On the $U-$module Structure of Unipotent Specht Modules of Finite General Linear Groups

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Vorgelegt von

Qiong Guo

aus China

Hauptberichter:	Prof. Dr. rer. nat. R. Dipper
Mitberichter:	Prof. Dr. rer. nat. S. König
	Dr. S. Lyle
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Institut für Algebra und Zahlentheorie der Universität Stuttgart

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Contents

Introduction

In [12], Gordon James investigated the Specht modules of the symmetric groups: For each partition λ of n there is a Specht module $S(\lambda)$, defined in terms of the intersection of the kernels of certain homomorphisms. The dimension of $S(\lambda)$ for $\mathfrak{S}_n$ can be determined in two different ways.

(1) Let h_{ij}^{λ} be the hook length for the (i,j) node in $[\lambda]$. Then

$$\dim S(\lambda) = \frac{n!}{\prod\limits_{(i,j)\in[\lambda]} h_{ij}^{\lambda}}.$$

(2) $\dim S(\lambda)$ equals the number of standard $\lambda-$tableaux. Indeed, there exists a basis of $S(\lambda)$ which is indexed by the standard $\lambda-$tableaux. This basis is called a standard basis of $S(\lambda)$.

Following the philosophy that $GL_n(q)$ is a $q-$analog to $\mathfrak{S}_n$, Gordon James defined the unipotent Specht modules S^{λ} over a field F for $GL_n(q)$ in [13]. If the characteristic of F is coprime to q then the dimension of S^{λ} is independent of F. Indeed one expects the representation theory for $GL_n(q)$ to translate into that for $\mathfrak{S}_n$ by setting $q = 1$. For $GL_n(q)$ we have

$$\dim S^{\lambda} = q^{\sum(k-1)\lambda_k} \frac{[n]!}{\prod\limits_{(i,j)\in[\lambda]} [h_{ij}^{\lambda}]}$$

where $[r] = 1 + q + q^2 + \cdots + q^{r-1}$. In the sense of the following conjecture by Richard Dipper and Gordon James, we have the analogous concept of standard basis.

Conjecture. For $\mathfrak{s} \in \mathrm{Std}(\lambda)$, there exist $r_{\mathfrak{s}}(t) \in \mathbb{Z}[t]$ with constant term 1 and $\mathcal{B}_{\mathfrak{s}} \subset S^{\lambda}$ of size $|\mathcal{B}_{\mathfrak{s}}| = r_{\mathfrak{s}}(q)$ such that $\mathcal{B} = \bigcup\limits_{\mathfrak{s}\in\mathrm{Std}(\lambda)} \mathcal{B}_{\mathfrak{s}}$ is a basis of S^{λ}. $\mathcal{B}$ is called the standard basis of S^{λ}.

This conjecture was proved by Marco Brandt, Richard Dipper, Gordon James and Sinéad Lyle for the case that λ is a 2-part partition. But the proof is rather combinational hence it seems the method there will not work for an arbitrary λ. This is our motivation to find a new method

which is more related to representation theory. In fact, we give a new proof of this conjecture for the case that λ is a 2-part partition and the characteristic of the field is zero. Unfortunately when we move to the arbitrary characteristic case, we appeal to BDJL's result at the moment. However, we do think we can provide an independent proof in the near future. We decompose the permutation module $M^{(n-m,m)}$ into irreducible FU_n-modules for the 2-part partition case where U_n is the group of (lower) unitriangular matrices in $GL_n(q)$, denoted by U for convenience. Thus we get a method to investigate the kernel of the homomorphisms between permutation modules, which gives the unipotent Specht module. This does not only give us the hope to solve this conjecture for general partitions, but also introduce a way possibly to solve some more problems. For example, we have found every irreducible component of the permutation module $M^{(n-m,m)}$ has a dimension of $q-$power, therefore, if it is true for an arbitrary partition, we have a good chance to give a new proof of a theorem given by Isaacs: Every irreducible complex character of U has $q-$power degree. There is also a good chance to solve a conjecture of Higman: The number of conjugate classes of U is a polynomial in q.

Our new strategy is to investigate the FU module structure of S^λ. The advantage of restriction to FU module is that FU is semisimple, since by general assumption the characteristic of the field is coprime to q. Indeed, we give a complete decomposition of $M^{(n-m,m)}$ into irreducible FU modules. We find every irreducible FU submodule of $M^{(n-m,m)}$ is labeled by some set S. We call it condition set, and the corresponding irreducible module has dimension q^c $(0 \leqslant c \in \mathbb{Z})$ where c is fixed by the positions of the entries in the condition set S. Hence the number of irreducible direct summands of $\mathrm{Res}^{FG}_{FU} M^{(n-m,m)}$ to a fixed dimension q^c is a polynomial in q.

Chapter 1 sets the scene and gives an overview of the fundamental definitions and propositions for compositions, partitions, λ-tableaux and Bruhat-decomposition.

In chapter 2, for an arbitrary composition, we introduce $\lambda-$flags. We give the original definitions of the permutation module $M^\lambda = \overline{P}_\lambda\big(FGL_n(q)\big)$ and the unipotent Specht module $S^\lambda = M^\lambda E^+_F(\lambda')FGL_n(q)$ where $E^+_F(\lambda')$ is an idempotent in U. But in fact in the following chapter, we use an equivalent definition. Define M^λ as vector space over F with $\lambda-$flags as its basis and define S^λ as the intersection of the kernels of certain homomorphisms.

From chapter 3, we focus our attention on 2-part partition $\lambda = (n-m, m)$. We start with the introduction of a notation $\Xi_{m,n}$ of the set of $\lambda-$flags. As we can assign to each $\lambda-$flag a $\lambda-$tableau and we have a total ordering on the set of row standard $\lambda-$tableaux. We define, for an element $v = \sum\limits_{X\in\Xi_{m,n}} C_X X$ in M^λ, $\mathrm{last}(v)$ as the last $\lambda-$tableau which can be assigned to a $\lambda-$flag X occurring in this sum with nonzero coefficient C_X; $\mathrm{top}(v)$ as the collection of all the $\lambda-$flag X occurring in this sum with

$\mathrm{tab}(X) = \mathrm{last}(v)$. Motivated by the fact that S^λ is a submodule of M^λ, we carefully investigate the operation of U on M^λ. We first decompose M^λ into $\mathfrak{t}-$batches $\mathfrak{M}_\mathfrak{t}$ where $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ by using Mackey Decomposition then we decompose each batch into direct summand of irreducible submodules. In fact, $\mathfrak{M}_\mathfrak{t}$ has a basis of orthogonal primitive idempotents $\mathcal{E}_\mathfrak{t} = \{e_{_L} \mid L \in \mathfrak{X}_\mathfrak{t}\}$ which is more adaptable to the $FU-$module structure. We show that the subgroup $U^w \cap U$ of U acts monomially on the set $\mathcal{E}_\mathfrak{t}$ where $\mathfrak{t} = \mathfrak{t}^\lambda w$. Then we prove the $U^w \cap U-$orbit module is an irreducible $FU-$module. Moreover we find each irreducible orbit module has a dimension of some $q-$power; and there is a uniquely determined matrix in each orbit, called a pattern matrix; and each orbit can be attached to a unique set, called a condition set. Moreover, we can prove when the condition set is the same, then the corresponding irreducible orbit modules are isomorphic.
Finding a standard basis of S^λ for a two part partition $\lambda = (n-m, m)$ is the goal of chapter 4. When λ is a 2-part partition, we have

$$S^\lambda = \bigcap_{i=0}^{m-1} \ker \phi_{_{1,\,i}}; \quad \dim\, S^\lambda = \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}.$$

Sinéad Lyle proves in [15] that for every element $0 \neq v \in S^\lambda$, $\mathrm{last}(v)$ is a standard $\lambda-$tableau. We show in characteristic zero case, $S^\lambda = \ker \phi_{_{1,\,m-1}}$. Thus after comparing the dimensions we obtain $\phi_{_{1,\,m-1}}$ is an epimorphism.

Finally we get the following theorem:

Theorem. (4.4.12) Let $\lambda = (n-m, m)$. For $L \in \Xi_{m,n}$, there exists $v_{_L} \in S^\lambda$ such that $\mathrm{last}(v_{_L}) = \mathrm{tab}(L)$ and $\mathrm{top}(v_{_L}) = e_{_L}$ if and only if

$$\mathrm{tab}(L) \setminus \{b_{i_k}, j_k \mid 1 \leqslant k \leqslant s\} \text{ is a shifted standard } \mu\text{-tableau},$$

where $e_{_L} \in \mathcal{O}$, $S = S(\mathcal{O}) = \{l_{_{b_{i_k} j_k}} \mid 1 \leqslant k \leqslant s\}, \mu = (n-m-s, m-s)$.

For every L satisfying the conditions above, we fix one element $v_{_L}$ (not uniquely determined) and let

$$\mathcal{B}_S^\lambda := \{v_{_L} \mid e_{_L} \in \mathcal{O} \subset M^\lambda, S(\mathcal{O}) = S, \mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ is standard}\}$$

and $\mathcal{B}^\lambda = \dot{\bigcup_S} \mathcal{B}_S^\lambda$. Then $\mathcal{B}_S^\lambda$ is a standard basis of the $S-$component $S^\lambda{\downarrow_S}$ and $\mathcal{B}^\lambda$ is a standard basis of S^λ.

Moreover we show the number of the basis elements v such that $\mathrm{last}(v) = \mathfrak{t} \in \mathrm{Std}(\lambda)$ is the rank polynomial $r_\mathfrak{t}(q)$ given by BDJL in [4], which provides a new proof of the fact that they add up to the generic degree of the unipotent Specht module S^λ where $\lambda = (n-m, m)$.

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Chapter 1

Preliminaries

1.1 Basic setting

Throughout this thesis, let p be a prime, q be a fixed power of p; in particular, it is never 1. Let F be a field whose characteristic is coprime to p and which contains a primitive p^{th} root of unity. F^* denotes the multiplicative group of F. Let n be a natural number and $GL_n(q)$ denote the group of invertible $n \times n$ matrices over $GF(q)$, the field of q elements.

Let V be a vector space over $GF(q)$ with basis $v_1, v_2, \cdots, v_n$. Then we can freely identify $GL_n(q)$ with the group of all automorphisms of V acting from the right. The automorphism given by the matrix (g_{ij}) is:

$$v_i \mapsto \sum_{j=1}^{n} v_j g_{ij}, \quad 1 \leqslant i \leqslant n.$$

If $v_1, v_2, \cdots, v_k$ are vectors in V, we let

$$\langle v_1, v_2, \cdots, v_k \rangle$$

denote the subspaces of V spanned by $v_1, v_2, \cdots, v_k$.

Fix, once and for all, a non-trivial group homomorphism

$$\theta : \big(GF(q), +\big) \to F^*.$$

Thus, θ is a linear $F-$character of the group $GF(q)$.

1.1.1 Definition.

(1) If m is a non-negative integer, let

$$[m] = 1 + q + q^2 + \cdots + q^{m-1}.$$

In particular, $[0] = 0, [1] = 1$ and

$$[m] = \begin{cases} \frac{q^m - 1}{q-1} & \text{if } q \neq 1 \\ m & \text{if } q = 1. \end{cases}$$

(2) If m is a non-negative integer, let

$$[m]! = [m][m-1]\cdots[1].$$

(3) If m, n are non-negative integers, let

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{[n]!}{[m]![n-m]!} & \text{if } n \geqslant m \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\begin{bmatrix} n \\ m \end{bmatrix}$ is a polynomial in q; it is known as a Gaussian polynomial. And it is a $q-$analogue of the binomial coefficient $\binom{n}{m}$; when we put $q = 1$, we get $\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}$.

1.1.2 Example.

$$\begin{aligned} \begin{bmatrix} 4 \\ 2 \end{bmatrix} &= \frac{[4]!}{[2]![2]!} = \frac{[4][3]}{[2][1]} = \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)} \\ &= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4. \end{aligned}$$

The Gaussian polynomials have the following property.

1.1.3 Proposition. Let m, n be non-negative integers. Then $\begin{bmatrix} n \\ m \end{bmatrix}$ counts the number of $m-$dimensional subspaces of an $n-$dimensional vector space $GF(q)^n$ over $GF(q)$.

Proof. See Theorem 13.1 in [2]. □

A well-known fact is the following proposition.

1.1.4 Proposition. Let n be a natural number. Then

$$|GL_n(q)| = \prod_{k=1}^{n} (q^n - q^{k-1}) = q^{\frac{n(n-1)}{2}} (q^n - 1)\cdots(q-1).$$

Proof. See, for example, Prop. 2 in page 41 of [1]. □

1.2 The symmetric group

1.2.1 Definition. A bijective function from $\{1, 2, \ldots, n\}$ into itself is called a permutation of n numbers, and the set of all permutations of n numbers, together with the usual composition of functions, is the symmetric group of degree n, which will be denoted by $\mathfrak{S}_n$.

We denote a permutation $\pi \in \mathfrak{S}_n$ either by

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1\pi & 2\pi & \cdots & n\pi \end{pmatrix}$$

or in cycle notation, where a cycle $(a_1, \ldots, a_k)$, $\quad (a_1, \ldots, a_k \in \{1, \cdots, n\}$ pairwise different), is the permutation, which maps a_i to a_{i+1} for $i = 1, \ldots, k-1$ and a_k to a_1. Note that each permutation may be written as product of disjoint cycles, and that the occuring cyclic factors are unique up to order. For example $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = (123)(45)$. Usually we omit cycles of length one in the cycle notation.

We embed $\mathfrak{S}_n$ into $GL_n(q)$ by assigning to a permutation π the appropriate permutation matrix $g = (g_{ij}) \in GL_n(q)$, where

$$g_{ij} := \begin{cases} 1 & \text{for } j = i\pi \\ 0 & \text{otherwise.} \end{cases}$$

A cycle of the form (i, j) is called a transposition and a cycle of the form $(i, i+1)$ is called a basic transposition, denoted by s_i, for $i = 1, 2, \ldots, n-1$. Then, as a Coxeter group, $\mathfrak{S}_n$ is generated by $s_1, s_2, \ldots, s_{n-1}$, subject to the relations

$$\begin{aligned} s_i^2 = 1, &\quad \text{for} \quad i = 1, 2, \ldots, n-1, \\ s_i s_j = s_j s_i, &\quad \text{for} \quad 1 \leqslant i < j-1 \leqslant n-2, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, &\quad \text{for} \quad i = 1, 2, \ldots, n-2. \end{aligned}$$

1.2.2 Definition. Suppose that $w \in \mathfrak{S}_n$ and write

$$w = s_{i_1} s_{i_2} \ldots s_{i_k}$$

where $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ are all basic transpositions. If k is minimal we say that w has length k and write $l(w) = k$. In this case, $s_{i_1} s_{i_2} \ldots s_{i_k}$ is called a reduced expression of w.

1.2.3 Definition.

(1) $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ is a composition of n, written as $\lambda \vDash n$, if $\lambda_1, \lambda_2, \lambda_3, \cdots$ are non-negative integers with

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i = n.$$

The non-zero λ_i are called the parts of λ.

(2) A partition of n, written as $\lambda \vdash n$, is a composition λ of n for which

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots .$$

For example: $(1,2,3,4) \vDash 10$, $(4,3,2,1) \vdash 10$.

1.2.4 Remark. In the notation of compositions we often suppress the zeros at the end, and indicate repeated parts by an index. For example,

$$(3,2,1,2,2,1,0,0,0) = (3,2,1,2^2,1).$$

1.2.5 Definition. Suppose λ is a composition. Define $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ by setting λ'_i to be the cardinality of $\{\lambda_j \,|\, \lambda_j \geqslant i\}$. Then λ' is obviously a partition. It is called the conjugate partition of λ.

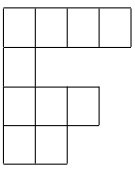
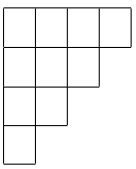
1.2.6 Definition. The diagram of a composition λ is the subset

$$[\lambda] = \{(i,j) \,|\, 1 \leqslant j \leqslant \lambda_i \text{ and } i \geqslant 1\}$$

of $\mathbb{N} \times \mathbb{N}$. If $(i,j) \in [\lambda]$, then (i,j) is called a node of λ. The $k-$th row (respectively, column) of a diagram consists of those nodes whose first (respectively, second) coordinate is k.

We represent the diagram as an array of boxes in the plane.

1.2.7 Example. If $\lambda = (4,1,3,2)$, then $\lambda' = (4,3,2,1)$. And we get

$[\lambda] =$  , $[\lambda'] =$  .

Obviously, for the conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ of λ, λ'_j equals the number of nodes in column j of $[\lambda]$.

1.2.8 Definition. Then dominance order $\trianglerighteq$ on the set of compositions of n is defined by $\lambda \trianglerighteq \mu$ if and only if, for all j,

$$\sum_{i=1}^{j} \lambda'_i \leqslant \sum_{i=1}^{j} \mu'_i$$

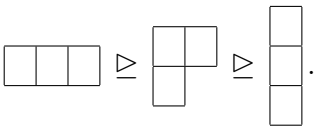
and $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ but $\lambda \neq \mu$.

1.2.9 Remark. Note that the dominance order on the set of compositions of n is not antisymmetric and hence no partial ordering. However, if we restrict $\trianglerighteq$ to partitions we obtain a partial ordering. And if λ and μ are partitions of n, then

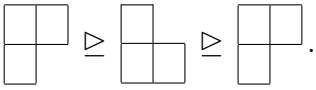
$$\lambda \trianglerighteq \mu \Leftrightarrow \sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i \text{ for all } j.$$

1.2.10 Example.

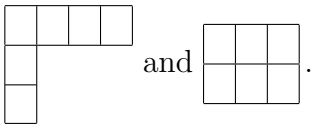
(1) We have the following order on the partitions of 3:



(2) It is possible to have $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$ but $\lambda \neq \mu$, so "$\trianglerighteq$" is not a partial order on the set of compositions of n. For example,



(3) However, obviously "$\trianglerighteq$" is a partial order on the partitions of n. For example, we can't order the elements



1.2.11 Definition. Let $\lambda \vdash n$ and $(i,j) \in [\lambda]$. Then:

(1) the $(i,j)-$*hook* of $[\lambda]$ consists of the $(i,j)-$node along with the $\lambda_i - j$ nodes to the right of it and the $\lambda'_j - i$ nodes below it;

(2) the length h^λ_{ij} of the $(i,j)-$*hook* is the number of nodes in (1). Then obviously,
$$h^\lambda_{ij} = \lambda_i + \lambda'_j + 1 - i - j;$$

(3) if we replace the $(i,j)-$node of $[\lambda]$ by the number h^λ_{ij} for each node, we obtain the hook graph.

1.2.12 Example. Let $\lambda = (4,3,2,1)$. Then the hook graph of λ is:

7	5	3	1
5	3	1	
3	1		
1			

.

1.3 Tableaux

1.3.1 Definition. Suppose λ is a composition of n. A $\lambda-$tableau is a bijection

$$\mathfrak{t} : [\lambda] \to \{1, 2, \cdots, n\};$$

we say that $\mathfrak{t}$ has shape λ and write $\mathrm{Shape}(\mathfrak{t}) = \lambda$.

We represent a tableau $\mathfrak{t}$ by a labeled diagram replacing every node (i, j) in $[\lambda]$ by it's image in $\{1, 2, \cdots, n\}$ under the map $\mathfrak{t}$.

1.3.2 Example. Let $\lambda = (2, 3, 2)$. Then

$$\mathfrak{t}_1 = \begin{array}{|c|c|c} \hline 1 & 2 & \\ \hline 3 & 4 & \multicolumn{1}{|c|}{5} \\ \hline 6 & 7 & \\ \hline \end{array} \text{ and } \mathfrak{t}_2 = \begin{array}{|c|c|c} \hline 1 & 4 & \\ \hline 2 & 5 & \multicolumn{1}{|c|}{7} \\ \hline 3 & 6 & \\ \hline \end{array}$$

are two $\lambda-$tableaux.

The symmetric group $\mathfrak{S}_n$ acts on the set of $\lambda-$tableaux from the right by permuting the integers $1, 2, 3, \ldots, n$. In Example 1.3.2,

$$\mathfrak{t}_1(2\,4\,5\,7\,6\,3) = \mathfrak{t}_2.$$

1.3.3 Definition. Let λ be a composition of n. The λ-tableau, where the nodes are replaced with the numbers $1, 2, \ldots, n$ in order along

(1) the rows is called the initial tableau and is denoted by $\mathfrak{t}^\lambda$.

(2) the columns is denoted by $\mathfrak{t}_\lambda$.

(3) the row stabilizer of $\mathfrak{t}^\lambda$ is called the standard Young subgroup of $\mathfrak{S}_n$ with respect to λ and is denoted by $\mathfrak{S}_\lambda$.

1.3.4 Definition. Let λ be a composition and $\mathfrak{s}$ be a $\lambda-$tableau. Define $d(\mathfrak{s}) \in \mathfrak{S}_n$ by letting $\mathfrak{t}^\lambda d(\mathfrak{s}) = \mathfrak{s}$.

1.3.5 Definition. For $\lambda \vDash n$, define $w_\lambda \in \mathfrak{S}_n$ such that

$$\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$$

1.3.6 Example. Let $\lambda = (2, 3, 2)$ and $\mathfrak{t}_1, \mathfrak{t}_2$ as in Example 1.3.2. Then

$$\mathfrak{t}^\lambda = \mathfrak{t}_1, \mathfrak{t}_\lambda = \mathfrak{t}_2 \text{ and } w_\lambda = d(\mathfrak{t}_\lambda) = (2\,4\,5\,7\,6\,3).$$

1.3.7 Notation. Suppose λ is a composition of n and $\mathfrak{t}$ is a $\lambda-$tableau. For $1 \leqslant b \leqslant n$ we write $\mathrm{row}_\mathfrak{t}(b) = i\left(\mathrm{col}_\mathfrak{t}(b) = j\right)$ of b occurs in row i (respectively column j) in $\mathfrak{t}$.

1.3.8 Definition. Let λ be a composition and $\mathfrak{t}$ a $\lambda-$tableau. Then $\mathfrak{t}$ is

(1) row-standard if the entries in $\mathfrak{t}$ increase from left to right in each row. We write $\mathrm{RStd}(\lambda)$ for the set of row-standard $\lambda-$tableaux.

(2) standard if λ is a partition and the entries in $\mathfrak{t}$ increase from left to right in each row and from top to bottom in each column. We write $\mathrm{Std}(\lambda)$ for the set of standard $\lambda-$tableaux.

1.3.9 Example. Let $\lambda = (2, 2)$. Then

$$\mathrm{Std}(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\};$$

$$\mathrm{RStd}(\lambda) = \left\{ \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\}.$$

1.3.10 Remark. Let $\mathcal{D}_\lambda = \{w \in \mathfrak{S}_n \,|\, \mathfrak{t}^\lambda w \in \mathrm{RStd}(\lambda)\}$. Then $\mathcal{D}_\lambda$ is a set of right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. We have

$$\mathcal{D}_\lambda = \{w \in \mathfrak{S}_n \,|\, iw < (i+1)w \text{ if } (i, i+1) \in \mathfrak{S}_\lambda\}.$$

1.4 Root subgroups of $GL_n(q)$

1.4.1 Definition. For $1 \leqslant i, j \leqslant n$. Define e_{ij} to be the $(i, j)-$th matrix unit, that is the $n \times n$ matrix over $GF(q)$, whose entries are all 0 except on place (i, j), where it is 1. And the symmetric group $\mathfrak{S}_n$ operates on those by conjugation:

$$e_{ij}.\pi := \pi^{-1} e_{ij} \pi = e_{i\pi, j\pi}.$$

1.4.2 Definition. For $1 \leqslant i, j \leqslant n$, and $i \neq j$. Define

$$x_{ij}(\alpha) = 1 + \alpha e_{ij},$$

where $1 = E_n = \sum_{i=1}^n e_{ii}$ is the unit matrix. Set

$$X_{ij} = \{x_{ij}(\alpha) \,|\, \alpha \in GF(q)\}.$$

Note that for $\alpha, \beta \in GF(q)$,

$$x_{ij}(\alpha) x_{ij}(\beta) = x_{ij}(\alpha + \beta).$$

Hence X_{ij} is a subgroup of $G = GL_n(q)$, the so called **root subgroup** to index (i, j). Moreover X_{ij} is isomorphic to $\big(GF(q), +\big)$, the additive group of the finite field $GF(q)$.

1.4.3 Definition. Let $\Pi = \{(i,j) \,|\, i \neq j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$. A subset Υ of Π is said to be closed if, for all $(i,j),(j,k) \in \Upsilon$, we have $(i,k) \in \Upsilon$.

If Υ is a closed subset of Π, let $G(\Upsilon)$ be the following subgroup of $GL_n(q)$:

$$G(\Upsilon) = \langle X_{ij} \,|\, (i,j) \in \Upsilon \,\rangle.$$

Then next result is very useful in the later Chapters.

1.4.4 Theorem. If Υ is a closed subset of Π, then

$$G(\Upsilon) = \prod_{(i,j)\in\Upsilon} X_{ij}$$

where the product can be taken in any order. The group $G(\Upsilon)$ consists of all those matrices which have 1's on the diagonal and zeros in those places $(i,j) \in \Pi \setminus \Upsilon$. Once the order of the product has been chosen, each element of $G(\Upsilon)$ has a unique expression of the form

$$\prod_{(i,j)\in\Upsilon} x_{ij}(\alpha_{ij}) \qquad \big(\alpha_{ij} \in GF(q)\big).$$

Proof. c.f. [13]. □

We remark that the root subgroups together with the diagonal matrices and permutation matrices generate $GL_n(q)$. For the general definition of root systems and their meaning for semisimple Lie algebras and associated groups of Lie type we refer to [5].

1.4.5 Definition.

(1) Let $\Delta^+ = \{(i,j) \,|\, 1 \leqslant i < j \leqslant n\}$. Define

$$U_n^+ = \prod_{(i,j)\in\Delta^+} X_{ij}.$$

In particular, U_n^+ is the group of upper unitriangular matrices.

(2) Let $\Delta^- = \{(i,j) \,|\, 1 \leqslant j < i \leqslant n\}$. Define

$$U_n^- = \prod_{(i,j)\in\Delta^-} X_{ij}.$$

In particular, U_n^- is the group of lower unitriangular matrices.

1.4.6 Definition. Let $1 \leqslant i \leqslant n$ and $0 \neq \beta \in GF(q)$. Define $h_i(\beta) \in GL_n(q)$ be the matrix with (i,i) entry equal to β, all other diagonal entries equal to 1 and zeros elsewhere. Thus:

$$h_i(\beta) = \begin{array}{cc} \begin{array}{ccccc} & & i & & \end{array} & \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \beta & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} & i \end{array}$$

Let $H = \langle h_i(\beta) \,|\, 1 \leqslant i \leqslant n,\, 0 \neq \beta \in GF(q) \,\rangle$. Then H is the group of diagonal matrices.

1.4.7 Proposition. H normalizes each root subgroup.

Proof. For any arbitrary $\alpha, \beta \in GF(q), \beta \neq 0$, we have

$$h_k^{-1}(\beta) x_{ij}(\alpha) h_k(\beta) = \begin{cases} x_{ij}(\alpha\beta) & \text{if } k = j \\ x_{ij}(\alpha\beta^{-1}) & \text{if } k = i \\ x_{ij}(\alpha) & \text{otherwise.} \end{cases}$$

Therefore, H normalizes each root subgroup. □

1.4.8 Remark. By the proposition above, it is easy to see: H normalizes U_n^+ and U_n^-. We write

$$B_n^+ = U_n^+ H = H U_n^+; \quad B_n^- = U_n^- H = H U_n^-.$$

These subgroups are called (upper, lower) Borel subgroups.

So:

$$(1)\ U_n^+ = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}; \quad B_n^+ = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n(q) \right\};$$

$$(2)\ U_n^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}; \quad B_n^- = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \in GL_n(q) \right\}.$$

1.4.9 Proposition. (Bruhat Decomposition)

$$GL_n(q) = \dot{\bigcup_{w \in W}} B_n^+ w B_n^- = \dot{\bigcup_{w \in W}} B_n^+ w H U_n^-$$

where $W \cong \mathfrak{S}_n$, the subgroup of all permutation matrices in $GL_n(q)$.

Proof. c.f. Theorem 65.4 and Theorem 65.10 in [6]. □

1.4.10 Notation. We shall work mostly with the subgroups U_n^- of lower triangular unipotent matrices of $GL_n(q)$. In order to keep notation simple we denote this group in the following by U.

Chapter 2

The Specht module S^λ

2.1 The permutation module M^λ

Firstly, we introduce the definition of $\lambda-$flags as in [13].

2.1.1 Definition. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_h)$ be a composition of n. Then a set of subspaces $V_0, V_1, V_2, \cdots, V_h$ of the vector space $GF(q)^n$ with the properties

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{h-1} \supseteq V_h = 0$$

such that

$$\dim(V_{i-1}/V_i) = \lambda_i, \forall\, 1 \leqslant i \leqslant h$$

is called a $\lambda-$flag. The set of $\lambda-$flags is denoted by $\mathcal{F}(\lambda)$.

Clearly, right multiplication of $GL_n(q)$ on V induces a permutation action of $GL_n(q)$ on $\mathcal{F}(\lambda)$. We may construct the corresponding permutation representation of $GL_n(q)$ by taking the vector space over F whose basis elements are all the $\lambda-$flags.

2.1.2 Definition. Let $\lambda \vDash n$. Define a parabolic subgroup of $GL_n(q)$:

$$P_\lambda = \langle H, X_{ij} \,|\, row_{\mathfrak{t}^\lambda}(i) \leqslant row_{\mathfrak{t}^\lambda}(j) \rangle,$$

where $\mathfrak{t}^\lambda$ is the initial $\lambda-$tableau. Clearly, if $\lambda = (1^n)$, then $P_\lambda = B_n^+$.

We have the following well known results:

2.1.3 Proposition. Suppose $\{v_1, v_2, \cdots, v_n\}$ is a basis of V. Consider the following $\lambda-$flag:

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{h-1} \supseteq V_h = 0,$$

where

$$V_i = \langle v_{\Lambda_i+1}, v_{\Lambda_i+2}, \cdots, v_n \rangle \quad \forall\, 0 \leqslant i \leqslant h,$$

and

$$\Lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

Then the stabilizer of this $\lambda-$flag in $GL_n(q)$ is P_λ.

2.1.4 Definition. Let $\lambda \vDash n$ and P_λ be the corresponding parabolic subgroup. Let

$$\overline{P}_\lambda = \sum_{g \in P_\lambda} g \in FGL_n(q).$$

Define

$$M^\lambda = \overline{P}_\lambda(FGL_n(q)).$$

Thus, M^λ is a right ideal of $FGL_n(q)$, and so it is a right $FGL_n(q)-$module.

By general theory, M^λ is the permutation representation arising from the permutation action of $GL_n(q)$ on $\mathcal{F}(\lambda)$.

2.1.5 Definition. Suppose that $\lambda \vDash n$ Let

$$U_\lambda^- = \langle X_{ij} \mid \mathrm{row}_{t^\lambda}(i) > \mathrm{row}_{t^\lambda}(j) \rangle.$$

2.1.6 Lemma. Suppose that $\lambda \vDash n, w \in \mathfrak{S}_n$. Then

$$w^{-1}P_\lambda w \cap U_n^- = \langle X_{ij} \mid i > j \text{ and } \mathrm{row}_{\mathfrak{t}^{\lambda_w}}(i) \leqslant \mathrm{row}_{\mathfrak{t}^{\lambda_w}}(j) \rangle.$$

Proof. c.f. [13] □

2.1.7 Proposition. Suppose that λ is a composition of n. Then every right coset of P_λ in $GL_n(q)$ has the form

$$P_\lambda wu, \text{ where } w \in \mathcal{D}_\lambda, \text{ and } u \in w^{-1}U_\lambda^- w \cap U_n^-.$$

Proof. c.f. [13] □

Next we introduce a subgroup of the lower unitriangular group, which will play an important role in the next chapter.

2.1.8 Lemma. Let $\lambda \vdash n$, $\mathfrak{s} = \mathfrak{t}^\lambda w \in \mathrm{RStd}(\lambda)$ where $w \in \mathfrak{S}_n$. Let $g = (g_{ij}) \in GL_n(q)$, $U_n = U_n^-$. Then $g \in U_n^w \cap U_n = w^{-1}U_n w \cap U_n$ if and only if

$$g_{ii} = 1, \forall\, 1 \leqslant i \leqslant n$$

and $\forall\, 1 \leqslant i, j \leqslant n$:

$$\big(i < j \text{ or } \mathrm{row}_{\mathfrak{s}}(i) < \mathrm{row}_{\mathfrak{s}}(j)\big) \Longrightarrow g_{ij} = 0.$$

So $U_n^w \cap U_n$ consists of all matrices, which are contained in U_n (hence have zeros above and 1's on the diagonal) and in addition have zeros at all places (i, j) with $i > j$ and $\mathrm{row}_{\mathfrak{s}}(i) < \mathrm{row}_{\mathfrak{s}}(j)$.

Proof. Let $h = (h_{kl}) \in GL_n(q)$ and $g = (g_{ij}) = w^{-1}hw$. Then

$$g_{ij} = h_{kl} \text{ for } i = kw, j = lw, \text{ and } \forall\, 1 \leqslant k, l \leqslant n.$$

Note that i occupies the place in $\mathfrak{s} = \mathfrak{t}^\lambda w$ which in occupied by k in $\mathfrak{t}^\lambda$ and j occupies the place in $\mathfrak{s}$ which in occupied by l in $\mathfrak{t}^\lambda$. In particular,

$$\mathrm{row}_{\mathfrak{t}^\lambda}(k) = \mathrm{row}_{\mathfrak{s}}(i) \text{ and } \mathrm{row}_{\mathfrak{t}^\lambda}(l) = \mathrm{row}_{\mathfrak{s}}(j).$$

(1) Suppose $g \in U_n^w \cap U_n$, then $h = wgw^{-1} \in U_n$.

1) If $i < j$ then $g_{ij} = 0$, since $g \in U_n$.
2) If $\mathrm{row}_{\mathfrak{s}}(i) < \mathrm{row}_{\mathfrak{s}}(j)$, then $\mathrm{row}_{\mathfrak{t}^\lambda}(k) < \mathrm{row}_{\mathfrak{t}^\lambda}(l)$, hence $k < l$ and $g_{ij} = h_{kl} = 0$, since $h \in U_n$.

This proves one direction ($\Rightarrow$) of the claim.

(2) Conversely, suppose $g \notin U_n^w \cap U_n$. Then either $g \notin U_n$ or $h \notin U_n$.

1) If $g \notin U_n$, we find $1 \leqslant i \leqslant n$ with $g_{ii} \neq 1$ or $1 \leqslant i < j \leqslant n$ with $g_{ij} \neq 0$ and hence we are done.
2) Let $h \notin U_n$. If $h_{kk} \neq 1$ for some $1 \leqslant k \leqslant n$, then $g_{ii} \neq 1$ for $i = kw$ and we are done as well. Now suppose $h_{kk} = 1$, $\forall\, 1 \leqslant k \leqslant n$. Then there exists $1 \leqslant k < l \leqslant n$ such that $h_{kl} \neq 0$. If $\mathrm{row}_{\mathfrak{t}^\lambda}(k) < \mathrm{row}_{\mathfrak{t}^\lambda}(l)$, then $\mathrm{row}_{\mathfrak{s}}(i) < \mathrm{row}_{\mathfrak{s}}(j)$, but $g_{ij} = h_{kl} \neq 0$ which is a contradiction, hence we are done in this case as well. If $\mathrm{row}_{\mathfrak{t}^\lambda}(k) = \mathrm{row}_{\mathfrak{t}^\lambda}(l)$, we have $\mathrm{row}_{\mathfrak{s}}(i) = \mathrm{row}_{\mathfrak{s}}(j)$. Since $k < l$, the entry k comes in $\mathfrak{t}^\lambda$ earlier then entry l, hence the same is true for i and j in that row of $\mathfrak{s}$. Therefore $i < j$ since $\mathfrak{s}$ is row standard, but $g_{ij} = h_{kl} \neq 0$ which is a contradiction as well.

This finishes the proof of the other direction ($\Leftarrow$).

□

2.2 The Specht module S^λ

Remember that we fixed a non trivial linear character

$$\theta : \big(GF(q), +\big) \to F^*$$

in Section 1.1.

2.2.1 Definition. Let $\mu \vDash n$. There is a linear $F-$character θ_μ of U_n^+, defined by

$$\theta_\mu(u) = \theta\big(\sum u_{i,i+1}\big) = \prod \theta(u_{i,i+1})$$

where $u \in U_n^+$, u_{ij} is the (i,j) entry in u, and the sum being over those i such that

$$\mathrm{row}_{\mathfrak{t}^\mu}(i) = \mathrm{row}_{\mathfrak{t}^\mu}(i+1).$$

2.2.2 Definition. Suppose that $\lambda \vDash n$. Let

$$E_F^+(\mu) = \frac{1}{|U_n^+|} \sum_{u \in U_n^+} \theta_\mu(u^{-1})u.$$

Then by general theory, $E_F^+(\mu)$ is an idempotent element of FU_n^+.

2.2.3 Definition. Suppose that $\lambda \vDash n$. Let

$$S^\lambda = M^\lambda E_F^+(\lambda')FGL_n(q).$$

This $FGL_n(q)-$module is called the unipotent Specht module of $GL_n(q)$ associated with λ.

It is known that for $F = \mathbb{C}$, the complex field, S^λ is an irreducible $\mathbb{C}GL_n(q)-$module. In fact in this way one has constructed all irreducible $\mathbb{C}GL_n(q)-$modules contained in the so called principal series of $GL_n(q)$. For details we refer to [13]. Gordon James showed in [13] the following kernel intersection theorem:

2.2.4 Theorem. If λ is a composition of n, then

$$S^\lambda = \bigcap_{\mu \rhd \lambda} \{\ker \Phi : \Phi \in \mathrm{Hom}_{FGL_n(q)}(M^\lambda, M^\mu)\}.$$

Proof. c.f. [13] □

If λ is a partition of n, we can restrict the set of homomorphisms in the kernel intersection:

2.2.5 Definition. Suppose that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_h)$ is a partition of n. Let $1 \leqslant d \leqslant h-1$ and $0 \leqslant i \leqslant \lambda_d$. We set

$$\mu := (\lambda_1, \lambda_2, \cdots, \lambda_{d-1}, \lambda_d + \lambda_{d+1} - i, i, \lambda_{d+2}, \cdots, \lambda_h).$$

Then the $FGL_n(q)-$homomorphism

$$\phi_{d,i} : M^\lambda \to M^\mu$$

sends a flag

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{d-1} \supseteq V_d \supseteq V_{d+1} \supseteq \cdots \supseteq V_h = 0$$

to the sum of all $\mu-$flags

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{d-1} \supseteq W_d \supseteq V_{d+1} \supseteq \cdots \supseteq V_h = 0$$

with the property that

$$V_d \supseteq W_d \text{ and } \dim(W_d/V_{d+1}) = i.$$

2.2.6 Theorem. The kernel intersection theorem for partitions: If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_h)$ is a partition of n, then

$$S^\lambda = \bigcap_{d=2}^{h} \bigcap_{i=0}^{\lambda_d - 1} \ker \phi_{d-1,\, i}, \quad \dim S^\lambda = q^{\sum (k-1)\lambda_k} \frac{[n]!}{\prod_{(i,j)\in[\lambda]} [h_{ij}^\lambda]}.$$

Moreover when λ is a two-part partition, we have:

$$S^\lambda = \bigcap_{i=0}^{m-1} \ker \phi_{1,\, i}, \quad \dim\, S^\lambda = \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}.$$

Proof. c.f. [13] □

2.3 Relations with Iwahori-Hecke algebras

For basic definitions and results on the representation theory of Hecke algebras of type A (i.e. associated with symmetric groups $\mathfrak{S}_n$), we refer to [9] and [16].

2.3.1 Definition. Let R be a commutative ring with a 1, and let q be an invertible element of R. The Iwahori-Hecke algebra $\mathcal{H} = \mathcal{H}_{R,\,q}(\mathfrak{S}_n)$ over R with respect to $\mathfrak{S}_n$ and q is defined as the associative $R-$algebra with generators $T_1, T_2, \cdots, T_{n-1}$and relations:

$$\begin{aligned} (T_i - q)(T_i + 1) = 0, &\quad \text{for} \quad i = 1, 2, \ldots, n-1, \\ T_i T_j = T_j T_i, &\quad \text{for} \quad 1 \leqslant i < j - 1 \leqslant n-2, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, &\quad \text{for} \quad i = 1, 2, \ldots, n-2. \end{aligned}$$

Note that when $q = 1$, $\mathcal{H}$ is isomorphic to the group ring $R\mathfrak{S}_n$ of the symmetric group. For this reason we call $\mathcal{H}$ a $q-$deformation of $R\mathfrak{S}_n$.

2.3.2 Definition. Suppose that $w \in \mathfrak{S}_n$. Let $s_{i_1}s_{i_2}\dots s_{i_k}$ be a reduced expression of w. Define

$$T_w = T_{i_1}T_{i_2}\dots T_{i_k}.$$

In particular, if w is the identity element of $\mathfrak{S}_n$ we identify T_w with $1 = 1_R$.

2.3.3 Remark. It is well known that if R is a commutative ring containing q, then

$$\mathcal{H}_{R,q}(\mathfrak{S}_n) \cong \mathrm{End}_{RG}(R_B^G),$$

where $G = GL_n(q)$, and $B = B_n^+$ is a Borel subgroup of G (See [14]).

Remember the definition of $\mathfrak{S}_\lambda$ in Definition 1.3.3 for $\lambda \vDash n$.

2.3.4 Definition. Let $\lambda \vDash n$. Define the elements x_λ and y_λ of $\mathcal{H}_{R,q}(\mathfrak{S}_n)$ by

$$\begin{aligned} x_\lambda &= \sum_{w\in\mathfrak{S}_\lambda} T_w; \\ y_\lambda &= \sum_{w\in\mathfrak{S}_\lambda} (-q)^{-l(w)} T_w. \end{aligned}$$

Remember the definition of w_λ in Definition 1.3.5.

