On the elementary theory of Heller triangulated categories

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0.1.1 Organisation

In Chapter I, Heller triangulated categories are defined and basic properties are derived. It has been published as [**36**].

In Chapter II, stability properties of the Heller formalism are established. It has been prepublished as [**40**].

In Chapter III, the difference between [**8,** 1.1.13] and [**36,** Def. 1.5.(ii.2)] is shown to be nonzero by way of counterexample. It has been published as [**39**].

In Chapter IV, we make some remarks on spectral sequences from the point of view of spectral objects, i.e. ∞-triangles. It has been published as [**37**].

0.1.2 Acknowledgements

I thank Sebastian Thomas for lively debates over the years and for keeping me informed about his work.

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0.2 An introduction to the general theory of Heller triangulated categories

0.2.1 Why derived categories?

0.2.1.1 Derived functors

Around 1960, GROTHENDIECK struggled with increasingly complicated spectral sequence comparisons ([1]). These spectral sequences arose as follows.

Given left exact functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ between abelian categories ([2]), under normal circumstances ([3]), the derived functor $\mathrm{R}^n(FG)$ of the composite FG can be approximated by the composites of the derived functors $(\mathrm{R}^i F)(\mathrm{R}^j G)$, where $i + j = n$. In other words, we have the Grothendieck spectral sequence, which has E_2-terms $(\mathrm{R}^i F)(\mathrm{R}^j G)$ and converges to $\mathrm{R}^n(FG)$.

For instance, roughly put, the first derivative

$$\mathrm{R}^1(FG)$$

consists of

$$\text{a part of } (\mathrm{R}^1 F)(\mathrm{R}^0 G) \qquad \text{plus} \qquad \text{a part of } (\mathrm{R}^0 F)(\mathrm{R}^1 G)\ ;$$

which can be read as an approximative "product rule for the first derivative" ([4]).

In practice, this is troublesome since it only yields an approximative relationship between

the derived functors of the composite   and   the composites of the derived functors ,

and since, moreover, this approximation is laborious.

Finally, if we want to compose three or more functors and relate their various derivatives, we are stuck.

0.2.1.2 Derived functors, renovated

The construction of such a derived functor $\mathrm{R}^i F$ proceeds in three steps.

(1) Resolve injectively.

(2) Apply the functor F.

(3) Take cohomology H^i.

---

[1] An example of an assertion of this kind may be seen in [**18,** 6.6.2]. According to ILLUSIE, GROTHENDIECK said: "The second part of EGA III is a mess, so, please, clean this up by introducing derived categories, write the Künneth formula in the general framework of derived categories." [**29,** p. 1108].

[2] An *abelian category* has direct sums, kernels, cokernels, and the homomorphism theorem holds.

[3] The categories $\mathcal{A}$ and $\mathcal{B}$ are supposed to have enough injectives, and F to map injectives to G-acyclics.

[4] More precisely, $\mathrm{R}^1(FG)$ has a two-step filtration, one subfactor of which being a subfactor of $(\mathrm{R}^1 F)(\mathrm{R}^0 G)$, the other being a subfactor of $(\mathrm{R}^0 F)(\mathrm{R}^1 G)$.

Grothendieck saw that the troubles were caused by the third step and that dropping the third step, one should get a smooth formalism, in which the spectral sequence approximation mentioned above is turned into the simple and precise rule

$$(*) \qquad \mathrm{R}(FG) \simeq (\mathrm{R}F)(\mathrm{R}G) .$$

The price to pay was the development of this formalism, undertaken by Verdier around 1963 in [**56**].

Since we have dropped taking cohomology H^i, the renovated derived functor $\mathrm{R}F$ now takes values in complexes (over $\mathcal{B}$). So in order to be able to compose, $\mathrm{R}F$ should also take as arguments complexes (over $\mathcal{A}$).

Moreover, in order to ensure the validity of the composition rule $(*)$, one has to formally invert morphisms of complexes that induce isomorphisms in cohomology, called *quasi-isomorphisms.* This process yields the *derived category*

$$\mathrm{D}^+(\mathcal{A}) ,$$

having as objects complexes ([5]) over $\mathcal{A}$, and as morphisms fractions

$$f/s ,$$

where the numerator f is a morphism of complexes and where the denominator s is a quasi-isomorphism of complexes.

So in full, the formula $(*)$ reads

$$(*') \qquad \big(\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}(FG)} \mathrm{D}^+(\mathcal{C})\big) \quad \simeq \quad \big(\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathrm{D}^+(\mathcal{B}) \xrightarrow{\mathrm{R}G} \mathrm{D}^+(\mathcal{C})\big) .$$

0.2.2 Why triangulated categories?

0.2.2.1 Verdier triangulated categories

The category $\mathcal{A}$ is abelian.

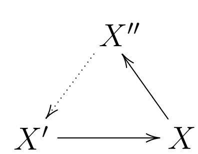
The derived category $\mathrm{D}^+(\mathcal{A})$ is not abelian ([6]).

There exist hardly any short exact sequences in $\mathrm{D}^+(\mathcal{A})$, only split ones.

As substitute, the image in $\mathrm{D}^+(\mathcal{A})$ of a short exact sequence of complexes $X' \rightarrowtail X \twoheadrightarrow X''$ fits into a diagram

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'^{+1} ,$$

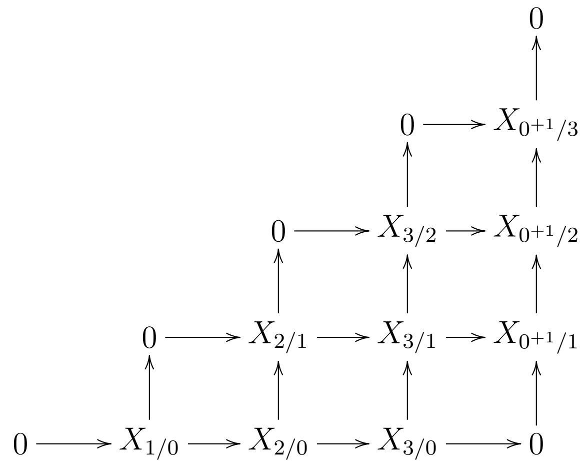
called a *distinguished triangle*, where X'^{+1} denotes the complex X', shifted one step to the left ([7]).



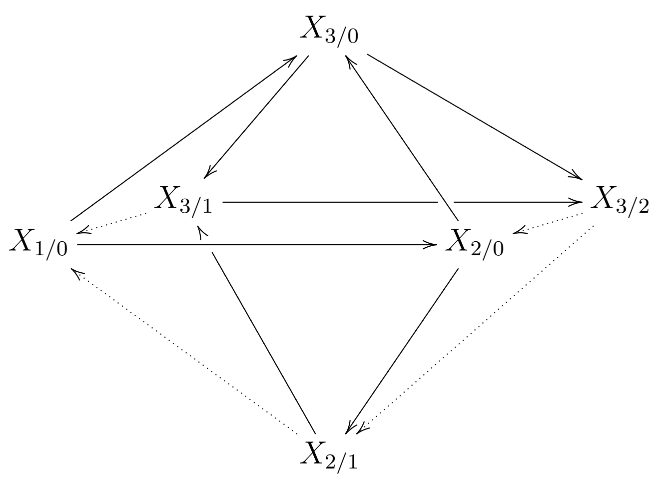
[5] Bounded to the left.
[6] Except if $\mathcal{A}$ is semisimple.
[7] And all differentials negated.

Now any morphism $X_{1/0} \longrightarrow X_{2/0}$ in $\mathrm{D}^+(\mathcal{A})$ fits into such a distinguished triangle $X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{2/1} \longrightarrow X_{1/0}^{+1}$, and this completion is unique up to isomorphism ([8]). We call $X_{2/1}$ the *cone* of the morphism $X_{1/0} \longrightarrow X_{2/0}$ ([9]).

The compatibility of taking cones with composition is expressed by the following *Verdier octahedron* ([10]), in which $X_{0^{+1}/i} = X_{i/0}^{+1}$ for $1 \leqslant i \leqslant 3$.



Here $X_{j/i}$ is the cone on $X_{i/0} \longrightarrow X_{j/0}$ for $1 \leqslant i < j \leqslant 3$. Moreover, $X_{3/2}$ is the cone on $X_{2/1} \longrightarrow X_{3/1}$.



A theory of Verdier triangulated categories was developed by Verdier **[56]**, which plays the same role for $\mathrm{D}^+(\mathcal{A})$ as the theory of abelian categories plays for $\mathcal{A}$.

Here, a *Verdier triangulated category* is a triple $(\mathcal{D}, \mathsf{T}, \Xi)$, consisting of an additive category $\mathcal{D}$, an automorphism $\mathsf{T} = (-)^{+1}$ of $\mathcal{D}$, called *shift,* and a set Ξ of distinguished triangles, satisfying a list of axioms, including the existence of a Verdier octahedron on each pair of composable morphisms. Then, Ξ is called a *Verdier triangulation* on $(\mathcal{D}, \mathsf{T})$.

For example, the derived category $\mathrm{D}^+(\mathcal{A})$ is Verdier triangulated. Also already the homotopy

[8] In contrast to what we are used to from kernels and cokernels in abelian categories, this isomorphism is not uniquely determined in general.

[9] This notion is motivated by the homotopy category of CW-complexes, which becomes a Verdier triangulated category after Spanier-Whitehead stabilisation, where this cone is an actual geometrically constructed cone.

[10] This alternative, non-octahedral form of this diagram was observed in [**8,** 1.1.14].

category $\mathrm{K}^+(\mathcal{A})$, obtained as the category of complexes ([11]) modulo split acyclic complexes, is Verdier triangulated. More generally, the stable category of a Frobenius category ([12]) is Verdier triangulated; cf. [**22,** Th. 2.6].

One of the axioms records a curious phenomenon, without parallel in the context of abelian categories. For every distinguished triangle

$$X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{2/1} \longrightarrow X_{1/0}^{+1} ,$$

we get the rotated distinguished triangle

$$X_{2/1}^{-1} \longrightarrow X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{2/1} \qquad ({}^{13}) .$$

In a Verdier triangulated category, the cone of a morphism is at the same time a substitute for its kernel and its cokernel, but only in a weak form ([14]). Therefore, a Verdier triangulated category is weakly abelian, i.e. it is an additive category in which each morphism is and has a weak kernel and a weak cokernel.

A set Ξ of distinguished triangles that satisfies the list of axioms except possibly for the existence of a Verdier octahedron on each diagram $X_{1/0} \longrightarrow X_{2/0} \longrightarrow X_{3/0}$, is called a *Puppe triangulation* on $(\mathcal{D}, \mathsf{T})$ [**51**]. Cf. §0.2.3.1 below.

0.2.2.2 Exact functors between Verdier triangulated categories

A *strictly exact functor* between Verdier triangulated categories is a shiftcompatible additive functor that maps distinguished triangles to distinguished triangles; i.e. that preserves cones.

Derived functors, such as the functor $\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathrm{D}^+(\mathcal{B})$ from §0.2.1.2, are strictly exact.

An *exact functor* between Verdier triangulated categories $(\mathcal{D}, \mathsf{T}, \Xi)$ and $(\mathcal{D}', \mathsf{T}', \Xi')$ is a pair (V, a) consisting of an additive functor $V : \mathcal{D} \longrightarrow \mathcal{D}'$ and an isotransformation $a : \mathsf{T}\, V \longrightarrow V\, \mathsf{T}'$ such that each distinghuished triangle in $\mathcal{D}$, mapped via V and isomorphically replaced via a, yields a distinguished triangle of $\mathcal{D}'$.

So $V : \mathcal{D} \longrightarrow \mathcal{D}'$ is strictly exact if and only if $(V, 1)$ is exact.

0.2.2.3 Stability properties of the Verdier formalism

Adjoints of exact functors are exact [**44,** App. 2, Prop. 11] [**34,** 1.6]. Already Grothendieck and Deligne observed in Algebraic Geometry the appearance of exact functors that are not derived functors, but adjoints to derived functors ([15]).

[11] Bounded to the left.

[12] A *Frobenius category* is an exact category with a sufficient supply of relatively bijective objects.

[13] With a sign inserted.

[14] In the notation above, $X_{2/0} \longrightarrow X_{2/1}$ is a weak cokernel of $X_{1/0} \longrightarrow X_{2/0}$, i.e. it satisfies the universal property of a cokernel, except for uniqueness of the induced morphism. Moreover, $X_{2/1}^{-1} \longrightarrow X_{1/0}$ is a weak kernel of $X_{1/0} \longrightarrow X_{2/0}$, i.e. satisfies the universal property of a kernel, except for uniqueness of the induced morphism.

[15] The functor $\mathrm{R}f^!$, constructed for certain morphisms f of schemes, is only abusively written with a "R"; cf. [**19,** Exp. XVIII, Th. 3.1.4].

The *Karoubi hull* of an additive category is the universal additive category whose idempotents split [**30,** III.II]. We may form Karoubi hulls within the context of Verdier triangulated categories and exact functors, as shown by Balmer and Schlichting [**2**].

The *localisation* of a category at a subset of its morphisms is the universal category such that morphisms of this subset become invertible. For instance, the derived category $\mathrm{D}^+(\mathcal{A})$ is the localisation of the homotopy category $\mathrm{K}^+(\mathcal{A})$ at the subset of quasiisomorphisms, i.e. at the subset of morphisms with acyclic cone. We may form the localisation of a Verdier triangulated category at the subset of morphisms with cone in a given thick subcategory ([16]) within the realm of Verdier triangulated categories and exact functors [**56**][**53,** Prop. 1.3].

0.2.3 Heller triangulated categories

0.2.3.1 Heller's original theorem

Let $\mathcal{D}$ be a weakly abelian category; cf. §0.2.2.1. The *Freyd category* $\hat{\mathcal{D}}$ is the universal abelian category containing $\mathcal{D}$ [**15**]. Reducing modulo the full additive subcategory of projective objects, we obtain the stable category $\underline{\hat{\mathcal{D}}}$, which is Verdier triangulated ([17]).

Now suppose $\mathcal{D}$ to carry a shift functor T. Then $\underline{\hat{\mathcal{D}}}$ carries two shift functors, a first one induced by T, a second one given by the Verdier triangulated structure on $\underline{\hat{\mathcal{D}}}$.

Heller discovered a bijection between the set of Puppe triangulations Ξ on $(\mathcal{D}, \mathsf{T})$ and the set of isomorphisms from the first shift functor to the third power of the second shift functor satisfying an extra condition ([18]) [**24,** Th. 16.4].

So such an isomorphism between these shift functors can be made responsable for a Puppe triangulation, as the extra datum needed to upgrade a weakly abelian category with shift $(\mathcal{D}, \mathsf{T})$ to a Puppe triangulated category $(\mathcal{D}, \mathsf{T}, \Xi)$.

So we could just as well include this isomorphism instead of Ξ in our data.

0.2.3.2 Extending Heller's theorem

Let $\mathcal{D}$ be a weakly abelian category. Let T be an automorphism of $\mathcal{D}$.

In order to extend Heller's result from Puppe triangulations to Verdier triangulations and beyond, all we need is a suitable replacement for $\underline{\hat{\mathcal{D}}}$.

A *weak square* in $\mathcal{D}$ is a commutative quadrangle that is at the same time a weak pullback and

[16] A *thick* subcategory is a full subcategory closed under shift, forming cones and taking summands.

[17] The reason being that $\mathcal{D}$ is a big enough full subcategory in $\hat{\mathcal{D}}$ consisting of bijective objects, so that $\hat{\mathcal{D}}$ is a Frobenius abelian category. Cf. [**22,** §2.1].

[18] Such an isomorphism can be pre- and postcomposed with the Verdier shift on $\underline{\hat{\mathcal{D}}}$; the condition is that the result of precomposition is the negative of the result of postcomposition.

a weak pushout ([19]). A weak square is marked as

$$\begin{array}{ccc} & \longrightarrow & \\ \uparrow & + & \uparrow \\ & \longrightarrow & \end{array} .$$

Alternatively, a commutative quadrangle is a weak square if and only if its diagonal sequence is exact in the middle when viewed in $\hat{\mathcal{D}}$.

Let $\mathcal{D}^+(\bar{\Delta}_2^\#)$ be the category of diagrams in $\mathcal{D}$ of the form

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & \longrightarrow & \cdots \\
 & & & & & & & & \uparrow & + & \\
 & & & & & & 0 & \longrightarrow & X_{0^{+1}/2} & \longrightarrow & \cdots \\
 & & & & & & \uparrow & + & \uparrow & + & \\
 & & & & 0 & \longrightarrow & X_{2/1} & \longrightarrow & X_{0^{+1}/1} & \longrightarrow & \cdots \\
 & & & & \uparrow & + & \uparrow & + & \uparrow & & \\
 & & 0 & \longrightarrow & X_{1/0} & \longrightarrow & X_{2/0} & \longrightarrow & 0 & & \\
 & & \uparrow & + & \uparrow & + & \uparrow & & & & \\
0 & \longrightarrow & X_{0/2^{-1}} & \longrightarrow & X_{1/2^{-1}} & \longrightarrow & 0 & & & & \\
\uparrow & + & \uparrow & + & \uparrow & & & & & & \\
\vdots & & \vdots & & \vdots & & & & & &
\end{array} ,$$

where we do not require any relation between $X_{0^{+1}/2}$ and $X_{2/0}^{+1}$ etc. Viewed in the abelian category $\hat{\mathcal{D}}$, this is just the category of acyclic complexes consisting of objects in $\mathcal{D}$.

Let $\mathcal{D}^+(\bar{\Delta}_3^\#)$ be the category of diagrams in $\mathcal{D}$ of the form

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & 0 & \longrightarrow & \cdots \\
 & & & & & & & & & & \uparrow & + & \\
 & & & & & & & & 0 & \longrightarrow & X_{0^{+1}/3} & \longrightarrow & \cdots \\
 & & & & & & & & \uparrow & + & \uparrow & + & \\
 & & & & & & 0 & \longrightarrow & X_{3/2} & \longrightarrow & X_{0^{+1}/2} & \longrightarrow & \cdots \\
 & & & & & & \uparrow & + & \uparrow & + & \uparrow & + & \\
 & & & & 0 & \longrightarrow & X_{2/1} & \longrightarrow & X_{3/1} & \longrightarrow & X_{0^{+1}/1} & \longrightarrow & \cdots \\
 & & & & \uparrow & + & \uparrow & + & \uparrow & + & \uparrow & & \\
 & & 0 & \longrightarrow & X_{1/0} & \longrightarrow & X_{2/0} & \longrightarrow & X_{3/0} & \longrightarrow & 0 & & \\
 & & \uparrow & + & \uparrow & + & \uparrow & + & \uparrow & & & & \\
0 & \longrightarrow & X_{0/3^{-1}} & \longrightarrow & X_{1/3^{-1}} & \longrightarrow & X_{2/3^{-1}} & \longrightarrow & 0 & & & & \\
\uparrow & + & \uparrow & + & \uparrow & + & \uparrow & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & & & &
\end{array} ,$$

[19] The respective universal property is supposed to hold, except for the uniqueness of the induced morphism.

where we do not require any relation between $X_{0^{+1}/3}$ and $X^{+1}_{3/0}$ etc.

Etc.

For $n \geqslant 0$, we let

$$\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)}$$

be the reduction of $\mathcal{D}^+(\bar{\Delta}_n^\#)$ modulo the full additive subcategory of diagrams all of whose morphisms split. This category carries two shift functors, the outer shift $[-]^{+1}$ and the inner shift $[-^{+1}]$, characterised by, respectively,

$$\begin{array}{rcl} ([X]^{+1})_{\beta/\alpha} & = & X_{\alpha^{+1}/\beta} \\ ([X^{+1}])_{\beta/\alpha} & = & (X_{\beta/\alpha})^{+1} \end{array}$$

for $X \in \operatorname{Ob}\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob}\mathcal{D}^+(\bar{\Delta}_n^\#)$. In other words, the outer shift pulls the whole diagram down left, the inner shift applies T pointwise.

Then

$$\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)} \qquad \simeq \qquad \underline{\hat{\mathcal{D}}}\,,$$

where

the outer shift	corresponds to	the third power of the Verdier shift ,
the inner shift	corresponds to	the functor induced by T .

So we can transport an isomorphism as in Heller's theorem from §0.2.3.1 to an isomorphism

$$[-]^{+1} \xrightarrow[\sim]{\vartheta_2} [-^{+1}]$$

from the outer to the inner shift functor on $\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)}$.

Using $\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)}$ as a replacement for $\underline{\hat{\mathcal{D}}}$ will enable us, in §0.2.3.3 below, to extend from $\underline{\mathcal{D}^+(\bar{\Delta}_2^\#)}$ to $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)}$ for $n \geqslant 0$, so as to include octahedra and bigger diagrams [**8,** 1.1.14], and to drop the extra condition on the isomorphism mentioned in §0.2.3.1.

0.2.3.3 Heller triangulated categories

Let $\mathcal{D}$ be a weakly abelian category. Let T be an automorphism of $\mathcal{D}$.

Let a *Heller triangulation* on $(\mathcal{D}, \mathsf{T})$ be a tuple $\vartheta = (\vartheta_n)_{n\geqslant 0}$ of isomorphisms $\vartheta_n : [-]^{+1} \longrightarrow [-^{+1}]$ from the outer shift $[-]^{+1}$ to the inner shift $[-^{+1}]$ on $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)}$ satisfying compatibilities with quasicyclic operations ([20]) and with folding ([21]).

---

[20]Deleting and doubling rows and columns in a periodic manner yield functors $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} \xrightarrow{p^\#} \underline{\mathcal{D}^+(\bar{\Delta}_m^\#)}$. We require that $X\vartheta_n \underline{p}^\# = X\underline{p}^\#\vartheta_m$ for $X \in \operatorname{Ob}\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob}\mathcal{D}^+(\bar{\Delta}_n^\#)$.

[21]Suppose given $n \geqslant 0$ and $X \in \operatorname{Ob}\underline{\mathcal{D}^+(\bar{\Delta}_{2n+1}^\#)} = \operatorname{Ob}\mathcal{D}^+(\bar{\Delta}_{2n+1}^\#)$. We can canonically ([22]) construct an object $X\underline{\mathfrak{f}}_n \in \operatorname{Ob}\underline{\mathcal{D}^+(\bar{\Delta}_{n+1}^\#)} = \operatorname{Ob}\mathcal{D}^+(\bar{\Delta}_{n+1}^\#)$ that has $(X\underline{\mathfrak{f}}_n)_{i/0} = X_{n+i/i-1}$ for $1 \leqslant i \leqslant n+1$; the diagram $X\underline{\mathfrak{f}}_n$ involves direct sums of objects occurring in X. The operation $\underline{\mathfrak{f}}_n$ can be turned into a functor from $\underline{\mathcal{D}^+(\bar{\Delta}_{2n+1}^\#)}$ to $\underline{\mathcal{D}^+(\bar{\Delta}_{n+1}^\#)}$. We require that $X\vartheta_{2n+1}\underline{\mathfrak{f}}_n = X\underline{\mathfrak{f}}_n\vartheta_{n+1}$. Cf. [**8,** 1.1.13].

[22]Up to sign.

A *Heller triangulated category* then is a triple $(\mathcal{D}, \mathsf{T}, \vartheta)$ consisting of a weakly abelian category $\mathcal{D}$, an automorphism T of $\mathcal{D}$ and a Heller triangulation ϑ on $(\mathcal{D}, \mathsf{T})$; cf. Def. I.5.(ii.1). Often, we write just $\mathcal{D} := (\mathcal{D}, \mathsf{T}, \vartheta)$.

For example, the derived category $\mathrm{D}^+(\mathcal{A})$ is Heller triangulated. Also the homotopy category $\mathrm{K}^+(\mathcal{A})$ is Heller triangulated. More generally, the stable category of a Frobenius category is Heller triangulated. Cf. Cor. I.33, Prop. II.36 ([23]).

0.2.3.4 n-triangles

Suppose given a Heller triangulated category $(\mathcal{D}, \mathsf{T}, \vartheta)$. Suppose given $n \geqslant 0$.

The *base* of a diagram $X \in \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^\#)$ is its subdiagram

$$(X_{1/0} \longrightarrow X_{2/0} \longrightarrow \cdots \longrightarrow X_{n-1/0} \longrightarrow X_{n/0}) \qquad \in \ \operatorname{Ob} \mathcal{D}(\dot{\Delta}_n)$$

on the linearly ordered set $\dot{\Delta}_n := \{1, 2, \ldots, n\}$.

A diagram $X \in \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^\#)$ is called an *n-triangle* if $X\vartheta_n = 1$. A morphism $X \xrightarrow{f} Y$ between n-triangles X and Y is called *periodic* if $[f]^{+1} = [f^{+1}]$.

The restriction functor to the base, mapping from the category of n-triangles and periodic morphisms to $\mathcal{D}(\dot{\Delta}_n)$, is full; cf. Lem. I.19. If all idempotents split in $\mathcal{D}$, then it is also surjective on objects; cf. Lem. I.18 ([24]).

Such triangles are stable under quasicyclic operations and under folding; cf. Lem. I.21.(1, 2).

0.2.3.5 Retrieving the Verdier context in the Heller context

Suppose given a Heller triangulated category $(\mathcal{D}, \mathsf{T}, \vartheta)$ in which all idempotents split.

Let Ξ be the set of 2-triangles in $\mathcal{D}$. Then the triple $(\mathcal{D}, \mathsf{T}, \Xi)$ is a Verdier triangulated category; cf. Prop. I.23 ([25]).

Each 3-triangle is a Verdier octahedron; cf. §0.2.2.1. However, not every Verdier octahedron is a 3-triangle; cf. Lem. III.6 ([26]).

$$\begin{array}{rcl} \{\text{distinguished triangles}\} & = & \{\text{2-triangles}\} \\ \{\text{Verdier octahedra}\} & \supseteq & \{\text{3-triangles}\} \end{array}$$

0.2.3.6 Exact functors between Heller triangulated categories

Suppose given Heller triangulated categories $(\mathcal{D}, \mathsf{T}, \vartheta)$ and $(\mathcal{D}', \mathsf{T}', \vartheta')$.

[23] Cf. also Prop. III.22.(1).

[24] More generally, this holds if $\mathcal{D}$ is a closed Heller triangulated category; cf. Lem. II.20. Cf. also Rem. I.20.

[25] More generally, this holds if $\mathcal{D}$ is a closed Heller triangulated category; cf. Rem. II.18.

[26] Not even when requiring that it contains the triangles described in [**8,** 1.1.13]; cf. Rem. III.7.

A *strictly exact functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a shiftcompatible additive functor $V : \mathcal{D} \longrightarrow \mathcal{D}'$ that respects weak squares, and that satisfies

$$X\vartheta_n \underline{V^+(\bar{\Delta}_n^\#)} \;=\; X\underline{V^+(\bar{\Delta}_n^\#)}\vartheta'_n$$

for all $n \geqslant 0$ and all $X \in \operatorname{Ob} \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^\#)$, where $\underline{V^+(\bar{\Delta}_n^\#)}$ acts by pointwise application of V.

An *exact functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a pair (V, a) consisting of an additive functor $V : \mathcal{D} \longrightarrow \mathcal{D}'$ respecting weak squares, and an isotransformation $a : \mathsf{T}\, V \longrightarrow V\, \mathsf{T}'$ such that

$$X\vartheta_n \underline{V^+(\bar{\Delta}_n^\#)} \cdot X\underline{a^+(\bar{\Delta}_n^\#)} \;=\; X\underline{V^+(\bar{\Delta}_n^\#)}\vartheta'_n$$

for all $n \geqslant 0$ and all $X \in \operatorname{Ob} \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob} \mathcal{D}^+(\bar{\Delta}_n^\#)$.

So $V : \mathcal{D} \longrightarrow \mathcal{D}'$ is strictly exact if and only if $(V, 1)$ is exact.

0.2.3.7 Stability properties of the Heller formalism

Adjoints of exact functors are exact; cf. Prop. II.28.

We may form the Karoubi hull within the context of Heller triangulated categories and exact functors; cf. Prop. II.12.

We may form the localisation at the subset of morphisms with cone in a given thick subcategory within the context of Heller triangulated categories and exact functors; cf. Prop. II.38.

The derived functor $\mathrm{D}^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathrm{D}^+(\mathcal{B})$ from §0.2.1.2 is exact, using that $\mathcal{A}$ is supposed to have enough injectives ([27]).

