta_{x,y-1} + \delta_{x,y+1} \right) \end{aligned} \tag{4.30}$$

Inserting this in Eq. (4.29) we get

$$\boxed{\frac{\partial}{\partial T} P_T(x,\omega) = 2\pi g(\omega) \Delta_x P_T(x,\omega)} \tag{4.31}$$

which is the desired discrete diffusion equation. It may appear strange that a system, which has translation invariance, exhibits diffusion. But if one would investigate the actual consequences of the symmetry, one would see that the conserved quantity does not correspond to the probability current operator, despite some similarity. The current operator does not commute with the Hamiltonian either. We refer to [29] where it was noticed that such a kind of model exhibits two time regimes with transition from diffusive to ballistic type of behavior.

5 Supplementary Details

5.1 Linking size and time, under one scaling: $t = N^{\gamma}$

In chapter 2 we have derived the dynamics by taking the limit $N \to \infty$ and then the limit $t \to \infty$, with the Van Hove type of scaling between the coupling constant, λ, and time, t. This is a double scaling limit. Essentially this means that although both N and t tend to ∞, N does so before t, because we first take this limit. Inverting the order of the limits does not guaranty that the same equations come out. Implicitly in the order of the limits is hidden the fact that N is always infinitely larger then t. The main point might be that N needs not be infinitely larger than t but just larger than a certain amount.

In this chapter we will show exactly this. We will show that there exist scaling relations between N and t, such as $t = N^{\gamma}$, with $\gamma < 1$, which in the limit $N \to \infty$ have the same behavior as the Van Hove limit. Thus we are reducing a two parameter limit procedure to a one parameter procedure. Scaling t like this with N is, of course, equivalent setting $\lambda^2 = N^{-\gamma}$ in $t = \frac{T}{\lambda^2}$.

Deriving the dynamics with just one parameter, under some scaling, does not only make more precise mathematically, when the equations are fullfilled. Remember that N represents the size of the Hilbert space, which in turn is related to the system we are considering. The scaling used between N and t may thus turn into a relation between time and whatever physical quantities which affect the size of the Hilbert space, such as the physical length of the system or the number of particles.

Our main theorem in this section is the following:

Theorem 5.1.1. *If $|\psi_t\rangle$ is the solution to the Schrödinger equation with the Hamiltonian 2.4 and $|\Psi_t^M\rangle$ is the Duhamel expansion solution truncated at the M^{th} term, that is, according to Eq. (2.21),*

$$|\Psi_t^M\rangle = \sum_{n=0}^{M-1} |\psi_t^n\rangle \tag{5.1}$$

and $\hat{O}$ is a bounded observable, then

$$\lim_{t\to\infty}^{\lambda^2 t = T} \lim_{N\to\infty} \mathbb{E}\left[\langle \Psi_t^M | \hat{O} | \Psi_t^M \rangle\right] = \lim_{N\to\infty}^{\lambda^2 t = T, tN^{-\gamma}=1} \mathbb{E}\left[\langle \Psi_t^M | \hat{O} | \Psi_t^M \rangle\right] \tag{5.2}$$

For the proof we will consider that the spectrum of the deterministic Hamiltonian, $\hat{H}_0$, is equidistant.

We go one step back here from chapter 2, i.e. we consider once again the evolution of the system, before taking any limits, and analyze with the help of what we have learned previously. Our strategy is then as follows. We will have an expression for the contribution of the graphs but two different limiting procedures. We will find a bound for the difference between the contribution of a graph in one limit and the contribution of a graph in the other limit. The sum over this difference is such that it tends to zero in the limits. The notation developed in the beginning of section 2.6 does not make reference to C, N, or S-graphs. It only relabels the independent variables as a function of their multiplicity in the propagators. This relabeling of a graph $C_\pi(n,m)$ depending on n and m by $C_\pi(\bar{n},\{k_j,p_j\})$ depending on $\{\bar{n},\{k_j,p_j\}\}$ happens to be useful in order to count simple graphs and sum them up because a set of variables $\{\bar{n},\{k_j,p_j\}\}$ uniquely determines a S-graph. But this new labeling of a graph can also be used for C-graphs for that matter, even if it does not uniquely specify the C-graph. We use this notation then in order to write down the contribution of a graph before taking any limit. By slightly rearranging Eq. (2.54) we can write

$$\begin{aligned} P_{a,t}(q_0,l_0) = \sum_{m,n=0}^{M} \sum_{\pi(n,m)\in\mathcal{G}_a} \lambda^{n+m} \sum_{\{q_i,q'_j\}_0} \sum_{\{l_i,l'_j\}_0} \psi_0^*(q'_m,l'_m)\psi_0(q_n,l_n) \\ \times K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_j}\})\, C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\}) \end{aligned} \tag{5.3}$$

The expression form the second line of this last equation to the end of the third, were it not for the λ^{n+m}, is exactly our definition of $Q(C_\pi,\{q_i,q'_j\}_\text{i},q_0,\omega_0)$ in Eq. (2.74) for NC-graphs, before taking the limit $N\to\infty$. We take this and define it as $\mathcal{F}_\pi$, the contribution of a graph, any graph. We have then, similar to Eq. (2.75):

$$P_{a,t}(q_0,l_0) = \sum_{m,n} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\mathcal{A}^q_{\pi 0}} \mathcal{F}_\pi\left(n,m,N,t,\{q_i,q'_j\}_\text{i},E_{l_0}\right) \tag{5.4}$$

with

$$\begin{aligned} \mathcal{F}_\pi\left(n,m,N,t,\{q_i,q'_j\}_\text{i},E_{l_0}\right) = \lambda^{n+m} \sum_{B^q_\pi} \sum_{\{l_i,l'_j\}_0} \psi_0^*(q'_m,l'_m)\psi_0(q_n,l_n) \\ \times K^n(t,\{E_{l_i}\})\bar{K}^m(t,\{E_{l'_j}\})\, C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\}) \end{aligned}$$

We can now express the contribution of $\mathcal{F}_\pi$ in terms of independent and dependent variables, just as we did in section 2.6. Eq. (5.5) is then discrete version of Eq. (2.82) times λ^{n+m}, that is before taking the limit $N\to\infty$. As for $\mathcal{W}(\{\omega_j,\omega'_l\},\{q_i,q'_j\}_\text{i})$ it is the same as in Eq. (2.81) but without taking the limit $N \to \infty$. We have then for $\mathcal{F}_{\pi}$

$$\mathcal{F}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N, t, \{q_i, q_j'\}_{\mathrm{i}}, \omega_0) = \lambda^{n' + \Sigma_j (k_j + p_j) - 2} \int\int d\alpha d\beta e^{i(\alpha - \beta)t} e^{2\eta t}$$
$$\times \prod_{l=1}^{n'} \sum_{\omega_l'} \frac{1}{N} \frac{-1}{\omega_l' - \gamma_l + i\eta}$$
$$\times \sum_{\omega_1, \dots \omega_{\bar{n}}} \frac{1}{N} \psi_0^*(\omega_1) \psi_0(\omega_2) \prod_{j=0}^{\bar{n}} \left(\frac{-1}{\omega_j - \beta + i\eta} \right)^{p_j} \left(\frac{-1}{\omega_j - \alpha - i\eta} \right)^{k_j} \mathcal{W}(\{\omega_j, \omega_l'\}, \{q_i, q_j'\}_{\mathrm{i}})$$
$$\times N^{-\frac{\Sigma_j \left(k_j + p_j\right) - n' - 2(\bar{n}+1)}{2}} \tag{5.5}$$

with the following relations and definitions:

l^{th} independent variable of multiplicity $1 : \omega_l'$

j^{th} independent variable of multiplicity higher then $1 : \omega_j$

Number of independent variables of multiplicity $1 : n'$

Number of independent variables of multiplicity higher then $1 : \bar{n} + 1$

Multiplicity on the left of the j^{th} variable : p_j

Multiplicity on the left of the j^{th} variable : k_j

$$\text{Number of propagators} = n + m + 2 = n' + \sum_{j=0}^{\bar{n}} k_j + p_j \tag{5.6}$$

$$\text{Number of independent variables} = n' + \bar{n} + 1 \leq \frac{n+m}{2} + 1 \tag{5.7}$$

In this notation we have

$$P_{a,t}(q_0, \omega_0) = \sum_{n,m} \sum_{\pi(n,m) \in \mathcal{G}_a} \sum_{\{q_i, q_j'\}} \mathcal{F}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N, t, \{q_i, q_j'\}_{\mathrm{i}}, \omega_0) \tag{5.8}$$

The sum over $\mathcal{F}_{\pi}(n,m)$ implicitly means that the variables of $\mathcal{F}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N, t, \{q_i, q_j'\}_{\mathrm{i}}, \omega_0)$ comply with the relationships given above. It can be verified that when taking all of the N factor together, and by use of relation (5.6), the result is a factor of $\left(\frac{1}{N}\right)^{\frac{n+m}{2}}$. We also set

$$\lambda^2 = T/t \tag{5.9}$$
$$t = N^{\gamma} \tag{5.10}$$

Eq. (5.6) comes from the number of propagators of a graph of order (n,m). Eq. (5.7) states that there are less then $\frac{n+m}{2} + 1$ independent variables for C-graphs and exactly $\frac{n+m}{2} + 1$ independent variables for Non C-graphs. Eqs. (5.9) and (5.10) are just the scaling relations and we have not yet specified γ. The whole point of this chapter is actually to find out for which γ

our single parameter procedure is equivalent to the double one. We shall omit the dependency on $\{q_i, q_j'\}_i$ of $\mathcal{F}_\pi$ for brevity.
$\mathcal{F}_\pi$ is the contribution of one graph, once the values of the graph structure, such as n', $\bar{n}$ and $\{k_j, p_j\}$ are given. We will show that if a certain scaling relationship is imposed between t and N (which means also between λ and N because of the Van Hove limit), then the difference between the contributions of $\mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N, t, \omega_0)$ and of $\mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N, N^\gamma, \omega_0)$ in the $N \to \infty$ limit and the $t \to \infty$ limit tends to zero.We show this by finding a proper N dependent bound. We will do this in two steps. First we will bound the difference between $\mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N, N^\gamma, \omega_0)$ and a $\tilde{\mathcal{F}}_\pi$ still to be defined. Then we shall bound the difference between this $\tilde{\mathcal{F}}_\pi$ and $\mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N', t, \omega_0)$. Together these results will provide us with a bound for the difference between
$\mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N, N^\gamma, \omega_0)$ and $\mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N', t, \omega_0)$. We will prove then the following theorem

Theorem 5.1.2. *With the previous definition of $\mathcal{F}_\pi$ we have*

$$\left| \lim_{N\to\infty} \sum_{\omega_0} \frac{1}{N} \mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N, N^\gamma, \omega_0) - \lim_{t\to\infty} \lim_{N'\to\infty} \sum_{\omega_0} \frac{1}{N'} \mathcal{F}_\pi(n', \bar{n}, \{k_j, p_j\}, N', t, \omega_0) \right| = 0 \tag{5.11}$$

Proof. Theorem 5.1.2
As we said we will consider the case where the spectrum of $\hat{H}_0$ is equidistant and bounded between 0 and 1. This means that the sums over each ω_j in Eq. (5.5) is a sum over the set $\left[\frac{1}{N}, \frac{2}{N}, \ldots 1\right]$. We have then the following:

$$\begin{aligned}
&\left| \int d\omega \left(\frac{-1}{\omega - \alpha - i\eta} \right) - \sum_{\omega_l = N^{-1}}^{1} \frac{1}{N} \frac{-1}{\omega_l - \alpha - i\eta} \right| \\
=&\left| \sum_{\omega_l = \frac{1}{N}}^{1} \left(\int_{\Omega(\omega_l)} d\omega \frac{-1}{\omega - \alpha - i\eta} - \frac{1}{N} \frac{-1}{\omega_l - \alpha - i\eta} \right) \right| \\
=&\left| \sum_{\omega_l = \frac{1}{N}}^{1} \int_{\Omega(\omega_l)} d\omega \left(\frac{-1}{\omega - \alpha - i\eta} - \frac{1}{N\,|\Omega(\omega_l)|} \frac{-1}{\omega_l - \alpha - i\eta} \right) \right|
\end{aligned}$$

Here $|\Omega(\omega_l)|$ refers to the length of the interval $\Omega(\omega_l)$. The $\Omega(\omega_l)$'s is an equidistant partition of Ω, the range of integration of the variable ω (in our case $[0, 1]$). As such each interval $\Omega(\omega_l)$ contains one eigenvalue ω_l of H_0.

$$\Omega(\omega_l) = \left(\frac{l-1}{N}, \frac{l}{N} \right] \tag{5.12}$$

$$|\Omega(\omega_l)| = N^{-1} \tag{5.13}$$

We have then

$$
\begin{aligned}
&\left|\int d\omega\left(\frac{-1}{\omega-\alpha-i\eta}\right)-\sum_{\omega_l=N^{-1}}^{1}\frac{1}{N}\frac{-1}{\omega_l-\alpha-i\eta}\right| \\
\leq &\sum_{\omega_l=N^{-1}}^{1}\int_{\Omega(\omega_l)} d\omega\left|\frac{N\left|\Omega(\omega_l)\right|(\omega_l-\alpha-i\eta)-(\omega-\alpha-i\eta)}{(\omega-\alpha-i\eta)N\left|\Omega(\omega_l)\right|(\omega_l-\alpha-i\eta)}\right| \\
\leq &\sum_{\omega_l=N^{-1}}^{1}\int_{\Omega(\omega_l)} d\Delta\omega\left|\frac{(1-N\left|\Omega(\omega_l)\right|)(\omega_l-\alpha-i\eta)-\Delta\omega}{(\omega-\alpha-i\eta)N\left|\Omega(\omega_l)\right|(\omega_l-\alpha-i\eta)}\right|
\end{aligned}
\tag{5.14}
$$

with $\omega = \omega_l + \Delta\omega$. We have then $\Delta\omega \leq N^{-1}$ inside each interval. If there is a scaling relation between t and N we also have one between η and N (remember η was set to t^{-1}), namely $\eta \sim N^{-\gamma}$. The denominator is then bigger then $\eta^2 = N^{-2\gamma}$. The first term in the numerator of Eq. (5.14) is zero because $N\left|\Omega(\omega_l)\right| = 1$ by Eq. (5.13). By bounding the rest of the numerator by its maximum we get

$$
\begin{aligned}
(5.14) \leq &\sum_{\omega_l=N^{-1}}^{1}\int_{\Omega(\omega_l)} d\Delta\omega\left(N^{2\gamma-1}\right) \\
\leq &N^{2\gamma-1}
\end{aligned}
\tag{5.15}
$$

Thus:

$$
\sum_{\omega_l=N^{-1}}^{1}\frac{1}{N}\frac{-1}{\omega_l-\alpha-i\eta} = \int d\omega\left(\frac{-1}{\omega-\alpha-i\eta}\right)+O(N^{2\gamma-1})
\tag{5.16}
$$

From this bound we can also deduce

$$
\sum_{\omega_l=N^{-1}}^{1}\frac{1}{N}\left|\frac{1}{\omega_l-\alpha-iN^{-\gamma}}\right|^{k}\left|\frac{1}{\omega_l-\beta+iN^{-\gamma}}\right|^{p} \leq CN^{\gamma(k+p-1)}\left(\log(N^\gamma)+CN^{2\gamma-1}\right)
\tag{5.17}
$$

This is done by bounding $k+p-1$ of the fractions by $N^{\gamma(k+p-1)}$ and using Eq. (5.16) to bound the rest. We define now $\tilde{\mathcal{F}}_\pi$.

$$
\begin{aligned}
&\tilde{\mathcal{F}}_\pi(n',\bar{n},\{k_j,p_j\},N,N^\gamma,\omega_0) \\
=&N^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}}\int\int d\alpha d\beta e^{i(\alpha-\beta)N^\gamma}e^2\prod_{l=1}^{n'}\int d\omega'_l\frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}\sum_{\omega_1,\dots\omega_{\bar{n}}}\frac{1}{N}\psi_0^*(\omega_1)\psi_0(\omega_2) \\
&\prod_{j=0}^{\bar{n}}\left(\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right)^{p_j}\left(\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right)^{k_j}\mathcal{W}(\{\omega_j,\omega'_l\},\{q_j,q'_l\}_\mathrm{i})N^{-\frac{n'+\sum_j\left(k_j+p_j\right)}{2}}
\end{aligned}
$$

This is equal to the expression of Eq.(5.5) except that we have substituted the sums over ω'_l by integrals. Using Eq. (5.16) in Eq.(5.5) we have

$$\begin{aligned}
&\mathcal{F}_\pi(n',\bar{n},\{k_j,p_j\},N,N^\gamma,\omega_0)\\
=&N^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}}\int\int d\alpha d\beta e^{i(\alpha-\beta)N^\gamma}e^2\prod_{l=1}^{n'}\left(\int d\omega'_l\frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}+O(N^{2\gamma-1})\right)\\
&\times\sum_{\omega_1\ldots\omega_{\bar{n}}}\frac{1}{N}\psi_0^*(\omega_1)\psi_0(\omega_2)\prod_{j=0}^{\bar{n}}\left(\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right)^{p_j}\left(\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right)^{k_j}\\
&\times\mathcal{W}(\{\omega_j,\omega'_l\},\{q_j,q'_l\})N^{-\frac{\sum_j\left(k_j+p_j\right)-n'-2(\bar{n}+1)}{2}}
\end{aligned}$$