2.3.5 Definition. For $\lambda \vDash n$, let $z_\lambda \in \mathcal{H} = \mathcal{H}_{R,q}(\mathfrak{S}_n)$ be defined by

$$z_\lambda = x_\lambda T_{w_\lambda} y_{\lambda'} = \sum_{u\in\mathfrak{S}_{\lambda'}} (-q)^{-l(u)} x_\lambda T_{w_\lambda u}.$$

2.3.6 Remark. The Specht module $S(\lambda)$ of the Hecke algebras $\mathcal{H}$ is defined to be the right ideal $z_\lambda\mathcal{H}$ of $\mathcal{H}$. It is well known (see [9] and [11]) that $\{S(\lambda) \mid \lambda \vdash n\}$ produces for fields R a complete set of non isomorphic irreducible $\mathcal{H}_{R,q}(\mathfrak{S}_n)-$modules provided $q \in R$ is not a $k-th$ root of unity for $k \leqslant n$. The dimension of of $S(\lambda)$ is given by

$$\dim S(\lambda) = |\,\mathrm{Std}(\lambda)| = n! \prod_{(i,j)\in[\lambda]} \frac{1}{h_{ij}^\lambda}.$$

For $R = \mathbb{C}$, the unipotent Specht module $S^\lambda = S_{\mathbb{C}}^\lambda$ is obtained as the submodule $S_{\mathbb{C}}(\lambda)(\mathbb{C}_B^G)$ of the permutation module $\mathbb{C}_B^G$ of $G = GL_n(q)$ on the cosets of the Borel subgroups B of G. This is, however, not true for arbitrary fields R. For details see e.g. [8].

Chapter 3

The permutation modules $M^{(n-m,m)}$

The representation theory of symmetric group is well developed. For instance one knows that the irreducible $\mathbb{C}\mathfrak{S}_n-$modules are parameterized by partitions of n. Those are called Specht modules. They are defined for arbitrary fields or even more generally for any commutative ring with identity element and a kernel intersection theorem holds. This generalizes to Hecke algebras associated with symmetric groups which were introduced in the last section. The dimension of the Specht module $S(\lambda)$ for $\mathfrak{S}_n$ (or $\mathcal{H} = \mathcal{H}_{R,q}(\mathfrak{S}_n), q$ invertible in R) is given by a hook formula. The connection between $S(\lambda)$ for $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ and S^λ for $\mathbb{C}GL_n(q)$ is given at the end of the last section: One obtains S^λ as $S(\lambda)(\mathbb{C}_B^G)$, $G = GL_n(q)$. It turns out that the dimension over any field F of the $FG-$module S^λ is another remarkable $q-$analogue of the hook formula for the dimension of $S(\lambda)$. It is obtained by replacing all natural numbers in the hook formula for $S(\lambda)$ by their $q-$analogue in Definition 1.1.1.

The Specht module $S(\lambda)$ for $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ has a remarkable basis indexed by standard $\lambda-$tableaux, called the standard basis of $S(\lambda)$. A conjecture of Dipper and James states that there should be polynomials $f_{\mathfrak{t}}(t)$ with integral coefficients for each standard $\lambda-$tableau $\mathfrak{t}$ and canonically constructed pairwise disjoint subsets $\mathcal{B}_{\mathfrak{t}}$ of S^λ with $|\mathcal{B}_{\mathfrak{t}}| = f_{\mathfrak{t}}(q)$ such that $f_{\mathfrak{t}}(1) = 1$ and the unions of the $\mathcal{B}_{\mathfrak{t}}$'s is a basis of S^λ for all field F such that $q \cdot 1_F \neq 0$. This conjecture was proved in the special case of two-part partitions in [7]. Our goal in this thesis is to provide a new proof of this result by inspecting the restriction of S^λ to the unipotent group $U(= U_n^-)$ for two-part partitions λ of n. The hope is that this provides a new method to prove the conjecture in general, that is for arbitrary partitions λ of n. In doing this we determine the precise $FU-$module structure of that restriction. Note that $q \cdot 1_F \neq 0$, by our general assumption. Since the order of U is $q^{\frac{n(n-1)}{2}}$, FU is semisimple hence $\mathrm{Res}_{FU}^{FG} S^\lambda$ is completely reducible.

The kernel intersection theorem 2.2.4 suggests that it may be a good idea, to inspect first the restriction of the permutation module M^λ to U, of which $\mathrm{Res}^{FG}_{FU} S^\lambda$ is a submodule. In this chapter we shall provide a complete decomposition of $\mathrm{Res}^{FG}_{FU} M^\lambda$ into irreducibles for two-part partitions λ. Moreover we shall show that each irreducible constituent of $\mathrm{Res}^{FG}_{FU} M^\lambda$ has $q-$power degree and that the number of irreducible constituents of $\mathrm{Res}^{FG}_{FU} M^\lambda$ of a given dimension q^c $(0 \leqslant c \in \mathbb{Z})$ is a polynomial in q with integral coefficients.

So let $\lambda = (n-m, m)$. From now on, we shall assume that λ is a partition of n (thus $m \leqslant n/2$) and denote $G = GL_n(q), U = U_n^-, B_n = B_n^+$.

3.1 A different description of $M^{(n-m,m)}$

Following the idea of Richard Dipper and Gordon James in [7], we provide in this section a different description of the permutation module $M^{(n-m,m)}$ as direct summands of group algebras of certain abelian groups on which G acts in a natural way.

We start with noting that every $\lambda-$flag with $\lambda = (n-m, m)$ in $V = GF(q)^n$ is given by an $m-$dimensional subspace V_1 of V, which can be described by choosing a $GF(q)-$ basis of V_1. Listing this basis as $m \times n-$matrix X over $GF(q)$, we remark that a base change in V_1 amounts to performing elementary row transformations on X. In this way we can produce a normal form of the basis of $m-$dimensional subspaces of V which is unique.

We proceed as follows: let $V_1 \subseteq V = GF(q)^n$ be of dimension m and let $v_1, \ldots, v_m$ be a basis of V_1 with $v_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in V_1 \subseteq V$. We may assume that $v_1, \ldots, v_m$ is ordered such that $\alpha_{mk} \neq 0$, for some $k \in \{1, \ldots, n\}$ but $\alpha_{il} = 0$ for all $1 \leqslant i \leqslant m$ and $k < l \leqslant n$. We replace v_m by $\frac{1}{\alpha_{mk}} v_m = v'_m$ and v_i by $v'_i = v_i - \alpha_{im} v'_m$ for $1 \leqslant i \leqslant m-1$. Now $v'_1, \ldots, v'_m$ is a new basis of V_1 with the following property, setting $v'_i = (\beta_{i1}, \ldots, \beta_{in}) \in V_1 \subseteq V$: In v'_m we have: $\beta_{mk} = 1, \beta_{ik} = 0$ for $1 \leqslant i \leqslant m-1$ and $\beta_{il} = 0$ for $1 \leqslant i \leqslant m$ and $l > k$. We set $b_m = k$ and repeat the procedure with the $(m-1)$ linear independent vectors $v'_1, \ldots, v'_m$ of V_1. After m steps we obtain a basis $w_1, \ldots, w_m$ of V_1 and integers $1 \leqslant b_1 < b_2 < \cdots < b_m \leqslant n$. Set $w_i = (\gamma_{i1}, \ldots, \gamma_{in}) \in V_1 \subseteq V$. At last, clearing up the entries under $\gamma_{b_i b_i}$ for $1 \leqslant i \leqslant m$, we obtain a basis $u_1, \ldots, u_m$ of V_1 such that the $m \times n$ matrix L with row vectors $u_1, \ldots, u_m$ fits the following definition:

3.1.1 Definition. Let m, n be integers with $0 \leqslant m \leqslant n$. Denote by $\Xi_{m,n}$ the set of $m \times n$ matrices $L = (l_{b_i j})$ over $GF(q)$ with the property that for some integers $b_1, \cdots, b_m$ with

$$1 \leqslant b_1 < b_2 < \cdots < b_m \leqslant n$$

the following holds for each i, with $1 \leqslant i \leqslant m$:

(1) $l_{b_i b_i} = 1$, and $l_{b_i j} = 0$ if $j > b_i$;

(2) $l_{b_k b_i} = 0$ if $k > i$.

3.1.2 Remark. Note in the above definition, we label the rows of the element in $\Xi_{m,n}$ by $b_1, b_2, \cdots, b_m$ instead of $1, 2, \cdots, m$. The reason for doing this will become apparent later on. Moreover, for each i, $l_{b_i b_i} = 1$ is the last nonzero entry in row b_i. We call it "last 1" for convenience.

Every $m \times n$ matrix over $GF(q)$ of rank m is row-equivalent to precisely one matrix in $\Xi_{m,n}$. Therefore $\Xi_{m,n}$ is in bijective correspondence with the set of m-dimensional subspaces of an n-dimensional vector space over $GF(q)$.

3.1.3 Example. There are $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4$ two-dimensional subspaces of a four-dimensional space over $GF(q)$, and the elements of $\Xi_{2,4}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{42} & l_{43} & 1 \end{pmatrix}, \begin{pmatrix} l_{21} & 1 & 0 & 0 \\ l_{41} & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} l_{21} & 1 & 0 & 0 \\ l_{41} & 0 & l_{43} & 1 \end{pmatrix}, \begin{pmatrix} l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 0 & 1 \end{pmatrix}$$

where $l_{31}, l_{32}, l_{41}, l_{42}, l_{43} \in GF(q)$.

3.1.4 Definition. Let $M^{(n-m,m)}$ be the $\begin{bmatrix} n \\ m \end{bmatrix}$-dimensional vector space over F with basis $\Xi_{m,n}$. If $L \in \Xi_{m,n}$ and $g \in G$ then Lg is row-equivalent to a matrix in $\Xi_{m,n}$, and we denote this matrix by $L \circ g$. Under the action $\circ$ of G, the vector space $M^{(n-m,m)}$ becomes an FG-module.

Obviously, this is isomorphic to the permutation module of G on the cosets of the parabolic subgroup for λ defined previously justifying the notation.

Remember U is the lower unitriangular subgroup of G. Hence $M^{(n-m,m)}$ can be regarded as an FU-module.

3.1.5 Definition. Suppose that $L = (l_{b_i j}) \in \Xi_{m,n}$, and let

$$1 \leqslant b_1 < b_2 < \cdots < b_m \leqslant n$$

be the integers which appear in Definition 3.1.1. Define tab(L) to be the row-standard $(n-m, m)$-tableau whose second row is $b_1, b_2, \cdots, b_m$. We refer to tab(L) as the tableau of L.

Remember RStd(λ) denotes the set of row-standard λ-tableaux.

3.1.6 Definition. For a two part partition $\lambda = (n-m, m)$ and $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ we denote the second row of $\mathfrak{t}$ by $\underline{\mathfrak{t}}$. Note that $\mathfrak{t}$ is completely determined by $\underline{\mathfrak{t}}$. Naturally, we can define the notation $\underline{\mathrm{tab}(L)}$ as the second row of $\mathrm{tab}(L)$:

$$\underline{\mathrm{tab}(L)} = (b_1, b_2, \cdots, b_m).$$

3.1.7 Example. (1) Suppose

$$L = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} l_{21} & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}.$$

Then

$$\mathrm{tab}(L) = \begin{array}{|c|c|c|} \hline 1 & 5 & 6 \\ \hline 2 & 3 & 4 \\ \hline \end{array}, \quad \underline{\mathrm{tab}(L)} = (2, 3, 4).$$

(2) Suppose

$$L = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} l_{31} & l_{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ l_{51} & l_{52} & 0 & l_{54} & 1 & 0 & 0 & 0 \\ l_{71} & l_{72} & 0 & l_{74} & 0 & l_{76} & 1 & 0 \\ l_{81} & l_{82} & 0 & l_{84} & 0 & l_{86} & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 5 \\ 7 \\ 8 \end{array}.$$

Then

$$\mathrm{tab}(L) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 & 8 \\ \hline \end{array}, \quad \underline{\mathrm{tab}(L)} = (3, 5, 7, 8).$$

When λ is a two part partition, we order the elements in $\mathrm{RStd}(\lambda)$ lexicographically by their second rows.

3.1.8 Example. (1) We refer to Example 3.1.3 for the elements in $\Xi_{2,4}$. The tableaux of the elements listed there are

$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} < \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} < \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} < \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} < \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} < \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

respectively.

(2) Let $\lambda = (3, 3)$. The elements in $\mathrm{RStd}(\lambda)$ are:

$$\begin{array}{|c|c|c|} \hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline 1 & 2 & 4 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 3 & 4 & 6 \\ \hline 1 & 2 & 5 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 1 & 2 & 6 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 2 & 5 & 6 \\ \hline 1 & 3 & 4 \\ \hline \end{array} <$$

$$\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 1 & 3 & 5 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 1 & 3 & 6 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 2 & 3 & 6 \\ \hline 1 & 4 & 5 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 1 & 4 & 6 \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 1 & 5 & 6 \\ \hline \end{array} <$$

$$\begin{array}{|c|c|c|}\hline 1&5&6\\\hline 2&3&4\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&3&5\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&3&6\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4&5\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} <$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4&5\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}.$$

The positions in a matrix $M \in \Xi_{m,n}$, which are not in columns of and not to the right of the last 1's will play an important role in the following sections. And for the matrices having the same tableau, these positions are also the same. Therefore we fix the following notation:

3.1.9 Definition. Suppose $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ and $\underline{\mathfrak{t}} = (b_1, b_2, \cdots, b_m)$. Set

$$\mathfrak{J}_{\mathfrak{t}} = \{(i,j) \,|\, i > j,\ i \in \underline{\mathfrak{t}},\ j \notin \underline{\mathfrak{t}}\}.$$

Since $\Xi_{m,n}$ is a basis of $M^{(n-m,m)}$, the following definition makes sense:

3.1.10 Definition. Let $\lambda = (n-m, m)$. Suppose that $v \in M^{\lambda}$, and write

$$v = \sum_{X \in \Xi_{m,n}} C_X X \text{ where } C_X \in F.$$

(1) For each $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ let

$$v(\mathfrak{t}) = \sum_{\mathrm{tab}(X)=\mathfrak{t}} C_X X.$$

(2) If $v \neq 0$, then let $\mathrm{last}(v)$ be the last $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ (with respect to the lexicographical order as above) such that $v(\mathfrak{t}) \neq 0$.

(3) For $v \neq 0$, define $\mathrm{top}(v) = v\big(\mathrm{last}(v)\big)$.

3.2 The idempotent basis of $M^{(n-m,m)}$

Our first goal is to investigate the $FU-$module structure of the permutation module $M^{(n-m,m)}$. Obviously, Mackey decomposition provides a first splitting of $\mathrm{Res}^{FG}_{FU} M^{\lambda}$. Note that $\mathcal{D}_{\lambda} = \{w \in \mathfrak{S}_n \,|\, \mathfrak{t}^{\lambda} w \in \mathrm{RStd}(\lambda)\}$ is a $P_{\lambda} - U$ double coset transversal in G, hence

$$\mathrm{Res}^{FG}_{FU} M^{\lambda} = \mathrm{Res}^{FG}_{FU} \mathrm{Ind}^{FG}_{FP_{\lambda}} F = \bigoplus_{w \in \mathcal{D}_{\lambda}} \mathrm{Ind}^{FU}_{F(P^w_{\lambda} \cap U)} F$$

is a direct sum decomposition of $\mathrm{Res}^{FG}_{FU} M^{\lambda}$. We call the $FU-$submodule $\mathrm{Ind}^{FU}_{F(P^w_{\lambda} \cap U)} F$ the $\mathfrak{t}-$batch of $\mathrm{Res}^{FG}_{FU} M^{\lambda}$, where $\mathfrak{t} = \mathfrak{t}^{\lambda} w \in \mathrm{RStd}(\lambda)$.

We now translate this notion into the setting of the last section:

3.2.1 Lemma. Let $\lambda = (n-m, m)$ and $\mathfrak{t} = \mathfrak{t}^\lambda w \in \mathrm{RStd}(\lambda)$. Set

$$\mathfrak{X}_\mathfrak{t} = \{L \in \Xi_{m,n} \mid \mathrm{tab}(L) = \mathfrak{t}\}.$$

Then for $L \in \mathfrak{X}_\mathfrak{t}$ and $u \in U$, we have $L \circ u \in \mathfrak{X}_\mathfrak{t}$. Moreover U acts transitively on $\mathfrak{X}_\mathfrak{t}$. Let $\mathfrak{M}_\mathfrak{t}$ be the corresponding permutation module with basis $\mathfrak{X}_\mathfrak{t}$. Then

$$\mathfrak{M}_\mathfrak{t} \cong \mathrm{Ind}_{F(P_\lambda^w \cap U)}^{FU} F\,, \text{ the } \mathfrak{t}\text{−batch of } \mathrm{Res}_{FU}^{FG} M^\lambda.$$

Proof. Let $\mathfrak{t} = (b_1, \cdots, b_m)$ be the second row of $\mathfrak{t}$ (see Definition 3.1.6). Let $g = (g_{ij}) \in U$, that is $g_{ii} = 1$ and $g_{ij} = 0$ for $1 \leqslant i < j \leqslant n$. Let

$$M = (m_{b_i j}) \in \mathfrak{X}_\mathfrak{t},\ N = (n_{b_i j}) = Mg.$$

Then $N' = (n'_{b_i j})$ is obtained by row reducing N. Now for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$,

$$n_{b_i j} = \sum_{r=1}^{n} m_{b_i r} g_{rj} = \sum_{r=j}^{b_i} m_{b_i r} g_{rj}, \tag{3.2.2}$$

since $g_{rj} = 0$ for $r < j$ and $m_{b_i r} = 0$ for $r > b_i$. Therefore

$$n_{b_i b_i} = m_{b_i b_i} g_{b_i b_i} = 1,$$

since $m_{b_i b_i} = n_{b_i b_i} = 1$ for $1 \leqslant i \leqslant m$, and

$$n_{b_i j} = 0, \text{ for } j > b_i.$$

This means that row b_i of N has 1 at position (b_i, b_i) and zeros to it's right. So, row reducing N amounts to clearing the nonzero entries below these last 1's, that is in positions (b_j, b_i) with $j > i$. The resulting matrix of $\Xi_{m,n}$ hence has last ones at positions $(b_i, b_i), 1 \leqslant i \leqslant m$ and therefore belong to $\mathfrak{X}_\mathfrak{t}$. This shows that $\mathfrak{M}_\mathfrak{t}$ is an FU−module under the operation $\circ$.

Next we show that U acts transitively on $\mathfrak{X}_\mathfrak{t}$. For this let $L = (l_{b_i j})$ be the matrix in $\mathfrak{X}_\mathfrak{t}$ whose only nonzero entries are the last 1's. Thus

$$l_{b_i b_i} = 1 \text{ and } l_{b_i j} = 0, \text{ for all } 1 \leqslant j \leqslant m, j \neq b_i \text{ and } 1 \leqslant i \leqslant m.$$

Let $M = (m_{b_i j}) \in \mathfrak{X}_\mathfrak{t}$. Define $u = (u_{rs}) \in U$ by

$$u_{rs} = \begin{cases} 1 & \text{for } r = s; \\ m_{rs} & \text{for } (r,s) \in \mathfrak{J}_\mathfrak{t} \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathfrak{J}_\mathfrak{t}$ is defined in (3.1.9). Then $N = (n_{b_i j}) = Lu$ is given by (3.2.2) as

$$n_{b_i j} = \sum_{r=j}^{b_i} l_{b_i r} u_{rj} = u_{b_i j} = \begin{cases} 1 & \text{for } b_i = j \\ m_{b_i j} & \text{for } (b_i, j) \in \mathfrak{J}_\mathfrak{t} \\ 0 & \text{otherwise.} \end{cases}.$$

Then $N = M$ is already given in row reduced form and hence $L \circ u = Lu = M$. This shows that U acts transitively on $\mathfrak{X}_{\mathfrak{t}}$. To finish the proof it suffices by general theory to show that the stabilizer $\mathrm{Stab}_U(L)$ of L in U is given as $P_\lambda^w \cap U$.

Note that for an arbitrary $g \in GL_n(q), Lg$ is obtained from g by deleting all rows with index $j \notin \underline{\mathfrak{t}} = \{b_1, \ldots, b_m\}$ of g. As we have seen above, for $u \in U, N = (n_{b_i j}) = Lu$ has already the property that $n_{b_i b_i} = 1$ and $n_{b_i j} = 0$ for $i = 1, \ldots, m, j > b_i$. Hence to obtain $L \circ u$ we have simply to replace all entries below the last 1's $n_{b_i b_i}$ by zeros. In particular $L \circ u = L$ if and only if the entries in rows b_i of u are zeros except the positions (b_i, b_j) where $i \geqslant j$. Then $\mathrm{Stab}_U(L)$ is generated by root subgroups X_{ij} with $1 \leqslant j < i \leqslant n$ where $i \notin \underline{\mathfrak{t}}$ or $i, j \in \underline{\mathfrak{t}}$. Since $\mathfrak{t}$ has precisely two rows this condition is equivalent to $1 \leqslant j < i \leqslant n$ and $\mathrm{row}_{\mathfrak{t}}(i) \leqslant \mathrm{row}_{\mathfrak{t}}(j)$. Using 2.1.6, we conclude $\mathrm{Stab}_U(L) = P_\lambda^w \cap U$. $\square$

Next for $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ fixed, we make $\mathfrak{X}_{\mathfrak{t}}$ into an abelian group through introducing an addition $\diamond$ on $\mathfrak{X}_{\mathfrak{t}}$ by componentwise adding the entries in $\mathfrak{J}_{\mathfrak{t}}$. So we add all entries componentwise besides the last one's.

3.2.3 Example. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in GF(q)$. Then

$$\begin{pmatrix} a_1 & 1 & 0 & 0 \\ b_1 & 0 & c_1 & 1 \end{pmatrix} \diamond \begin{pmatrix} a_2 & 1 & 0 & 0 \\ b_2 & 0 & c_2 & 1 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 1 & 0 & 0 \\ b_1 + b_2 & 0 & c_1 + c_2 & 1 \end{pmatrix}$$

Obviously $(\mathfrak{X}_{\mathfrak{t}}, \diamond)$ is an abelian group of order $q^{|\mathfrak{J}_{\mathfrak{t}}|}$. Therefore we can find $q^{|\mathfrak{J}_{\mathfrak{t}}|}$ linear irreducible $F-$characters of $\mathfrak{X}_{\mathfrak{t}}$. Such a character χ is a group homomorphism from $\mathfrak{X}_{\mathfrak{t}}$ to the multiplicative group F^*. In particular

$$\chi(M \diamond N) = \chi(M)\chi(N) \text{ for } M, N \in \mathfrak{X}_{\mathfrak{t}}.$$

We regard the set X of $F-$linear characters of $\mathfrak{X}_{\mathfrak{t}}$ as a vector space over $GF(q)$ by setting

$$(\chi_1 + \chi_2)(M) = \chi_1(M)\chi_2(M)$$
$$\alpha\chi(M) = \chi(\alpha M)$$

for all $M \in \mathfrak{X}_{\mathfrak{t}}, \alpha \in GF(q)$.

Remember that we fixed in Section 1.1, a non trivial linear character

$$\theta : \big(GF(q), +\big) \to F^*$$

in Section 1.1. For $(b_i, j) \in \mathfrak{J}_{\mathfrak{t}}$ we denote by $\xi_{b_i j}$ the (b_i, j) coordinate function from $\mathfrak{X}_{\mathfrak{t}}$ to $GF(q)$. Then

$$\{\theta\xi_{b_i j} \mid (b_i, j) \in \mathfrak{J}_{\mathfrak{t}}\}$$

is a basis of the $GF(q)-$vector space X.

Thus, if $\chi \in X$ then

$$\chi = \sum_{(b_i,j)\in\mathfrak{J}_\mathfrak{t}} l_{b_ij}\theta\xi_{b_ij}$$

for uniquely determined elements l_{b_ij} of $GF(q)$.

Vice versa, given a matrix $L = (l_{b_ij}) \in \mathfrak{X}_\mathfrak{t}$, we let

$$\chi_L = \sum_{(b_i,j)\in\mathfrak{J}_\mathfrak{t}} l_{b_ij}\theta\xi_{b_ij}$$

so that

$$X = \{\chi_L \mid L \in \mathfrak{X}_\mathfrak{t}\}$$

and for $M = (m_{b_ij}) \in \mathfrak{X}_\mathfrak{t}$, we have

$$\chi_L(M) = \prod_{(b_i,j)\in\mathfrak{J}_\mathfrak{t}} \theta(l_{b_ij}m_{b_ij}) \tag{3.2.4}$$

Since the characteristic of F is coprime to q and $|\mathfrak{X}_\mathfrak{t}|$ is a power of q, the $\mathfrak{t}-$batch $\mathfrak{M}_\mathfrak{t}$ is a semisimple $FU-$module. F is a splitting field for $\mathfrak{M}_\mathfrak{t}$ and $\mathfrak{M}_\mathfrak{t}$ has a basis of orthogonal primitive idempotents. This basis turns out to be very well adapted to the $FU-$module structure of $\mathfrak{M}_\mathfrak{t}$ as we shall show.

In order to not mix up the formal addition in the $F-$vector space $\mathfrak{M}_\mathfrak{t}$ and the matrix addition $\diamond$, we write $[M]$ if we consider the matrix M as a basis element of the $F-$vector space $\mathfrak{M}_\mathfrak{t}$.

3.2.5 Definition. Suppose that $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ and $L \in \mathfrak{X}_\mathfrak{t}$. Let

$$\begin{aligned} e_L &= \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M\in\mathfrak{X}_\mathfrak{t}} \chi_L(-M)[M] \\ &= \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M\in\mathfrak{X}_\mathfrak{t}} \prod_{(b_i,j)\in\mathfrak{J}_\mathfrak{t}} \theta(-l_{b_ij}m_{b_ij})[M]. \end{aligned}$$

By general theory e_L is the idempotent in $\mathfrak{M}_\mathfrak{t}$ affording the linear character χ_L. In fact,

$$\mathcal{E}_\mathfrak{t} = \{e_L \mid L \in \mathfrak{X}_\mathfrak{t}\}$$

is a complete set of primitive orthogonal idempotents in $F\mathfrak{X}_\mathfrak{t}$, and so

$$\mathfrak{M}_\mathfrak{t} = \bigoplus_{L\in\mathfrak{X}_\mathfrak{t}} Fe_L$$

is the decomposition of the regular module of $F\mathfrak{X}_\mathfrak{t}$ into pairwise non-isomorphic irreducible $F\mathfrak{X}_\mathfrak{t}-$modules.

3.2.6 Example. Let $L = \begin{pmatrix} l_{21} & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \Xi_{3,6}$. Then

$$e_L = \frac{1}{q^3} \sum \theta(-l_{21}m_{21})\theta(-l_{31}m_{31})\theta(-l_{41}m_{41}) \left[\begin{pmatrix} m_{21} & 1 & 0 & 0 & 0 & 0 \\ m_{31} & 0 & 1 & 0 & 0 & 0 \\ m_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right],$$

where m_{21}, m_{31}, m_{41} run over $GF(q)$.

3.3 Structure of M^λ as an $F(U^w \cap U)-$module

Throughout this section we fix a 2-part partition $\lambda = (n-m, m)$ of n and a row standard $\lambda-$tableau $\mathfrak{t}$. Let $w = d(\mathfrak{t})$ i.e. $\mathfrak{t}^\lambda w = \mathfrak{t}$. We now investigate the action of $U^w \cap U$ on $\mathcal{E}_\mathfrak{t} = \{e_L \mid L \in \mathfrak{X}_\mathfrak{t}\}$. Recall that the second row $\underline{\mathfrak{t}}$ of $\mathfrak{t}$ labels the rows of $L \in \mathfrak{X}_\mathfrak{t}$. So let $\underline{\mathfrak{t}} = (b_1, b_2, \cdots, b_m)$.

3.3.1 Lemma. $g = (g_{ij}) \in G$ is contained in $U^w \cap U$ if and only if $g \in U$ and the following holds:

$$i \notin \underline{\mathfrak{t}} \text{ and } j \in \underline{\mathfrak{t}} \text{ implies } g_{ij} = 0.$$

Proof. Lemma 2.1.8 says $g = (g_{ij}) \in U^w \cap U$ if and only if

$$g_{ii} = 1, \forall\, 1 \leqslant i \leqslant n$$

and $\forall\, 1 \leqslant i, j \leqslant n$:

$$i < j \text{ or } \operatorname{row}_\mathfrak{t}(i) < \operatorname{row}_\mathfrak{t}(j) \Rightarrow g_{ij} = 0.$$

Since λ has only two parts, this means

$$\operatorname{row}_\mathfrak{t}(i) < \operatorname{row}_\mathfrak{t}(j) \Leftrightarrow \operatorname{row}_\mathfrak{t}(i) = 1, \operatorname{row}_\mathfrak{t}(j) = 2$$

and the result follows. □

3.3.2 Example. Let $\lambda = (3,3), \mathfrak{t} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} = \mathfrak{t}^\lambda w$. Then

$$U^w \cap U = \{g = (g_{ij}) \in U_6 \mid g_{43} = 0\}.$$

3.3.3 Proposition. $U^w \cap U$ acts monomially on $\mathcal{E}_\mathfrak{t}$, that is given $L \in \mathfrak{X}_\mathfrak{t}$, $g \in U^w \cap U$, then there exist $K \in \mathfrak{X}_\mathfrak{t}$ and $0 \neq C(L, g) \in F$ such that

$$e_L \circ g = C(L, g) e_K.$$

Proof. From Lemma 2.1.6 and Lemma 3.3.1, it is easy to see that $U^w \cap U$ is generated by the root subgroups X_{ij} where $1 \leqslant j < i \leqslant n$ satisfy one of the following conditions: (1) $i \in \underline{\mathfrak{t}}$, $j \notin \underline{\mathfrak{t}}$; (2) i, $j \notin \underline{\mathfrak{t}}$; (3) i, $j \in \underline{\mathfrak{t}}$.

Hence it is enough to prove the result for matrices of the form $g = E + \alpha e_{ij}$, where E is the unit matrix, e_{ij} is the matrix unit to indices i and j, $0 \neq \alpha \in GF(q)$ and i, j satisfy one of the conditions above. So let $g = E + \alpha e_{ij}$. By Definition 3.2.5 expanding the action of g on $\mathfrak{X}_{\mathfrak{t}}$ by linearity to $\mathfrak{M}_{\mathfrak{t}}$ we have

$$e_L \circ g = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_M \chi_L(-M)[M \circ g].$$

Using general character theory we may write

$$[M \circ g] = \sum_{K \in \mathfrak{X}_{\mathfrak{t}}} \chi_K(M \circ g) e_K.$$

Hence

$$\begin{aligned} e_L \circ g &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_M \chi_L(-M) \sum_K \chi_K(M \circ g) e_K \\ &= \sum_K \Big(\frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_M \chi_L(-M) \chi_K(M \circ g) \Big) e_K. \end{aligned} \tag{3.3.4}$$

Our strategy will be, to determine which e_K occurs with non zero coefficient in this expression. We remark that in the course of the proof we use frequently

$$\sum_{\beta \in GF(q)} \theta(\beta) = 0 \tag{3.3.5}$$

which is a consequence of the orthogonality relations for irreducible character of the group $(GF(q), +)$.

Note that for $M \in \mathfrak{X}_{\mathfrak{t}}$, $g = E + \alpha e_{ij}$ as above, Mg is obtained from M by adding α times column i to column j of M. Moreover, in the conditions (1) and (2), we have $j \notin \underline{\mathfrak{t}}$, which implies that the columns of $M (\in \mathfrak{X}_{\mathfrak{t}})$ containing a last one are not changed by the action of g. Hence in these cases Mg is already in row reduced form that is we have $M \circ g = Mg$.

Case (1): $i \in \underline{\mathfrak{t}}$, $j \notin \underline{\mathfrak{t}}$. Say $i = b_r$ for some $1 \leqslant r \leqslant m$.

For $M \in \mathfrak{X}_{\mathfrak{t}}$, let $N = M \circ g = Mg = (n_{b_u v}) \in \mathfrak{X}_{\mathfrak{t}}$. In M the $i-$th column contains only one entry different from zero, namely at place b_r, b_r where $i = b_r$. This is the last one in row b_r of M. Hence multiplying column i of M by α and adding it to the $j-$th column of M amounts to adding α to

entry $m_{b_r j}$ of M. By (3.2.4), we have

$$\begin{aligned}
\chi_K(N) &= \prod_{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t}} \theta\left(k_{b_u v} n_{b_u v}\right) \\
&= \prod_{\substack{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t} \\ (u,v)\neq(r,j)}} \theta\left(k_{b_u v} m_{b_u v}\right)\theta\left(k_{b_r j}(m_{b_r j}+\alpha)\right) \\
&= \prod_{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t}} \theta\left(k_{b_u v} m_{b_u v}\right)\theta\left(k_{b_r j}\alpha\right) \\
&= \chi_K(M)\,\theta\left(k_{b_r j}\alpha\right).
\end{aligned}$$

Hence in formula (3.3.4) the coefficient of e_K is given as

$$\frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}}\Big(\sum_M \chi_L(-M)\,\chi_K(M)\Big)\,\theta\left(k_{b_r j}\alpha\right)$$

By the orthogonality relations for the characters of $(\mathfrak{X}_\mathfrak{t}, \diamond)$ this coefficient is zero unless $L = K$. In this case let $C(L,g) := \theta\left(l_{b_r j}\alpha\right)$ and we have therefore $e_L \circ g = C(L,g)e_K$.

Case (2): $i, j \notin \underline{\mathfrak{t}}$.

For $M \in \mathfrak{X}_\mathfrak{t}$, let $N = M \circ g = Mg = (n_{b_u v}) \in \mathfrak{X}_\mathfrak{t}$. Note that column j is the only column of M which is changed by the multiplication by g. The entries $n_{b_u j}$ of column j of N are given as

$$n_{b_u j} = m_{b_u j} + \alpha m_{b_u i}, \text{ for } u = 1, \ldots, m.$$

Hence for $K \in \mathfrak{X}_\mathfrak{t}$,

$$\begin{aligned}
\chi_K(N) &= \prod_{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t}} \theta\left(k_{b_u v} n_{b_u v}\right) \\
&= \prod_{\substack{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t} \\ v\neq j}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{u=1}^{m} \theta\left(k_{b_u j}(m_{b_u j}+\alpha m_{b_u i})\right).
\end{aligned}$$

Now

$$\theta\left(k_{b_u j}(m_{b_u j}+\alpha m_{b_u i})\right) = \theta\left(k_{b_u j} m_{b_u j}\right)\theta\left(\alpha k_{b_u j} m_{b_u i}\right)$$

and

$$m_{b_u i} = 0 \text{ for } (b_u, i) \notin \mathfrak{J}_\mathfrak{t}.$$

Hence we obtain

$$\begin{aligned}
\chi_K(N) &= \prod_{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{(b_u,\, i)\in\mathfrak{J}_\mathfrak{t}} \theta\left(\alpha k_{b_u j} m_{b_u i}\right) \\
&= \prod_{\substack{(b_u,\, v)\in\mathfrak{J}_\mathfrak{t} \\ v\neq i}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{(b_u,\, i)\in\mathfrak{J}_\mathfrak{t}} \theta\left(k_{b_u i} m_{b_u i}\right) \prod_{(b_u,\, i)\in\mathfrak{J}_\mathfrak{t}} \theta\left(\alpha k_{b_u j} m_{b_u i}\right).
\end{aligned}$$

Note that in this case $(b_u, i) \in \mathfrak{J}_{\mathfrak{t}}$ if and only $i < b_u, m_{b_u i} = 0$ for $i > b_u$, and

$$\theta\left(k_{b_u i} m_{b_u i}\right) \theta\left(\alpha k_{b_u j} m_{b_u i}\right) = \theta\left((k_{b_u i} + \alpha k_{b_u j}) m_{b_u i}\right).$$

Hence we obtain:

$$\chi_K(N) = \prod_{\substack{(b_u,\, v) \in \mathfrak{J}_{\mathfrak{t}} \\ v \neq i}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{(b_u,\, i) \in \mathfrak{J}_{\mathfrak{t}}} \theta\left((k_{b_u i} + \alpha k_{b_u j}) m_{b_u i}\right)$$

and hence in formula (3.3.4) the coefficient of e_K for $K \in \mathfrak{X}_{\mathfrak{t}}$ is given as

$$\begin{aligned}
& \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{X}_{\mathfrak{t}}} \chi_L(-M)\, \chi_K(M \circ g)) \\
= \; & \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{X}_{\mathfrak{t}}} \prod_{(b_u,\, v) \in \mathfrak{J}_{\mathfrak{t}}} \theta\left(-l_{b_u v} m_{b_u v}\right) \\
& \cdot \prod_{\substack{(b_u,\, v) \in \mathfrak{J}_{\mathfrak{t}} \\ v \neq i}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{(b_u,\, i) \in \mathfrak{J}_{\mathfrak{t}}} \theta\left((k_{b_u i} + \alpha k_{b_u j}) m_{b_u i}\right) \\
= \; & \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{X}_{\mathfrak{t}}} \prod_{\substack{(b_u,\, v) \in \mathfrak{J}_{\mathfrak{t}} \\ v \neq i}} \theta\left((-l_{b_u v} + k_{b_u v}) m_{b_u v}\right) \\
& \cdot \prod_{(b_u,\, i) \in \mathfrak{J}_{\mathfrak{t}}} \theta\left((-l_{b_u i} + k_{b_u i} + k_{b_u j} \alpha) m_{b_u i}\right).
\end{aligned}$$

Recall that by (3.3.5), for $0 \neq \beta \in GF(q)$,

$$\sum_{m_{b_u v}} \theta\left(\beta m_{b_u v}\right) = 0 \tag{3.3.6}$$

for any $(b_u, v) \in \mathfrak{J}_{\mathfrak{t}}$, where $m_{b_u v}$ runs through $GF(q)$. Since we sum in the equation above over all matrices $M \in \mathfrak{X}_{\mathfrak{t}}$, this means, that the coefficients in both product factors in this equations have to be zero, otherwise the product (and hence the coefficient of e_K in $e_L \circ g$) is zero. Therefore we obtain:

$$-l_{b_u v} + k_{b_u v} = 0 \quad \text{for all } (b_u,\, v) \in \mathfrak{J}_{\mathfrak{t}} \text{ with } v \neq i$$

and

$$-l_{b_u i} + k_{b_u i} + \alpha k_{b_u j} = 0 \quad \text{for all } (b_u,\, i) \in \mathfrak{J}_{\mathfrak{t}}.$$

From this we obtain immediately, since $i > j$ and $i, j \notin \underline{\mathfrak{t}}$, $(b_u,\, i) \in \mathfrak{J}_{\mathfrak{t}}$ implies $(b_u,\, j) \in \mathfrak{J}_{\mathfrak{t}}$ and therefore $k_{b_u j} = l_{b_u j}$ for all $(b_u,\, j) \in \mathfrak{J}_{\mathfrak{t}}$ by the first equation above:

$$k_{b_u v} = \begin{cases} l_{b_u v} & \text{for } (b_u,\, v) \in \mathfrak{J}_{\mathfrak{t}} \text{ and } v \neq i, \\ l_{b_u i} - \alpha l_{b_u j} & \text{for } v = i \text{ and } (b_u,\, i) \in \mathfrak{J}_{\mathfrak{t}}. \end{cases}$$

Note that this determines K uniquely. In particular note that $k_{b_u i} = 0$ automatically if $i > b_u$. Suppose $b_s < j < b_{s+1}$ and $b_t < i < b_{t+1}$. Then

$s \leqslant t$ and $b_s \leqslant b_t$, since by assumption $j < i$. Our formula says that we may obtain K from L by subtracting from column i α times that part of column j which consists of entries below entry $l_{b_t j}$. Then we may rewrite the formula as

$$k_{b_u v} = \begin{cases} l_{b_u v} & \text{for } v \neq i \\ l_{b_u i} - \alpha l_{b_u j} & \text{for } v = i \text{ and } u = t+1, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

We also have $e_L \circ g = e_K$ that is the coefficient of e_K in $e_L \circ g$ is 1.

Case (3): $i, j \in \underline{\mathfrak{t}}$, say $i = b_s, j = b_t$. since $j < i$ we have $t < s$ and $b_t < b_s$.