It is also possible to characterise exactness of a functor, in a manner similar to §0.2.2.2, by preservation of n-triangles; cf. Prop. II.25. The reason behind that possibility is that closed ([28]) Heller triangulated categories can, alternatively, be defined via sets of n-triangles for $n \geqslant 0$ with suitable preservation properties with respect to quasicyclic operations and folding, as Thomas informed me.

0.2.3.8 Advantages of ϑ

Having n-triangles at our disposal allows constructions that have not been possible within the Verdier context. For instance, given two 3-triangles, a morphism between the bases can be prolonged to a morphism between the 3-triangles. This is no longer true, in general, once we replace "3-triangles" by "Verdier octahedra"; cf. Lem. III.6.

But why should we work primarily with ϑ, and only secondarily with n-triangles? A possible answer is that usage of ϑ allows low-effort proofs of the stability properties of the Heller formalism explained in §0.2.3.7; cf. §II.2.2, §II.6, §II.5.2.

[27] Somewhat provisionally still, we may use Prop. II.28, Prop. II.36, Cor. III.21, Cor. I.35 to arrive there. It would be preferable to use the derived functor construction via ind-categories along the lines of [**19,** Exp. XVII, §1.2].

[28] A Heller triangulated category is called *closed* if it is closed under taking cones in its Karoubi hull.

Of course, the price to pay is to get accustomed to the administration of the n-triangles being done by a tuple of isomorphisms ϑ.

0.2.3.9 An amusing observation

Suppose given Heller triangulated category $(\mathcal{D}, \mathsf{T}, \vartheta)$ in which all idempotents split ([29]).

A commutative quadrangle

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle x}\uparrow & & \uparrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in $\mathcal{D}$ is called a *dweak square* ([30]) if its diagonal sequence

$$X \xrightarrow{(f\ x)} Y \oplus X' \xrightarrow{\binom{y}{-f'}} Y'$$

appears as part of a 2-triangle. So a dweak square is in particular a weak square; cf. §0.2.3.2.

Alternatively, a commutative quadrangle is a dweak square if and only if it appears in some n-triangle for some $n \geqslant 0$.

Any corner $\uparrow_{\rightarrow}$ can be completed to a dweak square. This completion is unique up to non-unique isomorphism. Accordingly in the dual situation.

Suppose given $n \geqslant 1$. Consider the set Chain_n of isoclasses of diagrams of the form $X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n$ in $\mathcal{D}$, i.e. the set of isoclasses in $\mathcal{D}(\dot{\Delta}_n)$. We obtain two bijections

$$\sigma,\, \tau \;:\; \mathrm{Chain}_n \xrightarrow{\sim} \mathrm{Chain}_n$$

as follows.

Let τ map the isoclass of $X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n$ to the isoclass of $X_1^{+1} \to X_2^{+1} \to \cdots \to X_{n-1}^{+1} \to X_n^{+1}$.

Let σ be defined as follows. Suppose given $X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n$. Prolong this diagram by $X_n \to 0$. Complete to dweak squares along $X_1 \to 0$, yielding a new row $0 \to X_2' \to \cdots \to X_{n-1}' \to X_n' \to W_1$. Complete to dweak squares along $X_2' \to 0$, yielding a new row $0 \to X_3'' \to \cdots \to X_{n-1}'' \to X_n'' \to W_2$. Etc. Then let σ map the isoclass of $X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n$ to the isoclass of $W_1 \to W_2 \to \cdots \to W_{n-1} \to W_n$.

Elementary properties of n-triangles force $\sigma = \tau$.

If we only require $\mathcal{D}$ to be Verdier triangulated, both σ and τ are still definable, but it is unclear to me whether they coincide ([31]).

[29] More generally, the following holds if $\mathcal{D}$ is a closed Heller triangulated category.

[30] An abbreviation for "distinguished weak square". Also known as *homotopy cartesian square*, as *homotopy bicartesian square*, or as *Mayer-Vietoris square.*

[31] Suppose that [**8,** 1.1.13] holds in our Verdier triangulated category. Then σ and τ coincide if $n \in \{1, 2, 3\}$.

0.2.4 Remarks on spectral sequences

0.2.4.1 Four indices

Suppose given an abelian category $\mathcal{A}$. Suppose given a filtered complex M with values in $\mathcal{A}$, i.e. a chain of monomorphisms

$$M(-\infty) \rightarrowtail \cdots \rightarrowtail M(i) \rightarrowtail M(i+1) \rightarrowtail \cdots \rightarrowtail M(+\infty),$$

indexed by $\{-\infty\} \sqcup \mathbf{Z} \sqcup \{+\infty\}$, that satisfies certain technical conditions ([32]).

We shall use the linearly ordered set

$$\bar{\mathbf{Z}}_\infty := \{ i^{+k} : i \in \{-\infty\} \sqcup \mathbf{Z} \sqcup \{+\infty\},\ k \in \mathbf{Z} \} ,$$

where formally i^{+k} is defined as the pair (i,k), and where $i^{+k} \leqslant j^{+\ell}$ if $k < \ell$ or ($k = \ell$ and $i \leqslant j$).

Taking M as a base, we can form a diagram that is, morally, an ∞-triangle. It consists of shifted subfactor complexes $M(\beta/\alpha)$ for $\beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$ and is called *spectral object* $\mathrm{Sp}(M)$ of M ([33]). For $\gamma/\alpha \leqslant \delta/\beta$, i.e. $\gamma \leqslant \delta$ and $\alpha \leqslant \beta$, the induced morphism

$$M(\gamma/\alpha) \longrightarrow M(\delta/\beta)$$

appears in this diagram $\mathrm{Sp}(M)$.

Let $M\mathrm{E}(\delta/\beta /\!\!/ \gamma/\alpha) \in \mathrm{Ob}\,\mathcal{A}$ be defined as the image of H^0 of this morphism, i.e.

$$M(\gamma/\alpha)\mathrm{H}^0 \twoheadrightarrow M\mathrm{E}(\delta/\beta /\!\!/ \gamma/\alpha) \rightarrowtail M(\delta/\beta)\mathrm{H}^0 \qquad (^{34}) .$$

These objects $M\mathrm{E}(\delta/\beta /\!\!/ \gamma/\alpha)$ assemble to a big diagram with values in $\mathcal{A}$, the *spectral sequence*

$$M\mathrm{E}$$

of M ([35]).

Suppose given $\varepsilon^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \varepsilon \leqslant \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$. We obtain the short exact sequence

$$M\mathrm{E}(\varepsilon/\beta /\!\!/ \gamma/\alpha) \rightarrowtail M\mathrm{E}(\varepsilon/\beta /\!\!/ \delta/\alpha) \twoheadrightarrow M\mathrm{E}(\varepsilon/\gamma /\!\!/ \delta/\alpha) ,$$

which can be made responsible for all exact sequences in general spectral sequences known to me. Cf. Lem. I.26, generalising a particular case of [**57,** §II.4.2.6].

Dropping certain "initial terms" ([36]) from the spectral sequence $M\mathrm{E}$, we obtain the *proper spectral sequence*

$$M\dot{\mathrm{E}}$$

of M.

[32] Viz. $M(-\infty) = 0$, $M(i) \longrightarrow M(i+1)$ being pointwise split and the whole filtration being pointwise almost everywhere constant. Cf. §IV.3.1.

[33] This term has been coined by Verdier; cf. [**57,** §II.4].

[34] This definition slightly generalises the definition given in [**12,** App.]. The original definition in [**57,** §II.4.2.3] was closer to classical terminology, as found in [**10,** §XV.1].

[35] The classical spectral sequence terms are amongst the terms $M\mathrm{E}(\delta/\beta /\!\!/ \gamma/\alpha)$; cf. §IV.3.5.

[36] E_1-terms and similar ones; cf. §IV.3.6.

0.2.4.2 Comparisons

0.2.4.2.1 Grothendieck spectral sequences

Maintain the situation of §0.2.1.1. So $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$.

Suppose given $X \in \operatorname{Ob}\mathcal{A}$. Resolve X injectively. This yields a complex with values in $\mathcal{A}$. Apply F pointwise. This yields a complex with values in $\mathcal{B}$. Resolve this complex injectively, via the method of Cartan-Eilenberg [**10,** §XVII.1]. This yields a double complex with values in $\mathcal{B}$. Apply G pointwise. This yields a double complex with values in $\mathcal{C}$.

The *total complex* of this resulting double complex is obtained by forming direct sums over its diagonals. Replacing an increasing number of rows in this double complex by zero rows, and then taking the total complex, we obtain a descending chain of subcomplexes filtering our original total complex.

This filtered complex gives rise to the Grothendieck spectral sequence $X\mathrm{E}^{\mathrm{Gr}}_{F,G}$ via the method of §0.2.4.1. This yields a functor $\mathrm{E}^{\mathrm{Gr}}_{F,G}$ on $\mathcal{A}$, mapping to the category of spectral sequences with values in $\mathcal{C}$.

So we had to "resolve X twice", with an intermediate application of F, and a final application of G, to carry out this construction.

0.2.4.2.2 First comparison

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{C}$ ([37]). Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $X' \in \operatorname{Ob}\mathcal{A}'$. Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor such that $(X,-)F$ and $(-,X')F$ are left exact. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be a left exact functor. Suppose further conditions to hold; see §IV.5.1.

$$\begin{array}{c} \scriptstyle X \quad\; X' \\ \mathcal{A} \times \mathcal{A}' \\ \downarrow F \\ \mathcal{B} \\ \downarrow G \\ \mathcal{C} \end{array}$$

We have Grothendieck spectral sequence functors,

$$\begin{array}{lll} \mathrm{E}^{\mathrm{Gr}}_{(X,-)F,\,G} & \text{for} & \mathcal{A}' \xrightarrow{(X,-)F} \mathcal{B} \xrightarrow{G} \mathcal{C}\ , \\ \mathrm{E}^{\mathrm{Gr}}_{(-,X')F,\,G} & \text{for} & \mathcal{A} \xrightarrow{(-,X')F} \mathcal{B} \xrightarrow{G} \mathcal{C}\ . \end{array}$$

We evaluate the former at X' and the latter at X. Then the proper Grothendieck spectral sequences are isomorphic, i.e.

$$X'\,\dot{\mathrm{E}}^{\mathrm{Gr}}_{(X,-)F,\,G} \;\simeq\; X\,\dot{\mathrm{E}}^{\mathrm{Gr}}_{(-,X')F,\,G}\ ;$$

cf. Th. IV.31. So instead of "resolving X' twice", we may just as well "resolve X twice".

[37] Of which $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ are supposed to have enough injectives.

0.2.4.2.3 Second comparison

Suppose given abelian categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{C}$ ([38]). Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}'$ be a left exact functor. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor such that $(Y,-)G$ is left exact.

Let $B \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y such that $(B^k,-)G$ is exact for all $k \geqslant 0$. Let $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ be an injective resolution of X. Suppose further conditions to hold; see §IV.6.1.

$$
\begin{array}{ccc}
 & & X \\
 & & \mathcal{A} \\
 & Y & \downarrow F \\
 & \mathcal{B} & \times\ \mathcal{B}' \\
 & & \downarrow G \\
 & & \mathcal{C}
\end{array}
$$

We have the Grothendieck spectral sequence functor

$$\mathrm{E}^{\mathrm{Gr}}_{F,(Y,-)G} \quad \text{for} \quad \mathcal{A} \xrightarrow{F} \mathcal{B}' \xrightarrow{(Y,-)G} \mathcal{C}\ ,$$

which we evaluate at X.

On the other hand, we can consider the double complex $(B, AF)G$, where the indices of B count rows and the indices of A count columns. As described in §0.2.4.2.1, we can associate a spectral sequence to a double complex, in this case named $\mathrm{E}_{\mathrm{I}}\big((B, AF)G\big)$.

Then the proper spectral sequences are isomorphic,

$$X\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,(Y,-)G} \simeq \dot{\mathrm{E}}_{\mathrm{I}}\big((B, AF)G\big)\ .$$

So instead of “resolving X twice”, we may just as well “resolve X once and Y once”.

0.2.4.2.4 Applications

The comparisons in §0.2.4.2.2 and §0.2.4.2.3 may be used to reprove the following two theorems of Beyl.

The first theorem allows acyclic objects to be alternatively used to calculate Grothendieck spectral sequences [**7,** Th. 3.4]; cf. Th. IV.40.

The second theorem allows the Hochschild-Serre-Hopf spectral sequence to be calculated with injective or, equivalently, with projective resolutions; the former fitting in the context of Grothendieck spectral sequences, the second being apt for manipulating concrete representing cocycles of cohomology classes; cf. [**7,** Th. 3.5], Th. IV.52, IV.53.

Further applications can be found in §IV.8 ([39]).

[38] Of which $\mathcal{A}$ and $\mathcal{B}'$ are supposed to have enough injectives.

[39] If we were to reduce complexity in the assertions of §0.2.4.2, then, in the spirit of §0.2.1.2, we should directly work with suitably defined derived categories of double complexes; I do not know how to do that. We would probably get an additional shift functor.

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Chapter I

Heller triangulated categories

I.0 Introduction

I.0.1 Heller's idea ([1])

I.0.1.1 Stable Frobenius categories and an isomorphism between outer and inner shift

Let $\mathcal{E}$ be a Frobenius category, i.e. an exact category with enough bijective objects. For instance, the category of complexes with values in an additive category, equipped with pointwise split exact sequences, is a Frobenius category.

Let $\underline{\underline{\mathcal{E}}}$ denote the stable category of $\mathcal{E}$; cf. §I.0.3. Assume that $\underline{\underline{\mathcal{E}}}$ has split idempotents.

A complex with entries in $\underline{\underline{\mathcal{E}}}$ is *acyclic* if any Hom functor turns it into an acyclic complex of abelian groups. Let $\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_2^\#)$ denote the category of acyclic complexes with entries in $\underline{\underline{\mathcal{E}}}$ ([2]). Let $\underline{\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_2^\#)}$ denote the homotopy category of the category $\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_2^\#)$ of acyclic complexes; that is, the quotient category of acyclic complexes modulo split acyclic complexes.

There is a shift automorphism T on $\underline{\underline{\mathcal{E}}}$. It induces a first, *inner shift* automorphism $\mathsf{T}^+(\bar{\Delta}_2^\#)$ on $\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_2^\#)$ by pointwise application.

There is also a shift automorphism T_2 on the diagram $\bar{\Delta}_2^\#$. It induces a second, *outer shift* automorphism $\underline{\underline{\mathcal{E}}}^+(\mathsf{T}_2)$ on $\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_2^\#)$, shifting a complex by three positions.

Both outer and inner shift induce automorphisms

$$\underline{\underline{\underline{\mathcal{E}}}^+(\mathsf{T}_2)} \quad \text{resp.} \quad \underline{\mathsf{T}^+(\bar{\Delta}_2^\#)} \qquad \text{on} \quad \underline{\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_2^\#)} \ .$$

Heller remarked that these functors are isomorphic. But there is no a priori given isomorphism. So he chose an isomorphism

$$\underline{\underline{\underline{\mathcal{E}}}^+(\mathsf{T}_2)} \xrightarrow[\sim]{\vartheta_2} \underline{\mathsf{T}^+(\bar{\Delta}_2^\#)} \ ,$$

satisfying, for technical reasons, still a further compatibility.

[1] Heller formulated his idea using Freyd categories. We will rephrase it using complexes, for this is the language we will use below. Cf. §§ I.0.2.2, I.0.2.4.

[2] The notation using the diagram $\bar{\Delta}_2^\#$ is chosen to fit into a larger framework; see §I.0.2.2 for more details.

Then he remarked that the choice of such an isomorphism ϑ_2 determines a triangulation on $\underline{\underline{\mathcal{E}}}$ in the sense of PUPPE [**51,** Sec. 2]; that is, it satisfies all the axioms of VERDIER [**56,** Def. 1-1] except possibly for the octahedral axiom. Namely, as distinguished triangles we take acyclic complexes on which outer and inner shift coincide (i.e., which are "3-periodic up to shift") and on which ϑ_2 is the identity.

Whether this observation now fathoms Puppe triangulations remains to be discussed. Whenever two objects are isomorphic but lack a nature-given isomorphism, it is at any rate not unusual to pick an isomorphism. Once a suitable isomorphism between our shift functors chosen, a Puppe triangulation ensues. In nontechnical terms, we may let the relation between the two shifts govern the Puppe triangulations. This is a possible point of view, which we shall adopt and put into a larger framework; cf. §I.0.2.2.

HELLER used this construction to parametrise Puppe triangulations on $\underline{\underline{\mathcal{E}}}$. The non-uniqueness of such a Puppe triangulation on $\underline{\underline{\mathcal{E}}}$, and hence the impossibility of an intrinsic definition of distinguished triangles, thus can be regarded as rooted in the possible nontriviality of the automorphism group of the inner shift functor $\underline{\mathsf{T}^+(\bar{\Delta}_2^\#)}$, or, by choice, of the outer shift functor $\underline{\underline{\mathcal{E}}^+(\mathsf{T}_2)}$. This is to be seen in contrast to the intrinsic characterisation of short exact sequences in an abelian category.

I.0.1.2 The stable Frobenius case models a general definition of Puppe triangulations

A *weak kernel* in an additive category is defined by the universal property of a kernel, except for the uniqueness of the induced morphism; dually a *weak cokernel*.

A *weakly abelian category* is an additive category in which each morphism has a weak kernel and a weak cokernel, and in which each morphism is a weak kernel and a weak cokernel. For instance, the stable category $\underline{\underline{\mathcal{E}}}$ appearing in §I.0.1.1 is a weakly abelian category.

Let $\mathcal{C}$ be a weakly abelian category with split idempotents carrying a shift automorphism T. Now Heller's construction yields an alternative, equivalent definition of a Puppe triangulation on $(\mathcal{C}, \mathsf{T})$ as being an isomorphism

$$\underline{\mathcal{C}^+(\mathsf{T}_2)} \overset{\vartheta_2}{\underset{\sim}{\longrightarrow}} \underline{\mathsf{T}^+(\bar{\Delta}_2^\#)}$$

satisfying still a further compatibility. In other words, a Puppe triangulated category can be defined to be such a triple $(\mathcal{C}, \mathsf{T}, \vartheta_2)$.

I.0.1.3 From Puppe to Verdier and beyond

In a Puppe triangulated category, Verdier's octahedral axiom [**56,** Def. 1-1] does not seem to hold in general ([3]).

[3] The author lacks an example of a category that is Puppe but not Verdier triangulated, but strongly suspects that such an example exists, i.e. that the octahedral axiom is not a consequence of Puppe's axioms; cf. Question I.6. In any case, such a deduction is unknown.

In a Verdier triangulated category, in turn, it seems to be impossible to derive the existence of the two extra triangles in a particular octahedron described in [**8,** 1.1.13], or to distinguish crosses as in [**28,** App.].

Moreover, to define a K-theory simplicial set of a triangulated category, one is inclined to take objects as 1-simplices, distinguished triangles as 2-simplices, distinguished octahedra as 3-simplices, etc.

So we enlarge the framework, generalising from $\mathcal{C}^+(\bar{\Delta}_2^\#)$ to $\mathcal{C}^+(\bar{\Delta}_n^\#)$, as described next in §I.0.2.

I.0.2 Definition of Heller triangulated categories

I.0.2.1 A diagram shape

Given $n \geqslant 0$, we let $\Delta_n := \{i \in \mathbf{Z} \ : \ 0 \leqslant i \leqslant n\}$, considered as a linearly ordered set. Let $\bar{\Delta}_n$ be the *periodic prolongation* of Δ_n, consisting of $\mathbf{Z}$ copies of Δ_n put in a row. This is a *periodic linearly ordered set*; that is, a linearly ordered set equipped with a shift automorphism $i \longmapsto i^{+1}$. For instance, $\bar{\Delta}_2 = \{\dots, 2^{-1}, 0, 1, 2, 0^{+1}, \dots\}$, equipped with $i \longmapsto i^{+1}$. Let $\bar{\mathbf{\Delta}}$ be the category consisting of periodic linearly ordered sets of the form $\bar{\Delta}_n$ as objects, and of monotone shiftcompatible maps as morphisms.

Let $\bar{\Delta}_n(\Delta_1)$ denote the category of morphisms in $\bar{\Delta}_n$, i.e. the category of $\bar{\Delta}_n$-valued diagrams of shape Δ_1. Given $\alpha,\ \beta \in \bar{\Delta}_n$ such that $\alpha \leqslant \beta$, the object $(\alpha \longrightarrow \beta)$ in $\bar{\Delta}_n(\Delta_1)$ is abbreviated by β/α.

Let $\bar{\Delta}_n^\#$ be the full subcategory of $\bar{\Delta}_n(\Delta_1)$ that consists of objects β/α within a single period, i.e. such that $\beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}$. For instance,

$$\bar{\Delta}_2^\# \qquad = \qquad \begin{array}{ccccccccc}
 & & & & & & 0^{+1}/0^{+1} & \longrightarrow & \cdots \\
 & & & & & & \uparrow & & \\
 & & & & 2/2 & \longrightarrow & 0^{+1}/2 & \longrightarrow & \cdots \\
 & & & & \uparrow & & \uparrow & & \\
 & & 1/1 & \longrightarrow & 2/1 & \longrightarrow & 0^{+1}/1 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0/0 & \longrightarrow & 1/0 & \longrightarrow & 2/0 & \longrightarrow & 0^{+1}/0 & & \\
\uparrow & & \uparrow & & \uparrow & & & & \\
\vdots & & \vdots & & \vdots & & & &
\end{array}$$

I.0.2.2 Heller triangulations

Let $\mathcal{C}$ be a weakly abelian category; cf. §I.0.1.2. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ is called *exact at Y* if f is a weak kernel of g, or, equivalently, if g is a weak cokernel of f. A commutative quadrangle in $\mathcal{C}$ whose diagonal sequence is exact at the middle object is called a *weak square*.

Let $\mathcal{C}^+(\bar{\Delta}_n^\#)$ be the category of $\mathcal{C}$-valued diagrams of shape $\bar{\Delta}_n^\#$ with a zero at α/α and at

α^{+1}/α for each $\alpha \in \bar{\Delta}_n$, and such that the quadrangle on $(\gamma/\alpha, \delta/\alpha, \gamma/\beta, \delta/\beta)$ is a weak square whenever $\delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1}$. Let $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ be the quotient of $\mathcal{C}^+(\bar{\Delta}_n^\#)$ modulo the full subcategory of diagrams therein that consist entirely of split morphisms.

For instance, $\mathcal{C}^+(\bar{\Delta}_2^\#)$ is the category of $\mathcal{C}$-valued acyclic complexes; and $\underline{\mathcal{C}^+(\bar{\Delta}_2^\#)}$ is its quotient modulo split acyclic complexes, i.e. the homotopy category of $\mathcal{C}$-valued acyclic complexes.

Furthermore, suppose given an automorphism T on $\mathcal{C}$. We obtain two shift functors on $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$, the *inner shift* given by pointwise application of T, and the *outer shift* induced by a diagram shift $j/i \longmapsto i^{+1}/j$.

A *Heller triangulation* on $(\mathcal{C}, \mathsf{T})$ is a tuple of isomorphisms $\vartheta = (\vartheta_n)_{n\geqslant 0}$, where ϑ_n is an isomorphism from the outer to the inner shift on $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$. This tuple is required to be compatible with the functors induced by periodic monotone maps between $\bar{\Delta}_n$ and $\bar{\Delta}_m$, where $m,\, n \geqslant 0$. Moreover, it is required to be compatible with an operation called *folding,* which emerges from the fact that a weak square

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle x}\uparrow & + & \uparrow{\scriptstyle y} \\ X' & \xrightarrow[\;f'\;]{} & Y' \end{array}$$

entails a *folded* weak square

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \uparrow & + & \uparrow{\scriptstyle \left(\begin{smallmatrix} f \\ -y \end{smallmatrix}\right)} \\ X' & \xrightarrow[(x\; f')]{} & X \oplus Y' \end{array} .$$

A *Heller triangulated category* is a triple $(\mathcal{C}, \mathsf{T}, \vartheta)$ as just described, often just denoted by $\mathcal{C}$.

An *n-triangle* in a Heller triangulated category $\mathcal{C}$ is an object X of $\mathcal{C}^+(\bar{\Delta}_n^\#)$ that is *periodic* in the sense that outer shift and inner shift coincide on X, and that satisfies $X\vartheta_n = 1$. The usual properties of 2-triangles generalise to n-triangles.

If $\mathcal{C}$ is a Heller triangulated category in which idempotents split, then, taking the 2-triangles as the distinguished triangles, it is also triangulated in the sense of Verdier [**56,** Def. 1-1]; see Proposition I.23.

I.0.2.3 Strictly exact functors

An additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ between Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$ is called *strictly exact* if, firstly, it respects weak kernels, or, equivalently, weak cokernels; if, secondly, $F\,\mathsf{T}' = \mathsf{T}\,F$; and if, thirdly, the functor

$$\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} \xrightarrow{\underline{F^+(\bar{\Delta}_n^\#)}} \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)} \ ,$$

induced by pointwise application of F, satisfies $\underline{F^+(\bar{\Delta}_n^\#)} \star \vartheta'_n \;=\; \vartheta_n \star \underline{F^+(\bar{\Delta}_n^\#)}$ for $n \geqslant 0$.

I.0.2.4 Enlarge to simplify

Let $\dot{\Delta}_n := \{i \in \mathbf{Z} \ : \ 1 \leqslant i \leqslant n\}$. We have an embedding $\dot{\Delta}_n \hookrightarrow \bar{\Delta}_n^{\#}$ via $\alpha \longmapsto \alpha/0$. Let $\mathcal{C}$ be a weakly abelian category. Let $\mathcal{C}(\dot{\Delta}_n)$ denote the category of $\mathcal{C}$-valued diagrams of shape $\dot{\Delta}_n$. Let $\underline{\mathcal{C}(\dot{\Delta}_n)}$ be the quotient of $\mathcal{C}(\dot{\Delta}_n)$ modulo the full subcategory of split diagrams. Restriction induces an equivalence

$$(*) \qquad \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})} \xrightarrow[\sim]{(-)|_{\dot{\Delta}_n}} \underline{\mathcal{C}(\dot{\Delta}_n)} \,,$$

which is also a useful technical tool; cf. Proposition I.12.

At first sight, one might be inclined to prefer $\underline{\mathcal{C}(\dot{\Delta}_n)}$ over $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. It contains smaller diagrams and has a less elaborate definition. By transport of structure along $(*)$, one obtains an outer shift on $\underline{\mathcal{C}(\dot{\Delta}_n)}$ as well. By pointwise application of the shift functor on $\mathcal{C}$, we also obtain an inner shift on $\underline{\mathcal{C}(\dot{\Delta}_n)}$. These could be compared in order to write down a definition of Heller triangulated categories.

So why then did we prefer to use $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ in our definition of Heller triangulated categories in §I.0.2.2? Working with $\underline{\mathcal{C}(\dot{\Delta}_n)}$, the indirect definition of the outer shift would cause problems. In practice, one would have to pass the equivalence $(*)$ back and forth. The "blown-up variant" $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ of $\underline{\mathcal{C}(\dot{\Delta}_n)}$ carries a directly defined outer shift functor and is thus easier to work with.

There is a further equivalence $\underline{\mathcal{C}(\dot{\Delta}_n)} \xrightarrow{\sim} \underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})}$, where $\hat{\mathcal{C}}$ denotes the Freyd category of $\mathcal{C}$, i.e. the universal abelian category containing $\mathcal{C}$, and where $\underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})}$ is the quotient of $\hat{\mathcal{C}}(\dot{\Delta}_{n-1})$ modulo split diagrams with entries in $\mathcal{C}$; cf. Proposition I.16. Originally, Heller worked with $\underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})}$ for $n = 2$, i.e. with $\hat{\mathcal{C}}/\mathcal{C}$.

I.0.3 A result to begin with

Let $\mathcal{E}$ be a Frobenius category. We define its *stable category* $\underline{\underline{\mathcal{E}}}$ to be the quotient category of the category of purely acyclic complexes with values in the bijective objects of $\mathcal{E}$, modulo the subcategory of split acyclic such complexes.