From this we deduce

$$\begin{aligned}
&\left|\sum_{\omega_0}\frac{1}{N}\left(\tilde{\mathcal{F}}_\pi(n',\bar{n},\{k_j,p_j\},N,N^\gamma,\omega_0)-\mathcal{F}_\pi(n',\bar{n},\{k_j,p_j\},N,N^\gamma,\omega_0)\right)\right|\\
&\leq CN^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}}O(N^{2\gamma-1})\sum_{v=1}^{n'}\left(\prod_{l=1,l\neq v}^{n'}\left|\int d\omega'_l\frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}+O(N^{2\gamma-1})\right|\right)\\
&\times\prod_{j=0}^{\bar{n}}\sum_{\omega_0\ldots\omega_{\bar{n}}}\frac{1}{N}\left|\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right|^{p_j}\left|\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right|^{k_j}N^{-\frac{\sum_j\left(k_j+p_j\right)-n'-2(\bar{n}+1)}{2}}\\
&\leq CN^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}-\frac{\sum_j\left(k_j+p_j\right)-n'-2(\bar{n}+1)}{2}}N^{2\gamma-1}n'\log^{n'-1}(N^\gamma)\\
&\times\prod_{j=0}^{\bar{n}}\sum_{\omega_0\ldots\omega_{\bar{n}}}\frac{1}{N}\left|\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right|^{p_j}\left|\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right|^{k_j}\\
&\leq CN^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}-\frac{\sum_j\left(k_j+p_j\right)-n'-2(\bar{n}+1)}{2}}N^{2\gamma-1}n'\log^{n'-1}(N^\gamma)\\
&\times\prod_{j=0}^{\bar{n}}N^{\gamma\left(k_j+p_j-1\right)}\left(\log(N^\gamma)+N^{2\gamma-1}\right)\\
&\leq CN^{\frac{\sum_j\left(k_j+p_j-2\right)-n'}{2}(\gamma-1)+3\gamma-1}\left(n'\log^{n'-1}(N^\gamma)\left(\log(N^\gamma)+N^{2\gamma-1}\right)^{\bar{n}+1}\right)
\end{aligned}\tag{5.18}$$

By Eqs. (5.6) and (5.7) we have:

$$\begin{aligned}
n'+\bar{n}+1\leq&\frac{n'+\sum_{j=0}^{\bar{n}}k_j+p_j}{2}\\
0\leq&\frac{\sum_{j=0}^{\bar{n}}(k_j+p_j-2)-n'}{2}
\end{aligned}$$

Thus, if we choose $\gamma<\frac{1}{3}$, the first power of N in Eq. (5.18) will be negative. This negative power of N decays faster then any power of the logarithm and so in the limit $N\to\infty$, the bound (5.18) would tend to zero. We have then

$$\lim_{N\to\infty}\left|\sum_{\omega_0}\frac{1}{N}\left(\tilde{\mathcal{F}}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)-\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)\right)\right| = 0 \quad (5.19)$$

We now show that

$$\lim_{N\to\infty}\sum_{\omega_0}\frac{1}{N}\tilde{\mathcal{F}}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0) - \lim_{t\to\infty}\lim_{N'\to\infty}\sum_{\omega_0}\frac{1}{N'}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N',t,\omega_0) = 0 \quad (5.20)$$

In the second term we make a change of variable $t = N^{\gamma}$ and so the limit $t \to \infty$ become $N \to \infty$. We then have to prove the following:

$$\lim_{N\to\infty}\lim_{N'\to\infty}\left(\sum_{\omega_0}\frac{1}{N}\tilde{\mathcal{F}}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0) - \sum_{\omega_0}\frac{1}{N'}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N',N^{\gamma},\omega_0)\right) = 0 \quad (5.21)$$

First we have by Eq. (5.5)

$$\lim_{N'\to\infty}\sum_{\omega_0}\frac{1}{N'}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N',N^{\gamma},\{q_i,q'_j\}_{\mathrm{i}},\omega_0) = N^{-\gamma\frac{\left(n'+\Sigma_j\left(k_j+p_j\right)-2\right)}{2}}\int\int d\alpha d\beta e^{i(\alpha-\beta)t}e^{2}$$
$$\times\prod_{l=1}^{n'}\int d\omega'_l\frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}$$
$$\times\prod_{j=0}^{\bar{n}}\int d\omega_j\psi_0^*(\omega_1)\psi_0(\omega_2)\prod_{j=0}^{\bar{n}}\left(\frac{-1}{\omega_j-\beta+i\eta}\right)^{p_j}\left(\frac{-1}{\omega_j-\alpha-i\eta}\right)^{k_j}\mathcal{W}(\{\omega_j,\omega'_l\},\{q_i,q'_j\}_{\mathrm{i}})$$
$$\times N^{-\frac{\Sigma_j\left(k_j+p_j\right)-n'-2(\bar{n}+1)}{2}} \quad (5.22)$$

We will show that the first term in Eq. (5.21) is equal to Eq. (5.22) plus some extra terms. These extra terms will then have a bound in N that will converge to zero.
Similar to the way we have bound the difference between the sum and the integral in Eq.

(5.16), we find the following bound:

$$\begin{aligned}
&\Big|\sum_{\omega_j}\frac{1}{N}\left(\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right)^{p_j}\left(\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right)^{k_j}\\
&-\int d\omega\left(\frac{-1}{\omega-\beta+iN^{-\gamma}}\right)^{p_j}\left(\frac{-1}{\omega-\alpha-iN^{-\gamma}}\right)^{k_j}\Big|\\
\leq&\Big|\sum_{\omega_j}\int_{\Omega(\omega_j)}d\omega\frac{1}{N\left|\Omega(\omega_j)\right|}\left(\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right)^{p_j}\left(\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right)^{k_j}\\
&-\left(\frac{-1}{\omega-\beta+iN^{-\gamma}}\right)^{p_j}\left(\frac{-1}{\omega-\alpha-iN^{-\gamma}}\right)^{k_j}\Big|\\
\leq&\Big|\sum_{\omega_j}\int_{\Omega(\omega_j)}d\omega\frac{1}{N\left|\Omega(\omega_j)\right|}\\
&\frac{N\left|\Omega(\omega_j)\right|(\omega_j-\beta+iN^{-\gamma})^{p_j}(\omega_j-\alpha-iN^{-\gamma})^{k_j}-(\omega-\alpha-iN^{-\gamma})^{k_j}(\omega-\beta-iN^{-\gamma})^{p_j}}{(\omega_j-\beta+iN^{-\gamma})^{p_j}(\omega_j-\alpha-iN^{-\gamma})^{k_j}(\omega-\alpha-iN^{-\gamma})^{k_j}(\omega-\beta-iN^{-\gamma})^{p_j}}\Big|
\end{aligned}\tag{5.23}$$

Because the integration over ω is in the interval $\Omega(\omega_j)$ we set $\omega=\omega_j+\Delta\omega$. Also because of Eq. (5.13) we have $N\left|\Omega(\omega_j)\right|=1$.The numerator of the integrand is then

$$\begin{aligned}
&\left(\omega_j-\beta+iN^{-\gamma}\right)^{p_j}\left(\omega_j-\alpha-iN^{-\gamma}\right)^{k_j}-\left(\omega-\alpha-iN^{-\gamma}\right)^{k_j}\left(\omega-\beta-iN^{-\gamma}\right)^{p_j}\\
=&\left(\omega_j-\beta+iN^{-\gamma}\right)^{p_j}\left(\omega_j-\alpha-iN^{-\gamma}\right)^{k_j}\\
&-\left(\omega_j+\Delta\omega-\alpha-iN^{-\gamma}\right)^{k_j}\left(\omega_j+\Delta\omega-\beta-iN^{-\gamma}\right)^{p_j}
\end{aligned}\tag{5.24}$$

In expanding the right hand part of Eq. (5.24) in powers of $\Delta\omega$ we have the following expression :

$$\begin{aligned}
&\left(\omega_j-\beta+iN^{-\gamma}\right)^{p_j}\left(\omega_j-\alpha-iN^{-\gamma}\right)^{k_j}\\
+&\sum_{n_j,m_j=0}^{k_j,p_j}\frac{k_j!p_j!}{n_j!m_j!(k_j-n_j)!(p_j-m_j)!}(\Delta\omega)^{n_j+m_j}\left(\omega_j-\beta+iN^{-\gamma}\right)^{p_j-m_j}\left(\omega_j-\alpha-iN^{-\gamma}\right)^{k_j-n_j}\\
=&\sum_{n_j,m_j=0;n_j+m_j\geq 1}^{k_j,p_j}\frac{k_j!p_j!}{n_j!m_j!(k_j-n_j)!(p_j-m_j)!}(\Delta\omega)^{n_j+m_j}\\
\times&\left(\omega_j-\beta+iN^{-\gamma}\right)^{p_j-m_j}\left(\omega_j-\alpha-iN^{-\gamma}\right)^{k_j-n_j}
\end{aligned}\tag{5.25}$$

If we insert this sum into the fraction of Eq. (5.23) and bound $|\omega-\alpha-iN^{-\gamma}|^{-1}$ by N^{γ}, we have the following bound for Eq. (5.23):

$$(5.23) \leq \sum_{\omega_j} \int_{\Omega(\omega_j)} d\Delta\omega \sum_{n_j,m_j=0; n_j+m_j\geq 1}^{k_j,p_j} \frac{k_j!p_j!}{n_j!m_j!(k_j-n_j)!(p_j-m_j)!} |\Delta\omega|^{n_j+m_j} N^{\gamma\left(k_j+p_j+m_j+n_j\right)} \tag{5.26}$$

Because $\Delta\omega \leq N^{-1}$ we get

$$\begin{aligned}(5.23) \leq& \sum_{\omega_j} \int_{\Omega(\omega_j)} d\Delta\omega N^{\gamma\left(k_j+p_j\right)} \sum_{n_j,m_j=0; n_j+m_j\geq 1}^{k_j,p_j} \frac{k_j!p_j!}{n_j!m_j!(k_j-n_j)!(p_j-m_j)!} N^{(\gamma-1)\left(m_j+n_j\right)} \\ \leq& N^{\gamma\left(k_j+p_j\right)} \left(\left(1+N^{\gamma-1}\right)^{k_j+p_j}-1\right) \end{aligned} \tag{5.27}$$

We then get

$$\begin{aligned}&\sum_{\omega_j} \frac{1}{N} \left(\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right)^{p_j} \left(\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right)^{k_j} \\ =& \int d\omega \left(\frac{-1}{\omega-\beta+iN^{-\gamma}}\right)^{p_j} \left(\frac{-1}{\omega-\alpha-iN^{-\gamma}}\right)^{k_j} + O\left(N^{\gamma\left(k_j+p_j\right)} \left(\left(1+N^{\gamma-1}\right)^{k_j+p_j}-1\right)\right)\end{aligned}$$

Using this estimate for $\tilde{\mathcal{F}}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N, N^{\gamma}, \omega_0)$ we get

$$\begin{aligned}&\sum_{\omega_0} \frac{1}{N} \tilde{\mathcal{F}}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N, N^{\gamma}, \omega_0) \\ =& N^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}} \int\int d\alpha d\beta e^{i(\alpha-\beta)N^{\gamma}} e^2 \prod_{l=1}^{n'} \left(\int d\omega'_l \frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}\right) \\ & \prod_{j=0}^{\bar{n}} \int d\omega_j \Big(\psi_0^*(\omega_1)\psi_0(\omega_2) \prod_{j=0}^{\bar{n}} \Big(\left(\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right)^{p_j} \left(\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right)^{k_j} \\ & + O\left(N^{\gamma\left(k_j+p_j\right)} \left(\left(1+N^{\gamma-1}\right)^{k_j+p_j}-1\right)\right)\Big) \mathcal{W}(\{\omega_j, \omega'_l\})\Big) N^{-\frac{-n'+\sum_j\left(k_j+p_j-2\right)}{2}}\end{aligned}$$

and thus

$$\begin{aligned}&\left|\sum_{\omega_0} \frac{1}{N} \tilde{\mathcal{F}}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N, N^{\gamma}, \omega_0) - \lim_{N'\to\infty} \sum_{\omega_0} \frac{1}{N'} \mathcal{F}_{\pi}(n', \bar{n}, \{k_j, p_j\}, N', N^{\gamma}, \omega_0)\right| \\ \leq& N^{-\gamma\frac{n'+\sum_j\left(k_j+p_j\right)-2}{2}} \int\int d\alpha d\beta \prod_{l=1}^{n'} \left|\int d\omega'_l \frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}\right| \\ \times& \sum_{v=0}^{\bar{n}} \left(O\left(N^{\gamma(k_v+p_v)} \left(\left(1+N^{\gamma-1}\right)^{k_v+p_v}-1\right)\right) \prod_{j=0,j\neq v}^{\bar{n}} \int d\omega_j \left|\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right|^{p_j} \left|\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right|^{k_j}\right) \\ \times& N^{-\frac{-n'+\sum_j\left(k_j+p_j-2\right)}{2}}\end{aligned} \tag{5.28}$$

By using the following bound for the integrals

$$\int d\omega_j \left|\frac{-1}{\omega_j-\beta+iN^{-\gamma}}\right|^{p_j}\left|\frac{-1}{\omega_j-\alpha-iN^{-\gamma}}\right|^{k_j} \leq N^{\gamma\left(k_j+p_j-1\right)} \tag{5.29}$$

$$\left|\int d\omega'_l \frac{-1}{\omega'_l-\gamma_l+iN^{-\gamma}}\right| \leq \log\left(N^{\gamma}\right) \tag{5.30}$$

we get

$$\begin{aligned}(5.28) \leq & CN^{-\gamma\frac{n'+\Sigma_j\left(k_j+p_j\right)-2}{2}}\log^{n'}\left(N^{\gamma}\right)N^{\gamma\left(\Sigma_j k_j+p_j-1\right)} \\ & \sum_{v=0}^{\bar{n}}\left(\left(\left(1+N^{\gamma-1}\right)^{k_v+p_v}-1\right)N^{\gamma}\right)N^{-\frac{-n'+\Sigma_j\left(k_j+p_j-2\right)}{2}} \\ \leq & CN^{\left(\Sigma_j\left(k_j+p_j-2\right)-n'\right)\frac{\gamma-1}{2}}\log^{n'}\left(N^{\gamma}\right)\sum_{v=0}^{\bar{n}}N^{2\gamma}\left(\left(1+N^{\gamma-1}\right)^{k_v+p_v}-1\right)\end{aligned} \tag{5.31}$$

If $gamma < \frac{1}{2}$ the first power of N in Eq. (5.21) will be negative and so tend to zero. If $\gamma < \frac{1}{3}$ the terms in the sum will all be negative power of N The right hand side of Eq. (5.31) will thus tend to zero in the limit $N \to \infty$ if $\gamma < \frac{1}{3}$ and so we get for Eq. (5.21)

$$\left|\lim_{N\to\infty}\sum_{\omega_0}\tilde{\mathcal{F}}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)-\lim_{t\to\infty}\lim_{N'\to\infty}\sum_{\omega_0}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N',t,\omega_0)\right|=0 \tag{5.32}$$

Finally we have

$$\begin{aligned}& \left|\lim_{N\to\infty}\sum_{\omega_0}\frac{1}{N}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)-\lim_{t\to\infty}\lim_{N\to\infty}\sum_{\omega_0}\frac{1}{N}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N,t,\omega_0)\right| \\ \leq & \left|\lim_{N\to\infty}\sum_{\omega_0}\frac{1}{N}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)-\lim_{N\to\infty}\sum_{\omega_0}\frac{1}{N}\tilde{\mathcal{F}}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)\right| \\ + & \left|\lim_{N\to\infty}\sum_{\omega_0}\frac{1}{N}\tilde{\mathcal{F}}_{\pi}(n',\bar{n},\{k_j,p_j\},N,N^{\gamma},\omega_0)-\lim_{t\to\infty}\lim_{N'\to\infty}\sum_{\omega_0}\frac{1}{N'}\mathcal{F}_{\pi}(n',\bar{n},\{k_j,p_j\},N',t,\omega_0)\right|\end{aligned}$$

the first term goes to zero by Eq. (5.19) whenever $\gamma < \frac{1}{3}$ and the second term goes to zero by Eq. (5.32). □

The proof of theorem (5.1.1) is now as follows:

Proof. Theorem 5.1.1
We define

$$\tilde{P}_{a,N^\gamma}(q_0,\omega_0) = \sum_{n,m=0}^{M} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\{q_i,q'_j\}_\text{i}} \mathcal{F}_\pi(n',\bar{n},\{k_j,p_j\},N,N^\gamma,\{q_i,q'_j\}_\text{i},q_0,\omega_0)$$

$$P_{a,t}(q_0,\omega_0) = \sum_{n,m=0}^{M} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\{q_i,q'_j\}} \mathcal{F}_\pi(n',\bar{n},\{k_j,p_j\},N,t,\{q_i,q'_j\},q_0,\omega_0) \tag{5.33}$$

so that we have the following for a diagonal observable

$$\lim_{t\to\infty}^{\lambda^2 t=T} \lim_{N\to\infty} \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right] = \lim_{t\to\infty}^{\lambda^2 t=T} \lim_{N\to\infty} \sum_{q_0,\omega_0} \frac{1}{N} O(q_0,\omega_0) P_{a,t}(q_0,\omega_0) \tag{5.34}$$

$$\lim_{N\to\infty}^{\lambda^2 t=T,tN^{-\gamma}=1} \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right] = \lim_{N\to\infty}^{\lambda^2 t=T,tN^{-\gamma}=1} \sum_{q_0,\omega_0} \frac{1}{N} O(q_0,\omega_0) \tilde{P}_{a,N^\gamma}(q_0,\omega_0) \tag{5.35}$$

For the difference between these two equations we have then

$$\begin{aligned}
&\left| \lim_{t\to\infty} \lim_{N\to\infty} \sum_{q_0,\omega_0} \frac{1}{N} O(q_0,\omega_0) P_{a,t}(q_0,\omega_0) - \lim_{N\to\infty} \sum_{q_0,\omega_0} \frac{1}{N} O(q_0,\omega_0) \tilde{P}_{a,N^\gamma}(q_0,\omega_0) \right| \\
&\le O_{max} \left| \lim_{t\to\infty} \lim_{N\to\infty} \sum_{q_0,\omega_0} \frac{1}{N} P_{a,t}(q_0,\omega_0) - \tilde{P}_{a,N^\gamma}(q_0,\omega_0) \right| \\
&\le C \sum_{n,m=0}^{M} \sum_{\pi(n,m)\in\mathcal{G}_a} \sum_{\{q_i,q'_j\}_\text{i}} \\
&\quad \Big| \lim_{t\to\infty} \lim_{N\to\infty} \sum_{\omega_0} \frac{1}{N} \mathcal{F}_\pi(n',\bar{n},\{k_j,p_j\},N,t,\{q_i,q'_j\}_\text{i},\omega_0) \\
&- \lim_{N\to\infty} \sum_{\omega_0} \frac{1}{N} \mathcal{F}_\pi(n',\bar{n},\{k_j,p_j\},N,N^\gamma,\{q_i,q'_j\}_\text{i},\omega_0) \Big| \\
&\le 0
\end{aligned}$$

To get to the last line we have used theorem 5.1.2 to the expression inside the absolute value and the fact that the sum is finite. We conclude that

$$\boxed{\lim_{t\to\infty}^{\lambda^2 t=T} \lim_{N\to\infty} \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right] - \lim_{N\to\infty}^{\lambda^2 t=T,tN^{-\gamma}=1} \mathbb{E}\left[\langle\Psi_t^M|\hat{O}|\Psi_t^M\rangle\right] = 0}$$

□

6 Bound of the remainder

We can summarize what we have done so far by referring to a simple kind of parameter-line. In chapter 2 we showed that the time evolution of the Schrödinger equation may be identified as a solution of a rate equation for the probability density. We did so by using the Duhamel expansion up to order M. We then sent N to ∞ which allowed us to discard certain graphs from $|\Psi_t^M\rangle$. Then we sent t to ∞ with the Van Hove limit, $\lambda^2 t = T$. This allowed us to discard even more graphs from $|\Psi_M(t)\rangle$. Finally when sending M to ∞ we arrived at the solution of the rate equation. This is depicted in the upper line of figure 6.1. The order in which the parameters are depicted in figure 6.1 symbolizes which one was taken first to its limit.