Note in this case $Mg \neq M \circ g$ in general, hence we need to row reduce Mg. In M, column $i(= b_s)$ contains a last 1 at position (b_s, b_s) and zero elsewhere. Hence multiplying column i of M by α and adding it to column $j(= b_t)$ amounts to changing the zero in place (b_s, b_t) of M to α. Thus $M \circ g = N$ is obtained from M subtracting from row b_s in M row b_t multiplied by α. Thus M and $N = (n_{b_u v}) \in \mathfrak{X}_{\mathfrak{t}}$ coincide everywhere except in row b_s which is given by

$$n_{b_s i} = \begin{cases} m_{b_s i} - \alpha m_{b_t i} & \text{for } 1 \leqslant i \leqslant b_t - 1 \\ 0 & \text{for } i = b_t \\ m_{b_s i} & \text{for } b_t + 1 \leqslant i \leqslant n. \end{cases}$$

Hence, for $K = (k_{b_u v}) \in \mathfrak{X}_{\mathfrak{t}}$:

$$\begin{aligned} \chi_K(N) &= \chi_K(M \circ g) = \prod_{(b_u, v) \in \mathfrak{J}_{\mathfrak{t}}} \theta\left(k_{b_u v} n_{b_u v}\right) \\ &= \prod_{\substack{(b_u, v) \in \mathfrak{J}_{\mathfrak{t}} \\ u \neq s}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t - 1} \theta\left(k_{b_s v}(m_{b_s v} - \alpha m_{b_t v})\right) \prod_{v=b_t+1}^{b_s - 1} \theta(k_{b_s v} m_{b_s v}) \end{aligned}$$

Now $\theta\left(k_{b_s v}(m_{b_s v} - \alpha m_{b_t v})\right) = \theta\left(k_{b_s v} m_{b_t v}\right) \theta(-\alpha k_{b_s v} m_{b_t v})$, hence

$$\begin{aligned} \chi_K(N) &= \prod_{\substack{(b_u, v) \in \mathfrak{J}_{\mathfrak{t}} \\ u \neq s}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t - 1} \theta\left(k_{b_s v} m_{b_t v}\right) \\ &\quad \cdot \prod_{v=1}^{b_t - 1} \theta(-\alpha k_{b_s v} m_{b_t v}) \prod_{v=b_t+1}^{b_s - 1} \theta(k_{b_s v} m_{b_s v}). \end{aligned}$$

Combining terms 1, 2 and 4, and then reordering the product we obtain:

$$\begin{aligned}
\chi_K(N) &= \prod_{(b_u,\,v)\in\mathfrak{J}_\mathfrak{t}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left(-\alpha k_{b_s v} m_{b_t v}\right) \\
&= \prod_{\substack{(b_u,v)\in\mathfrak{J}_\mathfrak{t}\\ u\neq t}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left(k_{b_t v} m_{b_t v}\right) \prod_{v=1}^{b_t-1} \theta\left(-\alpha k_{b_s v} m_{b_t v}\right) \\
&= \prod_{\substack{(b_u,v)\in\mathfrak{J}_\mathfrak{t}\\ u\neq t}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left((k_{b_t v} - \alpha k_{b_s v}) m_{b_t v}\right).
\end{aligned}$$

Hence in formula (3.3.4) the coefficient for e_K in $e_L \circ g$ is given as

$$\begin{aligned}
&\frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M\in\mathfrak{X}_\mathfrak{t}} \chi_L(-M)\, \chi_K(M\circ g) \\
=\ & \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M\in\mathfrak{X}_\mathfrak{t}} \prod_{(b_u,\,v)\in\mathfrak{J}_\mathfrak{t}} \theta\left(-l_{b_u v} m_{b_u v}\right) \\
&\cdot \prod_{\substack{(b_u,v)\in\mathfrak{J}_\mathfrak{t}\\ u\neq t}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left((k_{b_t v} - \alpha k_{b_s v}) m_{b_t v}\right) \\
=\ & \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M\in\mathfrak{X}_\mathfrak{t}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_\mathfrak{t}\\ u\neq t}} \theta\left(-l_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left(-l_{b_t v} m_{b_t v}\right) \\
&\cdot \prod_{\substack{(b_u,v)\in\mathfrak{J}_\mathfrak{t}\\ u\neq t}} \theta\left(k_{b_u v} m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left((k_{b_t v} - \alpha k_{b_s v}) m_{b_t v}\right) \\
=\ & \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M\in\mathfrak{X}_\mathfrak{t}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_\mathfrak{t}\\ u\neq t}} \theta\left((-l_{b_u v} + k_{b_u v}) m_{b_u v}\right) \prod_{v=1}^{b_t-1} \theta\left((-l_{b_t v} + k_{b_t v} - k_{b_s v}\alpha) m_{b_t v}\right),
\end{aligned}$$

where the last equality is obtained by combining terms 1 and 3, and terms 2 and 4 in the middle expression.
Arguing as in the previous case using again (3.3.6) we obtain:
The coefficient of e_K in formula (3.3.4) is zero unless

$$-l_{b_u v} + k_{b_u v} = 0, \text{ for } u \neq t$$

and

$$-l_{b_t v} + k_{b_t v} - \alpha k_{b_s v} = 0, \text{ for } v < b_t$$

Since $k_{b_s v} = l_{b_s v}$ for all v by the first equation, this can be solved and we obtain

$$k_{b_u v} = \begin{cases} l_{b_u v} & \text{for } u \neq t, \\ l_{b_t v} + l_{b_s v}\alpha & \text{for } u = t, v < b_t. \end{cases}$$

Note that again this determines K uniquely and $e_L \circ g = e_K$, that is the coefficient of e_K in $e_L \circ g$ is 1.We can obtain K from L by adding the left hand side part of row b_s up to entry $l_{b_s b_t - 1}$ of L multiplied by α to row b_t.

This finishes the proof of the proposition. □

3.3.7 Corollary. We collect the information from the proof of the previous proposition as follows: For $L \in \mathfrak{X}_{\mathfrak{t}}, g = E + \alpha e_{ij} \in U^w \cap U$:

$$e_L \circ g = \begin{cases} \theta\left(l_{ij}\alpha\right) e_L & \text{if } i \in \underline{\mathfrak{t}},\ j \notin \underline{\mathfrak{t}}; \\ e_K & \text{if } i,\ j \notin \underline{\mathfrak{t}}; \\ e_R & \text{if } i,\ j \in \underline{\mathfrak{t}}. \end{cases} \tag{3.3.8}$$

where $K = (k_{b_u v}) \in \mathfrak{X}_{\mathfrak{t}}, R = (r_{b_u v}) \in \mathfrak{X}_{\mathfrak{t}}$ satisfy:

$$k_{b_u v} = \begin{cases} l_{b_u v} & \text{if } v \neq i; \\ l_{b_u i} - \alpha l_{b_u j} & \text{if } v = i, i < b_u. \end{cases} \tag{3.3.9}$$

$$r_{b_u v} = \begin{cases} l_{b_u v} & \text{if } b_u \neq j; \\ l_{jv} + \alpha l_{iv} & \text{if } b_u = j, v < b_u. \end{cases} \tag{3.3.10}$$

From(3.3.9) follows that the action of $g = E+\alpha e_{ij} \in U^w \cap U$ on e_L under the condition $i,\ j \notin \underline{\mathfrak{t}}$ is equivalent to subtracting in L from the $i-$th column α times the $j-$th column ignoring the $(s,t)-$entries with $s \leqslant t$ and take the idempotent corresponding to the resulting matrix. Hence we call this a **truncated column operation**.
Similarly, by (3.3.10), the action of $g = E + \alpha e_{ij} \in U^w \cap U$ on e_L under the condition $i,\ j \in \underline{\mathfrak{t}}$ is equivalent to adding α times the $i-$th row to the $j-$th row of L ignoring the $(s,t)-$entries with $s \leqslant t$ and take the idempotent corresponding to the resulting matrix. We call this a **truncated row operation**.

3.3.11 Lemma. Define three different sets such that each of them corresponds to one case in (3.3.8):

$$\begin{aligned} \Upsilon_1 &= \{(i,j) \,|\, i > j \text{ and } i \in \underline{\mathfrak{t}},\ j \notin \underline{\mathfrak{t}}\}, \\ \Upsilon_2 &= \{(i,j) \,|\, i > j \text{ and } i, j \notin \underline{\mathfrak{t}}\}, \\ \Upsilon_3 &= \{(i,j) \,|\, i > j \text{ and } i, j \in \underline{\mathfrak{t}}\}. \end{aligned}$$

Remember the notation $\Pi = \{(i,j) \,|\, i \neq j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$ in Definition 1.4.3. Then $\Upsilon = \Upsilon_1 \cup \Upsilon_2 \cup \Upsilon_3$ is a closed subset of Π. Moreover,

$$U^w \cap U = G(\Upsilon) = \prod_{(i,j) \in \Upsilon} X_{ij}$$

where the product can be taken in any order. Hence the action of $U^w \cap U$ on e_L can be given as firstly acting with the root subgroups which do not change L, secondly acting with the truncated column operation and at last acting with the truncated row operation.

Proof. The second statement follows easily from Theorem 1.4.4 if we can prove that Υ is a closed subset of Π. By Definition 1.4.3, Υ is a closed subset if the follows hold:

(1) If $(i,j) \in \Upsilon_1, (j,k) \in \Upsilon$, then $(i,k) \in \Upsilon$.

(2) If $(i,j) \in \Upsilon_2, (j,k) \in \Upsilon$, then $(i,k) \in \Upsilon$.

(3) If $(i,j) \in \Upsilon_3, (j,k) \in \Upsilon$, then $(i,k) \in \Upsilon$.

In the first case, if $(i,j) \in \Upsilon_1$, $(j,k) \in \Upsilon$, then $i \in \underline{\mathfrak{t}}$, $j \notin \underline{\mathfrak{t}}$ and hence $(j,k) \in \Upsilon_2$ which leads to $k \notin \underline{\mathfrak{t}}$, moreover $i > j > k$, therefore $(i,k) \in \Upsilon_1 \subset \Upsilon$.

In the second case, if $(i,j) \in \Upsilon_2, (j,k) \in \Upsilon$, then $i,j \notin \underline{\mathfrak{t}}$ and hence $(j,k) \in \Upsilon_2$ which leads to $k \notin \underline{\mathfrak{t}}$, moreover $i > j > k$, therefore $(i,k) \in \Upsilon_2 \subset \Upsilon$.

In the third case, if $(i,j) \in \Upsilon_3, (j,k) \in \Upsilon$, then $i \in \underline{\mathfrak{t}}, j \in \underline{\mathfrak{t}}$ and hence we have two possibilities: one is $(j,k) \in \Upsilon_1$ which leads to $k \notin \underline{\mathfrak{t}}$, moreover $i > j > k$, therefore $(i,k) \in \Upsilon_1 \subset \Upsilon$; the other possibility is $(j,k) \in \Upsilon_3$, which leads $k \in \underline{\mathfrak{t}}$, moreover $i > j > k$, therefore $(i,k) \in \Upsilon_3 \subset \Upsilon$.

Now we finish the proof of Υ is a closed subset of Π. And the remaining statements follow easily. □

3.3.12 Lemma. Keep the notations of $\Upsilon_1, \Upsilon_2, \Upsilon_3$ as in Lemma 3.3.11. Set

$$U_0^w = \{\Pi\, x_{ij}(\alpha) \,|\, (i,j) \in \Upsilon_1, \alpha \in GF(q)\},$$

$$U_{\mathbf{C}}^w = \{\Pi\, x_{ij}(\alpha) \,|\, (i,j) \in \Upsilon_2, \alpha \in GF(q)\},$$

$$U_{\mathbf{R}}^w = \{\Pi\, x_{ij}(\alpha) \,|\, (i,j) \in \Upsilon_3, \alpha \in GF(q)\}$$

Then U_0^w is a normal subgroup of $U^w \cap U$ and we have

$$U^w \cap U = U_0^w \rtimes (U_{\mathbf{C}}^w \times U_{\mathbf{R}}^w).$$

Proof. For $(i,j),(s,t) \in \Upsilon_1$, we have

$$\begin{aligned} x_{ij}(\alpha)x_{st}(\beta) &= (E+\alpha e_{ij})(E+\beta e_{st}) \\ &= E+\alpha e_{ij}+\beta e_{st}+\alpha\beta e_{ij}e_{st}. \end{aligned}$$

Since $i,s \in \underline{\mathfrak{t}}$ and $j,t \notin \underline{\mathfrak{t}}$, then $e_{ij}e_{st} = e_{st}e_{ij} = 0$. Hence

$$x_{ij}(\alpha)x_{st}(\beta) \in U_0^w.$$

Moreover we have

$$\begin{aligned} x_{ij}(\alpha)x_{st}(\beta) &= E+\alpha e_{ij}+\beta e_{st} \\ &= E+\alpha e_{ij}+\beta e_{st}+\alpha\beta e_{st}e_{ij} \\ &= x_{st}(\beta)x_{ij}(\alpha) \end{aligned}$$

Therefore, U_0^w is an abelian subgroup of $U^w \cap U$.
For an arbitrary $g = (g_{st}) \in U^w \cap U, x_{ij}(\alpha) \in U_0^w$, set

$$m = (m_{st}) = gx_{ij}(\alpha)g^{-1}.$$

Hence

$$m = g(E+\alpha e_{ij})g^{-1} = gEg^{-1} + \alpha g e_{ij} g^{-1}.$$

Set $h = (h_{st}) = g^{-1}$. Then

$$m_{st} = \delta_{st} + \alpha g_{si} e_{ij} h_{jt}.$$

Since $(i,j) \in \Upsilon_1$, we have $i \in \underline{\mathfrak{t}}, j \notin \underline{\mathfrak{t}}$. By Lemma 3.3.1, $g_{si} = 0$ if $s \notin \underline{\mathfrak{t}}$ and $h_{jt} = 0$ if $t \in \underline{\mathfrak{t}}$, since $g,h \in U^w \cap U$. Hence $m_{st} = 0$ if $s \notin \underline{\mathfrak{t}}$ or $t \in \underline{\mathfrak{t}}$. Therefore, $m \in U_0^w$, which means U_0^w is an abelian normal subgroup of $U^w \cap U$.

It is apparent from the definitions of $U_{\mathbf{C}}^w$ and $U_{\mathbf{C}}^w$, they are both subgroups of $U^w \cap U$ and these two subgroups commute with each other. Hence

$$U^w \cap U = U_0^w \rtimes (U_{\mathbf{C}}^w \times U_{\mathbf{R}}^w).$$

□

3.3.13 Definition. Let $e_L \in \mathcal{E}_{\mathfrak{t}}$. Define the $U^w \cap U-$orbit of e_L as

$$\mathcal{O}_L = \{e_K \mid e_L \circ g = C(L,g)e_K \text{ for some } g \in U^w \cap U,\ 0 \neq C(L,g) \in F\}.$$

Moreover, we define the $F(U^w \cap U)-$submodule of the $\mathfrak{t}-$batch $\mathfrak{M}_{\mathfrak{t}}$ associated with the orbit $\mathcal{O}_L$ as:

$$M_{\mathcal{O}_L} = \bigoplus_{e_K \in \mathcal{O}_L} Fe_K.$$

3.3.14 Example. (1) Let $GF(q) = 2, L = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \in \Xi_{2,5}$. Then

$$\mathfrak{t} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2} \end{array} \quad , \quad U^{d(\mathfrak{t})} \cap U = \{g = (g_{ij}) \in U_5 \,|\, g_{43} = 0\}.$$

$$\mathcal{O}_L = \left\{ e_K \,\middle|\, K = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 1 & 1 & & \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & & \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \right\}.$$

(2) Let $L = \begin{pmatrix} 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 1 & 1 & \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \in \Xi_{3,6}$. Then

$$\mathfrak{t} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \quad , \quad U^{d(\mathfrak{t})} \cap U = \{g = (g_{ij}) \in U_5 \,|\, g_{43} = 0\}.$$

For some fixed $g = (g_{ij}) \in U^{d(\mathfrak{t})} \cap U$, using Corollary 3.3.7 step by step we obtain $e_L \circ g = C(L, g)e_K$, where

$$K = \begin{pmatrix} g_{53} & -g_{21}g_{53} + g_{63} & 1 & & & \\ 1 & -g_{21} + g_{65} & 0 & 1 - g_{41} - (-g_{21} + g_{65})g_{42} & 1 & \\ 0 & 1 & 0 & -g_{42} & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{O}_L = \left\{ e_K \,\middle|\, K = \begin{pmatrix} a & b & 1 & & & \\ 1 & c & 0 & d & 1 & \\ 0 & 1 & 0 & e & 0 & 1 \end{pmatrix}, \forall\, a, b, c, d, e \in GF(q) \right\}.$$

3.4 Pattern matrices and condition sets

In Section 3.3 we have seen that for $\lambda = (n-m, m) \vdash n, \mathfrak{t} = \mathfrak{t}^\lambda w \in \mathrm{RStd}(\lambda)$, the subgroup $U^w \cap U$ acts monomially on $\mathcal{E}_\mathfrak{t} = \{e_K \,|\, K \in \mathfrak{X}_\mathfrak{t}\}$. This means in particular that the restriction of the $\mathfrak{t}-$batch $\mathfrak{M}_\mathfrak{t}$ to the subgroup $U^w \cap U$ of U is a monomial module and $\mathcal{E}_\mathfrak{t}$ is a monomial basis of $\mathfrak{M}_\mathfrak{t}$. Therefore $\mathfrak{M}_\mathfrak{t}$ decomposes naturally into a direct sum of $U^w \cap U-$submodules $M_\mathcal{O}$, where $\mathcal{O}$ runs through the set of orbits of $U^w \cap U$ acting on $\mathcal{E}_\mathfrak{t}$:

$$\mathfrak{M}_\mathfrak{t} = \bigoplus_{\mathcal{O}} M_\mathcal{O}.$$

Our goal in this section is to classify the orbits $\mathcal{O}$, determine their size (and hence the $F-$dimension of $M_{\mathcal{O}}$) and count the number of orbits of a given fixed size. We shall show that this number is a polynomial in q with integral coefficients and the sizes of the orbits are powers of q. Later on we shall show that for a given orbit $\mathcal{O}$, the corresponding monomial $U^w \cap U-$module is irreducible and that the action of $U^w \cap U$ on $\mathcal{E}_{\mathfrak{t}}$ can be extended to U. Hence the decomposition

$$\mathfrak{M}_{\mathfrak{t}} = \bigoplus_{\mathcal{O}} M_{\mathcal{O}}$$

is a decomposition of the $\mathfrak{t}-$batch $\mathfrak{M}_{\mathfrak{t}}$ into irreducible $FU-$modules.

Throughout we fix $\mathfrak{t} \in \mathrm{RStd}(\lambda), \lambda = (n-m, m) \vdash n$ and let $w \in \mathfrak{S}_n$ be the permutation defined by $\mathfrak{t}^\lambda w = \mathfrak{t}$. Moreover let $\underline{\mathfrak{t}} = (b_1, \ldots, b_m)$.

3.4.1 Definition. Let $(b, j) \in \mathfrak{J}_{\mathfrak{t}}$. So $b \in \underline{\mathfrak{t}}, j \notin \underline{\mathfrak{t}}$ and $j < b$.

(1) The hook column $\Omega^{\mathbf{up}}_{(b,j)}$ of (b, j) consists of all positions in the column j above (b, j) belonging to $\mathfrak{J}_{\mathfrak{t}}$, that is

$$\Omega^{\mathbf{up}}_{(b,j)} = \{(u, j) \in \mathfrak{J}_{\mathfrak{t}} \,|\, u < b\}.$$

(2) The hook row $\Omega^{\mathbf{right}}_{(b,j)}$ of (b, j) consists of all positions in the row b to the right of (b, j) belonging to $\mathfrak{J}_{\mathfrak{t}}$, that is

$$\Omega^{\mathbf{right}}_{(b,j)} = \{(b, v) \in \mathfrak{J}_{\mathfrak{t}} \,|\, v > j\}.$$

(3) The $(b, j)-$hook $\Omega_{(b,j)} = \Omega^{\mathfrak{t}}_{(b,j)}$ (of $\mathfrak{t}$) is defined as

$$\Omega_{(b,j)} = \Omega^{\mathbf{up}}_{(b,j)} \cup \Omega^{\mathbf{right}}_{(b,j)} \cup \{(b, j)\}.$$

(4) We denote $\bar{\Omega}_{(b,j)} = \Omega_{(b,j)} \setminus \{(b, j)\} = \Omega^{\mathbf{up}}_{(b,j)} \cup \Omega^{\mathbf{right}}_{(b,j)}$.

(5) $(s, t) \in \mathfrak{J}_{\mathfrak{t}}$ is called an outside place of the $(b, j)-$hook, if either $s > b$ or $t < j$; denote all the outside places of the $(b, j)-$hook by $\mathrm{Out}(b, j)$.

(6) The number of different positions in $\Omega_{(b,j)}$ is called the residue of the $(b, j)-$hook $\Omega_{(b,j)}$ and denoted by $\mathrm{res}(b, j)$.

3.4.2 Lemma. Let $1 \leqslant j < b \leqslant n$ with $b \in \underline{\mathfrak{t}}$ and $j \notin \underline{\mathfrak{t}}$, that is $(b, j) \in \mathfrak{J}_{\mathfrak{t}}$. Then

$$\mathrm{res}(b, j) = b - j.$$

In particular $\mathrm{res}(b, j)$ is independent of $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ and independent of the two-part partition λ. More precisely: Let $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathrm{RStd}(\lambda), \mu = (n-l, l)$ and $\mathfrak{t}_3 \in \mathrm{RStd}(\mu)$. Let $b \in \mathfrak{t}_\rho, 1 \leqslant j < b$ with $j \notin \mathfrak{t}_\rho$ for $\rho = 1, 2, 3$. Then $\mathrm{res}(b, j) = |\Omega^{\mathfrak{t}_\rho}_{(b,j)}|$ is independent of ρ.

Proof. Note that the second claim follows immediately whence we have shown that $\operatorname{res}(b,j) = b - j$. But this formula easily from the fact that an entry belongs to the hook row if and only if its column index is any number $j \leqslant v \leqslant n$ such that $v \notin \underline{\mathfrak{t}}$ and to the hook column if and only if its row index $\tilde{b}$ satisfies $j \leqslant \tilde{b} \leqslant n$ with $\tilde{b} \in \underline{\mathfrak{t}}$. Since the hook row and column meet only in the hook corner, the hook has precisely $b - j$ many different elements. $\square$

We remark that this property of hooks is the deeper reason for labeling the rows of matrices in $\Xi_{m,n}$ in this unusual way. This will allow us later on to compare orbits for different row standard tableaux even for different two-part partitions.

3.4.3 Example. Let $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}$, $\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}$, $\mathfrak{u} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$.
Then

(1) $\Omega^{\mathfrak{t}}_{(6,2)} = \{(3,2),(5,2),(6,2),(6,4)\} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} & \times & 1 & & & \\ & \times & 0 & & 1 & \\ & \times & 0 & \times & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 5 \\ 6 \end{array}.$

$\operatorname{Out}(6,2) = \{(3,1),(5,1),(6,1)\}; \quad \operatorname{res}(6,2) = 4.$

(2) $\Omega^{\mathfrak{t}}_{(5,1)} = \{(3,1),(5,1),(5,2),(5,4)\} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \times & & 1 & & & \\ \times & \times & 0 & \times & 1 & \\ & & 0 & & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 5 \\ 6 \end{array}.$

$\operatorname{Out}(5,1) = \{(6,1),(6,2),(6,4)\}; \quad \operatorname{res}(5,1) = 4.$

(3) $\Omega^{\mathfrak{s}}_{(5,1)} = \{(2,1),(5,1),(5,3),(5,4)\} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \times & 1 & & & & \\ \times & 0 & \times & \times & 1 & \\ & 0 & & & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 2 \\ 5 \\ 6 \end{array}.$

$\operatorname{Out}(5,1) = \{(6,1),(6,3),(6,4)\}; \quad \operatorname{res}(5,1) = 4.$

(4) $\Omega^{\mathfrak{u}}_{(5,1)} = \{(2,1),(5,1),(5,3),(5,4)\} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \times & 1 & & & & \\ \times & 0 & \times & \times & 1 & \end{pmatrix} \end{array} \begin{array}{c} \\ 2 \\ 5 \end{array}.$

$\operatorname{Out}(5,1) = \emptyset; \quad \operatorname{res}(5,1) = 4.$

3.4.4 Definition. $L \in \Xi_{m,n}$ is called a **pattern matrix**, if each row and column of L has at most one non zero entry besides the last 1's.

3.4.5 Definition. Let L be a pattern matrix. The set of the nonzero indexed entries of L besides the last 1's is called the **condition set** of L, denoted by $S(L)$. And the set of position indices of the nonzero entries of L beside the last 1's is called the **frame of the condition set**, denoted by $\overline{S(L)}$.

3.4.6 Remark. We remark that it is possible that $S(L_1) = S(L_2)$ but $\mathrm{tab}(L_1) \neq \mathrm{tab}(L_2)$ or sometimes even $\mathrm{Shape}\big(\mathrm{tab}(L_1)\big) \neq \mathrm{Shape}\big(\mathrm{tab}(L_2)\big)$.

Next we show that each orbit $\mathcal{O}$ of $\mathcal{E}_{\mathfrak{t}}$ under the monomial action of $U^w \cap U$ contains precisely one pattern matrix and that the dimension of $M_{\mathcal{O}}$ is determined combinatorially by the frame of the corresponding condition set. This allows us to determine $\dim_F(M_{\mathcal{O}})$ as well as count the orbits of $U^w \cap U$ acting on $\mathcal{E}_{\mathfrak{t}}$.

3.4.7 Remark. Let L be a pattern matrix with condition set $S(L)$. Note that the truncated row and column operations in Corollary 3.3.7 work only from left to the right respectively from the bottom to the top of a matrices in $\mathfrak{X}_{\mathfrak{t}}$. This implies that action of $U^w \cap U$ on $e_{_L}$ will not change those elements $l_{bj} \in S(L)$ which are the outer south west rim of $S(L)$. More precisely: Say $(b, j) \in \overline{S(L)}$ belongs to the outer frame of the condition set $S^o(L)$ of L, if the following holds:

If $(c, k) \in \overline{S(L)}$ and $k < j$, then $c < b$.

In other words: If we order

$$\overline{S(L)} = \{(u_{_1}, v_{_1}), \ldots, (u_{_s}, v_{_s})\}$$

such that

$$v_1 < v_2 < \cdots < v_s,$$

then choose the unique maximal subset

$$\{(u_{\tau_{_1}}, v_{\tau_{_1}}), \ldots, (u_{\tau_{_r}}, v_{\tau_{_r}})\}$$

such that

$$u_{\tau_{_1}} < u_{\tau_{_2}} < \cdots < u_{\tau_{_r}}, \quad v_{\tau_{_1}} < v_{\tau_{_2}} < \cdots < v_{\tau_{_r}}$$

and for any $(u_\alpha, v_\alpha) \in \overline{S(L)}$ with $v_\alpha \neq v_{\tau_{_i}}$, $\forall\, 1 \leqslant i \leqslant s$, there exist at least one $i \in \{1, 2 \ldots, s\}$ such that $v_\alpha > v_{\tau_{_i}}$. Then this set is $S^o(L)$. Moreover,

- Given $(b, j) \in S^o(L)$, no truncated row or column operation can change it because there is no element $l_{ck} \in S(L)$ to the southwest of it. That is to say, for any $e_{_K} \in \mathcal{O}_L$, we have: $k_{bj} = l_{bj}$ for $(b, j) \in S^o(L)$.

- Similarly, the entries in those positions which are to the southwest of all positions $(b,j) \in S^o(L)$ are always zeros. More precisely, set

$$\mathrm{Out}(L) = \bigcap_{(b,j)\in S^o(L)} \mathrm{Out}(b,j).$$

Then for any $e_K \in \mathcal{O}_L$, we have: $k_{bj} = 0$ for all $(b,j) \in \mathrm{Out}(L)$.

3.4.8 Lemma. Each $U^w \cap U-$orbit $\mathcal{O}$ of $\mathcal{E}_{\mathfrak{t}}$ contains a unique pattern matrix. So we have a bijection between orbits of $\mathcal{E}_{\mathfrak{t}}$ under the action of $U^w \cap U$ and pattern matrices in $\mathfrak{X}_{\mathfrak{t}}$. Moreover, for a fixed frame $\overline{S}$ there are precisely $(q-1)^s$ many different pattern matrices L and orbits $\mathcal{O}_L$ with property $\overline{S(L)} = \overline{S}$ where s is the cardinality $|\overline{S}|$ of $\overline{S}$.

Proof. First we prove that every orbit contains a pattern matrix. So let $K \in \mathfrak{X}_{\mathfrak{t}}$ and $\mathcal{O}$ be the orbit containing e_K. We use 3.3.7 to find the pattern matrix in $\mathcal{O}$ by a series of truncated row and column operations. Let $1 \leqslant j \leqslant n, j \notin \underline{\mathfrak{t}}$ be the column index of the first nonzero column in K besides columns containing last 1's. Let k_{bj} be its last nonzero entry. Thus k_{bj} is the most southwest nonzero entry of K. Using truncated row operations (necessarily upward) and truncated column operations (necessarily to the right) we can obtain a matrix $M = (m_{cd}) \in \mathfrak{X}_{\mathfrak{t}}$ with $e_M \in \mathcal{O}$ such that $m_{bj} = k_{bj}$ and all entries m_{cd} with $(c,d) \in \bar{\Omega}_{(b,j)}$ are zeros. Next we choose $j < d \leqslant n, d \notin \underline{\mathfrak{t}}$ such that all columns to the left of column d besides the columns containing last 1's and column j in M are zero columns, and we choose $c \in \mathfrak{t}$ such that m_{cd} is the last nonzero entry in that column. Since $(b,d) \in \Omega_{(b,j)}$ we conclude that $c \neq b$. Again we can use truncated row and column operations to turn all entries on positions belong to $\bar{\Omega}_{(c,d)}$ into zeros. Continuing like this working from left to right we obviously get a pattern matrix L such that $e_L \in \mathcal{O}$.

Now we show the uniqueness. Suppose we have two different pattern matrices L and R in the orbit $\mathcal{O}$ with respectively condition $S(L)$ and $S(R)$. Assume $e_R = e_L g$ for some $g \in U^w \cap U$. Using Lemma 3.3.11, we can assume $g = g_1 g_2 g_3$ where $e_L g_1 = C(L, g_1) e_L$ and g_2 is a series of products of truncated column operations and g_3 is a series of products of truncated row operations. Now we have

$$e_L \circ g_1 g_2 = C(L, g_1) e_L \circ g_2 = e_R \circ g_3^{-1}, \tag{3.4.9}$$

where g_3^{-1} is again a series of products of truncated row operations. From Corollary 3.3.7 and Lemma 3.4.7, we can easily get $S^o(L) = S^o(R)$. If

we order the condition sets $S(L)$ and $S(R)$ by the column indices. Then without lose generality, we can choose the first $l_{uv} \in S(L)$ and $l_{uv} \notin S(R)$, such that $r_{st} \in S(R) \setminus S^o(R)$ has the property $t > v$. Since L is a pattern matrix, (3.3.9) shows that the truncated column operations only change the hook row on the $(i,j)-$hook where $l_{ij} \in S(L)$. Hence we get $m_{uv} = l_{uv}$ for any $e_M = e_L \circ g_2$. Similarly since R is a pattern matrix, the truncated row operations only change the hook column on the $(s,t)-$hook where $r_{st} \in S(R)$. Hence we get $m_{uv} = 0 \neq l_{uv}$, which means 3.4.9 never holds. Therefore, $S(L) = S(R)$ and hence $L = R$, which proves the uniqueness.

□

3.4.10 Definition. Let L be a pattern matrix, and assume $e_L \in \mathcal{O}$. Since we have proved each orbit has a unique pattern matrix, we define $\mathrm{tab}(\mathcal{O}) = \mathrm{tab}(L)$, $S(\mathcal{O}) = S(L)$, and $\overline{S(\mathcal{O})} = \overline{S(L)}$.

3.4.11 Notation. Let L be a pattern matrix and $S = S(L)$ be the condition set of L. Define:

(1) $S_I = \{i \mid (i,j) \in \overline{S}\}$, which collects all the row indices of the positions in the frame of the condition set S.

(2) $S_J = \{j \mid (i,j) \in \overline{S}\}$, which collects all the column indices of the positions in the frame of the condition set S.

(3) In particular, if $S = \emptyset$, then $S_I = S_J = \emptyset$.

3.4.12 Example. Let $L = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{pmatrix} 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & 1 & \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 3 \\ 5 \\ 6 \end{array} \end{array}$. Obviously, L is a pattern matrix. And

$$S := S(L) = \{l_{61} = 1, l_{54} = 1\};$$

$$\overline{S} = \{(6,1),(5,4)\};$$

$$S_I = \{5,6\};\ S_J = \{1,4\}.$$

Now we try to determine the size of an $U^w \cap U-$orbit. Since

$$U^w \cap U = U_0^w \rtimes (U_{\mathbf{C}}^w \times U_{\mathbf{R}}^w),$$

using Corollary 3.3.7 we see that every element in U_0^w is in the stabilizer of e_L, hence in order to compute the orbits of the action of $U^w \cap U$, it suffices to calculate the orbits of $U_{\mathbf{C}}^w \times U_{\mathbf{R}}^w$ on $\mathcal{E}_{\mathfrak{t}}$. We have seen that each orbit contains a pattern matrix. So let $\mathcal{O}$ be an $U^w \cap U$ orbit with pattern matrix L. We want to calculate $\mathrm{Stab}_{U_{\mathbf{C}}^w \times U_{\mathbf{R}}^w}(e_L)$.

3.4.13 Lemma. Let $\mathcal{O}$ be a $U^w \cap U$ orbit with pattern matrix L and condition set S. Then

$$\mathrm{Stab}_{U_{\mathbf{C}}^w}(e_L) = \langle X_{ij} \,|\, i, j \notin \underline{\mathfrak{t}}, j \notin S_J \,\rangle,$$

$$\mathrm{Stab}_{U_{\mathbf{R}}^w}(e_L) = \langle X_{ij} \,|\, i, j \in \underline{\mathfrak{t}}, i \notin S_I \,\rangle.$$

Proof. From (3.3.9) we see that, the truncated column action on e_L induced by $x_{ij}(\alpha)$ is just subtracting in L from the $i-$th column α times the $j-$th column ignoring in column i all zero entries to the right of a last one and taking the idempotent indexed by the resulting matrix. Hence X_{ij} is a generator of the stabilizer of e_L in $U_{\mathbf{C}}^w$ if and only if the $j-$th column of the pattern matrix L is a zero column, that is $j \notin \underline{\mathfrak{t}}$. Thus

$$\mathrm{Stab}_{U_{\mathbf{C}}^w}(e_L) = \langle X_{ij} \,|\, i, j \notin \underline{\mathfrak{t}}, j \notin S_J \,\rangle.$$

From (3.3.10), the truncated row action on e_L induced by $x_{ij}(\alpha)$ is just adding α times the $i-$th row to the $j-$th row for positions to the left of the last one in row j and taking the idempotent indexed by the resulting matrix. Hence X_{ij} is a generator of the stabilizer of e_L in $U_{\mathbf{R}}^w$ if and only if the $i-th$ row is a zero row of the pattern matrix L, that is $i \notin S_I$. Thus

$$\mathrm{Stab}_{U_{\mathbf{R}}^w}(e_L) = \langle X_{ij} \,|\, i, j \in \underline{\mathfrak{t}}, i \notin S_I \,\rangle.$$

□

3.4.14 Proposition. Let $\mathcal{O}$ be a $U^w \cap U$ orbit and assume

$$\overline{S} := \overline{S(\mathcal{O})} = \{(b_{u_i}, v_i) \,|\, 1 \leqslant i \leqslant s\}.$$

Then

$$\dim M_{\mathcal{O}} = q^{k-s},$$

where k is the number of places which are on the hooks whose corners belong to the condition set. Moreover,

$$k \;=\; \sum_{1\leqslant i\leqslant s} \big((b_{u_i} - v_i) - |Z_i|\big).$$

where $Z_i = \{j \,|\, b_{u_j} > b_{u_i} > v_j > v_i\,\}$ for $1 \leqslant i \leqslant s$.

Proof. Let L be the unique pattern matrix such that $e_L \in \mathcal{O}$. Now we calculate the stabilizer of e_L. We have already got three types of stabilizer of e_L by Corollary 3.3.7 and Lemma 3.4.13:

(1) $U_0^w = \langle X_{ij} \,|\, i \in \underline{\mathfrak{t}},\, j \notin \underline{\mathfrak{t}}\rangle$;

(2) $\mathrm{Stab}_{U_{\mathbf{C}}^w}(e_L) = \langle X_{ij} \,|\, i, j \notin \underline{\mathfrak{t}},\, j \notin S_J\,\rangle$;

(3) $\mathrm{Stab}_{U_{\mathbf{R}}^w}(e_L) = \langle X_{ij} \,|\, i, j \in \underline{\mathfrak{t}},\, i \notin S_I\,\rangle$.

Moreover, since we may have some common places which are both on some hook row and some hook column with those hooks whose corners belonging to the condition set S and again by Corollary 3.3.7 there exist pairs of row operations and column operations such that the product of these two operations acts trivially on e_L. More precisely, the pair has the form $x_{ij}(\alpha_{ij})x_{st}(\beta_{st})$ where $l_{si}, l_{tj} \in S$ and $\alpha_{ij}l_{tj} = \beta_{st}l_{si}$. That is

$$\mathbf{P} := \{x_{ij}(\alpha_{ij})x_{st}(\beta_{st}) \,|\, \alpha_{ij} \in GF(q), l_{si}, l_{tj} \in S, \alpha_{ij}l_{tj} = \beta_{st}l_{si}\}$$

is a set of some generators of the stabilizer of e_L.

For any $g \in U^w \cap U$, it can be written as a series product of $x_{ij}(\alpha_{ij})$. Lemma 3.3.11 shows that we can choose an arbitrary order of those $x_{ij}(\alpha_{ij})$. Now we fix an order like this: firstly those $x_{ij}(\alpha_{ij})$ belonging to U_0^w, $\mathrm{Stab}_{U_{\mathbf{C}}^w}(e_L)$ and $\mathrm{Stab}_{U_{\mathbf{R}}^w}(e_L)$, then those pairs $x_{ij}(\alpha_{ij})x_{st}(\beta_{st})$ belonging to the set $\mathbf{P}$, then the remaining truncated column operation, and at last the remaining truncated row operation. Since for those truncated row operation x_{st} in the pair set, there is a uniquely expression $x_{st}(\gamma_{st}) = x_{st}(\beta_{st})\, x_{st}(\gamma_{st} - \beta_{st})$ for any $\gamma_{st} \in GF(q)$, hence this order make sense.

Suppose u_1 is the rest truncated column operation of g, u_2 is the rest truncated row operation of g, then

$$u_1 \in \prod X_{ij} \text{ where } i, j \notin \underline{\mathfrak{t}}, i \notin S_J, j \in S_J,$$

$$u_2 \in \prod X_{st} \text{ where } s, t \in \underline{\mathfrak{t}}, s \in S_I.$$

We claim

$$e_L \circ u_1u_2 = e_L \Leftrightarrow u_1 = u_2 = 1. \tag{3.4.15}$$

Now

$$e_L \circ u_1 u_2 = e_L \Leftrightarrow e_L \circ u_1 = e_L \circ u_2^{-1}$$

where u_2^{-1} is again a truncated row operation and

$$u_2^{-1} \in \prod X_{st} \text{ where } s, t \in \underline{\mathfrak{t}}, s \in S_I,$$

since it is easy to see $\langle X_{st} \,|\, s, t \in \underline{\mathfrak{t}}, s \in S_I \rangle$ is a subgroup of $U^w \cap U$. Moreover, by (3.3.9), $e_L \circ u_1$ has only possible nonzero entries on the positions in rows $u \in S_I$ except those whose column indices belonging to S_J. And by (3.3.10), $e_L \circ u_2^{-1}$ has only possible nonzero entries on the positions in columns $v \in S_J$. It means that the action of u_1 and u_2^{-1} on e_L exactly influence different positions. Hence

$$e_L \circ u_1 = e_L \circ u_2^{-1} \Leftrightarrow u_1 = u_2^{-1} = 1 \Leftrightarrow u_1 = u_2 = 1.$$

Therefore

$$e_L \circ g = C(L, g) e_L \Leftrightarrow g = \prod_{(u,v)} x_{uv}(\gamma_{uv}) \prod_{\substack{(i,j) \\ (s,t)}} x_{ij}(\alpha_{ij}) x_{st}(\beta_{st}),$$

where

$$x_{uv}(\gamma_{uv}) \in U_0^w \cup \mathrm{Stab}_{U_{\mathbf{C}}^w} \cup \mathrm{Stab}_{U_{\mathbf{R}}^w},$$

and

$$x_{ij}(\alpha_{ij}) x_{st}(\beta_{st}) \in \mathbf{P}.$$

Now we argue for any

$$g = g_1 u_1 u_2, \quad h = h_1 v_1 v_2$$

where $g_1, h_1 \in U_0^w \cdot \mathrm{Stab}_{U_{\mathbf{C}}^w} \cdot \mathrm{Stab}_{U_{\mathbf{R}}^w} \cdot \mathbf{P}$ and u_1(resp. v_1) is the rest truncated column operation of g, u_2(resp. v_2) is the rest truncated row operation of h, we have

$$e_L \circ u_1 u_2 = e_L \circ v_1 v_2 \Leftrightarrow u_1 = v_1, u_2 = v_2.$$

From Lemma 3.3.12 follows that the column operations commute with row operations, then we have

$$e_L \circ u_1 u_2 = e_L \circ v_1 v_2 \Leftrightarrow e_L \circ u_1 u_2 = e_L \circ v_2 v_1 \Leftrightarrow e_L \circ u_1 v_1^{-1} = e_L \circ v_2 u_2^{-1}.$$

Since

$$u_1 v_1^{-1} \in \prod X_{ij} \text{ where } i, j \notin \underline{\mathfrak{t}}, i \notin S_J, j \in S_J,$$

$$v_2 u_2^{-1} \in \prod X_{st} \text{ where } s, t \in \underline{\mathfrak{t}}, s \in S_I.$$

Using the claim above in (3.4.15), we obtain

$$e_L \circ u_1 v_1^{-1} = e_L \circ v_2 u_2^{-1} \Leftrightarrow u_1 v_1^{-1} = v_2 u_2^{-1} = 1 \Leftrightarrow u_1 = v_1, u_2 = v_2.$$

Hence $e_L \circ u_1 u_2$ gives all the coset representatives of $\mathrm{Stab}_{U^w \cap U}(e_L)$ in $U^w \cap U$ where

$$u_1 \in \prod X_{ij} \text{ where } i, j \notin \underline{\mathfrak{t}}, i \notin S_J, j \in S_J,$$

$$u_2 \in \prod X_{st} \text{ where } s, t \in \underline{\mathfrak{t}}, s \in S_I.$$

Now we can calculate the size of the orbit, which is just the index of the stabilizer in $U^w \cap U$. Suppose

$$\mathfrak{t} = \mathrm{tab}(L) = \begin{array}{|c|c|c|c|c|c|} \hline a_1 & a_2 & \cdots & a_m & \cdots & a_{n-m} \\ \hline b_1 & b_2 & \cdots & b_m & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array}.$$

Set

$$Y_1 = \{(a_i, a_j) \mid a_i > a_j, a_j \in S_J\}.$$

$$Y_2 = \{(b_r, b_t) \mid b_r > b_t, b_r \in S_I\}.$$

$$Z_i = \{j \mid b_{u_j} > b_{u_i} > v_j > v_i\} \text{ for } 1 \leqslant i \leqslant s.$$

Obviously, Y_1 labels the root subgroups satisfying the second condition except those in the stabilizer by themselves. Y_2 labels the root subgroups satisfying the third condition except those in the stabilizer by themselves. Z_i denotes the number of the intersection points on $i-$th row and of the hooks with corners in the condition set. Therefore,

$$\dim_{M_{\mathcal{O}}} = q^d$$

where

$$\begin{aligned} d &= |Y_1| + |Y_2| - \sum_{i=1}^{s} |Z_i| \\ &= \sum_{a_j \in S_J} (n - m - j) + \sum_{b_r \in S_I} (r - 1) - \sum_{i=1}^{s} |Z_i|. \end{aligned}$$

It is easy to see $\sum_{a_j \in S_J}(n-m-j)$ is the number of all the hook row places except the corners which are in the condition set, and $\sum_{b_r \in S_I}(r-1)$ is the the number of all the hook column places except the corners which are in the condition set. Then by Remark 3.4.2,

$$\begin{aligned} d &= \sum_{1 \leqslant i \leqslant s} (b_{u_i} - v_i) - s - \sum_{i=1}^{s} |Z_i| \\ &= \sum_{1 \leqslant i \leqslant s} \big((b_{u_i} - v_i) - s - |Z_i|\big). \end{aligned}$$

Set

$$k = \sum_{1 \leqslant i \leqslant s} \big((b_{u_i} - v_i) - |Z_i|\big),$$

then obviously it is the number of places which are on the hooks whose corners belong to the condition set. Now we obtain $\dim_{M_{\mathcal{O}}} = q^{k-s}$.