Then $\underline{\underline{\mathcal{E}}}$ is equivalent to the *classical stable category* $\underline{\mathcal{E}}$ of $\mathcal{E}$, defined as the quotient category of $\mathcal{E}$ modulo bijective objects. But $\underline{\underline{\mathcal{E}}}$ carries a shift automorphism T (invertible), whereas $\underline{\mathcal{E}}$ carries, in general, only a shift autoequivalence (invertible up to isomorphism). In this sense, $\underline{\underline{\mathcal{E}}}$ is a "strictified version" of $\underline{\mathcal{E}}$.

Theorem (Corollary I.33, Corollary I.35). *Given a Frobenius category $\mathcal{E}$, there exists a Heller triangulation ϑ on $(\underline{\underline{\mathcal{E}}}, \mathsf{T})$. An exact functor $\mathcal{E} \xrightarrow{E} \mathcal{E}'$ between Frobenius categories that sends all bijective objects of $\mathcal{E}$ to bijective objects of $\mathcal{E}'$ induces a strictly exact functor $\underline{\underline{\mathcal{E}}} \xrightarrow{\underline{\underline{E}}} \underline{\underline{\mathcal{E}}}'$.*

The Verdier triangulated version of this theorem is due to Happel [**22,** Th. 2.6].

I.0.4 A quasicyclic category

Let $\mathcal{C}$ be a Heller triangulated category.

A *quasicyclic category* is a contravariant functor from $\bar{\mathbf{\Delta}}^\circ$ to the (1-)category of categories. Letting $\operatorname{qcyc}_n \mathcal{C}$ be the subcategory of isomorphisms in $\mathcal{C}^+(\bar{\Delta}_n^\#)$ for $n \geqslant 0$, we obtain a quasicyclic category $\operatorname{qcyc}_\bullet \mathcal{C}$. There is a quasicyclic subcategory $\operatorname{qcyc}_\bullet^{\vartheta=1} \mathcal{C}$ that consists only of n-triangles and their isomorphisms instead of all objects in $\mathcal{C}^+(\bar{\Delta}_n^\#)$ and their isomorphisms ([4]).

Restricting $\operatorname{qcyc}_\bullet^{\vartheta=1} \mathcal{C}$ along the functor $\mathbf{\Delta}^\circ \hookrightarrow \bar{\mathbf{\Delta}}^\circ$ of "periodic prolongation", this yields a simplicial category, hence a topological space; depending functorially on $\mathcal{C}$. This space is the author's tentative proposal for the definition of the K-theory of $\mathcal{C}$; cf. [**50,** Rem. 63]. Of course, this definition still needs to be justified by results one expects of such a K-theory, which has not yet been attempted.

I.0.5 Some remarks

A comparison of our theory to the derivator approach and related constructions in [**11**], [**26,** chap. V.1], [**25**], [**20**], [**32**], [**14**] and [**42**] would be interesting. One might ask whether the base category of a triangulated derivator in the sense of [**42**] carries a Heller triangulation; and if so, whether morphisms of triangulated derivators give rise to strictly exact functors.

Our approach differs from the derivator approach in that we consider a single category $\mathcal{C}$ with shift and an "exactness structure", i.e. a Heller triangulation, on it. The categories $\mathcal{C}^+(\bar{\Delta}_n^\#)$ needed to define this "exactness structure" on $\mathcal{C}$ consist of veritable $\mathcal{C}$-valued diagrams; cf. §I.0.2. In particular, a "structure preserving map" between two such categories $\mathcal{C}$ and $\mathcal{C}'$, i.e. a strictly exact functor, is a single additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ compatible with the "exactness structures" imposed on $\mathcal{C}$ and on $\mathcal{C}'$. In contrast, a "structure preserving map" of triangulated derivators is a compatible family of additive functors.

The generalised triangles in [**8,** 1.1.14] are, in our language, n-pretriangles for which the 2-pretriangle obtained by restriction along any periodic monotone map $\bar{\Delta}_2 \longrightarrow \bar{\Delta}_n$ is a 2-triangle. An n-triangle is such a generalised triangle, but the converse does not hold in general, as pointed out to me by A. Neeman. For an example, see §III.2.

It is conceivable that the concept of Heller triangulated categories is essentially equivalent to a direct axiomatisation via n-triangles, as worked out independently by G. Maltsiniotis [**43**] and myself [**35**]. To compare these approaches, on the one hand, the Heller triangulated category should be closed in the sense of §III.4.2, Def. III.13; on the other hand, the n-triangles should be stable under folding as in Lemma I.21.(2) ([5]).

Concerning the motivation to consider triangulated categories at all, and in particular derived categories, conceived by Grothendieck, we refer the reader to the introduction of the thesis of Verdier [**57**]; cf. also [**27**] and [**60,** p. 26].

I.0.6 Acknowledgements

I thank A. Wiedemann for an introduction to derived categories. I thank A. Beligiannis for directing me to Heller's parametrisation of Puppe triangulations, and for helpful remarks.

[4] Here, " $\vartheta = 1$ " is a mere symbol that should evoke the definition of n-triangles via ϑ.
[5] Jan. 2013: This is feasible, as I have been informed by S. Thomas.

I thank B. Keller for the hint how to "strictify" the classical stable category of a Frobenius category using acyclic complexes. I thank A. Neeman for the hint leading to the folding operation, and for corrections.

More than once I returned to A. Heller's original construction [**24,** p. 53–54], the reference not only for the basic idea, but also for arguments perfectly extendable to the more general framework used here.

I.0.7 Notations and conventions

(i) The disjoint union of sets X and Y is written $X \sqcup Y$.

(ii) Given a, b, $c \in \mathbf{Z}$, the assertion $a \equiv_c b$ is defined to hold if there exists a $z \in \mathbf{Z}$ such that $a - b = cz$.

(iii) For a, $b \in \mathbf{Z}$, we denote by $[a, b] := \{z \in \mathbf{Z} \;:\; a \leqslant z \leqslant b\}$ the integral interval. Similarly, we let $[a, b[\;:= \{z \in \mathbf{Z} \;:\; a \leqslant z < b\}$, $]a, b] := \{z \in \mathbf{Z} \;:\; a < z \leqslant b\}$, $\mathbf{Z}_{\geqslant 0} := \{z \in \mathbf{Z} \;:\; z \geqslant 0\}$ and $\mathbf{Z}_{\leqslant 0} := \{z \in \mathbf{Z} \;:\; z \leqslant 0\}$.

(iv) All categories are supposed to be small with respect to a sufficiently big universe.

(v) Given a category $\mathcal{C}$, and objects X, Y in $\mathcal{C}$, we denote the set of morphisms from X to Y by ${}_{\mathcal{C}}(X, Y)$, or simply by (X, Y), if unambiguous.

(vi) Given a poset P, we frequently consider it as a category, letting ${}_P(x, y)$ contain one element y/x if $x \leqslant y$, and letting it be empty if $x \not\leqslant y$, where $x, y \in \operatorname{Ob} P = P$.

(vii) Given $n \geqslant 0$, we denote by $\Delta_n := [0, n]$ the linearly ordered set with ordering induced by the standard ordering on $\mathbf{Z}$. Let $\dot{\Delta}_n := \Delta_n \smallsetminus \{0\} = [1, n]$, considered as a linearly ordered set.

(viii) Maps act on the right. Composition of maps, and of more general morphisms, is written on the right, i.e. $\xrightarrow{a}\xrightarrow{b} \;=\; \xrightarrow{ab}$.

(ix) Functors act on the right. Composition of functors is written on the right, i.e. $\xrightarrow{F}\xrightarrow{G} \;=\; \xrightarrow{FG}$. Accordingly, the entry of a transformation a between functors at an object X will be written Xa.

> The reason for this convention is that we will mainly consider functors of type "restriction to a subdiagram" or "shift", and such operations are usually written on the right.

(x) A functor is called *strictly dense* if its map on the objects is surjective. It is called *dense* if its induced map on the isoclasses is surjective.

(xi) Given transformations $\mathcal{C} \underset{G}{\overset{F}{\underset{\Downarrow a}{\rightrightarrows}}} \mathcal{C}' \underset{G'}{\overset{F'}{\underset{\Downarrow a'}{\rightrightarrows}}} \mathcal{C}''$, we write $a \star a'$ for the transformation from FF' to GG' given at $X \in \operatorname{Ob} \mathcal{C}$ by $X(a \star a') := (XFa')(XaG') = (XaF')(XGa')$. In this context, we also write the object F for the identity 1_F on this object, i.e. e.g. $X(F \star a') = X(1_F \star a') = (XF)a'$.

(xii) The inverse of an isomorphism f is denoted by f^-. Note that if we denote an iterated shift automorphism $f \longmapsto f^{+1}$ by $f \longmapsto f^{+z}$ for $z \in \mathbf{Z}$, then we have to distinguish f^- (inverse isomorphism if f is an isomorphism) and f^{-1} (inverse of the shift functor applied to f).

(xiii) In an exact category, pure monomorphy is indicated by $X \longrightarrow\!\!\!\!\!\!\bullet\;\; Y$, pure epimorphy by $X \longrightarrow\!\!\!\!\!\!+\;\; Y$.

(xiv) A morphism in an additive category $\mathcal{A}$ is *split* if it is isomorphic, in $\mathcal{A}(\Delta_1)$, to a morphism of the form $X \oplus Y \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} Y \oplus Z$. A morphism being split is indicated by $X \rightarrowtail Y$ (not to be confused with monomorphy). Accordingly, a morphism being a split monomorphism is indicated by $X \rightarrowtail\!\!\!\!\!\!\bullet\;\; Y$, a morphism being a split epimorphism by $X \rightarrowtail\!\!\!\!\!\!+\;\; Y$. Cf. §I.6.2.1.

(xv) We say that *idempotents split* in an additive category $\mathcal{A}$ if every endomorphism e in $\mathcal{A}$ that satisfies $e^2 = e$ is split.

(xvi) The category of functors and transformations from a category D to a category $\mathcal{C}$ is denoted by $[\![D,\mathcal{C}]\!]$ or by $\mathcal{C}(D)$. To objects in $\mathcal{C}(D)$, we also refer to as *diagrams on* D with *values* or *entries* in $\mathcal{C}$.

(xvii) If $\mathcal{C}$ and D are categories, and $X \in \operatorname{Ob}\mathcal{C}(D)$, we usually write $(d \xrightarrow{a} e)X =: (X_d \xrightarrow{X_a} X_e)$ for a morphism $d \xrightarrow{a} e$ in D. If the morphism a is unambiguously given by the context, we also use small letters to write $(X_d \xrightarrow{x} X_e) := (X_d \xrightarrow{X_a} X_e)$ (similarly $X'_d \xrightarrow{x'} X'_e$, $\tilde{Y}_d \xrightarrow{\tilde{y}} \tilde{Y}_e$, ...)

(xviii) Let Add denote the 2-category of additive categories.

(xix) Given an additive category $\mathcal{A}$ and a full additive subcategory $\mathcal{B} \subseteq \mathcal{A}$, we denote by $\mathcal{A}/\mathcal{B}$ the quotient of $\mathcal{A}$ by $\mathcal{B}$, having as objects the objects of $\mathcal{A}$ and as morphisms equivalence classes of morphisms of $\mathcal{A}$; where two morphisms f and f' are equivalent, written $f \equiv_{\mathcal{B}} f'$, if their difference factors over an object of $\mathcal{B}$.

(xx) In an exact category, an object P is called *projective* if $(P,-)$ turns pure epimorphisms into epimorphisms. An object I is called *injective* if $(-,I)$ turns pure monomorphisms into epimorphisms. It is called *bijective* if it is injective and projective. See §I.6.2 for details.

(xxi) In an additive category, a morphism $K \xrightarrow{i} X$ is called a *weak kernel* of a morphism $X \xrightarrow{f} Y$ if for every morphism $T \xrightarrow{t} X$ with $tf = 0$ there exists a morphism $T \xrightarrow{t'} K$ with $t'i = t$. A *weak cokernel* is defined dually. An additive category is called *weakly abelian* if every morphism has a weak kernel and a weak cokernel, and is a weak kernel and a weak cokernel.

(xxii) The Freyd category of a weakly abelian category $\mathcal{C}$ is written $\hat{\mathcal{C}}$. See §I.6.6.3 for details.

(xxiii) In an abelian category, a commutative quadrangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

is called a *square* if its *diagonal sequence* $X \xrightarrow{(x\ f)} X' \oplus Y \xrightarrow{\left(\begin{smallmatrix} f' \\ -y \end{smallmatrix}\right)} Y'$ is short exact. Being a square is indicated by a box sign "□" in the quadrangle.

The quadrangle (X, Y, X', Y') is called a *weak square* if its diagonal sequence is exact in the middle; cf. Definition I.49. Being a weak square is indicated by a "+"-sign in the quadrangle.

In an exact category, (X, Y, X', Y') is a *pure square* if it has a pure short exact diagonal sequence. Being a pure square is indicated by a box sign "□" in the quadrangle.

In a weakly abelian category, (X, Y, X', Y') is a *weak square* if it is a weak square in the Freyd category of that weakly abelian category.

(xxiv) In an abelian category, given a morphism $X \xrightarrow{f} Y$, we sometimes denote its kernel by K_f, and its cokernel by C_f.

I.1 Definition of a Heller triangulated category

I.1.1 Periodic linearly ordered sets and their strips

Without further comment, we consider a poset D as a category, whose set of objects is given by D, and for which $\#_D(\alpha,\beta) = 1$ if $\alpha \leqslant \beta$, and $\#_D(\alpha,\beta) = 0$ otherwise. If existent, i.e. if $\alpha \leqslant \beta$, the morphism from α to β is denoted by β/α. A full subposet of a category is a full subcategory that is a poset. In particular, a full subposet of a poset is just a full subcategory of that poset.

A *periodic poset* is a poset P together with an automorphism $\mathsf{T} : P \xrightarrow{\sim} P$, $\alpha \longmapsto \alpha\,\mathsf{T} =: \alpha^{+1}$. Likewise, we denote $\alpha\,\mathsf{T}^m =: \alpha^{+m}$ resp. $\alpha\,\mathsf{T}^{-m} =: \alpha^{-m}$ for $m \in \mathbf{Z}_{\geqslant 0}$. By abuse of notation, we denote a periodic poset (P, T) simply by P.

A morphism of periodic posets $P \xleftarrow{p} P'$ is a monotone map p of the underlying posets such that $(\alpha'^{+1})p = ((\alpha')p)^{+1}$ for all $\alpha' \in P'$. The category of periodic posets shall be denoted by $\mathfrak{pp}$.

A *periodic linearly ordered set* is a periodic poset the underlying poset of which is linearly ordered, i.e. such that $\#\big(\,_D(\alpha, \beta) \cup \,_D(\beta, \alpha)\big) = 1$ for all $\alpha,\ \beta\ \in\ D$.

To any linearly ordered set D we can attach a periodic linearly ordered set $\bar{D}$ by letting $\bar{D} := D \times \mathbf{Z}$, and $(\alpha, z) \leqslant (\beta, w)$ if $z \leqslant w$, or if ($z = w$ and $\alpha \leqslant \beta$ in D). We let $(\alpha, z)^{+1} := (\alpha, z+1)$. Sending $D \longrightarrow \bar{D}$, $\alpha \longmapsto (\alpha, 0)$, and identifying D with its image, we obtain $(\alpha, z) = \alpha^{+z}$, and the latter is the notation we will usually use. The periodic linearly ordered set $\bar{D}$ is called the *periodic repetition* of D. Likewise, the functor $D \longmapsto \bar{D}$ from the category of linearly ordered sets to the category of periodic linearly ordered sets is called *periodic repetition.*

Let $\mathbf{\Delta}$ be the full subcategory of the category of linearly ordered sets defined by $\operatorname{Ob}\mathbf{\Delta} := \{\Delta_n\ :\ n \in \mathbf{Z}_{\geqslant 0}\}$.

Let $\bar{\mathbf{\Delta}}$ be the full subcategory of the category of periodic linearly ordered sets defined by $\operatorname{Ob}\bar{\mathbf{\Delta}} := \{\bar{\Delta}_n\ :\ n \in \mathbf{Z}_{\geqslant 0}\}$ ([6]).

> The reason for considering periodic linearly ordered sets is that the functor periodic repetition from $\mathbf{\Delta}$ to $\bar{\mathbf{\Delta}}$ is dense and faithful but not full. We will require a naturality of a certain construction with respect to $P \in \operatorname{Ob}\bar{\mathbf{\Delta}}$, which is stronger than setting $P = \bar{D}$ and requiring naturality with respect to $D \in \operatorname{Ob}\mathbf{\Delta}$.

Given $n \geqslant 0$, the underlying linearly ordered set of $\bar{\Delta}_n$ is isomorphic to $\mathbf{Z}$ via $\alpha^{+z} \longmapsto \alpha + (n+1)z$. We use this isomorphism to define the operation

$$\bar{\Delta}_n \times \mathbf{Z} \longrightarrow \bar{\Delta}_n\ ,\quad (\alpha^{+z}, x) \longmapsto \alpha^{+z} + x\ :=\ (\overline{\alpha + x})^{+(z + \underline{\alpha+x})}\ ,$$

where we write $k = (n+1)\underline{k} + \overline{k}$ with $\underline{k} \in \mathbf{Z}$ and $\overline{k} \in [0, n]$ for $k \in \mathbf{Z}$. For instance, if $n = 3$, then $2^{+1} + 7 = 1^{+3}$.

To a periodic linearly ordered set P, we attach the poset

$$P^{\#} := \{\beta/\alpha \in P(\Delta_1)\ :\ \beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}\}$$

as a full subposet of $P(\Delta_1)$, called the *strip* of P. A morphism therein from β/α to δ/γ is written $\delta/\gamma /\!\!/ \beta/\alpha$, which is unique if it exists, i.e. if $\alpha \leqslant \gamma$ and $\beta \leqslant \delta$.

The strip $P^{\#}$ carries the automorphism $\beta/\alpha \longmapsto (\beta/\alpha)^{+1} := \alpha^{+1}/\beta$ in $\mathfrak{pp}$, where $\beta/\alpha \in P^{\#}$.

If $P = \bar{\Delta}_n$, we also write $\beta/\alpha \overset{\mathsf{T}_n}{\longmapsto} (\beta/\alpha)^{+1}$.

This construction defines a functor

$$\begin{array}{ccc} \bar{\mathbf{\Delta}} & \xrightarrow{(-)^{\#}} & \mathfrak{pp} \\ P & \longmapsto & P^{\#} \end{array}$$

[6] The category $\bar{\mathbf{\Delta}}$ is isomorphic to the category L defined by Elmendorf in [**13**].

which sends a morphism $P \xleftarrow{p} P'$ in $\bar{\mathbf{\Delta}}$ to

$$\begin{array}{ccc} P^{\#} & \xleftarrow{p^{\#}} & P'^{\#} \\ \beta' p/\alpha' p & \longleftarrow\!\shortmid & \beta'/\alpha' \end{array}$$

In fact, $p^{\#}$ is welldefined, since if $\beta'^{-1} \leqslant \alpha' \leqslant \beta' \leqslant \alpha'^{+1}$, then $(\beta' p)^{-1} \leqslant \alpha' p \leqslant \beta' p \leqslant (\alpha' p)^{+1}$. Moreover, $p^{\#}$ is monotone and compatible with shift.

Example I.1 The periodic poset $\bar{\Delta}_2^{\#}$, i.e. the strip of the periodic repetition of Δ_2, can be displayed as

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & 1^{+1}/1^{+1} & \to & \cdots \\
 & & & & & & & & & & \uparrow & & \\
 & & & & & & & & 0^{+1}/0^{+1} & \to & 1^{+1}/0^{+1} & \to & \cdots \\
 & & & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & 2/2 & \to & 0^{+1}/2 & \to & 1^{+1}/2 & \to & \cdots \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & 1/1 & \to & 2/1 & \to & 0^{+1}/1 & \to & 1^{+1}/1 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & 0/0 & \to & 1/0 & \to & 2/0 & \to & 0^{+1}/0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & & & \\
2^{-1}/2^{-1} & \to & 0/2^{-1} & \to & 1/2^{-1} & \to & 2/2^{-1} & & & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & & & & & \\
\vdots & & \vdots & & \vdots & & & & & & & &
\end{array}$$

I.1.2 Heller triangulated categories

Suppose given a weakly abelian category $\mathcal{C}$; cf. Definition I.66. From §I.1.2.1.3 on, we assume it to be equipped with an automorphism

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow[\sim]{\mathsf{T}} & \mathcal{C} \\ (X \xrightarrow{u} Y) & \longmapsto & (X\,\mathsf{T} \xrightarrow{u\,\mathsf{T}} Y\,\mathsf{T}) \;=:\; (X^{+1} \xrightarrow{u^{+1}} Y^{+1})\,. \end{array}$$

Similarly, we denote $(X\,\mathsf{T}^m \xrightarrow{u\,\mathsf{T}^m} Y\,\mathsf{T}^m) =: (X^{+m} \xrightarrow{u^{+m}} Y^{+m})$ for $m \in \mathbf{Z}$.

Recall that its Freyd category $\hat{\mathcal{C}}$ is an abelian Frobenius category, and that the image of $\mathcal{C}$ in $\hat{\mathcal{C}}$, identified with $\mathcal{C}$, is a sufficiently big subcategory of bijectives; cf. §I.6.6.3.

I.1.2.1 The stable category of pretriangles $\underline{\mathcal{C}^{+}(P^{\#})}$

I.1.2.1.1 Definition of $\underline{\mathcal{C}^{+}(P^{\#})}$

Concerning the Freyd category $\hat{\mathcal{C}}$ of $\mathcal{C}$, cf. §I.6.6.3. Concerning the notion of a weak square in $\hat{\mathcal{C}}$, see Definition I.49. A weak square in $\mathcal{C}$ is a weak square in $\hat{\mathcal{C}}$ that has all four objects in $\operatorname{Ob}\mathcal{C}$. Applying Remark I.67, we obtain an elementary way to characterise weak squares as having a diagonal sequence with first morphism being a weak kernel of the second; or, equivalently, with second morphism being a weak cokernel of the first.

Given a periodic linearly ordered set P, we let $\mathcal{C}^+(P^\#)$ be the full subcategory of $\mathcal{C}(P^\#)$ defined by

$$\mathrm{Ob}\,\mathcal{C}^+(P^\#) \;:=\; \left\{ X \in \mathrm{Ob}\,\mathcal{C}(P^\#) \quad : \quad \begin{array}{ll} 1) & X_{\alpha/\alpha} = 0 \text{ and } X_{\alpha^{+1}/\alpha} = 0 \text{ for all } \alpha \in P. \\ 2) & \text{For all } \delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1} \text{ in } P, \\ & \text{the quadrangle} \\ & \begin{array}{ccc} X_{\gamma/\beta} & \xrightarrow{x} & X_{\delta/\beta} \\ {\scriptstyle x}\uparrow & + & \uparrow{\scriptstyle x} \\ X_{\gamma/\alpha} & \xrightarrow{x} & X_{\delta/\alpha} \end{array} \\ & \text{is a weak square (as indicated by +).} \end{array} \right\} .$$

Note that we do not require that $(X_{\alpha^{+1}/\gamma} \xrightarrow{x} X_{\beta^{+1}/\delta}) = (X_{\gamma/\alpha} \xrightarrow{x} X_{\delta/\beta})^{+1}$ for $\gamma/\alpha,\ \delta/\beta \in P^\#$ with $\gamma/\alpha \leqslant \delta/\beta$.

An object of $\mathcal{C}^+(P^\#)$ is called a *P-pretriangle.* Given $n \geqslant 0$, an object of $\mathcal{C}^+(\bar{\Delta}_n^\#)$, i.e. a $\bar{\Delta}_n$-pretriangle, is also called an *n-pretriangle.*

Roughly put, an n-pretriangle is a diagram on the strip $\bar{\Delta}_n^\#$ of the periodic repetition $\bar{\Delta}_n$ of Δ_n consisting of weak squares with zeroes on the boundaries.

Example I.2 A 0-pretriangle consists of zero objects. A 1-pretriangle is just a sequence $\dots, X_{0/1^{-1}}, X_{1/0}, X_{0^{+1}/1}, \dots$ of objects of $\mathcal{C}$, decorated with some zero objects. A 2-pretriangle is a complex in $\mathcal{C}$ which becomes acyclic in $\hat{\mathcal{C}}$ – for short, which *is* acyclic –, decorated with some zero objects.

A morphism in $\mathcal{C}$ is split in $\hat{\mathcal{C}}$ if and only if it factors in $\hat{\mathcal{C}}$ into a retraction followed by a coretraction. Equivalently, its image, taken in $\hat{\mathcal{C}}$, is bijective as an object of $\hat{\mathcal{C}}$.

Let $\mathcal{C}^{+,\,\mathrm{split}}(P^\#)$ be the full subcategory of $\mathcal{C}^+(P^\#)$ defined by

$$\mathrm{Ob}\,\mathcal{C}^{+,\,\mathrm{split}}(P^\#) \;:=\; \left\{ X \in \mathrm{Ob}\,\mathcal{C}^+(P^\#) \;:\; \begin{array}{l} X_{\gamma/\alpha} \xrightarrow{x} X_{\delta/\beta} \text{ is split in } \hat{\mathcal{C}} \\ \text{for all } \gamma/\alpha,\ \delta/\beta \in P^\# \text{ with } \gamma/\alpha \leqslant \delta/\beta \end{array} \right\}$$

We denote the quotient category by

$$\underline{\mathcal{C}^+(P^\#)} \;:=\; \mathcal{C}^+(P^\#)/\mathcal{C}^{+,\,\mathrm{split}}(P^\#)\,,$$

called the *stable category of P-pretriangles.*

Example I.3 We have $\underline{\mathcal{C}^+(\bar{\Delta}_0^\#)} = \underline{\mathcal{C}^+(\bar{\Delta}_1^\#)} = 0$. The category $\underline{\mathcal{C}^+(\bar{\Delta}_2^\#)}$ can be regarded as the homotopy category of the category of acyclic complexes with entries in $\mathcal{C}$.

I.1.2.1.2 Naturality of $\underline{\mathcal{C}^+(P^\#)}$ in P

Suppose given periodic linearly ordered sets P, P', and a morphism $P^\# \xleftarrow{q} P'^\#$ of periodic posets such that either ($P = P'$ and $q = \mathsf{T}$, the shift functor on $P^\#$) or $q = p^\#$ for some morphism $P \xleftarrow{p} P'$ of periodic linearly ordered sets.

Recall that if $P = \bar{\Delta}_n$, then we write alternatively T_n for the shift functor T on $\bar{\Delta}_n^\#$.

We obtain an induced functor

$$\begin{array}{rcl} \mathcal{C}^+(P^\#) & \xrightarrow{\mathcal{C}^+(q)} & \mathcal{C}^+(P'^\#) \\ X & \longmapsto & X\big(\mathcal{C}^+(q)\big) \;:=\; q\,X\;, \end{array}$$

given by composition of q, followed by X.

In particular, the shift T on $P^\#$ induces a functor

$$\begin{array}{rcl} \mathcal{C}^+(P^\#) & \xrightarrow{\mathcal{C}^+(\mathsf{T})} & \mathcal{C}^+(P^\#) \\ X & \longmapsto & [X]^{+1} \;:=\; X\big(\mathcal{C}^+(\mathsf{T})\big)\;, \end{array}$$

called the *outer shift.* Note that if $P = \bar{\Delta}_n$, then $[X]^{+1}_{\beta/\alpha} = X_{(\beta/\alpha)^{+1}} = X_{\alpha^{+1}/\beta}$ for $\beta/\alpha \in \bar{\Delta}_n^\#$. On the stable category, this functor induces a functor

$$\begin{array}{rcl} \underline{\mathcal{C}^+(P^\#)} & \xrightarrow{\mathcal{C}^+(\mathsf{T})} & \underline{\mathcal{C}^+(P^\#)} \\ X & \longmapsto & [X]^{+1}, \end{array}$$

likewise called the *outer shift.*

Given a morphism $P \xleftarrow{p} P'$ in $\bar{\mathbf{\Delta}}$, we obtain an induced morphism $P^\# \xleftarrow{p^\#} P'^\#$, and hence an induced functor usually abbreviated by

$$\begin{array}{rcl} \mathcal{C}^+(P^\#) & \xrightarrow{p^\# := \mathcal{C}^+(p^\#)} & \mathcal{C}^+(P'^\#) \\ X & \longmapsto & Xp^\# := X\big(\mathcal{C}^+(p^\#)\big)\;. \end{array}$$

Likewise on the stable categories; we abbreviate $\underline{p}^\# \;:=\; \underline{\mathcal{C}^+(p^\#)}$.