In chapter 3 we showed that for fixed t and λ there exists an M and an N such that the variance of macro-observables ($\mathrm{Var}_N\left[\hat{O}\right]$ in the figure) would go to zero. This was our result of typicality. This means that in this case t is finite. This is depicted in middle line of figure 6.1.

In chapter 5 we showed that if $t = N^{\gamma}$, with $\gamma < \frac{1}{3}$, and send N to ∞ (and thus t at the same time, although slower) we would obtain the same rate equation as for the case where we take the limits separately. This is depicted in the lower line of figure 6.1. For the results of chapter 2 and 5 we assumed that in these limits the contribution of $R(M,t,\lambda,q_0,l_0)$, the remaining so to say of $|\Psi_t^M\rangle$, in Eq. (2.25) converges to zero. This is natural and expected but non trivial. The proof of this is the aim of this chapter. In order to do so we will show that the norm under the average of $|\phi_t^M\rangle$ tends to zero. If this is so it makes sense that $R(M,t,\lambda,q_0,l_0)$ will yield a zero contribution because of its dependency on $|\phi_t^M\rangle$. We recall here the following expressions:

$$
\begin{aligned}
|\Psi_t^M\rangle &= \sum_{n=0}^{M-1} |\psi_t^n\rangle \\
R(M,t,\lambda,q_0,l_0) =& \langle\phi_t^M|q_0,l_0\rangle\langle q_0,l_0|\tilde{\psi}_t^M\rangle + \langle\tilde{\psi}_t^M|q_0,l_0\rangle\langle q_0,l_0|\phi_t^M\rangle \\
&+ \langle\phi_t^M|q_0,l_0\rangle\langle q_0,l_0|\phi_t^M\rangle
\end{aligned} \tag{6.1}
$$

Because $|\phi_t^M\rangle$ contains the next order perturbation terms of $|\Psi_t^M\rangle$ we have to extract these contributions which we can bound and show that the rest does not matter. The idea to bound the norm is to divide the integral in Eq. (6.2) in κ smaller pieces and then to expand the

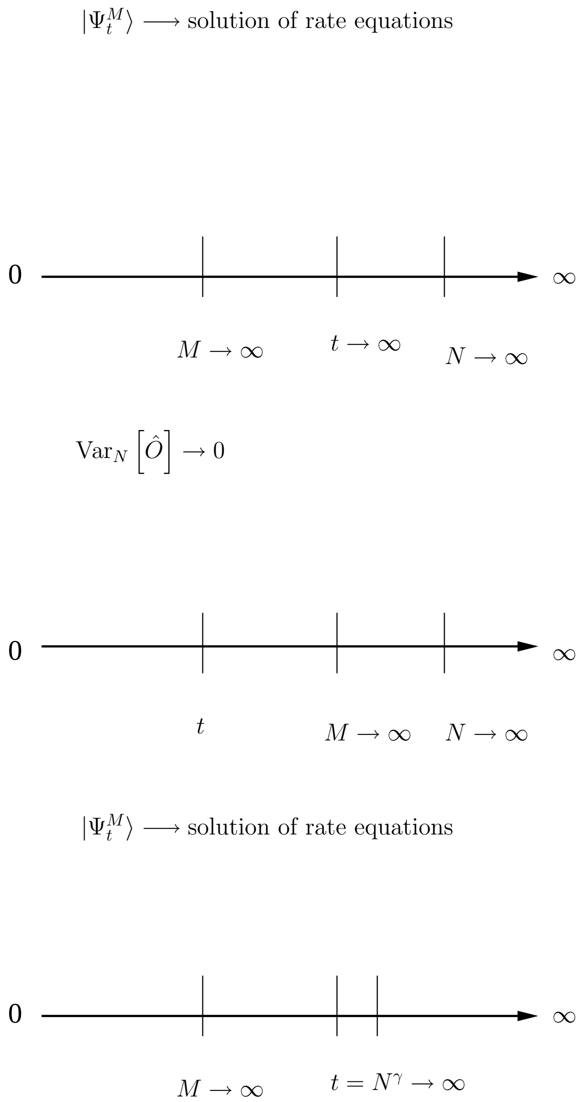

Figure 6.1: parameter line

evolution operator once again by using the Duhamel expansion.

$$
\begin{aligned}
|\phi_t^M\rangle &= \int_0^t ds e^{-i(t-s)H} (-i\lambda)\, V|\psi_s^M\rangle \\
&= \sum_{j=0}^{\kappa} e^{-i(t-\theta_{j+1})H} \int_{\theta_j}^{\theta_{j+1}} ds e^{-i(\theta_{j+1}-s)H} (-i\lambda)\, V|\psi_s^M\rangle
\end{aligned} \tag{6.2}
$$

with

$$
|\Psi_s^M\rangle = (-i\lambda)^M \int_0^s \dots \int_0^s [ds_M]\, e^{-is_0 H_0}\, V \dots e^{-is_M H_0} \delta\left(s - \sum_{j=0}^{M} s_j\right) |\psi_0\rangle
$$

$$
\begin{aligned}
& e^{-i\left(\theta_{j+1}-s\right)H} \\
=& \sum_{n=0}^{M_0-1} (-i\lambda)^n \int_0^{\theta_{j+1}-s} \dots \int_0^{\theta_{j+1}-s} [d\tilde{s}_n]\, e^{-i\tilde{s}_0 H_0}\, V \dots e^{-i\tilde{s}_n H_0} \delta\left(\theta_{j+1} - s - \sum_{j=0}^{n} \tilde{s}_j\right) \\
&+ (-i\lambda)^{M_0} \int_0^{\theta_{j+1}-s} \dots \int_0^{\theta_{j+1}-s} [d\tilde{s}_n]\, e^{-i\tilde{s}_0 H}\, V \dots e^{-i\tilde{s}_{M_0} H_0} \delta\left(\theta_{j+1} - s - \sum_{j=0}^{M_0} \tilde{s}_j\right)
\end{aligned}
$$

Inserting this in Eq. (6.2) we have

$$
\begin{aligned}
|\phi_t^M\rangle =& \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H} \sum_{n=0}^{M_0-1} |\psi_{M,n,\theta_j}(\theta_{j+1})\rangle \\
&+ \sum_{j}^{\kappa-1} e^{-i(t-\theta_{j+1})H} \int_{\theta_j}^{\theta_{j+1}} d\tilde{s} e^{-i(\theta_{j+1}-\tilde{s})H} (-i\lambda V)\, |\psi_{M,M_0,\theta_j}(\tilde{s})\rangle && (6.3) \\
=& |\psi^1_{M,M_0,\kappa}(t)\rangle + |\psi^2_{M,M_0,\kappa}(t)\rangle && (6.4)
\end{aligned}
$$

with

$$
\begin{aligned}
& |\psi_{M,m,\theta_j}(\tilde{s})\rangle \\
=& (-i\lambda)^m \int_{\theta_j}^{\tilde{s}} ds \int_0^{\tilde{s}-s} \dots \int_0^{\tilde{s}-s} [d\tilde{s}_m]\, e^{-i\tilde{s}_1 H_0}\, V \dots e^{-i\tilde{s}_m H_0} \delta\left(\tilde{s} - s - \sum_{j=1}^{m} \tilde{s}_j\right) \\
& \times V|\psi_s^M\rangle
\end{aligned} \tag{6.5}
$$

We will show that the norm of $|\psi^1_{M,M_0,\kappa}(t)\rangle$ and $|\psi^2_{M,M_0,\kappa}(t)\rangle$ indeed tends to zero which would imply that the norm of $|\phi_t^M\rangle$ goes to zero. We have then the following theorem:

Theorem 6.0.3.

$$
\lim_{N\to\infty} \mathbb{E}\left[\langle\psi^2_{M,M_0,\kappa}(t)\,|\psi^2_{M,M_0,\kappa}(t)\rangle\right] \le \frac{(2\,(M+M_0+1))!\,(T)^{M+M_0+1}\, t}{(M+M_0+1)!\kappa^{M_0-M}} \log^{n'+4}(t) \tag{6.6}
$$

Proof. The complicated part of bounding properly $|\psi^2_{M,M_0,\kappa}(t)\rangle$ comes from the fact that it still possesses a complete evolution term, that is e^{-itH}, which cannot be averaged over. Nevertheless, the operator being unitary, it does not influence the norm of the state to which it is applied and we can in this way get rid of it.
We have then the following bound for $\langle\psi^2_{M,M_0,\kappa}(t)|\psi^2_{M,M_0,\kappa}(t)\rangle$:

$$\begin{aligned}\langle\psi^2_{M,M_0,\kappa}(t)|\psi^2_{M,M_0,\kappa}(t)\rangle =&\lambda^2\sum_i^{\kappa-1}\sum_j^{\kappa-1}\int_{\theta_i}^{\theta_{i+1}}d\tau\int_{\theta_j}^{\theta_{j+1}}ds\langle\psi_{M,M_0,\theta_i}(\tau)|V^2|\psi_{M,M_0,\theta_j}(s)\rangle\\ \leq& t^2\lambda^2\mathrm{supp}_{j,s\in[\theta_j,\theta_{j+1}]}\langle\psi_{M,M_0,\theta_j}(s)|V^2|\psi_{M,M_0,\theta_j}(s)\rangle\end{aligned}\tag{6.7}$$

The right hand side of Eq. (6.7) does not have the complete evolution operator anymore, and so we can average over it as we did for the expressions coming from the expansion in section2.3. The price we payed have to pay is a factor $t^2\lambda^2$, which, in the limits considered, diverges as t. Using Eq. (6.5) we have:

$$\begin{aligned}&\langle\tilde{l}_0|V|\tilde{l}_1\rangle\langle\tilde{l}_1|\psi_{M,M_0,\theta_j}(\tilde{s})\rangle\\ =&(-i\lambda)^{M_0}\int_{\theta_j}^{\tilde{s}}ds\int_0^{\tilde{s}-s}\dots\int_0^{\tilde{s}-s}[d\tilde{s}_{M_0}]\,e^{-i\tilde{s}_1H_0}V\dots e^{-i\tilde{s}_{M_0}H_0}\delta\left(\tilde{s}-s-\sum_{j=1}^{M_0}\tilde{s}_j\right)\\ &\times V(-i\lambda)^M\int_0^s\dots\int_0^s[ds_M]\,e^{-is_0H_0}V\dots e^{-is_MH_0}\delta\left(s-\sum_{j=0}^{M}s_j\right)\\ =&(-i\lambda)^{M+M_0}\int_{\theta_j}^{\tilde{s}}ds\int_0^{\tilde{s}-s}[d\tilde{s}_n]\int_0^s[ds_M]\,\delta\left(s-\sum_{j=0}^{M}s_j\right)\delta\left(\tilde{s}-s-\sum_{j=1}^{M_0}\tilde{s}_j\right)\\ &\sum_{\{\tilde{l}_j,l_j\}_{0,1}=1}^{N}\left(\prod_{j=1}^{M_0}e^{-i\tilde{s}_j\tilde{\omega}_j}\right)\left(\prod_{j=0}^{M}e^{-is_j\omega_j}\right)L^{M_0+M+1}(\tilde{l}_j,l_j)\end{aligned}\tag{6.8}$$

with $\omega_j=\omega_{l_j}$, $\tilde{\omega}_j=\omega_{\tilde{l}_j}$ and

$$L^{M_0+M+1}(\tilde{l}_j,l_j)=\langle\tilde{l}_0|V|\tilde{l}_1\rangle\langle\tilde{l}_1|V|\tilde{l}_2\rangle\dots\langle\tilde{l}_{M_0-1}|V|\tilde{l}_{M_0}\rangle\langle\tilde{l}_{M_0}|V|l_0\rangle\dots\langle l_{M-1}|V|l_M\rangle\tag{6.9}$$

Notice that there is an l-variable that has no energy variable associated to it, namely $\tilde{l}_0$. Similarly to the procedure followed in section 2.3, we insert the identities of Eq. (6.10) and (6.11) in Eq. (6.8), which will allow us to perform the integrals over the s_j and $\tilde{s}_j$ variables and results in Eq. (6.12).

$$\delta\left(s-\sum_{j=0}^{M}s_j\right)=\int d\alpha e^{-i\alpha\left(s-\sum_{j=0}^{M}s_j\right)}e^{\eta\left(s-\sum_{j=0}^{M}s_j\right)}\tag{6.10}$$

$$\delta\left(\tilde{s}-s-\sum_{j=1}^{M_0}\tilde{s}_j\right)=\int d\tilde{\alpha}e^{-i\tilde{\alpha}\left(\tilde{s}-s-\sum_{j=1}^{M_0}\tilde{s}_j\right)}e^{\tilde{\eta}\left(\tilde{s}-s-\sum_{j=1}^{M_0}\tilde{s}_j\right)}\tag{6.11}$$

$$\langle \tilde{l}_0 | V | \tilde{l}_1 \rangle \langle \tilde{l}_1 | \psi_{M,M_0,\theta_j}(\tilde{s}) \rangle = (-i\lambda)^{M+M_0} \int_{\theta_j}^{\tilde{s}} ds \int d\alpha d\tilde{\alpha} e^{-i\alpha s} e^{\eta s} e^{-i\tilde{\alpha}(\tilde{s}-s)} e^{\tilde{\eta}(\tilde{s}-s)}$$
$$\sum_{\{\tilde{l}_j, l_j\}_{0,1}=1}^{N} \left(\prod_{j=1}^{M_0} \frac{1}{\tilde{\omega}_j - \tilde{\alpha} - i\tilde{\eta}} \right) \left(\prod_{j=0}^{M} \frac{1}{\omega_j - \alpha - i\eta} \right)$$
$$\times L^{M_0+M+1}(\tilde{l}_j, l_j) \tag{6.12}$$

We then have

$$\lambda^2 \langle \psi_{M,M_0,\theta_j}(\tilde{s}) | V^2 | \psi_{M,M_0,\theta_j}(\tilde{s}) \rangle = \lambda^{2(M+M_0+1)} \int_{\theta_j}^{\tilde{s}} d\tau ds \int d\beta d\tilde{\beta} \int d\alpha d\tilde{\alpha}$$
$$e^{-i\alpha s} e^{\eta s} e^{-i\tilde{\alpha}(\tilde{s}-s)} e^{\tilde{\eta}(\tilde{s}-s)} e^{i\beta\tau} e^{\eta\tau} e^{i\tilde{\beta}(\tilde{s}-\tau)} e^{\tilde{\eta}(\tilde{s}-\tau)} \sum_{\{\tilde{l}'_j, l'_j, \tilde{l}_j, l_j\}}$$
$$\times \left(\prod_{j=1}^{M_0} \frac{1}{\tilde{\omega}_j - \tilde{\alpha} - i\tilde{\eta}} \right) \left(\prod_{j=0}^{M} \frac{1}{\omega_j - \alpha - i\eta} \right) \left(\prod_{j=1}^{M_0} \frac{1}{\tilde{\omega}'_j - \tilde{\beta} + i\tilde{\eta}} \right) \left(\prod_{j=0}^{M} \frac{1}{\omega'_j - \beta + i\eta} \right)$$
$$\times \bar{L}^{M_0+M+1}(\tilde{l}'_j, l'_j) L^{M_0+M+1}(\tilde{l}_j, l_j)$$