□

3.4.16 Remark. By Proposition 3.4.14, if two orbits have the same frame of condition sets then the two orbits have the same dimension. Even in the case that the tableau of the two orbits have different shape, the statement still holds. Therefore, for a given frame of a condition set, the number of all the admissible orbits is a polynomial in q with integral coefficients and the sizes of the orbits are powers of q. Moreover, if the elements on the hooks with corners belonging to the condition set are fixed, then we have no choice of the other places, otherwise the dimension of the orbit will be increased.

3.4.17 Example. Let L be defined as Example 3.4.12. By Proposition 3.3.3, for $e_K \in \mathcal{O}_L$, we have

$$K = \begin{pmatrix} g_{63} & -g_{21}g_{63} & 1 & & & \\ g_{65} & -g_{21}g_{65} & 0 & 1+(-g_{41}+g_{21}g_{42})g_{65} & 1 & \\ 1 & -g_{21} & 0 & -g_{41}+g_{21}g_{42} & 0 & 1 \end{pmatrix}$$

where $g = (g_{ij}) \in U^w \cap U$. Then

$$\mathcal{O}_L = \left\{ e_K \,\middle|\, K = \begin{pmatrix} a & 0+\square & 1 & & & \\ b & 0+\square & 0 & 1+\square & 1 & \\ 1 & c & 0 & d & 0 & 1 \end{pmatrix}, \forall\, a,b,c,d \in GF(q) \right\}$$

where $\square$ are depended on a,b,c,d. It means that when a,b,c,d are fixed, then $\square$ is fixed. And we have

$$\dim M_{\mathcal{O}_L} = q^{(6-1-0)+(5-4-0)-2} = q^4.$$

In the next example, we change the entry of the position with index $(5,2)$.

3.4.18 Example. Let $L = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & & & \\ 0 & 1 & 0 & 1 & 1 & \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 3 \\ 5 \\ 6 \end{array}$. By Proposition 3.3.3, for $e_K \in \mathcal{O}_L$, we have

$$K = \begin{pmatrix} g_{63} & -g_{21}g_{63}+g_{53} & 1 & & & \\ g_{65} & 1-g_{21}g_{65} & 0 & 1+(-g_{41}+g_{21}g_{42})g_{65}-g_{42} & 1 & \\ 1 & -g_{21} & 0 & -g_{41}+g_{21}g_{42} & 0 & 1 \end{pmatrix},$$

where $g = (g_{ij}) \in U^w \cap U$. Obviously, when g runs over $U^w \cap U$, g_{53} and g_{42} will run over $GF(q)$, hence even the $(6,1)-$hook is fixed, k_{32}, k_{54} will still run over $GF(q)$, which means

$$\mathcal{O}_L = \left\{ e_K \,\middle|\, K = \begin{pmatrix} a & e & 1 & & & \\ b & 1+\square & 0 & f & 1 & \\ 1 & c & 0 & d & 0 & 1 \end{pmatrix}, \forall\, a,b,c,d,e,f \in GF(q) \right\},$$

where $\square$ depends on b, c. And we have

$$\dim M_{\mathcal{O}_L} = q^{(6-1-0)+(5-2-0)-2} = q^6.$$

3.5 The irreducibility of $M_{\mathcal{O}}$

In previous section, we have defined for each $U^w \cap U-$orbit $\mathcal{O}$, an $F(U^w \cap U)$ submodule $M_{\mathcal{O}}$. The goal of this section is to prove $M_{\mathcal{O}}$ is an irreducible $F(U^w \cap U)-$module.

3.5.1 Theorem. Let $\lambda = (n-m, m)$, $\mathcal{O}$ be a $U^w \cap U-$orbit with $\mathrm{tab}(\mathcal{O}) = \mathfrak{t}^\lambda w$. Then $M_{\mathcal{O}}$ is an irreducible $F(U^w \cap U)$ submodule of $\mathfrak{M}_{\mathfrak{t}}$.

Proof. Suppose the condition set of the orbit $\mathcal{O}$ is S. And the frame of the condition set is $\overline{S}$. Define

$$\Gamma = \bigcup_{(b,j)\in\overline{S}} \bar{\Omega}^{\mathfrak{t}}_{(b,j)}.$$

Let

$$x = \sum_{e_K \in \mathcal{O}} a_K e_K \in M_{\mathcal{O}} = e_L F(U^w \cap U).$$

We can reduce our problem to the simple case: L is a pattern matrix, which means

$$l_{bv} = 0, \forall (b, v) \in \Gamma. \tag{3.5.2}$$

Choose K such that $a_K \neq 0$. Multiplying by $u \in U^w \cap U$ such that $e_K \circ u = e_L$ and dividing by a_L we see that we may assume $a_L = 1$ that is

$$x = e_L + \sum_{K \neq L} a_K e_K.$$

Let

$$a_{ij} = \sum_{\alpha \in GF(q)} x_{ij}(\alpha) \text{ be a linear combination in } FU_0^w,$$

where $x_{ij}(\alpha) = E + \alpha e_{ij}$. By Corollary 3.3.7 we have:

$$e_L \circ x_{ij}(\alpha) = \theta\left(l_{ij}\alpha\right) e_L \text{ for } i \in \underline{\mathfrak{t}},\ j \notin \underline{\mathfrak{t}};$$

$$e_K \circ x_{ij}(\alpha) = \theta\left(k_{ij}\alpha\right) e_K \text{ for } i \in \underline{\mathfrak{t}},\ j \notin \underline{\mathfrak{t}}.$$

Note that for $(b,v) \in \Gamma$ we have by Definition 3.4.1 automatically $b \in \underline{\mathfrak{t}}$, $v \notin \underline{\mathfrak{t}}$. Set

$$a = \prod_{(b,v)\in\Gamma} a_{bv}.$$

Then

$$e_L \circ a = \prod_{(b,v)\in\Gamma} \sum_{\alpha\in GF(q)} \theta\left(l_{bv}\alpha\right) e_L \overset{(3.5.2)}{=} q^{|\Gamma|} e_L.$$

Since for any $K \neq L$, there exist at least one place $(b,v) \in \Gamma$ such that $k_{bv} \neq 0$. Therefore by (3.3.6),

$$e_K \circ a = \prod_{(b,v)\in\Gamma} \sum_{\alpha\in GF(q)} \theta\left(k_{bv}\alpha\right) e_K = 0.$$

Now we obtain

$$x \circ a = \big(e_L + \sum_{K\neq L} a_K e_K\big) g = q^{|\Gamma|} e_L.$$

This shows

$$xF(U^w \cap U) = e_L F(U^w \cap U) = M_{\mathcal{O}}.$$

Therefore $M_{\mathcal{O}}$ is an irreducible $F(U^w \cap U)$ submodule of $\mathfrak{M}_{\mathfrak{t}}$. □

3.5.3 Example. Let $L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ l_{41} & 0 & 0 & 1 \end{pmatrix}$ where $l_{41} \neq 0$. Let

$$K_i = \begin{pmatrix} u_i & 1 & 0 & 0 \\ l_{41} & 0 & v_i & 1 \end{pmatrix}, \ 1 \leqslant i \leqslant q^2 - 1,$$

where $(u_i, v_i) \in \big(GF(q), GF(q)\big) \setminus \{(0,0)\}$, $\forall\, 1 \leqslant i \leqslant q^2 - 1$. Let

$$g = \sum_{g_{21},\, g_{43} \in GF(q)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ g_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & g_{43} & 0 \end{pmatrix}$$

Then

$$e_{K_i} g = \sum_{g_{21},\, g_{43} \in GF(q)} \theta\left(u_i g_{21}\right) \theta\left(v_i g_{43}\right) e_{K_i}$$

Because $(u_i, v_i) \in \big(GF(q), GF(q)\big) \setminus \{(0,0)\}$, so when g_{21}, g_{43} run over $GF(q)$, we get

$$e_{K_i} g = 0, \ \forall\, 1 \leqslant i \leqslant q^2 - 1.$$

Moreover, we have $e_L g = q^2 e_L$. Hence for any $a_i \in F$, we have

$$e_L + \sum_i a_i e_{K_i} = q^2 e_L.$$

3.6 $U-$invariance of $M_{\mathcal{O}}$

In this section, we fix a $U^w \cap U-$orbit $\mathcal{O}$ of $\mathcal{E}_{\mathfrak{t}}$ with $\mathfrak{t} = \mathfrak{t}^\lambda w \in \mathrm{RStd}(\lambda)$, $\lambda = (n-m, m)$. Let $\underline{\mathfrak{t}} = (b_1, \cdots, b_m)$ and let L be the unique pattern matrix associated with $\mathcal{O}$. Now we show that the corresponding module $M_{\mathcal{O}}$ is invariant under the action of U not just of $U^w \cap U$. Since the result in the last section shows that $M_{\mathcal{O}}$ is an irreducible $F(U^w \cap U)-$module (c.f. 3.5.1), $M_{\mathcal{O}}$ is of course irreducible as $FU-$module as well. Decomposing M^λ into batches and those into orbit modules we have then obtained a complete decomposition of M^λ into irreducible $FU-$modules.

Obviously, to prove that $M_{\mathcal{O}}$ is invariant under the action of U it suffices to show that $e_K \circ g \in M_{\mathcal{O}}$ for all $e_K \in \mathcal{O}$, where g is an arbitrary generator of U. Of course we may choose $g = x_{ij}(\alpha)$ where $\alpha \in GF(q), g \notin U^w \cap U$. We first show that it suffices to prove $e_L \circ g \in M_{\mathcal{O}}$.

3.6.1 Lemma. Suppose $e_L \circ g \in M_{\mathcal{O}}$ for all $g \in U$. Then $M_{\mathcal{O}}$ is invariant under the action of U.

Proof. By Theorem 1.4.4, we can define a normal sequence:

$$U^w \cap U = U_0 \leqslant U_1 \leqslant \cdots \leqslant U_i \leqslant U_{i+1} \leqslant \cdots \leqslant U_k = U$$

such that $U_i \trianglelefteq U_{i+1}$, which means for each $0 \leqslant i \leqslant k-1$, U_i is a normal subgroup of U_{i+1}.

Now we do the induction on k. The fact that $M_{\mathcal{O}}$ is $U^w \cap U$ invariant is our induction basis. Inductively, suppose that $M_{\mathcal{O}}$ is U_i invariant. Let $g_1, \ldots, g_r$ be generators of U_{i+1}. Suppose

$$e_L \circ g_j \in M_{\mathcal{O}} \text{ for } j = 1, \ldots, r.$$

Choose an arbitrary $e_K \in \mathcal{O}$. Then there exist some $u \in U^w \cap U \leqslant U_i$ such that

$$e_L \circ u = e_K.$$

Hence

$$\begin{aligned} e_K \circ g_j &= e_L \circ ug_j = e_L \circ g_j g_j^{-1} ug_j \\ &= (e_L \circ g_j) \circ (g_j^{-1} ug_j). \end{aligned}$$

Since U_i is a normal subgroup in U_{i+1}, then $g_j^{-1} ug_j \in U_i$. Moreover, we have $e_L \circ g_j \in M_{\mathcal{O}}$ and $M_{\mathcal{O}}$ is U_i invariant by assumption. Hence for all $e_K \in \mathcal{O}$, and $j = 1, \ldots, n$ we have

$$e_K \circ g_j \in M_{\mathcal{O}}$$

which shows that $M_{\mathcal{O}}$ is U_{i+1} invariant. $\square$

3.6.2 Example. Let $L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ l_{51} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & l_{63} & 0 & 0 & 1 \end{pmatrix} \in \Xi_{3,6}$ be a pattern matrix where $l_{51}, l_{63} \neq 0$, and denote

$$\mathfrak{t} := \mathrm{tab}(L) = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} = \mathfrak{t}^{\lambda} w, \text{ with } \lambda = (3,3).$$

By Theorem 1.4.4 and Lemma 3.3.1, we have the the following normal sequence:

$$U_6^w \cap U_6 \trianglelefteq (U_6^w \cap U_6)X_{42} \trianglelefteq (U_6^w \cap U_6)X_{42}X_{32} = U_6.$$

In order to show that $M_{\mathcal{O}}$ is $U-$invariant it suffices to prove $e_L \circ g \in M_{\mathcal{O}}$ for all generators g of U. By Lemma 3.3.1, it suffices to prove

$$e_L \circ x_{ij}(\alpha) \in M_{\mathcal{O}}$$

for $i \notin \underline{\mathfrak{t}}, j \in \underline{\mathfrak{t}}, i > j$ and $\alpha \in GF(q)$. So choose $v \notin \underline{\mathfrak{t}}, b_{\mathfrak{t}} \in \underline{\mathfrak{t}}$ with $b_{\mathfrak{t}} < v$ and let

$$g = x_{vb_t}(\alpha) \in U.$$

There exist $1 \leqslant s \leqslant m$ such that $b_{s-1} < v < b_s$ then $1 \leqslant t < s \leqslant m$. By (3.3.4) we have

$$e_L \circ g = \sum_K \Big(\frac{1}{q^{|\mathfrak{Z}\mathfrak{t}|}} \sum_M \chi_L(-M)\chi_K(M \circ g)\Big) e_K. \tag{3.6.3}$$

To compute $e_L \circ g$ we need to calculate $M \circ g$ first. Obviously, Mg is obtained from M by adding column v times α to column b_t of M, which is a column with a last 1 at position (b_t, b_t). More precisely:

$$Mg = \begin{array}{c} \\ \\ b_t \\ \\ b_{s-1} \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} b_t & b_{s-1} & v & b_s & \end{array} \\ \begin{pmatrix} \boxed{\phantom{m_{b_i j}}} & \begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \alpha m_{b_s v} \\ \vdots \\ \alpha m_{b_m v} \end{array} & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ m_{b_s v} \\ \vdots \\ m_{b_m v} \end{array} & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ \cdots \\ 1 \\ \vdots \\ 0 \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{array} \end{pmatrix} \end{array}$$

Picture of Mg.

The row reduction of Mg is obtained by changing the entries on the b_t-column and under the position (b_{s-1}, b_t) to zeros. During this procedure, we also changed the entries to the southwest of the position (b_{s-1}, b_t) but the other entries are the same as M. This is the reason we use the box presentation for those positions. Now we obtain

$$M \circ g = \begin{array}{c} \\ \\ b_t \\ \\ b_{s-1} \\ \\ b_i \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} j & b_t & b_{s-1} & v & b_s & \end{array} \\ \begin{pmatrix} & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots m_{b_t j} \cdots & 1 & \vdots & \vdots & \vdots & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \cdots \\ & 0 & 1 & 0 & 0 & \cdots \\ & 0 & 0 & \vdots & 1 & \cdots \\ \boxed{m_{b_i j} - \alpha m_{b_i v} m_{b_t j}} & \vdots & \vdots & m_{b_i v} & \vdots & \cdots \\ & 0 & 0 & \vdots & 0 & \cdots \end{pmatrix} \end{array}$$

Picture of $M \circ g$.

Note that the entries outside the box all coincide in M and $M \circ g$.

We first calculate $e_R \circ g$ for arbitrary $R \in \mathfrak{X}_{\mathfrak{t}}$ not just for the pattern matrices.

3.6.4 Lemma. Let $R \in \mathfrak{X}_{\mathfrak{t}}, g = x_{vb_t}(\alpha)$ with $\alpha \in GF(q), v \notin \underline{\mathfrak{t}}, b_{\mathfrak{t}} \in \underline{\mathfrak{t}}$ and $b_{\mathfrak{t}} \leqslant b_{s-1} < v < b_s$. Write, using (3.6.3),

$$e_R \circ g = \sum_K C_K e_K$$

where

$$C_K = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_M \chi_R(-M)\chi_K(M \circ g).$$

Then $C_K \neq 0$ implies that for all $i \neq t, j \neq v, (b_i, j) \in \mathfrak{J}_{\mathfrak{t}}$, we have

$$k_{b_i j} = r_{b_i j}.$$

Thus if $C_K \neq 0$ then K and R coincide in all positions except possibly those in row b_t and column v.

Proof. Let $N = (n_{b_i j}) = M \circ g$. From **Picture of** $M \circ g$ above, we obtain:

$$n_{b_i j} = \begin{cases} m_{b_i j} & \text{for } 1 \leqslant i \leqslant s-1; \\ m_{b_i j} & \text{for } b_t < j < b_i; \\ m_{b_i j} - \alpha m_{b_i v} m_{b_t j} & \text{for } s \leqslant i \leqslant m, 1 \leqslant j < b_t. \end{cases}$$

So the coefficient C_K of $e_{_K}$ in the expansion $e_{_R} \circ g$ is

$$\begin{aligned} C_K &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M} \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}} \theta\left(-r_{_{b_ij}} m_{_{b_ij}}\right) \theta\left(k_{_{b_ij}} n_{_{b_ij}}\right) \\ &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M} \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}} \theta\left(k_{_{b_ij}} n_{_{b_ij}} - r_{_{b_ij}} m_{_{b_ij}}\right) \end{aligned}$$

Now we calculate the factor $\theta\left(-r_{_{b_ij}} m_{_{b_ij}} + k_{_{b_ij}} n_{_{b_ij}}\right)$ obtaining three types:

$$\theta\left(k_{_{b_ij}} n_{_{b_ij}} - r_{_{b_ij}} m_{_{b_ij}}\right) \tag{3.6.5}$$

$$= \begin{cases} \theta\left((k_{_{b_ij}} - r_{_{b_ij}}) m_{_{b_ij}}\right) & \text{for } 1 \leqslant i \leqslant s-1; \\ \theta\left((k_{_{b_ij}} - r_{_{b_ij}}) m_{_{b_ij}}\right) & \text{for } b_t < j < b_i; \\ \theta\left(k_{_{b_ij}} (m_{_{b_ij}} - \alpha m_{_{b_iv}} m_{_{b_tj}}) - r_{_{b_ij}} m_{_{b_ij}}\right) & \text{for } s \leqslant i \leqslant m, 1 \leqslant j < b_t. \end{cases}$$

In the third type, we have some products $m_{_{b_iv}} m_{_{b_tj}}$ where $m_{_{b_tj}}$ appears in the first type and $m_{_{b_iv}}$ appears in the second type. Hence row b_t and column v are our critical cases. Moreover the terms containing $m_{_{bl}}$ with $b \neq b_t, l \neq v$ can always be separated from those critical cases. Therefore if we fix the entries in the critical cases, that is we fix $m_{_{b_iv}}, m_{_{b_tj}}$ for $s \leqslant i \leqslant m$, $1 \leqslant j < b_t$, then the term

$$\prod_{\substack{i\neq t \\ j\neq v \\ (b_i,j)\in\mathfrak{J}_{\mathfrak{t}}}} \sum_{m_{_{b_ij}}\in GF(q)} \theta\left((k_{_{b_ij}} - r_{_{b_ij}}) m_{_{b_ij}}\right)$$

is a nonzero multiple factor of a summand of C_K. Hence in order to get $C_K \neq 0$, the term $k_{_{b_ij}} - r_{_{b_ij}}$ must be zero, which leads to

$$k_{_{b_ij}} = r_{_{b_ij}} \text{ for } i \neq t, j \neq v, (b_i, j) \in \mathfrak{J}_{\mathfrak{t}}.$$

□

Now we turn to our reduced case where $R = L$ is a pattern matrix. We want to prove $e_{_L} \circ g \in M_{\mathcal{O}}$, where g is the same as in the lemma above.

3.6.6 Lemma. Let L be a pattern matrix with condition set S, and let $g = x_{_{vb_t}}(\alpha)$ with $\alpha \in GF(q), v \notin \underline{\mathfrak{t}}, b_{\mathfrak{t}} \in \underline{\mathfrak{t}}$ and $b_{\mathfrak{t}} \leqslant b_{s-1} < v < b_s$. If

$$(b, j) \in \overline{S} \text{ implies } b < s \text{ or } j > b_t,$$

then

$$e_{_L} \circ g = e_{_L}.$$

In other words, if the southwest box in **Picture of** $M \circ g$ contains no positions belonging to the frame of the condition set S, then g acts trivially on $e_{_L}$.

Proof. By (3.6.3) we have $e_{_L} \circ g = \sum_K C_K e_{_K}$ where

$$C_K = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_M \chi_{_L}(-M)\chi_{_K}(M \circ g).$$

By Lemma 3.6.4 for $s \leqslant i \leqslant m, 1 \leqslant j < b_t$, if $C_K \neq 0$, then $k_{_{b_i j}} = l_{_{b_i j}}$. These (b_i, j) are all contained in the southwest box of $M \circ g$. But we have the assumption that the southwest box of $M \circ g$ contains no positions belonging to the frame of the condition set S, therefore, if $C_K \neq 0$, then

$$k_{_{b_i j}} = l_{_{b_i j}} = 0 \text{ for } s \leqslant i \leqslant m, 1 \leqslant j < b_t,$$

that is for all the positions in the third type of (3.6.5). Obviously when $k_{_{b_i j}} = 0$ in the third type, then the critical products $m_{_{b_i v}} m_{_{b_t j}}$ will disappear in the coefficient C_K. That means we can always separate m_{bu} from each other for all $(b, u) \in \mathfrak{J}_{\mathfrak{t}}$. Moreover the summation

$$\sum_{m_{_{bu}} \in GF(q)} \theta\big((k_{_{bu}} - l_{_{bu}})m_{_{bu}}\big) \text{ for all } (b, u) \in \mathfrak{J}_{\mathfrak{t}}$$

is a factor of C_K. Hence in order to get $C_K \neq 0$, the term $k_{_{bu}} - l_{_{bu}}$ must be zero, which leads to

$$k_{_{bu}} = l_{_{bu}} \text{ for all } (b, u) \in \mathfrak{J}_{\mathfrak{t}}.$$

Therefore in this case $K = L$, and $e_{_L} \circ g = e_{_L}$.

□

3.6.7 Lemma. Let L be a pattern matrix with condition set S. Let $g = x_{_{vb_{\mathfrak{t}}}}(\alpha)$ with $\alpha \in GF(q), v \notin \underline{\mathfrak{t}}, b_{\mathfrak{t}} \in \underline{\mathfrak{t}}$ and $b_{\mathfrak{t}} \leqslant b_{s-1} < v < b_s$. If there exist $(b_i, j) \in \overline{S}$ in the southwest box of $M \circ g$, that is $s \leqslant i \leqslant m, 1 \leqslant j < b_t$, then $e_{_L} \circ g \in M_{\mathcal{O}}$.

Proof. By (3.6.3) we have $e_{_L} \circ g = \sum_K C_K e_{_K}$ where

$$C_K = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_M \chi_{_L}(-M)\chi_{_K}(M \circ g).$$

By Lemma 3.6.4, if $C_K \neq 0$ then we have only possible different entries from L on row b_t or on column v, more precisely:

$$k_{_{b_i j}} = l_{_{b_i j}} \text{ for all } i \neq t, j \neq v, (b_i, j) \in \mathfrak{J}_{\mathfrak{t}}.$$

If there exist some $(b_i, j) \in \overline{S}$ in the southwest box of $M \circ g$ then

$$k_{_{b_i j}} = l_{_{b_i j}} \neq 0.$$

Obviously, these (b_i, j) are in the third type of (3.6.5). And when $k_{b_i j} \neq 0$ in the third type, then the critical products $m_{b_i v} m_{b_t j}$ may disappear in the coefficient C_K, which means we cannot separate the terms involving $m_{b_i v}$ and the terms involving $m_{b_t j}$ from each other. Hence in this case we can get possible nonzero entries on (b_i, v) and (b_t, j). Though we do not calculate these, we notice that (b_i, v) and (b_t, j) are both on the $(b_i, j)-$hook with nonzero entry $l_{b_i j}$ in the hook corner.

Suppose that $C_K \neq 0$ and K has nonzero entries on positions (b_i, v) and (b_t, j) for some $s \leqslant i \leqslant m, 1 \leqslant j < b_t$. Then by Corollary 3.3.7, using truncated column operations and truncated column operations, we can change all those nonzero entries on positions (b_i, v) and (b_t, j) to zeros. Hence if $C_K \neq 0$, then there exist $u \in U^w \cap U$ such that $e_L \circ u = e_K$. Therefore we obtain $e_L \circ g \in M_{\mathcal{O}}$. □

3.6.8 Theorem. Let $\lambda = (n-m, m) \vdash n, \mathfrak{t} = \mathfrak{t}^\lambda w$ where $w \in \mathfrak{S}_n$. Let $\mathcal{O}$ be a $U^w \cap U$orbit of $\mathcal{E}_\mathfrak{t}$. Then $M_{\mathcal{O}}$ is an irreducible FU_n-module.

Proof. It is just a consequent result from Lemma 3.6.1, Lemma 3.6.6 and Lemma 3.6.7.

□

3.6.9 Example. Let $L = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & l_{52} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 5 \\ 6 \end{array}$ with $S = \{l_{52} \neq 0\}$.

Then

$$e_L = \frac{1}{q^8} \sum_{m_{ij} \in GF(q)} \theta(-l_{52} m_{52}) \begin{pmatrix} m_{31} & m_{32} & 1 & 0 & 0 & 0 \\ m_{51} & m_{52} & 0 & m_{54} & 1 & 0 \\ m_{61} & m_{62} & 0 & m_{64} & 0 & 1 \end{pmatrix}.$$

Let $g = x_{43}(\alpha) \in U_6$. Then

$$e_L g = \frac{1}{q^8} \sum_{m_{ij} \in GF(q)} \theta(-l_{52} m_{52}) \begin{pmatrix} m_{31} & m_{32} & 1 & 0 & 0 & 0 \\ m_{51} & m_{52} & \alpha m_{54} & m_{54} & 1 & 0 \\ m_{61} & m_{62} & \alpha m_{64} & m_{64} & 0 & 1 \end{pmatrix}.$$

And we have

$$e_L \circ g = \frac{1}{q^8} \sum_{m_{ij} \in GF(q)} \theta(-l_{52}m_{52}) \begin{pmatrix} m_{31} & m_{32} & 1 & 0 & 0 & 0 \\ m_{51} - \alpha m_{54} m_{31} & m_{52} - \alpha m_{54} m_{32} & 0 & m_{54} & 1 & 0 \\ m_{61} - \alpha m_{64} m_{31} & m_{62} - \alpha m_{64} m_{32} & 0 & m_{64} & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{q^8} \sum_{m_{ij} \in GF(q)} \theta(-l_{52}m_{52}) \begin{pmatrix} m_{31} & m_{32} & 1 & 0 & 0 & 0 \\ m_{51} & m_{52} - \alpha m_{54} m_{32} & 0 & m_{54} & 1 & 0 \\ m_{61} & m_{62} & 0 & m_{64} & 0 & 1 \end{pmatrix}.$$

Now $e_L \circ g = \sum_K C_K e_K$ where

$$\begin{aligned} C_K &= \frac{1}{q^8} \sum_M \chi_L(-M)\chi_K(M \circ g) \\ &= \frac{1}{q^8} \sum_M \theta\left(-l_{52}m_{52}\right) \theta\left(k_{31}m_{31}\right) \theta\left(k_{51}m_{51}\right) \theta\left(k_{61}m_{61}\right) \\ &\quad \cdot \theta\left(k_{52}m_{52}\right) \theta\left(k_{62}m_{62}\right) \theta\left(k_{64}m_{64}\right) \\ &\quad \cdot \theta\left(k_{32}m_{32}\right) \theta\left(k_{54}m_{54}\right) \theta\left(-\alpha k_{52}m_{54}m_{32}\right). \end{aligned}$$

Hence if $C_K \neq 0$, then we have

$$k_{52} = l_{52}, \quad k_{31} = k_{51} = k_{61} = k_{62} = 0.$$

Moreover if $C_K \neq 0$, then K may have some possible nonzero entries on position $(3,2)$ or $(5,4)$. But position $(3,2)$ and $(5,4)$ are both on the $(5,2)-$ hook with $l_{52} \in \overline{S}$, then by Corollary 3.3.7, using truncated column operations and truncated column operations, we can change all those nonzero entries on positions $(3,2)$ and $(5,4)$ to zeros. Therefore if $C_K \neq 0$ then $e_K \in M_{\mathcal{O}}$ where $\mathcal{O}$ is the orbit containing e_L.

3.6.10 Remark. Using Proposition 3.4.14, we can determine all the $1-$dimensional irreducible orbit modules in M^λ for $\lambda = (n-m, m)$:

Let $\mathcal{O}$ be an orbit in M^λ with condition set S. Let $\overline{S}$ be the frame of S and suppose $(b_i, j) \in \overline{S}$ implies $b_i - j = 1$. Then all the positions in $\overline{S}$ are next to the left of a last one and hence all the hooks with corners in $\overline{S}$ consist of their corners alone. In this case the corresponding orbit module $M_{\mathcal{O}}$ is a $1-$dimensional module by Proposition 3.4.14. In particular, this applies to the irreducible orbit modules associated with orbits with empty condition set. These orbit modules are obviously trivial FU_n-submodules of M^λ. By general theory every batch $\mathfrak{M}_{\mathfrak{t}}$ of M^λ contains precisely one trivial component and this is the orbit module with empty condition set. More precisely, the unique matrix L in $\mathfrak{X}_{\mathfrak{t}}$, whose only nonzero entries are

the last ones, induces the unique trivial component of the $\mathfrak{t}-$batch $\mathfrak{M}_{\mathfrak{t}}$. This is given as $M_{\mathfrak{t}} = Fe_{_L}$.

3.6.11 Main results of this chapter.
For convenience of the reader we list the main results of this chapter:

(1) We decomposed $\mathrm{Res}^{FG}_{FU} M^{\lambda}$ where $\lambda = (n-m, m)$ into a direct sum of batches, labeled by row standard $\lambda-$tableau.

(2) We introduced the idempotent basis $\mathcal{E}_{\mathfrak{t}}$ where $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ for each batch $\mathfrak{M}_{\mathfrak{t}}$ on which $U^w \cap U$ acts monomially.

(3) We decomposed each batch into a direct sum of orbit modules which are irreducible $FU-$modules (for any field F whose characteristic is different from p). Each orbit is associated with an unique pattern matrix.

(4) We determined the dimension of $M_{\mathcal{O}}$ for each orbit $\mathcal{O}$ on $\mathcal{E}_{\mathfrak{t}}$ where $\mathfrak{t} \in \mathrm{RStd}(\lambda)$. This is a power of q, the exponent is given by the hooks of the unique pattern matrix L such that $e_{_L} \in \mathcal{O}$.

(5) We proved that the number of irreducible direct summands of $\mathrm{Res}^{FG}_{FU} M^{(n-m,m)}$ to a fixed dimension q^c $(0 \leqslant c \in \mathbb{Z})$ is a polynomial in q.

Chapter 4

The Specht modules $S^{(n-m,m)}$

Having completely decomposed $M^\lambda, \big(\lambda = (n-m,m)\big)$ into a direct sum of irreducible $FU-$modules we now turn our attention to the unipotent Specht module S^λ which is by James's kernel intersection theorem 2.2.6 a submodule of M^λ. Unfortunately the decomposition of M^λ into batches does not carry easily to the Specht module. More precisely we use the fact that over a field of characteristic zero, S^λ is the kernel of a single epimorphism $\Phi_m : M^{(n-m,m)} \to M^{(n-m+1,m-1)}$. Now Φ_m does not preserve batches. However, as we will show, Φ_m preserves condition sets. This will enable us to decompose $\mathrm{Res}^{FG}_{FU} S^\lambda$ completely into irreducibles, and we show that the number of all irreducibles occurring as constituents of $\mathrm{Res}^{FG}_{FU} S^\lambda$ of a fixed dimension q^c where $0 \leqslant c \in \mathbb{Z}$, is a polynomial in q. Moreover we shall construct a basis of S^λ independent of F, which is a standard basis of S^λ in the sense of the conjecture of Dipper and James mentioned in the introduction. In fact, we shall show that the polynomials $r_{\mathfrak{s}}(t) \in \mathbb{Z}[t]$ attached to $\mathfrak{s} \in \mathrm{Std}(\lambda)$ are precisely the rank polynomials defined in [4].

4.1 The homomorphism Φ_m

In this section we explain how the homomorphism $\Phi_m : M^\lambda \to M^\mu$, where $\lambda = (n-m,m)$, $\mu = (n-m+1, m-1)$ acts on the basis of M^λ given by row reduced $m \times n-$matrices. This will enable us to show the main result of this section, namely that Φ_m preserves condition sets. In particular, we show if $\mathrm{char}\, F = 0$ then S^λ is the kernel of a single epimorphism Φ_m.

We already observed in Remark 3.4.6 (see also examples 3.4.3) that condition sets may "fit" into different batches, that is in these batches, there exist orbits with the given condition set. This means that for a fixed condition set S there might exist pattern matrices L_1, L_2 in different batches $\mathfrak{M}_{\mathfrak{t}_1}, \mathfrak{M}_{\mathfrak{t}_2}$ even for different two-part partitions λ and μ, that is $\mathfrak{t}_1 \in \mathrm{RStd}(\lambda), \mathfrak{t}_2 \in \mathrm{RStd}(\mu)$. Thus it makes sense to ask, if Φ_m preserves condition sets. We begin by describing the homomorphisms in Theorem

2.2.6 in the special case of two-part partitions where the description of M^λ is given as permutation module whose basis consists of flags (compare Definition 2.2.5):

4.1.1 Definition. Assume that $0 \leqslant i \leqslant m$. Define $\phi_{1,i}$ to be the linear map from $M^{(n-m,m)}$ into $M^{(n-i,i)}$ which sends each m-dimensional subspace V to the formal linear combination of the i-dimensional subspaces contained in it. More precisely, let $X \subseteq V = GF(q)^n$ with $\dim X = m$. Then

$$\phi_{1,i}([X]) = \sum_{\substack{Y \subseteq X \\ \dim Y = i}} [Y]$$

where $[X]$ denotes the flag $X \subseteq V$ in $\mathcal{F}(\lambda)$.

For a subset S of a vector space V we denote the span of S by $\langle S \rangle$. The following examples illustrate how this is translated into the setting of matrices in $\mathfrak{X}_{m,n}$:

4.1.2 Example. (1) Let a two dimensional subspace of $V = GF(q)^n$ be given by a basis v_1, v_2. Thus v_1, v_2 are linear independent. Note that the flag $X = \langle v_1, v_2 \rangle \subseteq V$ is a basis element of $M^{(n-2,2)}$. Then $\phi_{1,1}$ sends X into the formal sum of all $1-$dimensional subspaces of X. But each $1-$dimensional subspaces of X is of the form $\langle v_1 \rangle$ or $\langle \alpha v_1 + v_2 \rangle$ for $\alpha \in GF(q)$. This description is unique, hence

$$\phi_{1,1} : \quad \langle v_1, v_2 \rangle \quad \mapsto \quad \langle v_1 \rangle + \sum_{\alpha \in GF(q)} \langle \alpha v_1 + v_2 \rangle.$$

Note that the image is a formal linear combination with coefficient $1 = 1_F$ for all summands.

(2) Similarly, if $v_1, v_2, v_3 \in V$ are linear independent, $V = GF(q)^n$. Then

$$\phi_{1,i} : M^{(n-3,3)} \to M^{(n-i,i)}, \quad i = 1, 2, 3$$

is given as:

$$\begin{aligned}
\phi_{1,0} \quad &: \quad \langle v_1, v_2, v_3 \rangle \quad \mapsto \quad \langle 0 \rangle. \\
\phi_{1,1} \quad &: \quad \langle v_1, v_2, v_3 \rangle \quad \mapsto \quad \langle v_1 \rangle + \sum_{\alpha \in GF(q)} \langle \alpha v_1 + v_2 \rangle \\
&\qquad\qquad\qquad\qquad + \sum_{\alpha, \beta \in GF(q)} \langle \alpha v_1 + \beta v_2 + v_3 \rangle \\
\phi_{1,2} \quad &: \quad \langle v_1, v_2, v_3 \rangle \quad \mapsto \quad \sum_{\alpha, \beta \in GF(q)} \langle \alpha v_1 + v_2, \beta v_1 + v_3 \rangle \\
&\qquad\qquad\qquad\qquad + \sum_{\gamma \in GF(q)} \langle v_1, \gamma v_2 + v_3 \rangle + \langle v_1, v_2 \rangle
\end{aligned}$$

And $\phi_{1,3}$ is obviously the identity map on $M^{(n-3,3)}$.

Here are some remarks on the second example in 4.1.2:

4.1.3 Remark.

(1) $\phi_{1,2}$ is obtained in the following way:

We have to describe all $2-$dimensional subspaces of the space spanned by v_1, v_2, v_3. By Definition 3.1.1 we can describe those by row reduced $2 \times 3-$matrices which are given as

$$\left\{ \begin{pmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in GF(q) \right\}$$

whereas the entries in the rows of these matrices are interpreted as coefficients of v_1, v_2 and v_3. In this example, the term $\langle \alpha v_1 + v_2, \beta v_1 + v_3 \rangle$ is given by the first matrix $\begin{pmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}$; the term $\langle v_1, \gamma v_2 + v_3 \rangle$ is given by the second matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 1 \end{pmatrix}$; the term $\langle v_1, v_2 \rangle$ is given by the third matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(2) Note that M^λ with λ equals the degenerate two-part partition $(n-0, 0)$ of n is the trivial $GL_n(q)-$ module. Its basis, written as flags, is the trivial flag $(0) \subseteq V$ and hence given as the null space $\langle 0 \rangle$ in V. Hence $\phi_{1,0}$ is an epimorphism and $M^{(n-0,0)} = F(0)$.

We now concentrate on one of these homomorphisms, namely

$$\phi_{1,m-1} : M^\lambda \to M^\mu$$

where $\lambda = (n-m, m), \mu = (n-m+1, m-1)$. Henceforth we denote this homomorphism by Φ_m. Thus, for $X \subseteq V$, $\dim_{GF(q)} X = m$:

$$\Phi_m([X]) = \sum_{\substack{Y \subseteq X \\ \dim Y = m-1}} [Y]$$

where $[X]$ denotes the flag $X \subseteq V$ in $\mathcal{F}(\lambda)$.

In section 3.1 we have seen that each subspace X of V of dimension m may be given uniquely by a row reduced $m \times n$ matrix $M \in \Xi_{m,n}$, and that under this identification the action of $G = GL_n(q)$ on $X \in \mathcal{F}(\lambda)$ turns into the action $\circ$ on M (see Definition 3.1.1), which means for $g \in G$ we row reduce the ordinary matrix product Mg to obtain $M \circ g \in \Xi_{m,n}$.

We next describe, how Φ_m acts on matrices $M \in \Xi_{m,n}$. So let $r_{b_1}, r_{b_2}, \ldots, r_{b_m} \in V$ be the row vectors of M. By Definition 3.1.4 the set

$\{r_{b_i} \mid 1 \leqslant i \leqslant m\}$ is (the unique row reduced) $GF(q)-$basis of the subspace $X = X(M)$ of V of dimension m corresponding to $M \in \Xi_{m,n}$.

Let $Y \subseteq X$ be of dimension $m-1$. Let $1 \leqslant i \leqslant m$ be maximal such that $r_{b_1}, \ldots, r_{b_{i-1}} \in Y$. Let

$$\{r_{b_1}, \ldots, r_{b_{i-1}}, y_{i+1}, \ldots, y_m\} \subseteq X$$

be a basis of Y. Then, for $i+1 \leqslant t \leqslant m$ we find coefficients $\lambda_{ts} \in GF(q)$, $s = 1, \ldots, m$ such that

$$y_t = \sum_{s=1}^{m} \lambda_{ts} r_{b_s}.$$

Set

$$y'_t = y_t - \sum_{s=1}^{i-1} \lambda_{ts} r_{b_s} = \sum_{s=i}^{m} \lambda_{ts} r_{b_s} \in X$$

and obviously

$$\{r_{b_1}, \ldots, r_{b_{i-1}}, y'_{i+1}, \ldots, y'_m\}$$

generates Y as well, hence is a basis of Y. Thus we may assume that

$$y_{i+1}, \ldots, y_m \in \langle r_{b_i}, \ldots, r_{b_m} \rangle \subseteq X,$$

$$y_t = \sum_{s=i}^{m} \lambda_{ts} r_{b_s}. \tag{4.1.4}$$

Consider the $(m-i-1) \times (m-i)-$matrix $\Lambda = (\lambda_{ts})$, where $i+1 \leqslant t \leqslant m$ and $i \leqslant s \leqslant m$. Since $y_{i+1}, \ldots, y_m$ are linearly independent, we may assume that Λ is row reduced. Since $m-i = (m-i-1)+1$, Λ contains exactly one column which does not contain a last 1. Suppose the first column of Λ contains a last 1. This is then the first entry of that column by construction, all other entries in that column are zero, and all entries of the first row, beside the first entry, are zeros as well. Inserting this in equation 4.1.4 we obtain

$$y_{i+1} = 1\, r_{b_i} + 0\, r_{b_{i+1}} + \cdots + 0\, r_{b_m} \in Y,$$

a contradiction. Hence the first column of Λ is the unique one, which does not contain a last 1. Therefore Λ has the form

$$\begin{pmatrix} \alpha_{i+1} & 1 & & 0 \\ \vdots & & \ddots & \\ \alpha_m & 0 & & 1 \end{pmatrix}$$

with $\alpha_t \in GF(q)$ for $t = i+1, \ldots, m$. Consequently

$$y_t = r_{b_t} + \alpha_t r_{b_i} \text{ for } t = i+1, \ldots, m.$$

Thus we obtain the row reduced $(m-1)\times n$ matrix $N \in \Xi_{n,m-1}$ corresponding to $Y \subseteq V$, by adding in the matrix M corresponding to X row b_i multiplied by α_t to row b_t for $t = i+1,\dots,m$ and omit row b_i from the resulting matrix. obviously the resulting matrix N is row reduced, and $\mathrm{tab}(N)$ is obtained from the tableau $\mathrm{tab}(M) = \mathfrak{t}$ by moving entry b_i from the second row $\underline{\mathfrak{t}}$ of $\mathfrak{t}$ into the first row at the appropriate place to make the resulting tableau row standard.