> So altogether, we have defined $Xp^\# := p^\# X$ ($Xp^\#$: operation induced by p, applied to X ; $p^\# X$: composition of $p^\#$ and X), which is a bit unfortunate, but convenient in practice.

Given a morphism $P \xleftarrow{p} P'$ in $\bar{\mathbf{\Delta}}$ and $X \in \operatorname{Ob} \mathcal{C}^+(P^\#)$, we have

$$[X]^{+1} p^\# \;=\; [Xp^\#]^{+1}\;,$$

natural in X. Likewise on the stable categories.

Given $P,\, P' \in \operatorname{Ob} \bar{\mathbf{\Delta}}$, a functor $\mathcal{C}^+(P^\#) \xrightarrow{F} \mathcal{C}^+(P'^\#)$ is called *strictly periodic* if

$$[XF]^{+1} \;=\; [X]^{+1} F\;,$$

natural in X. Likewise on the stable categories.

I.1.2.1.3 Naturality of $\underline{\mathcal{C}^+(P^\#)}$ in $\mathcal{C}$

An additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ is called *subexact* if the induced additive functor $\hat{\mathcal{C}} \xrightarrow{\hat{F}} \hat{\mathcal{C}}'$ is an exact functor of abelian categories; cf. §I.6.6.3. Alternatively, it is subexact if and only if it preserves weak kernels, or, equivalently, weak cokernels; cf. Remark I.67.

Suppose given a subexact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and $P \in \operatorname{Ob} \mathbf{\Delta}$. We obtain an induced functor

$$\begin{array}{ccc} \mathcal{C}^+(P^\#) & \xrightarrow{F^+(P^\#)} & \mathcal{C}'^+(P^\#) \\ X & \longmapsto & X\,F^+(P^\#)\,, \end{array}$$

where, writing $Y := X\,F^+(P^\#)$, we let

$$(Y_{\beta/\alpha} \xrightarrow{y} Y_{\delta/\gamma}) \;:=\; (X_{\beta/\alpha}F \xrightarrow{xF} X_{\delta/\gamma}F)$$

for $\beta/\alpha,\, \delta/\gamma \,\in\, P^\#$ with $\beta/\alpha \leqslant \delta/\gamma$.

In particular, the automorphism $\mathcal{C} \xrightarrow{\mathsf{T}} \mathcal{C}$ induces an automorphism

$$\begin{array}{ccc} \mathcal{C}^+(P^\#) & \xrightarrow{\mathsf{T}^+(P^\#)} & \mathcal{C}^+(P^\#) \\ X & \longmapsto & [X^{+1}] \;:=\; X\big(\mathsf{T}^+(P^\#)\big)\,, \end{array}$$

called the *inner shift.* Note that if $P = \bar{\Delta}_n$, then $[X^{+1}]_{\beta/\alpha} = X^{+1}_{\beta/\alpha}$ for $\beta/\alpha \in \bar{\Delta}_n^\#$.

On the stable category, this induces an automorphism

$$\begin{array}{ccc} \underline{\mathcal{C}^+(P^\#)} & \xrightarrow{\underline{\mathsf{T}^+(P^\#)}} & \underline{\mathcal{C}^+(P^\#)} \\ X & \longmapsto & [X^{+1}]\,, \end{array}$$

likewise called the *inner shift.*

I.1.2.2 Folding

The following construction arose from a hint of A. Neeman, who showed me a multitude of 2-triangles in an n-triangle similar to the two 2-triangles explained in [**8,** 1.1.13]; cf. Definition I.5.(ii) below.

I.1.2.2.1 Some notation

Given $P = (P, \mathsf{T}) \in \operatorname{Ob} \bar{\mathbf{\Delta}}$, we denote by $2P$ the periodic poset (P, T^2).

Given a linearly ordered set D, we let $\rho \sqcup D$ be the linearly ordered set having as underlying set $\{\rho\} \sqcup D$; and as partial order $\rho \leqslant_{\rho \sqcup D} \alpha$ for all $\alpha \in D$, and $\alpha \leqslant_{\rho\sqcup D} \beta$ if $\alpha,\, \beta \,\in\, D$ and $\alpha \leqslant_D \beta$.

Roughly put, $2P$ is P with doubled period, and $\rho \sqcup D$ is D with an added initial object ρ.

Let $n \geqslant 0$. We have an isomorphism of periodic linearly ordered sets

$$\begin{array}{ccl} 2\bar{\Delta}_n & \xrightarrow{\sim} & \bar{\Delta}_{2n+1} \\ k^{+l} & \longmapsto & \begin{cases} k^{+l/2} & \text{for } l \equiv_2 0 \\ (k+n+1)^{+(l-1)/2} & \text{for } l \equiv_2 1 \end{cases} \end{array}$$

and an isomorphism of linearly ordered sets

$$\begin{array}{ccll} \rho \sqcup \Delta_n & \xrightarrow{\sim} & \Delta_{n+1} & \\ k & \longmapsto & k+1 & \text{for } k \in [0,n] \\ \rho & \longmapsto & 0\,. & \end{array}$$

In order to remain inside $\bar{\mathbf{\Delta}}$ resp. inside $\mathbf{\Delta}$, we use these isomorphisms as identifications.

Then $P \longmapsto 2P$ is natural in P and therefore defines an endofunctor of $\bar{\mathbf{\Delta}}$, and $D \longmapsto \rho \sqcup D$ is natural in D and therefore defines an endofunctor of $\mathbf{\Delta}$.

Given a linearly ordered set D, we will need to consider the periodic posets $2\bar{D}$ and $\overline{\rho \sqcup D}$, formed using periodic repetition.

I.1.2.2.2 The folding operation

Let $n \geqslant 0$. Let the strictly periodic functor

$$\begin{array}{rcl} \mathcal{C}^+((2\bar{\Delta}_n)^{\#}) & \xrightarrow{\mathfrak{f}_n} & \mathcal{C}^+(\overline{\rho \sqcup \Delta_n}^{\#}) \\ X & \longmapsto & X\mathfrak{f}_n \end{array}$$

be determined on objects by the following data. Writing $Y = X\mathfrak{f}_n$, we let

$$\begin{array}{rcl} (Y_{\alpha/\rho} \xrightarrow{y} Y_{\beta/\rho}) & := & \left(X_{\alpha^{+1}/\alpha} \xrightarrow{x} X_{\beta^{+1}/\beta}\right) \\ (Y_{\beta/\rho} \xrightarrow{y} Y_{\beta/\alpha}) & := & \left(X_{\beta^{+1}/\beta} \xrightarrow{(x\ x)} X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta}\right) \\ (Y_{\beta/\alpha} \xrightarrow{y} Y_{\delta/\gamma}) & := & \left(X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta} \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)} X_{\delta^{+1}/\gamma^{+1}} \oplus X_{\gamma^{+2}/\delta}\right) \\ (Y_{\delta/\gamma} \xrightarrow{y} Y_{\rho^{+1}/\gamma}) & := & \left(X_{\delta^{+1}/\gamma^{+1}} \oplus X_{\gamma^{+2}/\delta} \xrightarrow{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} X_{\gamma^{+2}/\gamma^{+1}}\right) \end{array}$$

for $\alpha,\ \beta,\ \gamma,\ \delta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$, with $\gamma \leqslant \delta$ and with $\beta/\alpha \leqslant \delta/\gamma$. The remaining morphisms are given by composition.

Note that $X \in \operatorname{Ob} \mathcal{C}^+((2\bar{\Delta}_n)^{\#})$, so e.g. $X_{\beta^{+1}/\beta} \neq 0$ is possible, whereas $X_{\beta^{+2}/\beta} = 0$ for $\beta \in \Delta_n$.

We claim that $X\mathfrak{f}_n$ is an object of $\mathcal{C}^+(\overline{\rho \sqcup \Delta_n}^{\#})$.

In fact, by Lemma I.54, applied in the abelian category $\hat{\mathcal{C}}$, we are reduced to considering the quadrangles of Y on $(\gamma/\rho,\ \delta/\rho,\ \gamma/\beta,\ \delta/\beta)$ for $\beta,\ \gamma,\ \delta\ \in\ \Delta_n$ with $\beta \leqslant \gamma \leqslant \delta$; on $(\gamma/\alpha,\ \delta/\alpha,\ \gamma/\beta,\ \delta/\beta)$ for $\alpha,\ \beta,\ \gamma,\ \delta\ \in\ \Delta_n$ with $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta$; and on $(\gamma/\alpha,\ \rho^{+1}/\alpha,\ \gamma/\beta,\ \rho^{+1}/\beta)$ for $\alpha,\ \beta,\ \gamma\ \in\ \Delta_n$ with $\alpha \leqslant \beta \leqslant \gamma$.

The quadrangle of Y on $(\gamma/\alpha,\ \delta/\alpha,\ \gamma/\beta,\ \delta/\beta)$ is a weak square as the direct sum of two weak squares.

For the remaining quadrangles to be treated, Lemma I.57 reduces us to considering the quadrangles of Y on $(\alpha/\rho,\ \beta/\rho,\ \alpha/\alpha,\ \beta/\alpha)$, on $(\beta/\rho,\ \rho^{+1}/\rho,\ \beta/\alpha,\ \rho^{+1}/\alpha)$ and on $(\beta/\alpha,\ \rho^{+1}/\alpha,\ \beta/\beta,\ \rho^{+1}/\beta)$ for $\alpha,\ \beta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$. These are in fact weak squares, as ensues from Lemma I.58 and its dual assertion. This proves our claim.

This construction of $Y = X\mathfrak{f}_n$ is functorial in X.

To prove that the folding operation passes to the stable categories, we have to show that for an object X of $\mathcal{C}^{+,\mathrm{split}}((2\bar{\Delta}_n)^{\#})$, the folded object $X\mathfrak{f}_n$ is in $\mathcal{C}^{+,\mathrm{split}}(\overline{\rho \sqcup \Delta_n}^{\#})$. Denote $Y := X\mathfrak{f}_n$. Since $Y_{\alpha/\rho} \xrightarrow{y} Y_{\beta/\rho}$ is split in $\hat{\mathcal{C}}$ for all $\alpha,\ \beta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$, it suffices to prove the following lemma.

Lemma I.4 *Suppose given $m \geqslant 0$ and $Z \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_m^\#)$ such that $Z_{\alpha/0} \xrightarrow{z} Z_{\beta/0}$ is split in $\hat{\mathcal{C}}$ for all $\alpha,\ \beta \in \Delta_m$ with $0 < \alpha \leqslant \beta$. Then $Z \in \operatorname{Ob} \mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_m^\#)$.*

Proof. Consider the morphism $Z_{\gamma/\alpha} \xrightarrow{z} Z_{\delta/\beta}$ for $\gamma/\alpha \leqslant \delta/\beta$ in $\bar{\Delta}_m^\#$. We have to show that it is split in $\hat{\mathcal{C}}$, i.e. that its image, taken in $\hat{\mathcal{C}}$, is bijective there. Unless $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1}$, this morphism is zero, hence split in $\hat{\mathcal{C}}$. If this condition holds, it is the diagonal morphism of the weak square $(Z_{\gamma/\alpha},\ Z_{\delta/\alpha},\ Z_{\gamma/\beta},\ Z_{\delta/\beta})$.

So by Lemma I.59, applied in the abelian category $\hat{\mathcal{C}}$, we see that it suffices to show that the (horizontal) morphism $Z_{\beta/\alpha} \xrightarrow{z} Z_{\gamma/\alpha}$ is split in $\hat{\mathcal{C}}$ and that the (vertical) morphism $Z_{\gamma/\alpha} \xrightarrow{z} Z_{\gamma/\beta}$ is split in $\hat{\mathcal{C}}$ for all α, β, γ in $\bar{\Delta}_m$ with $\gamma^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \alpha^{+1}$.

The long exact sequence

$$\cdots \longrightarrow Z_{\alpha/\beta^{-1}} \longrightarrow Z_{\alpha/\gamma^{-1}} \longrightarrow Z_{\beta/\gamma^{-1}} \longrightarrow Z_{\beta/\alpha} \longrightarrow Z_{\gamma/\alpha} \longrightarrow Z_{\gamma/\beta} \longrightarrow Z_{\alpha^{+1}/\beta} \longrightarrow \cdots$$

in $\hat{\mathcal{C}}$ shows that it suffices to show that the morphism $Z_{\beta/\alpha} \xrightarrow{z} Z_{\gamma/\alpha}$ is split in $\hat{\mathcal{C}}$ for all $0 \leqslant \alpha \leqslant \beta \leqslant \gamma < 0^{+1}$. In fact, first of all we may assume that $0 \leqslant \alpha < 0^{+1}$, so that $0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \alpha^{+1} < 0^{+2}$. Hence either $0 \leqslant \alpha \leqslant \beta \leqslant \gamma < 0^{+1}$, or $0 \leqslant \gamma^{-1} \leqslant \alpha \leqslant \beta < 0^{+1}$, or $0 \leqslant \beta^{-1} \leqslant \gamma^{-1} \leqslant \alpha < 0^{+1}$.

Now we may assume that $0 < \alpha$ and apply Lemma I.59 to the weak square $(Z_{\beta/0},\ Z_{\gamma/0},\ Z_{\beta/\alpha},\ Z_{\gamma/\alpha})$, in which $Z_{\beta/0} \xrightarrow{z} Z_{\gamma/0}$ is split in $\hat{\mathcal{C}}$ by assumption, in which $Z_{\beta/0} \xrightarrow{z} Z_{\beta/\alpha}$ is split in $\hat{\mathcal{C}}$ since $Z_{\alpha/0} \xrightarrow{z} Z_{\beta/0}$ is split in $\hat{\mathcal{C}}$ by assumption, and in which $Z_{\gamma/0} \xrightarrow{z} Z_{\gamma/\alpha}$ is split in $\hat{\mathcal{C}}$ since $Z_{\alpha/0} \xrightarrow{z} Z_{\gamma/0}$ is split in $\hat{\mathcal{C}}$ by assumption. ▫

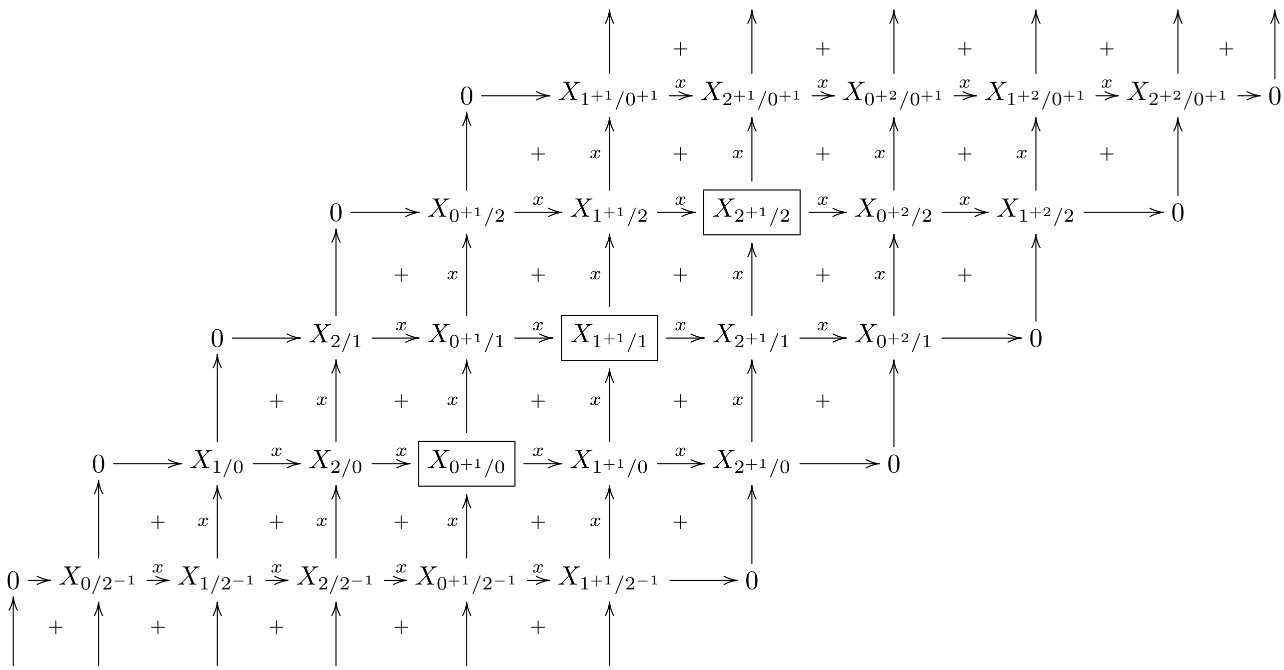
So the folding operation passes to an operation

$$\begin{array}{ccc} \underline{\mathcal{C}^+((2\bar{\Delta}_n)^\#)} & \xrightarrow{\underline{\mathfrak{f}}_n} & \underline{\mathcal{C}^+(\overline{\rho \sqcup \Delta_n}^{\,\#})} \\ X & \longmapsto & X\underline{\mathfrak{f}}_n \end{array}$$

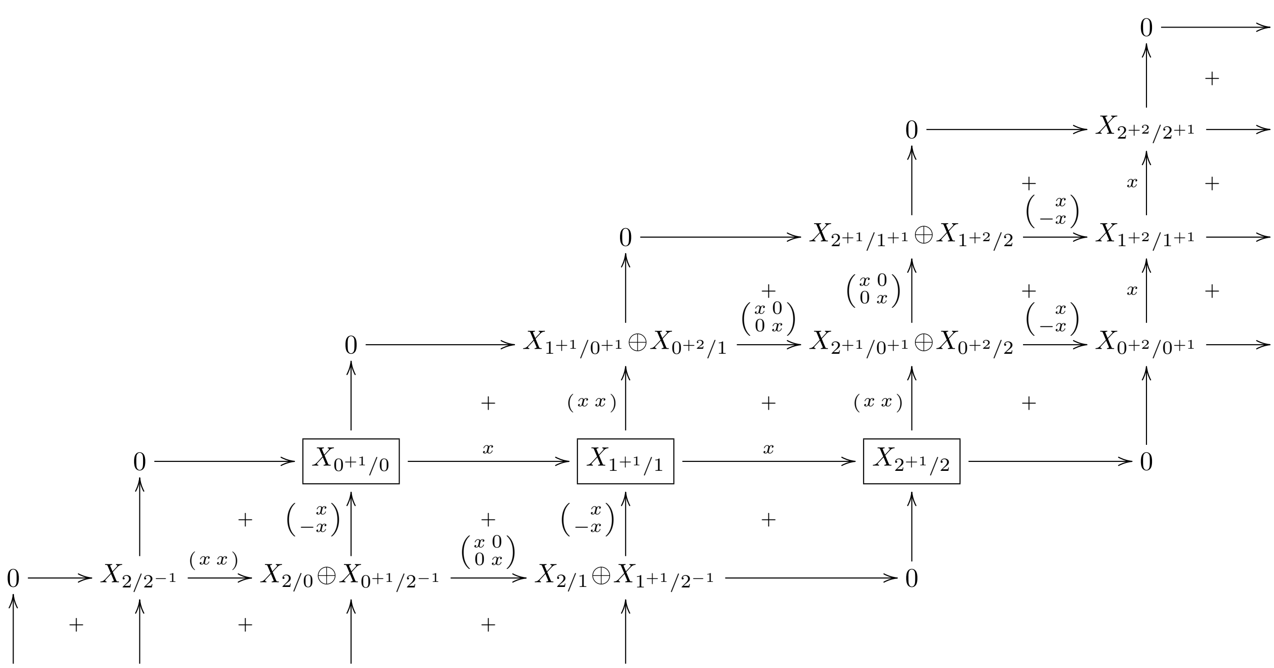
on the stable categories.

I.1.2.2.3 An example: folding from $\bar{\Delta}_5^{\#}$ to $\bar{\Delta}_3^{\#}$

Let $D = \Delta_2$. Note that $2\bar{\Delta}_2 \simeq \bar{\Delta}_5$. Let $X \in \mathrm{Ob}\, \mathcal{C}^+((2\bar{\Delta}_2)^{\#})$, depicted as follows.



Note that $\rho \sqcup \Delta_2 \simeq \Delta_3$. Folding turns X into $X\mathfrak{f}_2 \in \mathrm{Ob}\, \mathcal{C}^+(\overline{\rho \sqcup \Delta_2}^{\#})$, depicted as follows.



I.1.2.3 A definition of Heller triangulated categories and strictly exact functors

Recall that $\mathcal{C}$ is a weakly abelian category, and that $\mathsf{T} = (-)^{+1}$ is an automorphism of $\mathcal{C}$.

Suppose given $n \geqslant 0$. We have introduced the automorphisms

$$\begin{array}{rcll} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \xrightarrow[\sim]{\underline{\mathcal{C}^+(\mathsf{T}_n)}} & \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \text{(outer shift; §I.1.2.1.2)} \\ X & \longmapsto & [X]^{+1} & \end{array}$$

$$\begin{array}{rcll} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \xrightarrow[\sim]{\underline{\mathsf{T}^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \text{(inner shift; §I.1.2.1.3)} \\ X & \longmapsto & [X^{+1}]\,. & \end{array}$$

The outer shift shifts the whole diagram $X \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ one step downwards – the object $X_{\alpha^{+1}/\beta}$ is the entry of $[X]^{+1}$ at position β/α.

The inner shift applies the given shift automorphism $(-)^{+1}$ of $\mathcal{C}$ entrywise to a diagram $X \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Furthermore, we write $[X^{+a}]^{+b} := X\,\mathsf{T}^+(\bar{\Delta}_n^\#)^a\,\mathcal{C}^+(\mathsf{T}_n)^b = X\mathcal{C}^+(\mathsf{T}_n)^b\,\mathsf{T}^+(\bar{\Delta}_n^\#)^a$ for $a,\, b \in \mathbf{Z}_{\geqslant 0}$ and $X \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^\#)$; similarly for $a,\, b \in \mathbf{Z}$. Likewise in the stable case.

Definition I.5

(i) A *Heller triangulation* on $(\mathcal{C}, \mathsf{T})$ is a tuple of isomorphisms of functors

$$\vartheta \;=\; \Big(\, \underline{\mathcal{C}^+(\mathsf{T}_n)} \xrightarrow[\sim]{\vartheta_n} \underline{\mathsf{T}^+(\bar{\Delta}_n^\#)} \,\Big)_{n\geqslant 0} \;=\; \Big(\, [-]^{+1} \xrightarrow[\sim]{\vartheta_n} [-^{+1}] \,\Big)_{n\geqslant 0}$$

such that

(∗) $$\underline{p}^\# \star \vartheta_m \;=\; \vartheta_n \star \underline{p}^\#$$

for all $n,\, m \geqslant 0$ and all periodic monotone maps $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$ in $\bar{\mathbf{\Delta}}$, and such that

(∗∗) $$\underline{\mathfrak{f}}_n \star \vartheta_{n+1} \;=\; \vartheta_{2n+1} \star \underline{\mathfrak{f}}_n$$

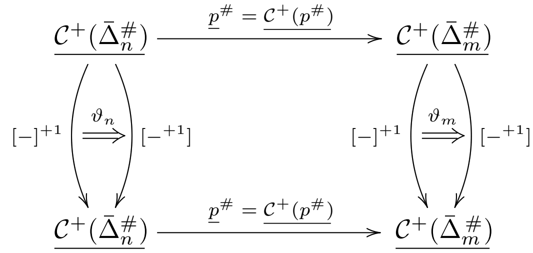
for all $n \geqslant 0$.

Note that given $n \geqslant 0$, the isomorphism ϑ_n consists of isomorphisms

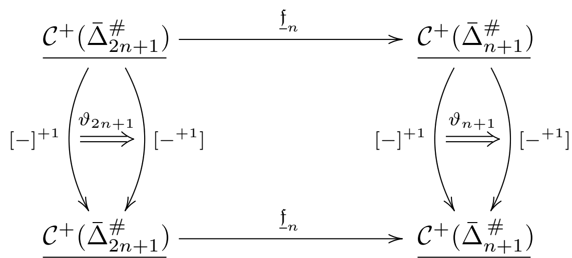
$$[X]^{+1} \xrightarrow[\sim]{X\vartheta_n} [X^{+1}]$$

in the stable category $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ of n-pretriangles, where X runs over the set $\operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ of n-pretriangles.

Condition (∗) asserts that the following diagram commutes in Add for all $n,\, m \geqslant 0$ and all periodic monotone maps $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$ in $\bar{\mathbf{\Delta}}$.



Condition $(**)$ asserts that the following diagram commutes in Add for all $n \geqslant 0$.



(ii) Given a Heller triangulation ϑ on $(\mathcal{C}, \mathsf{T})$, we use the following terminology.

(1) The triple $(\mathcal{C}, \mathsf{T}, \vartheta)$ forms a *Heller triangulated category,* usually just denoted by $\mathcal{C}$.

(2) Given $n \geqslant 0$, an *n-triangle* is an object X of $\mathcal{C}^+(\bar{\Delta}_n^\#)$ for which $[X^{+1}] = [X]^{+1}$ in Ob $\mathcal{C}^+(\bar{\Delta}_n^\#)$, and for which

$$X\vartheta_n \;=\; 1_{[X]^{+1}} \;=\; 1_{[X^{+1}]} \qquad \text{(equality in } \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}) \;.$$

A *morphism of n-triangles* is a morphism $X \xrightarrow{u} Y$ in $\mathcal{C}^+(\bar{\Delta}_n^\#)$ between n-triangles X and Y such that $[u]^{+1} = [u^{+1}]$.

The category of n-triangles and morphisms of n-triangles is denoted by $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^\#)$.

In the notation $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^\#)$, the index " $\vartheta{=}1$ " is to be read as a symbol, not as an actual equation.
The subcategory of n-triangles $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^\#)$ in the category of n-pretriangles $\mathcal{C}^+(\bar{\Delta}_n^\#)$ is not full in general.

(iii) An additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ between Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$ is called *strictly exact* if the following conditions hold.

(1) $F\,\mathsf{T}' = \mathsf{T}\,F$.

(2) F is subexact; cf. §I.1.2.1.3.

(3) We have

$$(***) \qquad\qquad \vartheta_n \star \underline{F^+(\bar{\Delta}_n^\#)} \;=\; \underline{F^+(\bar{\Delta}_n^\#)} \star \vartheta'_n$$

for all $n \geqslant 0$.

Such a functor F is called strictly exact because of the equality in (1).

Condition $(***)$ asserts that the following diagram commutes in Add for all $n \geqslant 0$.

$$\begin{array}{ccc} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{F^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)} \\ {\scriptstyle [-]^{+1}} \downarrow \overset{\vartheta_n}{\Longrightarrow} \downarrow {\scriptstyle [-^{+1}]} & & {\scriptstyle [-]^{+1}} \downarrow \overset{\vartheta'_n}{\Longrightarrow} \downarrow {\scriptstyle [-^{+1}]} \\ \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{F^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)} \end{array}$$

To summarise Definition I.5 roughly, a Heller triangulation is an isomorphism ϑ from the outer shift to the inner shift, varying with Δ_n, and compatible with folding. An n-triangle is a periodic n-pretriangle at which ϑ is an identity. A strictly exact functor respects the weakly abelian structure and is compatible with shift and ϑ.

Note that if ϑ is a Heller triangulation on $(\mathcal{C}, \mathsf{T})$, so is $-\vartheta$.

Definition I.5 would make sense for periodic, but not necessarily linearly ordered posets, generalising $\bar{\Delta}_n$. But then it is unknown whether, and, it seems to the author, not very probable that the stable category of a Frobenius category is triangulated in this generalised sense. More specifically, it seems to be impossible to generalise Proposition I.11 below accordingly, which is the technical core of our approach.

Question I.6 Does there exist an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ between Heller triangulated categories that, in Definition I.5.(iii), satisfies (1) and (2), but (3) only for $n \leqslant 2$? If F is an identity, this amounts to asking for the existence of two Heller triangulations ϑ and ϑ' on $(\mathcal{C}, \mathsf{T})$, $\mathcal{C}$ weakly abelian, T automorphism of $\mathcal{C}$, such that $\vartheta_n = \vartheta'_n$ only for $n \leqslant 2$.