Averaging we get

$$\lambda^2 \mathbb{E}\left[\langle \psi_{M,M_0,\theta_j}(\tilde{s}) | V^2 | \psi_{M,M_0,\theta_j}(\tilde{s}) \rangle \right] = \lambda^{2(M+M_0+1)}$$
$$\times \int d\beta d\tilde{\beta} \int d\alpha d\tilde{\alpha} i \frac{e^{-i\alpha\tilde{s}+\eta\tilde{s}} - e^{-i\alpha\theta_j+\eta\theta_j-i\tilde{\alpha}(\tilde{s}-\theta_j)+\tilde{\eta}(\tilde{s}-\theta_j)}}{\alpha-\tilde{\alpha}+i(\eta-\tilde{\eta})}$$
$$i \frac{e^{i\beta\tilde{s}+\eta\tilde{s}} - e^{i\beta\theta_j+\eta\theta_j+i\tilde{\beta}(\tilde{s}-\theta_j)+\tilde{\eta}(\tilde{s}-\theta_j)}}{\beta-\tilde{\beta}+i(\eta-\tilde{\eta})} \sum_{\{\tilde{l}'_j, l'_j, \tilde{l}_j, l_j\}}$$
$$\times \left(\prod_{j=1}^{M_0} \frac{1}{\tilde{\omega}_j - \tilde{\alpha} - i\tilde{\eta}} \right) \left(\prod_{j=0}^{M} \frac{1}{\omega_j - \alpha - i\eta} \right) \left(\prod_{j=1}^{M_0} \frac{1}{\tilde{\omega}'_j - \tilde{\beta} + i\tilde{\eta}} \right) \left(\prod_{j=0}^{M} \frac{1}{\omega'_j - \beta + i\eta} \right)$$
$$\times \mathbb{E}\left[\bar{L}^{M_0+M+1}(\tilde{l}'_j, l'_j) L^{M_0+M+1}(\tilde{l}_j, l_j) \right]$$

Once again the average will bring graphs into play.

$$\lambda^2\mathbb{E}\left[\langle\psi_{M,M_0,\theta_j}(\tilde{s})|\,V^2|\psi_{M,M_0,\theta_j}(\tilde{s})\rangle\right]=\lambda^{2(M+M_0+1)}$$
$$\times\int d\beta d\tilde{\beta}\int d\alpha d\tilde{\alpha} i\frac{e^{-i\alpha\tilde{s}+\eta\tilde{s}}-e^{-i\alpha\theta_j+\eta\theta_j-i\tilde{\alpha}(\tilde{s}-\theta_j)+\tilde{\eta}(\tilde{s}-\theta_j)}}{\alpha-\tilde{\alpha}+i(\eta-\tilde{\eta})}$$
$$i\frac{e^{i\beta\tilde{s}+\eta\tilde{s}}-e^{i\beta\theta_j+\eta\theta_j+i\tilde{\beta}(\tilde{s}-\theta_j)+\tilde{\eta}(\tilde{s}-\theta_j)}}{\beta-\tilde{\beta}+i(\eta-\tilde{\eta})}\sum_{\{\tilde{l}'_j,l'_j,\tilde{l}_j,l_j\}}$$
$$\times\left(\prod_{j=1}^{M_0}\frac{1}{\tilde{\omega}_j-\tilde{\alpha}-i\tilde{\eta}}\right)\left(\prod_{j=0}^{M}\frac{1}{\omega_j-\alpha-i\eta}\right)\left(\prod_{j=1}^{M_0}\frac{1}{\tilde{\omega}'_j-\tilde{\beta}+i\tilde{\eta}}\right)\left(\prod_{j=0}^{M}\frac{1}{\omega'_j-\beta+i\eta}\right)$$
$$\times\sum_{\pi(M+M_0+1,M+M_0+1)}C_\pi(M+M_0+1,M+M_0+1,\{\tilde{l}'_j,\tilde{l}_j,l'_i,l_i\})\tag{6.13}$$

According to theorem 2.4.6 the number of independent l-variables generated by a NC-graph in $C_\pi(M+M_0+1,M+M_0+1,\{\tilde{l}'_j,\tilde{l}_j,l'_i,l_i\})$ is M_0+M+2. Remember that $\tilde{\omega}_0$ has no propagator associated to it. We write Eq. (6.13) in short as

$$\lambda^2\mathbb{E}\left[\langle\psi_{M,M_0,\theta_j}(\tilde{s})|\,V^2|\psi_{M,M_0,\theta_j}(\tilde{s})\rangle\right]=\sum_{\pi(M+M_0+1,M+M_0+1)}Q_\pi(M,M_0,\theta_j,\tilde{s})\tag{6.14}$$

Just as in section 2.6 we can write down $Q_\pi(M,M_0,\theta_j,\tilde{s})$ as a function of the independent variables. Because propagators of the right hand side can depend on α or $\tilde{\alpha}$ the multiplicity of the right hand side, k_j, does not specify uniquely how many free propagators that are α or $\tilde{\alpha}$ there are for this specific independent variable. The same is valid for the left hand side. Therefore we define a_j as the multiplicity of the propagators that are $\tilde{\alpha}$ dependent, b_j as the multiplicity of the propagators that are α dependent, c_j as the multiplicity of the propagators that are $\tilde{\beta}$ dependent and d_j as the multiplicity of the propagators that are β dependent. With this we can write down, in the limit $N\to\infty$, the contribution for an S or an N-graph as

$$Q_\pi(M,M_0,\theta_j,\tilde{s})=\lambda^{2(M+M_0+1)}\int d\beta d\tilde{\beta}\int d\alpha d\tilde{\alpha} i\frac{e^{-i\alpha\tilde{s}+\eta\tilde{s}}-e^{-i\alpha\theta_j+\eta\theta_j-i\tilde{\alpha}(\tilde{s}-\theta_j)+\tilde{\eta}(\tilde{s}-\theta_j)}}{\alpha-\tilde{\alpha}+i(\eta-\tilde{\eta})}$$
$$\times i\frac{e^{i\beta\tilde{s}+\eta\tilde{s}}-e^{i\beta\theta_j+\eta\theta_j+i\tilde{\beta}(\tilde{s}-\theta_j)+\tilde{\eta}(\tilde{s}-\theta_j)}}{\beta-\tilde{\beta}+i(\eta-\tilde{\eta})}$$
$$\times\prod_{j=0}^{\bar{n}}\left(\int d\omega_j\left(\frac{1}{\omega_j-\tilde{\alpha}-i\tilde{\eta}}\right)^{a_j}\left(\frac{1}{\omega_j-\alpha-i\eta}\right)^{b_j}\left(\frac{1}{\omega_j-\tilde{\beta}+i\tilde{\eta}}\right)^{c_j}\left(\frac{1}{\omega_j-\beta+i\eta}\right)^{d_j}\right)$$
$$\times\left(\prod_{l=1}^{n'}\left(\int d\omega'_l\frac{1}{\omega'_l-\gamma_l-i\eta_l}\right)\right)\tag{6.15}$$

with

$$a_j+b_j=k_j$$
$$c_j+d_j=p_j$$

Because $\tilde{\omega}_0$ has no propagator associated to it the relationship between k_j, p_j, n' and $\bar{n}$ are a bit different then those from section 2.6. The number of propagators we have in Eq. (6.13) and Eq. (6.15) must be equal. Therefore

$$\sum_{j=0}^{\bar{n}} (k_j + p_j) + n' = 2(M_0 + M + 1) \tag{6.16}$$

Also since there must be $2M_0$ propagators dependent on $\tilde{\alpha}$ or $\tilde{\beta}$ (either as propagators of multiplicity one or higher) we must have

$$\sum_{j=0}^{\bar{n}} a_j + c_j + n' \geq 2M_0 \tag{6.17}$$

But because $\tilde{\omega}_0$ has no propagator associated it and because it could be independent of the rest the number of independent variables in the set of ω's could be one less then in the set of l-variables. Therefore

$$M_0 + M + 1 \leq n' + \bar{n} + 1 \leq M_0 + M + 2$$

We set $\eta = \theta_j^{-1}$ and $\tilde{\eta} = (\tilde{s} - \theta_j)^{-1}$. The absolute value of such a contribution is then bounded as follows:

$$\begin{aligned}
|Q_\pi| \leq{}& \lambda^{2(M+M_0+1)} \int d\beta d\tilde{\beta} \int d\alpha d\tilde{\alpha} \left| \frac{1}{\alpha - \tilde{\alpha} + i(\eta - \tilde{\eta})} \right| \left| \frac{1}{\beta - \tilde{\beta} + i(\eta - \tilde{\eta})} \right| \\
&\times \prod_{j=0}^{\bar{n}} \left(\int d\omega_j \left| \frac{1}{\omega_j - \tilde{\alpha} - i\tilde{\eta}} \right|^{a_j} \left| \frac{1}{\omega_j - \alpha - i\eta} \right|^{b_j} \left| \frac{1}{\omega_j - \tilde{\beta} + i\tilde{\eta}} \right|^{c_j} \left| \frac{1}{\omega_j - \beta + i\eta} \right|^{d_j} \right) \\
&\times \left(\prod_{l=1}^{n'} \left| \int d\omega'_l \frac{1}{\omega'_l - \gamma_l - i\eta_l} \right| \right)
\end{aligned}$$

Using Hölder's inequality we can bound the following type of integrals:

$$\begin{aligned}
I(\eta, \tilde{\eta}) &= \int d\omega_j \left| \frac{1}{\omega_j - \tilde{\alpha} - i\tilde{\eta}} \right|^{a_j} \left| \frac{1}{\omega_j - \alpha - i\eta} \right|^{b_j} \left| \frac{1}{\omega_j - \tilde{\beta} + i\tilde{\eta}} \right|^{c_j} \left| \frac{1}{\omega_j - \beta + i\eta} \right|^{d_j} \\
&\leq \left(\frac{1}{\tilde{\eta}} \right)^{a_j - 2 + c_j} \left(\frac{1}{\eta} \right)^{b_j + d_j} \int d\omega_j \left| \frac{1}{\omega_j - \tilde{\alpha} - i\tilde{\eta}} \right|^2 \\
&\leq \left(\frac{1}{\tilde{\eta}} \right)^{a_j + c_j - 1} \left(\frac{1}{\eta} \right)^{b_j + d_j}
\end{aligned} \tag{6.18}$$

We use this bound for all ω_j except ω_0.

$$
\begin{aligned}
|Q_\pi| \leq & \lambda^{2(M+M_0+1)} \int d\beta d\tilde{\beta} \int d\alpha d\tilde{\alpha} \int d\omega_0 \left|\frac{1}{\alpha-\tilde{\alpha}+i(\eta-\tilde{\eta})}\right| \left|\frac{1}{\beta-\tilde{\beta}+i(\eta-\tilde{\eta})}\right| \\
& \times \left|\frac{1}{\omega_0-\alpha-i\eta}\right| \left|\frac{1}{\omega_0-\beta+i\eta}\right| \\
& \times \prod_{j=1}^{\bar{n}} \left(\left(\frac{1}{\tilde{\eta}}\right)^{a_j+c_j-1} \left(\frac{1}{\eta}\right)^{b_j+d_j} \right) \left(\frac{1}{\tilde{\eta}}\right)^{a_0+c_0} \left(\frac{1}{\eta}\right)^{b_0+d_0-2} \log^{n'}(\eta)
\end{aligned}
$$

The integrals are bounded by a logarithm:

$$
\begin{aligned}
& \int d\beta d\tilde{\beta} \int d\alpha d\tilde{\alpha} \int d\omega_0 \left|\frac{1}{\alpha-\tilde{\alpha}+i(\eta-\tilde{\eta})}\right| \left|\frac{1}{\beta-\tilde{\beta}+i(\eta-\tilde{\eta})}\right| \left|\frac{1}{\omega_0-\alpha-i\eta}\right| \left|\frac{1}{\omega_0-\beta+i\eta}\right| \\
\leq & C \log^4(\eta)
\end{aligned} \tag{6.19}
$$

Since we have $\eta^{-1} = \theta_j \leq t$ and $\tilde{\eta}^{-1} = (\tilde{s}-\theta_j) \leq \kappa^{-1} t$ we can bound $|Q_\pi|$ as follows:

$$
\begin{aligned}
|Q_\pi| \leq & \lambda^{2(M+M_0+1)} \prod_{j=0}^{\bar{n}} \left(\left(\frac{t}{\kappa}\right)^{a_j+c_j-1} t^{b_j+d_j} \right) t^{-2} \frac{t}{\kappa} \log^{n'+4}(t) \\
\leq & \lambda^{2(M+M_0+1)} \prod_{j=0}^{\bar{n}} \left(t^{k_j+p_j-1} \right) \prod_{j=0}^{\bar{n}} \left(\left(\frac{1}{\kappa}\right)^{a_j+c_j-1} \right) t^{-1} \frac{1}{\kappa} \log^{n'+4}(t) \\
\leq & \left(\lambda^2 t\right)^{M+M_0+1} t^{-1} \prod_{j=0}^{\bar{n}} \left(\frac{1}{\kappa}\right)^{a_j+c_j-1} \frac{1}{\kappa} \log^{n'+4}(t)
\end{aligned} \tag{6.20}
$$

By Eqs. (6.16) and (6.17) we have

$$
\sum_{j=0}^{\bar{n}} a_j + c_j - 1 \geq M_0 - M - 1 \tag{6.21}
$$

Because of this inequality and because $n' < M + M_0 + 1$ the Q_π of such graphs is bounded by

$$
|Q_\pi| \leq \left(\lambda^2 t\right)^{M+M_0+1} t^{-1} \left(\frac{1}{\kappa}\right)^{M_0-M} \log^{M+M_0+5}(t)
$$

Using this bound in Eq. (6.14) and subsequently in Eq. (6.7) we have a bound for the average norm of $|\psi^2_{M,M_0,\kappa}(t)\rangle$.

$$
\begin{aligned}
& \mathbb{E}\left[\langle\psi^2_{M,M_0,\kappa}(t)|\psi^2_{M,M_0,\kappa}(t)\rangle\right] \\
\leq & t\left(\lambda^2 t\right)^{M+M_0+1} \left(\frac{1}{\kappa}\right)^{M_0-M} \log^{M+M_0+5}(t) \sum_{\pi(M+M_0+1,M+M_0+1)\in\mathcal{G}_{0,1}(M+M_0+1,M+M_0+1)} 1 \\
\leq & \frac{(2(M+M_0+1))!\,(T)^{M+M_0+1}}{(M+M_0+1)!\kappa^{M_0-M}} t \log^{M+M_0+5}(t)
\end{aligned} \tag{6.22}
$$

In going to the last line we used Eq. B.15. □

To bound the norm of $|\psi^1_{M,M_0,\kappa}(t)\rangle$ is much more straight forward, because there is no unitary time evolution inside. From Eq. (6.3) and the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
&\langle\psi^1_{M,M_0,\kappa}(t)|\psi^1_{M,M_0,\kappa}(t)\rangle \\
&\leq \sum_{i,j=0}^{\kappa-1}\sum_{n,m=0}^{M_0-1}\left|\langle\psi_{M,m,\theta_i}(\theta_{i+1})|e^{i(t-\theta_{i+1})H}e^{-i(t-\theta_{j+1})H}|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle\right| \\
&\leq \sum_{i,j=0}^{\kappa-1}\sum_{n,m=0}^{M_0-1}\sqrt{\langle\psi_{M,m,\theta_i}(\theta_{i+1})|\psi_{M,m,\theta_i}(\theta_{i+1})\rangle\langle\psi_{M,n,\theta_j}(\theta_{j+1})|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle} \\
&\leq \sum_{i,j=0}^{\kappa-1}\sum_{n,m=0}^{M_0-1}2supp_{\theta_j,\theta_{j+1}}\left(\langle\psi_{M,n,\theta_j}(\theta_{j+1})|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle\right)
\end{aligned}
$$

And so

$$
\mathbb{E}\left[\langle\psi^1_{M,M_0,\kappa}(t)|\psi^1_{M,M_0,\kappa}(t)\rangle\right] \leq \kappa^2 M_0^2 supp_{n,\theta_j,\theta_{j+1}}\left(\mathbb{E}\left[\langle\psi_{M,n,\theta_j}(\theta_{j+1})|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle\right]\right) \tag{6.23}
$$

The average $\mathbb{E}\left[\langle\psi_{M,n,\theta_j}(\theta_{j+1})|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle\right]$ can again be expanded in C, N, and S-graphs. In the limit $N\to\infty$ the C-graphs will vanish and in the Van Hove limit $t\to\infty$ the N-graphs will vanish. Now from theorem 2.9.1 we have a bound for S-graphs. Thus

$$
\mathbb{E}\left[\langle\psi_{M,n,\theta_j}(\theta_{j+1})|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle\right] \leq \sum_{\pi(M+n,M+n)\in\mathcal{G}_0}\frac{\left(C\lambda^2\theta_{j+1}\right)^{M+n}}{(M+n)!}
$$

Because we are summing over simple graphs we have that the number of graphs is less then some constant to the power of the order of the graphs.

$$
\sum_{\pi(M+n,M+n)\in\mathcal{G}_0} 1 \leq C'^{M+n}
$$

This yields then

$$
\mathbb{E}\left[\langle\psi_{M,n,\theta_j}(\theta_{j+1})|\psi_{M,n,\theta_j}(\theta_{j+1})\rangle\right] \leq \frac{\left(C\lambda^2 t\right)^{M+n}}{(M+n)!}
$$

and so

$$
\mathbb{E}\left[\langle\psi^1_{M,M_0,\kappa}(t)|\psi^1_{M,M_0,\kappa}(t)\rangle\right] \leq \kappa^2 M_0^2 supp_n\left(\frac{(C'T)^{M+n}}{(M+n)!}\right) \tag{6.24}
$$

We now have all the ingredients to prove that the rest vanishes.

