On torsion subgroups and their normalizers in integral group rings

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The small part of ignorance that we arrange and classify we give the name of knowledge.

(Ambrose Bierce)

Zusammenfassung (German summary)

Kapitel 2

In Kapitel 2 werden ganzzahlige Gruppenringe von endlichen Gruppen auf die Zassenhausvermutungen hin untersucht, das heißt ob endliche Untergruppen der normierten Einheiten des ganzzahligen Gruppenringes innerhalb des rationalen Gruppenringes konjugiert zu Untergruppen der Gruppenbasis sind. Nach einem Überblick über einige wichtige Resultate zu den Zassenhausvermutungen, wird als Erweiterung eines Resultates von Dokuchaev und Juriaans gezeigt, dass für eine Primzahl p für Gruppen, die p-beschränkt sind und eine abelsche p-Sylowgruppe haben, alle p-Untergruppen der normierten Einheiten des ganzzahligen Gruppenringes rational konjugiert zu einer Untergruppe der Gruppenbasis sind (Proposition 2.13). Darüber hinaus wird folgendes Resultat gezeigt, das einen Hinweis darauf gibt, dass man rationale Konjugiertheit auch zeigen kann, wenn die Gruppe nicht p-beschränkt ist.

> **Theorem 2.16.** *Sei G eine endliche Gruppe mit abelschen 2-Sylowgruppen von Ordnung höchstens 8. Dann sind 2-Untergruppen der normierten Einheiten von $\mathbb{Z}G$ rational zu Untergruppen von G konjugiert.*

Außerdem werden p-Untergruppen der Einheiten des ganzzahligen Gruppenringes der endlichen speziellen linearen Gruppen untersucht. Insbesondere wird auch eine positive Antwort für die Primgraphfrage für die Gruppen $\mathrm{SL}(2,q)$, wobei q eine ungerade Primzahlpotenz ist, hergeleitet (Proposition 2.21).

Kapitel 3

Im ersten Abschnitt dieses Kapitels wird ein ausführlicher Überblick über bekannte Ergebnisse für das »klassische« Normalisatorproblem gegeben: die Frage, ob die Gruppenbasis in den Einheiten des ganzzahligen Gruppenrings nur von Produkten aus Gruppenelementen und zentralen Einheiten normalisiert wird. Im Hauptteil des Kapitels wird dann eine entsprechende Frage für Untergruppen untersucht: wann gilt $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ für eine Untergruppe $H \leq G$ und geeignete Ringe R? Positive Antworten erhält man zum Beispiel wenn H zyklisch ist (Proposition 3.23). Ebenso werden für normale Untergruppen H folgender Typen positive Resultate erzielt: Gruppen von Ordnung pq, Diedergruppe von Ordnung $2m$ für ungerades m (Corollary 3.50) und endlich einfache Gruppen (Corollary 3.38). Auch wird eine positive Antwort für unendliche einfache Untergruppen (Proposition 3.19) und für torsionsfreie abelsche Normalteiler gegeben (Proposition 3.20). Außerdem erhält man folgendes Resultat:

> **Theorem 3.26.** *Sei G eine lokal nilpotente Torsionsgruppe. Dann gilt $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ für alle Untergruppen $H \leq G$ und G-adaptierte Ringe R.*

Ähnliche Resultate werden auch noch für weitere Klassen von nilpotenten Gruppen, welche nicht mehr zwingend Torsionsgruppen sein müssen, erzielt (Proposition 3.30, Corollary 3.33 und Proposition 3.34). Ein Hauptresultat des Kapitels behandelt metazyklische Gruppen:

> **Theorem 3.42.** *Sei G eine metazyklische Gruppe mit zyklischem Normalteiler N, so dass der Quotient ${}^{G}/_{N}$ zyklisch ist. Gilt eine der folgenden Bedingungen*
>
> 1. *$|N|$ und $\left|{}^{G}/_{N}\right|$ sind teilerfremd,*
> 2. *N ist von Primzahlordnung oder*
> 3. *${}^{G}/_{N}$ ist von Primzahlordnung,*
>
> *so gilt für jede Untergruppe $H \leq G$ und jeden Koeffizientenring R, dass $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$.*

Hiermit folgt, dass die Eigenschaft für alle Gruppen von quadratfreier Ordnung gilt (Corollary 3.44). Das Resultat dient auch als wichtiger Baustein für

> **Corollary 3.48.** *Sei G eine endliche Untergruppe von $\mathrm{O}(3,\mathbb{R})$, der dreidimensionalen orthogonalen Gruppe über $\mathbb{R}$. Dann gilt für alle Untergruppen $H \leq G$, dass $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ für G-adaptierte Ringe R.*

Weiterhin wird bewiesen, dass $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ für alle $H \leq G$ gilt, falls G eine abelsche Untergruppe vom Index 2 besitzt (Lemma 3.53). Mit diesen Resultaten wird auch überprüft, dass der Normalisator von Untergruppen

H von G in den Einheiten des Gruppenrings RG für alle Gruppen G von Ordnung höchstens 47 nur aus den »offensichtlichen« Einheiten besteht, wobei R ein G-adaptierter Ring ist (Proposition 3.54).

Im letzten Unterabschnitt des Kapitels werden für Einheiten des ganzzahligen Gruppenringes, welche eine Untergruppe H normalisieren, H-Trägergruppen eingeführt (entsprechend der Trägergruppen, welche bei der Untersuchung des »klassischen« Normalisatorproblems für unendliche Gruppen durch Jespers, Juriaans, de Miranda und Rogerio eine entscheidende Rolle gespielt haben). Außerdem werden einige Eigenschaften der Gruppe $\mathrm{Aut}_{RG}(H)$, einer speziellen Untergruppe der Automorphismengruppe von H, welche in natürlicher Weise bei den Untersuchungen dieses Kapitels auftritt, hergeleitet.

Kapitel 4

Im letzten Kapitel werden Zentralisatoren von Untergruppen einer Gruppenbasis untersucht und gezeigt, dass unter bestimmten Voraussetzungen nur die »offensichtlichen« Torsionseinheiten bestimmte Untergruppen zentralisieren:

> **Proposition 4.7.** *Sei $H \leq G$. Falls es eine Primzahl p und eine p-Torsionsgruppe $P \leq H$ gibt, für welche $\mathrm{C}_G(P)$ im Torsionsteil von H enthalten ist, dann sind alle H zentralisierenden normierten Torsionseinheiten von $\mathbb{Z}G$ schon im Zentrum von H enthalten.*

Die Voraussetzungen sind beispielsweise für Frobeniuskern und -komplement endlicher Frobeniusgruppen erfüllt. Es wird gezeigt, dass unter bestimmten Voraussetzungen maximale abelsche Untergruppen auch die maximalen Untergruppen mit der entsprechenden Eigenschaft in der Einheitengruppe des ganzzahligen Gruppenringes sind (Corollary 4.8, Proposition 4.10). Abschließend wird folgendes Resultat gezeigt:

> **Theorem 4.14.** *Angenommen U ist eine isolierte Untergruppe der endlichen Gruppe G. Dann stimmen die Primgraphen von $\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)$ und $\mathrm{N}_G(U)$ überein.*

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Summary

Chapter 2

In Chapter 2 we consider integral group rings of finite groups with respect to the Zassenhaus conjectures, i.e. whether finite subgroups of the group of normalized units of the integral group ring are conjugate within the rational group ring to subgroups of the group basis. Following an overview of important results concerning the Zassenhaus conjectures, we extend a result of Dokuchaev and Juriaans and show that for a fixed prime p, p-subgroups of the units of integral group rings of p-constrained groups with abelian Sylow p-subgroups are conjugate within the rational group ring to subgroups of the group (Proposition 2.13). In addition, we prove the following result which indicates that the conjugacy statement may hold more generally, even if the considered groups are not p-constrained.

> **Theorem 2.16 .** *Let G be a finite group having abelian Sylow 2-subgroups of order at most 8. Then 2-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are conjugate within $\mathbb{Q}G$ to subgroups of G.*

Furthermore we examine p-subgroups of the unit groups of the finite special linear groups. In particular we obtain a positive answer to the prime graph question for the groups $\mathrm{SL}(2,q)$, where q is a power of an odd prime (Proposition 2.21).

Chapter 3

In the first section of this chapter we give an extensive overview of known results concerning the 'classical' normalizer problem: i.e. the question, whether the group basis in the units of the integral group ring is only normalized by products of group elements and central units. In the main section of the chapter we investigate the corresponding question for subgroups: when does $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ hold for a subgroup $H \leq G$ and an appropriate ring R? We get at a positive answer, for example, if H is cyclic (Proposition 3.23). We also obtain positive results for normal subgroups H of the following isomorphism types: groups of order pq, dihedral groups of order $2m$ for odd m (Corollary 3.50), and finite simple groups (Corollary 3.38). We also give an affirmative answer if H is an infinite simple groups (Proposition 3.19) or a normal, torsion-free abelian subgroups (Proposition 3.20). A large class of groups is settled by the following result:

Theorem 3.26. *Let G be a locally nilpotent torsion group. Then $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ holds for all $H \leq G$ and all G-adapted rings R.*

Similar results are obtained for further classes of nilpotent groups, including groups that are not necessarily torsion (Proposition 3.30, Corollary 3.33, and Proposition 3.34). A main result of the chapter deals with metacyclic groups:

Theorem 3.42. *Let G be a metacyclic group with normal cyclic subgroup N, such that the quotient ${}^{G}/_{N}$ is cyclic. Assume that one of the following is true:*

1. *$|N|$ and $\left|{}^{G}/_{N}\right|$ are finite and coprime,*
2. *N is of prime order, or*
3. *${}^{G}/_{N}$ is of prime order.*

Then $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ for every subgroup $H \leq G$ and all coefficient rings R.

This implies that this property holds as well for all groups of squarefree order (Corollary 3.44). The result is also an important component of

Corollary 3.48. *Let G be a finite subgroup of $\mathrm{O}(3, \mathbb{R})$, the orthogonal group of degree 3 over the reals. Then for all subgroups H of G and all G-adapted coefficient rings R, $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ holds.*

Furthermore we prove that $\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$ holds for all subgroups $H \leq G$, provided G has an abelian subgroup of index 2 (Lemma 3.53). Using the results obtained so far, we verified that the normalizer of subgroups H of G in the unit group of RG, where R is a G-adapted ring, consists only of the 'obvious' units, provided G has order at most 47 (Proposition 3.54).

In the last subsection of the chapter we define for units of integral group rings, normalizing a subgroup H, an H-support group (corresponding to the support group, which played a crucial role when Jespers, Juriaans, de Miranda, and Rogerio considered the 'classical' normalizer problem for infinite groups). Furthermore we deduce certain properties of the group $\mathrm{Aut}_{RG}(H)$, a special subgroup of the automorphism group of H, which appears in a natural way within the investigations of the subgroup normalizer problem.

Chapter 4

In the last chapter we examine centralizers of subgroups of group bases and we show that under certain assumptions only the 'obvious' torsion units centralize these subgroups

> **Proposition 4.7.** *Let $H \leq G$ and assume that there is a prime p and a torsion p-subgroup $P \leq H$ such that $\mathrm{C}_G(P)$ is contained in the torsion part of H. Then the normalized torsion units of $\mathbb{Z}G$ centralizing H are precisely the torsion elements of $\mathrm{Z}(H)$.*

This result applies for example to Frobenius kernels and complements of finite Frobenius groups. Moreover we establish that under certain assumptions maximal abelian subgroups are also maximal with the corresponding properties in the unit group of the integral group ring (Corollary 4.8, Proposition 4.10). Finally we prove the following:

> **Theorem 4.14.** *Assume that U is an isolated subgroup of the finite group G. Then the prime graphs of $\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)$ and $\mathrm{N}_G(U)$ coincide.*

1 Introduction

Let G be a (multiplicatively written) group and let R be a ring, commutative with an identity 1. We can form from this the so-called *group ring* RG of G over R. It is defined as the free R-module with basis G, i.e. it consists of all formal R-linear combinations

$$\sum_{g\in G} r_g g, \qquad r_g \in R, \quad r_g \neq 0 \text{ for only finitely many } g \in G,$$

equipped with a 'pointwise' addition and multiplication induced by the multiplication of G. Clearly, if two groups G and H are isomorphic, so are their group rings RG and RH over every ring R. The reverse question, if a group G is determined by its group ring RG, is much harder to decide. Are there any reasonable restrictions we can put on the ring R such that we have the following implication for any two groups G and H

$$RG \simeq RH \quad \Rightarrow \quad G \simeq H\ ?$$

Richard Brauer included this question for group rings over fields in his famous expository lecture on modern mathematics [Bra63, Problem 2*]: 'If two groups G_1 and G_2 have isomorphic group algebras over every ground field Ω, are G_1 and G_2 isomorphic?' An example constructed by Everett Dade [Dad71] shows that fields are in a certain sense not 'suitable' to distinguish two groups by their group rings: he constructed two non-isomorphic metabelian groups of order p^3q^6 for two different primes p and q having isomorphic group rings over every field. However, a result of Albert Whitcomb [Whi68] assures that metabelian groups are determined by their group rings over $\mathbb{Z}$ (in this thesis group rings over $\mathbb{Z}$ are called *integral group rings*). Noting that for every commutative ring R and every group G we have $R \otimes_{\mathbb{Z}} \mathbb{Z}G \simeq RG$ as rings, we see that the most general version of the question whether a group is determined by its group ring is the so-called *isomorphism problem for integral group rings*, which asks if

$$\mathbb{Z}G \simeq \mathbb{Z}H \quad \Rightarrow \quad G \simeq H\,? \tag{IP}$$

An element x of a group X is called a *torsion element*, if it has finite order, i.e. there is a positive integer m such that $x^m = 1$. The isomorphism problem is equivalent to the question if all maximal torsion subgroups of the units of $\mathbb{Z}G$ are isomorphic to $\pm G$. This is therefore a very special aspect of the problem to determine all torsion subgroups and their properties. Hans Zassenhaus conjectured, roughly speaking, that all these groups can be taken by conjugation inside the larger ring $\mathbb{Q}G$ into subgroups of G. To state his conjectures accurately we need some notation.

When considering torsion of the units of a group ring, we are interested in the torsion that comes from the group-ring-interplay, rather than torsion coming purely from the coefficient ring. To make this more precise, we make use of the so called augmentation map. Let R be a commutative ring with 1, G a group, and RG the group ring of G over R, then the *augmentation map* of RG is

$$\varepsilon\colon\ RG \to R\colon\ \sum_{g\in G} r_g g \mapsto \sum_{g\in G} r_g.$$

This is a ring homomorphism. Denote by $\mathrm{U}(RG)$ the *group of units of the group ring* RG, and define the *group of normalized units* $\mathrm{V}(RG)$ by

$$\mathrm{V}(RG) = \{x \in \mathrm{U}(RG) \mid \varepsilon(x) = 1\}.$$

Identifying $R^\times$ with $\{\lambda 1_G \mid \lambda \in R^\times\} \le \mathrm{U}(RG)$, where 1_G is the identity element of the group G, we have $\mathrm{U}(RG) = R^\times \cdot \mathrm{V}(RG)$. Now we can state the conjectures of Hans Zassenhaus:

Conjecture 1.1 (Zassenhaus conjectures)**.** Let G be a finite group.

(ZC1) If $u \in \mathrm{V}(\mathbb{Z}G)$ is an element of finite order, then there is a unit $x \in \mathbb{Q}G$ such that $x^{-1}ux \in G$.

(ZC2) If $H \le \mathrm{V}(\mathbb{Z}G)$ with $|H| = |G|$, then there is a unit $x \in \mathbb{Q}G$ such that $x^{-1}Hx = G$.

(ZC3) If $H \le \mathrm{V}(\mathbb{Z}G)$ is a finite subgroup, then there is a subgroup $W \le G$ and a unit $x \in \mathbb{Q}G$ such that $x^{-1}Hx = W$.

A subgroup as in (ZC2), i.e. a subgroup H of $\mathrm{V}(\mathbb{Z}G)$ with $|H| = |G|$, is a $\mathbb{Z}$-basis of $\mathbb{Z}G$ and is called a *group basis*. Roggenkamp and Scott constructed a counterexample to (ZC2) according to which group bases don't need to be conjugate within $\mathrm{U}(\mathbb{Q}G)$. The conjecture (ZC1) is still an open problem. There will be a brief discussion about the Zassenhaus conjectures and related questions, including further known results, in Chapter 2. The main purpose of Chapter 2 is to present positive results for the Zassenhaus conjectures for some cases where a group basis G is non-solvable.

For the question (IP), if group bases are at least determined up to isomorphism by their integral group rings, Martin Hertweck constructed in 1998 an example of a group answering this in the negative (see Theorem 3.8). A crucial step to get there was the construction of a counterexample to the so-called *normalizer problem*. This is the question whether the group basis G is only normalized by units which obviously do so, i.e. if $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G)$ coincides with $G \cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$. Based on much theoretical insight into the construction of a possible solvable counterexample for the isomorphism problem gained by the end of the last century, Hertweck was able to employ his solution of the normalizer problem to obtain a finite group X, such that the group ring $\mathbb{Z}X$ has a group basis Y not isomorphic to X. This gives a hint that there is a certain amount information about the structure of torsion subgroups encoded in the structure of the normalizer in the unit group of a group ring. This was one of the motivations that the main part of this thesis, Chapter 3, deals with the corresponding question for subgroups, namely if for a subgroup $H \leq G$ only the units of $\mathbb{Z}G$ that come to mind immediately, $\mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(\mathbb{Z}G)}(H)$, normalize H. The second motivation to consider this question is that this fact has been known to be true for a long time for a 'big' class of groups, namely for all p-groups H, and this knowledge was useful in different proofs in the area of group rings. Besides the above result, there were hardly any results concerning the question about normalizers of subgroups. In Section 3.1 we will review what is known about normalizers in integral group rings, and in Section 3.2 we will present new results.

The torsion part of the centralizer of G in $\mathbb{Z}G$ is well understood, again, considering subgroups H of G much less is known. In Chapter 4 we will consider this question. The results we obtain there enables us to give a proof, which is a nice blend of the knowledge about normalizers and centralizers that in a certain setting the existence of torsion units of specific orders are already reflected in some subgroups of the group basis.

Notation. In this thesis we will always assume that the coefficient ring of group ring is commutative and has an identity 1. Homomorphisms will be denoted from the right (i.e. for a homomorphism $f\colon X \to Y$ and an element $x \in X$, the symbol xf denotes the image of x under f). An exception is the augmentation map of the group ring: it will be written in the traditional notation, i.e. $\varepsilon(x)$ denotes the augmentation of x. For a map $f\colon X \to Y$ and a subset $D \subseteq X$, $f|_D$ denotes the restriction of f to D. We use conjugation as a right action, i.e. $y^x = x^{-1}yx$, the conjugacy class of x in G is denoted by x^G. We write $\mathrm{conj}(x)$ for any conjugation homomorphism $y \mapsto y^x$. The commutator of two group elements x and y is defined as $[x,y] = x^{-1}y^{-1}xy$. For a subgroup H of the group G we denote by ${}^{G}\!/_{H}$ the set of right cosets Hg of H in G, and, if H is a normal subgroup of G, the quotient group.

2 Zassenhaus conjectures and related topics

Throughout this chapter G is assumed to be a finite group.