Conversely, given any $(m-1)\times n-$matrix N which is obtained from the row reduced matrix M corresponding to the $\lambda-$flag $X \subseteq V$ by deleting row b_i from M where $1 \leqslant i \leqslant m$ and adding multiples of row b_i to b_t for $t = i+1,\dots,m$. The rows of N span an $(m-1)-$dimensional subspace of V which is contained in X of dimension $m-1$. One checks easily that N is then reduced and $\mathrm{tab}(N)$ is obtained from $\mathrm{tab}(M)$ by moving $b_i \in \underline{\mathrm{tab}(M)}$ from the second row into the first row at the appropriate place to make the resulting tableau row standard.

Thus an immediate consequence, we obtain from the definition of Φ_m the following proposition:

4.1.5 Proposition. Let $\lambda = (n-m, m), \mu = (n-m+1, m-1)$, and let $M \in \Xi_{m,n}$, $\mathrm{tab}(M) = \mathfrak{t}$. Then $\Phi_m([M])$ is the formal linear combination $\sum[N] \in M^\mu$, where N runs through the set of all matrices obtained from M as follows: Choose a row b_i for $1 \leqslant i \leqslant m$ of M and add multiples of it to rows $b_{i+1},\dots,b_m$ below row b_i with coefficients in $GF(q)$. And then delete row b_i from M. Moreover all matrices N occurring in this process with a fixed $1 \leqslant i \leqslant m$ have associated $\mu-$tableau $\mathrm{tab}(N)$ obtained from $\mathrm{tab}(M)$ by moving $b_i \in \underline{\mathfrak{t}}$ from the second row into the first row at the appropriate place to make the resulting $\mu-$tableau row standard.

4.1.6 Example. (1) Let $\lambda = (2,2)$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m & n & 1 \end{pmatrix} \in \Xi_{2,4}$. Then

$$\Phi_2 : \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m & n & 1 \end{pmatrix}\right] \longmapsto \left[\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\right] + \sum_{a\in GF(q)} \left[\begin{pmatrix} a & m & n & 1 \end{pmatrix}\right].$$

(2) Let $\lambda = (3,3)$, $\begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \\ p & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \Xi_{3,6}$. Then

$$\Phi_3 : \left[\begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \\ p & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\right] \longmapsto \left[\begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \end{pmatrix}\right] +$$

$$\sum_{a,b\in GF(q)} \left[\begin{pmatrix} n+am & a & 1 & 0 & 0 & 0 \\ p+bm & b & 0 & 1 & 0 & 0 \end{pmatrix}\right] + \sum_{a\in GF(q)} \left[\begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ p+an & 0 & a & 1 & 0 & 0 \end{pmatrix}\right].$$

Now we fix $\lambda = (n-m, m)$, $\mu = (n-m+1, m-1)$, $M \in \Xi_{m,n}$, $\mathrm{tab}(M) = \mathfrak{t}$, $b_d \in \underline{\mathfrak{t}}$ and then introduce some notation for convenience:

Define $R_d(M)$ to be the set of all $(m-1) \times n-$matrices obtained from deleting row b_d from M and adding multiplies of row b_d to rows b_t for $t = d+1, \ldots, m$. Then $\mathrm{tab}(N) \in \mathrm{RStd}(\mu)$ for $N \in R_d(M)$ is obtained by moving b_d to the first row of $\mathfrak{t}$ at the appropriate place to make the resulting $\mu-$tableau row standard.

Thus each matrix $N \in R_d(M)$ is contained in $\mathfrak{u}_d-$batch of M^μ where

$$\mathfrak{u}_d = \mathrm{tab}(N).$$

Note in particular that for $1 \leqslant d_1, d_2 \leqslant m$ with $d_1 \neq d_2$ and $N_1 \in R_{d_1}(M)$, $N_2 \in R_{d_2}(M)$, we have $\mathrm{tab}(N_1) \neq \mathrm{tab}(N_2)$. Thus

$$\Phi_m([M]) = \sum_{d=1}^{m} \sum_{N \in R_d(M)} [N].$$

If we set

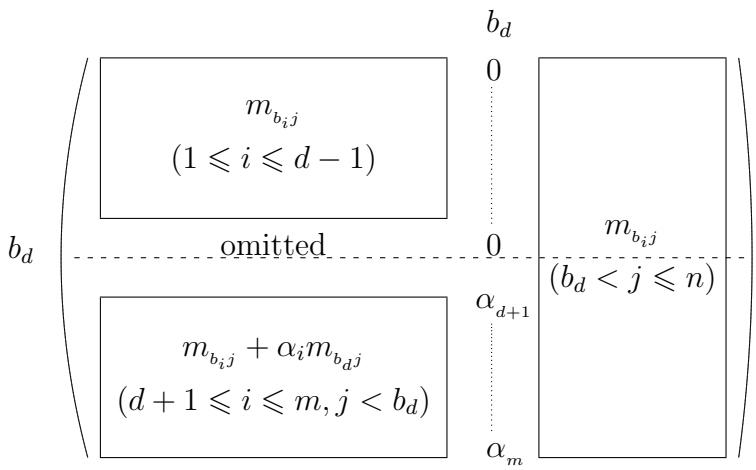
$$\Phi_m^d([M]) := \sum_{N \in R_d(M)} [N], \tag{4.1.7}$$

then

$$\Phi_m([M]) = \bigoplus_{d=1}^{m} \Phi_m^d([M]). \tag{4.1.8}$$

In fact, Φ_m^d is Φ_m composed with the projection from M^μ onto the $\mathfrak{u}_d-$batch of M^μ, and hence is $FU-$linear.

4.1.9 Remark. Keep the notations above. Now we use a picture to show the element $N \in R_d(M)$ for a fixed d :



Picture of $N \in R_d(M)$

Obviously $\underline{\mathfrak{u}_d} = \{b_1, \ldots, b_{d-1}, b_{d+1}, \ldots, b_m\}$, where $\underline{\mathfrak{t}} = \{b_1, \ldots, b_m\}$. By Definition 3.1.9, we have

$$\mathfrak{J}_{\mathfrak{t}} = \{(i,j) \,|\, i > j,\ i \in \underline{\mathfrak{t}},\ j \notin \underline{\mathfrak{t}}\},$$

$$\mathfrak{J}_{\mathfrak{u}_d} = \{(i,j) \,|\, i > j,\ i \in \underline{\mathfrak{u}_d},\ j \notin \underline{\mathfrak{u}_d}\}.$$

It can be seen easily from the **Picture of** $N \in R_d(M)$ that: $\mathfrak{J}_{\mathfrak{u}_d}$ is the set of all positions of $\mathfrak{J}_{\mathfrak{t}}$ except those in row b_d (which is omitted), and additionally the positions below the last one in column b_d. So:

$$\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d} = \{(i,j) \,|\, i > j,\ i \in \underline{\mathfrak{u}_d},\ j \notin \underline{\mathfrak{t}}\},$$

$$\mathfrak{J}_{\mathfrak{t}} = (\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}) \,\dot{\cup}\, \{(b_d, j) \,|\, j < b_d, j \notin \underline{\mathfrak{t}}\},$$

$$\mathfrak{J}_{\mathfrak{u}_d} = (\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}) \,\dot{\cup}\, \{(b_i, b_d) \,|\, i = d+1, \ldots, m\}.$$

So $\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}$ consists of all positions in $\mathfrak{J}_{\mathfrak{t}}$ besides row b_d and these are precisely the positions in $\mathfrak{J}_{\mathfrak{u}_d}$ besides those in column b_d. In other words, $\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}$ together with row b_d gives $\mathfrak{J}_{\mathfrak{t}}$ and together with column b_d gives $\mathfrak{J}_{\mathfrak{u}_d}$.

Next we shall show first Φ_m^d preserves condition sets. Then it follows that Φ_m preserves condition sets since $\Phi_m = \bigoplus_{d=1}^m \Phi_m^d$.

To begin with, we investigate the image of the idempotent $e_{_L}$ under the homomorphism Φ_m^d for $e_{_L} \in \mathfrak{M}_{\mathfrak{t}} \subset M^\lambda$. By Definition 3.2.5, we have

$$e_{_L} = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{X}_{\mathfrak{t}}} \chi_{_L}(-M)[M]$$

then

$$\Phi_m^d(e_{_L}) = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{X}_{\mathfrak{t}}} \chi_{_L}(-M)\Phi_m^d([M]). \tag{4.1.10}$$

By (4.1.7) we may write

$$\Phi_m^d([M]) = \sum_{N \in R_d(M)} [N]$$

where $\mathrm{tab}(N) = \mathfrak{u}_d$. Let $N = (n_{_{b_i j}}) \in \mathfrak{X}_{u_d}$. Then from the **Picture of** $N \in R_d(M)$, we obtain:

$$n_{_{b_i j}} = \begin{cases} m_{_{b_i j}} & \text{if } i \leqslant d-1 \text{ or } j > b_d; \\ \alpha_i \in GF(q) & \text{if } d+1 \leqslant i \leqslant m, j = b_d; \\ m_{_{b_i j}} + \alpha_i m_{_{b_d j}} & \text{if } d+1 \leqslant i \leqslant m, j < b_d. \end{cases} \tag{4.1.11}$$

Note that by Remark 4.1.9, the first case combined with the third case in (4.1.11) are just all positions in $\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}$. (4.1.12)

Obviously, different elements in $R_d(M)$ only have possible different entries α_i on places (b_i, b_d) for $d+1 \leqslant i \leqslant m$. Let $\underline{\alpha} = (\alpha_{d+1}, \ldots, \alpha_m)$ and we write $\underline{\alpha} \in GF(q)^{m-d}$ for $\alpha_i \in GF(q)$, $\forall\, d+1 \leqslant i \leqslant m$. Denote $N \in R_d(M)$ with $\underline{\alpha}$ fixed by $N_{\underline{\alpha}}$. Then

$$\Phi_m^d([M]) = \sum_{\underline{\alpha} \in GF(q)^{m-d}} [N_{\underline{\alpha}}]. \tag{4.1.13}$$

Inserting this formula into (4.1.10), we obtain

$$\begin{aligned} \Phi_m^d(e_{_L}) &= \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M \in \mathfrak{X}_\mathfrak{t}} \chi_{_L}(-M) \sum_{\underline{\alpha} \in GF(q)^{m-d}} [N_\alpha] \\ &= \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{\underline{\alpha} \in GF(q)^{m-d}} \sum_{M \in \mathfrak{X}_\mathfrak{t}} \chi_{_L}(-M)[N_{\underline{\alpha}}]. \end{aligned}$$

Now we fixed an $\underline{\alpha} \in GF(q)^{m-d}$. Using general character theory we may write

$$[N_{\underline{\alpha}}] = \sum_{K \in \mathfrak{X}_{\mathfrak{u}_d}} \chi_{_K}(N_{\underline{\alpha}}) e_{_K},$$

hence

$$\begin{aligned} \Phi_m^d(e_{_L}) &= \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{\underline{\alpha} \in GF(q)^{m-d}} \sum_{M \in \mathfrak{X}_\mathfrak{t}} \chi_{_L}(-M) \sum_{K \in \mathfrak{X}_{\mathfrak{u}_d}} \chi_{_K}(N_{\underline{\alpha}}) e_{_K} \\ &= \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{\underline{\alpha} \in GF(q)^{m-d}} \sum_{K \in \mathfrak{X}_{\mathfrak{u}_d}} \sum_{M \in \mathfrak{X}_\mathfrak{t}} \chi_{_L}(-M) \chi_{_K}(N_{\underline{\alpha}}) e_{_K}. \end{aligned} \tag{4.1.14}$$

For $K \in \mathfrak{X}_{u_d}$ the coefficient $C_K^{\underline{\alpha}}$ for a fixed $\underline{\alpha} \in GF(q)^{m-d}$ of $e_{_K}$ is

$$C_K^{\underline{\alpha}} = \frac{1}{q^{|\mathfrak{J}_\mathfrak{t}|}} \sum_{M \in \mathfrak{X}_\mathfrak{t}} \chi_{_L}(-M) \chi_{_K}(N_{\underline{\alpha}}). \tag{4.1.15}$$

Hence the coefficient C_K of $e_{_K}$ is

$$C_K = \sum_{\underline{\alpha} \in GF(q)^{m-d}} C_K^{\underline{\alpha}}. \tag{4.1.16}$$

Using Remark 4.1.9 we calculate $\chi_L(-M)\chi_K(N_{\underline{\alpha}})$:

$$\begin{aligned}
\chi_L(-M)\chi_K(N_{\underline{\alpha}}) &= \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}} \theta\left(-l_{b_ij}m_{b_ij}\right) \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{u}_d}} \theta\left(k_{b_ij}n_{b_ij}\right) \\
&= \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}\cap\mathfrak{J}_{\mathfrak{u}_d}} \theta\left(-l_{b_ij}m_{b_ij}\right) \prod_{\substack{1\leqslant j<b_d \\ j\notin\underline{\mathfrak{t}}}} \theta\left(-l_{b_dj}m_{b_dj}\right) \\
&\quad\cdot \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}\cap\mathfrak{J}_{\mathfrak{u}_d}} \theta\left(k_{b_ij}n_{b_ij}\right) \prod_{d+1\leqslant i\leqslant m} \theta\left(k_{b_ib_d}n_{b_ib_d}\right) \\
&= \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}\cap\mathfrak{J}_{\mathfrak{u}_d}} \theta\left(-l_{b_ij}m_{b_ij}\right)\theta\left(k_{b_ij}n_{b_ij}\right) \\
&\quad\cdot \prod_{\substack{1\leqslant j<b_d \\ j\notin\underline{\mathfrak{t}}}} \theta\left(-l_{b_dj}m_{b_dj}\right) \prod_{d+1\leqslant i\leqslant m} \theta\left(k_{b_ib_d}n_{b_ib_d}\right) \\
&\overset{(4.1.11)}{=} \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}\cap\mathfrak{J}_{\mathfrak{u}_d}} \theta\left(-l_{b_ij}m_{b_ij}\right)\theta\left(k_{b_ij}n_{b_ij}\right) \\
&\quad\cdot \prod_{\substack{1\leqslant j<b_d \\ j\notin\underline{\mathfrak{t}}}} \theta\left(-l_{b_dj}m_{b_dj}\right) \prod_{d+1\leqslant i\leqslant m} \theta\left(k_{b_ib_d}\alpha_i\right). \qquad (4.1.17)
\end{aligned}$$

Hence by (4.1.11) and (4.1.12), for $(b_i,j)\in\mathfrak{J}_t\cap\mathfrak{J}_{\mathfrak{u}_d}$ we have:

$$\begin{aligned}
&\theta\left(-l_{b_ij}m_{b_ij}\right)\theta\left(k_{b_ij}n_{b_ij}\right) \qquad (4.1.18)\\
&= \begin{cases} \theta\left(-l_{b_ij}m_{b_ij}\right)\theta\left(k_{b_ij}m_{b_ij}\right) & \text{if } i\leqslant d-1 \text{ or } j>b_d \\ \theta\left(-l_{b_ij}m_{b_ij}\right)\theta\left(k_{b_ij}(m_{b_ij}+\alpha_i m_{b_dj})\right) & \text{if } d+1\leqslant i\leqslant m, j<b_d \end{cases} \\
&= \begin{cases} \theta\left((k_{b_ij}-l_{b_ij})m_{b_ij}\right) & \text{if } i\leqslant d-1 \text{ or } j>b_d \\ \theta\left((k_{b_ij}-l_{b_ij})m_{b_ij}\right)\theta\left(\alpha_i k_{b_ij}m_{b_dj}\right) & \text{if } d+1\leqslant i\leqslant m, j<b_d\,. \end{cases}
\end{aligned}$$

Therefore by (4.1.17) and (4.1.18) the coefficient $C_K^{\underline{\alpha}}$ is

$$\begin{aligned}
C_K^{\underline{\alpha}} &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}}\sum_{M\in\mathfrak{X}_{\mathfrak{t}}} \chi_L(-M)\chi_K(N_{\underline{\alpha}}) \\
&= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}}\sum_{M\in\mathfrak{X}_{\mathfrak{t}}} \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}\cap\mathfrak{J}_{\mathfrak{u}_d}} \theta\left((k_{b_ij}-l_{b_ij})m_{b_ij}\right)\cdot \prod_{\substack{d+1\leqslant i\leqslant m \\ 1\leqslant j<b_d \\ j\notin\underline{\mathfrak{t}}}} \theta\left(\alpha_i k_{b_ij}m_{b_dj}\right) \\
&\quad\cdot \prod_{\substack{1\leqslant j<b_d \\ j\notin\underline{\mathfrak{t}}}} \theta\left(-l_{b_dj}m_{b_dj}\right) \prod_{d<i\leqslant m} \theta\left(k_{b_ib_d}\alpha_i\right).
\end{aligned}$$

Combining the last three factors together, we get

$$\begin{aligned}
C_K^{\underline{\alpha}} &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}}\sum_{M\in\mathfrak{X}_{\mathfrak{t}}} \prod_{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}}\cap\mathfrak{J}_{\mathfrak{u}_d}} \theta\left((k_{b_ij}-l_{b_ij})m_{b_ij}\right) \\
&\quad\cdot \prod_{\substack{d+1\leqslant i\leqslant m \\ 1\leqslant j<b_d \\ j\notin\underline{\mathfrak{t}}}} \theta\left(\alpha_i k_{b_ij}m_{b_dj}\right)\theta\left(-l_{b_dj}m_{b_dj}\right)\theta\left(k_{b_ib_d}\alpha_i\right).
\end{aligned}$$

Using $\mathfrak{J}_{\mathfrak{t}} = (\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}) \cup \{(b_d, j) \,|\, j < b_d, j \notin \underline{\mathfrak{t}}\}$ by Remark (4.1.9), we can rewrite the sum $\sum_{M \in \mathfrak{X}_{\mathfrak{t}}}$ and then we obtain:

$$\begin{aligned} C_K^{\underline{\alpha}} &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}} \sum_{m_{b_i j} \in GF(q)} \theta\left((k_{b_i j} - l_{b_i j}) m_{b_i j}\right) \\ &\quad \cdot \prod_{\substack{d+1 \leqslant i \leqslant m \\ 1 \leqslant j < b_d \\ j \notin \underline{\mathfrak{t}}}} \sum_{m_{b_d j} \in GF(q)} \theta\left(\alpha_i k_{b_i j} m_{b_d j}\right) \theta\left(-l_{b_d j} m_{b_d j}\right) \theta\left(k_{b_i b_d} \alpha_i\right). \end{aligned} \tag{4.1.19}$$

Note that N runs through $R_d(M)$ just means α_i runs over $GF(q)$ for $d+1 \leqslant i \leqslant m$. Hence by (4.1.16), we obtain:

$$\begin{aligned} C_K &= \sum_{\underline{\alpha} \in GF(q)^{m-d}} C_K^{\underline{\alpha}} \\ &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}} \sum_{m_{b_i j} \in GF(q)} \theta\left((k_{b_i j} - l_{b_i j}) m_{b_i j}\right) \\ &\quad \cdot \prod_{\substack{d+1 \leqslant i \leqslant m \\ 1 \leqslant j < b_d \\ j \notin \underline{\mathfrak{t}}}} \sum_{\alpha_i \in GF(q)} \sum_{m_{b_d j} \in GF(q)} \theta\left(\alpha_i k_{b_i j} m_{b_d j}\right) \theta\left(-l_{b_d j} m_{b_d j}\right) \theta\left(k_{b_i b_d} \alpha_i\right). \\ &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}} \sum_{m_{b_i j} \in GF(q)} \theta\left((k_{b_i j} - l_{b_i j}) m_{b_i j}\right) \\ &\quad \cdot \prod_{\substack{d+1 \leqslant i \leqslant m \\ 1 \leqslant j < b_d \\ j \notin \underline{\mathfrak{t}}}} \sum_{m_{b_d j} \in GF(q)} \left(\sum_{\alpha_i \in GF(q)} \theta\left(\alpha_i (k_{b_i j} m_{b_d j} + k_{b_i b_d})\right) \right) \theta\left(-l_{b_d j} m_{b_d j}\right). \end{aligned} \tag{4.1.20}$$

Obviously by (4.1.20) C_K contains the factor

$$\frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}} \sum_{m_{b_i j} \in GF(q)} \theta\left((k_{b_i j} - l_{b_i j}) m_{b_i j}\right) \tag{4.1.21}$$

and there is no other factor of C_K involving $m_{b_i j}$ with $(b_i, j) \in \mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}$. Hence if the coefficient $C_K \neq 0$, by (3.3.6) we must have:

$$k_{b_i j} = l_{b_i j}, \ \forall\, (b_i, j) \in \mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{\mathfrak{u}_d}. \tag{4.1.22}$$

That is, the entries in K are the same as L in the northwest, southwest and east boxes (c.f. Remark 4.1.9). Thus from (4.1.21), we get a factor

$$\frac{q^{|\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{u_d}|}}{q^{|\mathfrak{J}_{\mathfrak{t}}|}}.$$

And the remaining factor of C_K is:

$$\prod_{\substack{d+1\leqslant i\leqslant m\\ 1\leqslant j<b_d\\ j\notin \underline{\mathfrak{t}}}} \sum_{m_{b_dj}\in GF(q)} \left(\sum_{\alpha_i\in GF(q)} \theta\left(\alpha_i(k_{b_ij}m_{b_dj}+k_{b_ib_d})\right)\right)\theta\left(-l_{b_dj}m_{b_dj}\right).$$

$$= \prod_{\substack{d+1\leqslant i\leqslant m\\ 1\leqslant j<b_d\\ j\notin \underline{\mathfrak{t}}}} \sum_{m_{b_dj}\in GF(q)} \left(\sum_{\alpha_i\in GF(q)} \theta\left(\alpha_i(l_{b_ij}m_{b_dj}+k_{b_ib_d})\right)\right)\theta\left(-l_{b_dj}m_{b_dj}\right). \tag{4.1.23}$$

since $k_{b_ij} = l_{b_ij}$ for $d+1 \leqslant i \leqslant m$, $(b_d, j) \in \mathfrak{J}_{\mathfrak{t}}$ by (4.1.22).

Recall that for a given condition set S we denote the set of all occurring row indices in the frame of S by S_I and of all occurring column indices by S_J (c.f. Definition 3.4.11). Now we are ready for the following theorem:

4.1.24 Theorem. Let $\lambda = (n-m, m), \mu = (n-m+1, m-1)$. Then the homomorphism $\Phi_m : M^\lambda \to M^\mu$ preserves condition sets. More precisely, let $L \in \mathfrak{X}_{\mathfrak{t}}$, $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ and let $e_L \in \mathcal{O} \subset \mathfrak{M}_{\mathfrak{t}} \subset M^\lambda$, $S = S(O)$ be the condition set of $\mathcal{O}$. Then

$$\Phi_m(e_L) = \sum_K C_K e_K$$

where $K \in \Xi_{m-1,n}$ such that each e_K whose coefficient $C_K \neq 0$ is contained in an orbit $\tilde{\mathcal{O}}$ of some $\mathfrak{u}-$batch of M^μ such that $S(\tilde{\mathcal{O}}) = S$.

Proof. From (4.1.8) we know $\Phi_m = \bigoplus_{d=1}^m \Phi_m^d$. Hence it suffices to prove Φ_m^d preserves condition set. Since Φ_m^d is $FU-$linear and each orbit module is an irreducible $U-$module by Theorem 3.6.8, we can restrict our attention to the case that $L = (l_{b_ij}) \in \mathfrak{X}_{\mathfrak{t}}$ is a pattern matrix.

Assume

$$\Phi_m^d(e_L) = \sum_K C_K e_K$$

where $K \in \Xi_{m-1,n}, C_K \neq 0$. Since $b_d \notin \underline{\mathrm{tab}(K)}$, we shall consider the following case separately:

Case 1. $b_d \notin S_I = \{i \mid (i,j) \in \overline{S} \text{ for some } 1 \leqslant j \leqslant n\}$.

That is there exist no $(b_d, j) \in \mathfrak{J}_{\mathfrak{t}}$ such that $l_{b_dj} \in S$. Hence $\theta(-l_{b_dj}m_{b_dj}) = 1$ for all $(b_d, j) \in \mathfrak{J}_{\mathfrak{t}}$. Thus the remaining factor (4.1.23) becomes:

$$\prod_{\substack{d+1\leqslant i\leqslant m\\ 1\leqslant j<b_d\\ j\notin \underline{\mathfrak{t}}}} \sum_{\alpha_i\in GF(q)} \sum_{m_{b_dj}\in GF(q)} \theta\left(\alpha_i(l_{b_ij}m_{b_dj}+k_{b_ib_d})\right). \tag{4.1.25}$$

If there exist no $(b_i, j) \in \overline{S}$ for $d+1 \leqslant i \leqslant m, 1 \leqslant j < b_d$, then in this case $l_{b_i j} = 0$, thus the remaining factor (4.1.25) becomes

$$\prod_{\substack{d+1\leqslant i\leqslant m\\ 1\leqslant j<b_d\\ j\notin \underline{t}}} \sum_{m_{b_d j}\in GF(q)} \sum_{\alpha_i\in GF(q)} \theta\left(k_{b_i b_d}\alpha_i\right).$$

Therefore by (3.3.6), if $C_K \neq 0$, then this factor should be nonzero and hence we obtain $k_{b_i b_d} = 0$ for all $d \leqslant i \leqslant m$. Combining with (4.1.22), we obtain easily in this case K is a pattern matrix and $S(K) = S$. Moreover in this case the factor (4.1.25) of C_K is a power of q, namely, q^t where t is the number of choices for $m_{b_d j}$ and α_i i.e.

$$t = b_d - 1 - (d-1) + (m-d) = b_d - 2d + m.$$

If there exist $l_{b_u v} \in S$ for some $d+1 \leqslant u \leqslant m, 1 \leqslant v < b_d$, then we rewrite the factor (4.1.25) by separating the elements in the condition set from those which are not:

$$\begin{aligned}
&\prod_{\substack{d+1\leqslant i\leqslant m\\ 1\leqslant j<b_d\\ j\notin \underline{t}}} \sum_{m_{b_d j}\in GF(q)} \sum_{\alpha_i\in GF(q)} \theta\big((l_{b_i j}m_{b_d j} + k_{b_i b_d})\alpha_i\big)\\
=&\prod_{\substack{d+1\leqslant u\leqslant m\\ (b_u,v)\in\overline{S}}} \sum_{m_{b_d v}\in GF(q)} \sum_{\alpha_u\in GF(q)} \theta\big((l_{b_u v}m_{b_d v} + k_{b_u b_d})\alpha_u\big)\\
&\cdot \prod_{\substack{d+1\leqslant i\leqslant m\\ b_i\notin S_I\\ j\notin S_J}} \sum_{m_{b_d j}\in GF(q)} \sum_{\alpha_i\in GF(q)} \theta\left(k_{b_i b_d}\alpha_i\right).
\end{aligned}$$

If $C_K \neq 0$ then by (3.3.6), we must have

$$l_{b_u v}m_{b_d v} + k_{b_u b_d} = 0 \quad \text{for } d+1 \leqslant u \leqslant m,\ (b_u, v) \in \overline{S} \tag{4.1.26}$$

and

$$k_{b_i b_d} = 0 \quad \text{for } d+1 \leqslant i \leqslant m,\ b_i \notin S_I.$$

Therefore in this case

$$k_{b_u b_d} = -l_{b_u v}m_{b_d v} \text{ for } d+1 \leqslant u \leqslant m,\ (b_u, v) \in \overline{S}.$$

Note that (b_u, b_d) is on the $(b_u, v)-$hook row with nonzero entry $l_{b_u v}$ in the hook corner. Hence by Corollary 3.3.7, using truncated column operations, we can change all those possible nonzero entries on positions (b_u, b_d) to zeros. Combining with (4.1.22), we know that in this case if $C_K \neq 0$, we can find $u \in U^w \cap U$, $\big(\mathfrak{t}^\mu w = \mathrm{tab}(K)\big)$ such that $e_K \circ u = e_{\tilde{L}}$ where $\tilde{L}$ is a pattern matrix in $\Xi_{m-1,n}$ and $S(\tilde{L}) = S$. Hence, e_K is contained in an orbit $\tilde{\mathcal{O}}$ of $\mathrm{tab}(K)-$batch of M^μ such that $S(\tilde{\mathcal{O}}) = S$.

More precisely, suppose we have a many conditions in the southwest box of L, denoted by $S^{>d}$:

$$S^{>d} = \{l_{u_s v_s} \in S \,|\, u_s > b_d, 1 \leqslant s \leqslant a\}.$$

Similarly we define the corresponding row and column indices sets by:

$$S_I^{>d} = \{u \,|\, l_{uv} \in S^{>d} \text{ for some } v\};$$

$$S_J^{>d} = \{v \,|\, l_{uv} \in S^{>d} \text{ for some } u\}.$$

Moreover if K is fixed, then from (4.1.26), we know $m_{b_d v}$ are fixed for all $v \in S_J^{>d}$. Then the remaining factor 4.1.25 of C_K is a power of q namely $q^{t'}$, i.e.

$$\mathfrak{t}' = t - a = b_d - 2d + m - a.$$

Now we can write the image of e_L as the following:

$$\Phi_m^d(e_L) = \frac{q^{|\mathfrak{J}_\mathfrak{t} \cap \mathfrak{J}_{u_d}|}}{q^{|\mathfrak{J}_\mathfrak{t}|}} q^{b_d - 2d + m - a} \sum_{\underline{\beta} \in GF(q)^a} e_{N_{\underline{\beta}}} \tag{4.1.27}$$

where $\underline{\beta} = (\beta_{u_1}, \ldots, \beta_{u_a})$ with $\underline{\beta} \in GF(q)^a$ means $\beta_{u_s} \in GF(q)$, for all $u_s \in S_I^{>d}$, $s = 1, \ldots, a$ and $N_{\underline{\beta}}$ is obtained from L by first deleting row b_d and then replacing the $(u_s, b_d)-$entry by β_{u_s} for $u_s \in S_I^{>d}$, $s = 1, \ldots, a$.

Case 2. $l_{b_d v} \in S$ for some $(b_d, v) \in \mathfrak{J}_t$.

In this case we rewrite the factor (4.1.23) as follows

$$\begin{aligned}
&\prod_{\substack{d+1 \leqslant i \leqslant m \\ 1 \leqslant j < b_d \\ j \notin \mathfrak{t}}} \sum_{\alpha_i \in GF(q)} \sum_{m_{b_d j} \in GF(q)} \theta(\alpha_i l_{b_i j} m_{b_d j}) \theta(-l_{b_d j} m_{b_d j}) \theta(k_{b_i b_d} \alpha_i) \\
= &\prod_{\substack{d+1 \leqslant i \leqslant m \\ 1 \leqslant j < b_d \\ j \notin \mathfrak{t}, j \neq v}} \sum_{\alpha_i \in GF(q)} \sum_{m_{b_d j} \in GF(q)} \theta(\alpha_i l_{b_i j} m_{b_d j}) \theta(-l_{b_d j} m_{b_d j}) \theta(k_{b_i b_d} \alpha_i) \\
&\cdot \prod_{d+1 \leqslant i \leqslant m} \sum_{\alpha_i \in GF(q)} \sum_{m_{b_d v} \in GF(q)} \theta(\alpha_i l_{b_i v} m_{b_d v}) \theta(-l_{b_d v} m_{b_d v}) \theta(k_{b_i b_d} \alpha_i).
\end{aligned} \tag{4.1.28}$$

From the definition of pattern matrix, we know $l_{b_i v} = 0$ for $d + 1 \leqslant i \leqslant m$. Thus (4.1.28) becomes:

$$\prod_{\substack{d+1 \leqslant i \leqslant m \\ 1 \leqslant j < b_d \\ j \notin \mathfrak{t}, j \neq v}} \sum_{\alpha_i, m_{b_d j} \in GF(q)} \theta(\alpha_i l_{b_i j} m_{b_d j}) \theta(-l_{b_d j} m_{b_d j}) \theta(k_{b_i b_d} \alpha_i) \sum_{m_{b_d v} \in GF(q)} \theta(-l_{b_d v} m_{b_d v})$$

Note that $m_{b_d v}$ only occurs in the second sum, hence by 3.3.6, if $C_K \neq 0$ we must have $l_{b_d v} = 0$ which is a contradiction to $l_{b_d v} \in S$. It means that

if $C_{_K} \neq 0$, this case will never appear.

This finishes the proof. $\square$

We state an easy consequence of the **Case 2** in the proof of Theorem 4.1.24:

4.1.29 Corollary. Let $\mathcal{O}$ be an orbit in $M^{(n-m,m)}$. Let $S = S(\mathcal{O})$ be the condition set of $\mathcal{O}$. Then for any $e_{_K} \in \mathcal{O}$ we have:

$$\Phi_m^d(e_{_K}) = 0 \text{ if } d \in S_I = \{i \,|\, (i,j) \in \overline{S} \text{ for some } 1 \leqslant j \leqslant n\}.$$

Proof. The **Case 2** in the proof of Theorem 4.1.24 proves that if $L \in \mathcal{O}$ is a pattern matrix, the statement holds. Since for $K \in \mathcal{O}$, $e_{_K} = e_{_L} \circ u$ for some $u \in U^w \cap U$ and Φ_m is an $FU-$linear homomorphism, the statement holds for any $e_{_K} \in \mathcal{O}$. $\square$

Now we give an explicit expression of $\Phi_m^d(e_{_K})$ for an arbitrary $K \in \Xi_{m,n}$. Suppose $e_{_K} \in \mathcal{O}$ with $S(\mathcal{O}) = S$. From Corollary 4.1.29, we can restrict our attention to those d such that $b_d \notin S_I$.

Let L be the pattern matrix associated with $\mathcal{O}$. Keeping the notation in the proof of Theorem 4.1.24, from (4.1.27) we have

$$\Phi_m^d(e_{_L}) = \frac{q^{|\mathfrak{I}_{\mathfrak{t}} \cap \mathfrak{I}_{u_d}|}}{q^{|\mathfrak{I}_{\mathfrak{t}}|}} q^{b_d - 2d + m - a} \sum_{\underline{\beta} \in GF(q)^a} e_{_{N_{\underline{\beta}}}} \tag{4.1.30}$$

where $\underline{\beta} = (\beta_{u_1}, \ldots, \beta_{u_a})$ with $\underline{\beta} \in GF(q)^a$ means $\beta_{u_s} \in GF(q)$, for all $u_s \in S_I^{>d}$, $s = 1, \ldots, a$ and $N_{\underline{\beta}}$ is obtained from L by first deleting row b_d and then replacing the $(u_s, b_d)-$entry by β_{u_s} for $u_s \in S_I^{>d}$, $s = 1, \ldots, a$.

For $e_{_K} \in \mathcal{O}$, there exist some $u \in U^w \cap U$ such that $e_{_L} \circ u = e_{_K}$. Since Φ_m^d is a $FU-$linear map, we obtain

$$\Phi_m^d(e_{_K}) = \Phi_m^d(e_{_L} \circ u) = \Phi_m^d(e_{_L}) \circ u.$$

By Lemma 3.3.11 we can rewrite u as the following

$$u = \prod_{s=1}^{a} x_{_{u_s b_d}}(\alpha_s) \cdot u' \text{ where } u_s \in S_J^{>d},\ s = 1, \ldots, a.$$

Thus by 4.1.30, we have

$$\Phi_m^d(e_{_L}) \prod_{s=1}^{a} x_{_{u_s b_d}}(\alpha_s) = \frac{q^{|\mathfrak{I}_{\mathfrak{t}} \cap \mathfrak{I}_{u_d}|}}{q^{|\mathfrak{I}_{\mathfrak{t}}|}} q^{b_d - 2d + m - a} \sum_{\underline{\beta} \in GF(q)^a} e_{_{N_{\underline{\beta}}}} \circ \prod_{s=1}^{a} x_{_{u_s b_d}}(\alpha_s).$$

Since $u_s \in \underline{\mathfrak{u}_d}, b_d \notin \underline{\mathfrak{u}_d}$ from Corollary 3.3.7 we have:

$$e_{N_{\underline{\beta}}} \circ \prod_{s=1}^{a} x_{u_s b_d}(\alpha_s) = \prod_{s=1}^{a} \theta(\beta_{u_s}\alpha_s) e_{N_{\underline{\beta}}},$$

hence we obtain:

$$\Phi_m^d(e_L) \prod_{s=1}^{a} x_{u_s b_d}(\alpha_s) = \frac{q^{|\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{u_d}|}}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} q^{b_d - 2d + m - a} \sum_{\underline{\beta} \in GF(q)^a} \prod_{s=1}^{a} \theta(\beta_{u_s}\alpha_s) e_{N_{\underline{\beta}}}.$$

Denote

$$e_{N_{\underline{\beta}}} \circ u' = e_{N_{\underline{\beta},K}}.$$

Obviously, the entries of the positions of the southwest, northwest and east boxes of $N_{\underline{\beta},K}$ are the same as those entries of K and we get the image of e_K under Φ_m^d :

$$\Phi_m^d(e_K) = \frac{q^{|\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{u_d}|}}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} q^{b_d - 2d + m - a} \sum_{\underline{\beta} \in GF(q)^a} \prod_{s=1}^{a} \theta(\beta_{u_s}\alpha_s) e_{N_{\underline{\beta},K}}. \tag{4.1.31}$$

4.1.32 Remark. Note that $N_{\underline{\beta},K}$ in (4.1.31) can have possible nonzero entries on positions (b_u, b_d) even when $b_u \notin S_I^{>d}$, but if $\beta_{u_s} = 0, \forall\, s = 1, \ldots, a$ then in this case all the entries in column b_d are zeros except the last 1. Hence, we can rewrite (4.1.31) as:

$$\Phi_m^d(e_K) = \frac{q^{|\mathfrak{J}_{\mathfrak{t}} \cap \mathfrak{J}_{u_d}|}}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} q^{b_d - 2d + m - a} \Big(e_{N_{\underline{0},K}} + \sum_{\substack{\underline{\beta} \in GF(q)^a \\ \underline{\beta} \neq \underline{0}}} \prod_{s=1}^{a} \theta(\beta_{u_s}\alpha_s) e_{N_{\underline{\beta},K}} \Big). \tag{4.1.33}$$

where $\underline{0}$ is the $a-$tuple zero vector. Note that $N_{\underline{0},K}$ is obtained from K by removing row b_d and $b_d \notin S_I$.

From Theorem 4.1.24, we know the condition sets play an important role. This give us an idea of collecting all the orbits which contain the same condition set together:

4.1.34 Definition. Let $\lambda = (n-m, m)$, S be a condition set. Define:

$$\begin{aligned} \mathfrak{C}_S^\lambda &= \bigoplus_{S(\mathcal{O})=S} M_{\mathcal{O}} \\ &= \bigoplus_{S(\mathcal{O})=S} \bigoplus_{e_L \in \mathcal{O}} F e_L, \text{ where } M_{\mathcal{O}} \subset M^\lambda, \end{aligned}$$

the direct sum being over the different orbits in M^λ which have the same condition set.

Now we give some examples to explain the Definition 4.1.34 by using a short notation for the orbit module $M_{\mathcal{O}}$ for convenience.

4.1.35 Example. Let $L = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & l_{62} & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 5 \\ 6 \end{array}$ with $S = \{l_{62} \neq 0\}$. Suppose $e_L \in \mathcal{O}$, then

$$M_{\mathcal{O}} = \bigoplus_{a_i, b_i, c_i \in GF(q)} F e_{R_i} \text{ where } R_i = \begin{pmatrix} 0 & a_i & 1 & 0 & 0 & 0 \\ 0 & b_i & 0 & 0+\square & 1 & 0 \\ 0 & l_{62} & 0 & c_i & 0 & 1 \end{pmatrix}$$

and $\square$ is fixed when the $(6,2)-$hook is fixed.