Theorem 6.0.4.

$$\lim_{t\to\infty}\lim_{N\to\infty}\mathbb{E}\left[\langle\phi_M(t)|\phi_M(t)\rangle\right]\leq\sum_{n=M}^{\infty}\frac{(CT)^n}{n!} \tag{6.25}$$

Proof. Theorem 6.0.4 We will take advantage of the extra parameter κ which we used to subdivide the evolution and also of the parameter M_0 which is the number of times we have applied Duhamel's formula the second time. As we said previously $|\phi_M(t)\rangle$ contains the next order contributions of the evolution and some extra terms. The next order terms are those that will give rise to the bound $\frac{(CT)^M}{M!}$. In order to capture this we expand our rest terms up to an order that will be time dependent.

$$|\phi_t^M\rangle=\sum_{n=M}^{M(t)}|\psi_t^n\rangle+|\phi_t^{M(t)}\rangle \tag{6.26}$$

We have then

$$\langle\phi_t^M|\phi_t^M\rangle\leq\sum_{n,m=M}^{M(t)}\langle\psi_t^m|\psi_t^n\rangle+\langle\phi_t^{M(t)}|\phi_t^{M(t)}\rangle \tag{6.27}$$

The last factor is the one that will still contain the total evolution. We will apply our previous theorems in order to bound it. First we apply the decomposition of Eq. (6.4) to it and we will take a κ that is time dependent. We set

$$\kappa=\log(t)^{\alpha} \tag{6.28}$$

$$M(t)=\gamma\frac{\log(t)}{\log(\log(t))} \tag{6.29}$$

$$M_0=2M(t) \tag{6.30}$$

By theorem 6.0.3 and Eq. (6.24) we have

$$\begin{aligned}\mathbb{E}\left[\langle\phi_t^{M(t)}|\phi_t^{M(t)}\rangle\right]\leq&\frac{(2(3M(t)+1))!\,T^{3M(t)+1}t}{(3M(t)+1)!\kappa^{M(t)}}\log^{3M(t)+5}(t)\\&+\kappa(t)^2(2M(t))^2 supp_n\left(\frac{(C'T)^{M(t)+n}}{(M(t)+n)!}\right)\end{aligned} \tag{6.31}$$

We set $\Lambda=\log(t)$ and analyze the first term. For the two factorials we get

$$\begin{aligned}\frac{(6M(t)+2)!\,T^{3M(t)+1}}{(3M(t)+1)!}&\leq C^{3M(t)+1}(3M(t)+1)!\\&\leq C^{3\frac{\gamma\Lambda}{\log(\Lambda)}+1}\left(\frac{\gamma\Lambda}{\log(\Lambda)}\right)^{\frac{\gamma\Lambda}{\log(\Lambda)}}\\&\leq C^{3\frac{\gamma\Lambda}{\log(\Lambda)}+1}\left(\frac{\gamma}{\log(\Lambda)}\right)^{\frac{\gamma\Lambda}{\log(\Lambda)}}\left(e^{\gamma\Lambda}\right)\end{aligned} \tag{6.32}$$

As for the rest we get

$$\frac{\log^{3M(t)+5}(t)}{\kappa^{M(t)}} \leq \frac{\Lambda^{3\frac{\gamma\Lambda}{\log(\Lambda)}+5}}{e^{\alpha\gamma\Lambda}} \leq \frac{e^{3\gamma\Lambda+5\log(\Lambda)}}{e^{\alpha\gamma\Lambda}} \tag{6.33}$$

Grouping now the exponential terms from Eqs. (6.32) and (6.33) we get

$$\frac{(2(3M(t)+1))!\,T^{3M(t)+1}t}{(3M(t)+1)!\kappa^{M(t)}}\log^{3M(t)+5}(t) \leq C^{3\frac{\gamma\Lambda}{\log(\Lambda)}+1}\left(\frac{\gamma}{\log(\Lambda)}\right)^{\frac{\gamma\Lambda}{\log(\Lambda)}} e^{5\log(\Lambda)}\left(e^{(4-\alpha)\gamma\Lambda}\right)$$

By selecting $\alpha > 4$ the last exponential will dominate when $\Lambda \to \infty$, that is when $t \to \infty$. Therefore

$$\lim_{t\to\infty}\frac{(2(3M(t)+1))!\,T^{3M(t)+1}t}{(3M(t)+1)!\kappa^{M(t)}}\log^{3M(t)+5}(t) = 0 \tag{6.34}$$

For the second term in Eq. (6.31) the factorial will dominate

$$\kappa(t)^2(2M(t))^2\left(\frac{(C'T)^{M(t)+n}}{(M(t)+n)!}\right) \leq \Lambda^{2\alpha}\frac{4\gamma^2\Lambda^2}{\log^2(\Lambda)}\left(\frac{\gamma\Lambda}{\log(\Lambda)}+n\right)^{-\left(\frac{\gamma\Lambda}{\log(\Lambda)}+n\right)} \tag{6.35}$$

For $\Lambda \to \infty$ and thus $t \to \infty$ the right hand side of Eq. (6.35) goes to zero. In conclusion we get

$$\lim_{t\to\infty}\mathbb{E}\left[\langle\phi_t^{M(t)}|\phi_t^{M(t)}\rangle\right] = 0 \tag{6.36}$$

We now turn to the sum in Eq. (6.27). Using theorem 2.9.2 we get

$$\begin{aligned}\mathbb{E}\left[\sum_{n,m=M}^{M(t)}\langle\psi_t^m|\psi_t^n\rangle\right] &\leq C\left(\mathbb{E}\left[\langle\psi_t^M|\psi_t^M\rangle\right]+\mathbb{E}\left[\sum_{n,m=M+1}^{M(t)}\langle\psi_t^m|\psi_t^n\rangle\right]\right)\\ &\leq \sum_{n=M}^{M(t)}C^n\mathbb{E}\left[\langle\psi_t^n|\psi_t^n\rangle\right]\\ &\leq \sum_{n=M}^{\infty}\frac{(C'T)^n}{n!}\end{aligned}$$

□

From theorem 6.0.4 we have

$$\lim_{M\to}\lim_{t\to}\lim_{N\to\infty}\mathbb{E}\left[\langle\phi_t^M|\phi_t^M\rangle\right] = 0 \tag{6.37}$$

and thus

$$\lim_{M\to}\lim_{t\to}\lim_{N\to\infty}\mathbb{E}\left[\sum_{q_0,l_0} O(q_0,l_0)R(M,t,\lambda,q_0,l_0)\right]=0 \tag{6.38}$$

7 Conclusion and Outlook

In this work we have analyzed the time evolution of randomly coupled quantum systems. The Hamiltonians taken were composed of a deterministic part plus a random part. We presented a derivation for the coarse grained dynamics (dynamics of Macro-observables) starting directly from the Schrödinger equation. In doing this derivation we took some special limits such as the large size of the Hilbert space and the large time-weak coupling limit (Van Hove limit). We found that the dynamics in these limits converges to those given by rate equations, with Fermi's golden rule fulfilled (chapter 2, Eqs. (2.142) and (2.143)). An interesting case where the effective equations were applied is that of a lattice model, where each site has N energy levels. The quantum particle is allowed to jump with a random amplitude to nearest neighboring site levels. This interaction is then given by a random matrix. The effective rate equations turns out to be the diffusion equation (Eq. (4.31)). Thus the effective dynamics for the quantum particle is brownian motion.

We also analyzed how typical the dynamics are (chapter 3, theorem 3.2.3). We have shown that the system behaves typically the same up to finite time with finite coupling when the dimension of the Hilbert space is large. The larger the dimension of the Hilbert space is the longer the time for typical behavior becomes. In other words the dynamics of a Macro-observable will be the same regardless of the random interaction chosen from the ensemble. This result does not overlap with our derivation of the effective rate equations because of the large time-weak coupling limit that was taken to derive these. Nevertheless this does not mean that the effective equations are not the typical behavior. It only means that we have not been able to prove this. In order to prove typicality in this regime we have to use better estimates for the propagators. More refined bounds combining better estimates for sets of propagators and the properties of non-separable graphs would have to be used.

In the derivation of the effective equation two limits were taken, namely the large Hilbert space limit and the Van Hove limit. We have also analyzed the case where instead of taking separately two limits one takes them together by imposing a relation between time, t, and the Hilbert space size, $N\aleph$ (chapter 5, theorem 5.1.1). This was done by maintaining the Van Hove type of scaling. In this limit we showed that the effective equations would remain the same if the scaling between t and N is given by $t = N^{\gamma}$ with $\gamma < \frac{1}{3}$. Nevertheless it was not shown that the remainder would converge to zero when this scaling is taken.

It seems as if random interactions favor the emergence of stochastic type of equations, not only in the average but also as individual realisations of an ensemble.

The method used here relies extensively on the use of Feynman diagrams. These can become extremely complicated as higher orders are calculated but as one recognizes the different classes the use of these allows in some sense to see the "anatomy" of the solution of the

Schrödinger equation.

A Estimates for integrals

In this section we basically prove some estimates that are used for proofs in the main text. We prove the following bound

$$|\Theta(\alpha,\eta)| = \left|\int_0^1 dx g(x)\frac{1}{x-\alpha-i\eta}\right| \le C\log(\eta^{-1}) \tag{A.1}$$

Proof. For $\alpha < -1$ and $\alpha > 2$ the real part of $\frac{1}{x-\alpha-i\eta}$ is positive and negative respectively for any x. The imaginary part is always positive. Thus

$$\begin{aligned}
|\Theta(\alpha,\eta)| &\le \int_0^1 dx g(x)\left|\frac{1}{x-\alpha-i\eta}\right| \\
&\le g_{max}\int_0^1 dx\left|\frac{1}{x-\alpha}\right| \\
&\le Constant
\end{aligned}$$

For $-1 < \alpha < 2$ divergences appear. Integrating by parts we obtain

$$|\Theta(\alpha,\eta)| \le \left|g(x)\log(x-\alpha-i\eta)|_0^1\right| + \left|\int_0^1 dx g'(x)\log(x-\alpha-i\eta)\right|$$

For η very small we have

$$Max_{x,\alpha}|\log(x-\alpha-i\eta)| \le C\log(\eta^{-1})$$

Thus for bounded $g(E)$ and $g'(E)$ we have

$$|\Theta(\alpha,\eta)| \le C\log(\eta^{-1}) \tag{A.2}$$

□

$$\begin{aligned}
I(\alpha,\beta,\eta) &= \int_0^1 dx\left|\frac{1}{x-\alpha-i\eta}\right|^p\left|\frac{1}{x-\beta+i\eta}\right|^k \\
&\le C\left(\frac{1}{\eta}\right)^{k+p-1}
\end{aligned} \tag{A.3}$$

with p or k larger then 1.

Proof. By use of Hölder's inequality we have

$$I(\alpha,\beta,\eta) \leq \int_0^1 dx \left|\frac{1}{x-\alpha-i\eta}\right| \left|\frac{1}{x-\beta+i\eta}\right| \left(\frac{1}{\eta}\right)^{k+p-2}$$

$$I(\alpha,\beta,\eta) \leq \left(\int_0^1 dx \left|\frac{1}{x-\alpha-i\eta}\right|^2\right)^{\frac{1}{2}} \left(\int_0^1 dx \left|\frac{1}{x-\beta-i\eta}\right|^2\right)^{\frac{1}{2}} \left(\frac{1}{\eta}\right)^{k+p-2}$$

$$I(\alpha,\beta,\eta) \leq C\left(\frac{1}{\eta}\right)^{k+p-1} \tag{A.4}$$

□

$$I(\eta) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_0^1 dx \int_0^1 dy \frac{1}{|x-\alpha-i\eta|}\frac{1}{|x-\beta+i\eta|}\frac{1}{|y-\alpha-i\eta|}\frac{1}{|y-\beta+i\eta|} \tag{A.5}$$
$$\leq \frac{C}{\eta}\log^2\eta^{-1}$$

Proof. We use the following inequality:

$$\int_{-\infty}^{\infty} d\alpha \frac{1}{|x-\alpha-i\eta|}\frac{1}{|y-\alpha-i\eta|} \leq \frac{C}{|y-x-i\eta|}\left[1+\log\left|\frac{x-y}{\eta}\right|\right] \tag{A.6}$$

Bounding the integrations over α and β gives then

$$\begin{aligned} I(\eta) &\leq \int_0^1 dx \int_0^1 dy \frac{C}{|y-x-i\eta|^2}\left[1+\log\left|\frac{x-y}{\eta}\right|\right]^2 \\ &\leq \int_0^1 dx \int_0^1 dy \frac{C}{|y-x-i\eta|^2}\left[1+\log\frac{2}{\eta}\right]^2 \\ &\leq \int_0^1 dx \frac{C}{\eta}\left[1+C_1\log\frac{1}{\eta}\right]^2 \\ &\leq \frac{\tilde{C}}{\eta}\log^2\eta^{-1} \end{aligned} \tag{A.7}$$

□

$$I_2(\eta) = \int_{-\infty}^{\infty} d\alpha d\beta \int_0^1 dx dy \left|\frac{1}{z-\alpha-i\eta}\right|^k \left|\frac{1}{x-\beta+i\eta}\right| \left|\frac{1}{x-\alpha-i\eta}\right| \left|\frac{1}{y-\beta+i\eta}\right| \left|\frac{1}{y-\alpha-i\eta}\right| \tag{A.8}$$
$$\leq C\frac{1}{\eta^k}\log^3\left(\eta^{-1}\right)$$

with $1 \leq k$.

Proof. Using the inequality (A.6) over β and then y we obtain

$$I_2(\eta) \leq \frac{1}{\eta^{k-1}} \int_{-\infty}^{\infty} d\alpha \int_0^1 dxdy \left|\frac{1}{z-\alpha-i\eta}\right| \left|\frac{1}{x-\alpha-i\eta}\right| \left|\frac{1}{y-\alpha-i\eta}\right| \left|\frac{1}{x-y+i\eta}\right| \log\left|\frac{x-y}{\eta}\right|$$
$$\leq \frac{1}{\eta^{k-1}} \int_{-\infty}^{\infty} d\alpha \int_0^1 dx \left(\left|\frac{1}{z-\alpha-i\eta}\right|\right) \left|\frac{1}{x-\alpha-i\eta}\right| \left|\frac{1}{x-\alpha-i\eta}\right| \log\frac{2}{\eta} \log\left|\frac{x-\alpha}{\eta}\right|$$

In the region where α is much larger then x the integral is bounded by a constant. We thus have the following

$$I_2(\eta) \leq \frac{1}{\eta^{k-1}} C_1 \int_{-C_2}^{C_2} d\alpha \int_0^1 dx \left(\left|\frac{1}{z-\alpha-i\eta}\right|\right) \left|\frac{1}{x-\alpha-i\eta}\right|^2 \log^2 \eta^{-1}$$
$$\leq \frac{1}{\eta^{k-1}} C_1 \int_{-C_2}^{C_2} d\alpha \left(\left|\frac{1}{z-\alpha-i\eta}\right|\right) \frac{1}{\eta} \log^2 \eta^{-1}$$
$$\leq C \frac{1}{\eta^k} \log^3\left(\eta^{-1}\right) \tag{A.9}$$

□

$$I = \int_{-b_1}^{b_2} d\alpha d\beta \int dxdy \left|\frac{1}{x-\beta-i\eta}\right| \left|\frac{1}{y-\alpha-i\eta}\right| \left|\frac{1}{y-\beta+i\eta}\right| \left|\frac{1}{z-\alpha-i\eta}\right| \tag{A.10}$$

with $0 < b_1, 1 < b_2$.