Some of the most stimulating questions within the last 40 years in the area of integral group rings were, besides the Isomorphism problem, the conjectures of Hans Zassenhaus described in the introductory Chapter 1. We say that (ZC j) holds for a group ($j \in \{1,2,3\}$), if the corresponding statement is true for the group of normalized units in its integral group ring. The first Zassenhaus conjecture, (ZC1), was stated by Zassenhaus in [Zas74]. An overview over the conjectures and the facts as at 1993 can be found in [Seh93, Chapter 5]. We say that two elements u and v in $\mathrm{U}(\mathbb{Z}G)$ are *rationally conjugate*, if there is a unit $x \in \mathbb{Q}G$ such that $x^{-1}ux = v$. Similarly we say that two subgroups H and K in $\mathrm{U}(\mathbb{Z}G)$ are *rationally conjugate*, if there is a unit $x \in \mathbb{Q}G$ such that $x^{-1}Hx = K$.

There is a weakened version of the third Zassenhaus conjecture which does not require rational conjugacy, but only that the subgroups are isomorphic: the *subgroup question* asks

(SQ) If G is a finite group and $U \leq \mathrm{V}(\mathbb{Z}G)$ is a finite subgroup, is there a subgroup $W \leq G$ such that $U \simeq W$?

In 2007 Wolfgang Kimmerle proposed a question, which is a weakened version of the first Zassenhaus conjecture. To formulate this question, we need the notation of a prime graph.

Definition 2.1. Let X be a group. The *prime graph* or *Gruenberg-Kegel graph*

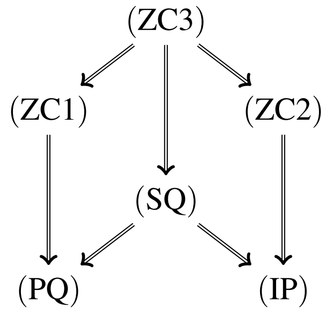
$\Gamma(X)$ of the group X is the undirected, loop-free graph having as vertices exactly those primes p, for which there is an element $x \in X$ of order p. Two different vertices p and q are joined by an edge if and only if there is an element of order pq in X.

Note that this definition makes sense for arbitrary groups.

The *prime graph question* posed in [Ari07, Problem 21] now is the following:

(PQ) If G is a finite group. Is $\Gamma(G) = \Gamma(\mathrm{V}(\mathbb{Z}G))$?

The connections between the conjectures and the questions are shown on the right (here, (PQ), (SQ), and (IP) denote that there is an affirmative answer to the corresponding questions).



Klaus W. Roggenkamp and Leonard Scott constructed a metabelian group of order $2880 = 2^6 \cdot 3^2 \cdot 5$ serving as a counterexample to (ZC2) and hence also to (ZC3), see [Rog91, Kli91, Sco92]. Nowadays there are examples known not fulfilling (ZC2) and being smaller in size. In [Bla01] Blanchard constructed three 'semi-local' counterexamples (i.e where the coefficient ring is the semi-localization $\mathbb{Z}_\pi$ at a set of primes dividing the order of G instead of $\mathbb{Z}$ – in Blanchard's examples $\pi = \{2,3\}$) to (ZC2) of order $96 = 2^5 \cdot 3$ and argued that there are no smaller ones. In his Habilitationsschrift [Her04, Section 10-12] Martin Hertweck showed that two of these examples lead to 'global' counterexamples (i.e with the coefficient ring $\mathbb{Z}$), but not the third one. There are important classes of groups for which (ZC3), and hence all the conjectures, are known to be true. One of the most striking examples is Alfred Weiss' proof that (ZC3) is true for all finite nilpotent groups [Wei91, Theorem 1].

The question if (ZC1) is always true is still an open problem. Luthar and Passi introduced an approach to deal with this question in an algorithmic way, which relies on the data of the group stored in the ordinary character table of the examined group, and proved with this method that (ZC1) holds for A_5, the alternating group of degree 5 [LP89]. This method was improved by Martin Hertweck, when he discovered that there is a p-modular analogue, which uses Brauer characters instead of ordinary characters [Her07]. He used this to verify the first Zassenhaus conjecture for certain projective special linear groups over prime fields. A celebrated result is Martin Hertweck's proof that (ZC1) holds for all metacyclic groups [Her08b]. Besides this there are affirmative results by Juriaans and Polcino Milies for all Frobenius groups of order $p^a q^b$ for distinct primes p and q [JPM00], by Höfert for all 'small groups' of order at most 71 [Höf04], and by Bovdi, Höfert, and Kimmerle for all 2- and 3-dimensional crystallographic groups [BHK04].

Asking if (ZC1) is valid for a group always includes the questions if all finite cyclic subgroups of the integral group rings group of units are isomorphic to subgroups of the group, or, if there is at least an affirmative answer to the prime graph question. Using the method developed by Luthar, Passi, and Hertweck it was proved that the integral group ring of the smallest group of the series of the Suzuki simple groups, $\mathrm{Sz}(8)$, only has finite cyclic subgroups isomorphic to subgroups of the group [Bäc08]. Affirmative answers to the prime graph question were obtained by this method for thirteen of the sporadic simple groups in a series of papers by Bovdi, Jespers, Konovalov, Linton, do Nascimento Marcos, and Siciliano [BK07d, BKS07, BKL08, BK08, BK07b, BJK07, BK07e, BK07c, BK07a, BKdNM08]. For solvable groups there is also a positive answer to (PQ) obtained by Christian Höfert and Wolfgang Kimmerle in [Kim06, Höf08]. This was improved by Martin Hertweck in [Her08a], when he showed that all isomorphism types of finite cyclic subgroups occurring in the units of the integral group ring of a finite solvable group already occur as a subgroup of the considered solvable group.

A question of Marciniak at a satellite conference of the ICM 2006 can be seen as an instance of a reverse of (SQ): If we fix an isomorphism type of a finite group, is there necessarily a subgroup isomorphic to this group, provided there is such a subgroup in $\mathrm{V}(\mathbb{Z}G)$? This is known for cyclic subgroups of prime power order by a result of Cohn and Livingstone [CL65] and was extended to arbitrary cyclic groups by [Her08a] if G is solvable. Zbigniew Marciniak illustrated how little is known for this question, when he asked if this is also true when we consider a Klein four group. Wolfgang Kimmerle observed that this can be settled by using the Brauer-Suzuki theorem [Kim07, Proposition]. The result was extended to subgroups isomorphic to $C_p \times C_p$ for all primes p by Martin Hertweck (cf. [Her08c, Theorem A]). These results are, of course, also very useful when considering (SQ).

2.1 Some fundamental results on torsion subgroups of integral group rings

One of the main ingredient when studying torsion elements in integral group rings was already discovered by Graham Higman in his thesis [Hig40] and, independently, by Samuil Davidovich Berman [Ber53]

Theorem 2.2 (Higman, Berman). *Let G be a finite group and $u = \sum u_g g \in \mathrm{V}(\mathbb{Z}G)$ a unit of finite order. If $u_1 \neq 0$, then $u = 1$.*

In the abovementioned work, Berman noticed that finite subgroups of $\mathrm{V}(\mathbb{Z}G)$ can not be arbitrarily large

Theorem 2.3 (Berman). *Let G be a finite group and $H \leq \mathrm{V}(\mathbb{Z}G)$ finite, then $|H|$ divides $|G|$.*

For finite cyclic subgroups this was improved in the following way:

Theorem 2.4 (Cohn, Livingstone [CL65]). *Let G be a finite group and $u \in \mathrm{V}(\mathbb{Z}G)$ a torsion unit, then $o(u) \mid \exp G$.*

Marciniak, Ritter, Sehgal, and Weiss discovered an equivalent formulation to verify conjugation by units of $\mathbb{Q}G$ which uses so-called partial augmentation (and forms the basis of the already mentioned Luthar-Passi-Hertweck method).

Definition 2.5. For a conjugacy class C of the group G define the *partial augmentation* map by

$$\varepsilon_C \colon RG \to R\colon \ \sum_{g\in G} u_g g \mapsto \sum_{g \in C} u_g.$$

If g is an element of G we write ε_g instead of ε_{g^G}, when considering the partial augmentation with respect to the conjugacy class g^G for reasons of clarity of the notation, if this causes no confusion. If $\mathrm{ccl}(G)$ denotes the set of all conjugacy classes of G, then $\varepsilon = \sum_{C \in \mathrm{ccl}(G)} \varepsilon_C$. The maps ε and ε_1 are independent of the chosen group basis of $\mathbb{Z}G$. The criterion for rational conjugacy now reads as follows

Theorem 2.6 (Marciniak, Ritter, Sehgal, Weiss [MRSW87, Theorem 2.5]). *Let $U \leq \mathrm{V}(\mathbb{Z}G)$ be a torsion subgroup. Then the following are equivalent*

1. *For every $u \in U$ there exists a group element $g \in G$ and a unit $x \in \mathrm{U}(\mathbb{Q}G)$ such that $x^{-1}ux = g$.*
2. *For every $u \in U$, there exists a group element $g_0 \in G$, unique up to conjugacy, such that $\varepsilon_{g_0}(u) \neq 0$.*

One important criterion (which has a weaker predecessors, cf. [MRSW87, Theorem 2.7]) for vanishing of partial augmentations is

Theorem 2.7 (Hertweck [Her07, Theorem 2.3]). *Let u be a torsion unit in $\mathrm{V}(\mathbb{Z}G)$. Then $\varepsilon_x(u) \neq 0$ is possible only for elements x of G whose order is a divisor of the order of u.*

To obtain that subgroups of the normalized units are isomorphic to subgroups of a group basis we have the following theorem

Theorem 2.8 (Hertweck, Kimmerle [Her08c, Corollary 1]). *Let G be a finite group having cyclic Sylow p-subgroups for some prime p. Then any finite p-subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.*

For a character χ of G and a subgroup $H \leq \mathrm{V}(\mathbb{Z}G)$ we obtain a character of H in the following way: If $\rho\colon G \to \mathrm{GL}(n,\mathbb{C})$ is a representation affording χ, then the induced ring homomorphism $\mathbb{Z}G \to \mathbb{C}^{n\times n}$ restricts to a group homomorphism $\mathrm{V}(\mathbb{Z}G) \to \mathrm{GL}(n,\mathbb{C})$ and hence affords a character χ_H for every subgroup $H \leq \mathrm{V}(\mathbb{Z}G)$. Note that if 1 denotes the trivial character of G, then 1_H is the trivial character of H. To guarantee rational conjugacy of subgroups we have

Lemma 2.9 ([Seh93, Lemma (37.6)]). *Suppose that H is a finite subgroup of $\mathrm{V}(\mathbb{Z}G)$ which is isomorphic to a subgroup G_0 of G; $G \supseteq G_0 \overset{\varphi}{\simeq} H \subset \mathrm{V}(\mathbb{Z}G)$. Suppose that $\chi(g_0) = \chi_H(g_0\varphi)$ for all $g_0 \in G_0$ and all $\chi \in \mathrm{Irr}(G)$. Then G_0 is conjugate within $\mathbb{Q}G$ to H.*

2.2 Groups with abelian Sylow p-subgroups

In [DJ96] Michael A. Dokuchaev and Stanley O. Juriaans considered a p-subgroup version of the third Zassenhaus conjecture, where they only look at p-subgroups:

Let G be a finite group and p a prime.

(p-ZC3) If $H \leq \mathrm{V}(\mathbb{Z}G)$ is a finite p-subgroup, then there is a subgroup $W \leq G$ and a unit $x \in \mathbb{Q}G$ such that $x^{-1}Hx = W$.

They proved in the article that (p-ZC3) holds for solvable groups with Sylow subgroups either abelian or generalized quaternion, solvable Frobenius groups, and nilpotent-by-nilpotent groups. Here we consider groups that are not necessarily solvable. The results in the remaining part of this section are joint work with Wolfgang Kimmerle and are also published in [BK11, Section 3].

Proposition 2.10. *If the Sylow 2-subgroups of the finite group G are elementary abelian, then all 2-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomorphic to subgroups of G.*

Proof. Let $H \leq \mathrm{V}(\mathbb{Z}G)$ be a 2-subgroup. By Theorem 2.4 the order of every element of H divides the exponent of G, hence non-trivial elements of H are involutions, so H is elementary abelian. Now Theorem 2.3 implies that the order of H divides the order of G. This completes the proof. □

If we want to decide on rational conjugacy of p-subgroups, the following theorem shows that we can eliminate any normal p'-subgroup:

Theorem 2.11 (Dokuchaev, Juriaans [DJ96, Theorem 2.2]). *Let $N \trianglelefteq G$, H be a finite subgroup of $\mathrm{V}(\mathbb{Z}G)$, and $G_0 \leq G$ with $(|H|,|N|) = (|G_0|,|N|) = 1$. Denote by bar the natural reduction homomorphism $\mathbb{Q}G \to \mathbb{Q}G/N$, then H is conjugate to G_0 in $\mathbb{Q}G$ if and only if $\overline{H}$ is conjugate to $\overline{G_0}$ in $\mathbb{Q}\overline{G}$.*

Definition 2.12. A finite group X is *p-constrained* for a prime p, if $\mathrm{C}_{\overline{X}}(\mathrm{O}_p(\overline{X})) \leq \mathrm{O}_p(\overline{X})$, where $\overline{X} = X/\mathrm{O}_{p'}(X)$.

Note that solvable groups are p-solvable and hence, by a Lemma of Hall and Higman [Hup67, VI. 6.5 Hilfsatz], p-constrained. The following proposition for solvable groups is contained in [DJ96] as Proposition 2.11. Here it is proved as a slight generalization for p-constrained groups.

Proposition 2.13. *Let G be p-constrained. Suppose that G has abelian Sylow p-subgroups, then p-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are rationally conjugate to subgroups of G.*

Proof. By Theorem 2.11 we can assume that $\mathrm{O}_{p'}(G) = 1$. The definition of p-constrained now implies that $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$. By assumption G has abelian Sylow p-subgroups, thus G has a normal Sylow p-subgroup. In this situation a theorem of Weiss [Seh93, Theorem (41.12)] implies that p-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are rationally conjugate to subgroups of G. □

Next we give a result where (2-ZC3) holds, even for groups that are not 2-constrained. For this purpose we make use of Walter's classification of groups having abelian Sylow 2-subgroups which we cite now ([Wal69, Theorem 1], see also [HB82, XI, 13.7 Theorem])

Theorem 2.14 (Walter)**.** *Suppose that the Sylow 2-subgroups of the group G are abelian. Then G has normal subgroups M, N with $M \leq N$ having the following properties*

1. *$|M|$ and $\left|G/N\right|$ are odd.*
2. *$N/M \simeq A_2 \times S_1 \times S_2 \times \ldots \times S_k$, where A_2 is an abelian 2-group and the S_j's are simple non-abelian groups, each isomorphic to one of the groups in the following list*
   a) *$\mathrm{PSL}(2, 2^f)$, (here the Sylow 2-subgroups are elementary abelian of order 2^f)*
   b) *$\mathrm{PSL}(2, q)$, where $q \equiv 3 \bmod 8$ or $q \equiv 5 \bmod 8$ (the Sylow 2-subgroups are elementary abelian of order 4)*
   c) *J_1, the first Janko group (where the Sylow 2-subgroups are elementary abelian of order 8)*
   d) *${}^2G_2(q)$, where $q = 3^{2m+1}$, $m \geq 1$, a Ree-group (the Sylow 2-subgroups are elementary abelian of order 8).*

Here, and later on, the following well-known consequence of the Higman-Berman result will appear to be very useful

Lemma 2.15. *Let G be a finite group and $u = \sum_{g \in G} u_g g \in \mathrm{V}(\mathbb{Z}G)$ an element of finite order. Then for every element $z \in \mathrm{Z}(G)$ we have $u_z \neq 0 \Leftrightarrow u = z$.*

Proof. Assume $u_z \neq 0$. The element $uz^{-1} \in \mathrm{V}(\mathbb{Z}G)$ is also of finite order and $\varepsilon_1(uz^{-1}) = u_z \neq 0$, so Theorem 2.2 implies that $u = z$. The reverse implication is obvious. □

Theorem 2.16. *Let G be a finite group having abelian Sylow 2-subgroups of order at most 8. Then the 2-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are rationally conjugate to subgroups of G.*

Proof. If G is solvable, then the result follows from Proposition 2.13 (or, of course, already from the version [DJ96, Proposition 2.11] for solvable groups). We may assume that $\mathrm{O}_{2'}(G) = 1$ by Theorem 2.11, hence G has no normal subgroup of odd order, and that G is non-solvable. Now, by Walter's classification Theorem 2.14, G has a normal subgroup N isomorphic to $A_2 \times S_1 \times S_2 \times \ldots \times S_k$, where A_2 is an abelian 2-group, the S_j's are simple groups listed in the theorem, and G/N is of odd order (note that here the Sylow 2-subgroups of G are elementary abelian).

1. If a Sylow 2-subgroup of G is isomorphic to $C_2 \times C_2$.
   We obtain from the list in Walter's classification that N is a simple non-abelian group isomorphic to $\mathrm{PSL}(2,q)$, where $q \equiv 3 \bmod 8$ or $q \equiv 5 \bmod 8$. N, and hence G, possesses precisely one conjugacy class of involutions. Consider an involution u of $\mathrm{V}(\mathbb{Z}G)$, then Theorem 2.7 together with the Higman-Berman result (Theorem 2.2) implies that there is exactly one non-vanishing partial augmentation of u, and hence u is rationally conjugate to a group element by Theorem 2.6. Let $H \leq \mathrm{V}(\mathbb{Z}G)$ be a 2-subgroup. From Proposition 2.10 we obtain that H is isomorphic to a subgroup G_0 of G. Let φ be any such isomorphism, then the assumptions of Lemma 2.9 are fulfilled and H is rationally conjugate to G_0.

2. If a Sylow 2-subgroup of G is isomorphic to $C_2 \times C_2 \times C_2$.
   According to the classification N is either simple non-abelian, or N is a direct product of a simple non-abelian group S and a cyclic group C of order 2.

   If N is simple, then N is isomorphic to $\mathrm{PSL}(2,8)$, the first Janko group, or a small Ree group, and all involutions of N are conjugate within N. Hence there is only one conjugacy class of elements of order 2 in G, and the same arguments as for the four subgroup case apply.