From now on, we use the short notation as the following:

$$M_{\mathcal{O}} = \begin{pmatrix} 0 & * & 1 & 0 & 0 & 0 \\ 0 & * & 0 & 0+\square & 1 & 0 \\ 0 & l_{62} & 0 & * & 0 & 1 \end{pmatrix}.$$

4.1.36 Example. Let $\mu = (4,2), S = \{l_{64} \neq 0\}$. Then:

$$\begin{aligned} \mathfrak{C}_S^\mu &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & * & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & * & 1 \end{pmatrix} \\ &\oplus \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & * & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & l_{64} & 0 & 1 \end{pmatrix}. \end{aligned}$$

4.1.37 Example. Let $\mu = (3,3), S = \{l_{41} \neq 0\}$. Then:

$$\begin{aligned} \mathfrak{C}_S^\mu &= \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ l_{41} & 0 & * & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ &\oplus \begin{pmatrix} * & 0+\square & 1 & 0 & 0 & 0 \\ l_{41} & * & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ l_{41} & 0 & * & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Using Definition 4.1.34, we obtain the following corollary:

4.1.38 Corollary. Let $\lambda = (n-m, m) \vdash n, \mu = (n-m+1, m-1) \vdash n$ and $\mathcal{O}$ be an orbit in M^λ with the condition set $S = S(\mathcal{O})$. Then

$$\Phi_m(\mathfrak{C}_S^\lambda) \subseteq \mathfrak{C}_S^\mu.$$

Proof. If we denote the orbit M^μ by $\tilde{\mathcal{O}}$ then Theorem 4.1.24 tells us

$$\Phi_m(M_{\mathcal{O}}) \subseteq \bigoplus M_{\tilde{\mathcal{O}}} \text{ where } S(\tilde{\mathcal{O}}) = S. \tag{4.1.39}$$

Hence by Definition 4.1.34 we have $\Phi_m(\mathfrak{C}_S^\lambda) \subseteq \mathfrak{C}_S^\mu$. □

4.1.40 Proposition. If $\mathrm{char}(F) = 0$, then $S^{(n-m,m)} = \ker \Phi_m$.

Proof. By James' kernel intersection theorem 2.2.6, we have

$$S^{(n-m,m)} = \bigcap_{i=0}^{m-1} \ker \phi_{1,i} = \Big(\bigcap_{i=0}^{m-2} \ker \phi_{1,i}\Big) \bigcap \ker \Phi_m.$$

Hence if we can prove $\ker \Phi_m \subset \ker \phi_{1,i}$ for all $0 \leqslant i \leqslant m-2$, then we are done. In fact, for $X \subseteq V$, $\dim_{GF(q)} X = m$, we have by Definition 4.1.1:

$$\Phi_m([X]) = \sum_{\substack{Y \subseteq X \\ \dim Y = m-1}} [Y], \qquad \phi_{1,i}([Y]) = \sum_{\substack{Z \subseteq Y \\ \dim Z = i}} [Z].$$

Hence

$$\phi_{1,i} \circ \Phi_m([X]) = \sum_{\substack{Y \subseteq X \\ \dim Y = m-1}} \sum_{\substack{Z \subseteq Y \\ \dim Z = i}} [Z].$$

Now we calculate the number of the $(m-1)-$dimensional subspace $Y \subseteq X$ which contains a fixed $i-$dimensional space $Z \subseteq X$. Obviously, this number equals the ways of choosing $(m-i-1)-$dimensional space from a $(m-i)-$dimensional space, that is

$$\begin{bmatrix} m-i \\ m-i-1 \end{bmatrix} = [m-i].$$

Hence for all $0 \leqslant i \leqslant m-2$, we have:

$$\phi_{1,i} \circ \Phi_m = [m-i]\, \phi_{1,i}.$$

Since we are in the characteristic zero case, we obtain for all $0 \leqslant i \leqslant m-2$:

$$\phi_{1,i} = \frac{1}{[m-i]}\, \phi_{1,i} \circ \Phi_m$$

and hence $\ker \Phi_m \subset \ker \phi_{1,i}$ for all $0 \leqslant i \leqslant m-2$. Therefore

$$S^{(n-m,m)} = \ker \Phi_m.$$

□

4.1.41 Corollary. If $\mathrm{char}(F) = 0$ then Φ_m is an epimorphism.

Proof. By Proposition 4.1.40, if $\mathrm{char}(F) = 0$ then $S^{(n-m,m)} = \ker \Phi_m$. Thus

$$\begin{aligned} \dim \Phi_m\big(M^{(n-m,m)}\big) &= \dim M^{(n-m,m)} - \dim \ker \Phi_m \\ &= \dim M^{(n-m,m)} - \dim S^{(n-m,m)}. \end{aligned}$$

By Theorem 2.2.6, we have $\dim S^{(n-m,m)} = \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}$. Thus

$$\begin{aligned} \dim \Phi_m\big(M^{(n-m,m)}\big) &= \begin{bmatrix} n \\ m \end{bmatrix} - \left(\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix} \right) \\ &= \begin{bmatrix} n \\ m-1 \end{bmatrix} = \dim M^{(n-m+1,m-1)}. \end{aligned}$$

Obviously, $\Phi_m\big(M^{(n-m,m)}\big) \subset M^{(n-m+1,m-1)}$, hence

$$\Phi_m\big(M^{(n-m,m)}\big) = M^{(n-m+1,m-1)},$$

that is Φ_m is an epimorphism. □

4.1.42 Corollary. Let $\lambda = (n-m, m) \vdash n, \mu = (n-m+1, m-1) \vdash n$ and $\mathcal{O}$ be an orbit in M^λ with the condition set $S = S(\mathcal{O})$. If $\mathrm{char}(F) = 0$ then

$$\Phi_m(\mathfrak{C}_S^\lambda) = \mathfrak{C}_S^\mu.$$

Proof. It is a easy consequence of Corollary 4.1.38 and Corollary 4.1.41 . □

4.1.43 Definition. Let $\lambda = (n-m, m) \vdash n, \mu = (n-m+1, m-1) \vdash n$. And let S be a condition set. Define:

$$\Phi_{m,S} : \mathfrak{C}_S^\lambda \to \mathfrak{C}_S^\mu$$

such that

$$\Phi_m = \bigoplus_S \Phi_{m,S}.$$

We give the following examples to explain Theorem 4.1.24 and Corollary 4.1.38.

4.1.44 Example. Let $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{42} & l_{43} & 1 \end{pmatrix} \in \Xi_{2,4}$. Then

$$e_{_L} = \frac{1}{q^2} \sum_{m,n \in GF(q)} \theta(-l_{42} \cdot m)\theta(-l_{43} \cdot n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m & n & 1 \end{pmatrix}.$$

(1) $S = \emptyset, L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$$\begin{aligned} \Phi_2(e_{_{L_0}}) &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{q^2} \sum_{a,m,n \in GF(q)} \begin{pmatrix} a & m & n & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + q \cdot \Big(\frac{1}{q^3} \sum_{a,m,n \in GF(q)} \begin{pmatrix} a & m & n & 1 \end{pmatrix} \Big) \\ &= e_{_{L_1}} + q \cdot e_{_{L_2}}, \end{aligned}$$

where $L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. Let

$$M_{L_0} = e_{_{L_0}} F U_4 = F e_{_{L_0}}.$$

Then

$$\Phi_2(M_{L_0}) = F - \{e_{_{L_1}} + q \cdot e_{_{L_2}}\} \subset \mathfrak{C}_S^{(3,1)}.$$

(2) $S = \{l_{43} \neq 0\}, L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & l_{43} & 1 \end{pmatrix}$.

$$\begin{aligned} \Phi_2(e_{_{L_0}}) &= \frac{1}{q^2} \sum_{a,m,n \in GF(q)} \theta(-l_{43} \cdot n) \begin{pmatrix} a & m & n & 1 \end{pmatrix} \\ &= q \cdot \Big(\frac{1}{q^3} \sum_{a,m,n \in GF(q)} \theta(-l_{43} \cdot n) \begin{pmatrix} a & m & n & 1 \end{pmatrix} \Big) \\ &= q \cdot e_{_{L_1}}, \end{aligned}$$

where $L_1 = \begin{pmatrix} 0 & 0 & l_{43} & 1 \end{pmatrix}$. Let

$$M_{L_0} = e_{_{L_0}} F U_4 = F e_{_{L_0}}.$$

Then

$$\Phi_2(M_{L_0}) = F e_{_{L_1}} = \mathfrak{C}_S^{(3,1)}.$$

(3) $S = \{l_{42} \neq 0\}, L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{42} & 0 & 1 \end{pmatrix}$.

$$\begin{aligned} \Phi_2(e_{_{L_0}}) &= \frac{1}{q^2} \sum_{a,m,n \in GF(q)} \theta(-l_{42} \cdot m) \begin{pmatrix} a & m & n & 1 \end{pmatrix} \\ &= q \cdot \Big(\frac{1}{q^3} \sum_{a,m,n \in GF(q)} \theta(-l_{42} \cdot m) \begin{pmatrix} a & m & n & 1 \end{pmatrix} \Big) \\ &= q \cdot e_{_{L_1}}, \end{aligned}$$

where $L_1 = \begin{pmatrix} 0 & l_{42} & 0 & 1 \end{pmatrix}$. Let

$$M_{L_0} = e_{L_0} F U_4, M_{L_1} = e_{L_1} F U_4.$$

Then from Proposition 3.3.3, we have

$$M_{L_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{42} & * & 1 \end{pmatrix}, \quad M_{L_1} = \begin{pmatrix} 0 & l_{42} & * & 1 \end{pmatrix}$$

and $\Phi_2(M_{L_0}) = M_{L_1} = \mathfrak{C}_S^{(3,1)}$.

4.1.45 Example. Let $L = \begin{pmatrix} l_{21} & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \Xi_{3,6}$. Then

$$e_L = \frac{1}{q^3} \sum_{m,n,p \in GF(q)} \theta(-l_{21} \cdot m)\theta(-l_{31} \cdot n)\theta(-l_{41} \cdot p) \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \\ p & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(1) $S = \emptyset, L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

$$\begin{aligned} \phi_3(e_{L_0}) &= \frac{1}{q^3} \sum_{m,n,p \in GF(q)} \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,m,n,p \in GF(q)} \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ p & 0 & a & 1 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,b,m,n,p \in GF(q)} \begin{pmatrix} n & a & 1 & 0 & 0 & 0 \\ p & b & 0 & 1 & 0 & 0 \end{pmatrix} \\ &= e_{L_1} + q \cdot e_{L_2} + q^2 \cdot e_{L_3}. \end{aligned}$$

where

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $M_{L_0} = e_{L_0} F U_6 = F e_{L_0}$ then

$$\Phi_3(M_{L_0}) = F - \{e_{L_1} + q \cdot e_{L_2} + q^2 \cdot e_{L_3}\} \subset \mathfrak{C}_S^{(4,2)}.$$

(2) $S = \{l_{21} \neq 0\}, L_0 = \begin{pmatrix} l_{21} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$

$$\begin{aligned} \phi_3(e_{L_0}) &= \frac{1}{q^3} \sum_{m,n,p \in GF(q)} \theta(-l_{21} \cdot m) \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,b,m,n,p \in GF(q)} \theta(-l_{21} \cdot m) \begin{pmatrix} n & a & 1 & 0 & 0 & 0 \\ p & b & 0 & 1 & 0 & 0 \end{pmatrix} \\ &= e_{L_1} + q \cdot e_{L_2}. \end{aligned}$$

where

$$L_1 = \begin{pmatrix} l_{21} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} l_{21} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $M_{L_0} = e_{L_0} F U_6 = F e_{L_0}$ then

$$\Phi_3(M_{L_0}) = F - \{e_{L_1} + q \cdot e_{L_2}\} \subset \mathfrak{C}_S^{(4,2)}.$$

(3) $S = \{l_{31} \neq 0\}, L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$

$$\begin{aligned} \phi_3(e_{L_0}) &= \frac{1}{q^3} \sum_{m,n,p \in GF(q)} \theta(-l_{31} \cdot n) \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,b,m,n,p \in GF(q)} \theta(-l_{31} \cdot n) \begin{pmatrix} n + am & a & 1 & 0 & 0 & 0 \\ p & b & 0 & 1 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{q^3} \sum_{m,n,p \in GF(q)} \theta(-l_{31} \cdot n) \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,b,m,n,p \in GF(q)} \theta(-l_{31} \cdot (n - am)) \begin{pmatrix} n & a & 1 & 0 & 0 & 0 \\ p & b & 0 & 1 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{q^3} \sum_{m,n,p \in GF(q)} \theta(-l_{31} \cdot n) \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ n & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,b,m,n,p \in GF(q)} \theta(-l_{31} \cdot n) \theta(l_{31} \cdot m \cdot a) \begin{pmatrix} n & a & 1 & 0 & 0 & 0 \\ p & b & 0 & 1 & 0 & 0 \end{pmatrix} \\ &= e_{L_1} + \sum_m e_{R_m}. \end{aligned}$$

where

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, R_m = \begin{pmatrix} l_{31} & m & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $M_{L_0} = e_{L_0} FU_6, M_{L_1} = e_{L_1} FU_6, M_{L_2} = (\sum_m e_{R_m}) FU_6$. Then from Proposition 3.3.3 and Theorem 3.6.8, we have :

$$\begin{aligned} M_{L_0} &= \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\ M_{L_1} &= \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ l_{31} & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ M_{L_2} &= \begin{pmatrix} l_{31} & * & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\Phi_3(M_{L_0}) \subset M_{L_1} \oplus M_{L_2} \subset \mathfrak{C}_S^{(4,2)}.$$

(4) $S = \{l_{41} \neq 0\}, L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$

$$\begin{aligned} \phi_3(e_{L_0}) &= \frac{1}{q^3} \sum_{a,b,m,n,p \in GF(q)} \theta(-l_{41} \cdot p) \begin{pmatrix} n & a & 1 & 0 & 0 & 0 \\ p+bm & b & 0 & 1 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{q^3} \sum_{a,m,n,p \in GF(q)} \theta(-l_{41} \cdot p) \begin{pmatrix} m & 1 & 0 & 0 & 0 & 0 \\ p+an & 0 & a & 1 & 0 & 0 \end{pmatrix} \\ &= q \cdot \sum_m e_{L_m} + \sum_n e_{R_n} \end{aligned}$$

where

$$L_m = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ l_{41} & m & 0 & 1 & 0 & 0 \end{pmatrix}, R_n = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ l_{41} & 0 & n & 1 & 0 & 0 \end{pmatrix}.$$

Let $M_{L_0} = e_{L_0} FU_6, M_{L_1} = (\sum_m e_{L_m}) FU_6, M_{L_2} = (\sum_n e_{R_n}) FU_6$. Then from Proposition 3.3.3 Remark 3.4.16 and Theorem 3.6.8, we have :

$$\begin{aligned} M_{L_0} &= \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\ M_{L_1} &= \begin{pmatrix} * & 0+\square & 1 & 0 & 0 & 0 \\ l_{41} & * & 1 & 0 & 0 & 0 \end{pmatrix}, \\ M_{L_2} &= \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ l_{41} & 0 & * & 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\Phi_3(M_{L_0}) \subset M_{L_1} \oplus M_{L_2} \subset \mathfrak{C}_S^{(4,2)}.$$

4.1.46 Theorem. Let λ, μ be two-part partitions of n. Let $\mathcal{O}$ (resp. O') be an orbit in M^λ (resp. M^μ). If the condition set of this two orbits are the same, then the corresponding irreducible modules are isomorphic. More precisely,

$$S(\mathcal{O}) = S(O') \quad \text{implies} \quad M_{\mathcal{O}} \cong M_{\mathcal{O}'}.$$

Proof. Let $m = [\frac{n}{2}]$. With respect of the dominance order $\trianglerighteq$ of partitions, we have

$$(n-m, m) \trianglerighteq (n-m+1, m-1) \trianglerighteq \cdots \trianglerighteq (n, 0)$$

Let S be a condition set of some orbit in $M^{n-m,m}$ and let $s = |S|$. It is obvious that this condition set only fits the following partitions:

$$(n-m, m) \trianglerighteq (n-m+1, m-1) \trianglerighteq \cdots \trianglerighteq (n-s, s).$$

Moreover this condition set S only fits one $\mathfrak{t}-$batch $\mathfrak{M}_{\mathfrak{t}}$ of $M^{(n-s,s)}$ since the elements in the set $S_I = \{i \,|\, (i,j) \in \overline{S} \text{ for some } 1 \leqslant j \leqslant n\}$ should be in the second row $\underline{\mathfrak{t}}$ of $\mathfrak{t}$ but $|S_I| = s$ hence these are all elements in $\underline{\mathfrak{t}}$, which leads to $\mathfrak{t}$ is fixed. Thus, we obtain

$$\mathfrak{C}_S^{(n-s,s)} = M_{\mathcal{O}}$$

where $\mathcal{O}$ is the unique orbit in $\mathfrak{M}_{\mathfrak{t}}$ such that $S(\mathcal{O}) = S$.

Suppose O' is an arbitrary orbit in $M^{n-s-1,s+1}$ such that $S(\mathcal{O}') = S$. Then by (4.1.39), we get

$$\Phi_{s+1}(M_{\mathcal{O}'}) \subseteq \mathfrak{C}_S^{(n-s,s)} = M_{\mathcal{O}}$$

where $M_{\mathcal{O}'}$ is the orbit module corresponding to $\mathcal{O}'$.

By Theorem 3.6.8, $M_{\mathcal{O}}$ is an irreducible FU_n-module and obviously $\Phi_m(M_{\mathcal{O}'}) \neq 0$ hence we obtain

$$\Phi_{s+1}(M_{\mathcal{O}'}) = M_{\mathcal{O}}.$$

Since $M_{\mathcal{O}'}$ is an irreducible module, the image of it is either zero or a copy of itself. Hence

$$M_{\mathcal{O}'} \cong M_{\mathcal{O}}.$$

Suppose $\mathcal{O}''$ is an arbitrary orbit in $M^{(n-i-1,i+1)}$ such that $S(\mathcal{O}'') = S$, for some $i \geqslant s+1$. Then by Corollary 4.1.38, we get

$$\Phi_{i+1}(M_{\mathcal{O}''}) \subseteq \mathfrak{C}_S^{(n-i,\,i)} = \bigoplus M_{\tilde{\mathcal{O}}}$$

where $M_{\tilde{\mathcal{O}}} \subseteq M^{(n-i,i)}$ and $S(\tilde{\mathcal{O}}) = S$. Assume $M_{\tilde{\mathcal{O}}} \cong M_{\mathcal{O}}$ for all $\tilde{\mathcal{O}} \subset M^{(n-i,i)}$ such that $S(\tilde{\mathcal{O}}) = S$. Since $M_{\mathcal{O}''}$ is an irreducible FU_n-module by Theorem 3.6.8 and $\Phi_{i+1}(M_{\mathcal{O}''}) \neq 0$, we obtain

$$M_{\mathcal{O}''} \cong \Phi_{i+1}(M_{\mathcal{O}''}) \subseteq \bigoplus M_{\tilde{\mathcal{O}}} = \bigoplus M_{\mathcal{O}}.$$

Since $M_{\mathcal{O}}$ is irreducible, we have $M_{\mathcal{O}''} \cong M_{\mathcal{O}}$.

$\square$

Note that we only show in this Theorem that orbit modules are isomorphic if their condition sets are the same but we did not show whether two orbit modules are isomorphic or not if they have the same size but different condition sets.

4.2 Special orbits in $M^{(n-m,m)}$

In this section we investigate two special orbits in $M^{(n-m,m)}$ which can easily give us some elements in $S^{(n-m,m)}$.

The first one concerns special orbits in M^λ for $\lambda = (n-m, m)$, namely those, whose condition sets are full in the sense that their size is m.

4.2.1 Proposition. Let $\lambda = (n-m, m)$. Let $\mathcal{O}$ be an orbit in the $\mathfrak{t}-$batch $\mathfrak{M}_\mathfrak{t}$ of M^λ. Let $S = S(\mathcal{O})$ be the condition set of $\mathcal{O}$. If $|S| = m$, then

$$M_\mathcal{O} \subset \ker \Phi_m \quad \text{ and } \mathfrak{t} \in \mathrm{Std}(\lambda).$$

More precisely, for any $e_L \in \mathcal{O}$ we have

$$\Phi_m(e_L) = 0 \text{ and } \mathrm{tab}(L) \in \mathrm{Std}(\lambda).$$

Proof. By (4.1.8) and Corollary 4.1.29, for any $e_L \in \mathcal{O}$ we have:

$$\Phi_m^d(e_L) = \bigoplus_{d \notin S_I} \Phi_m^d(e_L).$$

But in our case, for any $1 \leqslant d \leqslant m$, we have $d \in S_I$ since $|S| = m$. Hence

$$\Phi_m^d(e_L) = 0 \quad \text{ for all } e_L \in \mathcal{O}.$$

Now we can assume L is a pattern matrix. Assume

$$\mathrm{tab}(L) = \begin{array}{|c|c|c|c|c|c|} \hline a_1 & a_2 & \cdots & a_m & \cdots & a_{n-m} \\ \hline b_1 & b_2 & \cdots & b_m & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array}.$$

In order to make $S(L) = S$, we need to have $i-$many columns $a_1, \ldots, a_i$ before column b_i, hence $a_i < b_i$ and $\mathrm{tab}(L)$ is a standard $\lambda-$tableau. More precisely, if $\mathrm{tab}(L) \in \mathrm{RStd}(\lambda)$ is nonstandard, then we assume $a_i < b_i$ for $1 \leqslant i \leqslant r$ and $a_r > b_r$. Then we can only choose $1 \leqslant j < r$ such that $(b_r, a_j) \in \overline{S}$. Hence there should be some $(b_s, a_t) \in \overline{S}$ for $s < r$ and $t \geqslant r$. Thus we have $b_r > b_s > a_t \geqslant a_r$ which is a contradiction of our assumption. Hence we must have $\mathrm{tab}(L) \in \mathrm{Std}(\lambda)$.

$\square$

From the definition of those series of homomorphisms $\phi_{1,i}$ for $0 \leqslant i \leqslant m-2$, we know the orbit modules with full condition sets also live in $\ker \phi_{1,i}$ for all $0 \leqslant i \leqslant m-2$, then they are in the Specht module $S^{(n-m,m)}$ for any arbitrary field.

Note that the result $\mathrm{tab}(L) \in \mathrm{Std}(\lambda)$ in Proposition 4.2.1 coincides with an important result by Sinéad Lyle, which we will use very often in the later sections. First we introduce an order which was used in Lyle's theorem:

4.2.2 Definition. Let $\lambda = (n-m, m)$. Define a partial order $\trianglelefteq$ on $\mathrm{RStd}(\lambda)$ by

$$\begin{array}{|c|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_m & \cdots & a_{n-m} \\ \hline b_1 & b_2 & \cdots & b_m \\ \cline{1-4}\end{array} \trianglelefteq \begin{array}{|c|c|c|c|c|c|}\hline a'_1 & a'_2 & \cdots & a'_m & \cdots & a'_{n-m} \\ \hline b'_1 & b'_2 & \cdots & b'_m \\ \cline{1-4}\end{array}$$

if and only if

$$b_i \leqslant b'_i \text{ for all } 1 \leqslant i \leqslant m.$$

4.2.3 Theorem. Suppose that $0 \neq v \in S^{(n-m,m)}$, and write

$$v = \sum_{X \in \Xi_{m,n}} C_X X \text{ where } C_X \in F.$$

Say that X occurs in v if $C_X \neq 0$. Suppose that X' occurs in v and is such that for every X with $X \neq X'$ and $\mathrm{tab}(X') \trianglelefteq \mathrm{tab}(X)$ we have that X does not occur in v . Then the tableau $\mathrm{tab}(X')$ is standard.

Proof. [15] □

Our work relies heavily on the following corollary of the theorem above:

4.2.4 Corollary. Suppose that $0 \neq v \in S^{(n-m,m)}$. Then $\mathrm{last}(v)$ is standard.

Proof. Write

$$v = \sum_{X \in \Xi_{m,n}} C_X X \text{ where } C_X \in F.$$

Suppose $\mathrm{tab}(X') = \mathrm{last}(v)$, then obviously for any $X \neq X'$ and $\mathrm{tab}(X') \trianglelefteq \mathrm{tab}(X)$ we have that X does not occur in v. (Otherwise, $\mathrm{tab}(X') \trianglelefteq \mathrm{tab}(X) \Rightarrow \mathrm{tab}(X') < \mathrm{tab}(X)$, then X' can not be the last tableau of v.) By Theorem 4.2.3, $\mathrm{last}(v) = \mathrm{tab}(X')$ is standard .

□

Now we investigate the second kind of orbits having empty condition set. First we prove an easy lemma which will be very useful later on.

4.2.5 Lemma. For $\lambda = (n-m, m) \vdash n, \mu = (n-m+1, m-1) \vdash n$, note

$$P_m = \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda), \quad Q_m = \mathrm{RStd}(\mu),$$

then $|P_m| = |Q_m|$.

Proof. Note that

$$|\,\mathrm{RStd}(\lambda)| = \binom{n}{m}; \quad |\,\mathrm{RStd}(\mu)| = \binom{n}{m-1};$$

$$|\,\mathrm{Std}(\lambda)| = \binom{n}{m} - \binom{n}{m-1},$$

hence the statement holds. □

Recall that for $v = \sum_{X \in \Xi_{m,n}} C_X X$, $\mathrm{top}(v)$ is the collection of all X occurring in this sum with $\mathrm{tab}(X)$ being the last tableau respect to the lexicographic order.

4.2.6 Proposition. Let $\lambda = (n-m, m)$, $\mathfrak{t} \in \mathrm{Std}(\lambda)$. Suppose $M_\emptyset = Fe_{_L}$ is the unique trivial orbit in the $\mathfrak{t}-$batch $\mathfrak{M}_\mathfrak{t}$. If char $F = 0$ then there exist $v \in S^\lambda$ such that $\mathrm{top}(v) = e_{_L}$.

Proof. We know each batch of M^λ has a unique orbit. For each $\mathfrak{s}-$batch we denote the basis element in the empty orbit by $L^\mathfrak{s}_\emptyset$. We claim the set

$$R := \{\Phi_m(e_{_{L^\mathfrak{s}_\emptyset}}) \,|\, \mathfrak{s} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda)\}$$

is linearly independent. In fact if we have a linear combination

$$\sum_\mathfrak{s} a_\mathfrak{s} \Phi_m(e_{_{L^\mathfrak{s}_\emptyset}}) = 0$$

then

$$\Phi_m(\sum_\mathfrak{s} a_\mathfrak{s} e_{_{L^\mathfrak{s}_\emptyset}}) = 0$$

hence

$$\sum_\mathfrak{s} a_\mathfrak{s} e_{_{L^\mathfrak{s}_\emptyset}} \in \ker \Phi_m.$$

Since we are in the characteristic zero case, $\ker \Phi_m = S^\lambda$. Hence

$$\sum_\mathfrak{s} a_\mathfrak{s} e_{_{L^\mathfrak{s}_\emptyset}} \in S^\lambda \text{ but } \mathrm{tab}(L^\mathfrak{s}_\emptyset) = \mathfrak{s} \text{ is nonstandard for all } L^\mathfrak{s}_\emptyset.$$

By Corollary 4.2.4 we must have $a_{\mathfrak{s}} = 0$ for all $\mathfrak{s} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda)$.

Since $\Phi_m(\mathfrak{C}^{\lambda}_{\emptyset}) \subset \mathfrak{C}^{\mu}_{\emptyset}$, we get $R \subset \mathfrak{C}^{\mu}_{\emptyset}$. Moreover by Lemma 4.2.5, we know

$$\dim R = P_m = Q_m = \dim \mathfrak{C}^{\mu}_{\emptyset}.$$

Thus we obtain

$$FR = \mathfrak{C}^{\mu}_{\emptyset}.$$

Now suppose $\mathfrak{t} \in \mathrm{Std}(\lambda)$, then by Corollary 4.1.38

$$\Phi_m(e_{L^{\mathfrak{t}}_{\emptyset}}) \subset \mathfrak{C}^{\mu}_{\emptyset} = FR$$

thus

$$\Phi_m(e_{L^{\mathfrak{t}}_{\emptyset}}) = \sum_{\mathfrak{s}} a_{\mathfrak{s}} \Phi_m(e_{L^{\mathfrak{s}}_{\emptyset}})$$

where $\mathfrak{s} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda)$, $a_{\mathfrak{s}} \in F$. Let

$$v = e_{L^{\mathfrak{t}}_{\emptyset}} - \sum_{\mathfrak{s}} a_{\mathfrak{s}} e_{L^{\mathfrak{s}}_{\emptyset}}.$$

Then $v \in \ker \Phi_m$. Since we are in the characteristic zero case, $\ker \Phi_m = S^{\lambda}$. Hence $v \in S^{\lambda}$ with $\mathrm{last}(v) = \mathfrak{t}$ since by Corollary we know $\mathrm{last}(v)$ must be standard. Thus we have $\mathrm{top}(v) = e_{L^{\mathfrak{t}}_{\emptyset}}$. □

4.3 Standard basis of $S^{(n-m,m)}$

In this section we first find a basis for the unipotent Specht module $S^{(n-m,m)}$ over a field with characteristic zero and [4] shows that this is an integral basis for any arbitrary field.

In the previous section, the second kind of special elements in $S^{(n-m,m)}$ gives us an idea of reducing nonempty condition set case to the simple case that the condition set is empty. In this sense, we define the following map $\mathfrak{R}_S$ where S is a condition set. This map removes every row and column related to the condition set S. More precisely:

4.3.1 Definition. Let $L \in \Xi_{m,n}$. Suppose $e_L \in \mathcal{O}$ with condition set $S = S(\mathcal{O})$. We define an $F-$linear map $\mathfrak{R}_S$ extended by linearity to M^{λ}. The map:

$$\mathfrak{R}_S : \Xi_{m,n} \longrightarrow \Xi_{m-s,\, n-2s}$$

is given by

$$\mathfrak{R}_S(L) = \tilde{L}.$$

where $\tilde{L} \in \Xi_{m-s,n-2s}$ is the $(m-s,\, n-2s)-$matrix obtained from L by removing rows b_i such that $b_i \in S_I$ (that are rows which contain a condition) and the columns with $b_i \in S_I$ and the columns $j \in S_J$(that are columns which contain a condition).

4.3.2 Remark. Note that if L is a pattern matrix, then $\tilde{L} = \mathfrak{R}_S(L)$ is a pattern matrix with empty condition set.

4.3.3 Example. Let

$$L = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{52} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_{86} & 0 & 1 \end{pmatrix} \begin{matrix} 3 \\ 5 \\ 7 \\ 8 \end{matrix}$$

be a pattern matrix with condition set $S = \{l_{52} \neq 0, l_{86} \neq 0\}$. Then

$$\tilde{L} = \mathfrak{R}_S(L) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Xi_{2,4}, \quad S(\tilde{L}) = \emptyset;$$

$$\mathrm{tab}(L) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 & 8 \\ \hline \end{array}, \quad \mathrm{tab}(\tilde{L}) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

4.3.4 Remark. By the definition of $\mathfrak{R}_S$, we can easily get $\mathrm{tab}(\tilde{L})$ from $\mathrm{tab}(L)$ in the following way:

First delete the numbers i for all $i \in S_I \cup S_J$ in $\mathrm{tab}(L)$; then we omit the gaps. Thus we get a tableau $\tilde{\mathfrak{t}}$ of shape $\mu = (n-m-s, m-s)$ filled by numbers in the set

$$\{1, 2, ..., n\} \setminus (S_I \cup S_J),$$

denoted by

$$\tilde{\mathfrak{t}} = \mathrm{tab}(L) \setminus (S_I \cup S_J),$$

called **shifted $\mu-$tableau**.

Assume

$$\{1, 2, ..., n\} \setminus (S_I \cup S_J) = \{a_1, a_2, ..., a_{n-2s}\}$$

with order

$$a_1 < a_2 < ... < a_{n-2s}.$$

Then we replace the numbers a_i in $\tilde{\mathfrak{t}}$ by i instead, and we get a $\mu-$tableau $\mathfrak{s}$ filled by numbers $1, 2, \ldots, n-2s$ with $\mathfrak{s} = \mathrm{tab}(\tilde{L})$.

Obviously $\mathfrak{s}$ and $\tilde{\mathfrak{t}}$ are 1-1 correspondence if we fixed the frame $\overline{S}$ of the condition set. We say $\mathfrak{s}$ and $\tilde{\mathfrak{t}}$ are $\overline{S}$**-similar**, denoted by

$$\mathfrak{s} \overset{\overline{S}}{\sim} \tilde{\mathfrak{t}}.$$

Of course, $\mathfrak{s}$ is standard if and only if $\tilde{\mathfrak{t}}$ is standard.

4.3.5 Example. In Example 4.3.3,

$$\mathfrak{t} = \mathrm{tab}(L) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 & 8 \\ \hline \end{array};$$

$$S_I \cup S_J = \{2, 5, 6, 8\};$$

$$\{1, 2, ..., 8\} \setminus (S_I \cup S_J) = \{1, 3, 4, 7\}.$$

After deleting and omitting the gaps, we get a shifted tableau

$$\tilde{\mathfrak{t}} = \mathfrak{t} \setminus (S_I \cup S_J) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 7 \\ \hline \end{array}.$$

Now we denote

$$a_1 = 1, a_2 = 3, a_3 = 4, a_4 = 7.$$

Rewrite

$$\tilde{\mathfrak{t}} = \begin{array}{|c|c|}\hline a_1 & a_3 \\ \hline a_2 & a_4 \\ \hline \end{array}.$$

Replacing a_i by index i for $i = 1, \ldots, n - 2s$, we obtain a tableau

$$\mathfrak{s} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array},$$

the same as $\mathrm{tab}(\tilde{L})$ in Example 4.3.3. So $\mathfrak{s} \overset{\overline{S}}{\sim} \tilde{\mathfrak{t}}$, more precisely:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 7 \\ \hline \end{array} \overset{\overline{S}}{\sim} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

4.3.6 Definition. Let S be a condition set with frame $\overline{S}$. Let $\lambda = (n - m, m)$, $\nu = (n - m - |S|, m - |S|)$. Denote $T^{\lambda}_{\overline{S}}$ as the set of row-standard but non-standard shifted $\nu-$tableaux, which are filled by numbers in $\{1, 2, ..., n\} \setminus (S_I \cup S_J)$. In particular, if $S = \emptyset$, then $T^{\lambda}_{\emptyset}$ is the set of row-standard but non-standard tableaux of shape λ.

4.3.7 Example. Let $\lambda = (3,3), \overline{S} = \{(6,4)\}$, hence $S_I \cup S_J = \{4,6\}$ and

$$T_{\overline{S}}^{\lambda} = \left\{ \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline \end{array} \right\}.$$

4.3.8 Corollary. Let $\lambda = (n-m, m)$, S be a condition set. Then

$$T_{\overline{S}}^{\lambda} = |\operatorname{RStd}(\mu)| \text{ where } \mu = (n-m-s+1, m-s-1).$$

Proof. It is a easy consequence of Lemma 4.2.5. □

4.3.9 Lemma. Let S be a condition set, $s = |S|, 0 \leqslant s \leqslant m-1$. Let $\lambda = (n-m, m)$. If

$$\{\Phi_m(e_{_L}) \,|\, e_{_L} \in \mathfrak{C}_S^{\lambda}\}$$

is a linearly dependent set, then

$$\{\Phi_{m-s}(e_{_{\mathfrak{R}_S(L)}}) \,|\, e_{_L} \in \mathfrak{C}_S^{\lambda}\}$$

is a linearly dependent set.

Proof. Let $\lambda = (n-m, m)$. And let $e_{_{L_1}}, \ldots, e_{_{L_k}} \in \mathfrak{C}_S^{\lambda}$ where L_r are pairwise different for $r = 1, \ldots, k$. Denote $\operatorname{tab}(L_r) = \mathfrak{t}_r$ for $r = 1, \ldots, k$. Suppose we have

$$\sum_{r=1}^{k} \gamma_r \, \Phi_m(e_{_{L_r}}) = 0$$

with at least one $r \in \{1, \ldots, k\}$ such that $\gamma_r \neq 0$.

We may assume L_1 is a pattern matrix with $\gamma_1 \neq 0$. Otherwise we find $u \in U^w \cap U$ where $\mathfrak{t}_1 = \mathfrak{t}^{\lambda} w$ such that

$$e_{_{L_1}} \circ u = e_{_{L_1'}} \in \mathfrak{C}_S^{\lambda}$$

where L_1' is a pattern matrix and

$$\sum_{r=1}^{k} \gamma_r \, \Phi_m(e_{_{L_r}} \circ u) = 0.$$

Hence we obtain

$$\gamma_1 \, \Phi_m(e_{_{L_1'}}) + \sum \rho_K \, \Phi_m(e_{_K}) = 0 \tag{4.3.10}$$

where $K \neq L_1', \rho_K \in F^*$ and $e_{_K} \in \mathfrak{C}_S^\lambda$.

So let L_1 be a pattern matrix with $\gamma_1 \neq 0$. In order to keep notation simple, we denote

$$e_r = e_{_{L_r}} \text{ for } r = 1, \ldots, k$$

and

$$\tilde{e}_r = e_{_{\tilde{L}_r}} \text{ for } r = 1, \ldots, k \tag{4.3.11}$$

where

$$\tilde{L}_r = \mathfrak{R}_S(L_r) \text{ for } r = 1, \ldots, k$$

and the corresponding shifted tableau

$$\tilde{\mathfrak{t}}_r = \mathfrak{t}_r \setminus S_I \cup S_J \text{ for } r = 1, \ldots, k.$$

Thus $\tilde{\mathfrak{t}}_r$ is filled by numbers in $\{1, \ldots, n\} \setminus S_I \cup S_J$ for $r = 1, \ldots, k$.

For $r \in \{1, \ldots, k\}$, by Definition 3.2.5 we have:

$$\begin{aligned} e_r &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}_{\mathfrak{t}_r}} \chi_{_{L_r}}(-M_r)[M_r] \\ &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}_{\mathfrak{t}_r}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}_r}} \theta(-l^r_{_{b_i j}} m^r_{_{b_i j}})[M_r] \end{aligned}$$

where $L_r = (l^r_{_{b_i j}})$ and $M_r = (m^r_{_{b_i j}}) \in \mathfrak{X}_{\mathfrak{t}_r}$. Hence

$$\Phi^d_m(e_r) = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}_{\mathfrak{t}_r}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}_r}} \theta(-l^r_{_{b_i j}} m^r_{_{b_i j}}) \Phi^d_m([M_r]). \tag{4.3.12}$$

Using similar notation as in (4.1.13) we may write

$$\Phi^d_m([M_r]) = \sum_{\underline{\alpha} \in GF(q)^{m-d}} [N_{\underline{\alpha}}(M_r)] \tag{4.3.13}$$

where $\underline{\alpha} = (\alpha_{_{d+1}}, \ldots, \alpha_{_m})$. If we denote

$$N_{\underline{\alpha}}(M_r) = (n^r_{_{b_i j}}) \in \mathfrak{X}_{\mathfrak{u}^r_d}$$

where $\mathfrak{u}^r_d$ is a $\mu-$tableau $\big(\mu = (n-m+1, m-1)\big)$ obtained from $\mathfrak{t}_r$ by moving the number b_d to the first row at the appropriate place to make the resulting tableau row-standard, then from (4.1.11) we have:

$$n^r_{_{b_i j}} = \begin{cases} m^r_{_{b_i j}} & \text{if } i \leqslant d-1 \text{ or } j > b_d; \\ \alpha_i \in GF(q) & \text{if } d+1 \leqslant i \leqslant m, j = b_d; \\ m^r_{_{b_i j}} + \alpha_i m^r_{_{b_d j}} & \text{if } d+1 \leqslant i \leqslant m, j < b_d. \end{cases} \tag{4.3.14}$$

We split the summation in (4.3.13) as follows:

$$\Phi_m^d([M_r]) = \sum_{\substack{\alpha_i \in GF(q) \\ d+1 \leqslant i \leqslant m \\ b_i \in S_I}} \sum_{\substack{\alpha_u \in GF(q) \\ d+1 \leqslant u \leqslant m \\ b_u \notin S_I}} [N_{\underline{\alpha}}(M_r)]. \tag{4.3.15}$$

where $\underline{\alpha} = (\alpha_{d+1}, \ldots, \alpha_m) \in GF(q)^{m-d}$.

For fixed $\alpha_i \in GF(q)$ for $d+1 \leqslant i \leqslant m$ with $b_i \in S_I$, let

$$\overline{N_{\underline{\check{\alpha}}}}(M_r) = \sum_{\substack{\alpha_u \in GF(q) \\ d+1 \leqslant u \leqslant m \\ b_u \notin S_I}} [N_{\underline{\alpha}}(M_r)] \tag{4.3.16}$$

where $\check{\alpha} = (\alpha_{i_1}, \ldots, \alpha_{i_h})$ with $b_{i_1}, \ldots, b_{i_h} \in S_I$ for some $0 \leqslant h \leqslant m-d$. Using (4.3.16) we rewrite (4.3.15):

$$\Phi_m^d([M_r]) = \sum_{\check{\alpha} \in GF(q)^h} \overline{N_{\underline{\check{\alpha}}}}(M_r) \tag{4.3.17}$$

where $\check{\alpha} = (\alpha_{i_1}, \ldots, \alpha_{i_h})$ with $b_{i_1}, \ldots, b_{i_h} \in S_I$ for some $0 \leqslant h \leqslant m-d$.

Note that if all entries of b_i-th row for all $b_i \in S_I$ in $N_{\underline{\alpha}}$ are zeros except the last 1's then $\alpha_i = 0$ for all $d+1 \leqslant i \leqslant m$ such that $b_i \in S_I$ (α_i is the entry at position (b_i, b_d) of $N_{\underline{\alpha}}$) and hence by (4.3.14), we obtain $m^r_{b_i j} = 0$ for $(b_i, j) \in \mathfrak{J}_{\mathfrak{t}}$ and $b_i \in S_I$. In this case N_α is a summand of $\overline{N_{\underline{\check{0}}}}(M_r)$.

Inserting (4.3.17) into (4.3.12), we obtain

$$\begin{aligned} \Phi_m^d(e_r) &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}_{\mathfrak{t}_r}} \prod_{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}_r}} \theta(-l^r_{b_i j} m^r_{b_i j}) \sum_{\check{\alpha} \in GF(q)^h} \overline{N_{\underline{\check{\alpha}}}}(M_r) \\ &= \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}^0_{\mathfrak{t}_r}} \prod_{\substack{(b_i,j) \in \mathfrak{J}_{\mathfrak{t}_r} \\ b_i \notin S_I}} \theta(-l^r_{b_i j} m^r_{b_i j}) \overline{N_{\underline{\check{0}}}}(M_r) + y_r \end{aligned} \tag{4.3.18}$$

where $\mathfrak{X}^0_{\mathfrak{t}_r}$ is the matrices $M \in \mathfrak{X}_{\mathfrak{t}_r}$ such that $m^r_{b_i j} = 0$ for all $(b_i, j) \in \mathfrak{J}_{\mathfrak{t}_r}$ with $b_i \in S_I$ and y_r is a linear combination of matrices in $\mathfrak{X}_{\mathfrak{u}^r_d}$ with at least one nonzero entry at a position $(b_i, j) \in \mathfrak{J}_{\mathfrak{u}^r_d}$ with $b_i \in S_I$; moreover, we used $\theta(-l^r_{b_i j} m^r_{b_i j}) = 1$ for $m^r_{b_i j} = 0$.