Proof. Using successively Eq. (A.6)

$$I \leq C \int_{-c_1}^{c_1} d\alpha \int dxdy \left|\frac{1}{x-y-i\eta}\right| \left|\frac{1}{y-\alpha-i\eta}\right| \left|\frac{1}{z-\alpha-i\eta}\right| \left(1+\log\left(C_1\eta^{-1}\right)\right)$$
$$\leq C \int_{-c_1}^{c_1} d\alpha \int dx \left|\frac{1}{x-\alpha-i\eta}\right| \left|\frac{1}{z-\alpha-i\eta}\right| \left(1+\log\left(C_1\eta^{-1}\right)\right)^2$$
$$\leq C \int dx \left|\frac{1}{x-z-i\eta}\right| \left(1+\log\left(C_1\eta^{-1}\right)\right)^3$$
$$\leq \tilde{C} \log^4 \eta^{-1} \tag{A.11}$$

□

$$I_3(\eta) = \int_{-c_1}^{c_1} d\alpha d\beta \int dxdy \left|\frac{1}{x-\alpha-i\eta}\right| \left|\frac{1}{x-\beta-i\eta}\right| \left|\frac{1}{y-\alpha-i\eta}\right| \left|\frac{1}{y-\beta+i\eta}\right|$$
$$\times \left|\int_\Gamma d\omega \int dE \frac{g'(E)}{E-\omega}\right| \tag{A.12}$$
$$\leq C \log^3(\eta^{-1})\eta^{\frac{1}{2}} \tag{A.13}$$

Proof. We define the following:

$$I = \left| \int_{\Gamma(z,z')} d\omega \int dE \frac{g'(E)}{E-\omega} \right| \tag{A.14}$$

Γ is a path in the complex plane from z to z'. We have then:

$$\begin{aligned} I &\leq \int_{\Gamma(z,z')} d|\omega| \int dE \left| \frac{g'(E)}{E-\omega} \right| \\ &\leq \int_{\Gamma(z,z')} d|\omega| \int dE \frac{1}{|\omega|^{\frac{1}{2}}} \\ &\leq |z|^{\frac{1}{2}} - \left|z'\right|^{\frac{1}{2}} \\ &\leq \left|z-z'\right| \eta^{-\frac{1}{2}} \end{aligned} \tag{A.15}$$

Using this estimate in the following and applying Eq. (A.6) we obtain

$$I_3(\eta) \leq C \log^3(\eta^{-1}) \eta^{\frac{1}{2}} \tag{A.16}$$

□

We now want to prove inequality (2.104).

$$\int_z^{z'} d|\chi| \int d\omega \left| \frac{G'(\omega)}{\omega-\chi} \right| \leq \left|z-z'\right| \log(|\eta|) \tag{A.17}$$

with $z = \alpha - i\eta$ and $z' = \omega_{\bar{n}} - i\eta'$ The path we choose the integrate over in going from z to z' are two straight paths, first from $\alpha - i\eta$ to $z'' = \omega_{\bar{n}-i\eta}$ and then from z'' to $z' = \omega_{\bar{n}} - i\eta'$. We have then

$$\begin{aligned} \int_z^{z'} d|\chi| \int d\omega \left| \frac{G'(\omega)}{\omega-\chi} \right| &\leq G_{max} \int_z^{z'} d|\chi| \log\left(\omega - \chi_r + \sqrt{\chi_i^2 + (\omega-\chi_r)^2}\right)\Big|_0^1 \\ &\leq G_{max} \int_\alpha^{\omega_{\bar{n}}} d\chi_r \log\left(\omega - \chi_r + \sqrt{\eta^2 + (\omega-\chi_r)^2}\right)\Big|_0^1 \\ &+ G_{max} \int_\eta^{\eta'} d\chi_i \log\left(\omega - \omega_{\bar{n}} + \sqrt{\chi_i^2 + (\omega-\omega_{\bar{n}})^2}\right)\Big|_0^1 \end{aligned}$$

We can bound the first term as follows:

$$G_{max} \int_\alpha^{\omega_{\bar{n}}} d\chi_r \log\left(\omega - \chi_r + \sqrt{\eta^2 + (\omega-\chi_r)^2}\right)\Big|_0^1 \leq C_1 |\alpha - \omega_{\bar{n}}| \log|\eta| \tag{A.18}$$

The second term can be integrated again

$$\int_{\eta}^{\eta'} d\chi_i \log\left(\omega-\omega_{\bar{n}}+\sqrt{\chi_i^2+(\omega-\omega_{\bar{n}})^2}\right)\Big|_0^1$$
$$=-\chi_i+\chi_i\log\left(\omega-\omega_{\bar{n}}+\sqrt{\chi_i^2+(\omega-\omega_{\bar{n}})^2}\right)+(\omega-\omega_{\bar{n}})\log\left(\chi_i+\sqrt{\chi_i^2+(\omega-\omega_{\bar{n}})^2}\right)\Big|_0^1\Big|_{\eta}^{\eta'} \tag{A.19}$$

In the limit $\eta' \to 0$ we have then:

$$\begin{aligned}
&\left|\int_{\eta}^{\eta'} d\chi_i \log\left(\omega-\omega_{\bar{n}}+\sqrt{\chi_i^2+(\omega-\omega_{\bar{n}})^2}\right)\Big|_0^1\right| \\
\leq&\eta\left(1+\log|\eta|\right)+(\omega-\omega_{\bar{n}})\left(\log\left(\eta+\sqrt{\eta+(\omega-\omega_{\bar{n}})^2}\right)-\log|\omega-\omega_{\bar{n}}|\right)\Big|_0^1 \\
\leq&\eta\left(1+\log|\eta|+C_2\left(\omega-\omega_{\bar{n}}\right)\Big|_0^1\right)
\end{aligned}$$

We have then the bound

$$\begin{aligned}
\left|\int_{z}^{z'} d|\chi|\int d\omega\left|\frac{G'(\omega)}{\omega-\chi}\right|\right| \leq\, & C_1\,|\alpha-\omega_{\bar{n}}|\log|\eta|+C_2\eta\left(1+\log|\eta|\right) \\
\leq\, & C\,|\alpha-\omega_{\bar{n}}-i\eta|\log|\eta| \\
\leq\, & C\left|z-z'\right|\log|\eta|
\end{aligned} \tag{A.20}$$

In the following we prove

$$\left|\int_0^1 d\omega\int_{-B_1}^{B_1} db e^{-ib\omega} b^n F(\omega)\right| \leq CB_1^n \tag{A.21}$$

where $F(\omega)$ and its derivative are bounded by some constant. Integrating by parts in ω we have

$$\begin{aligned}
\int_0^1 d\omega\int_{-B_1}^{B_1} db e^{-ib\omega} b^n F(\omega) =& \int_{-B_1}^{B_1} db\frac{e^{-ib\omega}}{-i}b^{n-1}F(\omega)\Big|_0^1-\int_0^1 d\omega\int_{-B_1}^{B_1} db\frac{e^{-ib\omega}}{-i}b^{n-1}F'(\omega) \\
\leq& \int_{-B_1}^{B_1} db\,|b|^{n-1}\left(|F(1)|+|F(0)|\right)+\int_0^1 d\omega\int_{-B_1}^{B_1} db\,|b|^{n-1}\left|F'(\omega)\right| \\
\leq& CB_1^n
\end{aligned} \tag{A.22}$$

In the case that $n=0$ we have

$$\begin{aligned}
\int_0^1 d\omega\int_{-B_1}^{B_1} db e^{-ib\omega} F(\omega) =& \int_0^1 d\omega 2\frac{\sin(B_1)}{\omega}F(\omega) \\
\leq& C
\end{aligned} \tag{A.23}$$

B Graph structures and properties

In this appendix we want to prove some theorems about graph structures. In section 2.4 graphs were introduced as we wanted to average over a product of random variables. This was motivated by Wick's theorem ([32], [14], [33], [26]) for random variables, which we recall here

Theorem B.0.5. *Wick's theorem*
Say we have $2n$ random Gaussian variables denoted by X_i, $1 \leq i \leq 2n$, and say we have $Y = X_1 X_2 \ldots X_{2n}$. Denote by $\pi(2n)$ a set of pairings between all the elements of the set s, $s = (1, 2 \ldots, 2n)$, i.e. $\pi(2n)$ is a list of pairs of elements of s. Then

$$\mathbb{E}[Y] = \sum_{\pi(2n)} \prod_{(i,j)\in\pi(2n)} \mathbb{E}[X_i X_j] \tag{B.1}$$

where (i,j) refers to a pair of $\pi(2n)$. $\prod_{(i,j)\in\pi(2n)}$ is the product of all the pairs in $\pi(2n)$ and $\sum_{\pi(2n)}$ is a sum over all possible sets of pairings. A list $\pi(2n)$ can be seen as a graph on s.

In our case the random variables X_i come from the random matrix interaction. Each X_i being a matrix element is labeled by two number indicating the row and the column of course. An average of two matrix elements will induce a relation on those labels. Taking the random matrix to be a $N \times N$ complex gaussian random matrix, the average of two elements reads:

$$\begin{aligned} \mathbb{E}\left[X_i X_j\right] =& \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right] \\ =& \frac{\delta_{l_i,l_{j+1}} \delta_{l_{i+1},l_j}}{N} \end{aligned} \tag{B.2}$$

Our matrix elements are actually labeled as $V_{q_1,q_2}(l_1, l_2)$ but the same applies.
Starting from section 2.3 we introduced L^n which is just a product of random variables that comes from the matrix multiplication as more interaction terms are included in the expansion. We have then the following type of random variable products:

$$X_1 X_2 \ldots X_{2n} = V_{l_1,l_2} V_{l_2,l_3} \ldots V_{l_{2n},l_{2n+1}} \tag{B.3}$$

Upon averaging and using Wick's theorem we have then a sum over all possible pairings of the matrix elements. For example, if there were four random variables in the product, we would have:

$$\begin{aligned} \mathbb{E}\left[X_1 X_2 X_3 X_4\right] =& \mathbb{E}\left[V_{l_1,l_2} V_{l_2,l_3} V_{l_3,l_4} V_{l_4,l_5}\right] \\ =& \mathbb{E}\left[V_{l_1,l_2} V_{l_2,l_3}\right] \mathbb{E}\left[V_{l_3,l_4} V_{l_4,l_5}\right] \\ &+ \mathbb{E}\left[V_{l_1,l_2} V_{l_3,l_4}\right] \mathbb{E}\left[V_{l_2,l_3} V_{l_4,l_5}\right] \\ &+ \mathbb{E}\left[V_{l_1,l_2} V_{l_4,l_5}\right] \mathbb{E}\left[V_{l_2,l_3} V_{l_3,l_4}\right] \end{aligned}$$

This gives three different possibilities for the average to be non zero. Graphically we represent

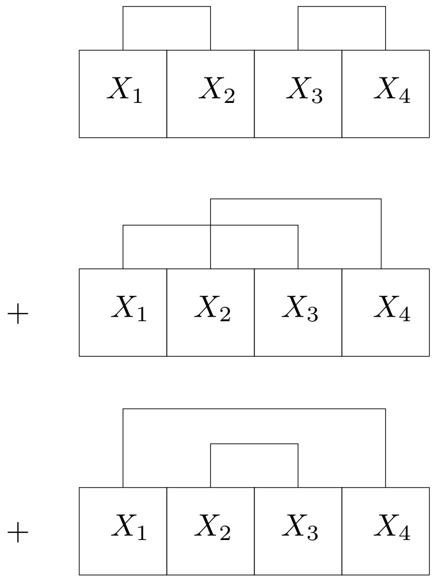

Figure B.1: Graphical representation of the contributions to the average of a product of four random variables

this as in figure B.1, where each box represents an X_i. The fact that the last index of a random variable X_i is equal to the first index of the next random variable X_{i+1} is represented by the fact that the boxes are right next to each other. The difference in the contributions comes into play when one sums up over the indices. Say we would want to compute the trace over the product of four random matrices. Using Eq.(B.2) we get

$$\begin{aligned}\mathrm{Tr}\left[V^4\right] &= \sum_{l_1,l_2,l_3,l_4} \mathbb{E}\left[V_{l_1,l_2} V_{l_2,l_3} V_{l_3,l_4} V_{l_4,l_1}\right] \\ &= \sum_{l_1,l_2,l_3,l_4} \frac{\delta_{l_1,l_3}}{N}\frac{\delta_{l_3,l_1}}{N} + \sum_{l_1,l_2,l_3,l_4} \frac{\delta_{l_1,l_4}\delta_{l_2,l_3}}{N}\frac{\delta_{l_2,l_1}\delta_{l_3,l_4}}{N} + \sum_{l_1,l_2,l_3,l_4} \frac{\delta_{l_2,l_4}}{N}\frac{\delta_{l_2,l_4}}{N} \\ &= N + N^{-1} + N \end{aligned} \tag{B.4}$$

The difference in the three contributions comes from the fact that the relations among the indices are more restricted for some than others. The δ-relations for the second contribution leave much less independent variables than for the other two. Characterizing how many independent variables different type of graphs possess is the main point of this section. As already mentioned the average over a pair $V_{l_i,l_{i+1}}(q_i,q_{i+1})\,V_{l_j,l_{j+1}}(q_j,q_{j+1})$ will generate a set of δ-identities among the l-variables and the q-variables, (see Eqs. (2.42) and (2.43)). Since from the identities on the l-variables we can deduce the identities on the q-variables we will focus on the l-variables since, the results for q-variables are analogues. We will refer to a random variable $V_{l_i,l_{i+1}}(q_i,q_{i+1})$ as X_i for brevity and when an average is made between two random variables, or equivalently when two random variables are paired up, it is called a contraction. In Eq. (B.1) we then two sets

$$R_I = \{X_1, \ldots, X_{2k}\}$$
$$I = \{l_1, \ldots l_{2k+1}\}$$

and a graph $\pi(2k)$ on the set R_I. The graph on R_I yields a set of relationships on I. We define the following:

Definition B.0.6. *Independent sets*
Say we have a set of variables, I, with some relationships among them given through a graph, $\pi(2k)$, on R_I. A subset S of a set I is called independent when the variables in S have no relationship (δ-relation) with the variables in I/S.

Definition B.0.7. *Totally independent sets and independent variables*
A subset S of a set I is called totally independent when S is independent and possesses no subsets that are independent. This means that all variables in S are equal.
Any variable of a totally independent set is then called an independent variable. They are all equivalent since they are all equal.

Based on these we add the following definition (similar to definitions 2.4.5)

Definition B.0.8. *For a graph $\pi(2k)$ on R_I that generates δ-relations on I we define:*
$\mathcal{A}_\pi$ = The set of independent variables of the set I given by the graph $\pi(2k)$
$\mathcal{B}_\pi$ = The set of dependent variables of the I given by the graph $\pi(2k)$ that is complementary to $\mathcal{A}_\pi$

For the definition of $\mathcal{A}_\pi$ we referred to "The set of independent variables". It does not really matter which variable we choose out of a totally independent set since they are all equal in the end. We denote by S_α sets which are independent and by I_α sets which are totally independent. W_β will denote sets with an index continuously running, that is $W_\beta = \{l_{i_\beta}, l_{i_\beta+1}, \ldots l_{j_\beta-1}, l_{j_\beta}\}$. R_{W_β} are sets associated to a set W_β as follows:

$$W_\beta = \{l_{i_\beta}, l_{i_\beta+1}, \ldots l_{j_\beta-1}, l_{j_\beta}\}$$
$$R_{W_\beta} = \{X_{i_\beta}, \ldots X_{j_\beta-1}\}$$

R_{S_α} are sets associated to S_α as follows:

$$S_\alpha = \bigcup_\beta W_\beta^\alpha$$
$$R_{S_\alpha} = \bigcup_\beta R_{W_\beta^\alpha}$$

We recall the definitions of Crossing, Non-Crossing graphs here.

Definition B.0.9. *Say we have the set R_I and a graph π on it.*

1 *A contraction, or pairing, between $X_i$and X_j is called a next neighboring contraction (nn-contraction) if $j = i+1$.*

2 *If we have a contraction between X_i and X_j, and a contraction between X_k and X_l, and $i < k < j < l$ then we call this a crossing.*

3 *If we have a contraction between X_i and X_j ($i < j$), and a no contraction between an X_k, with $i < k < j$, and an X_l, with $j < l$ or $l < i$ then we call this a non-crossing contraction, nc-contraction.*

4 *If a graph has at least 1 crossing it is called a Crossing graph, C-graph. If not it is called a Non-crossing graph, NC-graph.*

Definition B.0.10. *For any set S and any set of contractions on R_S we set*

$$M_S = \text{Number of independent variables in } S$$
$$N_S = \text{Number of elements in } S$$
$$C_S = \text{Number of contractions on } R_S$$

We now prove the following theorem:

Lemma B.0.11. *An NC-graph $\pi(2k)$ on the set $R_I = \{X_1, \dots X_{2k}\}$ will generate $k+1$ totally independent sets and thus $k+1$ independent variables on I.*

Proof. Lemma B.0.11
We represent R_I just as a set of connected boxes (see figure B.2). Since the graph is an NC-

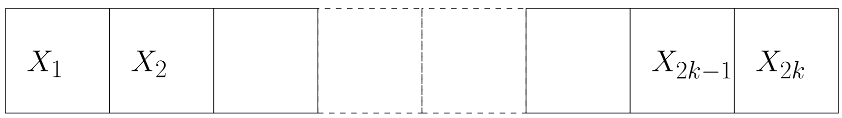

Figure B.2: Graphical representation of R_I

graph, all of the contractions are nc-contractions. We shall prove the theorem by applying successively the nc-contractions. If we have an nc-contraction between X_a and X_b we have the following

$$\begin{aligned}
I &= S_1 \cup S_2 \\
S_1 &= \{l_{a+1}, \dots l_b\} \\
R_{S_1} &= \{X_{a+1}, \dots X_{b-1}\} \\
S_2 &= \{l_1, \dots, l_a\} \cup \{l_{b+1}, \dots l_{2k+1}\} \\
S_2 &= W_1 \cup W_2 \\
R_{S_2} &= \{X_1, \dots, X_{a-1}, X_{b+1}, \dots, X_{2k}\} \\
&= \{X_1, \dots, X_{a-1}\} \cup \{X_{b+1}, \dots, X_{2k}\}
\end{aligned}$$

with S_1 and S_2 being independent and

$$\begin{aligned}
l_a &= l_{b+1} \\
l_{a+1} &= l_b
\end{aligned}$$

R_{S_1} and R_{S_2} are sets of random variables, which are still to be contracted. We wrote them separately because it is an nc-contraction and so no random variables of R_{S_1} can contract with those of R_{S_2}. The effect of a nc-contraction is thus to divide the set in two independent parts and to enforce an equality among the border terms. If we continue to apply nc-contraction to the sets S_1 and S_2 we will have each time one more independent set and border equations appearing. After multiple nc-contractions we will end up with independent sets of the following form

$$S = \bigcup_{\alpha=1}^{P_S} W_\alpha \tag{B.5}$$

$$W_\alpha = \{l_{i_\alpha}, \dots l_{j_\alpha}\} \tag{B.6}$$

$$\begin{aligned}
R_S &= \{X_{i_1}, \dots X_{j_1-1}, X_{i_2}, \dots X_{j_2-1}, \dots, X_{i_{P_S}}, \dots X_{j_{P_S}-1}\} \\
&= \bigcup_{\alpha=1}^{P_S} \{X_{i_\alpha}, \dots X_{j_\alpha-1}\} \\
&= \bigcup_{\alpha=1}^{P_S} R_{W_\alpha}
\end{aligned}$$

and with $P_S - 1$ equations for the border terms

$$l_{j_\alpha} = l_{i_{\alpha+1}} \tag{B.7}$$

for $1 \leq \alpha \leq P_S - 1$. When all nc-contractions have been made I will be split in $k+1$ independent sets.

$$I = \bigcup_{\alpha=1}^{k+1} S_\alpha \tag{B.8}$$

Each S_α has the form

$$S_\alpha = \bigcup_\beta W^\alpha_\beta$$

but since all the random variables X_i have been eliminated we see that each W^α_β contains exactly 1 element for if it did not, R_{I_α} would still contain some random variables to contract. In addition to all W^α_β containing just 1 element we have the border identities from Eq. (B.7) and so not only are the sets S_α independent but also totally independent. Thus the S_α's in Eq. (B.8) are actually I_α's. Thus an NC-graph splits I into $k+1$ totally independent sets and so has generated $k+1$ independent variables. □

We now prove the converse.