I.2 Some equivalences

Suppose given $n \geqslant 0$. Suppose given a weakly abelian category $\mathcal{C}$, together with an automorphism $\mathsf{T} : \mathcal{C} \longrightarrow \mathcal{C}$, $X \longmapsto X^{+1}$. Concerning the Freyd category $\hat{\mathcal{C}}$ of $\mathcal{C}$, we refer to §I.6.6.3.

We shall show in Proposition I.12 that the functor $\underline{\mathcal{C}(\bar{\Delta}_n^\#)} \longrightarrow \underline{\mathcal{C}(\dot{\Delta}_n)}$, induced on the stable categories by restriction from $\bar{\Delta}_n^\#$ to $\dot{\Delta}_n := [1, n]$, is an equivalence.

I.2.1 Some notation

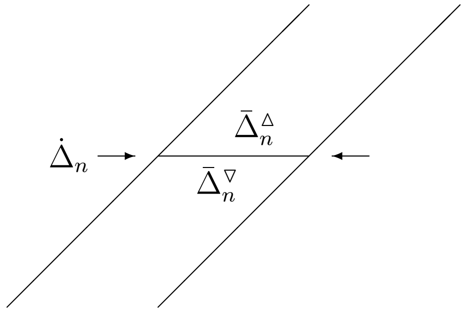
I.2.1.1 Some posets

Let $\dot{\Delta}_n := \Delta_n \smallsetminus \{0\} = [1, n]$, considered as a linearly ordered set. We have an injection $\dot{\Delta}_n \longrightarrow \bar{\Delta}_n^\#$, $i \longmapsto i/0$, and identify $\dot{\Delta}_n$ with its image in $\bar{\Delta}_n^\#$.

We define two subposets of $\bar{\Delta}_n^\#$ by

$$\begin{array}{rcl} \bar{\Delta}_n^{\vartriangle} & := & \{\beta/\alpha \in \bar{\Delta}_n^\# \;:\; 0 \leqslant \alpha\} \\ \bar{\Delta}_n^{\triangledown} & := & \{\beta/\alpha \in \bar{\Delta}_n^\# \;:\; \alpha \leqslant 0\} \,. \end{array}$$

Then $\dot{\Delta}_n = (\bar{\Delta}_n^{\vartriangle} \cap \bar{\Delta}_n^{\triangledown}) \smallsetminus \{0/0,\, 0^{+1}/0\}$.



I.2.1.2 Fixing parametrisations $\kappa^{\vartriangle}$, $\kappa^{\triangledown}$

There exists a bijective morphism $\bar{\Delta}_n^{\vartriangle} \longrightarrow \mathbf{Z}_{\geqslant 0}$ of posets ("refining the partial to a linear order"). We fix such a morphism and denote by $\mathbf{Z}_{\geqslant 0} \xrightarrow{\kappa^{\vartriangle}} \bar{\Delta}_n^{\vartriangle}$ its inverse (as a map of sets; in general, $\kappa^{\vartriangle}$ is not monotone). So whenever $\kappa^{\vartriangle}(\ell) \leqslant \kappa^{\vartriangle}(\ell')$, then $\ell \leqslant \ell'$. In particular, $\kappa^{\vartriangle}(0) = 0/0$.

There exists a bijective morphism $\bar{\Delta}_n^{\triangledown} \longrightarrow \mathbf{Z}_{\leqslant 0}$ of posets. We fix such a morphism and denote by $\mathbf{Z}_{\geqslant 0} \xrightarrow{\kappa^{\triangledown}} \bar{\Delta}_n^{\triangledown}$ its inverse (as a map of sets). So whenever $\kappa^{\triangledown}(\ell) \leqslant \kappa^{\triangledown}(\ell')$, then $\ell \leqslant \ell'$. In particular, $\kappa^{\triangledown}(0) = 0^{+1}/0$.

I.2.1.3 The categories $\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\vartriangle})$, $\mathcal{C}^{+}(\bar{\Delta}_n^{\vartriangle})$ etc.

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B} \subseteq \mathcal{A}$ be a full subcategory. Let $E \subseteq \bar{\Delta}_n^{\#}$ be a full subposet.

> For example, for E we may take the subposets $\bar{\Delta}_n^{\vartriangle}$, $\bar{\Delta}_n^{\triangledown}$ or $\bar{\Delta}_n^{\vartriangle} \cap \bar{\Delta}_n^{\triangledown,+1}$ of $\bar{\Delta}_n^{\#}$.
>
> Moreover, for example, we may take $\mathcal{A} = \hat{\mathcal{C}}$ and for $\mathcal{B}$ either $\hat{\mathcal{C}}$ or $\mathcal{C}$.

Let $\mathcal{B}^{+,*}(E)$ be the full subcategory of $\mathcal{B}(E)$ defined by

$$\operatorname{Ob} \mathcal{B}^{+,*}(E) := \left\{ X \in \operatorname{Ob} \mathcal{B}(E) \quad : \quad \begin{array}{l} \text{For all } \delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1} \text{ in } \bar{\Delta}_n \\ \text{such that } \gamma/\alpha,\ \gamma/\beta,\ \delta/\alpha \text{ and } \delta/\beta \text{ are in } E, \\ \text{the quadrangle} \\ \begin{array}{ccc} X_{\gamma/\beta} & \xrightarrow{x} & X_{\delta/\beta} \\ {\scriptstyle x}\uparrow & + & \uparrow{\scriptstyle x} \\ X_{\gamma/\alpha} & \xrightarrow{x} & X_{\delta/\alpha} \end{array} \\ \text{is a weak square (as indicated by } +\text{).} \end{array} \right\} .$$

The symbol $*$ should remind us of the fact that we still allow $X_{\alpha/\alpha}$ resp. $X_{\alpha^{+1}/\alpha}$ to be arbitrary for $\alpha \in \bar{\Delta}_n$ such that $\alpha/\alpha \in E$ resp. $\alpha^{+1}/\alpha \in E$.

In turn, let $\mathcal{B}^{+}(E)$ be the full subcategory of $\mathcal{B}^{+,*}(E)$ defined by

$$\operatorname{Ob} \mathcal{B}^{+}(E) := \left\{ X \in \operatorname{Ob} \mathcal{B}^{+,*}(E) \quad : \quad \begin{array}{l} X_{\alpha/\alpha} = 0 \text{ for } \alpha \in \bar{\Delta}_n \text{ such that } \alpha/\alpha \in E \text{ , and} \\ X_{\alpha^{+1}/\alpha} = 0 \text{ for } \alpha \in \bar{\Delta}_n \text{ such that } \alpha^{+1}/\alpha \in E. \end{array} \right\} .$$

I.2.1.4 Reindexing

Given a subposet $E \subseteq \bar{\Delta}_n^{\#}$, we have a *reindexing equivalence*

$$\begin{array}{ccc} \mathcal{C}(E) & \xrightarrow{\sim} & \mathcal{C}(E^{+1}) \\ X & \longmapsto & X^{(-1)} \\ X^{(+1)} & \longleftarrow & X \end{array}$$

defined by

$$(X^{(-1)})_{\beta/\alpha} := X_{(\beta/\alpha)^{-1}} = X_{\alpha/\beta^{-1}} ,$$

where $\beta/\alpha \in E^{+1}$; and inversely by

$$(X^{(+1)})_{\beta/\alpha} := X_{(\beta/\alpha)^{+1}} = X_{\alpha^{+1}/\beta} ,$$

where $\beta/\alpha \in E$. This equivalence restricts to an equivalence between $\mathcal{C}^+(E)$ and $\mathcal{C}^+(E^{+1})$.

For instance, if $E = \bar{\Delta}_n^{\#}$, then $X^{(+1)} = [X]^{+1}$. The outer shift and reindexing will play different roles, and so we distinguish in notation.

I.2.2 Density of the restriction functor from $\bar{\Delta}_n^{\#}$ to $\dot{\Delta}_n$

I.2.2.1 Upwards and downwards spread

Let the *upwards spread* $S^{\vartriangle}$ be defined by

$$\begin{array}{ccc} \hat{\mathcal{C}}(\dot{\Delta}_n) & \xrightarrow{S^{\vartriangle}} & \hat{\mathcal{C}}^+(\bar{\Delta}_n^{\vartriangle}) \\ X & \longmapsto & XS^{\vartriangle} , \end{array}$$

where $XS^{\vartriangle}$ is given by

$$\begin{array}{llll} (XS^{\vartriangle})_{0/0} & := & 0 & \\ (XS^{\vartriangle})_{\beta/0} & := & X_\beta & \text{for } \beta \in \dot{\Delta}_n \\ (XS^{\vartriangle})_{\beta/\alpha} & := & \operatorname{Cokern}(X_\alpha \xrightarrow{x} X_\beta) & \text{for } \alpha,\, \beta \in \dot{\Delta}_n \text{ with } \alpha \leqslant \beta \\ (XS^{\vartriangle})_{\beta/\alpha} & := & 0 & \text{for } \alpha,\, \beta \in \bar{\Delta}_n \text{ with } 0^{+1} \leqslant \beta \leqslant \alpha^{+1} \leqslant \beta^{+1} , \end{array}$$

the diagram being completed with the induced morphisms between the cokernels and zero morphisms elsewhere.

This construction is functorial in X. The functor $S^{\vartriangle}$ is left adjoint to the restriction functor from $\hat{\mathcal{C}}^+(\bar{\Delta}_n^{\vartriangle})$ to $\hat{\mathcal{C}}(\dot{\Delta}_n)$, with unit being the identity, i.e. $X = XS^{\vartriangle}|_{\dot{\Delta}_n}$.

Dually, let the *downwards spread* $S^{\triangledown}$ be

$$\begin{array}{ccc} \hat{\mathcal{C}}(\dot{\Delta}_n) & \xrightarrow{S^{\triangledown}} & \hat{\mathcal{C}}^+(\bar{\Delta}_n^{\triangledown}) \\ X & \longmapsto & XS^{\triangledown} , \end{array}$$

where $XS^\triangledown$ is given by

$$\begin{array}{lcll} (XS^\triangledown)_{0^{+1}/0} & := & 0 & \\ (XS^\triangledown)_{\alpha/0} & := & X_\alpha & \text{for } \alpha \in \dot{\Delta}_n \\ (XS^\triangledown)_{\alpha/\beta^{-1}} & := & \operatorname{Kern}(X_\alpha \xrightarrow{x} X_\beta) & \text{for } \alpha,\ \beta \in \dot{\Delta}_n \text{ with } \alpha \leqslant \beta \\ (XS^\triangledown)_{\alpha/\beta^{-1}} & := & 0 & \text{for } \alpha,\ \beta \in \bar{\Delta}_n \text{ with } \alpha^{-1} \leqslant \beta^{-1} \leqslant \alpha \leqslant 0\,, \end{array}$$

the diagram being completed with the induced morphisms between the kernels and zero morphisms elsewhere.

This construction is functorial in X. The functor $S^\triangledown$ is right adjoint to the restriction functor from $\hat{\mathcal{C}}^+(\bar{\Delta}_n^\triangledown)$ to $\hat{\mathcal{C}}(\dot{\Delta}_n)$, with counit being the identity, i.e. $XS^\triangledown|_{\dot{\Delta}_n} = X$.

I.2.2.2 Resolutions

I.2.2.2.1 A stability under pointwise pushouts and pullbacks

Let $E \subseteq \bar{\Delta}_n^\#$ be a full subposet. Moreover, assume that E is a *convex* subposet, i.e. that whenever given $\xi,\ \zeta \in E$ and $\eta \in \bar{\Delta}_n^\#$ such that $\xi \leqslant \eta \leqslant \zeta$, then $\eta \in E$.

An element $\delta/\beta \in E$ is *on the left boundary of* E if we may conclude from $\gamma/\beta \in E$ and $\gamma \leqslant \delta$ that $\gamma = \delta$. It is *on the lower boundary of* E if we may conclude from $\delta/\alpha \in E$ and $\alpha \leqslant \beta$ that $\alpha = \beta$.

An element $\gamma/\alpha \in E$ is *on the right boundary of* E if we may conclude from $\delta/\alpha \in E$ and $\gamma \leqslant \delta$ that $\gamma = \delta$. It is *on the upper boundary of* E if we may conclude from $\gamma/\beta \in E$ and $\alpha \leqslant \beta$ that $\alpha = \beta$.

Let $\mathcal{A}$ be an abelian category. Concerning pointwise pullbacks and pointwise pushouts, we refer to §I.6.7.

Lemma I.7 *Suppose given $\varepsilon \in E$ and an object X of $\mathcal{A}^{+,*}(E)$.*

(1) *Given a monomorphism $X_\varepsilon \xrightarrow{x'} X'$ in $\mathcal{A}$, the pointwise pushout $X{\uparrow}^{x'}$ of X along x' is an object of $\mathcal{A}^{+,*}(E)$ again.*

(2) *Given an epimorphism $X_\varepsilon \xleftarrow{x'} X'$ in $\mathcal{A}$, the pointwise pullback $X{\uparrow}_{x'}$ of X along x' is an object of $\operatorname{Ob}\mathcal{A}^{+,*}(E)$ again.*

(3) *Suppose that ε is on the left boundary or on the lower boundary of E. Given a morphism $X_\varepsilon \xrightarrow{x'} X'$ in $\mathcal{A}$, the pointwise pushout $X{\uparrow}^{x'}$ of X along x' is an object of $\mathcal{A}^{+,*}(E)$ again.*

(4) *Suppose that ε is on the right boundary or on the upper boundary of E. Given a morphism $X_\varepsilon \xleftarrow{x'} X'$ in $\mathcal{A}$, the pointwise pullback $X{\uparrow}_{x'}$ of X along x' is an object of $\operatorname{Ob}\mathcal{A}^{+,*}(E)$ again.*

Proof. Ad (1). First we remark that by Lemma I.55, the quadrangle $(X_{\beta/\alpha}\,,\ X_{\delta/\gamma}\,,\ (X{\uparrow}^{x'})_{\beta/\alpha}\,,\ (X{\uparrow}^{x'})_{\delta/\gamma})$ is a pushout for $\varepsilon \leqslant \beta/\alpha \leqslant \delta/\gamma$ in E.

We have to show that the quadrangle of $X{\uparrow}^{x'}$ on $(\gamma/\alpha, \delta/\alpha, \gamma/\beta, \delta/\beta)$, where $\delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1}$ in $\bar{\Delta}_n$, is a weak square, provided its corners have indices in E. Using Lemmata I.54, I.56 and convexity of E, we are reduced to the case $\varepsilon \leqslant \gamma/\alpha$. In this case, the assertion follows by Lemma I.55.

Ad (3). Here we need only Lemma I.54 and convexity of E to reduce to the case $\varepsilon \leqslant \gamma/\alpha$, the rest of the argument is as in (1). Hence the morphism x' may be arbitrary. ▫

I.2.2.2.2 Upwards and downwards resolution

Remark I.8

(1) Given a direct system $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$ in $\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\vartriangle})$ such that its restriction to any finite full subposet $E \subseteq \bar{\Delta}_n^{\vartriangle}$ eventually becomes constant, then the direct limit $\varinjlim_i X_i$ exists in $\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\vartriangle})$.

(2) Given an inverse system $X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$ in $\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\triangledown})$ such that its restriction to any finite subposet $E \subseteq \bar{\Delta}_n^{\triangledown}$ eventually becomes constant, then the inverse limit $\varprojlim_i X_i$ exists in $\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\triangledown})$.

For $k \geqslant 0$, we let

$$\begin{array}{ccc} \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\vartriangle}) & \xrightarrow{R_k^{\vartriangle}} & \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\vartriangle}) \\ X & \longmapsto & X{\uparrow}^{x'(k)} , \end{array}$$

where

$$x'(k) := \begin{cases} (X_{\kappa^{\vartriangle}(k)} \longrightarrow 0) & \text{if } \kappa^{\vartriangle}(k) \in \{\alpha/\alpha,\ \alpha^{+1}/\alpha\} \text{ for some } \alpha \in \bar{\Delta}_n \text{ with } 0 \leqslant \alpha \\ (X_{\kappa^{\vartriangle}(k)} \xrightarrow{\iota} X_{\kappa^{\vartriangle}(k)}\mathsf{I}) & \text{if } \kappa^{\vartriangle}(k) = \beta/\alpha \text{ for some } \alpha,\ \beta \in \bar{\Delta}_n \text{ with } 0 \leqslant \alpha < \beta < \alpha^{+1} \end{cases}$$

Define the *upwards resolution* map by

$$\begin{array}{ccc} \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\vartriangle}) & \xrightarrow{R^{\vartriangle}} & \operatorname{Ob}\mathcal{C}^{+}(\bar{\Delta}_n^{\vartriangle}) \\ X & \longmapsto & XR^{\vartriangle} := \varinjlim_m XR_0^{\vartriangle}\cdots R_m^{\vartriangle} , \end{array}$$

the direct system being given by the transition morphisms

$$XR_0^{\vartriangle}\cdots R_m^{\vartriangle} \xrightarrow{i} (XR_0^{\vartriangle}\cdots R_m^{\vartriangle})R_{m+1}^{\vartriangle} .$$

We have $XR^{\vartriangle} = X$ for $X \in \operatorname{Ob}\mathcal{C}^{+}(\bar{\Delta}_n^{\vartriangle})$.

Note that we apparently cannot turn the upwards resolution into a functor unless we are in a particular case in which the map I on objects can be turned into a functor.

Dually, for $k \geqslant 0$, we let

$$\begin{array}{ccc} \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\triangledown}) & \xrightarrow{R_k^{\triangledown}} & \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar{\Delta}_n^{\triangledown}) \\ X & \longmapsto & X{\uparrow}_{x''(k)} , \end{array}$$

where

$$x''(k) \;:=\; \begin{cases} (X_{\kappa^\triangledown(k)} \longleftarrow 0) & \text{if } \kappa^\triangledown(k) \in \{\alpha/\alpha,\, \alpha^{+1}/\alpha\} \text{ for some } \alpha \in \bar\Delta_n \text{ with } \alpha \leqslant 0 \\ (X_{\kappa^\triangledown(k)} \xleftarrow{\pi} X_{\kappa^\triangledown(k)}\mathsf{P}) & \text{if } \kappa^\triangledown(k) = \beta/\alpha \text{ for some } \alpha,\, \beta \,\in\, \bar\Delta_n \text{ with } \alpha^{-1} < \beta^{-1} < \alpha \leqslant 0 \end{cases}$$

Define the *downwards resolution* map by

$$\begin{array}{rcl} \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar\Delta_n^\triangledown) & \xrightarrow{R^\triangledown} & \operatorname{Ob}\mathcal{C}^{+}(\bar\Delta_n^\triangledown) \\ X & \longmapsto & XR^\triangledown \;:=\; \varprojlim_m XR_0^\triangledown\cdots R_m^\triangledown \,, \end{array}$$

the inverse system being given by the transition morphisms

$$XR_0^\triangledown\cdots R_m^\triangledown \xleftarrow{p} (XR_0^\triangledown\cdots R_m^\triangledown)R_{m+1}^\triangledown \,.$$

We have $XR^\triangledown = X$ for $X \in \operatorname{Ob}\mathcal{C}^{+}(\bar\Delta_n^\triangledown)$.

Lemma I.9

(1) *Given a morphism* $Y \xrightarrow{g} X$ *in* $\hat{\mathcal{C}}^{+}(\bar\Delta_n^\vartriangle)$ *with* $X \in \operatorname{Ob}\mathcal{C}^{+}(\bar\Delta_n^\vartriangle)$, *there exists a factorisation*

$$(Y \xrightarrow{g} X) \;=\; (Y \longrightarrow YR^\vartriangle \longrightarrow X)\,.$$

(2) *Given a morphism* $Y \xleftarrow{g} X$ *in* $\hat{\mathcal{C}}^{+}(\bar\Delta_n^\triangledown)$ *with* $X \in \operatorname{Ob}\mathcal{C}^{+}(\bar\Delta_n^\triangledown)$, *there exists a factorisation*

$$(Y \xleftarrow{g} X) \;=\; (Y \longleftarrow YR^\triangledown \longleftarrow X)\,.$$

Proof. Ad (1). Since the entries of X are injective in $\hat{\mathcal{C}}$ and since $X_{\alpha/\alpha} = 0$ and $X_{\alpha^{+1}/\alpha} = 0$ for $\alpha \geqslant 0$, we obtain, using the universal property of the pointwise pushout, a factorisation

$$(Y \xrightarrow{g} X) \;=\; (Y \longrightarrow YR_0^\vartriangle\cdots R_m^\vartriangle \longrightarrow X)$$

for every $m \geqslant 0$, compatible with the transition morphisms, resulting in a factorisation over $YR^\vartriangle = \varinjlim_m YR_0^\vartriangle\cdots R_m^\vartriangle$. ▫

I.2.2.2.3 Both-sided resolutions

Let the *resolution* map

$$\begin{array}{rcl} \operatorname{Ob}\mathcal{C}(\dot\Delta_n) & \xrightarrow{R} & \operatorname{Ob}\mathcal{C}^{+}(\bar\Delta_n^{\#}) \\ X & \longmapsto & XR \end{array}$$

be defined by gluing an upper and a lower part along $\dot\Delta_n$ as follows.

$$\begin{array}{rcl} XR|_{\bar\Delta_n^\vartriangle} & := & XS^\vartriangle R^\vartriangle \\ XR|_{\bar\Delta_n^\triangledown} & := & XS^\triangledown R^\triangledown \end{array}$$

This is welldefined, since $XS^\vartriangle R^\vartriangle|_{\dot\Delta_n} = X = XS^\triangledown R^\triangledown|_{\dot\Delta_n}$. In particular, we obtain

$$XR|_{\dot\Delta_n} \;=\; X\,.$$

We summarise.

Proposition I.10 *The restriction functor*

$$\begin{array}{ccc} \mathcal{C}^+(\bar{\Delta}_n^{\#}) & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \mathcal{C}(\dot{\Delta}_n) \\ Y & \longmapsto & Y|_{\dot{\Delta}_n} \end{array}$$

is strictly dense, i.e. it is surjective on objects.

I.2.3 Fullness of the restriction functor from $\bar{\Delta}_n^{\#}$ to $\dot{\Delta}_n$

Proposition I.11 *The restriction functors*

$$\begin{array}{ccc} \mathcal{C}^+(\bar{\Delta}_n^{\#}) & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \mathcal{C}(\dot{\Delta}_n) \\ Y & \longmapsto & Y|_{\dot{\Delta}_n} \end{array}$$

and

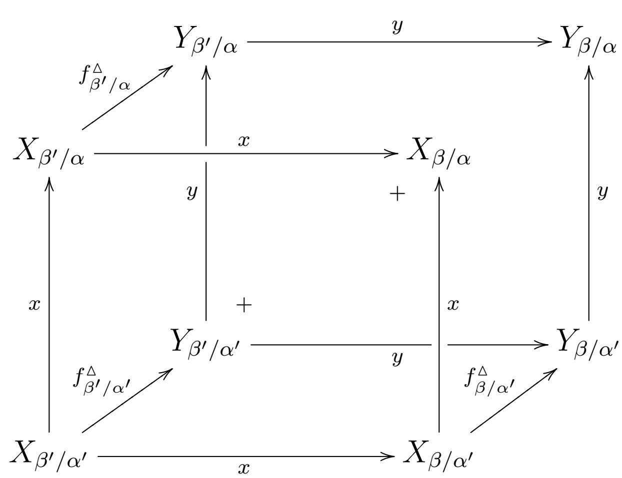
$$\begin{array}{ccc} \mathcal{C}^+(\bar{\Delta}_n^{\vartriangle}) & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \mathcal{C}(\dot{\Delta}_n) \\ Y & \longmapsto & Y|_{\dot{\Delta}_n} \end{array}$$

are full.

Proof. By duality and gluing along $\dot{\Delta}_n$, it suffices to consider the restriction from $\bar{\Delta}_n^{\vartriangle}$ to $\dot{\Delta}_n$. So suppose given $X, Y \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\vartriangle})$ and a morphism $X|_{\dot{\Delta}_n} \xrightarrow{f} Y|_{\dot{\Delta}_n}$. We have to find a morphism $X \xrightarrow{f^{\vartriangle}} Y$ such that $f^{\vartriangle}|_{\dot{\Delta}_n} = f$.

We construct $f^{\vartriangle}_{\kappa^{\vartriangle}(\ell)}$ for $\ell \geqslant 0$ by induction on ℓ.

At $\kappa^{\vartriangle}(0)$, we let $f^{\vartriangle}_{\kappa^{\vartriangle}(0)} := 1_0$. Suppose given $\ell \geqslant 1$. If $\kappa^{\vartriangle}(\ell) \in \dot{\Delta}_n$, we let $f^{\vartriangle}_{\kappa^{\vartriangle}(\ell)} := f_{\kappa^{\vartriangle}(\ell)}$. If $\kappa^{\vartriangle}(\ell) \in \{\alpha/\alpha,\, \alpha^{+1}/\alpha\}$ for some $\alpha \geqslant 0$, we let $f^{\vartriangle}_{\kappa^{\vartriangle}(\ell)} := 1_0$. If $\kappa^{\vartriangle}(\ell) =: \beta/\alpha$ with $0 < \alpha < \beta < \alpha^{+1}$, then we let $\alpha' := \alpha - 1$ be the predecessor of α in $\bar{\Delta}_n$, and we let $\beta' := \beta - 1$ be the predecessor of β in $\bar{\Delta}_n$, using that $\bar{\Delta}_n$ is linearly ordered. We may complete the diagram



to a commutative cuboid, inserting a morphism $X_{\beta/\alpha} \xrightarrow{f^{\vartriangle}_{\beta/\alpha}} Y_{\beta/\alpha}$. □

Since we need the restriction functor $\mathcal{C}^+(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ to be full, we are not able to generalise from linearly ordered periodic posets to arbitrary periodic posets.

I.2.4 The equivalence between $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ and $\underline{\mathcal{C}(\dot{\Delta}_n)}$

Let $\mathcal{C}^{\text{split}}(\dot{\Delta}_n)$ be the full subcategory of $\mathcal{C}(\dot{\Delta}_n)$ defined by

$$\text{Ob } \mathcal{C}^{\text{split}}(\dot{\Delta}_n) \ := \ \{X \in \text{Ob } \mathcal{C}(\dot{\Delta}_n) \ : \ X_\alpha \longrightarrow X_\beta \text{ is split in } \hat{\mathcal{C}} \text{ for all } \alpha,\, \beta \in \dot{\Delta}_n \text{ with } \alpha \leqslant \beta\}\ .$$

We denote the quotient category by

$$\underline{\mathcal{C}(\dot{\Delta}_n)} \ := \ \mathcal{C}(\dot{\Delta}_n)/\mathcal{C}^{\text{split}}(\dot{\Delta}_n)\ .$$

Proposition I.12 *The functor*

$$\begin{array}{ccc} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \underline{\mathcal{C}(\dot{\Delta}_n)} \\ X & \longmapsto & X|_{\dot{\Delta}_n}\ , \end{array}$$

induced by restriction from $\bar{\Delta}_n^\#$ to $\dot{\Delta}_n$, is an equivalence.

Proof. By Propositions I.10 and I.11, we may invoke Lemma I.41. Moreover, Lemma I.4 gives the inverse image of $\text{Ob } \mathcal{C}^{\text{split}}(\dot{\Delta}_n)$ under $\mathcal{C}^+(\bar{\Delta}_n^\#) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ as $\text{Ob } \mathcal{C}^{+,\,\text{split}}(\bar{\Delta}_n^\#)$.

Consider a morphism $X \xrightarrow{f} X'$ in $\mathcal{C}^+(\bar{\Delta}_n^\#)$ such that $(X \xrightarrow{f} X')|_{\dot{\Delta}_n}$ is zero in $\underline{\mathcal{C}(\dot{\Delta}_n)}$. We have to prove that it factors over an object of $\mathcal{C}^{+,\,\text{split}}(\bar{\Delta}_n^\#)$.