If $N = S \times C$, where S a simple non-abelian group and $C = \langle t \rangle$ is a cyclic group of order 2. Then $C \leq \mathrm{Z}(G)$. Let $s \in S$ be any involution, then the three conjugacy classes of involutions in G are represented by s, t, and st, respectively. Assume that $u \in \mathrm{V}(\mathbb{Z}G)$ is an involution. As above, non-vanishing partial augmentations are only possible at classes of involutions. We can assume by Lemma 2.15 that $\varepsilon_t(u) = 0$. Consider the homomorphism

$$\sigma\colon \mathrm{V}(\mathbb{Z}G) \to \mathrm{V}(\mathbb{Z}{}^{G}/_{S})$$

induced by the reduction homomorphism $G \to {}^{G}/_{S}$. Then $\varepsilon_1(u\sigma) = \varepsilon_s(u)$. Hence by Higman-Berman (Theorem 2.2) $\varepsilon_s(u) \in \{0,1\}$, and u has exactly one non-vanishing partial augmentation, implying that it is rationally conjugate to an involution of the group. This proves that the first Zassenhaus conjecture for 2-elements holds in this group.
Let $H \leq \mathrm{V}(\mathbb{Z}G)$ be a subgroup isomorphic to $C_2 \times C_2$. Assume that $t \in H$ (t the central involution of G), and let $u \in H$ be another involution. Then $H = \langle t, u \rangle$. By the foregoing discussion u is rationally conjugate to an element of G. The same conjugation takes H into a subgroup of G. Now assume that $t \notin H$. If H contains two involutions h_1 and h_2 having the same partial augmentations as s, then, applying the homomorphism σ defined above to these elements, we see that h_1 and h_2 are contained in $\operatorname{Ker}\sigma$, and thus also h_1h_2. Let $P = \langle s_1, s_2 \rangle$ be a Sylow 2-subgroup of S, then the homomorphism $H \to P$ determined by $h_1 \mapsto p_1$ and $h_2 \mapsto p_2$ meets the requirements of Lemma 2.9 and H is rationally conjugate to $P \leq G$. Assume now, that H contains two involutions h_1 and h_2 with the same partial augmentations as st. Note that h_1h_2 has partial augmentations coinciding with those of s. With the generators of the group P above define $Q = \langle s_1t, s_2t \rangle$. The isomorphism $H \to Q$ induced by $h_1 \mapsto s_1t$ and $h_2 \mapsto s_2t$ fulfills the assumptions of Lemma 2.9. This completes the discussion for the case of Klein four-groups H.
Finally, let $H \leq \mathrm{V}(\mathbb{Z}G)$ be a subgroup of order 8. Consider the natural homomorphism $\mathrm{V}(\mathbb{Z}G) \to \mathrm{V}(\mathbb{Z}{}^{G}/_{C})$ induced by the reduction $G \to {}^{G}/_{C}$. The image of H under this homomorphism has order at most 4 by Theorem 2.3, hence there must be a non-trivial element of H in the kernel of this homomorphism. This element has necessarily a non-vanishing partial augmentation at the class of the central element t, hence coincides with t by Lemma 2.15, so $H = H_0 \times \langle t \rangle$, where H_0 is a Klein four-group, which is rationally conjugate to a subgroup of G by the previous discussion. The

same conjugation takes H into a subgroup of G. □

Remark 2.17. By the same way of arguing, it can be proved that if a Sylow 2-subgroup of G is an elementary-abelian group of order 16, then 2-elements of $\mathrm{V}(\mathbb{Z}G)$ are rationally conjugate to group elements, except in the cases where the group N occurring in the last proof is of the form $S\times C\times D$ or $S\times T$ (S,T simple non-abelian groups, and C,D groups of order 2).

2.3 Special linear groups

In this section we consider the special linear groups over finite fields, which are (in general) not p-constrained, and obtain results concerning some isomorphism and conjugacy questions for finite subgroups of the units of the corresponding integral group rings.

If $K=\mathbb{F}_q$, a finite field with $q=p^f$ elements, p a prime, the group of invertible $n\times n$-matrices with determinant 1 over such finite fields will be denoted by $\mathrm{SL}(n,q)$ or by $\mathrm{SL}(n,\mathbb{F}_q)$. Note that we have $|\mathrm{SL}(2,q)|=(q-1)q(q+1)$. A Sylow p-subgroup of $\mathrm{SL}(2,q)$ is an elementary abelian p-group and given by

$$P=\left\{ \begin{pmatrix} 1 & x \\ . & 1 \end{pmatrix} \middle| x\in\mathbb{F}_q \right\}$$

(dot indicates a zero). We see that the Sylow r-subgroups of $\mathrm{SL}(2,q)$ are cyclic for $r\notin\{2,p\}$ as follows. There is a subgroup of $\mathrm{SL}(2,q)$ isomorphic to $\mathbb{F}_q^\times$ given by the diagonal matrices $\mathrm{diag}(\zeta,\zeta^{-1})$, where ζ runs through $\mathbb{F}_q^\times$. Additionally, there is a cyclic subgroup of order $q+1$, induced by a so-called 'Singer cycle' (cf. [Hup67, II. 7.3 Satz b)]). Noting that an odd prime r cannot divide both, $q-1$ and $q+1$, implies that all r-subgroups occurring in $\mathrm{SL}(2,q)$ are cyclic.

In [Her07, Section 6] Martin Hertweck achieved quite satisfactory results concerning the first Zassenhaus conjecture for the projective special linear groups $\mathrm{PSL}(2,q)$, q a prime power. Some of the techniques can be adapted for the special linear groups.

Proposition 2.18. *Let* $G=\mathrm{SL}(2,q)$ *for* $q=p^f$, p *an odd prime and* $f\in\mathbb{N}$. *Let* $u\in\mathrm{V}(\mathbb{Z}G)$ *be a torsion unit of order* r, *a prime. If* r *is different from* p, *or* $r=p$

and $f \leq 2$, then u is rationally conjugate to a group element.

Proof. If $r = 2$, there is only one non-vanishing partial augmentation of u, as there is only one conjugacy class of involutions in G.
Now assume $r \neq 2$ and consider the homomorphism $\mathrm{SL}(2,q) \to \mathrm{SL}(2,q)/Z = \mathrm{PSL}(2,q)$, where $Z = \mathrm{Z}(\mathrm{SL}(2,q))$, the center of $\mathrm{SL}(2,q)$ of order 2. By [Her07, Proposition 6.1, Proposition 6.4] units of $\mathrm{V}(\mathbb{Z}\mathrm{PSL}(2,q))$ of orders given in the proposition are conjugate within the corresponding rational group ring to elements of $\mathrm{PSL}(2,q)$. By Theorem 2.11 the same is valid for elements in the group ring of $\mathbb{Z}\mathrm{SL}(2,q)$. □

Martin Hertweck used in his proof that elements of prime order $r \neq p$ in the integral group ring of projective special linear groups $\mathrm{PSL}(2,q)$, $q = p^f$, are rationally conjugate to elements of the group a Brauer character of degree 3. For the groups $\mathrm{SL}(2,q)$ this can already be done with a character of degree 2:

Alternative proof of Proposition 2.18 for $r \neq p$. Let $\overline{\mathbb{F}_q}$ denote an algebraic closure of $\mathbb{F}_q$ and let $\xi \in \overline{\mathbb{F}_q}^{\times}$ be a primitive r-th root of unity. Every element of order r in G is diagonalizable in $\mathrm{SL}(2,\overline{\mathbb{F}_q})$. There are $(r-1)/2$ conjugacy classes of elements of order r in G, the representants (in $\mathrm{SL}(2,\overline{\mathbb{F}_q})$) can be chosen as

$$\mathbf{j} = \begin{pmatrix} \xi^j & \cdot \\ \cdot & \xi^{-j} \end{pmatrix}, \qquad 1 \leq j \leq \frac{r-1}{2}.$$

On the degree 2 Brauer character φ afforded by the natural p-modular representation of G (i.e. the representation that is given by the inclusion map $G \hookrightarrow \mathrm{GL}(2,q)\colon\ A \mapsto A$.) the character values are

$$\varphi(\mathbf{j}) = \zeta^j + \zeta^{-j}, \qquad \zeta \in \mathbb{C}^{\times} \text{ a fixed primitive } r\text{-th root of unity.}$$

Now let $u \in \mathrm{V}(\mathbb{Z}G)$ be an element of order r. By Theorem 2.7 all partial augmentations of u vanish, except maybe those on the classes of the $\mathbf{j}$'s introduced above. Calculating the value of the character φ on u we get

$$\varphi(u) = \sum_{1 \leq j \leq \frac{r-1}{2}} \varepsilon_{\mathbf{j}}(u)\varphi(\mathbf{j}) = \sum_{1 \leq j \leq \frac{r-1}{2}} \varepsilon_{\mathbf{j}}(u)\left(\zeta^j + \zeta^{-j}\right)$$

On the other hand $\varphi(u)$ is a sum of two r-th roots of unity, $\varphi(u) = \zeta^k + \zeta^\ell$ with $1 \leq k, \ell \leq r$. We see that $k = -\ell = j_0$ for one $j_0 \in \left\{1, ..., \frac{r-1}{2}\right\}$ and hence $\varepsilon_{\mathbf{j}}(u) = 0$ for all $\mathbf{j}$, except $\varepsilon_{\mathbf{j}_0}(u) = 1$. Consequently u is rationally conjugate to an element of G by Theorem 2.6. □

We will need small parts of the generic character table of $\mathrm{SL}(2,q)$, which will be given in Table 2.1. The character λ given here coincides in the notation of [DM91, 15.9] with any of the characters $-R^{\mathrm{G}}_{\mathrm{T}_s}(\omega)$ on the conjugacy classes considered here. Note that there are exactly two conjugacy classes of elements of order p in $\mathrm{SL}(2,q)$, denoted by `pa` and `pb`, respectively. The conjugacy class of the identity is denoted by `1a`.

	1a	pa	pb
λ	$q-1$	-1	-1

Table 2.1: Parts of the generic character table of $\mathrm{SL}(2,q)$, $q = p^f$ for an odd prime p and $f \in \mathbb{N}$.

Proposition 2.19. *Let $G = \mathrm{SL}(2,q)$, where p is any prime, $q = p^f$ for some $f \in \mathbb{N}$. Let r be a prime.*

- *If $p = 2$, then all finite r-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomorphic to subgroups of G.*
- *If $p \neq 2$, then all elementary-abelian r-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomorphic to subgroups of G.*

Proof.

- Assume that $p = 2$. As Sylow 2-subgroups of G are elementary abelian Proposition 2.10 implies that all 2-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomorphic to subgroups of G. The Sylow r-subgroups of G are cyclic, provided r is odd. So by Theorem 2.8 all r-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomorphic to subgroups of G.

- Assume that $p \neq 2$. Let $r = 2$. There is a unique involution t in G, which is central, hence there is also only one involution in $\mathrm{V}(\mathbb{Z}G)$. Now let H be an elementary abelian p-subgroup of $\mathrm{V}(\mathbb{Z}G)$. Considering the ordinary, irreducible character λ of degree $q-1$, taking the value -1 on all classes of p-elements yields
$$\langle\, 1_H \mid \lambda_H \,\rangle_H = \frac{1}{|H|}\sum_{h\in H}\lambda(h) = \frac{1}{|H|}\left((q-1)-(|H|-1)\right).$$
Hence $k \leq f$. Let $P \in \mathrm{Syl}_p(G)$, then $P \simeq C_p^f$. Hence there is an isomorphism $\varphi\colon G_0 \to H$ for a subgroup $G_0 \leq P$ fulfilling the requirements of Lemma 2.9. Now assume that $r \notin \{2,p\}$, then the Sylow r-subgroups of G are cyclic and r-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomorphic to subgroups of G by Theorem 2.8. □

Combining the last two results we have

Corollary 2.20. *Let p be an odd prime, $q = p^f$ for some $f \in \mathbb{N}$ and r a prime different from 2 and p. Then (r-ZC3) holds for* $\mathrm{SL}(2,q)$ *provided* $r^2 \nmid |\mathrm{SL}(2,q)|$.

From Hertweck's affirmative answer to the prime graph question for $\mathrm{PSL}(2,q)$ we can derive one for the corresponding special linear groups:

Proposition 2.21. *Let p be an odd prime, $q = p^f$ for some $f \in \mathbb{N}$. Then the prime graphs of the group* $\mathrm{SL}(2,q)$ *and the group of normalized units of the corresponding integral group ring coincide:* $\Gamma(\mathrm{SL}(2,q)) = \Gamma(\mathrm{V}(\mathbb{Z}\mathrm{SL}(2,q)))$.

Proof. Let $G = \mathrm{SL}(2,q)$. The vertices of $\Gamma(\mathrm{V}(\mathbb{Z}G))$ and $\Gamma(G)$ coincide by Theorem 2.4, so we only have to show that if there is an edge between two primes r and s in $\Gamma(\mathrm{V}(\mathbb{Z}G))$, then there is also one in $\Gamma(G)$. There is an edge between 2 and any prime r dividing $|G|$, as G contains a central involution t. So we may assume that both, r and s, are odd. Let $\pi\colon \mathrm{V}(\mathbb{Z}\mathrm{SL}(2,q)) \to \mathrm{V}(\mathbb{Z}\mathrm{PSL}(2,q))$ denote homomorphism induces by the reduction homomorphism $\mathrm{SL}(2,q) \to \mathrm{PSL}(2,q)$. The torsion part of $\mathrm{Ker}\,\pi$ is the subgroup generated by t (the central involution): assume x is a torsion element in the kernel of π, then $1 = \varepsilon_1(x\pi) = \varepsilon_1(x) + \varepsilon_t(x)$, so either $x = 1$ or $x = t$, by Lemma 2.15. Hence if there is an edge between odd primes r and s in $\Gamma(\mathrm{V}(\mathbb{Z}\mathrm{SL}(2,q)))$, there is also one in $\Gamma(\mathrm{V}(\mathbb{Z}\mathrm{PSL}(2,q)))$. By [Her07, Proposition 6.3, Proposition 6.7] there is an affirmative answer to (PQ) for the groups $\mathrm{PSL}(2,q)$, so there is a corresponding edge in $\Gamma(\mathrm{PSL}(2,q))$, and hence also in $\Gamma(\mathrm{SL}(2,q))$. □

3 The normalizer problem

Throughout the chapter G denotes a (possibly infinite) group and R a commutative ring with 1.

Considering a subgroup H of G as a subgroup of $\mathrm{U}(RG)$ leads to the question if only the 'obvious' units of RG normalize H, i.e. if

$$\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H).$$

If $H = G$ this is a classical question, the so called *'normalizer problem'*. We give a survey on that in the first section. Afterwards we consider the question where we allow H to be a (proper) subgroup. There we prove that the above equation holds, if H is a cyclic group. Assuming additionally that H is normal in G, we show the above-stated equation holds, if H is of order pq, a dihedral group of order $2m$ for odd m, simple, or torsion-free abelian. Fixing G, we prove that the normalizers of every subgroup of G in the units of group rings contain only the 'obvious' units, provided G belongs to certain classes of nilpotent groups or to certain classes of metacyclic groups, including all dihedral groups (under some 'natural' restrictions on the coefficient ring). In the last paragraph of the chapter we discuss an adaption of the support subgroup in order to get results for subgroups of a group basis.

3.1 The 'classical' normalizer problem

If we consider G as a subgroup of $\mathrm{U}(RG)$, then G is clearly normalized by itself and all central units, $\mathrm{Z}(\mathrm{U}(RG))$, and thus by all elements in $G \cdot \mathrm{Z}(\mathrm{U}(RG))$. These elements are the *'obvious' normalizing units*. In [JM87, 3.7. Question] Jackowski and Marciniak ask whether for a finite group G the obvious units are always the only units of the integral group ring $\mathbb{Z}G$ that normalize the group basis G after they provided some supporting evidence, namely proving this property for finite groups having a normal Sylow 2-subgroup. This question also made it into the list of research problems in Sudarshan Sehgal's fundamental book 'Units in integral group rings' [Seh93, problem 43]. It can also be considered for coefficients from arbitrary commutative coefficient rings R with 1. Then we say that G together with the coefficient ring R has the *normalizer property*, if the equation

$$\mathrm{N}_{\mathrm{U}(RG)}(G) = G \cdot \mathrm{Z}(\mathrm{U}(RG)) \qquad \mathrm{NP}(G, R)$$

holds. Note that this takes two objects into account: the group G and the ring R. Some authors (e.g. in [Her04]) say that a group has the normalizer property if the statement holds for all *G-adapted coefficient rings R*. These are integral domains of characteristic zero in which a rational prime p is not invertible whenever G contains an element of order p.

This question if $\mathrm{NP}(G,R)$ holds can be translated into the language of automorphisms. Define the group of automorphisms of G induced by units of RG,

$$\mathrm{Aut}_{RG}(G) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi = \mathrm{conj}(u), \text{ for some } u \in \mathrm{N}_{\mathrm{U}(RG)}(G)\}^{1},$$

and set $\mathrm{Out}_{RG}(G) = {}^{\mathrm{Aut}_{RG}(G)}/_{\mathrm{Inn}(G)}$.

Then we have: $\mathrm{NP}(G,R)$ holds $\Leftrightarrow$ $\mathrm{Out}_{RG}(G) = 1$.

One of the most classical results concerning the normalizer problem, which turned out to be very useful in proofs in this area, is due to Ward and Coleman (it was initially proved for fields of characteristic p, but the same proof works for general coefficient rings, as long as p is not invertible)

Lemma 3.1 (Coleman lemma). *Let R be a ring with $pR \neq R$ for a rational prime p and G a finite group with p-subgroup P. Then* $\mathrm{N}_{\mathrm{U}(RG)}(P) = \mathrm{N}_G(P) \cdot \mathrm{C}_{\mathrm{U}(RG)}(P)$.

This lemma is usually referred to as Coleman lemma, although it was first proved by Ward (cf. [War61], [Col64]).

[1]this group is often denoted by $\mathrm{Aut}_R(G)$ but for the subgroup case, which appears later, we want to distinguish the two 'G' that occure here in the notation, so that we decided to stay consistent with the notation within this work and differ at this point from the classical name of the group

The last result asserts that the normalizer property holds for finite p-groups and rings R with $p \notin R^\times$, and, more generally, for finite nilpotent groups G and G-adapted rings R. This was extended by Stefan Jackowski and Zbigniew Marciniak to

Theorem 3.2 (Jackowski, Marciniak [JM87, 3.6. Theorem]). *If the finite group G has a normal Sylow 2-subgroup, then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

Their proof works only for the coefficient ring $\mathbb{Z}$ as they made use of the following crucial observation due to Jan Krempa, which makes use of a so called 'star argument' (which depends on the anti-involution of the group ring induced by the group anti-automorphism $g \mapsto g^{-1}$, having some exceptional properties when considering the coefficient ring $\mathbb{Z}$):

Theorem 3.3 (Krempa [JM87, 3.2. Theorem]). *Let G be a group.* $\mathrm{Out}_{\mathbb{Z}G}(G)$ *is an elementary abelian 2-group.*

Marcin Mazur discovered a connection between the normalizer problem and the isomorphism problem for infinite groups. He proved the following:

Theorem 3.4 (Mazur [Maz95]). *Let G be a group, α_0, $\beta_0 \in \mathrm{Aut}(G)$, and let α, $\beta \colon C_\infty = \langle x \rangle \to \mathrm{Aut}(G)$ be the homomorphisms determined by $x \mapsto \alpha_0$ and $x \mapsto \beta_0$, respectively. Then*

1. *$G \rtimes_\alpha C_\infty \simeq G \rtimes_\beta C_\infty$ if and only if $\alpha_0\beta_0^{-1}$ is an inner automorphism of G.*

2. *$R[G \rtimes_\alpha C_\infty] \simeq R[G \rtimes_\beta C_\infty]$ if and only if $\alpha_0\beta_0^{-1}$ is an inner ring automorphism of RG.*

This was used by Roggenkamp and Zimmermann together with their discovery that outer group automorphisms may become inner in a particular semi-local group ring ([RZ95b]) to find a counterexample to the isomorphism problem for (infinite) polycyclic groups over such rings ([RZ95a]).