Note that

$$\sum_{r=1}^{k} \gamma_r \, \Phi_m(e_r) = 0 \text{ if and only if } \sum_{r=1}^{k} \gamma_r \, \Phi_m^d(e_r) = 0, \, \forall \, d \notin S_I.$$

Hence for a fixed $d \notin S_I$, by (4.3.18) we have

$$\begin{aligned} 0 &= \sum_{r=1}^{k} \gamma_r \Phi_m^d(e_r) \\ &= \sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{I}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}^0_{\mathfrak{t}_r}} \prod_{\substack{(b_i,j)\in\mathfrak{I}_{\mathfrak{t}_r} \\ b_i \notin S_I}} \theta(-l^r_{b_ij} m^r_{b_ij}) \overline{N_{\underline{\check{0}}}}(M_r) + \sum_{r=1}^{k} \gamma_r y_r. \end{aligned}$$

Note that all matrices involved in y_r are linearly independent of those involved in $\overline{N_{\underline{\check{0}}}}(M_r)$ for some $r \in \{1, \ldots, k\}$ since they differ in some row $b_i \in S_I$. Hence we have

$$\sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{I}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}^0_{\mathfrak{t}_r}} \prod_{\substack{(b_i,j)\in\mathfrak{I}_{\mathfrak{t}_r} \\ b_i \notin S_I}} \theta(-l^r_{b_ij} m^r_{b_ij}) \overline{N_{\underline{\check{0}}}}(M_r) = 0. \tag{4.3.19}$$

We calculate $\sum_{r=1}^{k} \gamma_r \, \Phi_m^d(\tilde{e}_r)$. $\big($c.f. (4.3.11)$\big)$. Similarly as (4.3.12) and (4.3.13) we have

$$\begin{aligned} \Phi^d_{m-s}(\tilde{e}_r) &= \frac{1}{q^{|\mathfrak{I}_{\tilde{\mathfrak{t}}_r}|}} \sum_{\tilde{M}_r \in \mathfrak{X}_{\tilde{\mathfrak{t}}_r}} \prod_{(b_i,j)\in\mathfrak{I}_{\tilde{\mathfrak{t}}_r}} \theta(-l^r_{b_ij} \tilde{m}^r_{b_ij}) \Phi^d_{m-s}([\tilde{M}_r]) \\ &= \frac{1}{q^{|\mathfrak{I}_{\tilde{\mathfrak{t}}_r}|}} \sum_{\substack{\tilde{m}^r_{b_ij} \in GF(q) \\ (b_i,j)\in\mathfrak{I}_{\tilde{\mathfrak{t}}_r}}} \prod_{(b_i,j)\in\mathfrak{I}_{\tilde{\mathfrak{t}}_r}} \theta(-l^r_{b_ij} \tilde{m}^r_{b_ij}) \Phi^d_{m-s}([\tilde{M}_r]) \end{aligned} \tag{4.3.20}$$

and

$$\Phi^d_{m-s}([\tilde{M}_r]) = \sum_{\underline{\beta} \in GF(q)^{m-d-h}} [\tilde{N}_{\underline{\beta}}(\tilde{M}_r)] \tag{4.3.21}$$

where $\underline{\beta} = (\beta_{i_1}, \ldots, \beta_{i_{m-d-h}})$ with $d+1 \leqslant i_t \leqslant m$ such that $b_{i_t} \notin S_I$.

Recall from (4.3.16) we have

$$\overline{N_{\underline{\check{0}}}}(M_r) = \sum_{\substack{\alpha_u \in GF(q) \\ d+1 \leqslant u \leqslant m \\ b_u \notin S_I}} [N_{\underline{\alpha}}(M_r)]$$

where $\underline{\alpha} = (\alpha_{d+1}, \ldots, \alpha_m)$ such that $\alpha_i = 0$ for all $d+1 \leqslant i \leqslant m$ with $b_i \in S_I$.

Hence for $\tilde{m}^r_{b_ij} = m^r_{b_ij}$, $\forall\, (b_i, j) \in \mathfrak{J}_{\tilde{\mathfrak{t}}_r}$, we have:

$$\mathfrak{R}_S\big(\overline{N_{\underline{\check{0}}}}(M_r)\big) = \sum_{\underline{\beta} \in GF(q)^{m-d-h}} [\tilde{N}_{\underline{\beta}}(\tilde{M}_r)] = \Phi^d_{m-s}([\tilde{M}_r]). \tag{4.3.22}$$

We act by the $F-$linear map $\mathfrak{R}_S$ on both sides of equation (4.3.19). By (4.3.22) we get:

$$\begin{aligned} 0 &= \sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}^0_{\mathfrak{t}_r}} \prod_{\substack{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_i \notin S_I}} \theta(-l^r_{b_i j} m^r_{b_i j}) \mathfrak{R}_S\big(\overline{N_{\underline{0}}}(M_r)\big) \\ &= \sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}^0_{\mathfrak{t}_r}} \prod_{\substack{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_i \notin S_I}} \theta(-l^r_{b_i j} m^r_{b_i j}) \Phi^d_{m-s}([\tilde{M}_r]). \end{aligned} \tag{4.3.23}$$

where

$$\tilde{m}^r_{b_i j} = m^r_{b_i j},\ \forall\, (b_i, j) \in \mathfrak{J}_{\tilde{\mathfrak{t}}_r}. \tag{4.3.24}$$

We split the product in (4.3.23) along the column indices as the following:

$$0 = \sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \sum_{M_r \in \mathfrak{X}^0_{\mathfrak{t}_r}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_u \notin S_I,\, v\in S_J}} \theta(-l^r_{b_u v} m^r_{b_u v}) \prod_{\substack{(b_i,j)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_i \notin S_I,\, j\notin S_J}} \theta(-l^r_{b_i j} m^r_{b_i j}) \Phi^d_{m-s}([\tilde{M}_r]). \tag{4.3.25}$$

Since $\Phi^d_{m-s}([\tilde{M}_r])$ is independent of $m^r_{b_u v}$ for all $(b_u, v) \in \mathfrak{J}_{\mathfrak{t}_r}$, $b_u \notin S_I$, $v \in S_J$, and

$$\{(b_i, j) \in \mathfrak{J}_{\mathfrak{t}_r},\ |\, b_i \notin S_I,\ j \notin S_J\} = \{(b_i, j) \in \mathfrak{J}_{\tilde{\mathfrak{t}}_r}\},$$

we can separate the summation in (4.3.25) as follows:

$$\begin{aligned} 0 &= \sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_u \notin S_I,\, v\in S_J}} \sum_{m^r_{b_u v}\in GF(q)} \theta(-l^r_{b_u v} m^r_{b_u v}) \\ &\quad \cdot \sum_{\substack{m^r_{b_i j}\in GF(q) \\ (b_i,j)\in\mathfrak{J}_{\tilde{\mathfrak{t}}_r}}} \prod_{(b_i,j)\in\mathfrak{J}_{\tilde{\mathfrak{t}}_r}} \theta(-l^r_{b_i j} m^r_{b_i j}) \Phi^d_{m-s}([\tilde{M}_r]) \\ &\overset{(4.3.24)}{=} \sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_u \notin S_I,\, v\in S_J}} \sum_{m^r_{b_u v}\in GF(q)} \theta(-l^r_{b_u v} m^r_{b_u v}) \\ &\quad \cdot \sum_{\substack{\tilde{m}^r_{b_i j}\in GF(q) \\ (b_i,j)\in\mathfrak{J}_{\tilde{\mathfrak{t}}_r}}} \prod_{(b_i,j)\in\mathfrak{J}_{\tilde{\mathfrak{t}}_r}} \theta(-l^r_{b_i j} \tilde{m}^r_{b_i j}) \Phi^d_{m-s}([\tilde{M}_r]). \end{aligned} \tag{4.3.26}$$

Using (4.3.20), we rewrite (4.3.26):

$$\sum_{r=1}^{k} \frac{\gamma_r}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_u \notin S_I,\, v\in S_J}} \sum_{m^r_{b_u v}\in GF(q)} \theta(-l^r_{b_u v} m^r_{b_u v})\, q^{|\mathfrak{J}_{\tilde{\mathfrak{t}}_r}|}\, \Phi^d_{m-s}(\tilde{e}_r) = 0. \tag{4.3.27}$$

For $1 \leqslant r \leqslant k$, let

$$\delta_r = \gamma_r \frac{q^{|\mathfrak{J}_{\tilde{\mathfrak{t}}_r}|}}{q^{|\mathfrak{J}_{\mathfrak{t}_r}|}} \prod_{\substack{(b_u,v)\in\mathfrak{J}_{\mathfrak{t}_r} \\ b_u\notin S_I,\, v\in S_J}} \sum_{m^r_{b_u v}\in GF(q)} \theta(-l^r_{b_u v} m^r_{b_u v}). \tag{4.3.28}$$

Note that δ_r is independent of d. Using (4.3.28) in (4.3.27) we obtain:

$$\sum_{r=1}^{k} \delta_r \Phi_{m-s}(\tilde{e}_r) = 0. \tag{4.3.29}$$

Since L_1 is a pattern matrix, we have $l^1_{b_u v} = 0$ for all $(b_u, v) \in \mathfrak{J}_{\mathfrak{t}_1}$ with $b_u \notin S_I$, $v \in S_J$. Hence

$$\theta(-l^1_{b_u v} m^1_{b_u v}) = 1 \text{ for all } (b_u, v) \in \mathfrak{J}_{\mathfrak{t}_1} \text{ with } b_u \notin S_I,\ v \in S_J.$$

Therefore

$$\delta_1 = \gamma_1 \frac{q^{|\mathfrak{J}_{\tilde{\mathfrak{t}}_1}|}}{q^{|\mathfrak{J}_{\mathfrak{t}_1}|}} q^c \neq 0$$

where $c = \big|\{(b_u, v) \in \mathfrak{J}_{\mathfrak{t}_1} \,|\, b_u \notin S_I,\ v \in S_J\}\big|$.

Hence by 4.3.29, the set

$$\{\Phi_{m-s}(\tilde{e}_r) \,|\, 1 \leqslant r \leqslant k\}$$

is linearly dependent. This finishes the proof. □

4.3.30 Corollary. If L is a pattern matrix with $S(L) = S$ and

$$\Phi_m(e_L) + \sum_K \gamma_K\, \Phi_m(e_K) = 0$$

where $S(\mathcal{O}_K) = S$, then there exist $\delta_K \in F$ such that

$$\Phi_{m-s}(e_{\mathfrak{R}_S(L)}) + \sum_K \delta_K\, \Phi_{m-s}(e_{\mathfrak{R}_S(K)}) = 0.$$

Proof. It is a easy consequence from the proof Lemma 4.3.9. □

4.3.31 Corollary. Let S be a condition set, $s = |S|, 0 \leqslant s \leqslant m-1$. Let $\lambda = (n-m, m)$. If

$$\{\Phi_m(e_L) \,|\, e_L \in \mathfrak{C}^\lambda_S,\ \mathrm{tab}(L) \setminus (S_I \cup S_J) \in T^\lambda_{\overline{S}}\}$$

is a linearly dependent set, then

$$\{\Phi_{m-s}(e_{\mathfrak{R}_S(L)}) \,|\, e_L \in \mathfrak{C}^\lambda_S,\ \mathrm{tab}(L) \setminus (S_I \cup S_J) \in T^\lambda_{\overline{S}}\}$$

is a linearly dependent set.

Proof. Obviously every K occurring in (4.3.10) has the property:

$$\mathrm{tab}(K) \setminus (S_I \cup S_J) \in T^{\lambda}_{\overline{S}}.$$

Hence the statement holds by Lemma 4.3.9. □

4.3.32 Proposition. Let $\lambda = (n-m, m), \mu = (n-m+1, m-1)$, and S be a condition set, with $0 \leqslant |S| \leqslant m-1$. Suppose

$$\mathfrak{C}^{\mu}_{S} = \bigoplus_{S(\tilde{\mathcal{O}})=S} \bigoplus_{R \in \tilde{\mathcal{O}}} F e_{R}.$$

If $\mathrm{char}(F) = 0$, then

$$\mathfrak{C}^{\mu}_{S} = F\text{-}\mathrm{span}\{\Phi_m(e_L) \mid S(\mathcal{O}_L) = S, \mathrm{tab}(L) \setminus (S_I \cup S_J) \in T^{\lambda}_{\overline{S}}\}$$

as $F-$vector space.

Proof. Let $M_S = \{\Phi_m(e_L) \mid S(\mathcal{O}_L) = S, \mathrm{tab}(L) \setminus (S_I \cup S_J) \in T^{\lambda}_{\overline{S}}\}$. Obviously by Corollary 4.1.38 we have

$$M_S \subset \mathfrak{C}^{\mu}_{S}.$$

We prove first that M_S is a linearly independent set.

Suppose M_S is a linearly dependent set then by Corollary 4.3.31,

$$\{\Phi_{m-s}(e_{\mathfrak{R}_S(L)}) \mid S(\mathcal{O}_L) = S, \mathrm{tab}(L) \setminus (S_I \cup S_J) \in T^{\lambda}_{\overline{S}}\}$$

is linearly dependent. Assume

$$\sum \delta_L \Phi_{m-s}(e_{\mathfrak{R}_S(L)}) = 0$$

where $S(\mathcal{O}_L) = S$, $\mathrm{tab}(L) \setminus (S_I \cup S_J) \in T^{\lambda}_{\overline{S}}$ and there exist at least one L such that $\delta_L \neq 0$. Hence

$$\sum \delta_L e_{\mathfrak{R}_S(L)} \in \ker \Phi_{m-s}.$$

Since $\mathrm{char}(F) = 0$, we have by Proposition 4.1.40,

$$S^{(n-m-s,m-s)} = \ker \Phi_{m-s}.$$

Thus

$$\sum \delta_L e_{\mathfrak{R}_S(L)} \in S^{(n-m-s,m-s)}$$

with at least one L such that $\delta_L \neq 0$.

But by Remark 4.3.4, we know $\operatorname{tab}\big(\mathfrak{R}_S(L)\big)$ is row-standard but non-standard since $\operatorname{tab}(L) \setminus (S_I \cup S_J) \in T^{\lambda}_{\overline{S}}$. This is a contradiction to Corollary 4.2.4.

Thus M_S is a linearly independent set. Hence if we can prove

$$|M_S| = \dim_F(\mathfrak{C}^{\mu}_S),$$

we are done.

Let $M_{\mathcal{O}}$ (resp. $M_{\tilde{\mathcal{O}}}$) be orbit modules in M^{λ} (resp. M^{μ}). For any $\mathcal{O}$ (resp. $\tilde{\mathcal{O}}$) such that $S(\mathcal{O}) = S$ (resp. $S(\tilde{\mathcal{O}}) = S$), by Theorem 4.1.46 we have

$$\dim M_{\mathcal{O}} = \dim M_{\tilde{\mathcal{O}}}.$$

Then

$$|M_S| = |T^{\lambda}_{\overline{S}}| \dim M_{\mathcal{O}}$$

and

$$\dim_F(\mathfrak{C}^{\mu}_S) = |\operatorname{RStd}(\mu)| \dim M_{\tilde{\mathcal{O}}}.$$

By Corollary 4.3.8, we have

$$|T^{\lambda}_{\overline{S}}| = |\operatorname{RStd}(\mu)|$$

hence

$$|M_S| = \dim_F(\mathfrak{C}^{\mu}_S).$$

This finishes the proof. □

4.3.33 Example. Let $\lambda = (3,3), \mu = (4,2), S = \{l_{64} \neq 0\}$. As in Example 4.3.7,

$$T^{\lambda}_{\overline{S}} = \left\{ \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline \end{array} \right\}.$$

So each possible orbit $\mathcal{O}$ with $S(\mathcal{O}) = S$ has one of the following tableaux:

$$\begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 1 & 3 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 5 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 3 & 6 \\ \hline \end{array}.$$

We investigate each different $\operatorname{tab}(\mathcal{O})$:

(1) $\operatorname{tab}(\mathcal{O}) = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 & 6 \\ \hline \end{array}$, for any $L_a \in \mathcal{O}$:

$$L_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & a & 1 \end{pmatrix} \text{ where } a \text{ runs over } GF(q).$$

$$\Phi_3(e_{L_a}) = e_{L_a^1} \cdot q^2 + e_{L_a^2} \cdot q, \text{ where}$$

$$L_a^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & a & 1 \end{pmatrix}, \; L_a^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & a & 1 \end{pmatrix}.$$

(2) $\mathrm{tab}(\mathcal{O}) = \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 1 & 3 & 6 \\ \hline \end{array}$, for any $N_b \in \mathcal{O}$:

$$N_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & b & 1 \end{pmatrix} \text{ where } b \text{ runs over } GF(q).$$

$$\Phi_3(e_{N_b}) = e_{N_b^1} \cdot q^2 + e_{N_b^2} \cdot q, \text{ where}$$

$$N_b^1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & b & 1 \end{pmatrix}, \; N_b^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & b & 1 \end{pmatrix}.$$

(3) $\begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 1 & 5 & 6 \\ \hline \end{array}$, for any $P_c \in \mathcal{O}$:

$$P_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & l_{64} & 0 & 1 \end{pmatrix} \text{ where } c \text{ runs over } GF(q).$$

$$\Phi_3(e_{P_c}) = e_{P_c^1} \cdot q^2 + \sum_{m \in GF(q)} \theta(-cm) e_{P_m^2} \cdot q, \text{ where}$$

$$P_c^1 = \begin{pmatrix} 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & l_{64} & 0 & 1 \end{pmatrix}, \; P_m^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & -l_{64}m & 1 \end{pmatrix}.$$

(4) $\mathrm{tab}(\mathcal{O}) = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 3 & 6 \\ \hline \end{array}$, for any $Q_d \in \mathcal{O}$:

$$Q_d = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & d & 1 \end{pmatrix} \text{ where } d \text{ runs over } GF(q).$$

$$\Phi_3(e_{Q_d}) = e_{Q_d^1} \cdot q^2 + e_{Q_d^2} \cdot q, \text{ where}$$

$$Q_d^1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & d & 1 \end{pmatrix}, \; N_b^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & d & 1 \end{pmatrix}.$$

In Example 4.1.36, we have:

$$\begin{aligned} \mathfrak{C}_S^\mu &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & * & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & * & 1 \end{pmatrix} \\ &\oplus \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{64} & * & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & l_{64} & 0 & 1 \end{pmatrix} \end{aligned}$$

By Proposition 4.3.32 we have:

$$F - \{\Phi_3(e_{L_a}), \Phi_3(e_{N_b}), \Phi_3(e_{P_c}), \Phi_3(e_{Q_d}) | a, b, c, d \in GF(q)\} = \mathfrak{C}_S^\mu.$$

4.3.34 Example. We use notation of Example 4.3.33. If $\mathcal{O}$ is an orbit in M^λ such that $S(\mathcal{O}) = S$, and

$$\mathrm{tab}(\mathcal{O}) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}.$$

Then $\forall\, e_R \in \mathcal{O}$, we have

$$\Phi_3(e_R) \in \mathfrak{C}_S^\mu = F - \{\Phi_3(e_{L_a}), \Phi_3(e_{N_b}), \Phi_3(e_{P_c}), \Phi_3(e_{Q_d}) | a, b, c, d \in GF(q)\}.$$

Assume

$$\Phi_3(e_R) = \sum_{a,b,c,d \in GF(q)} \big(r_a \Phi_3(e_{L_a}) + r_b \Phi_3(e_{N_b}) + r_c \Phi_3(e_{P_c}) + r_d \Phi_3(e_{Q_d})\big).$$

Let

$$v = e_R - \sum_{a,b,c,d \in GF(q)} \big(r_a e_{L_a} + r_b e_{N_b} + r_c e_{L_c} + r_d e_{Q_d}\big).$$

Then $\Phi_3(v) = 0$ and if $\mathrm{char}(F) = 0$, we have $v \in S^\lambda$.

In this example, we have a very easy total order relation:

$$\mathrm{tab}(L_a) < \mathrm{tab}(N_b) < \mathrm{tab}(P_c) < \mathrm{tab}(Q_d) < \mathrm{tab}(R).$$

Thus in this case, for all $e_R \in \mathcal{O}$, we say

$$e_R \text{ appears as the leading term of some element in } S^\lambda.$$

4.3.35 Remark. If an element in the orbit $\mathcal{O}$ appears as the leading term of some element in S^λ, then of course every element in the orbit $\mathcal{O}$ can appear as the leading term of some element in S^λ. In this sense, we say

$$M_{\mathcal{O}} \text{ appears as the leading term of } S^\lambda.$$

In general, there exist some L such that $S(\mathcal{O}_L) = S$ and $\mathrm{tab}(L) \setminus S_I \cup S_J$ is nonstandard but $\mathrm{tab}(L)$ is standard.

4.3.36 Lemma. Let $\lambda = (n-m, m)$ and S be a condition set. If S fits some $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ and $\mathfrak{t}$ is non-standard, then $\tilde{\mathfrak{t}} = \mathfrak{t} \setminus (S_I \cup S_J)$ is non-standard.

Proof. If $s = |S| = 0$, the lemma holds obviously. Now we assume $s > 0$. Let

$$\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|} \hline a_1 & a_2 & \cdots & a_m & \cdots & a_{n-m} \\ \hline b_1 & b_2 & \cdots & b_m & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda).$$

If we can prove for any $(b_i, j) \in \overline{S}$ that $\mathfrak{t} \setminus \{b_i, j\}$ is non-standard, then the lemma holds inductively. For $(b_i, j) \in \overline{S}$ we have the following three cases:

(1) $\mathrm{col}_{\mathfrak{t}}(b_i) = \mathrm{col}_{\mathfrak{t}}(j)$.
Obviously in this case $\tilde{\mathfrak{t}} = \mathfrak{t} \setminus \{b_i, j\}$ is non-standard since $\mathfrak{t}$ is non-standard.

(2) $\mathrm{col}_{\mathfrak{t}}(b_i) < \mathrm{col}_{\mathfrak{t}}(j)$.
Recall that $j < b_i$, since $(b_i, j) \in \mathfrak{J}_{\mathfrak{t}}$ (see Definition 3.1.9). Thus $b_t > a_t$ for all $i \leqslant t \leqslant \mathrm{col}_{\mathfrak{t}}(j)$. Hence we must have some $b_r < a_r$ where $r < i$ or $r > col_{\mathfrak{t}}(j)$ since $\mathfrak{t}$ is non-standard. Then of course $\tilde{\mathfrak{t}} = \mathfrak{t} \setminus \{b_i, j\}$ is non-standard. (Observe that columns 1 up to $i-1$ and $\mathrm{col}_{\mathfrak{t}}(j) + 1$ up to $n - m$ are preserved when entries b_i and j are removed from $\mathfrak{t}$).

(3) $\mathrm{col}_{\mathfrak{t}}(b_i) > \mathrm{col}_{\mathfrak{t}}(j)$.

1) if there is some $r < \mathrm{col}_{\mathfrak{t}}(j)$ or $r > i$ such that $b_r < a_r$, then $\tilde{\mathfrak{t}} = \mathfrak{t} \setminus \{b_i, j\}$ is non-standard by the same argument as in (2).
2) if there is some $r \geqslant \mathrm{col}_{\mathfrak{t}}(j)$ and $r < i$ such that $b_r < a_r$, then $b_r < a_{r+1}$, hence $\tilde{\mathfrak{t}} = \mathfrak{t} \setminus \{b_i, j\}$ is non-standard since column r in $\mathfrak{t}$ has entries a_{r+1} and b_r.
3) if $b_i < a_i$, then $b_{i-1} < b_i < a_i$, hence $\tilde{\mathfrak{t}} = \mathfrak{t} \setminus \{b_i, j\}$ is non-standard, since column $i-1$ in $\mathfrak{t}$ has entries a_i and b_{i-1}.

□

4.3.37 Lemma. Let $\lambda = (n-m, m)$ and S be a condition set and let $e_L, e_R \in \mathfrak{C}_S^\lambda$ such that $S(\mathcal{O}_L) = S(\mathcal{O}_R) = S$, then

$$\mathrm{tab}(R) < \mathrm{tab}(L) \Rightarrow \mathrm{tab}(\tilde{R}) < \mathrm{tab}(\tilde{L}).$$

Proof. Let $\underline{\mathfrak{t}_1} = \underline{\mathrm{tab}(L)} = (t_1, t_2, ..., t_m)$, $\underline{\mathfrak{t}_2} = \underline{\mathrm{tab}(R)} = (r_1, r_2, ..., r_m)$ and assume $\mathfrak{t}_2 < \mathfrak{t}_1$. Working step by step, by removing one condition at each step we may assume that $S = \{l_{kj}\}$ consists of one element $0 \neq l_{kj} \in GF(q)$. Since S fits $\mathfrak{t}_1$ and $\mathfrak{t}_2$, $k \in \{t_1, t_2, \ldots, t_m\} \cap \{r_1, r_2, \ldots, r_m\}$. Note that

$$\underline{\tilde{\mathfrak{t}}_1} = (t_1, t_2, \ldots, t_m) \setminus \{k\}, \quad \underline{\tilde{\mathfrak{t}}_2} = (r_1, r_2, \ldots, r_m) \setminus \{k\}.$$

Assume i is the smallest number satisfying $r_i < t_i$. Then by the minimality of i, we obtain:

$$k < r_i \text{ or } k \geqslant t_i.$$

(1) If $k = r_j$ with $j < i$ then $k = t_j = r_j$ and

$$\underline{\tilde{\mathfrak{t}}_1} = (r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_{i-1}, t_i, \ldots, t_m),$$

$$\underline{\tilde{\mathfrak{t}}_2} = (r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_{i-1}, r_i, \ldots, r_m).$$

Hence we obtain $\tilde{\mathfrak{t}}_2 < \tilde{\mathfrak{t}}_1$.

(2) If $k > t_i$, a similar argument shows that $\tilde{\mathfrak{t}}_2 < \tilde{\mathfrak{t}}_1$.

(3) If $k = t_i$, then $t_{i+1} > t_i > r_i$. And we get

$$\underline{\tilde{\mathfrak{t}}_1} = (r_1, \ldots, r_{i-1}, t_{i+1}, \ldots, t_m),$$

$$\underline{\tilde{\mathfrak{t}}_2} = (r_1, \ldots, r_{i-1}, r_i, \ldots, r_m) \setminus \{k\}$$

where $k = r_j$ such that $j > i$. Hence we obtain $\tilde{\mathfrak{t}}_2 < \tilde{\mathfrak{t}}_1$.

□

4.3.38 Theorem. Let $\mathrm{char}(F) = 0$, $\lambda = (n-m, m)$. For $e_{_L} \in \mathcal{O} \subset M^\lambda$ with $S = S(\mathcal{O})$, there exists $v \in S^\lambda$ such that $\mathrm{last}(v) = \mathrm{tab}(L)$ and $\mathrm{top}(v) = e_{_L}$ if and only if

$$\mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ is a shifted standard } \mu\text{−tableau},$$

where $\mu = (n-m-|S|, m-|S|)$ and "shifted" again means the tableau is filled by numbers in $\{1, 2, \ldots, n\} \setminus (S_I \cup S_J)$.

Proof. For any condition set S, we denote $s = |S|$. In particular, we have discussed two special types of orbits in Section 4.2. One is the case of orbits with full condition: For any $e_{_L} \in \mathcal{O} \subset M^\lambda$ with $S = S(\mathcal{O})$ and $s = m$, by Proposition 4.2.1 and Proposition 4.1.40, we have

$$e_{_L} \in S^\lambda \text{ and } \mathrm{tab}(L) \in \mathrm{Std}(\lambda).$$

Hence in this case

$$S_I = \underline{\mathrm{tab}(L)},$$

hence $\mathrm{tab}(L) \setminus (S_I \cup S_J)$ is a shifted standard $(n-2m, 0)$−tableau. The other case of special orbits are those with $s = 0$. In this case the sufficiency is just Proposition 4.2.6 and the necessity is just the Corollary 4.2.4. Now we assume $1 \leqslant s \leqslant m-1$.

(1) ($\Longleftarrow$) Assume $e_{_L} \in \mathcal{O} \subset M^\lambda$ with $S = S(\mathcal{O})$ and $\mathrm{tab}(L) \setminus (S_I \cup S_J)$ is a shifted standard $\mu-$tableau. By Lemma 4.3.36, we know $\mathrm{tab}(L)$ is standard. By Lemma 4.3.32, there exists $a_{_R}$'s $\in F$ and $e_{_R} \in \mathcal{O}_R \subset M^\lambda$ with $S(\mathcal{O}_R) = S$, $\mathrm{tab}(R) \setminus (S_I \cup S_J) \in T^\lambda_{\overline{S}}$ such that

$$\Phi_m(e_{_L}) = \sum a_{_R} \Phi_m(e_{_R}) \in \mathfrak{C}^\mu_S.$$

We claim that all occurring R with $a_{_R} \neq 0$ has the property:

$$\mathrm{tab}(R) < \mathrm{tab}(L).$$

Otherwise, assume there exist some R such that

$$a_{_R} \neq 0 \text{ and } \mathrm{tab}(R) > \mathrm{tab}(L).$$

We choose some $u \in U^w \cap U$ where $\mathfrak{t}^\lambda w = \mathrm{tab}(R)$ such that

$$e_{_{R_0}} = e_{_R} \circ u$$

where R_0 is a pattern matrix. Hence we obtain:

$$\Phi_m(e_{_L} \circ u) - a_{_R} \Phi_m(e_{_{R_0}}) - \Phi_m\big(\sum_{R' \neq R} a_{_{R'}} e_{_{R'}} \circ u\big) = 0.$$

Suppose

$$e_{_L} \circ u = \sum_K \alpha_{_K} e_{_K}, \quad \sum_{R' \neq R} a_{_{R'}} e_{_{R'}} \circ u = \sum_N \beta_{_N} e_{_N}.$$

Then we have

$$\sum_K \alpha_{_K} \Phi_m(e_{_K}) - a_{_R} \Phi_m(e_{_{R_0}}) - \sum_N \beta_{_N} \Phi_m(e_{_N}) = 0$$

where $\mathrm{tab}(K) = \mathrm{tab}(L)$, $\mathrm{tab}(N) \setminus (S_I \cup S_J) \in T^\lambda_{\overline{S}}$ for all occurring K and N.

Denote $\tilde{K} = \mathfrak{R}_S(K)$, $\tilde{R}_0 = \mathfrak{R}_S(R_0), \tilde{N} = \mathfrak{R}_S(N)$, by Corollary 4.3.30 we obtain

$$\sum_K \delta_{_K} \Phi_{m-s}(e_{_{\tilde{K}}}) - a_{_R} \Phi_{m-s}(e_{_{\tilde{R}_0}}) - \sum_N \delta_{_N} \Phi_{m-s}(e_{_{\tilde{N}}}) = 0$$

where $\mathrm{tab}(\tilde{K}) = \mathrm{tab}(\tilde{L}), \mathrm{tab}(\tilde{R}_0) = \mathrm{tab}(\tilde{R})$ and $\mathrm{tab}(\tilde{N})$ is nonstandard.

Now we have

$$0 \neq \sum_K \delta_{_K} e_{_{\tilde{K}}} - a_{_R} e_{_{\tilde{R}_0}} - \sum_N \delta_{_N} e_{_{\tilde{N}}} \in \ker \Phi_{m-s}.$$

Since $\mathrm{char}(F) = 0$, by Proposition 4.1.40 we have

$$S^{\mu} = \ker \Phi_{m-s}$$

where $\mu = (n-m-s, m-s)$ and then

$$0 \neq v = \sum_{K} \delta_{K} e_{\tilde{K}} - a_{R} e_{\tilde{R}_0} - \sum_{N} \delta_{N} e_{\tilde{N}} \in S^{\mu}.$$

By assumption we have $\mathrm{tab}(R) > \mathrm{tab}(L)$ then from Lemma 4.3.37, we obtain $\mathrm{tab}(\tilde{R}) > \mathrm{tab}(\tilde{L})$ hence

$$\mathrm{tab}(\tilde{R}_0) > \mathrm{tab}(\tilde{K}).$$

Moreover we know $\mathrm{tab}(\tilde{R}_0)$ and $\mathrm{tab}(\tilde{N})$ are non-standard. Hence we obtain that $\mathrm{last}(v)$ is non-standard, which is a contradiction to Corollary 4.2.4. Let

$$v = e_{L} - \sum a_{K} e_{K}.$$

Then by Proposition 4.1.40 we obtain

$$v \in \ker \Phi_m = S^{\lambda} \text{ with } \mathrm{last}(v) = \mathrm{tab}(L),\ \mathrm{top}(v) = e_{L}.$$

This finishes the proof of the sufficiency.

(2) ($\Longrightarrow$) Suppose $e_{L} \in \mathcal{O} \subset M^{\lambda}$ with $S = S(\mathcal{O})$ such that there exists $v \in S^{\lambda}$ such that $\mathrm{last}(v) = \mathrm{tab}(L)$ and $\mathrm{top}(v) = e_{L}$. Assume

$$0 \neq v = e_{L} - \sum a_{R} e_{R} \in S^{\lambda} = \ker \Phi_m \tag{4.3.39}$$

where $\mathrm{tab}(L) > \mathrm{tab}(R)$, $\forall R$ with $a_{R} \neq 0$. Thus

$$\Phi_m(e_{L}) - \sum a_{R} \Phi_m(e_{R}) = 0.$$

By Corollary 4.1.38, $\Phi_m(\mathfrak{C}_S^{\lambda}) = \mathfrak{C}_S^{\mu}$, hence $\Phi_m(e_{L}) \in \mathfrak{C}_S^{\mu}$, that is if $R \in \Xi_{m,n}$ with $a_{R} \neq 0$ then $S(\mathcal{O}_R) = S$ where $\mathcal{O}_R$ denotes the orbit containing R.

Since we have proved in (1) that for an arbitrary $e_{K} \in \mathcal{O}_{L}$ with $\mathrm{tab}(K) \setminus (S_I \cup S_J)$ is standard, there exists $v_{K} \in S^{\lambda}$ such that

$$\mathrm{last}(v_{K}) = \mathrm{tab}(K) \text{ and } \mathrm{top}(v_{K}) = e_{K}.$$

We can assume in (4.3.39) that all occurring R have the property that

$$\mathrm{tab}(R) \setminus (S_I \cup S_J) \in T_{\bar{S}}^{\lambda}.$$

It suffices to prove for L is a pattern matrix since $\mathrm{tab}(K) = \mathrm{tab}(L)$ for all $e_K \in \mathcal{O}_L$. Assume L is a pattern matrix then by Corollary 4.3.30,

$$\Phi_{m-s}(e_{\tilde{L}}) = \sum b_R \Phi_{m-s}(e_{\tilde{R}})$$

for some $b_R \in F$. Hence by Proposition 4.1.40 we have

$$v_{\tilde{L}} := e_{\tilde{L}} - \sum b_R e_{\tilde{R}} \in \ker \Phi_{m-s} = S^{\mu}$$

where $\mu = (n-m-s, m-s)$. By Lemma 4.3.37, we have $\mathrm{tab}(\tilde{R}) < \mathrm{tab}(\tilde{L})$. Hence from Corollary 4.2.4, we obtain

$$\mathrm{tab}(\tilde{L}) = \mathrm{last}(v_{\tilde{L}}) \in \mathrm{Std}(\mu).$$

Thus by Remark 4.3.4, we obtain

$$\mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ is a shifted standard } \mu\text{−tableau.}$$

This finishes the proof of the necessity.

□

4.3.40 Definition. Let $\mathrm{char}\, F = 0$, $\lambda = (n-m, m) \vdash n$ and $e_L \in \mathcal{O} \subset M^{\lambda}$ with $S(\mathcal{O}) = S$. Suppose $\mathrm{tab}(L) \setminus (S_I \cup S_J)$ is a shifted standard μ−tableau where $\mu = (n-m-|S|, m-|S|)$. Choose $v_L \in S^{\lambda}$ such that $\mathrm{last}(v_L) = \mathrm{tab}(L)$ and $\mathrm{top}(v_L) = e_L$. By Theorem 4.3.38 there exist such an v_L. Let

$$\mathcal{B}_S^{\lambda} := \mathcal{B}_{S,F}^{\lambda} = \{v_L \mid e_L \in \mathcal{O} \subset M^{\lambda}, S(\mathcal{O}) = S, \mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ is standard}\}.$$

Finally take

$$\mathcal{B}^{\lambda} = \mathcal{B}_F^{\lambda} = \dot{\bigcup_S} \mathcal{B}_S^{\lambda}.$$

Note that this union is disjoint, since its elements are distinguished by their leading term $\mathrm{top}(v_L) = e_L$.

We choose now a suitable principal ideal domain Λ (containing a primitive p−th root of unity), with quotient field Q of characteristic zero. Moreover we assume that $q = q \cdot 1_{\Lambda} \in \Lambda$ is invertible. Then from Theorem 4.3.38, we see that $\mathcal{B}_S^{\lambda}$ is a standard basis of the S−component $S_Q^{\lambda}{\downarrow}_S$ of $\mathrm{Res}_{FU}^{FG} S_Q^{\lambda}$ and $\mathcal{B}^{\lambda}$ is a standard basis for S_Q^{λ}.

We assume that our field F is epimorphic image of Λ and has characteristic l coprime to q. Note that

$$M_R^{\lambda} = M_{\Lambda}^{\lambda} \otimes_{\Lambda} R \text{ and } S_R^{\lambda} = S_{\Lambda}^{\lambda} \otimes_{\Lambda} R \quad \text{for } R = Q \text{ or } F.$$

4.3.41 Proposition. If there exists $v \in S^\lambda_Q$ such that $\mathrm{last}(v) = \mathrm{tab}(L)$ and $\mathrm{top}(v) = e_L$ with $v \in \mathcal{B}^\lambda_{\emptyset,Q}$, the trivial-component $S^\lambda_Q{\downarrow}_\emptyset$ of $\mathrm{Res}^{FG}_{FU} S^\lambda_Q$, then there exists $\hat{v} \in S^\lambda_F$ such that $\mathrm{last}(\hat{v}) = \mathrm{tab}(L)$ and $\mathrm{top}(\hat{v}) = e_L$ with $\hat{v} \in \mathcal{B}^\lambda_{\emptyset,F}$, the trivial-component $S^\lambda_F{\downarrow}_\emptyset$ of $\mathrm{Res}^{FG}_{FU} S^\lambda_F$ (compare Proposition 4.2.6).

Proof. For $e_L \in \mathcal{O} \in M^\lambda$ with $S(\mathcal{O}) = \emptyset, \mathrm{tab}(L) \in \mathrm{Std}(\lambda)$ by Proposition 4.3.32 we may write uniquely:

$$\Phi_{m,Q}(e_L) - \sum_K \alpha_K \Phi_{m,Q}(e_K) = 0$$

hence

$$v_L := e_L - \sum_K \alpha_K e_K \in S^\lambda_Q{\downarrow}_S = \ker \Phi_{m,Q}$$

where K runs through all matrices with $S(\mathcal{O}_K) = \emptyset$, $\mathrm{tab}(K)$ is non-standard and $\alpha_K \in Q$. Multiplying this equation by the least common denominator of the coefficients we obtain an expression

$$\hat{v}_L := \beta_L e_L - \sum_K \beta_K e_K \in S^\lambda_\Lambda{\downarrow}_\emptyset = \ker \Phi_{m,\Lambda}$$

where $\beta_K \in \Lambda$, $\forall\, K$ and $\beta_L \in \Lambda$. Moreover, we may assume that the greatest common divisor of the coefficients β_L is 1. Note that $\hat{v} \in S^\lambda_\Lambda$, hence

$$\hat{v}_{L,F} = \hat{v}_L \otimes_\Lambda 1_F \in S^\lambda_F.$$

Let

$$\hat{v}_{L,F} = \hat{v}_L \otimes_\Lambda 1_F = \overline{\beta_L} e_L - \sum_K \overline{\beta_K} e_K,$$

where for $c \in \Lambda$, $\overline{c}$ denoted the corresponding residue class of c in F. Here we identify $M^\lambda_F = M^\lambda_\Lambda / l M^\lambda_\Lambda$ and $M^\lambda_F = M^\lambda_\Lambda \otimes_\Lambda F$ by the canonical isomorphism, where $l \in \Lambda$ generates the kernel of the epimorphism from Λ onto F.

Note that $\hat{v}_{L,F} \neq 0$ and

$$\hat{v}_{L,F} = \overline{\beta_L} e_L - \sum_K \overline{\beta_K} e_K \in S^\lambda_F = S^\lambda_\Lambda \otimes_\Lambda F$$

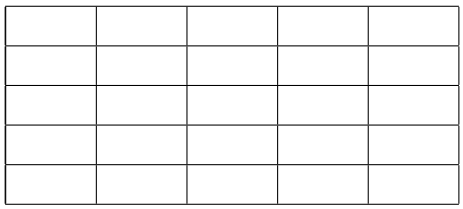
We now argue that $\overline{\beta_L} \neq 0$ in F. Otherwise, $0 \neq \sum_K \overline{\beta_K} e_K \in S^\lambda_F$ where every K has the property that $\mathrm{tab}(K)$ is non standard, which is a contradiction of Corollary 4.2.4. Hence we can assume $\overline{\beta_L} = 1$. That is: There exist an nonzero element $\hat{v}_{L,F} \in S^\lambda_F$ which has the leading term $\mathrm{top}(\hat{v}_{L,F}) = e_L$ and $\mathrm{last}(\hat{v}_{L,F}) = \mathrm{tab}(L)$. $\square$

4.4 Rank polynomials $r_{\mathfrak{t}}(q)$

In the last section, we have constructed a standard basis for $S^{(n-m,m)}$ in the characteristic zero case. And we proved for $e_{_L} \in \mathcal{O} \subset M^{\lambda}$ with $S(\mathcal{O}) = \emptyset$, $\mathrm{tab}(\mathcal{O}) \in \mathrm{Std}(\lambda)$ and over any arbitrary field F whose characteristic is coprime to q and which contains a $p-$th root of unity, there exist $v \in S^{\lambda}{\downarrow}_{\emptyset}$ such that $\mathrm{last}(v) = \mathrm{tab}(L)$ and $\mathrm{top}(v) = e_{_L}$. Now we want to get similar statement for $S^{\lambda}{\downarrow}_{S}$ where S is not the empty set.
In [4], Brandt-Dipper-James-Lyle introduced a kind of polynomials in q attached to each standard $\lambda-$tableau $\mathfrak{t}$, $\big(\lambda = (n-m, m)\big)$, called "rank polynomials", denoted by $r_{\mathfrak{t}}(q)$ such that $r_{\mathfrak{t}}(1) = 1$. He showed that and there exist $r_{\mathfrak{t}}(q)$ many elements of S^{λ} for every standard $\lambda-$tableau $\mathfrak{t}$, such that all the elements form a basis of S^{λ}. In this section we will show that the number of our basis elements B_Q^{λ} in the $\mathfrak{t}-$batch $\mathfrak{M}_{\mathfrak{t}}$ with leading term in $\mathcal{E}_{\mathfrak{t}}$ is the rank polynomial $r_{\mathfrak{t}}(q)$.