Let $Y^\vartriangle$ be the cokernel in $\hat{\mathcal{C}}(\bar{\Delta}_n^\vartriangle)$ of the counit $X|_{\dot{\Delta}_n} S^\vartriangle \longrightarrow X|_{\bar{\Delta}_n^\vartriangle}$ at $X|_{\bar{\Delta}_n^\vartriangle}$. Note that $Y^\vartriangle|_{\dot{\Delta}_n} = 0$. By Lemma I.61, we have $Y^\vartriangle \in \text{Ob } \hat{\mathcal{C}}^+(\bar{\Delta}_n^\vartriangle)$. Consider the following diagram in $\hat{\mathcal{C}}^+(\bar{\Delta}_n^\vartriangle)$.

$$\begin{array}{ccccccc} X|_{\dot{\Delta}_n} S^\vartriangle & \longrightarrow & X|_{\bar{\Delta}_n^\vartriangle} & \longrightarrow & Y^\vartriangle & \longrightarrow & Y^\vartriangle R^\vartriangle \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle f|_{\bar{\Delta}_n^\vartriangle}} & \swarrow & & \swarrow & \\ X'|_{\dot{\Delta}_n} S^\vartriangle & \longrightarrow & X'|_{\bar{\Delta}_n^\vartriangle} & & & & \end{array}$$

The indicated factorisation

$$(X|_{\bar{\Delta}_n^\vartriangle} \xrightarrow{f|_{\bar{\Delta}_n^\vartriangle}} X'|_{\bar{\Delta}_n^\vartriangle}) \ = \ (X|_{\bar{\Delta}_n^\vartriangle} \longrightarrow Y^\vartriangle \longrightarrow X'|_{\bar{\Delta}_n^\vartriangle})$$

ensues from the universal property of the cokernel $Y^\vartriangle$. By Lemma I.9.(1), we can factorise further to obtain

$$(*^\vartriangle) \qquad (X|_{\bar{\Delta}_n^\vartriangle} \xrightarrow{f|_{\bar{\Delta}_n^\vartriangle}} X'|_{\bar{\Delta}_n^\vartriangle}) \ = \ (X|_{\bar{\Delta}_n^\vartriangle} \longrightarrow Y^\vartriangle R^\vartriangle \longrightarrow X'|_{\bar{\Delta}_n^\vartriangle})\ .$$

Dually, we obtain a factorisation

$$(*^\triangledown) \qquad (X|_{\bar{\Delta}_n^\triangledown} \xrightarrow{f|_{\bar{\Delta}_n^\triangledown}} X'|_{\bar{\Delta}_n^\triangledown}) \ = \ (X|_{\bar{\Delta}_n^\triangledown} \longrightarrow Y^\triangledown R^\triangledown \longrightarrow X'|_{\bar{\Delta}_n^\triangledown})$$

for some $Y^\triangledown \in \text{Ob } \hat{\mathcal{C}}^+(\bar{\Delta}_n^\triangledown)$ such that $Y^\triangledown|_{\dot{\Delta}_n} = 0$.

Since $Y^\vartriangle R^\vartriangle|_{\dot{\Delta}_n} = 0 = Y^\triangledown R^\triangledown|_{\dot{\Delta}_n}$, there is a unique $N \in \text{Ob } \mathcal{C}^+(\bar{\Delta}_n^\#)$ such that $N|_{\bar{\Delta}_n^\vartriangle} = Y^\vartriangle R^\vartriangle$ and $N|_{\bar{\Delta}_n^\triangledown} = Y^\triangledown R^\triangledown$. By Lemma I.4, we have $N \in \text{Ob } \mathcal{C}^{+,\,\text{split}}(\bar{\Delta}_n^\#)$.

Moreover, since both factorisations $(*^\vartriangle)$ and $(*^\triangledown)$ restrict to the factorisation

$$(X|_{\dot{\Delta}_n} \xrightarrow{0} X'|_{\dot{\Delta}_n}) \ = \ (X|_{\dot{\Delta}_n} \longrightarrow 0 \longrightarrow X'|_{\dot{\Delta}_n})$$

in $\mathcal{C}(\dot{\Delta}_n)$, we may glue to a factorisation

$$(*) \qquad (X \xrightarrow{f} X') \ = \ (X \longrightarrow N \longrightarrow X')$$

that restricts to $(*^\vartriangle)$ in $\mathcal{C}^+(\bar{\Delta}_n^\vartriangle)$ and to $(*^\triangledown)$ in $\mathcal{C}^+(\bar{\Delta}_n^\triangledown)$. $\square$

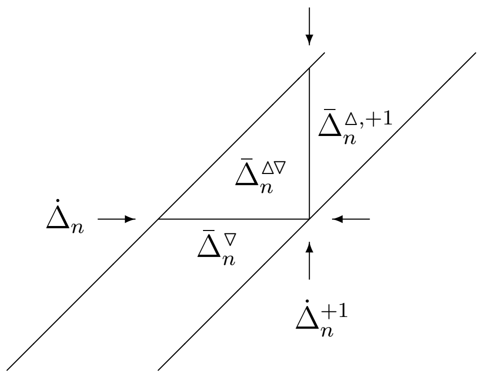
I.2.5 Auxiliary equivalences

We shall extend the equivalence $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} \xrightarrow{\sim} \underline{\mathcal{C}(\dot{\Delta}_n)}$ to a diagram of equivalences

$$\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} \xrightarrow{\sim} \underline{\mathcal{C}^+(\Delta_n^{\vartriangle\triangledown})} \xrightarrow{\sim} \underline{\mathcal{C}(\dot{\Delta}_n)} \xrightarrow{\sim} \underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})}\,.$$

I.2.5.1 Factorisation into two equivalences

Abbreviate $\bar{\Delta}_n^{\vartriangle,+1} := (\bar{\Delta}_n^{\vartriangle})^{+1} \subseteq \bar{\Delta}_n^\#$, $\bar{\Delta}_n^{\triangledown,+1} := (\bar{\Delta}_n^{\triangledown})^{+1} \subseteq \bar{\Delta}_n^\#$ and $\dot{\Delta}_n^{+1} := (\dot{\Delta}_n)^{+1} \subseteq \bar{\Delta}_n^\#$.
Abbreviate $\bar{\Delta}_n^{\vartriangle\triangledown} := \bar{\Delta}_n^{\vartriangle} \cap \bar{\Delta}_n^{\triangledown,+1} = \{\beta/\alpha \;:\; \alpha,\, \beta \in \bar{\Delta}_n,\; 0 \leqslant \alpha \leqslant \beta \leqslant 0^{+1}\} \subseteq \bar{\Delta}_n^\#$.



Let $\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\vartriangle\triangledown})$ be the full subcategory of $\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})$ defined by

$$\mathrm{Ob}\;\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\vartriangle\triangledown}) \;:=\; \left\{ X \in \mathrm{Ob}\;\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown}) \;:\; \begin{array}{l} X_{\gamma/\alpha} \xrightarrow{x} X_{\delta/\beta} \text{ is split in } \hat{\mathcal{C}} \\ \text{for all } \gamma/\alpha,\, \delta/\beta \in \bar{\Delta}_n^{\vartriangle\triangledown} \\ \text{with } \gamma/\alpha \leqslant \delta/\beta \end{array} \right\}$$

We denote the quotient category by

$$\underline{\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})} \;:=\; \mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})/\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\vartriangle\triangledown})$$

Lemma I.13

(1) *The restriction functors*

$$\begin{array}{ccccc} \mathcal{C}^+(\bar{\Delta}_n^\#) & \xrightarrow{(-)|_{\bar{\Delta}_n^{\vartriangle\triangledown}}} & \mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown}) & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \mathcal{C}^+(\dot{\Delta}_n) \\ X & \longmapsto & X|_{\bar{\Delta}_n^{\vartriangle\triangledown}} & & \\ & & Y & \longmapsto & Y|_{\dot{\Delta}_n} \end{array}$$

are full and strictly dense.

(2) *The functors*

$$\begin{array}{ccccc} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} & \xrightarrow{(-)|_{\bar{\Delta}_n^{\vartriangle\triangledown}}} & \underline{\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})} & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \underline{\mathcal{C}^+(\dot{\Delta}_n)} \\ X & \longmapsto & X|_{\bar{\Delta}_n^{\vartriangle\triangledown}} & & \\ & & Y & \longmapsto & Y|_{\dot{\Delta}_n}\,, \end{array}$$

induced by restriction, are equivalences.

Proof. Ad (1). The composition

$$\begin{array}{ccccc} \mathcal{C}^+(\bar{\Delta}_n^{\#}) & \longrightarrow & \mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown}) & \longrightarrow & \mathcal{C}(\dot{\Delta}_n) \\ X & \longmapsto & X|_{\bar{\Delta}_n^{\vartriangle\triangledown}} & \longmapsto & X|_{\dot{\Delta}_n} \end{array}$$

is strictly dense by Proposition I.10 and full by Proposition I.11. Therefore, the restriction functor from $\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})$ to $\mathcal{C}^+(\dot{\Delta}_n)$ is full and strictly dense.

We claim that the restriction functor from $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ to $\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})$ is strictly dense. Let

$$\begin{array}{ccc} \mathcal{C}(\dot{\Delta}_n^{+1}) & \xrightarrow{S^{\vartriangle,+1}} & \hat{\mathcal{C}}^+(\bar{\Delta}_n^{\vartriangle,+1}) \\ X & \longmapsto & (X^{(+1)}S^{\vartriangle})^{(-1)} , \end{array}$$

cf. §I.2.1.4. Similarly,

$$\begin{array}{ccc} \mathrm{Ob}\,\hat{\mathcal{C}}^+(\bar{\Delta}_n^{\vartriangle,+1}) & \xrightarrow{R^{\vartriangle,+1}} & \mathrm{Ob}\;\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle,+1}) \\ X & \longmapsto & (X^{(+1)}R^{\vartriangle})^{(-1)} . \end{array}$$

Given $X \in \mathrm{Ob}\;\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})$, we may define $X' \in \mathrm{Ob}\;\mathcal{C}^+(\bar{\Delta}_n^{\#})$ letting

$$\begin{array}{rcl} X'|_{\bar{\Delta}_n^{\vartriangle,+1}} & := & X|_{\dot{\Delta}_n^{+1}} S^{\vartriangle,+1} R^{\vartriangle,+1} \\ X'|_{\bar{\Delta}_n^{\vartriangle\triangledown}} & := & X \\ X'|_{\bar{\Delta}_n^{\triangledown}} & := & X|_{\dot{\Delta}_n} S^{\triangledown} R^{\triangledown} . \end{array}$$

We claim that the restriction functor from $\bar{\Delta}_n^{\#}$ to $\bar{\Delta}_n^{\vartriangle\triangledown}$ is full. Suppose given $X,\, Y \in \mathrm{Ob}\;\mathcal{C}^+(\Delta_n^{\#})$ and a morphism $X|_{\bar{\Delta}_n^{\vartriangle\triangledown}} \xrightarrow{f} Y|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$. By Proposition I.12 and a shift, there exists a morphism $X|_{\Delta_n^{\vartriangle,+1}} \xrightarrow{f^{\vartriangle}} Y|_{\Delta_n^{\vartriangle,+1}}$ such that $f^{\vartriangle}|_{\dot{\Delta}_n^{+1}} = f|_{\dot{\Delta}_n^{+1}}$. By Proposition I.12 and by duality, there exists a morphism $X|_{\Delta_n^{\triangledown}} \xrightarrow{f^{\triangledown}} Y|_{\Delta_n^{\triangledown}}$ such that $f^{\triangledown}|_{\dot{\Delta}_n} = f|_{\dot{\Delta}_n}$. We may define a morphism $X \xrightarrow{f'} Y$ letting

$$\begin{array}{rcl} f'|_{\bar{\Delta}_n^{\vartriangle,+1}} & := & f^{\vartriangle} \\ f'|_{\bar{\Delta}_n^{\vartriangle\triangledown}} & := & f \\ f'|_{\bar{\Delta}_n^{\triangledown}} & := & f^{\triangledown} . \end{array}$$

Ad (2). The composition

$$\begin{array}{ccccc} \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})} & \longrightarrow & \underline{\mathcal{C}^+(\bar{\Delta}_n^{\vartriangle\triangledown})} & \longrightarrow & \underline{\mathcal{C}(\dot{\Delta}_n)} \\ X & \longmapsto & X|_{\bar{\Delta}_n^{\vartriangle\triangledown}} & \longmapsto & X|_{\dot{\Delta}_n} \end{array}$$

is an equivalence by Proposition I.12. Therefore, the functor induced by restriction from $\bar{\Delta}_n^{\#}$ to $\bar{\Delta}_n^{\vartriangle\triangledown}$ is faithful. By (1), it is full and dense, and so it is an equivalence. Therefore, also the functor induced by restriction from $\bar{\Delta}_n^{\vartriangle\triangledown}$ to $\dot{\Delta}_n$ is an equivalence. □

I.2.5.2 Cutting off the last object

Putting $n = 2$, the equivalence given in Lemma I.16, composed with the equivalence in Proposition I.12, can be used to retrieve HELLER's original isomorphism, called $\delta(\Delta)$ in **[24,** p. 53].

In this section, we suppose that $n \geqslant 2$. Consider the functor

$$\begin{array}{rcl} \mathcal{C}(\dot{\Delta}_n) & \xrightarrow{K} & \hat{\mathcal{C}}(\dot{\Delta}_{n-1}) \\ X & \longmapsto & XK := (X\!\uparrow_0)|_{\dot{\Delta}_{n-1}} , \end{array}$$

where 0 denotes the morphism $0 \xrightarrow{0} X_n$; cf. §I.6.7.

Explicitly, we have $(XK)_i := \operatorname{Kern}(X_i \xrightarrow{x} X_n)$ for $i \in [1, n-1]$, taken in $\hat{\mathcal{C}}$, equipped with the induced morphisms $(XK)_i \longrightarrow (XK)_j$ for $i,\, j \in [1, n-1]$ with $i \leqslant j$, fitting into a pullback $((XK)_i, (XK)_j, X_i, X_j)$.

Let $\mathcal{C}^{\text{split}}(\dot{\Delta}_n) \subseteq \mathcal{C}(\dot{\Delta}_n)$ be the full subcategory defined by

$$\operatorname{Ob} \mathcal{C}^{\text{split}}(\dot{\Delta}_n) := \{X \in \operatorname{Ob} \mathcal{C}(\dot{\Delta}_n) \,:\, (X_i \longrightarrow X_j) \text{ is split in } \hat{\mathcal{C}} \text{ for all } i,\, j \in [1, n] \text{ with } i \leqslant j\} ,$$

and let $\underline{\mathcal{C}(\dot{\Delta}_n)} := \mathcal{C}(\dot{\Delta}_n)/\mathcal{C}^{\text{split}}(\dot{\Delta}_n)$ and $\underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})} := \hat{\mathcal{C}}(\dot{\Delta}_{n-1})/\mathcal{C}^{\text{split}}(\dot{\Delta}_{n-1})$.

For $Y \in \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_n)$ and $i \in [1, n]$, we let $YR_i := Y\!\uparrow^{Y_i\iota} \in \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_n)$.

We have a *resolution map*

$$\begin{array}{rcl} \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_n) & \xrightarrow{R'_n} & \operatorname{Ob} \mathcal{C}(\dot{\Delta}_n) \\ Y & \longmapsto & YR'_n := YR_0 \cdots R_n . \end{array}$$

If $Y \in \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_n)$ consists of monomorphisms, then so does YR'_n, whence $YR'_n \in \operatorname{Ob} \mathcal{C}^{\text{split}}(\dot{\Delta}_n)$.

Given a morphism $Y \longrightarrow Y'$ in $\hat{\mathcal{C}}(\dot{\Delta}_n)$ with Y' having bijective entries, this morphism factors over $Y \rightarrowtail YR'_n$ by injectivity of the entries of Y' and by the universal property of the pointwise pushout.

Lemma I.14 *The functor K is dense.*

Proof. Suppose given $X \in \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_{n-1})$. Let $X' \in \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_n)$ be defined by $X'|_{\dot{\Delta}_{n-1}} := X$ and $X'_n := 0$. Then $X'R'_n \in \operatorname{Ob} \mathcal{C}(\dot{\Delta}_n)$ has $(X'R'_n)K \simeq X$. ▫

Lemma I.15 *The functor K is full.*

Proof. Suppose given $X,\, Y \in \operatorname{Ob} \mathcal{C}(\dot{\Delta}_n)$ and a morphism $XK \xrightarrow{f} YK$. We claim that there exists a morphism $X \xrightarrow{\tilde{f}} Y$ such that $\tilde{f}K = f$. We construct its components $\tilde{f}_\ell$ by induction on ℓ. For $\ell = 1$, we obtain a morphism $X_1 \xrightarrow{\tilde{f}_1} Y_1$ such that $((XK)_1, (YK)_1, X_1, Y_1)$ commutes, by injectivity of Y_1 in $\hat{\mathcal{C}}$. For $\ell \geqslant 2$, we obtain a morphism $X_\ell \xrightarrow{\tilde{f}_\ell} Y_\ell$ such that $((XK)_\ell, (YK)_\ell, X_\ell, Y_\ell)$ and $(X_{\ell-1}, Y_{\ell-1}, X_\ell, Y_\ell)$ commute, by the fact that $((XK)_{\ell-1}, (XK)_\ell, X_{\ell-1}, X_\ell)$ is a weak square and by injectivity of Y_ℓ. ▫

Proposition I.16 *The functor K induces an equivalence*

$$\begin{array}{rcl} \underline{\mathcal{C}(\dot{\Delta}_n)} & \xrightarrow[\sim]{K} & \underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})} \\ X & \longmapsto & XK . \end{array}$$

Proof. Let $\tilde{\mathcal{C}} \subseteq \hat{\mathcal{C}}$ denote the full subcategory of bijective objects in $\hat{\mathcal{C}}$. Every object in $\tilde{\mathcal{C}}$ is a direct summand of an object in $\mathcal{C}$. Let $\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1}) \subseteq \tilde{\mathcal{C}}(\dot{\Delta}_{n-1})$ be the full subcategory defined by

$$\operatorname{Ob}\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1}) \;:=\; \{X \in \tilde{\mathcal{C}}(\dot{\Delta}_{n-1}) \;:\; (X_i \longrightarrow X_j) \text{ is split for all } i,\, j \in [1, n-1] \text{ with } i \leqslant j\}\;,$$

Let Y be an object of $\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1})$. Then YR'_{n-1} is an object of $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_{n-1})$ that has Y as a direct summand since the identity on Y factors over $Y \twoheadrightarrow YR_{n-1}$.

Therefore, any morphism that factors over an object of $\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1})$ already factors over an object of $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_{n-1})$. We infer that

$$\underline{\hat{\mathcal{C}}(\dot{\Delta}_{n-1})} \;=\; \hat{\mathcal{C}}(\dot{\Delta}_{n-1})/\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1}).$$

Suppose given $X \in \operatorname{Ob}\, \mathcal{C}(\dot{\Delta}_n)$. Denote $X' := XK \in \operatorname{Ob}\hat{\mathcal{C}}(\dot{\Delta}_{n-1})$.

We *claim* that if $X \in \operatorname{Ob}\, \mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$, then $X' \in \operatorname{Ob}\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1})$. First of all, X'_i is bijective for $i \in [1, n-1]$, since the image of $X_i \longrightarrow X_n$ is bijective, and since X'_i is the kernel of this morphism. Now suppose given $i,\, j \in [1, n-1]$ with $i < j$. In $\hat{\mathcal{C}}$, we let B be the image of $X_i \longrightarrow X_j$ and form a pullback (B', X'_j, B, X_j). Then there is an induced morphism $X'_i \longrightarrow B'$ turning (X'_i, B', X_i, B) into a commutative quadrangle, which is a pullback by composition to a pullback (X'_i, X'_j, X_i, X_j). We insert the common kernel Z of $X_i \longrightarrow X_j$ and $X'_i \longrightarrow X'_j$.

$$\begin{array}{ccccccc}
Z & \rightarrowtail & X_i & \twoheadrightarrow & B & \rightarrowtail & X_j \\
\| & & \uparrow & \llcorner & \uparrow & \llcorner & \uparrow \\
Z & \rightarrowtail & X'_i & \twoheadrightarrow & B' & \rightarrowtail & X'_j
\end{array}$$

Hence $Z \rightarrowtail X'_i$ is split monomorphic, and therefore $X' \twoheadrightarrow B'$ is split epimorphic. Thus B' is bijective, and so finally $B' \rightarrowtail X'_j$ is split monomorphic. This proves the *claim.*

We *claim* that if $X' \in \operatorname{Ob}\tilde{\mathcal{C}}^{\mathrm{split}}(\dot{\Delta}_{n-1})$, then $X \in \operatorname{Ob}\, \mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$. Suppose given $i,\, j \in [1, n-1]$ with $i < j$. We have to show that $X_i \longrightarrow X_j$ is split in $\hat{\mathcal{C}}$. In $\hat{\mathcal{C}}$, we insert the image B of $X_i \longrightarrow X_j$ and form a pullback (B', X'_j, B, X_j). Since (X'_i, B', X_i, B) is a square, and since X'_i is bijective, its diagonal sequence is split short exact. Hence B is bijective as a direct summand of $X_i \oplus B'$, which proves the *claim.*

Invoking Lemma I.41 to prove the equivalence, it remains to show that given $X \xrightarrow{f} Y$ in $\mathcal{C}(\dot{\Delta}_n)$ such that $fK = 0$, there exists an object V in $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$ such that there exists a factorisation

$$(X \xrightarrow{f} Y) \;=\; (X \longrightarrow V \longrightarrow Y)\;.$$

Denote by $XK' \in \operatorname{Ob}\, \mathcal{C}(\dot{\Delta}_n)$ the object that restricts to XK on $\dot{\Delta}_{n-1}$ and that has $(XK')_n := 0$. Let U be the cokernel of $XK' \rightarrowtail X$ and consider the following diagram.

$$\begin{array}{ccccccc}
XK' & \rightarrowtail & X & \twoheadrightarrow & U & \rightarrowtail & UR_n \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & \swarrow & & \swarrow & \\
YK' & \rightarrowtail & Y & & & &
\end{array}$$

The morphism $U \longrightarrow Y$ is induced by the universal property of the cokernel. Its factorisation $(U \longrightarrow Y) = (U \rightarrowtail UR_n \longrightarrow Y)$ exists since Y consists of bijective objects.

Since the morphism $XK' \rightarrowtail X$ consists of pullbacks, its cokernel U consists of monomorphisms. Hence so does $V := UR_n$, which is therefore in $\operatorname{Ob}\, \mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$, as required. □

I.2.5.3 Not quite an equivalence

Let $\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ be the subcategory of $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ that consists of morphisms $X \xrightarrow{f} Y$ for which

$$([X^{+1}] \xrightarrow{[f^{+1}]} [Y^{+1}]) \;=\; ([X]^{+1} \xrightarrow{[f]^{+1}} [Y]^{+1})\;;$$

which is in general not a full subcategory. The objects $\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ are called *periodic* n-pretriangles, the morphisms are called *periodic* morphisms of periodic n-pretriangles. Let $\mathcal{C}^{+,\,\mathrm{split},\,\mathrm{periodic}}(\bar{\Delta}_n^{\#}) := \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#}) \cap \mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\#})$.

For instance, if $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a Heller triangulated category, then $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#}) \subseteq \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ is a full subcategory.

For $Y \in \mathrm{Ob}\;\mathcal{C}(\dot{\Delta}_n)$, we define $YS \in \mathrm{Ob}\;\mathcal{C}^+(\bar{\Delta}_n^{\#})$ by

$$\begin{array}{rcl} YS|_{\bar{\Delta}_n^{\vartriangle}} & := & YS^{\vartriangle} \\ YS|_{\bar{\Delta}_n^{\triangledown}} & := & YS^{\triangledown}\,, \end{array}$$

and similarly on morphisms. If $Y \in \mathrm{Ob}\;\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$, then $YS \in \mathrm{Ob}\;\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\#})$ by Lemma I.4, since $YS|_{\dot{\Delta}_n} = Y \in \mathrm{Ob}\;\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$.

To any $X \in \mathrm{Ob}\;\mathcal{C}^+(\bar{\Delta}_n^{\#})$ for which $X_{\beta/\alpha}$ is zero for all but finitely many $\beta/\alpha \in \bar{\Delta}_n^{\#}$, we can assign its *periodification*

$$\bar{X} \;:=\; \bigoplus_{i\in\mathbf{Z}} [X^{+i}]^{-i} \;\in\; \mathrm{Ob}\;\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})\,,$$

and similarly for morphisms between such objects.

If $X \in \mathrm{Ob}\;\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\#})$, then $\bar{X} \in \mathrm{Ob}\;\mathcal{C}^{+,\,\mathrm{split},\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$.

We have the restriction functor

$$\begin{array}{rcl} \mathcal{C}^{+,\,\mathrm{split},\,\mathrm{periodic}}(\bar{\Delta}_n^{\#}) & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n) \\ X & \longmapsto & X|_{\dot{\Delta}_n} \end{array}$$

which is not faithful in general, as the case $n = 2$ shows. In the inverse direction, we dispose of the functor

$$\begin{array}{rcl} \mathcal{C}^{+,\,\mathrm{split},\,\mathrm{periodic}}(\bar{\Delta}_n^{\#}) & \xleftarrow{\bar{S}} & \mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n) \\ \overline{YS} \;=:\; Y\bar{S} & \longleftarrow\!\!\shortmid & Y\,. \end{array}$$

Lemma I.17 *For $X \in \mathrm{Ob}\;\mathcal{C}^{+,\,\mathrm{split},\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$, we have $X \simeq X|_{\dot{\Delta}_n}\bar{S}$.*

Note that we do not claim that $1 \simeq (-)|_{\dot{\Delta}_n}\bar{S}$ as endofunctors of $\mathcal{C}^{+,\,\mathrm{split},\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$.

Proof. We have a short exact sequence

$$X|_{\dot{\Delta}_n}S|_{\bar{\Delta}_n^{\vartriangle\triangledown}} \longrightarrow X|_{\bar{\Delta}_n^{\vartriangle\triangledown}} \longrightarrow [X|_{\dot{\Delta}_n}S^{+1}]^{-1}|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$$

in $\hat{\mathcal{C}}(\bar{\Delta}_n^{\vartriangle\triangledown})$, and it suffices to show that it splits. Write $C := X|_{\dot{\Delta}_n}S|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$.

It suffices to show that there exists a retraction to $C \longrightarrow X|_{\bar{\Delta}_n^{\Delta\nabla}}$, which we will construct by induction. Suppose given $0 < \alpha < \beta \leqslant 0^{+1}$. We may assume that after restriction of $C \longrightarrow X|_{\bar{\Delta}_n^{\Delta\nabla}}$ to $\{\delta/\gamma \in \bar{\Delta}_n^{\Delta\nabla} \;:\; \delta/\gamma < \beta/\alpha\}$, there exists a retraction. Let $\alpha' := \alpha - 1$ be the predecessor of α, and let $\beta' := \beta - 1$ be the predecessor of β, using that $\bar{\Delta}_n$ is linearly ordered. It suffices to show that the morphism from the quadrangle $(C_{\beta'/\alpha'}, C_{\beta'/\alpha}, C_{\beta/\alpha'}, C_{\beta/\alpha})$ to the quadrangle $(X_{\beta'/\alpha'}, X_{\beta'/\alpha}, X_{\beta/\alpha'}, X_{\beta/\alpha})$ has a retraction.

Let $(X_{\beta'/\alpha'}, X_{\beta'/\alpha}, X_{\beta/\alpha'}, T)$ be the pushout in $\hat{\mathcal{C}}$. The quadrangle $(C_{\beta'/\alpha'}, C_{\beta'/\alpha}, C_{\beta/\alpha'}, C_{\beta/\alpha})$ is a pushout. The induced morphism from $(C_{\beta'/\alpha'}, C_{\beta'/\alpha}, C_{\beta/\alpha'}, C_{\beta/\alpha})$ to $(X_{\beta'/\alpha'}, X_{\beta'/\alpha}, X_{\beta/\alpha'}, T)$ has a retraction by functoriality of the pushout. The morphism $T \longrightarrow X_{\beta/\alpha}$ induced by pushout is a monomorphism in $\hat{\mathcal{C}}$, since $(X_{\beta'/\alpha'}, X_{\beta'/\alpha}, X_{\beta/\alpha'}, X_{\beta/\alpha})$ is a weak square. Note that $(C_{\beta/\alpha}, T, X_{\beta/\alpha})$ is a commutative triangle.