Lemma B.0.12. *A graph* $\pi(2k)$ *on the set* $R_I = \{X_1, \ldots X_{2k}\}$ *that generates more then, or equal to,* $k+1$ *totally independent sets and thus* $k+1$ *independent variables on* I *is in fact an NC-graph.*

For this we will need the following lemma:

Lemma B.0.13. *If a graph* $\pi(2k)$ *on the set* $R_S = \{X_1, \ldots X_{2k}\}$ *is such that the number of independent sets on* S, M_S *is greater then* $C_S + 1 = k+1$, *then at least one of the contractions is an nc-contraction.*

Proof. Suppose we have no nc-contractions. We have $S = \sum_{\alpha=1}^{M_S} I_\alpha$. Each I_α has N_{I_α} elements. Since there are no nc-contractions, there are nn-contractions (next neighboring). Therefore $N_\alpha \geq 2$. We have then for N_S:

$$\begin{aligned} N_S = &\sum_{\alpha=1}^{M_S} N_{I_\alpha} \\ \geq &2M_S \\ \geq &2\,(C_S+1) \end{aligned} \tag{B.9}$$

Since $C_S = \frac{N_S - 1}{2}$, this is a contradiction. □

We now turn to lemma B.0.12

Proof. Lemma B.0.12
Suppose we have a graph on $R_I = \{X_1, \ldots X_{2k}\}$ and that $M_I \geq k+1$. By Lemma B.0.13 there is at least one nc-contraction. Thus we can split I and $R_I = \{X_1, \ldots X_{2k}\}$ by performing the

contraction thus obtaining

$$
\begin{aligned}
I &= S_1 \cup S_2 \\
S_1 &= \{l_{a+1}, \dots l_b\} \\
R_{S_1} &= \{X_{a+1}, \dots X_{b-1}\} \\
S_2 &= \{l_1, \dots, l_a\} \cup \{l_{b+1}, \dots l_{2k+1}\} \\
S_2 &= W_1 \cup W_2 \\
R_{S_2} &= \{X_1, \dots, X_{a-1}, X_{b+1}, \dots, X_{2k}\} \\
&= \{X_1, \dots, X_{a-1}\} \cup \{X_{b+1}, \dots, X_{2k}\}
\end{aligned}
$$

Since $M_S = M_{S_1} + M_{S_2} \geq C_S + 1 = C_{S_1} + C_{S_2} + 3$, either $M_{S_1} \geq C_{S_1} + 1$ or $M_{S_2} \geq C_{S_2} + 1$. Notice that if $M_{S_1} = C_{S_1} + 1$ then $M_{S_2} \geq C_{S_2} + 1$. We can thus use lemma B.0.13 again and iterate this process and keep on splitting either S_1 or S_2. Since finally we must arrive at a situation where there are no more contractions to be performed for a certain set S_1', we have then $M_{S_1'} = C_{S_1'} + 1$ with $C_{S_1'} = 0$. This means that $M_{S_2'} \geq C_{S_2'} + 1$. We can thus continue this procedure until we reach the point where all contractions have been made. Thus the graph must have been a NC-graph. By lemma B.0.11 no graph can generate more then $k+1$ independent variables. □

When combining lemma B.0.12 and B.0.11 we have proved theorems 2.4.6 and 2.4.7.

Lemma B.0.14. *Any graph $\pi(2k)$ on the set $R_I = \{X_1, \dots X_{2k}\}$ will generate on set $I = \{l_1, \dots l_{2k+1}\}$ the identity $\delta_{l_1, l_{2k+1}}$. In other words l_1 and l_{2k+1} belong to the same totally independent set.*

Proof. Lemma B.0.14
Each variable X_i stands for a random variable $V_{l_i, l_{i+1}}$. A graph $\pi(2k)$ pairs up each $V_{l_i, l_{i+1}}$ with a $V_{l_j, l_{j+1}}$. We define $\Delta_i = l_i - l_{i+1}$ and thus by Eq. (B.2) a graph imposes $\Delta_i + \Delta_j = 0$. This means that the graph will impose

$$
\begin{aligned}
&\sum_{j=1}^{2k} \Delta_j = 0 \\
&\sum_{j=1}^{2k} l_j - l_{j+1} = 0 \\
&\sum_{j=1}^{2k} l_j = \sum_{j=1}^{2k} l_{j+1} \\
&l_1 = l_{2k+1}
\end{aligned}
$$

□

The previous lemmas help us to compute the averages over random matrices of which we give an example here. Suppose we want to calculate the average of the expectation value of a power of the random matrix.

$$
\begin{aligned}
\mathbb{E}\left[\langle\psi|\, V^{2k}|\psi\rangle\right] &= \sum_{l_1,\ldots l_{2k+1}} \psi^*(l_1)\psi(l_{2k+1})\mathbb{E}\left[V_{l_1,l_2}\ldots V_{l_{2k},l_{2k+1}}\right] \\
&= \sum_{l_1,\ldots l_{2k+1}} |\psi(l_1)|^2 \sum_{\pi(2k)} \prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right] \\
&= \sum_{l_1,\ldots l_{2k+1}} |\psi(l_1)|^2 \sum_{\pi(2k)} \prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right] \\
&= \sum_{\mathcal{A}_\pi}\sum_{\mathcal{B}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)} \prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right] \\
&= \sum_{\mathcal{A}_\pi}\sum_{\mathcal{B}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)\in\mathcal{G}_{0,1}} \prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right] \\
&+ \sum_{\mathcal{A}_\pi}\sum_{\mathcal{B}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)\in\mathcal{G}_{2}} \prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right]
\end{aligned}
$$

The product $\prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right]$ will have a weighing factor of $\frac{1}{N^k}$ and the sum over the dependent variables $\mathcal{B}_\pi$ will impose the δ identity of the dependent variables with the independent variables. Thus

$$
\sum_{\mathcal{B}_\pi} \prod_{(i,j)\in\pi(2k)} \mathbb{E}\left[V_{l_i,l_{i+1}} V_{l_j,l_{j+1}}\right] = \frac{1}{N^k} \tag{B.10}
$$

Therefore

$$
\mathbb{E}\left[\langle\psi|\, V^{2k}|\psi\rangle\right] = \sum_{\mathcal{A}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)\in\mathcal{G}_{0,1}} \frac{1}{N^k} + \sum_{\mathcal{A}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)\in\mathcal{G}_{2}} \frac{1}{N^k} \tag{B.11}
$$

Through the previous lemmas we know that for an NC-graph, $\pi(2k)$, the number of independent variables is $k+1$ and for an C-graph it is less then $k+1$. Thus

$$
\begin{aligned}
\sum_{\mathcal{A}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)\in\mathcal{G}_{0,1}} \frac{1}{N^k} &= \sum_{\pi(2k)\in\mathcal{G}_{0,1}} 1 \\
\sum_{\mathcal{A}_\pi} |\psi(l_1)|^2 \sum_{\pi(2k)\in\mathcal{G}_{2}} \frac{1}{N^k} &\leq \frac{1}{N} \sum_{\pi(2k)\in\mathcal{G}_{2}} 1
\end{aligned}
$$

We therefore get

$$
\mathbb{E}\left[\langle\psi|\, V^{2k}|\psi\rangle\right] \leq \sum_{\pi(2k)\in\mathcal{G}_{0,1}} 1 + \frac{1}{N} \sum_{\pi(2k)\in\mathcal{G}_{2}} 1
$$

We now prove a lemma similar to lemma B.0.11. The difference will be that the set R_I will not be the same and there will be condition imposed on the graph $\pi(2k)$.

Lemma B.0.15. *An NC-graph* $\pi(2k)$ *on the set* $R_I = \{X_1,\ldots,X_{p-1},X_{p+1},\ldots X_{2k+1}\}$*, with thus* k *pairings, where there is at least one pairing between an* X_j *with* $j \leq p-1$ *and a* X_i *with* $i \geq p+1$*, will generate* $k+1$ *totally independent sets and thus* $k+1$ *independent variables on the set* $I = \{l_1,\ldots,l_p,l_{p+1},\ldots,l_{2k+2}\}$*.*

Graphs of this sort will be called Non-Separable in Chapter 3.

Proof. Lemma B.0.15
Because the X_{p-1} variable is not connected to the X_{p+1} we represent in this case R_I as in figure B.3. Similar to our previous lemma we can start dividing our set I in independent sets

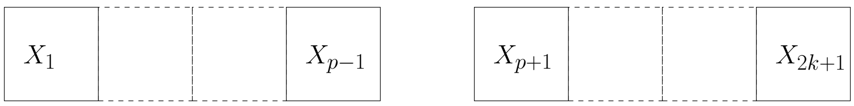

Figure B.3: Graphical representation of R_I

by applying the NC-contractions. We start then by imposing this contraction between the X_a with $a < p-1$ and X_b with $b > p+1$. We then have two independent sets.

$$\begin{aligned}
S_1 &= \{l_{a+1},\ldots,l_p,l_{p+1},\ldots,l_b\} \\
R_{S_1} &= \{X_{a+1},\ldots X_{p-1},X_{p+1},\ldots X_{b-1}\} \\
S_2 &= \{l_1,\ldots,l_a\} \cup \{l_{b+1},\ldots l_{2k+2}\} \\
&= W_1 \cup W_2 \\
R_{S_2} &= \{X_1,\ldots,X_{a-1},X_{b+1},\ldots,X_{2k+1}\} \\
&= \{X_1,\ldots,X_{a-1}\} \cup \{,X_{b+1},\ldots,X_{2k+1}\}
\end{aligned}$$

with S_1 containing l_p and l_{p+1}.

$$\begin{aligned}
l_a &= l_{b+1} \\
l_{a+1} &= l_b
\end{aligned}$$

Note that if multiple NC-contractions are applied to the set S_1, they will always generate sets, for which their most outer borders are equal. We can rearrange our set S_1 and R_{S_1} as

$$\begin{aligned}
S_1 =& \{l_{p+1},\ldots,l_{b-1},l_{a+1},\ldots,l_p\} \cup \{l_b\} \\
=& \tilde{S}_1 \cup \{l_b\} \\
R_{S_1} =& \{X_{p+1},\ldots X_{b-1},X_{a+1},\ldots X_{p-1}\}
\end{aligned}$$

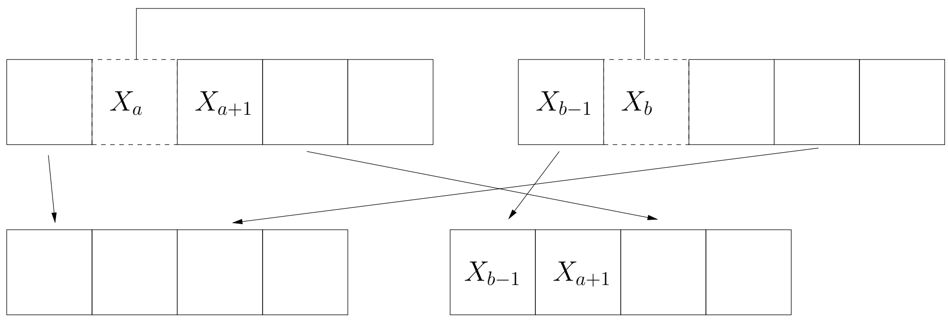

Figure B.4: Graphical representation of R_I

with $l_b = l_{a+1}$. This rearrangement is depicted in figure B.4. The following contractions on S_1 will not affect l_b, since this one is already equal to l_{a+1} and so will belong to the same totally independent set. We can thus apply lemma B.0.11 to $\tilde{S}_1$ and R_{S_1} to state that $\tilde{S}_1$, and thus S_1, has $\frac{b-a}{2}$ independent variables. Similarly we have for S_2 and R_{S_2}

$$\begin{aligned} S_2 =& \{l_1, \ldots, l_{a-1}, l_{b+1}, \ldots, l_{2k+2}\} \cup \{l_a\} \\ =& \tilde{S}_2 \cup \{l_a\} \\ R_{S_1} =& \{X_1, \ldots X_{a-1}, X_{b+1}, \ldots X_{2k+1}\} \end{aligned}$$

with $l_a = l_{b+1}$, and so l_a belongs to the same totally independent set as l_{b+1}. We can apply lemma B.0.11 to $\tilde{S}_2$ and R_{S_2} to state that $\tilde{S}_2$, has $\frac{2k+b-a}{2} + 1$ independent variables. S_2 has then $\frac{2k+b-a}{2} + 1$ independent variables. Since the number of independent variables of I is equal to those of S_1 plus those of S_2, I has $k+1$ independent variables. □

We have proved that an NC-graph on a product of $2k$ random variables produces $k+1$ independent variables. We now prove the converse, i.e. any graph that on a product of $2k$ random variables produces $k+1$ independent variables, is in fact an NC-graph. We shall prove this by showing that an nc-contraction can be applied each time. First we prove the following:

Lemma B.0.16. *Say we have a set $S = \bigcup_{\alpha=1}^{P_S} W_\alpha$ with $W_\alpha = \{l_{i_\alpha}, \ldots l_{j_\alpha}\}$ and $P_S - 1$ identities of the form $l_{j_\alpha} = l_{i_{\alpha+1}}$ ($l_{i_{P_S+1}} = l_{i_1}$) and a set of contractions among the random variables $R_S = \bigcup_{\alpha=1}^{P_S} \{X_{i_\alpha}, \ldots, X_{j_\alpha - 1}\} = \bigcup_{\alpha=1}^{P_S} R_{W_\alpha}$. We take S independent of the rest so $S = \cup_{\beta=1}^{M_S} I_\beta$ where the I_β are the totally independent sets produced by a graph. Each I_β possesses N_{I_β} elements, S possesses N_S elements and so there are $C_S = \frac{N_S - P_S}{2}$ contractions still to be performed.*
Then if $M_S \geq C_S + 1$ and $M_S \geq 2$ ($M_S = 1$ means there are no more contractions to be performed) there must exist an nc-contraction. That is we can divide the set in to two independent

sets S_1 and S_2 of the form

$$S_1 = \{l_i | l_i \in S, l_a \le l_i \le l_b\}$$
$$S_2 = \{l_i | l_i \in S, l_i < l_a, l_b < l_i\}$$

S_1 and S_2 have the form of S with the border identities applied. This would be a NC-contraction.

Proof. Lemma B.0.16
Suppose it not to be true. Clearly each $2 \le N_{I_\beta}$, for if it where not so for a certain I_γ, we could divide S into $S_1 = I_\gamma$ and S/S_1, S_1 having just one element and this would be a nc-contraction. In addition, the I_β that contains a border element of some W_α must have $3 \le N_{I_\beta}$ for if it where not the case it would have 2 elements and thus we would have $I_\beta = \{l_{j_\alpha}, l_{i_{\alpha+1}}\}$. Then we could divide S into $S_1 = I_\beta$ and S/S_1 which would be caused by a nc-contraction and would be a contradiction. For the total number of elements we have $N_S = \sum_{\beta=1}^{M_S} N_{I_\beta}$ and there are $P_S - 1$ sets I_β which have more then 3 elements, and thus

$$\begin{aligned} N_S &\ge 2M_S + P_S - 1 \\ N_S &\ge 2\left(C_S + 1\right) + P_S - 1 \\ N_S &\ge 2\left(\frac{N_S - P_S}{2} + 1\right) + P_S - 1 \\ N_S &\ge N_S + 1 \end{aligned}$$

which is a contradiction. Therefore if $C_S + 1 \le M_S$ we can divide the set into two independent sets, S_1 and S_2. This is then a nc-contraction. □

Now we prove the following

Lemma B.0.17. *Out of the sets S_1 and S_2 of lemma B.0.16, at least one has $C_{S_i} + 1 \le M_{S_i}$ and if $C_{S_1} + 1 = M_{S_1}$ then $C_{S_2} + 1 \le M_{S_2}$. This statement is of course symmetric under exchange of S_1 and S_2.*

Proof. Lemma B.0.17
If a NC-contraction divides the set S into S_1 and S_2 we have

$$\begin{aligned} S_1 &= \cup_{\alpha=1}^{P_{S_1}} W_\alpha^1 = \cup_{j=1}^{M_{S_1}} I_{\beta_j} \\ S_2 &= \cup_{\alpha=1}^{P_{S_2}} W_\alpha^2 = \cup_{j=1}^{M_{S_2}} I_{\gamma_j} \\ C_S &= C_{S_1} + C_{S_2} + 1 \\ M_I &= M_{S_1} + M_{S_2} \end{aligned}$$

Since $C_S + 1 \le M_S$ we have

$$\begin{aligned} C_S + 1 &\le M_S \\ C_{S_1} + C_{S_2} + 2 &\le M_{S_1} + M_{S_2} \end{aligned} \tag{B.12}$$

Thus either $C_{S_1}+1 \leq M_{S_1}$ or $C_{S_2}+1 \leq M_{S_2}$. If $C_{S_1}+1 = M_{S_1}$ we see by Eq. (B.12) we have $C_{S_2}+1 \leq M_{S_2}$. □

Lemma B.0.18. *Say we have graph on the set $R_I = \{X_1,\ldots,X_{p-1},X_{p+1},\ldots X_{2k+1}\}$ generating graph on the set*
$I = \{l_1,\ldots,l_p,l_{p+1},\ldots,l_{2k+2}\}$, where there is at least one pairing between an X_j with $j \leq p-1$ and a X_i with $i \geq p+1$, so that $I = \cup_{\alpha=1}^{M_I} I_\alpha$ with the I_α totally independent sets.
If $M_I \geq C_I+1$ then the graph is a NC-graph.