4.4.1 Definition. (Brandt, Dipper, James and Lyle [4])

(1) Consider a rectangular $a \times b$ of boxes with $a \leqslant b$, for example



where $a = 5$ and $b = 6$. We call a route from the north west corner to the south east corner a **path**, denoted by π. Define $P(a, b)$ to be the set of all paths in an $a \times b$ array of boxes.

(2) Given a corner (i, j) let $r(i, j) = j - i$. Suppose that Y is a filling of the boxes to the south of some path π with elements of $GF(q)$. Say that Y is good if for each corner (i, j) through which the path passes, the matrix whose bottom left hand corner is $(a+1, 1)$ and whose top right hand corner is (i, j) has rank at most $r(i, j)$.

(3) We define the rank polynomial $r(\pi)$ of the path to be the number of ways of filling the boxes below the path with elements of $GF(q)$ such that the filling is good.

4.4.2 Remark. (Brandt, Dipper, James and Lyle [4])

(1) If π passes through a corner with $i > j$ then $r(\pi) = 0$. In particular, if $r(\pi) \neq 0$ then the path must start with a east move.

(2) Note that in the definition of a good filling, we may replace 'for each corner (i, j) through which the path passes' by 'for each corner (i, j) through which the path passes and which has the property that $(i-1, j)$ and $(i, j+1)$ are on the path' since all the other restrictions follow from these.

4.4.3 Remark. Given a path and then filling of the boxes below the path with elements of $GF(q)$ is a way of encoding an $a-$dimensional subspace of a given $(a+b)-$dimensional vector space over $GF(q)$. In fact, this can be shown by definition of $\Xi_{a,\,a+b}$ (c.f. Definition 3.1.1) if we change the labeling of the rows back to the original label $1, 2 \ldots, a$ and delete the columns of the last 1's. To each $\pi \in P(m, n-m)$ we attach a unique $\mathfrak{t} \in \mathrm{RStd}(\lambda)$ with $\lambda = (n-m, m)$. We use the notation $\pi_{\mathfrak{t}}$ and let $r_{\mathfrak{t}} = r(\pi_{\mathfrak{t}})$.

4.4.4 Lemma. Let λ be a two part partition and $\mathfrak{t} \in \mathrm{RStd}(\lambda)$. Then $\mathfrak{t} \in \mathrm{Std}(\lambda)$ if and only if all the corners (i, j) of $\pi_{\mathfrak{t}}$ satisfying $i \leqslant j$.

Proof. Suppose $\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|} \hline a_1 & a_2 & \cdots & a_m & \cdots & a_{n-m} \\ \hline b_1 & b_2 & \cdots & b_m & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array} \in \mathrm{RStd}(\lambda)$. Then

$$\mathfrak{t} \in \mathrm{Std}(\lambda) \Leftrightarrow b_i > a_i,\ \forall\, i = 1, \ldots, m. \tag{4.4.5}$$

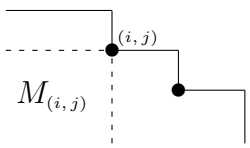
If we label the boxes by their left top corner labeling, then (4.4.5) is equivalent to say box (i, i) appears in the south of the path $\pi_{\mathfrak{t}}$, that is all the corners (i, j) of $\pi_{\mathfrak{t}}$ satisfying $i \leqslant j$.

□

4.4.6 Theorem. Let λ be a two part partition and $\mathfrak{t} \in \mathrm{RStd}(\lambda)$. Let $e_{_L} \in \mathcal{O} \subset \mathfrak{M}_{\mathfrak{t}} \subset M^{\lambda}$ with $S(O) = S$. Then

$$L \text{ is a good filling of path } \pi_{\mathfrak{t}} \Leftrightarrow \mathfrak{t} \setminus (S_I \cup S_J) \text{ is standard.}$$

Proof. By Remark 4.4.2, the path $\pi_{\mathfrak{t}}$ must start with a east move, hence we can draw the following picture for it:



Picture of $\pi_{\mathfrak{t}}$

Note that $\mathfrak{t} \setminus (S_I \cup S_J)$ is a shifted tableau, filled by numbers in $\{1, 2, \ldots, n\} \setminus (S_I \cup S_J)$. Remember the definition of $\overline{S}-$similar in Remark 4.3.4. We denote

$$\mathfrak{s} \overset{\overline{S}}{\sim} \mathfrak{t} \setminus (S_I \cup S_J).$$

Thus after deleting the rows and columns which contain positions in $\overline{S}$, and closing the gaps, we obtain the path $\pi_{\mathfrak{s}}$.

Again by Remark 4.4.2, it is sufficient to investigate those kind of corners labeled by black dots in the Picture of $\pi_{\mathfrak{t}}$ above. Choose an arbitrary corner of this kind, say (i, j).

Note that L is obtained from a pattern matrix L_0 by truncated row and column operations. Furthermore note that such operations preserve the ranks of the sub-matrices determined by the relevant corners of the path $\pi_{\mathfrak{t}}$. In particular L is a good filling if and only if L_0 is a good filling. Thus we may assume that L is a pattern matrix.

Assume there are $\alpha_{(i,j)}$ many positions in the north west boxes (u, v) of the corner (i, j) such that $(u, v) \in \overline{S}$, and $\beta_{(i,j)}$ many positions in the south west boxes (s, t) of the corner (i, j) such that $(s, t) \in \overline{S}$. If we denote the south west part of the corner (i, j) by $M_{(i,j)}$, then by Corollary 3.3.7, we obtain that:

$$\operatorname{rank} M_{(i,j)} = \beta_{(i,j)}. \tag{4.4.7}$$

Hence after deleting the rows and columns which contain positions in $\overline{S}$, and closing the gaps, the corner (i, j) has a new labeling (i', j') namely

$$i' = i - \alpha_{(i,j)} \text{ and } j' = j - \alpha_{(i,j)} - \beta_{(i,j)}.$$

Hence

$$j' - i' = j - \alpha_{(i,j)} - \beta_{(i,j)} - \big(i - \alpha_{(i,j)}\big) = j - i - \beta_{(i,j)}. \tag{4.4.8}$$

By the definition of good filling,

$$L \text{ is a good filling of } \pi_{\mathfrak{t}} \Leftrightarrow \operatorname{rank} M_{(i,j)} \leqslant j - i$$

for all black dots (i, j). By (4.4.7), we get

$$\operatorname{rank} M_{(i,j)} \leqslant j - i \Leftrightarrow \beta_{(i,j)} \leqslant j - i. \tag{4.4.9}$$

Combining (4.4.8) and (4.4.9), we get

$$\operatorname{rank} M_{(i,j)} \leqslant j - i \Leftrightarrow j' - i' \geqslant 0 \Leftrightarrow i' \leqslant j'. \tag{4.4.10}$$

By Lemma 4.4.4, we get that $\mathfrak{s}$ is a standard tableau. Then by Remark 4.3.4, we obtain that $\mathfrak{t} \setminus (S_I \cup S_J)$ is standard. $\square$

Now we come to our main goal: Find all the appearing leading terms for elements in S^λ over any possible field F. Unfortunately, at the moment we appeal to the following results by Brandt, Dipper, James and Lyle. However, we are confident to provide an independent proof in the near future.

4.4.11 Theorem. (Brandt, Dipper, James and Lyle [4])
If L is a good filling for $\pi_{\mathfrak{t}}$ where $\mathfrak{t} = \mathrm{tab}(L)$, then there exist $v_L \in S^\lambda$ such that $\mathrm{top}(v_L) = e_L$ and $\mathrm{last}(v_L) = \mathrm{tab}(L)$. Moreover, if we choose some appropriate v_L for each L which is a good filling, then

$$\{v_L \mid L \text{ is a good filling}\}$$

is a standard basis of S^λ.

Now we can state the main result of this thesis.

4.4.12 Theorem. Let $\lambda = (n-m, m)$. Then $\mathcal{B}^\lambda$ is an integral standard basis for S^λ and $\mathcal{B}^\lambda_S$ is an integral standard basis of the $S-$component $S^\lambda{\downarrow_S}$ of $\mathrm{Res}^{FG}_{FU} S^\lambda$. Moreover for $0 \leqslant c \in \mathbb{Z}$, there exist polynomials $f_c(t) \in \mathbb{Z}[t]$ such that the number of irreducible components of $\mathrm{Res}^{FG}_{FU} S^\lambda$ of dimension q^c is $f_c(q)$. Here S^λ is over any field F with characteristic coprime to p containing a primitive $p-$th root of unity.

Proof. From Theorem 4.4.6, we know

$$L \text{ is a good filling of path } \pi_{\mathfrak{t}} \Leftrightarrow \mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ is standard}$$

where $S = S(L)$. Hence by Theorem 4.4.11, we obtain: If $\mathrm{tab}(L) \setminus (S_I \cup S_J)$ is standard, then there exist $v_L \in S^\lambda$ such that $\mathrm{top}(v_L) = e_L$ and $\mathrm{last}(v_L) = \mathrm{tab}(L)$. So we may write

$$v_L = e_L + \sum_K \alpha_K e_K \in S^\lambda_Q = \ker \Phi_{m,Q}$$

where K runs through $\mathrm{tab}(K) \setminus (S_I \cup S_J)$ is non standard and α_K is integral elements over any field F with characteristic coprime to p containing a primitive p^{th} root of unity. Hence we get

$$\Phi_{m,Q}(e_L) + \sum_K \alpha_K \Phi_{m,Q}(e_K) = 0.$$

But by Proposition 4.3.32, these α_K are uniquely determined. That is $\mathcal{B}^\lambda_S$ is an integral standard basis of the $S-$component $S^\lambda{\downarrow_S}$ of $\mathrm{Res}^{FG}_{FU} S^\lambda$. Furthermore $\mathcal{B}^\lambda$ is an integral standard basis for S^λ.

Moreover, if $M_{\mathcal{O}}$ appears as the leading term of S^λ then varying the nonzero entries in the condition set $S = S(\mathcal{O})$, we will get $(q-1)^{|S|}$ many orbit

$M_{\mathcal{O}'}$ appearing as the leading term of S^λ. And obviously the dimensions of these orbits are the same, given by the hook length. That is for $0 \leqslant c \in \mathbb{Z}$, there exist polynomials $f_c(t) \in \mathbb{Z}[t]$ such that the number of irreducible components of $\mathrm{Res}^{FG}_{FU} S^\lambda$ of dimension q^c is $f_c(q)$. $\square$

4.4.13 Conclusion. Theorem 4.4.6 in connection with Theorem 4.3.38 gives a new proof for the fact that the rank polynomials add up to the generic degree of S^λ. Moreover Theorem 4.4.6 also provides more detailed information, how good filling of a path $\pi_{\mathfrak{t}}$ (to a standard tableau $\mathfrak{t}$) arise: Choose a condition set S and fill the path by the entries of S to obtain a pattern matrix L_S. If this pattern matrix satisfies the rank condition and hence is a good filling, then this is equivalent to $\mathfrak{t}\backslash(S_I \cup S_J)$ is still standard by Theorem 4.4.6. Then all the matrices obtained from L_S by truncated row and column operations as defined in Corollary 3.3.7 are good fillings of $\pi_{\mathfrak{t}}$. Taking idempotents we obtain the leading terms of precisely one irreducible component of $\mathrm{Res}^{FG}_{FU} S^\lambda$ belonging to the given condition set S. Varying the condition set fillings of $\pi_{\mathfrak{t}}$ we obtaion all irreducible $FU-$components of $\mathrm{Res}^{FG}_{FU} S^\lambda$ in the $\mathfrak{t}-$batch of S^λ. Theorem 4.3.38 provides the standard basis of this part of S^λ. Note that filling $\pi_{\mathfrak{t}}$ with a condition set amounts to placing at most one nonzero entry in each column and each row of the array of boxes.

4.5 Main results

(1) $\Phi_m = \bigoplus_S \Phi_{m,S}$. If $\mathrm{char}(F) = 0$, then $\Phi_{m,S}(\mathfrak{C}^\lambda_S) = \mathfrak{C}^\mu_S$. Note that $\mathfrak{C}^\lambda_S$, $\mathfrak{C}^\mu_S$ are homogeneous modules. For a fixed condition set S, the irreducible $FU-$module I_S to this condition set appears with multiplicity equivalent to the number of the row standard tableaux $\mathfrak{t}$ such that S fits batch $\mathfrak{t}$. Let this number for $\mathfrak{C}^\lambda_S$ (resp. $\mathfrak{C}^\mu_S$) be k_λ (resp. k_μ). Then the kernel $\Phi_{m,S}$ contains $k_\lambda - k_\mu$ many copies of I_S.

(2) Let S be a condition set. Then any orbit module associated with an orbit T such that the frame of the condition set $\overline{T} = \overline{S}$ has the same dimension q^c where c is given by the hook length and hence depends only on the frame of the condition set. Hence the number of irreducibles occurring as constituents of $\mathrm{Res}^{FG}_{FU} S^\lambda$ of a fixed dimension q^c $(0 \leqslant c \in \mathbb{Z})$ is a polynomial in q.

(3) We constructed a standard basis for S^λ, $\lambda = (n-m, m)$. We showed the number of the basis elements v such that $\mathrm{last}(v) = \mathfrak{t} \in \mathrm{Std}(\lambda)$ is the rank polynomial in [4], which gives a new proof the fact that they add up to the generic degree of the unipotent Specht module S^λ.

4.6 Examples

Using the language in Remark 4.3.35 and the short notations for orbit modules $M_{\mathcal{O}}$ in Example 4.1.35, we give the following examples:

4.6.1 Example. Let $\lambda = (2,2)$. $\mathrm{Std}(\lambda) = \left\{ \mathfrak{t}_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array},\ \mathfrak{t}_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\}$.

(1) For $\mathfrak{t}_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$:

1) $S = \emptyset : \mathfrak{t}_1$ is standard.
Hence $\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

2) $S = \{l_{21} \neq 0\} : \mathfrak{t}_1 \setminus \{1,2\} = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}$ is standard.
Hence $\begin{pmatrix} l_{21} & 1 & & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

3) $S = \{l_{43} \neq 0\} : \mathfrak{t}_1 \setminus \{4,3\} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ is standard.
Hence $\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & l_{43} & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

4) $S = \{l_{21} \neq 0, l_{43} \neq 0\} : \mathfrak{t}_1 \setminus \{1,2,3,4\} = \emptyset$.
Hence $\begin{pmatrix} l_{21} & 1 & & \\ 0 & 0 & l_{43} & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

5) $S = \{l_{41} \neq 0\} : \mathfrak{t}_1 \setminus \{4,1\} = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array}$ is non-standard.
Hence $\begin{pmatrix} * & 1 & & \\ l_{41} & 0 & * & 1 \end{pmatrix}$ can not appear as the leading term of S^λ.

(2) For $\mathfrak{t}_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$:

1) $S = \emptyset : \mathfrak{t}_2$ is standard.
Hence $\begin{pmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

2) $S = \{l_{32} \neq 0\} : \mathfrak{t}_2 \setminus \{2,3\} = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline \end{array}$ is standard.
Hence $\begin{pmatrix} 0 & l_{32} & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

3) $S = \{l_{31} \neq 0\} : \mathfrak{t}_2 \setminus \{1,3\} = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array}$ is standard.

Hence $\begin{pmatrix} l_{31} & * & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

4) $S = \{l_{42} \neq 0\} : \mathfrak{t}_2 \setminus \{2,4\} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$ is standard.

Hence $\begin{pmatrix} 0 & * & 1 & \\ 0 & l_{42} & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

5) $S = \{l_{31} \neq 0, l_{42} \neq 0\} : \mathfrak{t}_2 \setminus \{1,2,3,4\} = \emptyset$.

Hence $\begin{pmatrix} l_{31} & * & 1 & \\ 0 & l_{42} & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

6) $S = \{l_{41} \neq 0\} : \mathfrak{t}_2 \setminus \{1,4\} = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$ is standard.

Hence $\begin{pmatrix} * & 0+\square & 1 & \\ l_{41} & * & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

7) $S = \{l_{41} \neq 0, l_{32} \neq 0\} : \mathfrak{t}_2 \setminus \{1,2,3,4\} = \emptyset$.

Hence $\begin{pmatrix} * & l_{32}+\square & 1 & \\ l_{41} & * & 0 & 1 \end{pmatrix}$ can appear as the leading term of S^λ.

Chapter 5

German summary

In [12] definiert Gordon James Spechtmoduln für symmetrische Gruppen: Für jede Partition λ von n gibt es einen Spechtmodul $S(\lambda)$ von $\mathfrak{S}_n$, der als Schnitt von Kernen gewisser Homomorphismen definiert ist. Die Dimension von $S(\lambda)$ von $\mathfrak{S}_n$ wird auf zwei Arten bestimmt.

(1) Sei h_{ij}^{λ} die Hakenlänge für den Knoten (i,j) in $[\lambda]$. Dann ist

$$\dim S(\lambda) = \frac{n!}{\prod\limits_{(i,j)\in[\lambda]} h_{ij}^{\lambda}}.$$

(2) $\dim S(\lambda)$ ist gleich der Anzahl von standard $\lambda-$Tableaux. In der Tat gibt es eine Basis von $S(\lambda)$, die durch standard $\lambda-$Tableaux indiziert ist. Diese Basis heißt Standardbasis von $S(\lambda)$.

In der Philosophie, dass $GL_n(q)$ ein $q-$Analog zu $\mathfrak{S}_n$ ist, definiert Gordon James den unipotenten Spechtmoduln S^{λ} über einem Körper F für $GL_n(q)$ in [13]. Die Dimension von S^{λ} ist unabhängig von F, falls die Charakteristik von F koprim zu q ist. Wir hoffen, die Darstellungstheorie von $GL_n(q)$ zu der von $\mathfrak{S}_n$ übersetzen zu können. Für $GL_n(q)$ gilt

$$\dim S^{\lambda} = q^{\sum(k-1)\lambda_k} \frac{[n]!}{\prod\limits_{(i,j)\in[\lambda]} [h_{ij}^{\lambda}]}$$

wobei $[r] = 1 + q + q^2 + \cdots + q^{r-1}$ ist. Im Sinne der folgenden Vermutung von Richard Dipper und Gordon James haben wir den analogen Begriff der Standardbasis.

Vermutung. Für $\mathfrak{s} \in \mathrm{Std}(\lambda)$ gibt es $r_{\mathfrak{s}}(t) \in \mathbb{Z}[t]$ mit konstantem Term 1 und $\mathcal{B}_{\mathfrak{s}} \subset S^{\lambda}$ mit Größe $|\mathcal{B}_{\mathfrak{s}}| = r_{\mathfrak{s}}(q)$, so dass $\mathcal{B} = \bigcup\limits_{\mathfrak{s}\in Std(\lambda)} \mathcal{B}_{\mathfrak{s}}$ eine Basis von S^{λ} ist. $\mathcal{B}$ heißt Standardbasis von S^{λ}.

Marco Brandt, Richard Dipper, Gordon James und Sinéad Lyle zeigen diese Vermutung für den Fall, dass λ eine 2-Teile-Partition ist. Da der Beweis sehr kombinatorisch ist, scheint es unwahrscheinlich, dass diese Methoden für beliebige λ funktionieren. Dies motiviert uns neue Methoden zu finden. Unsere neue Strategie ist: Untersuchung der $FU-$Modul Struktur von S^λ. Der Vorteil der Einschränkung zu $FU-$Moduln ist, dass FU halbeinfach ist, wenn die Charakteristik von F koprim zu q ist.

Kapitel 1

Das erste Kapitel dient dazu, die grundlegenden Begriffe und Hilfsmittel einzuführen. Dabei starten wir mit der Beschreibung der Ausgangslage: In der ganzen Arbeit ist n eine natürliche Zahl, p eine Primzahl und q eine Potenz von p. Weiter soll F ein Körper sein, dessen Charakteristik koprim zu p ist und der eine primitive $p-$te Einheitswurzel enthält. Zudem wählen und fixieren wir einen nichttrivialen linearen $F-$Charakter θ der Gruppe $(GF(q), +)$.

In den folgenden Unterkapiteln geben wir dann einen Überblick über die für uns wichtigen Definitionen und Sätze auf dem Gebiet der Kompositionen, Partitionen und Tableaux. Desweiteren stellen wir Wurzel-Untergruppen, unipotente Untergruppen und Borel-Untergruppen von der endlichen generellen linearen Gruppe $GL_n(q)$ vor und erhalten somit die sogenannte Bruhatzerlegung.

Kapitel 2

Für eine Komposition λ von n führen wir $\lambda-$Fahnen und parabolische Untergruppen ein. Dann stellen wir den $FGL_n(q)-$Modul M^λ vor, der als $F-$Basis die $\lambda-$Fahnen besitzt, auf denen die generelle lineare Gruppe $GL_n(q)$ in kanonischer Form operiert. Diese Operation schreiben wir mit $\circ$.
Der unipotent Spechtmodul S^λ ist ein Untermodul von M^λ :

$$S^\lambda = M^\lambda E_F^+(\lambda')FGL_n(q),$$

wobei $E_F^+(\lambda')$ ein Idempotent in FU_n^+ ist. Gordon James gab eine andere Beschreibung von dem unipotenten Spechtmodul als Schnitt von Kernen gewisser $FGL_n(q)-$Homomorphismen.

Theorem 1. (2.2.4) Sei λ eine Komposition von n. Dann gilt:

$$S^\lambda = \bigcap_{\mu \rhd \lambda} \{\ker \Phi : \Phi \in \mathrm{Hom}_{FGL_n(q)}(M^\lambda, M^\mu)\}.$$

Kapitel 3

In diesem Kapital widmen wir uns der Untersuchung des Permutationsmoduls M^λ für eine beliebige aber feste 2-Teile-Partition $\lambda = (n-m, m)$. Dabei ist uns behilflich, dass wir jede $\lambda-$Fahne X als $m \times n$ Matrix schreiben.

Definition 1. (3.1.1) Seien m, n ganze Zahlen mit $0 \leqslant m \leqslant n$. Bezeichne $\Xi_{m,n}$ die Menge der $m \times n$ Matrizen $L = (l_{b_i j})$ über $GF(q)$, so dass für $b_1, \cdots, b_m \in \mathbb{Z}$ mit

$$1 \leqslant b_1 < b_2 < \cdots < b_m \leqslant n$$

die folgenden Bedingungen für alle $1 \leqslant i \leqslant m$ erfüllt sind:

(1) $l_{b_i b_i} = 1$ und $l_{b_i j} = 0$, falls $j > b_i$;

(2) $l_{b_k b_i} = 0$, falls $k > i$.

In dieser Definition indizieren wir die Zeilen der Elemente in $\Xi_{m,n}$ mit $b_1, \cdots, b_m$ anstelle von $1, \cdots, m$. Außerdem ist für jedes $1 \leqslant i \leqslant m$, $l_{b_i b_i} = 1$ der letzte von Null verschiedene Eintrag in Zeile b_i. Wir nennen ihn letzte 1.
Da wir auf den $\lambda-$Tableaux eine totale Ordnung (lexikographische Ordnung) haben, macht es Sinn, für ein Element $v \in M^\lambda$ mit $\mathrm{last}(v)$ das größte $\lambda-$Tableau zu bezeichnen, das in der Darstellung

$$v = \sum_{X \in \Xi_{m,n}} C_X X \qquad (C_X \in F)$$

einer Fahne X mit $C_X \neq 0$ zugeordnet werden kann und wir setzen

$$\mathrm{top}(v) = \sum_{\mathrm{tab}(X) = \mathrm{last}(v)} C_X X.$$

Unser erste Ziel ist die Untersuchung der Struktur von $M^{(n-m,m)}$ als FU_n-Modul. Wir haben mittels Mackey Zerlegung

$$\mathrm{Res}^{FG}_{FU} M^\lambda = \mathrm{Res}^{FG}_{FU} \mathrm{Ind}^{FG}_{FU} F = \bigoplus_{w \in \mathcal{D}_\lambda} \mathrm{Ind}^{FU}_{F(P^w_\lambda \cap U)} F.$$

$\mathrm{Ind}^{FU}_{F(P^w_\lambda \cap U)} F$ heißt der $\mathfrak{t}-$batch von $\mathrm{Res}^{FG}_{FU} M^\lambda$, wobei $\mathfrak{t} = \mathfrak{t}^\lambda w \in \mathrm{RStd}(\lambda)$ ist.

Lemma 1. (3.2.1) Sei $\lambda = (n-m, m)$ und $\mathfrak{t} \in \mathrm{RStd}(\lambda)$. Wir setzen

$$\mathfrak{X}_\mathfrak{t} = \{L \in \Xi_{m,n} \mid \mathrm{tab}(L) = \mathfrak{t}\}.$$

$\mathfrak{X}_\mathfrak{t}$ wird zur abelschen Gruppe indem man $L \diamond M$ als gewöhnliche summe von Matrizen definiert, wobei man neu Einträge, die nicht in Spalten mit letzten Einsen addiert.
Sei $\mathfrak{M}_\mathfrak{t}$ die Gruppenalgebra von $\mathfrak{X}_\mathfrak{t}$. Dann gilt

$$\mathfrak{M}_\mathfrak{t} = \mathrm{Ind}^{FU}_{F(P^w_\lambda \cap U)} F\,, \text{ der } \mathfrak{t}-\text{batch of } \mathrm{Res}^{FG}_{FU} M^\lambda.$$

Definition 2. Ist $\underline{\mathfrak{t}} = (b_1, b_2, \cdots, b_m)$, dann setzen wir

$$\mathfrak{J}_{\mathfrak{t}} = \{(b_i, j) \,|\, 1 \leqslant i \leqslant m, j < b_i, j \notin \{b_1, \cdots, b_m\}\}.$$

$\mathfrak{X}_{\mathfrak{t}}$ mit Addition $\diamond$ ist abelsche Gruppe der Ordnung $q^{|\mathfrak{J}_{\mathfrak{t}}|}$ (siehe Beispiel 3.2.3).

Bezeichnen wir für $(i, j) \in \mathfrak{J}_{\mathfrak{t}}$ mit $\xi_{b_i j}$ die $(i, j)-$Koordinatenfunktion von $\mathfrak{X}_{\mathfrak{t}}$ nach $GF(q)$, so erhalten wir für jedes $L = (l_{b_i j}) \in \mathfrak{X}_{\mathfrak{t}}$ mit

$$\chi_L := \sum_{(b_i, j) \in \mathfrak{J}_{\mathfrak{t}}} l_{b_i j} \theta \xi_{b_i j}$$

einen linearen Charakter von $(\mathfrak{X}_{\mathfrak{t}}, \diamond)$, der ein $M = (M_{b_i j}) \in \mathfrak{X}_{\mathfrak{t}}$ auf

$$\chi_L(M) = \prod_{(b_i, j) \in \mathfrak{J}_{\mathfrak{t}}} \theta(l_{b_i j} m_{b_i j})$$

abbildet. Für jedes $L =\in \mathfrak{X}_{\mathfrak{t}}$ bekommen wir dann mittels

$$e_L = \frac{1}{q^{|\mathfrak{J}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{X}_{\mathfrak{t}}} \chi_L(-M)[M]$$

ein Idempotent der Gruppenalgebra $\mathfrak{M}_{\mathfrak{t}}$ von $\mathfrak{X}_{\mathfrak{t}}$ über F, das zum linearen Charakter χ_L gehört. $\{[M] \,|\, M \in \mathfrak{X}_{\mathfrak{t}}\}$ ist die natürliche Basis von $\mathfrak{M}_{\mathfrak{t}}$. $\mathcal{E}_{\mathfrak{t}} = \{e_L \,|\, L \in \mathfrak{X}_{\mathfrak{t}}\}$ is eine weitere Basis die aus Idempotenten besteht.

Die Tatsache, dass die unipotente Gruppe $U_n(q)$ ein Untermodul von $GL_n(q)$ ist, motiviert uns, die Struktur von $M^{(n-m,m)}$ als FU_n-Modul zu untersuchen. Zuerst betrachten wir die Wirkung der Untergruppe $U_n^w \cap U_n$ auf $\mathcal{E}_{\mathfrak{t}} = \{e_L \,|\, L \in \mathfrak{X}_{\mathfrak{t}}\}$, wobei $\mathfrak{t} = \mathfrak{t}^{\lambda} w$ ist.

Proposition 1. (3.3.3) Sei $\lambda = (n - m, m), \mathfrak{t} = \mathfrak{t}^{\lambda} w \in \mathrm{RStd}(\lambda)$. Dann gilt für $L = (l_{b_i j}) \in \mathfrak{X}_{\mathfrak{t}}$, $g \in U_n^w \cap U_n$:

$$e_L \circ g = C(L, g) e_K$$

wobei $0 \neq C(L, g) \in F, K = (k_{b_i j}) \in \mathfrak{X}_{\mathfrak{t}}$ ist.

Hierdurch ist auf natürliche Weise der Begriff des $F(U_n^w \cap U_n)-$Orbits gegeben. Um mehr über den Orbit zu verstehen, führen wir die folgenden Definitionen ein.

Definition 3. (3.4.4) $L \in \Xi_{m,n}$ heißt eine Muster-Matrix, falls jede Zeile und Spalte von L höchstens einen von Null verschiedenen Eintrag außer der letzten 1 hat.

Definition 4. (3.4.5) Sei L eine Muster-Matrix. Die Menge von den nicht Null indiziert Eintragen von L außer dem letzten 1'n heißt eine Bedingungsmenge von L, mit $S(L)$ bezeichnet. Und die Menge von Indices der nicht Null Eintragen von L außer dem letzten 1'n heißt Rahmen von der Bedingungmenge, und wird mit $\overline{S(L)}$ bezeichnet.

Definition 5. (3.4.11) Sei $\lambda = (n-m, m), L \in \Xi_{m,n}$ eine Muster-Matrix und S die Bedingungsmenge von L. Wir definieren:

(1) $S_I = \{i \mid (i,j) \in \overline{S}\}, S_J = \{j \mid (i,j) \in \overline{S}\}, s = |S| = |S_I| = |S_J|$;

(2) Die Menge der zeilenstandard aber nicht standard $(n-m-s, m-s)-$Tableaux mit Einträgen aus der Menge $\{1, 2, ..., n\} \setminus (S_I \cup S_J)$ wird mit T_S bezeichnet.

In der Tat hat jeder $F(U_n^w \cap U_n)-$Orbit $\mathcal{O}$ eine eindeutige bestimmte Muster-Matrix L, und auch eine eindeutige bestimmte Bedingungsmenge. Wir definieren

$$M_{\mathcal{O}} := e_L F(U_n^w \cap U_n) = \bigoplus_{e_K \in \mathcal{O}} F e_K;$$

$$\mathrm{tab}(\mathcal{O}) = \mathrm{tab}(L), S(\mathcal{O}) = S(L) \text{ und } \overline{S(\mathcal{O})} = \overline{S(L)}.$$

Nun erweitern wir die Operation zu FU_n.

Proposition 2. (3.6.8) Sei $\lambda = (n-m, m), \mathcal{O}$ ein $F(U_n^w \cap U_n)-$Orbit in M^λ. Dann ist $M_{\mathcal{O}}$ invariant unter der Operation von U_n.

Nun ist $M_{\mathcal{O}}$ ein FU_n-Modul, deshalb können wir die $F(U_n^w \cap U_n)-$Orbits auch FU_n-Orbits nennen. Außerdem haben wir:

Theorem 2. (3.5.1) Sei $\lambda = (n-m, m), \mathcal{O}$ ein FU_n-Orbit in M^λ. Dann ist $M_{\mathcal{O}}$ ein irreduzibeler FU_n-Untermodul von M^λ.

Der letzte Begriff, den wir in diesem Kapital einführen, ist die Vereinigung aller FU_n-Orbits mit derselben BedingungsMenge.

Definition 6. (4.1.34) Sei $\lambda = (n-m, m)$, S eine BedingungsMenge. Wir definieren:

$$\begin{aligned} \mathfrak{C}_S^\lambda &= \bigoplus_{S(\mathcal{O})=S} M_{\mathcal{O}} \\ &= \bigoplus_{S(\mathcal{O})=S} \bigoplus_{e_R \in \mathcal{O}} F e_R, \text{ wobei } M_{\mathcal{O}} \subset M^\lambda. \end{aligned}$$

Kapitel 4

In Kapital 4 untersuchen wir den Spechtmodul S^λ für eine beliebige aber feste 2-Teile-Partition $\lambda = (n-m, m)$. Dabei verfolgen wir das Ziel, eine Standardbasis von S^λ zu finden.
Wenn λ eine 2-Teile-Partition ist, haben wir eine einfachere Version von Theorem 1.

Theorem 3. (2.2.6) Sei $\lambda = (n-m, m)$. Dann gilt:

$$S^\lambda = \bigcap_{i=0}^{m-1} \ker \phi_{1,i}; \quad \dim S^\lambda = \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}.$$

Hierbei ist $\phi_{1,i}$ eine lineare Abbildung von $M^{(n-m,m)}$ nach $M^{(n-i,i)}$, die jeden m-dimensionalen Untervektorraum V auf eine Linearkombination der in V enthaltenen i-dimensionalen Untervektorräume schickt.

Wir setzen $\Phi_m = \phi_{1,m-1}$ und fokussieren uns anschließend auf diesem Homomorphismus.

Lemma 2. (4.1.24) Sei $\lambda = (n-m, m)$, $\mathcal{O}$ ein Orbit in M^λ. Dann gilt:

$$\Phi_m(M_{\mathcal{O}}) \subset \mathfrak{C}_{S(\mathcal{O})}^{(n-m+1,m-1)}.$$

Eine wichtige Resultate folgen aus diesem Lemma:

Theorem 4. (4.1.46) $S(\mathcal{O}_1) = S(\mathcal{O}_2)$ impliziert $M_{\mathcal{O}_1} \cong M_{\mathcal{O}_2}$.

Sinéad Lyle zeigt eine interessante Eigenschaft der Elemente von S^λ.

Theorem 5. (4.2.4) Sei $0 \neq v \in S^\lambda$. Dann ist $\mathrm{last}(v)$ ein standard $\lambda-$Tableau.

Wir benutzen diese Eigenschaft häufig in den Beweisen der folgenden Behauptungen. Ein anderes wichtiges Hilfsmittel in den Beweisen ist die folgende Abbildung.

Definition 7. (4.3.1) Sei S eine Bedingungsmenge und $s = |S|$. Sei $\lambda = (n-m, m)$, $\mu = (n-m-s, m-s)$. Wir definieren eine Abbildung

$$\mathfrak{R}_S : \{L \mid L \in \Xi_{m,n}\} \longrightarrow \{\tilde{L} \mid \tilde{L} \in \Xi_{m-s,n-2s}\}$$

wobei $\mathfrak{R}_S(L) = \tilde{L}$ bezeichnet. Wir erhalten $\tilde{L}$, indem wir für jedes $i \in S_I, j \in S_J$ Zeile i, Spalte i, j aus L entfernen.

Zuerst fokussieren wir auf den Fall: $\mathrm{char}(F) = 0$.

Lemma 3. (4.1.40) Ist $\mathrm{char}(F) = 0$, dann gilt $S^\lambda = \ker \Phi_m$.

Lemma 4. (4.1.41) Ist $\mathrm{char}(F) = 0$, dann erhalte wir: Φ_m ist ein Epimorphismus.

Lemma 5. (4.3.32) Sei $\mathrm{char}(F) = 0, \lambda = (n-m, m), \mu = (n-m+1, m-1)$ und S eine Bedingungsmenge mit $0 \leqslant |S| \leqslant m-1$. wir bezeichnen mit $\mathcal{O}$(bzw. $\tilde{\mathcal{O}}$) die FU_n-Orbits in M^λ (bzw. M^μ). Setze

$$\mathfrak{C}_S^\mu = \bigoplus_{S(\tilde{\mathcal{O}})=S} \bigoplus_{e_R \in \tilde{\mathcal{O}}} F e_R.$$

Dann gilt

$$\mathfrak{C}_S^\mu = F\text{-span}\{\Phi_m(M_{\mathcal{O}}) \mid S(\mathcal{O}) = S, \mathrm{tab}(\mathcal{O}) \setminus (S_I \cup S_J) \text{ ist non-standard}\}$$

als $F-$Vektorraum.

Nun sind wir bereit, das folgende Theorem zu beweisen.

Theorem 6. (4.3.38) Sei $\lambda = (n-m, m)$. Für $L \in \Xi_{m,n}$ existiert $v_L \in S^\lambda$, so dass $\mathrm{last}(v_L) = \mathrm{tab}(L)$ und $\mathrm{top}(v_L) = e_L$ genau dann, wenn

$$\mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ ein standard } \mu-\text{Tableau ist,}$$

wobei $e_L \in \mathcal{O}, S = S(\mathcal{O}), \mu = (n-m-|S|, m-|S|)$.

Für jedes L, das diese Bedingung erfüllt, fixieren wir nun ein derart konstruiertes Element v_L (die Konstruktion ist bei weitem nicht eindeutig) und setzen wir

$$\mathcal{B}_S^\lambda := \{v_L \mid e_L \in \mathcal{O} \subset M^\lambda, S(\mathcal{O}) = S, \mathrm{tab}(L) \setminus (S_I \cup S_J) \text{ ist standard}\}$$

und

$$\mathcal{B}^\lambda = \bigcup_S \mathcal{B}_S^\lambda.$$

Dann ist $\mathcal{B}_S^\lambda$ eine Standardbasis von $S^\lambda{\downarrow}_S$. Und $\mathcal{B}^\lambda$ ist eine Standardbasis von S^λ.

Hauptergebnisse

Sei $\lambda = (n-m, m)$.

(1) Wir geben ein vollständig Zerlegung den Permutationsmoduls $\mathrm{Res}_{FU}^{FG} M^\lambda$ in eine direkte Summe irreduzibler $FU-$Moduln.

(2) Sei $0 \leqslant c \in \mathbb{Z}$. Dann ist die Anzahl der irreduziblen direkten Summanden von $\mathrm{Res}^{FG}_{FU} M^\lambda$ ein ganzzahliges Polynom in q.

(3) Wir geben ein vollständig Zerlegung der Einschränkung des unipotenten Spechtmoduls S^λ auf U in eine direkte summe von irreduziblen $FU-$Moduln.

(4) Wir konstruieren eine standard Basis von S^λ.

(5) Der S^λ mit den Ergebnissen von Marco Brandt, Richard Dipper, Gordon James und Sinéad Lyle wird hergestellt und ein neuer Beweise erbracht, dass die Rangpolynome sich zum generischen Rang von S^λ aufsummieren.

(6) Sei $0 \leqslant c \in \mathbb{Z}$. Dann ist die Anzahl der irreduziblen Komponenten von $\mathrm{Res}^{FG}_{FU} S^\lambda$ für eine festgelegt Dimension q^c ein ganzzahliges Polynom in q.

Notation

Symbol	Description	Page
B_n	Borel subgroup	9
F	field, $p \nmid \mathrm{char}(F)$, contains a pth root of unity	1
F^*	multiplicative group of F	1
$GF(q)$	finite field of q elements	1
$GL_n(q)$	group of invertible $n \times n$ matrices over $GF(q)$	1
M^λ	permutation module	12
$M_{\mathcal{O}}$	$U^w \cap U-$orbit module	33
$S(L)$	condition set of a pattern matrix L	37
$S(\mathcal{O})$	condition set of an orbit $\mathcal{O}$	39
S^λ	Specht module	14
S_I	row indices of positions in frame $\overline{S}$	39
S_J	column indices of positions in frame $\overline{S}$	39
$T^\lambda_\emptyset$	set of shifted tableau	83
U	lower triangular unipotent matrices of $GL_n(q)$	10
$U^w \cap U$	subgroup of U	25
X_{ij}	root subgroup	7
$[m]!$	polynomial in q	2
$[m]$	polynomial in q	2
$\mathfrak{J}_\mathfrak{t}$	index set	21
$\mathfrak{M}_\mathfrak{t}$	$\mathfrak{t}-$batch of M^λ	22
$\mathcal{O}$	$U^w \cap U-$orbit	33
$\Omega_{(b,j)}$	$(b,j)-$hook of a matrix	35
$\Omega^{\mathbf{right}}_{(b,j)}$	$(b,j)-$hook column of a matrix	35
$\Omega^{\mathbf{up}}_{(b,j)}$	$(b,j)-$hook column of a matrix	35
Φ_m	$FGL_n(q)-$homomorphism $\phi_{1,m-1}$	57
Φ^d_m	$d-$component of Φ_m	60
$\Phi_{m,S}$	$S-$component of Φ_m	72
$\mathrm{RStd}(\lambda)$	set of row-standard $\lambda-$tableaux	7
$\mathfrak{S}_n$	symmetric group of degree n	3
Stab	stabilizer	40
$\mathrm{Std}(\lambda)$	set of standard $\lambda-$tableaux	7
$\mathfrak{X}_\mathfrak{t}$	basis element of $\mathfrak{M}_\mathfrak{t}$	22
$\Xi_{m,n}$	notation for the $(n-m,m)$-flags	18
χ_L	character of $\mathfrak{X}_\mathfrak{t}$	24
last	function from M^λ to $\mathrm{Std}(\lambda)$	21

Symbol	**Description**	**Page**
$\mathcal{E}_{\mathfrak{t}}$	the set of idempotents in $\mathfrak{M}_{\mathfrak{t}}$	24
$\mathcal{F}(\lambda)$	set of $\lambda-$flags	11
$\mathfrak{R}_S$	$F-$linear map	81
$\overline{S(L)}$	the frame of a condition set $S(L)$	37
$\phi_{d,i}$	$FGL_n(q)-$homomorphism	15
π	path	100
$\mathfrak{t}^\lambda$	initial $\lambda-$tableau	6
$\mathrm{tab}(L)$	tableau associated with L	19
θ	linear F-character of the group $\big(GF(q),+\big)$	1
$\tilde{\mathfrak{t}}$	shifted tableau	82
e_L	idempotent in $\mathfrak{M}_{\mathfrak{t}}$	24
e_{ij}	$(i,j)-$th matrix unit	7
$r_{\mathfrak{t}}(q)$	rank polynomial	100
v_L	(basis) element of S^λ	98
$\mathrm{res}(b,j)$	residue of $\Omega_{(b,j)}$	35
$\mathfrak{C}_S^\lambda$	collection of orbit modules	69
$\underline{\mathfrak{t}}$	the second row of $\mathfrak{t}$	20

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