The morphism $T \longrightarrow C_{\beta/\alpha}$ that is part of the retraction of quadrangles, factors as

$$(T \longrightarrow C_{\beta/\alpha}) \;=\; (T \rightarrowtail X_{\beta/\alpha} \longrightarrow C_{\beta/\alpha})\;,$$

since $C_{\beta/\alpha}$ is injective in $\hat{\mathcal{C}}$ as a summand of $C_{\beta/0} = X_{\beta/0}$. Now $X_{\beta/\alpha} \longrightarrow C_{\beta/\alpha}$ completes the three morphisms on the other vertices to a retraction of quadrangles from $(X_{\beta'/\alpha'}, X_{\beta'/\alpha}, X_{\beta/\alpha'}, X_{\beta/\alpha})$ to $(C_{\beta'/\alpha'}, C_{\beta'/\alpha}, C_{\beta/\alpha'}, C_{\beta/\alpha})$ as sought. □

I.3 Verification of Verdier's axioms

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a Heller triangulated category.

I.3.1 Restriction from $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ to $\mathcal{C}(\dot{\Delta}_n)$ is dense and full

Let $n \geqslant 1$. Let

$$\begin{array}{rcl} \mathcal{C}(\dot{\Delta}_n^{+1}) & \xrightarrow{S'^{\nabla}} & \hat{\mathcal{C}}^{+}(\bar{\Delta}_n^{\nabla,+1}) \\ U & \longmapsto & US'^{\nabla} \;:=\; (U^{(+1)}S^{\nabla})^{(-1)} \end{array}$$

be the conjugate by reindexing, i.e. a "shifted version" of S^{∇}; cf. §I.2.1.4. Note that

$$(US'^{\nabla})_{\beta/\alpha} \;=\; \mathrm{Kern}(U_{0^{+1}/\alpha} \xrightarrow{u} U_{0^{+1}/\beta})$$

for $\alpha,\ \beta \in \bar{\Delta}_n$ with $0 \leqslant \alpha \leqslant \beta \leqslant 0^{+1}$.

Lemma I.18 *Suppose that idempotents split in* $\mathcal{C}$. *Given* $X \in \mathrm{Ob}\,\mathcal{C}(\dot{\Delta}_n)$, *there exists an* n*-triangle* $\tilde{X} \in \mathrm{Ob}\,\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ *that restricts to*

$$\tilde{X}|_{\dot{\Delta}_n} \;=\; X\,.$$

In other words, the restriction functor $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ *is strictly dense.*

Proof. Let $Y := XR \in \mathrm{Ob}\,\mathcal{C}^{+}(\bar{\Delta}_n^{\#})$; cf. §I.2.2.2.3. We have an isomorphism $[Y]^{+1} \underset{\sim}{\xrightarrow{Y\vartheta_n}} [Y^{+1}]$ in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$. Let $[Y^{+1}] \xrightarrow{\theta} [Y]^{+1}$ be a representative in $\mathcal{C}^{+}(\bar{\Delta}_n^{\#})$ of the inverse isomorphism $(Y\vartheta_n)^{-}$

in $\underline{\mathcal{C}^+(\bar\Delta_n^\#)}$. Consider the morphism [7]

$$[Y^{+1}]^{-1}|_{\dot\Delta_n^{+1}} \xrightarrow{[\theta]^{-1}|_{\dot\Delta_n^{+1}}} [Y]^{+1-1}|_{\dot\Delta_n^{+1}} = Y|_{\dot\Delta_n^{+1}} .$$

We have an induced pointwise epimorphism

$$Y|_{\bar\Delta_n^{\Delta\nabla}} \twoheadrightarrow Y|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}} ,$$

which we may use to form the pullback

$$\begin{array}{ccc} Z & \twoheadrightarrow & [Y^{+1}]^{-1}|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}} \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle [\theta]^{-1}|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}}} \\ Y|_{\bar\Delta_n^{\Delta\nabla}} & \twoheadrightarrow & Y|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}} \end{array}$$

in the abelian category $\hat{\mathcal{C}}(\bar\Delta_n^{\Delta\nabla})$, i.e. pointwise. An application of Lemma I.61.(2) to the diagonal sequence of this pullback shows that $Z \in \operatorname{Ob}\hat{\mathcal{C}}^{+,*}(\bar\Delta_n^{\Delta\nabla})$. We obtain $Z_{\alpha/\alpha} = 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$; and we obtain $Z_{0^{+1}/0} = 0$. Hence we have $Z \in \operatorname{Ob}\hat{\mathcal{C}}^+(\bar\Delta_n^{\Delta\nabla})$.

Suppose given $\beta/\alpha \in \bar\Delta_n^{\Delta\nabla}$. We *claim* that $Z_{\beta/\alpha} \xrightarrow{f_{\beta/\alpha}} Y_{\beta/\alpha}$ represents an isomorphism in $\hat{\mathcal{C}}/\mathcal{C}$. By Lemma I.64, it suffices to show that

$$([Y^{+1}]^{-1}|_{\dot\Delta_n^{+1}} S'^\nabla)_{\beta/\alpha} \xrightarrow{([\theta]^{-1}|_{\dot\Delta_n^{+1}} S'^\nabla)_{\beta/\alpha}} (Y|_{\dot\Delta_n^{+1}} S'^\nabla)_{\beta/\alpha}$$

represents an isomorphism in $\hat{\mathcal{C}}/\mathcal{C}$. Since evaluation at β/α induces a functor from $\hat{\mathcal{C}}^+(\bar\Delta_n^{\Delta\nabla})/\mathcal{C}^{+,\,\mathrm{split}}(\bar\Delta_n^{\Delta\nabla})$ to $\hat{\mathcal{C}}/\mathcal{C}$, where $\mathcal{C}^{+,\,\mathrm{split}}(\bar\Delta_n^{\Delta\nabla})$ denotes the full subcategory of $\mathcal{C}^+(\bar\Delta_n^{\Delta\nabla})$ consisting of diagrams all of whose morphisms split (in $\hat{\mathcal{C}}$ or, equivalently, in $\mathcal{C}$), it suffices to show that

$$[Y^{+1}]^{-1}|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}} \xrightarrow{[\theta]^{-1}|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}}} Y|_{\dot\Delta_n^{+1}} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}}$$

represents an isomorphism in $\hat{\mathcal{C}}^+(\bar\Delta_n^{\Delta\nabla})/\mathcal{C}^{+,\,\mathrm{split}}(\bar\Delta_n^{\Delta\nabla})$.

Now $(-)^{(-1)} S'^\nabla|_{\bar\Delta_n^{\Delta\nabla}}$ induces a functor from $\underline{\mathcal{C}(\dot\Delta_n)}$ to $\hat{\mathcal{C}}^+(\bar\Delta_n^{\Delta\nabla})/\mathcal{C}^{+,\,\mathrm{split}}(\bar\Delta_n^{\Delta\nabla})$, since it maps $\mathcal{C}^{\mathrm{split}}(\dot\Delta_n)$ to $\mathcal{C}^{+,\,\mathrm{split}}(\bar\Delta_n^{\Delta\nabla})$ by Lemma I.4, using that idempotents are assumed to split in $\mathcal{C}$. Therefore, it suffices to show that

$$([Y^{+1}]^{-1}|_{\dot\Delta_n^{+1}})^{(+1)} \xrightarrow{([\theta]^{-1}|_{\dot\Delta_n^{+1}})^{(+1)}} (Y|_{\dot\Delta_n^{+1}})^{(+1)}$$

represents an isomorphism in $\underline{\mathcal{C}(\dot\Delta_n)}$. Since $([-]^{-1}|_{\dot\Delta_n^{+1}})^{(+1)} = (-)|_{\dot\Delta_n}$, this means that it suffices to show that

$$[Y^{+1}]|_{\dot\Delta_n} \xrightarrow{\theta|_{\dot\Delta_n}} [Y]^{+1}|_{\dot\Delta_n}$$

represents an isomorphism in $\underline{\mathcal{C}(\dot\Delta_n)}$. Since $(-)|_{\dot\Delta_n}$ induces a functor from $\underline{\mathcal{C}^+(\Delta_n^\#)}$ to $\underline{\mathcal{C}(\dot\Delta_n)}$, it suffices to show that

$$[Y^{+1}] \xrightarrow{\theta} [Y]^{+1}$$

represents an isomorphism in $\underline{\mathcal{C}^+(\bar\Delta_n^\#)}$. This, however, follows by choice of θ. This proves the *claim.*

[7] We recall the convention that the inverse of the outer shift applied to a morphism f is written $[f]^{-1}$, whereas f^- denotes the inverse morphism, if existent.

Since idempotents are assumed to split in $\mathcal{C}$, we can conclude from the claim that for all β/α in $\bar{\Delta}_n^{\Delta\nabla}$, the entry $Z_{\beta/\alpha}$ is isomorphic in $\hat{\mathcal{C}}$ to an object in $\mathcal{C}$, hence without loss of generality, the entry $Z_{\beta/\alpha}$ is an object of $\mathcal{C}$. So $Z \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla})$.

We remark that

$$(*) \qquad \begin{array}{rclcl} Z|_{\dot{\Delta}_n} & = & Y|_{\dot{\Delta}_n} & = & X \\ Z|_{\dot{\Delta}_n^{+1}} & = & [Y^{+1}]^{-1}|_{\dot{\Delta}_n^{+1}} & = & (X^{+1})^{(-1)} , \end{array}$$

where X^{+1} arises from X by pointwise application of $(-)^{+1}$. Concerning morphisms, we remark that

$$(**) \qquad \begin{array}{rcl} (Z \xrightarrow{f} Y|_{\bar{\Delta}_n^{\Delta\nabla}})|_{\dot{\Delta}_n} & = & (X \xrightarrow{1_X} X) \\ (Z \xrightarrow{f} Y|_{\bar{\Delta}_n^{\Delta\nabla}})|_{\dot{\Delta}_n^{+1}} & = & ([Y^{+1}]^{-1} \xrightarrow{[\theta]^{-1}} Y)|_{\dot{\Delta}_n^{+1}} . \end{array}$$

In fact, on $\dot{\Delta}_n$, the right hand side column of our pullback vanishes; and on $\dot{\Delta}_n^{+1}$, the lower row of our pullback is an identity.

Now, $(*)$ allows to define the periodic prolongation $\bar{Z} \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\#})$ of $Z \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla})$ by $\bar{Z}|_{\bar{\Delta}_n^{\Delta\nabla}} := Z$ and by the requirement that $[\bar{Z}]^{+1} = [\bar{Z}^{+1}]$.

We *claim* that $\bar{Z}\vartheta_n = 1_{[\bar{Z}]^{+1}}$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. Let $\bar{Z} \xrightarrow{\hat{f}} Y$ be an inverse image of $Z \xrightarrow{f} Y|_{\bar{\Delta}_n^{\Delta\nabla}}$ under $\mathcal{C}^+(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\bar{\Delta}_n^{\Delta\nabla}}} \mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla})$; cf. Lemma I.13.(1). By $(**)$, we get

$$(**') \qquad \begin{array}{rcl} (\bar{Z} \xrightarrow{\hat{f}} Y)|_{\dot{\Delta}_n} & = & (X \xrightarrow{1_X} X) \\ (\bar{Z} \xrightarrow{\hat{f}} Y)|_{\dot{\Delta}_n^{+1}} & = & ([Y^{+1}]^{-1} \xrightarrow{[\theta]^{-1}} Y)|_{\dot{\Delta}_n^{+1}} . \end{array}$$

We consider the commutative quadrangle

$$\begin{array}{ccc} [\bar{Z}]^{+1} & \xrightarrow{\bar{Z}\vartheta_n} & [\bar{Z}^{+1}] \\ {\scriptstyle [\hat{f}]^{+1}}\downarrow & & \downarrow{\scriptstyle [\hat{f}^{+1}]} \\ [Y]^{+1} & \xrightarrow{Y\vartheta_n} & [Y^{+1}] \end{array}$$

in $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. We restrict it to $\dot{\Delta}_n$ to obtain the commutative quadrangle

$$\begin{array}{ccc} [\bar{Z}]^{+1}|_{\dot{\Delta}_n} & \xrightarrow{\bar{Z}\vartheta_n|_{\dot{\Delta}_n}} & [\bar{Z}^{+1}]|_{\dot{\Delta}_n} \\ {\scriptstyle [\hat{f}]^{+1}|_{\dot{\Delta}_n}}\downarrow & & \downarrow{\scriptstyle [\hat{f}^{+1}]|_{\dot{\Delta}_n}} \\ [Y]^{+1}|_{\dot{\Delta}_n} & \xrightarrow{Y\vartheta_n|_{\dot{\Delta}_n}} & [Y^{+1}]|_{\dot{\Delta}_n} \end{array}$$

in $\underline{\mathcal{C}(\dot{\Delta}_n)}$, which, using $(**')$, can be rewritten as

$$\begin{array}{ccc} X^{+1} & \xrightarrow{\bar{Z}\vartheta_n|_{\dot{\Delta}_n}} & X^{+1} \\ {\scriptstyle \theta|_{\dot{\Delta}_n}}\downarrow & & \downarrow{\scriptstyle 1_{X^{+1}}} \\ [Y]^{+1}|_{\dot{\Delta}_n} & \xrightarrow{Y\vartheta_n|_{\dot{\Delta}_n}} & X^{+1} , \end{array}$$

where we did not distinguish in notation between $\theta|_{\dot{\Delta}_n}$ and its residue class in $\underline{\mathcal{C}(\dot{\Delta}_n)}$, etc.

Since $\theta(Y\vartheta_n) = 1_{[Y^{+1}]}$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$, we have $\theta|_{\dot{\Delta}_n}(Y\vartheta_n|_{\dot{\Delta}_n}) = 1_{X^{+1}}$ in $\underline{\mathcal{C}(\dot{\Delta}_n)}$. Thus the last quadrangle shows that $\bar{Z}\vartheta_n|_{\dot{\Delta}_n} = 1_{X^{+1}} = 1_{[\bar{Z}]^{+1}}|_{\dot{\Delta}_n}$ in $\underline{\mathcal{C}(\dot{\Delta}_n)}$ as well. Since $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} \xrightarrow[\sim]{(-)|_{\bar{\Delta}_n^{\Delta\nabla}}} \underline{\mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla})}$ is an equivalence, we conclude that $\bar{Z}\vartheta_n = 1_{[\bar{Z}]^{+1}}$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$; cf. Proposition I.12. This proves the *claim;* i.e. we have shown that $\bar{Z}$ is an n-triangle.

Since $\bar{Z}|_{\dot{\Delta}_n} = X$ by $(*)$, this proves the lemma. □

In the proof of Lemma I.18, we needed the assumption that idempotents split in $\mathcal{C}$ in the equivalent form that the residue class functor $\hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}/\mathcal{C}$ maps precisely the objects isomorphic to objects of $\mathcal{C}$ to zero – just as HELLER did at that point.

Lemma I.19 *Given n-triangles X and Y and a morphism*

$$X|_{\dot{\Delta}_n} \xrightarrow{f} Y|_{\dot{\Delta}_n}$$

in $\mathcal{C}(\dot{\Delta}_n)$, there exists a morphism $X \xrightarrow{\tilde{f}} Y$ of n-triangles such that $\tilde{f}|_{\dot{\Delta}_n} = f$. In other words, the restriction functor $\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^\#) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ is full.

Proof. Since the restriction functor $\mathcal{C}^+(\bar{\Delta}_n^\#) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ is full by Proposition I.11, we find a morphism $X \xrightarrow{g} Y$ in $\mathcal{C}^+(\bar{\Delta}_n^\#)$ such that $g|_{\dot{\Delta}_n} = f$.

Let $\underline{g}$ denote the residue class of g in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$. Since ϑ_n is a transformation, we have $[\underline{g}]^{+1}(Y\vartheta_n) = (X\vartheta_n)[\underline{g}^{+1}]$. Since X and Y are n-triangles, both $X\vartheta_n$ and $Y\vartheta_n$ are identities, and this equality amounts to $[\underline{g}]^{+1} = [\underline{g}^{+1}]$, i.e. the difference $[g]^{+1} - [g^{+1}]$ factors over an object of $\mathcal{C}^{+,\mathrm{split}}(\bar{\Delta}_n^\#)$. Restricting to $\dot{\Delta}_n$, the difference

$$([g]^{+1} - [g^{+1}])|_{\dot{\Delta}_n} \;=\; (g|_{\dot{\Delta}_n^{+1}})^{(+1)} - f^{+1}$$

factors over an object of $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$. Therefore, $g|_{\dot{\Delta}_n^{+1}} - (f^{+1})^{(-1)}$ factors over an object Z of $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n^{+1})$, say, as

$$\Big(X|_{\dot{\Delta}_n^{+1}} \xrightarrow{g|_{\dot{\Delta}_n^{+1}} - (f^{+1})^{(-1)}} Y|_{\dot{\Delta}_n^{+1}}\Big) \;=\; \Big(X|_{\dot{\Delta}_n^{+1}} \xrightarrow{a} Z \xrightarrow{b} Y|_{\dot{\Delta}_n^{+1}}\Big)\,.$$

By periodic continuation, it suffices to find a morphism $X|_{\bar{\Delta}_n^{\Delta\nabla}} \xrightarrow{\check{f}} Y|_{\bar{\Delta}_n^{\Delta\nabla}}$ in $\mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla})$ such that $\check{f}|_{\dot{\Delta}_n} = f$ and such that $\check{f}|_{\dot{\Delta}_n^{+1}} = (f^{+1})^{(-1)}$. I.e. we have to find a morphism $X|_{\bar{\Delta}_n^{\Delta\nabla}} \xrightarrow{h} Y|_{\bar{\Delta}_n^{\Delta\nabla}}$ such that $h|_{\dot{\Delta}_n} = 0$ and such that $h|_{\dot{\Delta}_n^{+1}} = ab$, for then we may take $\check{f} := g|_{\bar{\Delta}_n^{\Delta\nabla}} - h$.

Note that $(ZS'^{\nabla}|_{\bar{\Delta}_n^{\Delta\nabla}})|_{\dot{\Delta}_n} = 0$. Note that $ZS'^{\nabla}|_{\bar{\Delta}_n^{\Delta\nabla}}$ is in $\mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla})$, hence in $\mathcal{C}^{+,\mathrm{split}}(\bar{\Delta}_n^{\Delta\nabla})$ by Lemma I.4.

Since $S'^{\nabla}|_{\bar{\Delta}_n^{\Delta\nabla}}$ is right adjoint to restriction to $\dot{\Delta}_n^{+1}$, we have a morphism $X|_{\bar{\Delta}_n^{\Delta\nabla}} \xrightarrow{a'} ZS'^{\nabla}|_{\bar{\Delta}_n^{\Delta\nabla}}$ such that $a'|_{\dot{\Delta}_n^{+1}} = a$.

Since $\mathcal{C}^+(\bar{\Delta}_n^{\Delta\nabla}) \xrightarrow{(-)|_{\dot{\Delta}_n^{+1}}} \mathcal{C}(\dot{\Delta}_n^{+1})$ is full by the dual and shifted assertion of Lemma I.13.(1), there is a morphism $ZS'^{\nabla}|_{\bar{\Delta}_n^{\Delta\nabla}} \xrightarrow{b'} Y|_{\bar{\Delta}_n^{\Delta\nabla}}$ such that $b'|_{\dot{\Delta}_n^{+1}} = b$.

We may take $h := a'b'$. □

In Lemmata I.18 and I.19, we do not claim the existence of a coretraction from $\mathcal{C}(\dot{\Delta}_n)$ to $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ to restriction. The construction made in the proof of Lemma I.19 involves e.g. a choice of a lift b' of b. Cf. [**57,** II.1.2.13].

The fullness used in the proof of Lemma I.19 to lift b, can also be used to lift a. We have used the direct argument and thus seen that the lift a' of a does not involve a choice.

Remark I.20 Suppose that idempotents split in $\mathcal{C}$. By Lemmata I.18 and I.19, the restriction functor

$$\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$$

is full and strictly dense. By Proposition I.12, the restriction functor

$$\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})} \xrightarrow[\sim]{(-)|_{\dot{\Delta}_n}} \underline{\mathcal{C}(\dot{\Delta}_n)}$$

is an equivalence. Denoting by $\underline{\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})}$ the image of $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$, we obtain a full and strictly dense functor $\underline{\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})} \xrightarrow{(-)|_{\dot{\Delta}_n}} \underline{\mathcal{C}(\dot{\Delta}_n)}$. Since it factors as a faithful embedding $\underline{\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})} \hookrightarrow \underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$ followed by an equivalence, it is also faithful. We end up with equivalences

$$\underline{\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})} \xrightarrow[\sim]{(-)|_{\dot{\Delta}_n}} \underline{\mathcal{C}(\dot{\Delta}_n)}\;, \qquad \underline{\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})} \xrightarrow{\sim} \underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}\;.$$

I.3.2 An omnibus lemma

Suppose given $n,\, m \geqslant 1$. Concerning the category $\mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ of periodic n-pretriangles and its full subcategory $\mathcal{C}^{+,\,\text{split, periodic}}(\bar{\Delta}_n^{\#})$, cf. §I.2.5.3; concerning the category $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ of n-triangles, cf. Definition I.5.(ii). Note that

$$\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#}) \;\subseteq\; \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#}) \;\subseteq\; \mathcal{C}^{+}(\bar{\Delta}_n^{\#})\;,$$

and that the first inclusion is full.

Lemma I.21

(1) *Let X be an n-triangle, and let $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$ be a morphism of periodic linearly ordered sets. Then $Xp^{\#}$, obtained by "restriction along p", is an m-triangle.*

(2) *Let X be a $(2n+1)$-triangle. Then $X\mathfrak{f}_n$, obtained by folding, is an $(n+1)$-triangle.*

(3) *The category $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ of n-triangles is a full additive subcategory of the category $\mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ of periodic n-pretriangles, closed under direct summands.*

(4) *Suppose given an isomorphism $X \xrightarrow{f} Y$ in $\mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$. If X is an n-triangle, then Y is an n-triangle.*

(5) *Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ such that $f|_{\dot{\Delta}_n}$ is an isomorphism. Then f is an isomorphism.*

(6) *Let X and Y be n-triangles. Suppose given an isomorphism $X|_{\dot{\Delta}_n} \xrightarrow[\sim]{u} Y|_{\dot{\Delta}_n}$ in $\mathcal{C}(\dot{\Delta}_n)$. Then there exists an isomorphism $X \xrightarrow[\sim]{\tilde{u}} Y$ in $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ such that $\tilde{u}|_{\mathcal{C}(\dot{\Delta}_n)} = u$.*

(7) *If* $X \in \operatorname{Ob} \mathcal{C}^{+,\,\mathrm{split,\,periodic}}(\bar{\Delta}_n^{\#})$*, then* X *is an* n*-triangle.*

Note that Lemma I.21.(5) applies in particular to n-triangles and a morphism of n-triangles.

Proof. Ad (1). In $\underline{\mathcal{C}^{+}(\bar{\Delta}_m^{\#})}$, we have

$$(X\underline{p}^{\#})\vartheta_m \;=\; (X\vartheta_n)\underline{p}^{\#} \;=\; (1_{[X]^{+1}})\underline{p}^{\#} \;=\; 1_{[X\underline{p}^{\#}]^{+1}}\;.$$

Ad (2). In $\underline{\mathcal{C}^{+}(\bar{\Delta}_{n+1}^{\#})}$, we have

$$(X\underline{\mathfrak{f}}_n)\vartheta_{n+1} \;=\; (X\vartheta_{2n+1})\underline{\mathfrak{f}}_n \;=\; (1_{[X]^{+1}})\underline{\mathfrak{f}}_n \;=\; 1_{[X\underline{\mathfrak{f}}_n]^{+1}}\;.$$

Ad (3). We have to show that

$$X,\,Y \;\in\; \operatorname{Ob} \mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#}) \qquad\Longleftrightarrow\qquad X\oplus Y \;\in\; \operatorname{Ob} \mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})\;\;.$$

But since ϑ_n is a morphism between additive functors, we have $(X\oplus Y)\vartheta_n = 1_{[X\oplus Y]^{+1}}$ if and only if $X\vartheta_n = 1_{[X]^{+1}}$ and $Y\vartheta_n = 1_{[Y]^{+1}}$. In fact, $(X\oplus Y)\vartheta_n$ identifies with $\left(\begin{smallmatrix} X\vartheta_n & 0 \\ 0 & Y\vartheta_n \end{smallmatrix}\right)$.

Ad (4). Since $f|_{\dot{\Delta}_n}$ is an isomorphism in $\mathcal{C}(\dot{\Delta}_n)$, so is its image in $\underline{\mathcal{C}(\dot{\Delta}_n)}$. Hence the image of f in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$ is an isomorphism by Proposition I.12. Consider the commutative quadrangle

$$\begin{array}{ccc} [X]^{+1} & \xrightarrow[\sim]{[f]^{+1}} & [Y]^{+1} \\ {\scriptstyle X\vartheta_n}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle Y\vartheta_n} \\ [X^{+1}] & \xrightarrow[\sim]{[f^{+1}]} & [Y^{+1}] \end{array}$$

in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$. Since $[f]^{+1} = [f^{+1}]$ by assumption, we conclude from $X\vartheta_n = 1_{[X]^{+1}}$ that $Y\vartheta_n = 1_{[Y]^{+1}}$.

Ad (5). It suffices to show that given $0 \leqslant i \leqslant j \leqslant n$, the morphism $f_{j/i}$ is an isomorphism in $\mathcal{C}$. In fact, we have a morphism of exact sequences

$$(\; f_{i/0},\; f_{j/0},\; f_{j/i},\; f_{i/0}^{+1},\; f_{j/0}^{+1}\;)$$

in $\hat{\mathcal{C}}$, whose entries except possibly $f_{j/i}$ are isomorphisms; hence also $f_{j/i}$ is isomorphic.

Ad (6). This follows by Lemma I.19 using (5).

Ad (7). We have $[X]^{+1} \simeq 0$ in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$, whence $X\vartheta_n = 1_{[X]^{+1}} \in {}_{\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}}([X]^{+1}, [X]^{+1})$. ▫

I.3.3 Turning n-triangles

Let $n \geqslant 2$.

Lemma I.22 *Given an* n*-triangle* $X \in \operatorname{Ob} \mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$*, we define* $Y \in \operatorname{Ob} \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ *by letting*

$$(Y_{j/i} \xrightarrow{\;y\;} Y_{j'/i'}) \;:=\; (X_{i^{+1}/j} \xrightarrow{\;x\;} X_{i'^{+1}/j'})$$

for $0 \leqslant i \leqslant j \leqslant n$ *and* $0 \leqslant i' \leqslant j' \leqslant n$ *such that* $i \leqslant i'$ *and* $j \leqslant j'$*, and by letting*

$$(Y_{n/i} \xrightarrow{\;y\;} Y_{0^{+1}/i}) \;:=\; (X_{i^{+1}/n} \xrightarrow{\;-x\;} X_{i^{+1}/0^{+1}})$$

for $0 \leqslant i \leqslant n$*. Then* $[X]^{+1}_{-} := Y$ *is an* n*-triangle.*

Proof. Let

$$\begin{array}{ccc} 2\bar{\Delta}_{n-1} & \xrightarrow{h_n} & \bar{\Delta}_n \\ i^{+j} & \longmapsto & \begin{cases} (i+1)^{+j/2} & \text{if } j \equiv_2 0 \\ 0^{+(j+1)/2} & \text{if } j \equiv_2 1\,, \end{cases} \end{array}$$

where $i \in [0, n-1]$ and $j \in \mathbf{Z}$. The map h_n is a morphism of periodic posets. We claim that

$$Y \;=\; X h_n^{\#} \mathfrak{f}_{n-1}\,.$$

Once this claim is shown, we are done by Lemma I.21.(1, 2).