Proof. Lemma B.0.18
We shall show that we can always apply a nc-contraction which shall divide the previous set in 2 independent sets, S_1 and S_2

$$S_1 = \{l_i | l_i \in I, l_a \leq l_i \leq l_b\}$$
$$S_2 = \{l_i | l_i \in I, l_i < l_a \, or l_b < l_i\}$$

which would then amount to a nc-contraction. The reason why lemma B.0.16 cannot be applied is because there are no border identities present at this point. Notice that $C_I = \frac{N_1-2}{2}$. We suppose thus that we cannot divide the set I into S_1 and S_2 and thus that there are no nc-contractions. If this were the case each $N_{I_\alpha} \geq 2$. Suppose now that at least one I_α has $N_{I_\alpha} \geq 3$. This will be proved later. If this were the case we would have

$$\begin{aligned} N_I &= \sum_{\alpha_1}^{M_I} N_\alpha \\ &\geq 2M_I + 1 \\ &\geq N_I + 1 \end{aligned}$$

Which is a contradiction.
We now show that at least one I_α has $N_{I_\alpha} \geq 3$. Suppose all $N_{I_\alpha} = 2$. Say I_1 contains the variable l_{p+1} and as said that $N_1 = 2$. Clearly we cannot have $I_1 = \{l_p, l_{p+1}\}$ for this would be a NC-contraction between X_{p-1} and X_{p+1}.
If X_{p+1} would contract with any other element then X_{2k+1}, say X_i, then $l_{p+1} = l_{i+1}$. The contraction of X_{i+1} with any other X_j would then give $l_{i+1} = l_{j+1}$ and thus $N_{I_1} \geq 3$ which is a contradiction. Thus $I_1 = \{l_{p+1}, l_{2k+2}\}$.
The contraction generating this also related $l_{p+2} = l_{2k+1}$. The set I_2 containing l_{p+2} has already at least two elements l_{p+2} and l_{2k+1}. In order for $N_{I_2} = 2$ we need the X_{p+2} to contract with X_{2k} for if it did not l_{p+2} would be related to another variable and $N_{I_2} \geq 3$. We see we can apply the same argument to X_{p+3} and so on. Thus we would need X_i, with $i \geq p+1$ to contract with $X_{2k+p-i+2}$. Since at least one variable has to contract with a X_j with $j \leq p-1$ it is impossible to realize this procedure and therefore at least one $N_{I_\alpha} \geq 3$ which leads to a contradiction.

The set I can thus be divided in S_1 and S_2 which have the following characteristics.

$$\begin{aligned} S_1 &= \{l_1, \dots, l_a\} \cup \{l_b, \dots, l_{2k+2}\} \\ S_2 &= \{l_{a+1}, \dots l_{b-1}\} \\ C_I &= C_{S_1} + C_{S_2} + 1 \\ M_I &= M_{S_1} + M_{S_2} \\ l_a &= l_b \\ l_{a+1} &= l_{b-1} \end{aligned}$$

We have

$$\begin{aligned} M_I &\geq C_I + 1 \\ M_{S_1} + M_{S_2} &\geq C_{S_1} + C_{S_2} + 2 \end{aligned}$$

By applying successively lemma B.0.16 and B.0.17 to S_1 and S_2 we can perform all contractions by applying nc-contractions and so the graphs is an nc-graph. □

When combining lemma B.0.15 and B.0.18 we have proved theorems 2.4.6 and 2.4.7. Finally we give the following equalities for the amount of graphs of a certain order.

$$\sum_{\pi(n,m)\in\mathcal{G}(n,m)} 1 = \frac{(n+m)!}{2^{\frac{n+m}{2}} \frac{n+m}{2}!} \tag{B.13}$$

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}(n,m,\bar{n},\bar{m})} 1 = \frac{(n+m+\bar{n}+\bar{m})!}{2^{\frac{n+m+\bar{n}+\bar{m}}{2}} \frac{n+m+\bar{n}+\bar{m}}{2}!} \tag{B.14}$$

$$\sum_{\pi(n,m)\in\mathcal{G}_{0,1}(n,m)} 1 = \frac{(n+m)!}{\left(\frac{n+m}{2}\right)!\left(\frac{n+m}{2}+1\right)!} \tag{B.15}$$

$$\sum_{\pi(n,m,\bar{n},\bar{m})\in\mathcal{G}_{0,1}(n,m,\bar{n},\bar{m})} 1 = \frac{(n+m+\bar{n}+\bar{m})!}{\left(\frac{n+m+\bar{n}+\bar{m}}{2}\right)!\left(\frac{n+m+\bar{n}+\bar{m}}{2}+1\right)!} \tag{B.16}$$

Eqs. (B.15) and (B.16) can be shown by realizing that there is a one to one correspondence between these NC-graphs of order (n,m) or $(n,m,\bar{n},\bar{m})$ and Catalan paths of length $n+m$ or $n+m+\bar{n}+\bar{m}$. Thus the number of NC-graphs equals the Catalan number.

C From the discrete sum to the integration

In this appendix we want to explain how the factor N^{-1} appears in section 2.2 in Eq. (2.14) when we wish to calculate the limit $N \to \infty$.

Initially our Hilbert space is finite and of dimension $N\aleph$ and our system is in state $|\psi_0\rangle$. This give us a discrete probability distribution $P_0(q_0, l_0)$ which fullfills

$$\sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} P_0(q_0, l_0) = \sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} |\psi_0(q_0, l_0)|^2 = 1 \tag{C.1}$$

As $N \to \infty$ and as l_0 tend to a continues variable this property must be maintained. We can rewrite our equality as

$$\sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} \frac{1}{N} N P_0(q_0, l_0) = \sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} \frac{1}{N} \tilde{P}_0(q_0, l_0) \tag{C.2}$$
$$= 1$$

$$\lim_{N\to\infty} \sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} \frac{1}{N} N P_0(q_0, l_0) = \lim_{N\to\infty} \sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} \frac{1}{N} \tilde{P}_0(q_0, l_0)$$
$$= \sum_{q_0=1}^{\aleph} \int dE_0 \, g(E_0) P_0(q_0, E_0) \tag{C.3}$$
$$= 1$$

By demanding that $\tilde{P}_0(q_0, l_0) \xrightarrow{N\to\infty} P_0(q_0, E_0)$, remains bounded and becomes a function of a continues variable E_0 (that now stand for l_0), we have the definition of the Riemann integral. $g(E_0)$ is a function representing the density of states in the limit $N \to \infty$.

Another way of viewing this (equivalent) is to imagine that as our Hilbert space grows, the number of populated states grows as well and so the number of $P_0(q_0, l_0) \neq 0$ grows. Since the normalization condition (Eq. (C.1)) has to be satisfied $P_0(q_0, l_0)$ has to decrease as N grows and so we end up with a distribution $\tilde{P}_0(q_0, l_0) = NP_0(q_0, l_0)$ that remains finite. A consequence of this is

$$P_0\left(q_0, l_0\right) \leq \frac{C}{N} \tag{C.4}$$

$$\left|\psi_0\left(q_0, l_0\right)\right| \leq \sqrt{\frac{C}{N}} \tag{C.5}$$

We note that for many proofs (chapter 3) the factor N^{-1} seems crucial but is not. We could also perform the proofs by using the normalization condition of Eq. (C.1).

Equivalent to this is to consider from the start our normalization condition to be

$$\begin{aligned} \sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} \frac{1}{N} P_0\left(q_0, l_0\right) &= \sum_{q_0=1}^{\aleph} \sum_{l_0=1}^{N} \frac{1}{N} \left|\psi_0\left(q_0, l_0\right)\right|^2 \\ &= 1 \end{aligned} \tag{C.6}$$

such that $P_0\left(q_0, l_0\right) \xrightarrow{N \to \infty} P_0\left(q_0, E_0\right)$ and of course

$$\begin{aligned} \lim_{N \to \infty} \sum_{q_0=1}^{\aleph} \frac{1}{N} P_0\left(q_0, l_0\right) &= \int dE_0 g(E_0) \sum_{q_0=1}^{\aleph} P_0\left(q_0, E_0\right) \\ &= 1 \end{aligned} \tag{C.7}$$

D German Summary: Zusammenfassung

Diese Arbeit behandelt die Entstehung von Transportgleichungen in Quantensystemen. Für die Dynamik komplexer Systeme gibt es oftmals keine Lösung. Deshalb werden Annahmen und Näherungen vorgenommen, um die Komplexität zu reduzieren und um die Systeme entweder numerisch oder analytisch zu lösen. Einige dieser Annahmen sind gut begründet, andere wiederum können verblüffend wirken obwohl sie physikalisch sinnvoll sind. Die hier vorgebrachten und analysierten Modelle enthalten eine Zufallskomponente in ihrem Hamiltonian. Dieser zufällige Teil kann dazu verwendet werden, um einige chaotische Quantensysteme oder Systeme mit Unordnung zu modellieren. Die grundlegende Schwierigkeit liegt in der mathematischen Behandlung des zufälligen Teils, weshalb sie auch das grundlegende Ergebnis dieser Analyse darstellt. Eine solche Zufallskomponente im Hamiltonian und folglich die Beschreibung der Dynamik dieser Systeme ist hauptsächlich deshalb kompliziert, weil wir nicht ein einzelnes System betrachten sondern ein Ensemble. Zufallsmatrizen haben wesentlich zur Beschreibung der statistischen Eigenschaften komplexer Systeme beigetragen. Daher stellt die Analyse ihrer Effekte in der Dynamik von zufälligen Hamiltonians in einem System einen wichtigen Schritt für das Verständnis komplexer Quantensysteme und ihrer Transporteigenschaften dar. Wir führen eine mathematische Analyse der Zeitentwicklung der Schrödinger Gleichung durch für grosse zufällig gekoppelte Quantensysteme. Die Zufälligkeit ist durch ein Ensemble von Interaktionsmatrizen gegeben. Mit grossen Systemen meinen wir, dass die Dimension des Hilbertraums der gegebenen Systeme gross ist, sodass es sich also nicht unbedingt um ein räumlich ausgedehntes System handelt. Ähnlich definieren wir Makroobservable als diejenigen Observablen, die eine grosse Anzahl an Hilbertraum-Projektoren enthalten. Die Notwendigkeit der Analyse von Makroobservablen in diesem Sinne entsteht durch die Tatsache, dass die Evolution für "kleine Observablen" nicht aufzulösen ist. Denn die Besetzungszahl eines einzelnen Niveaus in einem gross dimensionerten Hilbertraum, der grösstenteils an der Systemevolution teilnimmt, ist annähernd Null. Man könnte grob sagen: wenn in einem D-dimensionalen Raum alle Basiszustände ein wenig an der Evolution teilnehmen und wenn keiner davon favorisiert wird, dann sollte die Besetzungszahl um D^{-1} oszillieren, was extrem gering für grosse D ist. Um die Zeitentwicklung des Systems zu ermitteln, verwenden wir eine Feynman-Diagramm-Expansion der Lösung der Schrödinger Gleichung. Dann kann die Relevanz der jeweiligen Diagramme geschätzt werden. Wir teilen die Feynman Diagramme in drei Kategorien ein und zeigen, dass nur eine davon beibehalten werden muss, um die Systementwicklung zu beschreiben. Wir zeigen, dass diese Schlussfolgerungen innerhalb bestimmter Limites gültig sind. Zuerst stellen wir fest, dass die Dimension des Systems und damit auch der gekoppelten Zufallsmatrix gross ist. Mit “gross” meinen wir, dass wir den Limes betrachten, an dem die Dimension gegen unendlich geht. Der zweite Limes, den wir benutzen, ist der Van Hove Limes, ein schwach gekoppelter Limes mit langer Zeit. Es sind diese Limites, die den verschiedenen Kategorien von Feynman Diagrammen mehr oder weniger Wichtigkeit in der Zeitentwicklung des Systems geben. Jedes Diagramm steht für einen möglichen Weg, den das System nehmen kann, und ist damit eine mögliche Geschichte des Systems. Wir zeigen, dass die relevanten Diagrammpaare diejenigen sind, die einander ähneln. Es ist bekannt, dass Interferenz-Effekte auftauchen, wenn verschiedene Feynman Wege sich überlappen. Daher kann man sagen, dass die Interferenz-Effekte irrelevant sind. So überrascht es nicht, dass man eine klassischere Beschreibung in der Sprache von stochastischen Gleichungen für das System erhalten kann. Wir zeigen, dass die Zeitentwicklung, die durch diese Klasse von Diagrammen beschrieben wird, mit der Lösung der Ratengleichungen identisch ist, die Fermis Goldene Regel vorhersagt. Wenn man die Frage nach der Zeitentwicklung des Systems beantworten möchte, bzw. präziser ausgedrückt, die nach der Zeitentwicklung des Ensembles, dann kann man sagen, dass sie durch eine Reihe von Ratengleichungen beschrieben wird, die sich an Fermis Goldene Regel halten. Da es nicht immer gültig ist, die Schlussfolgerungen über ein Ensemble auf ein beliebiges Mitglied des Ensembles zu übertragen, analysieren wir im Anschluss die Frage nach Typikalität. Hier geht es um die Frage, ob die Zeitentwicklung der meisten Mitglieder des Ensembles die gleiche ist wie die Zeitentwicklung des Ensembles. Wenn dies der Fall ist, sprechen wir von dynamischer Typikalität. Um festzustellen, ob dynamische Typikalität vorliegt oder nicht, muss die Varianz der Zeitentwicklung einer Observable analysiert werden. Wiederum wird die Feynman-Diagramm-Expansion der Lösung verwendet und man kann feststellen, dass im Limes des grossen Systems die Varianz verschwindet. Somit haben wir es mit dynamischer Typikalität zu tun. Wir betonen, dass in diesem Teil der Analyse, sowohl die Zeit als auch die Kopplungskonstante als endlich angenommen werden müssen. Hier nehmen wir nicht den Van Hove Limes, was bedeutet, dass wir nicht formal darauf schliessen können, dass das typische Verhalten durch die Ratengleichungen gegeben ist. Diese Möglichkeit ist natürlich nicht ausgeschlossen. Wir schliessen also, dass für die meisten Mitglieder des Ensembles die Makroobservablen die gleiche Zeitentwicklung für endliche Zeit und für die endliche Kopplungskonstante haben. Dieses Ergebnis lässt stark vermuten, dass dies auch im Van Hove Limes gelten sollte, jedoch ist eine raffiniertere Analyse notwendig, um dies zu beweisen.

Symbols

$\tilde{\Gamma}_M(t)$	Integral operator of the of the whole evolution after the M^{th} order perturbation
$\bar{n}$	Number of outer contraction for a graph $\pi(n,m)$
γ	scaling exponent between t and N
$\Gamma_n(t)$	Integral operator of the n^{th} order perturbation term of the perturbation theory
$\hat{H}_0$	Unperturbed Hamiltonian
$\hat{O}$	Macro-observable
$\hat{V}$	Perturbation Hamiltonian
κ_π	Number of independent variables
λ	Coupling constant
$\mathbb{E}[\cdot]$	Refers to the average over the random matrix ensemble
$\mathcal{F}_\pi(n,m)$	Product of free propagators evaluated on a certain contraction function given by $\pi(n,m)$, times λ^{n+m}
$\mathcal{G}_0(n,m)$	Set of Simple graphs of order (n,m)
$\mathcal{G}_1(n,m)$	Set of Nested graphs of order (n,m)
$\mathcal{G}_2(n,m)$	Set of Crossing graphs of order (n,m)
$\mathcal{G}_{ns}$	Set of Non-Separable graphs
$\mathcal{G}_s$	Set of Separable graphs
ω_j	Frequency or energy variable
$\pi(n,m)$	Graph of order (n,m)
$\mathrm{Var}[\cdot]$	Refers to the variance over the random matrix ensemble
$\mathrm{Var}[\cdot]$	Variance with respect to the random matrix distribution
$\tilde{C}_\pi(n,m)$	Non weighed contraction function on the sets $\{l_i, l'_j\}$ and $\{q_i, q'_j\}$, given by a graph $\pi(n,m)$
$\{l_j\}$	Refers to the list of variables $\{l_0, \dots l_n\}$
$\{l_j\}_0$	Refers to the list of variables $\{l_1 \dots l_n\}$, that is $\{l_j\}$ without l_0
$\{l_j\}_{\mathrm{i}}$	Refers to the independent variables of the list of variables $\{l_j\}$
A_π	Set of independent variables
B_π	Set of dependent variables
$C_\pi(n,m,\{l_i,l'_j\},\{q_i,q'_j\})$	Contraction function on the sets $\{l_i, l'_j\}$ and $\{q_i, q'_j\}$, given by a graph $\pi(n,m)$
$K^n(t,\{E_{l_j}\})$	Product of n successive propagators that depend $\{E_{l_j}\}$

k_j	Multiplicity on the right-hand side of the free propagator that is ω_j dependent
$L^n(\{q_j\},\{l_j\})$	Product of n successive random matrix elements that depend on $\{q_j\}$ and $\{l_j\}$
M	Order of the expansion
N	Number of different energy eigenstates of $\hat{H}_0$
p_j	Multiplicity on the left-hand side of the free propagator that is ω_j dependent
$Q_\pi(n,m)$	Product of free propagators evaluated on a certain contraction function given by $\pi(n,m)$
q_j	Variable denoting quantum numbers other then the energy
T	Macroscopic time ($\lambda^2 t$)

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Ehrenwörtliche Erklärung

Ich erkläre, daß ich diese Dissertation, abgesehen von den ausdrücklich bezeichneten Hilfsmitteln und den Ratschlägen von den jeweils namentlich aufgeführten Personen, selbständig verfaßt habe.

Stuttgart, July 9, 2009 *Pedro Alejandro Vidal Miranda*