There are several results asserting that the normalizer property holds for certain classes of metabelian groups, including the following

Theorem 3.5 (Marciniak, Roggenkamp [MR01, Proposition 12.3]). *If G is a finite metabelian group with abelian Sylow 2-subgroups, then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

Theorem 3.6 (Li, Parmenter, Sehgal [LPS99, Theorem 2]). *If G has an abelian subgroup of index 2, then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

But there is a counterexample provided by Martin Hertweck in his PhD-thesis that $\mathrm{Out}_{\mathbb{Z}G}(G) = 1$, and hence $\mathrm{NP}(G,\mathbb{Z})$, can fail even for metabelian groups:

Theorem 3.7 (Hertweck [Her98, Theorem A]). *There is a metabelian group G of order $2^{25} \cdot 97^2$ with a non-inner automorphism τ such that there is a unit t of $\mathbb{Z}G$ with $\tau = \mathrm{conj}(t)$.*

Using this result he gave his well-known counterexample for the isomorphism problem for integral group rings of finite groups:

Theorem 3.8 (Hertweck [Her98, Theorem B]). *There are two non-isomorphic groups X and Y of order $2^{21} \cdot 97^{28}$ with isomorphic integral group rings.*

Further positive results for the normalizer problem were obtained, including the following

Theorem 3.9 (Li, Parmenter, Sehgal [LPS99, Theorem 1]). *If G is a finite group and the intersection of all non-normal subgroups is non-trivial, then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

Theorem 3.10 (Petit Lobão, Polcino Milies [PLPM02, Theorem 3.1]). *Let G be a finite Frobenius group then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

By investigating the group of Coleman automorphisms (cf. Subsection 3.2.1 for a definition of Coleman automorphisms) Hertweck and Kimmerle proved

Theorem 3.11 (Hertweck, Kimmerle [HK02, Corollary 16, Corollary 20]). *If G is finite quasi-nilpotent, then* $\mathrm{NP}(G,R)$ *holds for G-adapted rings R. If G is solvable and no chief factor of ${}^{G}/_{\mathrm{O}_2(G)}$ is of order 2, then* $\mathrm{NP}(G,R)$ *holds for G-adapted rings R. This applies in particular to groups having only non-abelian composition factors.*

For important classes of infinite groups the normalizer problem has an affirmative answer:

Theorem 3.12 (Jespers, Juriaans, de Miranda, Rogerio [JJdMR02, Theorem 2.4])**.** *Let G be a locally nilpotent group, then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

Theorem 3.13 (Hertweck [Her04, 19.11 Corollary])**.** *Let G be a group whose finite normal subgroups have a normal Sylow 2-subgroup. Then* $\mathrm{NP}(G,\mathbb{Z})$ *holds.*

Hertweck also gave an alternative proof for Theorem 3.10 for G-adapted rings and extended it to locally finite Frobenius group (cf. [Her04, 19.17 Corollary]).

Using [GAP08] one could verify $\mathrm{NP}(G,R)$ for groups G of order at most 161 and G-adapted rings R by calculating their groups of class-preserving and Coleman automorphisms (cf. Subsection 3.2.1 for definitions).

3.2 The normalizer problem for subgroups

If we consider instead of the group basis G subgroups thereof, we can ask the corresponding question: how rigid is this subgroup embedded in the units of the group ring, i.e. are there only the 'obvious' units normalizing it? So does the equation

$$\mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H) \qquad \mathrm{NP}(H \leq G,R)$$

hold? We say that $H \leq G$ together with the coefficient ring R has the *normalizer property*, if $\mathrm{NP}(H \leq G,R)$ is valid. (The notation $H \leq G$ should remind us that we have to keep in mind how H is embedded into G.)

We can look at this question from different points of view. We can fix an isomorphism type of the group H and ask, whether there is always a positive answer to $\mathrm{NP}(H \leq G,R)$, whenever we have a group G into which H embeds (and R is a ring with appropriate properties). An other approach is to fix the group G and ask if the property $\mathrm{NP}(H \leq G,R)$ holds for all subgroups H of G, i.e.

$$\forall\, H \leq G\colon \qquad \mathrm{N}_{\mathrm{U}(RG)}(H) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H). \qquad \mathrm{SNP}(G,R)$$

If this is the case we say that G together with a coefficient ring R has the *subgroup normalizer property*.

We will achieve results for both viewpoints of this problem.

The question if $\mathrm{NP}(H \leq G, R)$ holds can be restated in different ways. For the version with automorphism groups we first need a definition:

Definition 3.14. Let $H \leq G$ and R be an arbitrary ring. Then set

$$\begin{aligned} \mathrm{Aut}_G(H) &= \{\varphi \in \mathrm{Aut}(H) \mid \varphi = \mathrm{conj}(g), \text{ for some } g \in \mathrm{N}_G(H)\}, \\ \mathrm{Aut}_{RG}(H) &= \{\varphi \in \mathrm{Aut}(H) \mid \varphi = \mathrm{conj}(u), \text{ for some } u \in \mathrm{N}_{\mathrm{U}(RG)}(H)\}, \end{aligned}$$

the automorphisms of H induced by elements of G and units of RG, respectively.

Lemma 3.15. *For $H \leq G$ and a ring R the following conditions are equivalent:*

1. $\mathrm{NP}(H \leq G, R)$ *holds*
2. $\mathrm{Aut}_{RG}(H) = \mathrm{Aut}_G(H)$
3. *for every $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ there exists a $g \in \mathrm{N}_G(H)$ such that $[gu, H] = 1$.* □

We use this equivalence freely hereafter.

Remark 3.16. Hertweck's counterexample to $\mathrm{NP}(G, \mathbb{Z})$ (Theorem 3.7) shows that both, $\mathrm{NP}(H \leq G, \mathbb{Z})$ and $\mathrm{SNP}(G, \mathbb{Z})$, are not true in general. From any counterexample H to $\mathrm{NP}(H, \mathbb{Z})$ we can form a counterexample to $\mathrm{NP}(H \leq G, \mathbb{Z})$, in which H is a proper subgroup, in the following way. Let A be an abelian group together with a group homomorphism $\alpha\colon H \to \mathrm{Aut}(A)$, which is one-to-one (take, for example, the regular kH-module for a finite filed k as A), then set $G = A \rtimes_\alpha H$. We have $\mathrm{N}_G(H) = H$, and thus $\mathrm{Aut}_G(H) = \mathrm{Inn}(H) < \mathrm{Aut}_{\mathbb{Z}H}(H) \leq \mathrm{Aut}_{\mathbb{Z}G}(H)$.

3.2.1 Some differences and some similarities between the 'classical' and the subgroup version

In contrast to the 'classical' case, where $\mathrm{Aut}_G(G) = \mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}_{RG}(G)$, so that we can form the quotient group $\mathrm{Out}_{RG}(G)$, there seems to be in general no reason for $\mathrm{Aut}_G(H)$ to be normal in $\mathrm{Aut}_{RG}(H)$, so that we can not restate $\mathrm{NP}(H \leq G, R)$ by vanishing of an outer automorphism group. Nevertheless if $\mathrm{Out}(H) = 1$, then $\mathrm{NP}(H \leq G, R)$ clearly holds for all G with $H \leq G$ and all rings R (this happens for example if $H \simeq S_m$ with $m \neq 6$ or, more generally, if $H \simeq \mathrm{Aut}(S)$ for a simple non-abelian group S).

One important way to attack the 'classical' normalizer problem was to study certain subgroups of the automorphism group of G, namely the groups of class-preserving (or central) automorphisms and the group of Coleman automorphisms. These groups are defined by

$$\mathrm{Aut}_{\mathrm{c}}(G) = \{\varphi \in \mathrm{Aut}(G) \mid \forall\, x \in G\colon x\varphi \in x^G\},$$
$$\mathrm{Aut}_{\mathrm{Col}}(G) = \left\{ \varphi \in \mathrm{Aut}(G) \;\middle|\; \begin{array}{r} \forall P \in \mathrm{Syl}_*(G)\ \exists g \in G\colon \\ \varphi|_P = \mathrm{conj}(g)|_P \end{array} \right\}$$

(where, in the second case, G is assumed to be finite and $\mathrm{Syl}_*(G)$ denotes the collection of all Sylow p-subgroups of G where p varies through all primes dividing the order of G). The importance of these groups for the normalizer problem stems from the fact that $\mathrm{Aut}_{RG}(G) \leq \mathrm{Aut}_{\mathrm{c}}(G)$ for arbitrary groups (if all conjugacy classes of G are finite this can easily be seen by considering the class-sums; for the general case see [Her04, 17.3 Theorem]) and $\mathrm{Aut}_{RG}(G) \leq \mathrm{Aut}_{\mathrm{Col}}(G)$ if G is a finite group and R is a G-adapted ring (this is a consequence of the Coleman lemma 3.1). The groups of class-preserving or Coleman automorphisms, respectively, are only of group-theoretical nature and seem, in many cases, easier to analyze than $\mathrm{Aut}_{RG}(H)$. As these groups are of interest on their own right there are results concerning these groups, which could be used to obtain answers for the normalizer problem (for example, Feit and Seitz showed – using the classification of the finite simple groups – that class-preserving automorphisms of finite simple groups are inner [FS89, Theorem C], which immediately implies that the normalizer property holds for this class of groups).

If we are considering subgroups H of the group basis G it becomes apparent that $\mathrm{Aut}_{RG}(H) \leq \mathrm{Aut}_{\mathrm{c}}(H)$ and $\mathrm{Aut}_{RG}(H) \leq \mathrm{Aut}_{\mathrm{Col}}(H)$ will not hold in general (just consider $H = A_3 \leq G = S_3$). Nevertheless, putting some restrictions on how H is embedded into G some results could be achieved. For example, if we require $\mathrm{ccl}(H) \subseteq \mathrm{ccl}(G)$ for $H \leq G$, then $\mathrm{Aut}_{RG}(H) \leq \mathrm{Aut}_{\mathrm{c}}(H)$. If H is finite and for all primes p dividing $|H|$ and for all Sylow p-subgroups P of H, $\mathrm{N}_G(P) \leq H$ holds, then $\mathrm{Aut}_{RG}(H) \leq \mathrm{Aut}_{\mathrm{Col}}(H)$ for all H-adapted rings R. The class-preserving and Coleman automorphism groups could also be replaced by 'relative' versions. What seems to be more promising is looking at so-called p-central automorphisms (see Subsection 3.2.5), where we require special assumptions for only one prime dividing the group order. Some other tools, which were used in the 'classical' case are also of great impact when looking at subgroups of a group basis, see for example the next paragraph.

3.2.2 An action on the support

For $H \leq G$ it turns out that a particular action of H on a certain finite subset of G provides a lot of information about the normalizer of H in the units of the group ring RG, so that this action will be discussed here in detail. We need the notation of the support of an element of the group ring.

Definition 3.17. For an element $u = \sum u_g g \in RG$ of the group ring RG the *support* is defined as $\operatorname{supp}(u) = \{g \in G \mid u_g \neq 0\}$.

Let $H \leq G$ and $u = \sum_{g \in G} u_g g \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Then for every $h \in H$ we have

$$\sum_{g \in G} u_g g = u = h^{-1} u u^{-1} h u = \sum_{g \in G} u_g h^{-1} g h^u, \tag{3.1}$$

and hence

$$g \in \operatorname{supp}(u) \quad \Leftrightarrow \quad \forall\, h \in H\colon h^{-1} g h^u \in \operatorname{supp}(u). \tag{3.2}$$

In particular, we have the following right action of the group H on the (finite) support of u:

$$\begin{aligned} \operatorname{supp}(u) \times H &\to \operatorname{supp}(u) \\ (x,h) &\mapsto h^{-1} x h^u. \end{aligned} \tag{3.3}$$

This action was used to prove the classical Coleman lemma 3.1. We now prove an extension of [Her04, 19.4 Lemma] for subgroups H of the group basis G following the line of that proof:

Lemma 3.18 (Coleman lemma, relative version)**.** *Let $H \leq G$ and R a ring with $p \notin R^\times$ for a rational prime p. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Then there exists $P \leq H$ with $|H:P| < \infty$, $p \nmid |H:P|$ and $x \in \operatorname{supp}(u) \cap \mathrm{N}_G(P)$ such that $x^{-1}u \in \mathrm{C}_{\mathrm{U}(RG)}(P)$.*

Proof. We examine the action of H on the support of $u = \sum_g u_g g$ given in (3.3) more closely. The coefficients u_g of u under the action are constant by (3.1). Now let

$$K = \{h \in H \mid \forall\, x \in \operatorname{supp}(u)\colon h^{-1} x h^u = x\} \trianglelefteq H$$

be the kernel of this action. Then H/K is isomorphic to a subgroup of the finite group $\operatorname{Sym}(\operatorname{supp}(u))$, and hence finite. Let $K \leq P \leq H$, such that P/K is a Sylow p-subgroup of H/K. The induced action of the p-group P/K on $\operatorname{supp}(u)$ must have a fixed point $x \in \operatorname{supp}(u)$, as $\varepsilon(u) \in R^\times$ and $p \notin R^\times$ by assumption on R. But this implies $x^{-1}u \in \mathrm{C}_{\mathrm{U}(RG)}(P)$. □

We get some immediate positive results from this lemma:

Proposition 3.19. *If all finite quotients of H have p-power order for some fixed rational prime p, then* $\mathrm{NP}(H \leq G, R)$ *holds for all $H \leq G$ and all rings R with $p \notin R^\times$.*

Proof. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. We have to show that $u \in \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$. By the relative Coleman lemma 3.18 we have $P \leq H$ with finite index in H, $p \nmid |H:P|$ and an element $x \in \mathrm{N}_G(P)$, such that $x^{-1}u \in \mathrm{C}_{\mathrm{U}(RG)}(P)$. We show that $H = P$. To this end set $C = \mathrm{core}_H(P) = \bigcap_{h \in H} P^h \trianglelefteq H$. We have $|H:C| \mid (|H:P|)! < \infty$, as C is the kernel of the homomorphism $H \to \mathrm{Sym}(H/P)\colon h \mapsto (Ph' \mapsto Ph'h)$, the right multiplication on the right cosets. Hence by assumption $|H:P| = \frac{|H:C|}{|P:C|}$ is a power of p. As we already noted that $|H:P|$ is not divisible by p, this implies $H = P$. It follows that $u = x(x^{-1}u) \in \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$. □

This implies that the normalizer property holds for $H \leq G$ and rings R if we have for a fixed rational prime p that $p \notin R^\times$ and H is a torsion p-group (i.e. the orders of all elements are finite and powers of p), or H is an infinite simple group.

Proposition 3.20. *Let H be a torsion-free abelian group and $H \trianglelefteq G$. Then* $\mathrm{NP}(H \leq G, R)$ *holds for all rings R.*

Proof. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Choose $x \in \mathrm{supp}(u)$ and set $v = x^{-1}u$. As $H \trianglelefteq G$ we have $v \in \mathrm{N}_{\mathrm{U}(RG)}(H)$, and $1 \in \mathrm{supp}(v)$. Let h be an arbitrary element of H By setting $g = 1 \in \mathrm{supp}(v)$ and plugging in all powers of h in (3.2) we obtain $\{(h^{-n}(h^n)^v) \mid n \in \mathbb{Z}\} = \{(h^{-1}h^v)^n \mid n \in \mathbb{Z}\} \subseteq \mathrm{supp}(v)$. The finiteness of the support implies that $h^{-1}h^v \in H$ is a torsion element of the torsion-free group H. Hence $h = h^v = h^{x^{-1}u}$ for all $h \in H$, implying $u = x(x^{-1}u) \in \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H)$. □

3.2.3 Cyclic subgroups

We prove that $\mathrm{NP}(H \leq G, R)$ holds for all rings R, provided H is cyclic with a similar calculation that occurred in the proof of [PLPM02, Theorem 3.1] and the simplified proof of [Her04, 17.3 Theorem] due to I. B. S. Passi. To this end we need the notation of an additive commutator:

Definition 3.21. For $x, y \in RG$ set $[x,y]_L = xy - yx$, the *additive commutator* or *Lie commutator* of x and y. Let $[RG,RG]_L$ be the R-submodule of RG generated by all $[x,y]_L$, for $x,y \in RG$.

The map $[-,=]_L \colon RG \times RG \to RG \colon (x,y) \mapsto [x,y]_L$ is R-bilinear. Using this, we see that $[RG,RG]_L$ is already generated by all the elements $[g,h]_L$ for $g,h \in G$. Direct calculations show that

$$[RG,RG]_L = \{u \in RG \mid \forall\, C \in \mathrm{ccl}(G) \colon \varepsilon_C(u) = 0\},$$

where $\varepsilon_C \colon RG \to R$ denotes the partial augmentation map with respect to the conjugacy class C and $\mathrm{ccl}(G)$ denotes the collection of all conjugacy classes of G.

Lemma 3.22. *Let $H \leq G$, R a ring, $\sigma \in \mathrm{Aut}_{RG}(H)$. Then $x\sigma \in x^G$ for every $x \in H$.*

Proof. There is an element $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ such that $\sigma = \mathrm{conj}(u)$. We have

$$x\sigma - x = u^{-1}xu - x = [u^{-1}x, u]_L \in [RG,RG]_L,$$

and consequently $\varepsilon_C(x\sigma) = \varepsilon_C(x)$ for all $C \in \mathrm{ccl}(G)$. Together with $x\sigma \in H \leq G$ this implies $x\sigma \in x^G$. □

Proposition 3.23. *Let $H \leq G$. If H is cyclic, then $\mathrm{NP}(H \leq G, R)$ holds for arbitrary rings R.*

Proof. Let $H = \langle x \rangle$ and $\sigma \in \mathrm{Aut}_{RG}(H)$, then $x\sigma \in x^G$ by the previous Lemma 3.22 and hence there is a $g \in G$ such that $x\sigma = x\mathrm{conj}(g)$, so $\sigma = \mathrm{conj}(g)|_H \in \mathrm{Aut}_G(H)$. □

3.2.4 Nilpotent groups

In [LPS99, Proposition 3] it was proved that the 'classical' normalizer property behaves well with respect to (finite) direct products: it holds for a direct product of two groups if and only if it holds for both factors. Altering their line of proof at the points this is needed, we now show that the same is true for the subgroup normalizer property.

Let G_1 and G_2 be groups and $j \in \{1,2\}$. The natural projections $\pi_j \colon G_1 \times G_2 \to G_j$ give rise to ring homomorphisms $R[G_1 \times G_2] \to RG_j$ by the universal property of the group ring, which can be restricted to homomorphisms of the unit groups, $\Pi_j\colon \mathrm{U}(R[G_1 \times G_2]) \to \mathrm{U}(RG_j)$ (the box brackets are included for better readability). With the obvious inclusion maps we have the following diagram (with exact rows)

$$\begin{array}{ccccc} G_1 & \underset{\pi_1}{\overset{\hookrightarrow}{\longleftarrow}} & G_1 \times G_2 & \overset{\pi_2}{\underset{\hookleftarrow}{\longrightarrow}} & G_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{U}(RG_1) & \underset{\Pi_1}{\overset{\hookrightarrow}{\longleftarrow}} & \mathrm{U}(R[G_1 \times G_2]) & \overset{\Pi_2}{\underset{\hookleftarrow}{\longrightarrow}} & \mathrm{U}(RG_2) \end{array}$$

We fix this notation for the next lemma and proposition. Inclusion maps will be omitted whenever they are clear from the context.

Lemma 3.24. *Let $G = G_1 \times G_2$, R an arbitrary ring, and $H \leq G$. Assume that* $\mathrm{NP}(H\pi_1 \leq G\pi_1, R)$ *and* $\mathrm{NP}(H\pi_2 \leq G\pi_2, R)$ *hold, then* $\mathrm{NP}(H \leq G, R)$ *holds.*

Proof. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Set $H_j = H\pi_j \leq G_j$ and $u_j = u\Pi_j$. For every $x_1 \in H_1$ there exists a $x_2 \in H_2$, such that $x_1x_2 \in H$. Now

$$x_1^{u_1} = (x_1x_2)\pi_1^{u\Pi_1} = (x_1x_2)^u\pi_1 \in H_1,$$

hence $u_1 \in \mathrm{N}_{\mathrm{U}(RG_1)}(H_1)$. Analogously we see that $u_2 \in \mathrm{N}_{\mathrm{U}(RG_2)}(H_2)$, so we get from the assumption elements $g_j \in \mathrm{N}_{G_j}(H_j)$ and $z_j \in \mathrm{C}_{\mathrm{U}(RG_j)}(H_j)$ such that $u_j = g_jz_j$. To construct the corresponding units in the group ring of the direct product set $w = u_1^{-1}u_2^{-1}u$. This unit is centralizing H: To see this, pick any $x \in H$, then

$$\begin{aligned} x^w &= ((x\pi_1)(x\pi_2))^{(u^{-1}\Pi_1)(u^{-1}\Pi_2)u} = \left((x\pi_1)^{(u^{-1}\Pi_1)}(x\pi_2)^{(u^{-1}\Pi_2)}\right)^u \\ &= \left((x^{u^{-1}})\pi_1(x^{u^{-1}})\pi_2\right)^u = x. \end{aligned}$$

A similar calculation shows that the element g defined as $g = g_1g_2$ acts on H by conjugation like u does, in particular $g \in \mathrm{N}_G(H)$.

Obviously we have $z_j \in \mathrm{C}_{\mathrm{U}(RG)}(H)$. Hence

$$u = u_1u_2u_1^{-1}u_2^{-1}u = (g_1g_2)(z_1z_2w) \in \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{U}(RG)}(H). \qquad \square$$

Proposition 3.25. *If $G = G_1 \times G_2$ and R is an arbitrary ring, then*

$$\mathrm{SNP}(G,R) \text{ holds} \quad \Leftrightarrow \quad \mathrm{SNP}(G_1,R) \text{ and } \mathrm{SNP}(G_2,R) \text{ hold.}$$

Proof. '$\Rightarrow$': Let $H_1 \leq G_1$ and $u_1 \in \mathrm{N}_{\mathrm{U}(RG_1)}(H_1)$. Set $H = H_1 \times 1 \leq G$. Using the inclusion map, $u_1 \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. By assumption there are $g \in \mathrm{N}_G(H)$ and $z \in \mathrm{C}_{\mathrm{U}(RG)}(H)$ such that $u_1 = gz$. Now $g_1 = g\pi_1 \in \mathrm{N}_{G_1}(H_1)$ and $z_1 = z\Pi_1 \in \mathrm{C}_{\mathrm{U}(RG_1)}(H_1)$. Hence $u_1 = u_1\Pi_1 = (gz)\Pi_1 = g_1 z_1 \in \mathrm{N}_{G_1}(H_1) \cdot \mathrm{C}_{\mathrm{U}(RG_1)}(H_1)$. In the same way we can verify the normalizer property for the second factor.

'$\Leftarrow$': Follows from Lemma 3.24. □

A group is called *locally nilpotent*, if every finite subset is contained in a nilpotent subgroup or, equivalently, every finitely generated subgroup is nilpotent.

Theorem 3.26. *Let G be a locally nilpotent torsion group. Then $\mathrm{SNP}(G,R)$ holds for G-adapted rings R. In particular $\mathrm{SNP}(G,R)$ holds for finite nilpotent groups G and G-adapted rings R.*

Proof. By [Rob96, 12.1.1] we know that $G = \mathrm{Dr}_p\, G_p$, where p runs through all primes, G_p denotes the unique maximal p-subgroup of G and Dr stands for the restricted direct product (i.e. the subgroup of the direct product containing those elements where only finitely many elements are non-trivial). Let $H \leq G$ and $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Define the set P of primes 'occurring' in the support of u:

$$P = \{p \in \mathbb{N} \mid p \text{ a prime and } \exists g \in \mathrm{supp}(u)\colon\, p \mid o(g)\}.$$

Now G decomposes into $G = X \times Y$, where $X = \prod_{p \in P} G_p$ and $Y = \mathrm{Dr}_{p \notin P}\, G_p$. As $[X,Y] = 1$ and $\mathrm{supp}(u) \subseteq X$ it follows that $[u,Y] = 1$. Let $\kappa\colon RG \to RX$ denote the natural projection. By Lemma 3.24 it is enough to show that $u\kappa$ acts on $H\kappa$ like an element of $G\kappa$. But this follows by induction on the (finite) number of primes in P, using the relative Coleman lemma 3.18 and Lemma 3.24. □

This result can now be extended to finite subgroups of finitely generated nilpotent groups. For this purpose we need two lemmas.

Lemma 3.27. *Let G be a group, $K \leq G$ and $C \leq \mathrm{Z}(G)$ a central subgroup and R a ring. Assume that $\mathrm{NP}(KC \leq G,R)$ holds, then so does $\mathrm{NP}(K \leq G,R)$.*

Proof. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(K)$. As $C \leq \mathrm{Z}(G)$ we have

$$u \in \mathrm{N}_{\mathrm{U}(RG)}(K) \subseteq \mathrm{N}_{\mathrm{U}(RG)}(KC) = \mathrm{N}_G(KC) \cdot \mathrm{C}_{\mathrm{U}(RG)}(KC).$$

Hence there is a factorization $u = gz$ with $g \in \mathrm{N}_G(KC)$, $z \in \mathrm{C}_{\mathrm{U}(RG)}(KC)$. For every $k \in K$ we get $K \ni k^u = k^{gz} = (k^g)^z = k^g$, so in fact $g \in \mathrm{N}_G(K)$. Furthermore $\mathrm{C}_{\mathrm{U}(RG)}(KC) = \mathrm{C}_{\mathrm{U}(RG)}(K)$, and hence $u = gz \in \mathrm{N}_G(K) \cdot \mathrm{C}_{\mathrm{U}(RG)}(K)$. □

For the next lemma we need some notation: Let $1 \to K \to E \to Q \to 1$ be a short exact sequence of groups (i.e. an extension of Q by K), where K is an abelian group. For an automorphism φ of E fixing K as set denote by $\varphi|_Q$ the automorphism induced on the quotient group Q. Set

$$\mathrm{Aut}(Q,K) = \{\varphi \in \mathrm{Aut}(E) \mid \varphi|_K = \mathrm{id}_K \text{ and } \varphi|_Q = \mathrm{id}_Q\},$$

the group of automorphisms of E restricting to the identity on K and inducing the identity on the quotient group Q. The inner automorphisms of E contained therein form the normal subgroup $\mathrm{Inn}(Q,K) = \mathrm{Aut}(Q,K) \cap \mathrm{Inn}(E)$. Note that after having fixed a Q-module structure on K the group $\mathrm{Aut}(Q,K)$ does not depend on the concrete extension (cf. [Rot07, Corollary 9.17]). The crucial result for the next lemma is

Proposition 3.28 ([Rot07, Corollary 9.20]). *For every group Q and Q-module K, $\mathrm{H}^1(Q,K) \simeq {}^{\mathrm{Aut}(Q,K)}/_{\mathrm{Inn}(Q,K)}$, where H^1 denotes the first cohomology group.*

Lemma 3.29. *Let $1 \to C \hookrightarrow G \to \overline{G} \to 1$ be a central extension, where C is torsion-free and let R be a ring. Assume that $L \leq G$ with $C \leq L$ such that $|{}^{L}/_{C}| < \infty$. If $\mathrm{NP}({}^{L}/_{C} \leq {}^{G}/_{C}, R)$ holds, then $\mathrm{NP}(L \leq G, R)$ as well.*

Proof. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(L)$. Denote by bar the ring homomorphism $RG \to R\overline{G}$ induced by the natural homomorphism $G \to \overline{G}$. From $\overline{u} \in \mathrm{N}_{\mathrm{U}(R\overline{G})}(\overline{L})$ and the assumption we get that there is a $g \in G$ with $\mathrm{conj}(\overline{u})|_{\overline{L}} = \mathrm{conj}(\overline{g})|_{\overline{L}}$. We have $g \in C \cdot \mathrm{N}_G(L) = \mathrm{N}_G(L)$. Let $\sigma = \mathrm{conj}(ug^{-1}) \in \mathrm{Aut}_{RG}(L)$. As σ induces the identity on ${}^{L}/_{C}$ and $C \leq \mathrm{Z}(G)$ we have the following commuting diagram with the obvious horizontal maps

$$\begin{array}{ccccc} C & \hookrightarrow & L & \longrightarrow & \overline{L} \\ \Vert\ \mathrm{id} & & \downarrow \sigma & & \Vert\ \mathrm{id} \\ C & \hookrightarrow & L & \longrightarrow & \overline{L} \end{array}$$

So $\sigma \in \mathrm{Aut}(\overline{L},C)$. Now $1 \to C \hookrightarrow L \to \overline{L} \to 1$ is a central extension of a finite group by a torsion-free group, hence $\mathrm{H}^1(\overline{L},C) = \mathrm{Hom}(\overline{L},C) = 1$. Now Proposition 3.28 implies that $\sigma = \mathrm{conj}(\ell) \in \mathrm{Inn}(\overline{L},C)$ for some $\ell \in L$, so that $\mathrm{conj}(u)|_L = \mathrm{conj}(\ell g)|_L \in \mathrm{Aut}_G(L)$. □

Proposition 3.30. *Let G be finitely generated nilpotent and $H \leq G$ a torsion subgroup. Then $\mathrm{NP}(H \leq G,R)$ holds for G-adapted rings R.*

Proof. First define a special central series of G in the following way. [Rob96, 5.2.22 (ii)] states that if G is a finitely generated nilpotent group, then G is infinite if and only if there is an element of infinite order in the center of G. Let $U_0 = 1$. Assume by induction that $U_j \trianglelefteq G$ is already defined and G/U_j is finitely-generated nilpotent. If this quotient is infinite then there is an element $y_{j+1} \in \mathrm{Z}\left(G/U_j\right)$ of infinite order, the pre-image U_{j+1} of $\langle y_{j+1}\rangle$ is normal in G; if G/U_j is finite, define $U_{j+1} = U_j$. In any case the resulting quotient group is finitely generated nilpotent and consequently has the desired property for the induction process. This yields a chain

$$1 = U_0 \leq U_1 \leq U_2 \leq U_3 \leq \ldots$$

of normal subgroups of G. Define $U = \bigcup_j U_j$. This is finitely generated as a subgroup of a finitely generated nilpotent group [Rob96, 3.1.6], hence the above chain becomes stationary, i.e. there is a minimal $n \in \mathbb{N}$ such that $U_n = U_{n+k}$ for all $k \geq 0$. But then G/U_n is finite nilpotent.
For $0 \leq j \leq n$ define $G_j = G/U_j$. The natural map $G_j \to G_{j+1}$ is injecive on torsion elements, as the kernel is torsion-free. Hence the torsion subgroup $\mathrm{Tor}(G)$ of G is isomorphic to a subgroup of the finite group G_n and hence finite. Thus, the torsion subgroup H is finite.
Set $C_j = U_{j+1}/U_j \leq \mathrm{Z}(G_j)$. Additionally, let $K_j = U_jH/U_j \leq G_j$, a finite subgroup, and $L_j = U_{j+1}H/U_j \leq G_j$. Note that $L_j = K_jC_j$. For every $0 \leq j \leq n-1$ we get a short exact sequence

$$1 \to C_j \to G_j \to G_{j+1} \to 1.$$

$\mathrm{NP}(K_n \leq G_n,R)$ holds by Theorem 3.26, as G_n is a finite nilpotent group. Using Lemma 3.29 and Lemma 3.27 while proceeding inductively along the sequence

$$G = G_0 \to G_1 \to \ldots \to G_{n-1} \to G_n$$

shows that $\mathrm{NP}(K_0 \leq G_0,R)$ holds. □

If we want to get rid of the constraint that we can only handle finite subgroups we can use the following 'stronger' lifting: we prove under some assumptions that the conjugation action of the unit u on H coincides with the conjugation of an element of the support of u, provided this is true on a particular quotient group.

Lemma 3.31. *Let G be a group, $H \leq G$, R an arbitrary ring, and $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Assume there is a normal subgroup N of G with $N \leq \mathrm{C}_G(H)$. Set $\overline{G} = {}^{G}/_{N}$ and denote by bar the reduction homomorphism, $^{-}\colon RG \to R\overline{G}$. Assume that there is $\overline{x} \in \mathrm{supp}(\overline{u})$ with $\mathrm{conj}(\overline{u})|_{\overline{H}} = \mathrm{conj}(\overline{x})|_{\overline{H}}$. If N is torsion-free, then there is $g \in \mathrm{supp}(u)$ with $\mathrm{conj}(u)|_H = \mathrm{conj}(g)|_H$.*

Proof. We have $\overline{u} \in \mathrm{N}_{\mathrm{U}(R\overline{G})}(\overline{H})$, so by assumption there is a $g \in \mathrm{supp}(u)$ such that $\mathrm{conj}(\overline{u})|_{\overline{H}} = \mathrm{conj}(\overline{g})|_{\overline{H}}$ (we may take any $g \in Hx \cap \mathrm{supp}(u) \neq \varnothing$). Hence for all $h \in H$ we have $\overline{h^{-1}}\overline{g}\overline{h^u} = \overline{g}$, this means

$$h^{-1}gh^u = z_h g, \qquad \text{for some } z_h \in N. \tag{3.4}$$

Using $z_h \in N \leq \mathrm{C}_G(H)$ for all $h \in H$ we have for $h_1, h_2 \in H$

$$\begin{aligned} z_{h_1h_2}g &= (h_1h_2)^{-1}g(h_1h_2)^u = h_2^{-1}(h_1^{-1}gh_1^u)h_2^u = h_2^{-1}z_{h_1}gh_2^u \\ &= z_{h_1}h_2^{-1}gh_2^u = z_{h_1}z_{h_2}g, \end{aligned}$$

and hence $z\colon H \to N\colon h \mapsto z_h$ is a homomorphism. Now (3.2) on page 38 shows that $z_h g \in \mathrm{supp}(u)$ for all $h \in H$. As N is torsion-free, the finiteness of the support implies $z_h = 1$ for all $h \in H$. Considering (3.4) we obtain $\mathrm{conj}(u)|_H = \mathrm{conj}(g)|_H$. □

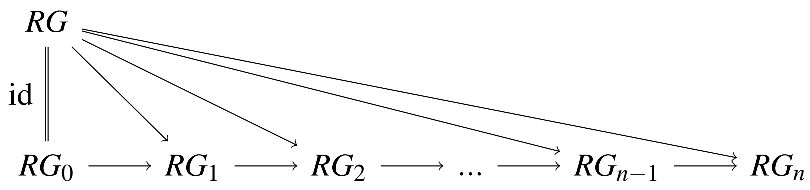
Remark 3.32. The lemma obviously stays true, if we replace the condition N torsion-free by $\mathrm{Hom}(H,N) = 1$.

Corollary 3.33. *If G is nilpotent of class 2 and the center of G is torsion-free, then $\mathrm{SNP}(G,R)$ holds for every ring R.*

Proof. Set $N = \mathrm{Z}(G)$ in Lemma 3.31 and note that the assumption implies that $G' \leq \mathrm{Z}(G)$, and hence that ${}^{G}/_{\mathrm{Z}(G)}$ is abelian. □

Proposition 3.34. *Let G be finitely-generated torsion-free nilpotent, then $\mathrm{SNP}(G,R)$ holds for all rings R.*

Proof. Let $H \leq G$ and $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$. Let $1 = U_0 < U_1 < \ldots < U_{n+1} = G$ be a central series of G, such that the factors $C_j = {}^{U_{j+1}}/_{U_j}$ are all infinite cyclic groups (such a series exists by [Rob96, 5.2.20]). Define $G_j = {}^{G}/_{U_j}$ and note that there is a natural exact sequence $1 \to C_j \to G_j \to G_{j+1} \to 1$ for every $j \in \{1, ..., n-1\}$. The natural projection maps make the following diagram commute



Denote by H_j the image of H in G_j and by u_j the image of u in $\mathrm{U}(RG_j)$. Obviously every element of the support of u_n acts on H_n like the element u_n, so fix any such support element g_n. Using induction together with Lemma 3.31 yields an element $g_j \in \mathrm{supp}(u_j)$ acting like u_j on H_j for every $0 \le j \le n$. Consequently there is an element $g = g_0$ in $\mathrm{supp}(u)$ such that $\mathrm{conj}(u)|_H = \mathrm{conj}(g)|_H$. □

Remark 3.35. The last proposition also follows from [MR77, Theorem 2.2.4] (torsion-free nilpotent groups are ordered) and [Seh93, Lemma (45.3)] (all units in integral group rings of ordered groups are trivial).

3.2.5 p-central automorphisms

Inspired by the work of Hertweck and Kimmerle in [HK02] we were led to look at p-central automorphisms rather than at Coleman automorphisms to settle some cases of the normalizer problem for subgroups.

Definition 3.36. Let p be a prime and X a finite group. An automorphism $\varphi \in \mathrm{Aut}(X)$ is called *p-central*, if there is a Sylow p-subgroup P of X such that $\varphi|_P = \mathrm{id}_P$.

Lemma 3.37. *Let $H \le G$ with H finite, and p a rational prime. Let $P \in Syl_p(H)$. Assume that all p-central automorphisms of H are induced by elements of G and we have $\mathrm{N}_G(P) \le \mathrm{N}_G(H)$. Then $\mathrm{NP}(H \le G, R)$ holds for rings R with $p \notin R^\times$.*

Proof. Let $\sigma \in \mathrm{Aut}_{RG}(H)$. Further let $P \in \mathrm{Syl}_p(H)$. Then there is $h \in H$ such that $P\sigma\mathrm{conj}(h) = P$. By the Coleman lemma 3.1 we get a $g \in \mathrm{N}_G(P) \le \mathrm{N}_G(H)$ such that $\sigma\mathrm{conj}(hg)$ is p-central, hence $\sigma \in \mathrm{Aut}_G(H)$. □

In Proposition 3.19 we have seen that $\mathrm{NP}(H \leq G,R)$ holds if H is an infinite simple group. The same is true for finite simple groups, provided they appear as normal subgroups:

Corollary 3.38. *Let $H \trianglelefteq G$ and H be a finite simple group. Then* $\mathrm{NP}(H \leq G,R)$ *holds for H-adapted rings R.*

Proof. By [HK02, Theorem 14] there is a prime p dividing $|H|$ such that p-central automorphisms of H are inner, in particular contained in $\mathrm{Aut}_G(H)$. Now Lemma 3.37 yields the result. □

Recall that a finite group X is *p-constrained* for a prime p, if $\mathrm{C}_{\overline{X}}(\mathrm{O}_p(\overline{X})) \leq \mathrm{O}_p(\overline{X})$, where $\overline{X} = X/\mathrm{O}_{p'}(X)$.

Corollary 3.39. *Let H be a p-constrained group with $\mathrm{O}_{p'}(H) = 1$ for some prime p, and $H \trianglelefteq G$ or $H = \mathrm{N}_G(P)$ for a $P \in \mathrm{Syl}_p(G)$. Then* $\mathrm{NP}(H \leq G,R)$ *holds for rings R with $p \notin R^\times$.*

Proof. In both of the two possible cases of how H is embedded into G we have $\mathrm{N}_G(P) \leq \mathrm{N}_G(H)$. By [Gro82, 2.4 Corollary] all p-central automorphisms of H are inner and the result follows from Lemma 3.37. □

3.2.6 Metacyclic groups and groups of small order

In this section we prove certain 'small' groups fulfill the normalizer property for subgroups. To this end we first prove this property for certain metacyclic groups and groups. Here a group is called *metacyclic* if it possesses a cyclic normal subgroup with cyclic quotient[2].

We have seen that elements from $\mathrm{Aut}_{RG}(H)$ preserve G-conjugacy classes. Even if H is not cyclic the conjugating elements can be 'glued together' in some cases:

[2]Some authors call a group G metacyclic, if G', the commutator-subgroup, and G/G', the commutator quotient group, are cyclic. Groups being metacyclic according to this definition are, of course, also metacyclic in our notation, but not necessarily vice versa (consider e.g. $G = Q_8$, the quaternion group of order 8).

Lemma 3.40. *Let $H \leq G$ be a subgroup generated by two elements, $H = \langle c, d \rangle$, such that $c^G = c^{C_G(d)}$, then $\mathrm{NP}(H \leq G, R)$ holds for all rings R.*

Proof. Let $\sigma \in \mathrm{Aut}_{RG}(H)$. As $d\sigma \in d^G$ by Lemma 3.22, there exists a $g \in G$ such that $d\sigma\mathrm{conj}(g) = d$. Now $c\sigma\mathrm{conj}(g) \in c^G = c^{C_G(d)}$ and hence there exists a $g' \in C_G(d)$ such that $\sigma = \mathrm{conj}(gg')^{-1} \in \mathrm{Aut}_G(H)$. □

Lemma 3.41. *Let G be a finite group such that $1 \to A \hookrightarrow G \to B \to 1$ is a short exact sequence where A and B are abelian groups of coprime order. Let $H \leq G$ be metacyclic with $H = \langle c \rangle \rtimes \langle d \rangle$ and $H \cap A = \langle c \rangle$. Then $\mathrm{NP}(H \leq G, R)$ holds for all rings R.*

Proof. Let $\sigma \in \mathrm{Aut}_{RG}(H)$. By assumption $(|A|, |B|) = 1$, so by Schur-Zassenhaus ([Rob96, 9.1.2]) there exists a complement of A in G isomorphic to B. As $|\langle d \rangle|$ divides $|B|$ and the complement is a Hall-subgroup, a theorem of Philip Hall ([Rob96, 9.1.7]) implies that there exists a complement B_1 of A in G containing $\langle d \rangle$. Now Lemma 3.40 completes the proof, as $c^G = c^{B_1} = c^{C_G(d)}$. □

Now we can prove the subgroup normalizer property for certain metacyclic groups:

Theorem 3.42. *Let G be a metacyclic group admitting a short exact sequence of the form $1 \to C_m \to G \xrightarrow{\pi} C_n \to 1$ with $m, n \in \mathbb{N} \cup \{\infty\}$ and let R be any ring. Then $\mathrm{SNP}(G, R)$ holds provided one of the following is true:*

1. *$m, n \in \mathbb{N}$ are coprime,*
2. *n is a prime,*
3. *m is a prime.*

Proof. Let $G = \langle a, b \rangle$, where a generates the kernel of π, a normal cyclic subgroup of order m of G. For a subgroup H there exist $c, d \in G$ with $H \cap \langle a \rangle = \langle c \rangle$ and $H = \langle c, d \rangle$.

1. Follows from Lemma 3.41.
2. Assume that $n = p$ is a prime. If $H \leq \langle a \rangle$, then H is cyclic and the statement follows from Proposition 3.23. Otherwise $\pi|_H$ is onto and we may assume that $d = a^k b$. Now $c^{C_G(d)} \supseteq c^{\langle d \rangle} \supseteq c^{\langle b \rangle} \supseteq c^G$ and Lemma 3.40 applies.

3. Now let $m = p$ be a prime. If $H \cap \operatorname{Ker} \pi = 1$, then H is cyclic and Proposition 3.23 implies the desired statement. Otherwise we can assume $c = a$, hence $H = \langle a, b^k \rangle$. Again $c^{C_G(b^k)} \supseteq c^{\langle b \rangle} \supseteq c^G$ and Lemma 3.40 completes the proof. □

Corollary 3.43. *The subgroup normalizer property holds for dihedral groups and all coefficient rings.* □

Corollary 3.44. *The subgroup normalizer property holds for groups of square-free order and all coefficient rings.*

Proof. By a theorem of Hölder, Burnside, and Zassenhaus a finite group whose Sylow subgroups are cyclic is metacyclic [Rob96, 10.1.10]. The normal cyclic subgroup and the cyclic complement are in that case clearly of coprime order, so Theorem 3.42 applies. □

Corollary 3.45. *Let G be a group admitting a short exact sequence of the form $1 \to C_n \to G \to Q_8 \to 1$, for $n = p^k$, p some rational prime and Q_8 the quaternion group of order 8. Let R be a ring, where $2 \notin R^\times$ if $p = 2$. Then $\mathrm{SNP}(G, R)$ holds.*

Proof. If $p = 2$, then G is a 2-group and the statement follows from Coleman's lemma. Now let p be odd, then the exact sequence is split by Schur-Zassenhaus and we can identify G with $C_n \rtimes_\tau Q_8$ for some $\tau\colon Q_8 \to \mathrm{Aut}(C_n)$. Now $\operatorname{Im} \tau \leq \mathrm{Aut}(C_n) \simeq C_{\phi(n)}$ is cyclic (ϕ denotes Euler's totient function), hence $|\operatorname{Ker} \tau| \geq 4$. Let $b \in Q_8$ with $o(b) = 4$, such that $\langle b \rangle \leq \operatorname{Ker} \tau \leq Q_8$ and $C_n = \langle a \rangle$. Then $\langle ab \rangle$ is a cyclic subgroup of G of index 2 and the statement follows from Theorem 3.42. □

The following lemma could also be proved by direct calculations, we will give some easy applications of the results about p-central automorphisms obtained in Subsection 3.2.5

Lemma 3.46. *If $G \in \{A_4, S_4, A_5\}$, then $\mathrm{SNP}(G, R)$ holds for G-adapted rings R.*

Proof. It suffices to check only one representative of each conjugacy class of subgroups.

- If $G = A_4$. Besides p-groups, which are settled by the Coleman lemma 3.1, there is only one subgroup, namely G itself. Noting that A_4 is 2-constrained with $\mathrm{O}_{2'}(A_4) = 1$ we see that $\mathrm{NP}(G \leq G, R)$ follows from Corollary 3.39. The Sylow 2-subgroups of $G \simeq (C_2 \times C_2) \rtimes C_3$ are abelian and G is metabelian, so we see that the property can also be derived from Theorem 3.5 of Marciniak-Roggenkamp.

- If $G = S_4$. Considering the list $1,\ C_2,\ C_3,\ C_2 \times C_2,\ C_4,\ S_3,\ D_4,\ A_4,\ S_4$ of isomorphism types of subgroups of G we see that, besides p-groups, there are only S_3, A_4, and S_4. For subgroups H isomorphic to S_3 or S_4 we have $\mathrm{Aut}(H) = \mathrm{Inn}(H)$ and for $H = A_4$ we get from $\mathrm{N}_{S_4}(A_4) = S_4$ and $\mathrm{C}_{S_4}(A_4) = 1$ that $\mathrm{Aut}_G(H) = \mathrm{Aut}(H)$.

- If $G = A_5$. The conjugacy classes of subgroups are of isomorphism types 1, $C_2,\ C_3,\ C_2 \times C_2,\ C_5,\ S_3,\ D_5,\ A_4,\ A_5$. Apart from p-groups there are S_3, $D_5,\ A_4$, and A_5. For $H \simeq S_3$ we have again $\mathrm{Aut}(H) = \mathrm{Inn}(H)$.

  Assume that $H \simeq D_5$. Let $P \in \mathrm{Syl}_5(H)$, then P is also a Sylow 5-subgroup of G and $H = \mathrm{N}_G(P)$. The group H is 5-constrained and $\mathrm{O}_{5'}(H) = 1$. So we can apply Corollary 3.39 to see that $\mathrm{NP}(H \leq G, R)$ holds.

  If $H \simeq A_4$ we can use again that A_4 is 2-constrained with $\mathrm{O}_{2'}(A_4) = 1$. Let $P \in \mathrm{Syl}_2(H)$, then $P \in \mathrm{Syl}_2(G)$ and $H = \mathrm{N}_G(P)$, so the assumptions of Corollary 3.39 are fulfilled.

  The last remaining case $H = G$ is of course already settled by theorems for the 'classical' case e.g. Theorem 3.11. The (virtually same) proof that $\mathrm{NP}(G \leq G, R)$ holds can also be derived by using Corollary 3.38. □

Remark 3.47. With similar methods we were able to check that $\mathrm{SNP}(G,R)$ holds for the simple groups $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,9)$, $\mathrm{PSL}(2,8)$, and $\mathrm{PSL}(2,11)$ of order 168, 360, 504, and 660, respectively and G-adapted rings R.

Corollary 3.48. *Let G be a finite subgroup of $\mathrm{O}(3,\mathbb{R})$. Then $\mathrm{SNP}(G,R)$ holds for all G-adapted rings R.*

Proof. Finite subgroups of $\mathrm{O}(3,\mathbb{R})$ are of the following isomorphism types (cf. [Kim08, Satz 8.3.15], [Kle82, §9], or [Sch80, 4.2, p.47]): cyclic groups C_n of order n $(n \in \mathbb{N})$, dihedral groups D_m of order $2m$ $(m \in \mathbb{N})$, the tetrahedral group A_4, the octahedral group S_4, the icosahedral group A_5, and, for any of these groups G also the direct product $G \times C_2$. By Proposition 3.25 it is enough to check the property for the groups from the first list. For dihedral groups this is Corollary 3.43. For the other three groups this is Lemma 3.46. □

For certain isomorphism types of subgroups occurring as normal subgroups we have the following reduction result:

Proposition 3.49. *Let H be a finite normal subgroup of a group G. If H has an abelian characteristic subgroup N with a complement W in H, which acts faithfully on N, and* $\mathrm{H}^1(W,N)=1$, *then* $\mathrm{NP}(H \le G,R)$ *holds, provided* $\mathrm{NP}(N \le G,R)$ *holds.*

Proof. Let $\sigma \in \mathrm{Aut}_{RG}(H)$. By assumption we can modify σ by an inner automorphism of G and assume that $\sigma|_N = \mathrm{id}_N$. For all $w \in W$ we have $w\sigma = xv$ for some $x \in N$ and $v \in W$. We get, using that N is abelian, for all $y \in N$

$$y\mathrm{conj}(w) = y^w = (y^w)\sigma = (y\sigma)^{(w\sigma)} = y^{xv} = y^v = y\mathrm{conj}(v).$$

Hence $\mathrm{conj}(w)|_N = \mathrm{conj}(v)|_N$. As the action of W on N was assumed to be faithful, we conclude that $w=v$ and hence $w\sigma = xw$. The subgroup W is a right transversal of N in H, so we deduce that the automorphism of H/N induced by σ is the identity. From the assumption on the cohomology together with its interpretation in Proposition 3.28 we obtain $\sigma = \mathrm{conj}(z)$ for some $z \in N$. □

Using that $\mathrm{H}^1(W,N)=1$ if W and N are of coprime order (see, for example, [Rot07, Proof of Theorem 9.42]) we get

Corollary 3.50. $\mathrm{NP}(H \le G,R)$ *for $H \trianglelefteq G$ and a ring R holds true, provided one of the following holds*

1. *$H \simeq C_n \rtimes C_p$ for a prime p such that $p \nmid n$.*

2. *H is finite and has an abelian normal Sylow p-subgroup (p a rational prime) with a complement acting faithfully on the Sylow p-subgroup and $p \notin R^\times$.*

In particular $\mathrm{NP}(H \le G,R)$ *holds for any ring R, if $H \trianglelefteq G$ and $|H| = pq$ for two primes p and q or $H \simeq D_m$, a dihedral group of order $2m$.* □

Proposition 3.51. *Let G be a group of order p^2q for two rational primes p and q. Then* $\mathrm{SNP}(G,R)$ *holds for rings R with $p \notin R^\times$.*

Proof. Let $H \le G$. Using Sylow's theorem we can assume that $G = P \rtimes Q$ or $G = Q \rtimes P$, where $Q \simeq C_q$ and P is either cyclic or elementary abelian of order p^2. In addition we can assume that G is not abelian, i.e. the product is not direct. For cyclic subgroups the result follows from Proposition 3.23, for elementary abelian

p-groups the result follows from Coleman lemma 3.1. So the only subgroups we have to deal with are of order pq and order p^2q (so $H = G$). The following two tables show which results apply (note that if $Q \trianglelefteq G$, subgroups H of order pq are normal in G):

	$G \simeq C_{p^2} \rtimes C_q$	$G \simeq (C_p \times C_p) \rtimes C_q$
$\lvert H\rvert = pq$	Lemma 3.41	Lemma 3.41
$\lvert H\rvert = p^2q$	Corollary 3.50 or Lemma 3.41	Corollary 3.50

	$G \simeq C_q \rtimes C_{p^2}$	$G \simeq C_q \rtimes (C_p \times C_p)$
$\lvert H\rvert = pq$	Corollary 3.50 or Lemma 3.41	Corollary 3.50 or Lemma 3.41
$\lvert H\rvert = p^2q$	Corollary 3.50 or Lemma 3.41	see below

In the remaining case $G \simeq C_q \rtimes_\varphi (C_p \times C_p)$ and $H = G$ note that the defining homomorphism $\varphi\colon C_p \times C_p \to \mathrm{Aut}(C_q) \simeq C_{q-1}$ can not be one-to-one, so in fact $G \simeq C_p \times (C_q \rtimes C_p)$. For the subgroup $C_q \rtimes C_p \trianglelefteq C_p \times (C_q \rtimes C_p)$ the normalizer property holds by Corollary 3.50 and so it holds also for the whole group as this subgroup has a central direct complement. □

Remark 3.52. The case $H = G$ in the last proposition is of course also settled by the theorem of Marciniak and Roggenkamp (Theorem 3.5).

Theorem 3.6 of Li, Parmenter, and Sehgal states us that the normalizer property holds for groups containing an abelian subgroup of index 2. The same is true, when considering all subgroups (reasoning in a similar way like [HIJJ07, Example 3.1]):

Lemma 3.53. *Assume the group G has an abelian subgroup of index 2, then* SNP(G,R) *holds for G and all rings R.*

Proof. We have a short exact sequence $1 \to A \hookrightarrow G \xrightarrow{\pi} C_2 \to 1$, where A is abelian. Let $H \leq G$ and $\sigma \in \mathrm{Aut}_{RG}(H)$. Now either $H \cap A = H$ or $H\pi \neq 1$. First we show that, also in the second case, it is enough to study the impact of σ on $A\cap H$. If $H\pi \neq 1$, then there is $g\in H$ with $G=\langle A,g\rangle$ and $H=\langle A\cap H,g\rangle$. By Lemma 3.22 $g\sigma \in g^G = g^A$, so there is $a\in A$ with $g\sigma\mathrm{conj}(a)=g$. So we may assume that g is fixed by σ – possibly after modification with a conjugation by an element of A.

To complete the proof it is enough to show that $\sigma|_{A\cap H} \in \mathrm{Aut}_G(A\cap H)$. Let either g be the element we chose above, or – if we are in the case where $H\leq A$ – let g be an arbitrary element of $G\smallsetminus A$. Note that for every $z\in A$ the G-conjugacy class consists of at most two different elements, namely z and z^g. If $\sigma|_{A\cap H} \in \{\mathrm{id}_{A\cap H}, \mathrm{conj}(g)|_{A\cap H}\}$, we are done. Otherwise there exist $x,y\in A\cap H$ with $x\sigma = x \neq x^g$ and $y\sigma = y^g \neq y$. Now

$$xy^g = (x\sigma)(y\sigma) = (xy)\sigma \in \{xy, (xy)^g\}.$$

But this yields in both possible cases a contradiction: $y^g=y$ or $x=x^g$. □

Using the results obtained so far and direct calculations, we were able to check the following:

Proposition 3.54. *All groups G of order at most 47 satisfy* $\mathrm{SNP}(G,\mathbb{Z})$.

Considering the same question for groups of order less or equal 71, there are only 10 isomorphism classes of groups of order 48 left where not all subgroups can be handled.

A simple isomorphism type of subgroup for which the above results don't provide a positive answer is of the form $H=C_p\times C_p\times C_q$ for two different primes p and q. However, for certain groups where subgroups H of this form occur not having index 2 (e.g. in $G=(C_7\times C_2\times C_2)\rtimes C_3$) we have the following proposition which is joint work with Wolfgang Kimmerle (in the just mentioned case it can be applied with $H=N$).

Proposition 3.55. *Let G be a group, H char $N\trianglelefteq G$ with $\mathrm{C}_G(P)=N$ for all non-trivial $P\in \mathrm{Syl}_p(H)$. Let R be an H-adapted ring. Assume that the only normalized units of RG/N are the elements of G/N, then* $\mathrm{NP}(H\leq G,R)$ *holds.*

Proof. We need the following observation: if $u\in \mathrm{C}_{\mathrm{V}(RG)}(P)$ for a finite p-subgroup P, then we can write $u=v+w$, where $v\in R\mathrm{C}_G(P)$ and $w\in RG$ with $\mathrm{supp}(w)\cap \mathrm{C}_G(P)=\varnothing$ and $p\mid \varepsilon(w)$: for $x\in P$ we obtain from $u^x=u$, that $u_{g^x}=u_g$ for all

$g \in G$. By noting that the orbit of an element $g \in G \setminus \mathrm{C}_G(P)$ under the conjugation action of P has length p^α for an $\alpha \geq 1$ we obtain the claimed decomposition.

To prove the proposition we may assume that $H \neq 1$. Obviously $H = \prod P$, where the product runs over the different Sylow p-subgroups of H. Using the Coleman lemma 3.1 and $\mathrm{N}_G(P) = G = \mathrm{N}_G(H)$ we have

$$\mathrm{N}_{\mathrm{V}(RG)}(H) \subseteq \mathrm{N}_{\mathrm{V}(RG)}(P) = \mathrm{N}_G(P) \cdot \mathrm{C}_{\mathrm{V}(RG)}(P) = \mathrm{N}_G(H) \cdot \mathrm{C}_{\mathrm{V}(RG)}(P)$$

for every $P \in \mathrm{Syl}_p(H)$. From now on let p be a fixed prime with $p \mid |H|$ and $P \in \mathrm{Syl}_p(H)$. Every $u \in \mathrm{N}_{\mathrm{V}(RG)}(H)$ can be decomposed as

$$u = gz, \qquad g \in \mathrm{N}_G(H), \quad z \in \mathrm{C}_{\mathrm{V}(RG)}(P) \cap \mathrm{N}_{\mathrm{V}(RG)}(H)$$

by the previous observation. Let q be another prime dividing $|H|$, if there is one. We have to show $z \in \mathrm{C}_{\mathrm{V}(RG)}(Q)$ for $Q \in \mathrm{Syl}_q(H)$. If this is not the case then we can write z as

$$z = g'z', \qquad g' \in \mathrm{N}_G(H) \setminus N, \quad z' \in \mathrm{C}_{\mathrm{V}(RG)}(Q).$$

Recall that $N = \mathrm{C}_G(P)$ and set $\overline{G} = {}^{G}/_{N}$. Consider the natural homomorphism $\theta\colon \mathrm{V}(RG) \to \mathrm{V}(R\overline{G})$. As $z \in \mathrm{V}(RG)$ we have $z\theta \in \mathrm{V}(R\overline{G}) = \overline{G}$.

1. $p \mid \varepsilon(z\theta) - \varepsilon_1(z\theta)$.
   Write $z \in \mathrm{C}_{\mathrm{V}(RG)}(P)$ as in the remark at the beginning of the proof: $z = v + w$, where $v \in R\mathrm{C}_G(P) = RN$ and $\mathrm{supp}(w) \cap N = \varnothing, p \mid \varepsilon(w)$. Now $z\theta = v\theta + w\theta$, where θ maps the elements of the support of v to $1 \in \overline{G}$ and the elements of the support of w to $\overline{G} \setminus \{1\}$, hence $p \mid \varepsilon(w) = \varepsilon(w\theta) = \varepsilon(z\theta) - \varepsilon_1(z\theta)$.

2. $q \mid \varepsilon_1(z\theta)$.
   Now decompose $z' \in \mathrm{C}_{\mathrm{V}(RG)}(Q)$ according to the remark at the beginning: $z' = v' + w'$, where $v' \in R\mathrm{C}_G(Q) = RN$, $\mathrm{supp}(w') \cap N = \varnothing$ and $q \mid \varepsilon(w')$. As $\mathrm{supp}(v') \subseteq N$ and $g' \notin N$ we have

   $$\mathrm{supp}(g'v') \cap N = g' \,\mathrm{supp}(v') \cap N = \varnothing. \tag{3.5}$$

   The element w' consists of summands of the form $s = \lambda(t_1^{-1}yt_1 + \ldots + t_n^{-1}yt_n)$, where $\lambda \in R$, $y \in G$, $t_j \in Q$ and $q \mid n$. For every $j \in \{1, \ldots, n\}$ we have

   $$g't_j^{-1}yt_j = g't_j^{-1}g'^{-1}(g'y)t_j$$

   and, as g' normalizes N, we see that $g't_j^{-1}yt_j \in N$ if and only if $g'y \in N$, hence

   $$\mathrm{supp}(g's) \cap N \neq \varnothing \;\Leftrightarrow\; \mathrm{supp}(g's) \subseteq N. \tag{3.6}$$

Write $g'w' = a + b$ with $a \in RN$ and $\mathrm{supp}(b) \cap N = \varnothing$. By (3.6) we have $q \mid \varepsilon(a)$. Consequently $z\theta = (g'z')\theta = (g'v')\theta + (g'w')\theta = (g'v')\theta + a\theta + b\theta$. Using (3.5) we have $\varepsilon_1(z\theta) = \varepsilon(a)$ and therefore $q \mid \varepsilon_1(z\theta)$.

These two divisibility statements together contradict the fact that $z\theta \in \overline{G}$. □

Remark 3.56. Higman classified the finite groups G with $\mathrm{V}(\mathbb{Z}G) = G$ [Seh93, Theorem (2.7)]: these groups are either abelian of exponent dividing 4 or 6, or isomorphic to $Q_8 \times E$, where Q_8 denotes the Quaternion group of order 8 and E is an elementary abelian 2-group.

3.2.7 The H-support group and some properties of the group $\mathrm{Aut}_{RG}(H)$

Let $u \in RG$, and assume that u is written as a R-linear combination of elements of G, $u = \sum u_g g$, $u_g \in R$ with $u_g \neq 0$ only for finitely many. Recall that the set $\mathrm{supp}(u) = \{g \in G \mid u_g \neq 0\}$ is the support of u. Jespers, Juriaans, de Miranda, Rogerio in [JJdMR02] and later Hertweck in [Her04, Section 17 & 18] discovered a lot properties of the subgroup generated by the support of an element $u \in \mathrm{N}_{\mathrm{U}(RG)}(G)$. It was proved that this subgroup enjoys many nice properties provided $1 \in \mathrm{supp}(u)$, among others it is normal in G, and, under some natural weak assumptions on the ring, it is finite. This served as a key to discover a lot of structure of the normalizer of G in $\mathrm{U}(RG)$ for infinite G as many questions could be boiled down with this knowledge to the finite group case. However, if $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ for a subgroup $H \leq G$ the structure of the subgroup of G generated by the support elements doesn't seem to have the same nice structure as in the 'classical' case. It turns out that a good substitute candidate is obtained from the H-support:

Definition 3.57. For a subgroup $H \leq G$ and an element $u = \sum u_g g \in RG$, expressed as a linear combination with respect to G, define the *H-support of u* as

$$\mathrm{supp}_H(u) = \mathrm{supp}(u) \cap H = \{h \in H \mid u_h \neq 0\}.$$

For an element $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ the subgroup $\langle \mathrm{supp}_H(u) \rangle$ is called the *H-support group of u*.

In the proof of the next proposition we will make use of the FC-center, so that we give a brief overview of the definition and properties needed (for details cf. [Pas77, Chapter 4]). For a group X the *FC-center* is the characteristic subgroup $\Delta(X) = \{x \in X \mid |X : \mathrm{C}_X(x)| < \infty\}$. For every finitely-generated subgroup Y of $\Delta(X)$ the torsion elements form a finite normal subgroup Y_0 of Y such that ${}^{Y}/_{Y_0}$ is a finitely-generated torsion-free abelian group [Pas77, 4, Lemma 1.5].

We get the following adaption of results of Hertweck's Habilitationsschrift [Her04] for the case when considering subgroups:

Proposition 3.58. *Let $H \leq G$, $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ and assume that $1 \in \mathrm{supp}(u)$. Define $S = \langle \mathrm{supp}_H(u) \rangle$, the H-support group of u. Then $S \trianglelefteq H$ and there is a characteristic subgroup T of S, such that T is finite and ${}^{S}/_{T}$ is a finitely generated, torsion-free abelian group. The unit u normalizes the subgroups S and T.*

Proof. Let $u = \sum u_g g \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ and $\sigma = \mathrm{conj}(u)$. We reestablish some of the properties already obtained in Subsection 3.2.2. For $h \in H$ we have $u(h^{-1}\sigma) = h^{-1}u$, and, by comparing coefficients, $u_{hg} = u_{g(h\sigma)}$ for all $g \in G$ and all $h \in H$. Substituting g by $h^{-1}g$ gives

$$\forall\, g \in G \quad \forall\, h \in H: \qquad u_g = u_{h^{-1}g(h\sigma)}.$$

In particular $g \in \mathrm{supp}(u) \Leftrightarrow \forall\, h \in H\colon h^{-1}g(h\sigma) \in \mathrm{supp}(u)$. Specializing $g = 1$ gives $\{h^{-1}(h\sigma) \mid h \in H\} \subseteq \mathrm{supp}_H(u)$. To check that S is normal in H let $s \in \mathrm{supp}_H(u)$ and $h \in H$, then

$$s^h = h^{-1}s(h\sigma) \cdot (h^{-1}(h\sigma))^{-1} \in S.$$

As $\mathrm{supp}_H(u)$ is a generating set of S, this implies the normality of S in H. Let $x \in \mathrm{supp}_H(u)$, then for all $h \in H$ the element $h^{-1}x(h\sigma) = x^h h^{-1}(h\sigma)$ is in the finite set $\mathrm{supp}_H(u)$, but there are also only finitely many possibilities for $h^{-1}(h\sigma)$. Hence x^H is finite, too. This implies that S is a subgroup of $\Delta(H) = \{x \in H \mid |H : \mathrm{C}_H(x)| < \infty\}$, the FC-center of H. $T = \Delta^+(S) = \{x \in \Delta(S) \mid o(x) < \infty\}$ is a finite subgroup of S, which is in fact characteristic in S, and ${}^{S}/_{T}$ is finitely-generated torsion-free abelian. Next we show that σ restricts to an automorphism of S. To this end let $s \in S$ and $s\sigma = h \in H$. Then $su = uh$ and, by looking at the support of both sides, we conclude that $h \in \mathrm{supp}(uh) \cap H = \mathrm{supp}(su) \cap H \subseteq S$, so $s\sigma \in S$ and $u \in \mathrm{N}_{\mathrm{U}(RG)}(S)$. As T is characteristic in S we obtain $u \in \mathrm{N}_{\mathrm{U}(RG)}(T)$. □

Remark 3.59. If $H \trianglelefteq G$, then, by modifying an element $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ by a multiplication with an inverse of a support-element, we can always assume that

$1 \in \mathrm{supp}(u)$. Having $1 \in \mathrm{supp}(u)$ we get that $\mathrm{conj}(u) = \mathrm{id}_H \Leftrightarrow \langle \mathrm{supp}_H(u) \rangle = 1$: this follows from the observation $h^{-1}(h\mathrm{conj}(u)) \in \langle \mathrm{supp}_H(u) \rangle$ for all $h \in H$ in the last proof.

Proposition 3.60. *Let $H \leq G$, $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ with $1 \in \mathrm{supp}(u)$. Let $S = \langle \mathrm{supp}_H(u) \rangle$ be the H-support group of u. The automorphism $\mathrm{conj}(u)|_S$ has finite order. If H is finitely-generated abelian, then $\mathrm{conj}(u)$ has finite order. If, in addition, $H \trianglelefteq G$ then every coset of $\mathrm{Aut}_{RG}(H) / \mathrm{Aut}_G(H)$ contains an element of finite order.*

Proof. Let $\sigma = \mathrm{conj}(u)$ and let T be the finite torsion subgroup of S obtained in Proposition 3.58, then S/T is finitely generated torsion-free abelian. Consider the natural homomorphism $\varphi\colon S \to S/T$. First assume that H is an arbitrary group and $h \in S$, then we have $h^{-1}, (h\sigma) \in S$ and we get for all $k \in \mathbb{N}$

$$(h^{-k}(h^k\sigma))\varphi = (h^{-k}\varphi)(h^k\sigma\varphi) = \left[(h^{-1}(h\sigma))\varphi\right]^k.$$

So, if for some $h_0 \in S$ the element $h_0^{-1}(h_0\sigma)$ is not contained in the kernel of φ, then $\mathrm{supp}_H(u)$ is infinite, as all $h_0^{-k}(h_0^k\sigma)$ are contained in the H-support and necessarily different, a contradiction. Hence

$$h^{-1}(h\sigma) = t(h) = t \in \mathrm{Ker}\,\varphi = T. \tag{3.7}$$

Induction shows that $h\sigma^k = ht(t\sigma)(t\sigma^2) \cdot \ldots \cdot (t\sigma^{k-1})$ for $k \in \mathbb{N}$. As T is finite, for every $h \in S$ there is $n \in \mathbb{N}$ such that $t(t\sigma)(t\sigma^2) \cdot \ldots \cdot (t\sigma^{n-1}) = 1$, but then $h\sigma^n = h$. As S is finitely generated, there is a $p \in \mathbb{N}$ such that $(\sigma|_S)^p = \mathrm{id}_S$.

From now on assume that H is finitely-generated abelian, then we get as in the proof of Proposition 3.58 that $h^{-1}(h\sigma) \in S$ for all $h \in H$ and the equation $(h^{-k}(h^k\sigma))\varphi = \left[(h^{-1}(h\sigma))\varphi\right]^k$ holds as the elements commute already in H and the rest of the proof can be carried over for this case.

To prove the last statement, let $\sigma = \mathrm{conj}(u) \in \mathrm{Aut}_{RG}(H)$. After modification with an element of $\mathrm{Aut}_G(H)$ we can assume that $1 \in \mathrm{supp}(u)$. Now for every such σ there is a finite group $T = T(\sigma)$ such that σ induces a homomorphism given by the formula in (3.7):

$$\delta = \delta(\sigma)\colon\ H \to T(\sigma) \hookrightarrow \mathrm{Tor}(H)\colon\ h \mapsto h^{-1}(h\sigma),$$

where $\mathrm{Tor}(H)$ denotes the torsion subgroup of H, which is finite as H was assumed to be finitely-generated abelian. We have $\sigma = \tau \Leftrightarrow \delta(\sigma) = \delta(\tau)$. But there are only finitely many homomorphisms from a finitely-generated abelian group to a finite group. □

For infinite H we always have a 'big' normal subgroup on which a normalizing unit u acts like all its support elements:

Proposition 3.61. *Let $H \leq G$ and R an arbitrary ring. For all $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ there is $Y \trianglelefteq H$ with $|H : Y| < \infty$ such that for all $g \in \mathrm{supp}(u)$: $\mathrm{conj}(u)|_Y = \mathrm{conj}(g)|_Y$.*

Proof. Let $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ and set $\sigma = \mathrm{conj}(u)$. Fix some $g \in \mathrm{supp}(u)$ and set $X(g) = \{h \in H \mid h\sigma = h^g\} \leq H$. To see that this is a subgroup of finite index in H we note that by (3.2) on page 38

$$\mathrm{supp}(u) \times H \to \mathrm{supp}(u)\colon\ (h,g) \mapsto h^{-1}g(h\sigma)$$

is an action of H on the finite support of u and $X(g)$ is the subgroup fixing g under this action.

Define

$$Y = \bigcap_{g \in \mathrm{supp}(u)} \mathrm{core}_H(X(g)),$$

this is a normal subgroup of H which is clearly of finite index in H. From the definition of $X(g)$ it follows that $y\sigma = y^g$ for all $y \in Y$ and all $g \in \mathrm{supp}(u)$. □

Remark 3.62. We get for $u \in \mathrm{N}_{\mathrm{U}(RG)}(H)$ that $\langle g_1 g_2^{-1} | g_1, g_2 \in \mathrm{supp}(u)\rangle \leq \mathrm{C}_G(Y)$. So for every $g_0 \in \mathrm{supp}(u)$ we can write $u = vg_0$ for some $v \in R\mathrm{C}_G(Y)$.

For certain kinds of groups we know more about the group $\mathrm{Aut}_{RG}(H)$. For example, if G is finite then we can show – using the idea of [HK02, Proposition 1] – that all prime divisors of $|\mathrm{Aut}_{RG}(H)|$ stem from $|G|$:

Proposition 3.63. *Let G be a finite group, $H \leq G$, then, for an H-adapted ring R, the prime divisors of $|\mathrm{Aut}_{RG}(H)|$ are also prime divisors of $|G|$.*

Proof. Let $\sigma \in \mathrm{Aut}_{RG}(H)$ with $o(\sigma) \mid r$ for a prime r with $(r, |G|) = 1$. We will show $\sigma = \mathrm{id}_H$. Let p be a prime divisor of $|H|$ and let $\langle t \rangle$ be a cyclic group of order r, then we have an action $\mathrm{Syl}_p(H) \times \langle t \rangle \to \mathrm{Syl}_p(H)$ defined by $S \cdot t = S\sigma$. By Sylow's theorem we know $|\mathrm{Syl}_p(H)| \mid |H|$, by assumption $r \nmid |H|$ and hence there is a Sylow p-subgroup $S = S(p)$ fixed by σ. Using Coleman lemma 3.1 we get $\sigma|_S = \mathrm{conj}(x)|_S$ for some $x \in \mathrm{N}_G(P)$, hence the order of $\sigma|_S$ divides r and divides the order of $x \in G$. This forces $\sigma|_S = \mathrm{id}_S$. Now p was an arbitrary prime divisor of $|H|$ and H is generated by Sylow p-subgroups, one for each prime divisor of $|H|$, so $\sigma = \mathrm{id}_H$. □

Considering $G = \mathrm{Hol}(H)$, i.e. $H \rtimes \mathrm{Aut}(H)$ with the 'natural' action of $\mathrm{Aut}(H)$ on H, shows that we cannot expect a better statement in the general case.

A group X is called a *FC-group* if it coincides with its FC-center, i.e. if $X = \Delta(X) = \{x \in X \mid |X : \mathrm{C}_X(x)| < \infty\}$.

Proposition 3.64. *Let G be a FC-group and $H \leq G$ finitely-generated, then the group $\mathrm{Aut}_{RG}(H)$ is a torsion group.*

Proof. Let $h \in H$ and $\sigma \in \mathrm{Aut}_{RG}(H)$. By Lemma 3.22 we have for all integers r that $h\sigma^r \in h^G$, a finite set, so for every $h \in H$ there is $s \in \mathbb{N}$ with $h\sigma^s = h$. By assumption H is finitely-generated and hence there is $p \in \mathbb{N}$ with $\sigma^p = \mathrm{id}_H$. □

Theorem 3.3 of Jan Krempa states that there is a strong obstruction for $\mathrm{Aut}_{\mathbb{Z}G}(G)$, namely that $\mathrm{Aut}_{\mathbb{Z}G}(G)/\mathrm{Inn}(G)$, the quotient group modulo the inner automorphisms, is an elementary abelian 2-group. A building block of the proof is the statement that the membership of an element to $\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(G)$ can be reformulated in terms of central elements. For this we need a definition and some remarks.

Definition 3.65. The anti-automorphism $g \mapsto g^{-1}$ of the group G gives rise to the *'standard' anti-involution*

$$^*\colon RG \to RG\colon\ \sum u_g g \mapsto \sum u_g g^{-1}$$

of the group ring RG.

For $u \in \mathrm{U}(RG)$ we have $(u^*)^{-1} = (u^{-1})^*$, justifying the abbreviation $u^{-*} = (u^*)^{-1}$. If $R = \mathbb{Z}$ then this anti-involution has the useful property that it can detect group elements in the following way: $u \in G \Leftrightarrow uu^* = 1$. To see this, note that we have $\varepsilon_1(uu^*) = \sum u_g^2$ for every element $u = \sum u_g g$. Using this, one can prove that $u \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G) \Leftrightarrow uu^* \in \mathrm{Z}(\mathrm{U}(\mathbb{Z}G)) = \mathrm{C}_{\mathrm{U}(\mathbb{Z}G)}(G)$ for every $u \in \mathrm{U}(\mathbb{Z}G)$ (cf. [JM87, 3.1 Proposition]). The analogue statement also holds true for normal subgroups (although we were forced to adapt the proof for the reverse implication a little):

Proposition 3.66. *Let H be a normal subgroup of G and $u \in \mathrm{U}(\mathbb{Z}G)$. Then*

$$u \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(H) \Leftrightarrow uu^* \in \mathrm{C}_{\mathrm{U}(\mathbb{Z}G)}(H).$$

Proof. '⇒': Let $u \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(H)$ and $x \in H$. Then $x\mathrm{conj}(u) \in H \leq G$. Using this we get

$$x\mathrm{conj}(u) = (x\mathrm{conj}(u))^{-*} = ((x^{-1})\mathrm{conj}(u))^{*} = (u^{-1}x^{-1}u)^{*} = u^{*}xu^{-*}$$
$$= x\mathrm{conj}(u^{-*}).$$

Hence $x = x\mathrm{conj}(u)\mathrm{conj}(u^{*}) = u^{-*}u^{-1}xuu^{*}$. As $x \in H$ was arbitrary, we conclude that $uu^{*} \in \mathrm{C}_{\mathrm{U}(\mathbb{Z}G)}(H)$.

'⇐': Let $u \in \mathrm{U}(\mathbb{Z}G)$ and assume that $uu^{*} \in \mathrm{C}_{\mathrm{U}(\mathbb{Z}G)}(H)$. Take any $x \in H$, then

$$(x\mathrm{conj}(u))(x\mathrm{conj}(u))^{*} = (u^{-1}xu)(u^{-1}xu)^{*} = u^{-1}xuu^{*}x^{-1}u^{-*}$$
$$= u^{-1}uu^{*}xx^{-1}u^{-*} = 1,$$

implying $x\mathrm{conj}(u) \in G$. Now we have to prove that in fact $x\mathrm{conj}(u) \in H$. To this end we can employ the same argument as in Lemma 3.22. We have

$$x\mathrm{conj}(u) - x = [u^{-1}x, u]_L \in [\mathbb{Z}G, \mathbb{Z}G]_L = \{v \in \mathbb{Z}G \mid \forall\, C \in \mathrm{ccl}(G) \colon \varepsilon_C(v) = 0\},$$

and hence $x\mathrm{conj}(u) \in x^G$. We conclude that

$$H\mathrm{conj}(u) \subseteq H^G = H,$$

and consequently $u \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(H)$. □

4 Centralizers of subgroups of a group basis

Let throughout the chapter G be an arbitrary group.

In this chapter we investigate how the torsion parts of the centralizers of certain kinds of subgroups H of G in the units of the integral group ring $\mathbb{Z}G$ look like. We prove under restrictions on a p-part of subgroups that there are only the 'obvious' torsion units centralizing this subgroup. We also prove under some assumptions that specific maximal abelian torsion subgroups of a group basis are also maximal in the normalized unit group of the group ring.

A corollary of one of the fundamental observation for torsion units of integral group rings gives a final answer concerning the torsion elements of the centralizer of a group basis

Proposition 4.1 (Higman; Berman; Bass, Passman)**.** *Let G be a group, then the normalized central torsion units of $\mathbb{Z}G$ are elements of $\mathrm{Z}(G)$.*

This result was extended by Martin Hertweck to $\Delta^+(G)$-adapted coefficient rings (cf. [Her04, 18.6 Corollary]), where $\Delta^+(G) = \{g \in G \mid |G : \mathrm{C}_G(g)| < \infty,\ o(g) < \infty\}$ denotes the torsion subgroup of the FC-center of G.

To prove the proposition above one makes use of one of the fundamental observation for torsion units of integral group rings, which is – for finite group G and a ring of algebraic integers R – due to Higman and Berman. For arbitrary groups there is the following version due to Bass and Passman, (cf. [Seh93, Theorem (45.8)], for a proof [Seh78, Chapter II, Corollary 1.3]).

Theorem 4.2 (Higman; Berman; Bass, Passman). *If G is an arbitrary group and $u \in \mathrm{V}(\mathbb{Z}G)$ is a torsion unit with $\varepsilon_1(u) \neq 0$, then $u = 1$.*

We will make use of this theorem to obtain a modification of Lemma 2.15, which serves here for our purpose of investigating centralizers:

Lemma 4.3. *Let $H \leq G$ and $u = \sum_{g\in G} u_g g \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(H)$ a unit of finite order. Then for every torsion element $h \in H$ we have $u_h \neq 0 \Leftrightarrow u = h$.*

Proof. We only have to prove one implication. Assume $u_h \neq 0$. The element uh^{-1} is also of finite order and $\varepsilon_1(uh^{-1}) = u_h \neq 0$, so Theorem 4.2 implies that $u = h$. □

We can define augmentations with respect to subsets of group bases:

Definition 4.4. Let B be a group basis of RG, $X \subseteq B$, and $u \in RG$. Express $u = \sum_{b\in B} u_b b$ with respect to the group basis B. Then set $\varepsilon_X^B(u) = \sum_{b\in X} u_b$.

Note that $\varepsilon_X^B(u)$ may depend on the chosen group basis B with respect to which it is defined, whereas the augmentation ε does not depend on the group basis used to calculate it. Further note that for a disjoint union $B = X \cup Y$ of the group basis B we have $\varepsilon = \varepsilon_X^B + \varepsilon_Y^B$. In what follows we will only consider the maps ε_X^G with respect to the fixed group basis G, and we will omit the superscript G, so $\varepsilon_X = \varepsilon_X^G$ for $X \subseteq G$.

An observation already used in the proof of Proposition 3.55 holds in a more general setting:

Lemma 4.5. *Let $H \leq G$, R a ring and $u \in \mathrm{C}_{\mathrm{V}(RG)}(H)$. Assume that $P \leq H$ is a torsion p-group for a prime p (i.e. every element of P has order p^a for some non-negative integer a) and $X \subseteq G$ is closed under conjugation by elements of P (i.e. $x^y \in X$ for all $x \in X$ and all $y \in P$), then $\varepsilon_{X \smallsetminus \mathrm{C}_G(P)}(u) \in pR$.*

Proof. The p-group P acts on X by conjugation. Express u with respect to the group basis G, $u = \sum_{g\in G} u_g g$. Then $u^y = u$ for all $y \in P \leq H$ implies that the coefficients of u are constant under the conjugation action, i.e. $u_{x^y} = u_x$ for all $x \in G$ and all $y \in P$. Any element x of $(X \smallsetminus \mathrm{C}_G(P)) \cap \mathrm{supp}(u)$ has an orbit of finite length $p^{b(x)}$, with $b(x) \geq 1$ under this action. Let T be a system of representatives of the orbits of $(X \setminus \mathrm{C}_G(P)) \cap \mathrm{supp}(u)$ under this action. Then clearly

$$\varepsilon_{X \setminus \mathrm{C}_G(P)}(u) = \sum_{g \in X \setminus \mathrm{C}_G(P)} u_g = p \sum_{h \in T} p^{b(h)-1} u_h \in pR. \qquad \square$$

As Theorem 2.4 in [BK11] appears the following result

Proposition 4.6. *Let G be a finite group, H a group basis of $\mathbb{Z}G$ and let W be a Hall subgroup of H. Assume that for each prime p dividing the order of W, there is $w \in W$ of order p such that $\mathrm{C}_H(w) \leq W$. Then the normalized torsion elements of the centralizer ring $\mathrm{C}_{\mathbb{Z}G}(W)$ are the central elements of W.*

We now can prove the following generalization[1]:

Proposition 4.7. *Let $H \leq G$ and assume that there is a prime p and a torsion p-subgroup $P \leq H$ (i.e. every element of P has order p^a for some non-negative integer a) such that $\mathrm{C}_G(P)$ is contained in the torsion part of H. Then the torsion elements of $\mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(H)$ are precisely the torsion elements of $\mathrm{Z}(H)$.*

Proof. Note that we have $\mathrm{C}_G(H) = \mathrm{Z}(H)$. Hence, to prove the statement, it is enough to show that the torsion elements of $\mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(H)$ are contained in G. Assume the contrary: there is an element $u \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(H) \setminus G$ of finite order. We have to reach a contradiction. If we express $u = \sum u_g g$ as a linear combination of elements of G, then, by Lemma 4.3, we have $u_x = 0$ for all torsion elements $x \in H$, in particular $\varepsilon_{\mathrm{C}_G(P)}(u) = 0$. Applying Lemma 4.5 yields $\varepsilon_{G \setminus \mathrm{C}_G(P)}(u) \in p\mathbb{Z}$, and hence

$$1 = \varepsilon(u) = \varepsilon_{\mathrm{C}_G(P)}(u) + \varepsilon_{G \setminus \mathrm{C}_G(P)}(u) \in p\mathbb{Z},$$

the desired contradiction. $\square$

Corollary 4.8. *Let $H \leq G$ be a maximal abelian torsion subgroup. If $\mathrm{C}_G(P) \leq H$ for some torsion p-subgroup P of H (p a prime), then there is no abelian torsion subgroup in $\mathrm{V}(\mathbb{Z}G)$ properly containing H.*

Proof. Applying Proposition 4.7 yields that the torsion part of $\mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(H)$ coincides with H. $\square$

[1] concerning arbitrary group basis see Remark 4.15

Remark 4.9. There is also an analogue result of the Higman-Berman result Theorem 4.2 for group rings of finite groups over G-adapted coefficient rings due to Saksonov, see [Kar80, Theorem 2.1] (according to this overview article the original of this work can be found in the russian article [Sak71, p. 191]). Using this, Lemma 4.3, and then all the results obtained so far in this section can also be proved for group rings of finite groups over G-adapted coefficient rings.

Proposition 4.10. *Let G be a finite group and $P \leq G$ be a maximal abelian p-subgroup. Then there is no abelian p-group in $\mathrm{V}(\mathbb{Z}G)$ properly containing P.*

Proof. Assume, by way of contradiction, that there is a $u \in \mathrm{V}(\mathbb{Z}G)$ generating together with P an abelian p-group strictly containing P. Then $u = \sum_{g\in G} u_g g \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P) \setminus G$. By Lemma 4.3, $u_g = 0$ for all $g \in P$. Set

$$X = \{g \in G \setminus \mathrm{C}_G(P) \mid o(g) = p^a, \text{ for some } a \in \mathbb{N}\},$$

the set of p-elements in $G \setminus \mathrm{C}_G(P)$. Then X is clearly invariant under conjugation by elements of P. Note that all p-elements of G are either contained in P or in X, as $\mathrm{C}_G(P) \setminus P$ cannot contain p-elements by the maximality of P. By Theorem 2.7 the element u can only have non-vanishing partial augmentations on classes of p-elements, hence we have reached by applying Lemma 4.5 at

$$1 = \varepsilon(u) = \varepsilon_P(u) + \varepsilon_X(u) \ \in p\mathbb{Z},$$

and this is clearly a contradiction. □

In Theorem 4.14 we provide a result of joint work with Wolfgang Kimmerle, that already appeared in [BK11, Theorem A], and shows that the results of the last and the current chapter can turn out to be useful when considering questions arising in the context of the Zassenhaus conjectures.

For this purpose we need another lemma about centralizers.

Lemma 4.11. *Let G be a finite group, let p and q be two different primes and $P \leq G$ a p-subgroup. Assume that there is an element $u \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$ of order q, then $q \mid |\mathrm{C}_G(P)|$.*

Proof. Let $u \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$ be a unit of order q. Assume that there is no element of order q in $\mathrm{C}_G(P)$ and set $Y = \{g \in G \mid o(g) = q\}$, the elements of order q in G. Then $Y \cap \mathrm{C}_G(P) = \varnothing$, so that Theorem 2.7 and Lemma 4.5 imply

$$1 = \varepsilon(u) = \varepsilon_Y(u) \in p\mathbb{Z}.$$

This contradiction proves the lemma. □

We need the following theorem, see [Kim06] (for a proof see also [BK11, Proposition 2.5]).

Theorem 4.12 (Kimmerle). *Let G be a finite Frobenius group and C be a Frobenius complement of G. Then $\Gamma(\mathrm{V}(\mathbb{Z}G)) = \Gamma(G)$ and $\Gamma(\mathrm{V}(\mathbb{Z}C)) = \Gamma(C)$.*

Definition 4.13. Let X be a group. A proper subgroup $U \leq X$ is called *isolated*, if $U^x \cap U \in \{1, U\}$ for all $x \in X$, and $\mathrm{C}_X(u) \leq U$ for all $u \in U \setminus \{1\}$.

Note that isolated subgroups of finite groups are Hall subgroups.

Theorem 4.14. *Assume that U is an isolated subgroup of the finite group G. Then $\Gamma(\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)) = \Gamma(\mathrm{N}_G(U))$.*

Proof. If $\mathrm{N}_G(U) = U$, then G is a Frobenius group and U is a Frobenius complement in G. By Theorem 4.12 we get that $\Gamma(\mathrm{V}(\mathbb{Z}G)) = \Gamma(G)$. As U is isolated, $\Gamma(U)$ is a full subgraph of $\Gamma(G)$, hence $\Gamma(\mathrm{N}_G(U))$ and $\Gamma(\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U))$ must coincide.

If $\mathrm{N}_G(U)$ contains U properly. Then every element of $\mathrm{N}_G(U) \setminus U$ induces a fixed point free automorphism of U, and $\mathrm{N}_G(U)$ is a Frobenius group with kernel U. By a famous result of Thompson [Hup67, V. 8.7 Hauptsatz a)] the Frobenius kernel U is nilpotent. Assume there is an edge between two different primes p and q in $\Gamma(\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U))$. We consider three cases separately to show that there is also an edge between those primes in $\Gamma(\mathrm{N}_G(U))$.

- Suppose that both, p and q, divide $|U|$. Then the nilpotency of U implies that p and q are linked by an edge even in $\Gamma(U)$.

- Suppose that exactly one of the two primes, say p, divides $|U|$. There is an edge between p and q in $\Gamma(\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U))$, so there is $x \in \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)$ of order pq. Clearly we may write $x = x_p x_q$ with $x_p,\ x_q \in \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)$, commuting elements of order p and q, respectively. Let $P \in \mathrm{Syl}_p(U)$. Because U is isolated we have $\mathrm{N}_G(P) = \mathrm{N}_G(U)$. Using the Coleman lemma 3.1 we obtain

$$\mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U) \subseteq \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(P) = \mathrm{N}_G(U) \cdot \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P). \tag{4.1}$$

  So we can factorize any element $y \in \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)$ as $y = gz$, with $g \in \mathrm{N}_G(U)$ and $z \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$. Mapping y to the coset $\mathrm{C}_G(P)g$ gives rise to a map

$$\varphi \colon \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U) \to {}^{\mathrm{N}_G(U)}/{}_{\mathrm{C}_G(P)},$$

which is well-defined as $\mathrm{N}_G(U) \cap \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P) \leq \mathrm{C}_G(P)$. Using that U is isolated, we have $\mathrm{C}_G(P) \trianglelefteq \mathrm{N}_G(U)$, so that the co-domain of φ is a group. $\mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$ is left invariant under conjugation by elements of $\mathrm{N}_G(U)$, implying that φ is a group homomorphism. Clearly φ is onto and $\mathrm{Ker}\,\varphi = \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U) \cap \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$.

By (4.1) we can write $x_p = g_p c_p$ and $x_q = g_q c_q$, where $g_p,\ g_q \in \mathrm{N}_G(U)$ and $c_p,\ c_q \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P) \cap \mathrm{N}_{\mathrm{V}(\mathbb{Z}G)}(U)$. Now $x\varphi = (g_p\varphi)(g_q\varphi)$. If p and q are not connected in $\Gamma(\mathrm{N}_G(U))$, then $g_p\varphi = 1$ or $g_q\varphi = 1$. Assume that $g_q \in \mathrm{Ker}\,\varphi$, then x_q is a torsion unit in $\mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$ of order q, hence by Lemma 4.11 there are elements of order q in $\mathrm{C}_G(w)$ for some $w \in P \setminus \{1\}$. As U is isolated $\mathrm{C}_G(w) \leq U$, and hence $q \mid |U|$, a contradiction. Now assume that $g_p \in \mathrm{Ker}\,\varphi$, then $x_p \in \mathrm{C}_{\mathrm{V}(\mathbb{Z}G)}(P)$ is a torsion element and by Proposition 4.7 we see that $x_p \in \mathrm{Z}(P)$. Now Lemma 4.11 implies that there is an element of order q in $\mathrm{C}_G(x_p) \leq U$, yielding a final contradiction for this case.

- Suppose that none of the two primes p and q divides $|U|$. With the same notation and the same arguments as in the preceding case it follows that $g_p\varphi \neq 1$ and $g_q\varphi \neq 1$. Then $x\varphi = (g_p\varphi)(g_q\varphi)$ yields an element of order pq in $\mathrm{N}_G(U)$, and there is an edge between p and q in $\Gamma(\mathrm{N}_G(U))$. □

Remark 4.15. All the statements of this chapter can be proved in the same manner if we replace G by an arbitrary group basis of $\mathbb{Z}G$ and consider subgroups thereof.

Nomenclature

x^y	$= y^{-1}xy$ conjugation of the group element x with the group element y
x^Y	$= \{y^{-1}xy \mid y \in Y\}$ conjugates of an element x of a group G under the set $Y \subseteq G$
x^G	conjugacy class of the element $x \in G$
$[x,y]$	$= x^{-1}y^{-1}xy$ commutator of the two group elements x and y
G'	commutator subgroup of the group G
$[x,y]_L$	$= xy - yx$ Lie commutator of the two ring elements x and y, page 40
$\langle X \rangle$	subgroup generated by X
G/H	set of right cosets or quotient group, respectively
$N \rtimes_\alpha C$	semi-direct product, where N is a normal subgroup, and α, if present, the homomorphism $C \to \mathrm{Aut}(N)$ via which the semi-direct product is defined
$R^\times$	unit group of the ring R
A_n	alternating group of degree n, where $n \in \mathbb{N}$
$\mathrm{Aut}(G)$	group of automorphisms of the group G
$\mathrm{Aut}_c(G)$	$= \{\varphi \in \mathrm{Aut}(G) \mid \forall\, x \in G\colon x\varphi \in x^G\}$ group of class-preserving automorphisms of the group G, page 37
$\mathrm{Aut}_{\mathrm{Col}}(G)$	group of Coleman automorphisms of the finite group G, page 37

$\mathrm{Aut}_G(H)$	$= \{\varphi \in \mathrm{Aut}(H) \mid \varphi = \mathrm{conj}(g),\ \text{for some } g \in \mathrm{N}_G(H)\}$ group of automorphisms of the subgroup H of G induced by elements of G, page 36
$\mathrm{Aut}(Q,K)$	group of automorphisms of E restricting to the identity of K and inducing the identity on Q for $1 \to K \to E \to Q \to 1$, a short exact sequence of groups, page 43
$\mathrm{Aut}_{RG}(H)$	$= \{\varphi \in \mathrm{Aut}(H) \mid \varphi = \mathrm{conj}(u),\ \text{for some } u \in \mathrm{N}_{\mathrm{U}(RG)}(H)\}$ group of automorphisms of the subgroup H of G induced by elements of $\mathrm{U}(RG)$, page 36
$\mathrm{ccl}(G)$	collection of all conjugacy classes of the group G
$\mathrm{C}_G(X)$	$= \{g \in G \mid \forall x \in X : x^g = x\}$ centralizer of a subset $X \subseteq G$ in the group G
C_n	cyclic group of order n, where $n \in \mathbb{N} \cup \{\infty\}$
$\mathrm{conj}(u)$	$= (x \mapsto x^u)$ conjugation automorphism induced by the element u
$\mathrm{core}_G(H)$	$= \bigcap_{g \in G} H^g \trianglelefteq G$ G-core of the subgroup $H \leq G$
D_m	dihedral group of order $2m$, with $m \in \mathbb{N}$
$\Delta(X)$	the FC-center of the group X, page 56
$\Delta^+(X)$	torsion subgroup of the FC-center $\Delta(X)$ of the group X
$\mathrm{Dr}_{j \in J}\ G_j$	restricted direct product of the groups G_j (subgroup of the direct product $\prod_{j \in J} G_j$ consisting of those tuples where all but finitely many coordinates are equal to the identity element)
$\varepsilon(u)$	Augmentation of u, page 14
$\varepsilon_1(u)$	coefficient u_1 at the group element 1 of a group ring element $\sum u_g g$, page 20
$\varepsilon_C(u)$	partial augmentation of u with respect to the conjugacy class C, page 20
$\varepsilon_X(u)$	augmentation map with respect to a subset X of a fixed group basis, page 62
$\exp G$	exponent of the group G
$\Gamma(X)$	prime graph of the group X, page 18

$\mathrm{H}^1(Q,K)$	first cohomology group, cf. [Rot07, Sections 9.1 and 9.2]
id_X	$=(x \mapsto x)$ identity map on the set X
$\mathrm{Inn}(G)$	group of inner automorphisms of the group G
$\mathrm{Inn}(Q,K)$	$=\mathrm{Aut}(Q,K)\cap\mathrm{Inn}(E)$, page 43
$\mathrm{N}_G(X)$	$=\{g\in G \mid \forall x\in X: x^g\in X\}$ normalizer of a subset $X\subseteq G$ in the group G
$\mathrm{NP}(G,R)$	'classical' normalizer property, page 32
$\mathrm{NP}(H\leq G,R)$	normalizer property, page 35
$o(x)$	order of an element x
$\mathrm{O}_p(G)$	largest normal p-subgroup of G
$\mathrm{O}_{p'}(G)$	largest normal p'-subgroup of G
$\mathrm{Out}_{RG}(G)$	$=\mathrm{Aut}_{RG}(G)/\mathrm{Inn}(G)$, page 32
S_n	symmetric group of degree n, where $n\in\mathbb{N}$
$\mathrm{SNP}(G,R)$	subgroup normalizer property, page 35
$\mathrm{supp}(u)$	support of the group ring element u, page 38
$\mathrm{supp}_H(u)$	H-support for a group ring element $u\in RG$ and a subgroup $H\leq G$, page 55
$\mathrm{U}(RG)$	unit group of the group ring RG, page 14
$\mathrm{V}(RG)$	group of normalized units of the group ring RG, page 14
$\mathrm{Z}(G)$	center of the group G

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