Note that $(Xh_n^{\#})_{l/k} = X_{lh_n/kh_n}$ for $k,\, l \in 2\bar{\Delta}_{n-1}$ with $k \leqslant l$. For $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 0^{+1}$, we obtain

$$\begin{aligned} (Xh_n^{\#}\mathfrak{f}_{n-1})_{j/i} \;&=\; \begin{cases} (Xh_n^{\#})_{(j-1)^{+1}/(j-1)} & \text{for } i = 0 \text{ and } 1 \leqslant j \leqslant n \\ (Xh_n^{\#})_{(i-1)^{+2}/(i-1)^{+1}} & \text{for } 1 \leqslant i \leqslant n \text{ and } j = 0^{+1} \\ (Xh_n^{\#})_{(j-1)^{+1}/(i-1)^{+1}} \oplus (Xh_n^{\#})_{(i-1)^{+2}/(j-1)} & \text{for } 1 \leqslant i \leqslant j \leqslant n \\ 0 & \text{for } i = 0 \text{ and } j = 0^{+1} \end{cases} \\ &=\; \begin{cases} X_{0^{+1}/j} & \text{for } i = 0 \text{ and } 1 \leqslant j \leqslant n \\ X_{i^{+1}/0^{+1}} & \text{for } 1 \leqslant i \leqslant n \text{ and } j = 0^{+1} \\ X_{i^{+1}/j} & \text{for } 1 \leqslant i \leqslant j \leqslant n \\ 0 & \text{for } i = 0 \text{ and } j = 0^{+1}\,, \end{cases} \end{aligned}$$

and also the morphisms result as claimed. □

I.3.4 Application to the axioms of Verdier

Recall that $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a Heller triangulated category.

Proposition I.23 *Suppose that idempotents split in $\mathcal{C}$. The tuple $(\mathcal{C}, \mathsf{T})$, equipped with the set of 2-triangles as the set of distinguished triangles, is a triangulated category in the sense of Verdier* [**56,** Def. 1-1].

Proof. We number the axioms of Verdier as in loc. cit.

Ad (TR 1). Stability under isomorphism of the set of distinguished triangles follows from Lemma I.21.(4).

The possible extension of a morphism to a distinguished triangle follows by Lemma I.18.

The distinguished triangle $(X, X, 0)$ on the identity of an object X in $\mathcal{C}$ follows by Lemma I.21.(7). Alternatively, one can use that each morphism is contained in a distinguished triangle and the fact that a distinguished triangle is a long exact sequence in $\hat{\mathcal{C}}$.

Ad (TR 2). Suppose given a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X^{+1}\,.$$

By Lemma I.22, we obtain the distinguished triangle

$$X^{+1} \xrightarrow{u^{+1}} Y^{+1} \xrightarrow{v^{+1}} Z^{+1} \xrightarrow{-w^{+1}} X^{+2} .$$

By Lemma I.21.(1), applied to the morphism $\bar{\Delta}_2 \longleftarrow \bar{\Delta}_2$ that sends 0 to 2^{-1}, 1 to 0 and 2 to 1, we obtain the distinguished triangle

$$Y \xrightarrow{v} Z \xrightarrow{-w} X^{+1} \xrightarrow{u^{+1}} Y^{+1} .$$

By Lemma I.21.(4), we obtain the distinguished triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} X^{+1} \xrightarrow{-u^{+1}} Y^{+1} .$$

Ad (TR 3). The possible completion of a morphism in $\mathcal{C}(\dot{\Delta}_2)$ to a morphism of distinguished triangles follows from Lemma I.19.

Ad (TR 4). The octahedral axiom, i.e. the compatibility of forming cones with composition of morphisms, follows from Lemma I.18, applied to the case $n = 3$, for by Lemma I.21.(6), we may arbitrarily choose completions to distinguished triangles. ▫

Note that 3-triangles are particular octahedra, in the language of [**8,** 1.1.6]. Using 3-triangles, we will now verify the axiom proposed in [**8,** 1.1.13].

Lemma I.24 *Suppose given a 3-triangle T in* $\operatorname{Ob} \mathcal{C}^{+,\vartheta=1}((2\bar{\Delta}_1)^{\#})$*, depicted as follows.*

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \uparrow \\
 & & & & & & 0 & \longrightarrow & Z^{+1} \\
 & & & & & & \uparrow & + & \uparrow {\scriptstyle v^{+1}} \\
 & & & & 0 & \longrightarrow & Z'' & \xrightarrow{w''} & Y^{+1} \\
 & & & & \uparrow & + & \uparrow {\scriptstyle z'} & + & \uparrow {\scriptstyle u^{+1}} \\
 & & 0 & \longrightarrow & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X^{+1} \\
 & & \uparrow & + & \uparrow {\scriptstyle y} & + & \uparrow {\scriptstyle z} & + & \uparrow \\
0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0
\end{array}$$

Then

$$T\mathfrak{f}_1 \;=\; (\,Y \xrightarrow{vz} Z' \xrightarrow{(z'\; w')} Z'' \oplus X^{+1} \xrightarrow{\left(\begin{smallmatrix} w'' \\ -u^{+1} \end{smallmatrix}\right)} Y^{+1}\,)$$

and

$$Ts^{\#}\mathfrak{f}_1 \;=\; (Z'^{-1} \xrightarrow{w'^{-1}u} Y \xrightarrow{(y\; v)} Y' \oplus Z \xrightarrow{\left(\begin{smallmatrix} v' \\ -z \end{smallmatrix}\right)} Z'\,)$$

are distinguished triangles, where $2\bar{\Delta}_1 \xleftarrow{s} 2\bar{\Delta}_1$ *is the morphism of periodic posets determined by* $0s = 1^{-1}$, $1s = 0$, $0^{+1}s = 1$ *and* $1^{+1}s = 0^{+1}$.

Proof. This follows by Lemma I.21.(1, 2). ▫

I.3.5 n-triangles and strictly exact functors

Let $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ be a strictly exact functor between Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$. Let $n \geqslant 0$.

Lemma I.25 *Given an n-triangle $X \in \operatorname{Ob} \mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^{\#})$, the diagram $X(F^+(\bar{\Delta}_n^{\#}))$, obtained by pointwise application of F to X, is an n-triangle, i.e. an object of $\mathcal{C}'^{+,\vartheta'=1}(\bar{\Delta}_n^{\#})$.*

Proof. Using $F\,\mathsf{T}' = \mathsf{T}\,F$ as well as $[X]^{+1} = [X^{+1}]$, we obtain

$$\begin{array}{rclcl} [XF^+(\bar{\Delta}_n^{\#})]^{+1} & = & X(F^+(\bar{\Delta}_n^{\#}))(\mathcal{C}^+(\mathsf{T}_n)) & = & X(F^+(\mathsf{T}_n)) \\ & = & X(\mathcal{C}^+(\mathsf{T}_n))(F^+(\bar{\Delta}_n^{\#})) & = & [X]^{+1}(F^+(\bar{\Delta}_n^{\#})) \\ & = & [X^{+1}](F^+(\bar{\Delta}_n^{\#})) & = & X(\mathsf{T}^+(\bar{\Delta}_n^{\#}))(F^+(\bar{\Delta}_n^{\#})) \\ & = & X((\mathsf{T}\,F)^+(\bar{\Delta}_n^{\#})) & = & X((F\,\mathsf{T}')^+(\bar{\Delta}_n^{\#})) \\ & = & X(F^+(\bar{\Delta}_n^{\#}))(\mathsf{T}'^+(\bar{\Delta}_n^{\#})) & = & [(X(F^+(\bar{\Delta}_n^{\#})))^{+1}]\,. \end{array}$$

Moreover,

$$X(F^+(\bar{\Delta}_n^{\#}))\vartheta'_n \;=\; X\vartheta_n(F^+(\bar{\Delta}_n^{\#})) \;=\; 1_{[X]^{+1}}(F^+(\bar{\Delta}_n^{\#})) \;=\; 1_{[XF^+(\bar{\Delta}_n^{\#})]^{+1}}\,.$$

□

I.3.6 A remark on spectral sequences

Verdier calls certain pretriangles *objets spectraux* (spectral objects); cf. [**57,** Sec. II.4]. We shall explain the connection to spectral sequences in our language.

Consider the linearly ordered set $\mathbf{Z}_\infty := \{-\infty\} \sqcup \mathbf{Z} \sqcup \{+\infty\}$. Let $\bar{\mathbf{Z}}_\infty^{\#\#}$ be the subposet of $\bar{\mathbf{Z}}_\infty^{\#}(\Delta_1)$ consisting of those $\delta/\beta /\!\!/ \gamma/\alpha$ for which

$$\delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1}\,,$$

where $\alpha, \beta, \gamma, \delta \in \bar{\mathbf{Z}}_\infty$. A *spectral object,* in a slightly different sense from [**57,** II.4.1.2], is an object of $\mathcal{C}^+(\bar{\mathbf{Z}}_\infty^{\#})$. The *spectral sequence functor*

$$\begin{array}{ccc} \mathcal{C}^+(\bar{\mathbf{Z}}_\infty^{\#}) & \xrightarrow{\mathrm{E}} & \hat{\mathcal{C}}(\bar{\mathbf{Z}}_\infty^{\#\#}) \\ X & \longmapsto & X\mathrm{E}\,, \end{array}$$

is defined by

$$X\mathrm{E}(\delta/\beta /\!\!/ \gamma/\alpha) \;:=\; \mathrm{Im}(X_{\gamma/\alpha} \longrightarrow X_{\delta/\beta})$$

for $\delta/\beta /\!\!/ \gamma/\alpha \in \bar{\mathbf{Z}}_\infty^{\#\#}$, equipped with the induced morphisms.

Lemma I.26 *Given $\alpha, \beta, \gamma, \delta, \varepsilon \in \bar{\mathbf{Z}}_\infty$ such that*

$$\varepsilon^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \varepsilon \leqslant \alpha^{+1}\,,$$

and given $X \in \operatorname{Ob} \mathcal{C}^+(\bar{\mathbf{Z}}_\infty^{\#})$, the morphisms appearing in $X\mathrm{E}$ form a short exact sequence

$$X\mathrm{E}(\varepsilon/\beta /\!\!/ \gamma/\alpha) \rightarrowtail X\mathrm{E}(\varepsilon/\beta /\!\!/ \delta/\alpha) \twoheadrightarrow X\mathrm{E}(\varepsilon/\gamma /\!\!/ \delta/\alpha)\,.$$

Proof. This follows by Lemma I.62, applied to the diagram

$$(X_{\gamma/\alpha} \, , \, X_{\delta/\alpha} \, , \, X_{\varepsilon/\alpha} \, , \, X_{\gamma/\beta} \, , \, X_{\delta/\beta} \, , \, X_{\varepsilon/\beta} \, , \, \underbrace{X_{\gamma/\gamma}}_{=0} , \, X_{\delta/\gamma} \, , \, X_{\varepsilon/\gamma} \,) \; .$$

□

Note that we may apply a shift $\beta/\alpha \longmapsto \alpha^{+1}/\beta$ to the indices, i.e. an outer shift to X, before applying Lemma I.26, to get another short exact sequence.

The usual exact sequences of spectral sequence terms can be derived from Lemma I.26. Cf. [**57,** II.4.2.6], [**12,** App.].

I.4 The stable category of a Frobenius category is Heller triangulated

Let $\mathcal{F} = (\mathcal{F}, \mathsf{T}, \mathsf{I}, \iota, \mathsf{P}, \pi)$ be a functorial Frobenius category; cf. Definition I.45.(3). Let $\mathcal{B} \subseteq \mathcal{F}$ denote the full subcategory of objects in the image of I, coinciding with the full subcategory of the objects in the image of P; then $\mathcal{B}$ is a sufficiently big full subcategory of bijectives in $\mathcal{F}$.

We shall prove in Theorem I.32 below that the classical stable category $\underline{\mathcal{F}}$ carries a Heller triangulation.

I.4.1 Definition of $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#})$, modelling $\underline{\mathcal{F}^{+}(\bar{\Delta}_n^{\#})}$

We shall model, in the sense of Proposition I.31 below, the category $\underline{\mathcal{F}}^{+}(\bar{\Delta}_n^{\#})$ by a category $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#})$. Morally, we represent weak squares (+) in $\underline{\mathcal{F}}$ by pure squares (□) in $\mathcal{F}$. To do so, we have to represent the zeroes on the boundaries by bijective objects.

Let $n \geqslant 0$. Concerning the notion of a pure square, see §I.6.4. Let $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) \subseteq \mathcal{F}(\bar{\Delta}_n^{\#})$ be the full subcategory defined by

$$\operatorname{Ob}\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) := \left\{ X \in \operatorname{Ob}\mathcal{F}(\bar{\Delta}_n^{\#}) \quad : \quad \begin{array}{ll} 1) & X_{\alpha/\alpha} \text{ and } X_{\alpha^{+1}/\alpha} \text{ are in } \operatorname{Ob}\mathcal{B} \text{ for all } \alpha \in \bar{\Delta}_n \\ 2) & \text{For all } \delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1} \text{ in } \bar{\Delta}_n \,, \\ & \text{the quadrangle} \\ & \begin{array}{ccc} X_{\gamma/\beta} & \xrightarrow{x} & X_{\delta/\beta} \\ {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle x} \\ X_{\gamma/\alpha} & \xrightarrow[x]{} & X_{\delta/\alpha} \end{array} \\ & \text{is a pure square.} \end{array} \right\} .$$

Given $n, m \geqslant 0$, a morphism $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$ induces a morphism $\mathcal{F}^{\square}(p^{\#})$, usually, and by abuse of notation, denoted by $p^{\#}$.

Given an exact functor $\mathcal{F} \xrightarrow{F} \tilde{\mathcal{F}}$ between functorial Frobenius categories that sends bijectives to bijectives, we obtain an induced functor $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) \xrightarrow{F^{\square}(\bar{\Delta}_n^{\#})} \tilde{\mathcal{F}}^{\square}(\bar{\Delta}_n^{\#})$ by pointwise application of F.

Denote by

$$\begin{array}{rcl} \mathcal{F}^{\Box}(\bar\Delta_n^{\#}) & \xrightarrow{M} & \underline{\underline{\mathcal{F}^{+}(\bar\Delta_n^{\#})}} \\ \mathcal{F}^{\Box}(\bar\Delta_n^{\#}) & \xrightarrow{M'} & \underline{\mathcal{F}}^{+}(\bar\Delta_n^{\#}) \\ \underline{\mathcal{F}}^{+}(\bar\Delta_n^{\#}) & \xrightarrow{M''} & \underline{\underline{\mathcal{F}^{+}(\bar\Delta_n^{\#})}} \\ \mathcal{F} & \xrightarrow{N} & \underline{\mathcal{F}} \\ \underline{\mathcal{F}}(\dot\Delta_n) & \xrightarrow{N'} & \underline{\underline{\mathcal{F}}(\dot\Delta_n)} \end{array}$$

the respective residue class functors, welldefined by Lemma I.71. In particular, $M = M'M''$.

I.4.2 Folding for $\mathcal{F}^{\Box}(\bar\Delta_n^{\#})$

We model, in the sense of Remark I.27, the folding operation $\mathfrak{f}$ introduced in §I.1.2.2.

Suppose given $n \geqslant 0$. Let the periodic functor

$$\begin{array}{rcl} \mathcal{F}^{\Box}((2\bar\Delta_n)^{\#}) & \xrightarrow{\tilde{\mathfrak{f}}_n} & \mathcal{F}^{\Box}(\overline{\rho \sqcup \Delta_n}^{\#}) \\ X & \longmapsto & X\tilde{\mathfrak{f}}_n \end{array}$$

be determined by the following data. Writing $Y := X\tilde{\mathfrak{f}}_n$, we let

$$\begin{array}{rcl} (Y_{\alpha/\rho} \xrightarrow{y} Y_{\beta/\rho}) & := & \Big(X_{\alpha^{+1}/\alpha} \xrightarrow{x} X_{\beta^{+1}/\beta}\Big) \\ Y_{\rho/\rho} & := & 0 \\ Y_{\rho^{+1}/\rho} & := & 0 \\ (Y_{\beta/\rho} \xrightarrow{y} Y_{\beta/\alpha}) & := & \Big(X_{\beta^{+1}/\beta} \xrightarrow{(x\ x)} X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta}\Big) \\ (Y_{\beta/\alpha} \xrightarrow{y} Y_{\delta/\gamma}) & := & \Big(X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta} \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)} X_{\delta^{+1}/\gamma^{+1}} \oplus X_{\gamma^{+2}/\delta}\Big) \\ (Y_{\delta/\gamma} \xrightarrow{y} Y_{\rho^{+1}/\gamma}) & := & \Big(X_{\delta^{+1}/\gamma^{+1}} \oplus X_{\gamma^{+2}/\delta} \xrightarrow{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} X_{\gamma^{+2}/\gamma^{+1}}\Big) \end{array}$$

for $\alpha,\ \beta,\ \gamma,\ \delta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$, with $\gamma \leqslant \delta$ and with $\beta/\alpha \leqslant \delta/\gamma$. The remaining morphisms are given by composition.

We claim that $X\tilde{\mathfrak{f}}_n$ is an object of $\mathcal{F}^{\Box}(\overline{\rho \sqcup \Delta_n}^{\#})$.

In fact, by Lemma I.52, we are reduced to considering the quadrangles of Y inside $\Delta_n^{\Delta\nabla}$, i.e. the quadrangles

(i) on $(\gamma/\alpha,\ \delta/\alpha,\ \gamma/\beta,\ \delta/\beta)$ for $\alpha,\ \beta,\ \gamma,\ \delta\ \in\ \Delta_n$ with $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta$;

(ii) on $(\gamma/\rho,\ \delta/\rho,\ \gamma/\beta,\ \delta/\beta)$ for $\beta,\ \gamma,\ \delta\ \in\ \Delta_n$ with $\beta \leqslant \gamma \leqslant \delta$;

(iii) on $(\gamma/\alpha,\ \rho^{+1}/\alpha,\ \gamma/\beta,\ \rho^{+1}/\beta)$ for $\alpha,\ \beta,\ \gamma\ \in\ \Delta_n$ with $\alpha \leqslant \beta \leqslant \gamma$;

(iv) and on $(\beta/\rho,\ \rho^{+1}/\rho,\ \beta/\alpha,\ \rho^{+1}/\alpha)$ for $\alpha,\ \beta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$.

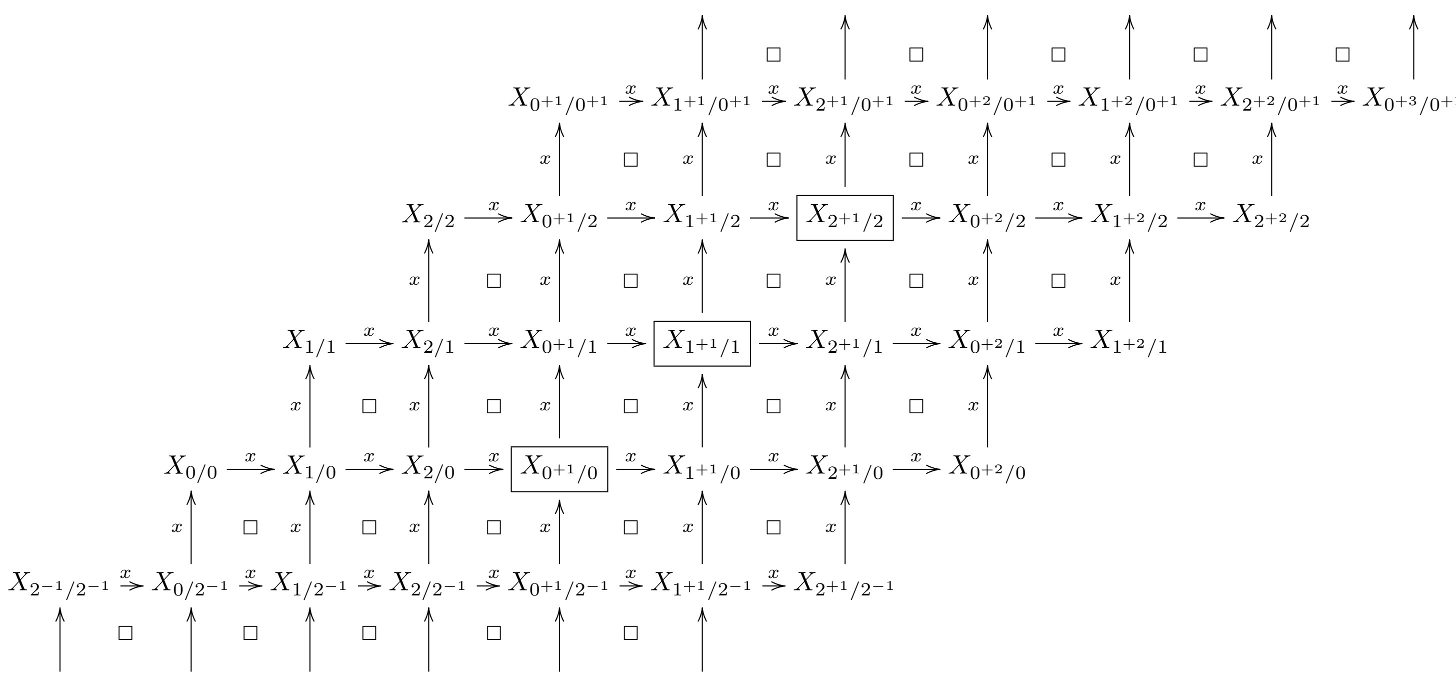
Another application of loc. cit. reduces case (i) to case (ii) (or (iii)). Still another application of loc. cit. reduces the cases (ii) and (iii) to case (iv). Now the quadrangle in case (iv) is in fact a pure square, as follows from $X \in \operatorname{Ob} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})$ and the definition of a pure square via its pure short exact diagonal sequence.

The construction of Y is functorial in X.

Remark I.27 We have $\tilde{\mathfrak{f}}_n M' = M' \mathfrak{f}_n$, and thus $\tilde{\mathfrak{f}}_n M = M \underline{\mathfrak{f}}_n$ for $n \geqslant 0$.

$$\begin{array}{ccc}
\mathcal{F}^{\Box}((2\bar{\Delta}_n)^{\#}) & \xrightarrow{\tilde{\mathfrak{f}}_n} & \mathcal{F}^{\Box}(\overline{\rho \sqcup \Delta_n}^{\#}) \\
\Big\downarrow {\scriptstyle M'} & & \Big\downarrow {\scriptstyle M'} \\
\underline{\mathcal{F}}^{+}((2\bar{\Delta}_n)^{\#}) & \xrightarrow{\mathfrak{f}_n} & \underline{\mathcal{F}}^{+}(\overline{\rho \sqcup \Delta_n}^{\#}) \\
\Big\downarrow {\scriptstyle M''} & & \Big\downarrow {\scriptstyle M''} \\
\underline{\underline{\mathcal{F}^{+}((2\bar{\Delta}_n)^{\#})}} & \xrightarrow{\underline{\mathfrak{f}}_n} & \underline{\underline{\mathcal{F}^{+}(\overline{\rho \sqcup \Delta_n}^{\#})}}
\end{array}$$

Example I.28 Let $n = 2$. Note that $2\bar{\Delta}_2 \simeq \bar{\Delta}_5$. Let $X \in \operatorname{Ob} \mathcal{F}^{\Box}((2\bar{\Delta}_2)^{\#})$, depicted as follows.

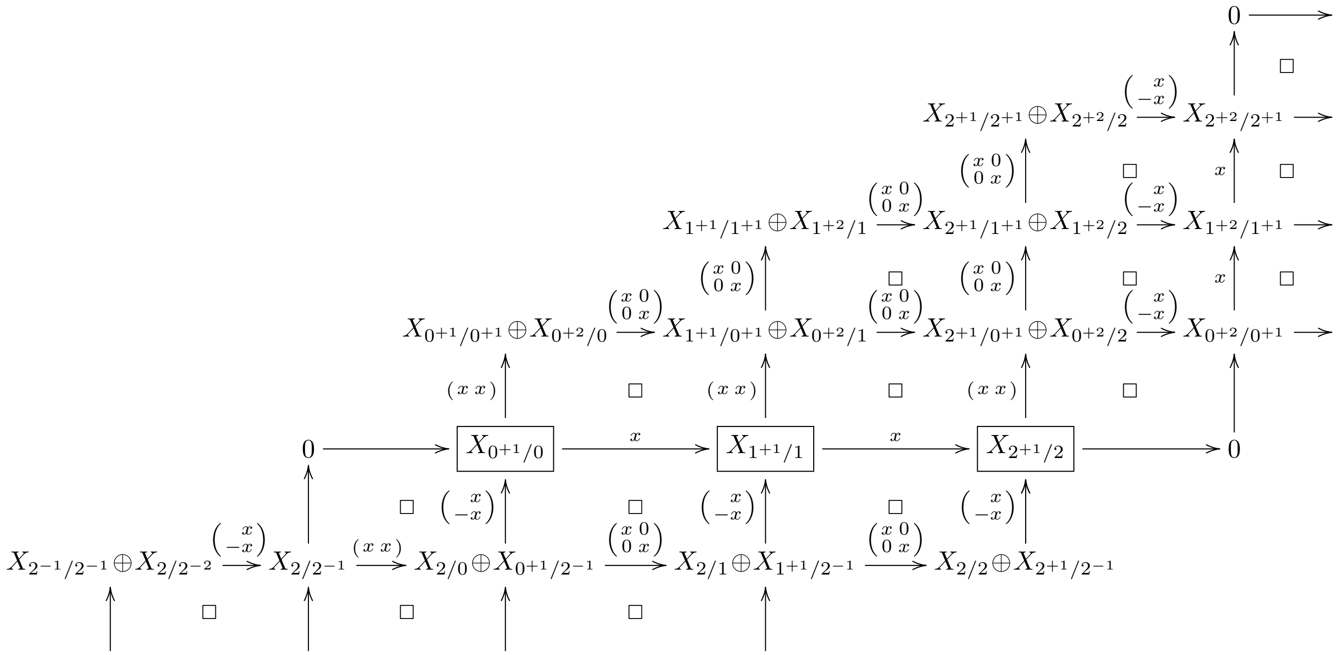


Note that the objects on the boundary of the diagram,

$$\begin{array}{c}
\ldots\,,\ X_{2^{-1}/2^{-1}}\,,\ X_{0/0}\,,\ X_{1/1}\,,\ X_{2/2}\,,\ X_{0^{+1}/0^{+1}}\,,\ \ldots \\
\ldots\,,\ X_{2^{+1}/2^{-1}}\,,\ X_{0^{+2}/0}\,,\ X_{1^{+2}/1}\,,\ X_{2^{+2}/2}\,,\ X_{0^{+3}/0^{+1}}\,,\ \ldots
\end{array}$$

are all supposed to be in $\operatorname{Ob} \mathcal{B}$.

Note that $\rho \sqcup \Delta_2 \simeq \Delta_3$. Folding turns X into $X\tilde{\mathfrak{f}}_2 \in \operatorname{Ob} \mathcal{F}^{\Box}(\overline{\rho \sqcup \Delta_2}^{\#})$, depicted as follows.



I.4.3 Some 1-epimorphic functors

Let $n \geqslant 0$. Concerning 1-epimorphy, cf. §I.6.8.

Lemma I.29 *The restriction functor*

$$\begin{array}{ccc} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#}) & \xrightarrow{(-)|_{\dot{\Delta}_n}} & \mathcal{F}(\dot{\Delta}_n) \\ X & \longmapsto & X|_{\dot{\Delta}_n} \end{array}$$

is 1*-epimorphic.*

Proof. We claim that the functor $(-)|_{\dot{\Delta}_n}$ satisfies the requirements (i, ii) of Corollary I.76, which then implies that it is 1-epimorphic.

Suppose given $Y \in \operatorname{Ob} \mathcal{F}(\dot{\Delta}_n)$. We construct an object $\tilde{Y}$ of $\mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})$ such that $\tilde{Y}|_{\dot{\Delta}_n} = Y$ by the following procedure.

Write $\bar{\Delta}_n^{\vartriangle,\cdot} := \bar{\Delta}_n^{\vartriangle} \smallsetminus \{0/0\}$ and $\bar{\Delta}_n^{\triangledown,\cdot} := \bar{\Delta}_n^{\triangledown} \smallsetminus \{0^{+1}/0\}$; cf. §I.2.1.1.

On $\bar{\Delta}_n^{\vartriangle,\cdot}$, we proceed by induction to construct a diagram for which, moreover, the morphisms $\tilde{Y}_{\gamma/\alpha} \longrightarrow \tilde{Y}_{\gamma/\beta}$ are purely monomorphic for all $\alpha, \beta, \gamma \in \bar{\Delta}_n$ with $0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \alpha^{+1}$, and, moreover, for which $\tilde{Y}_{\alpha^{+1}/\alpha} = 0$ for all $0 \leqslant \alpha$.

First of all, let $\tilde{Y}|_{\dot{\Delta}_n} := Y$.

Assume given $\ell \geqslant 0$ such that $\tilde{Y}_{\kappa^{\vartriangle}(\ell')}$, together with all diagram morphisms pointing to position $\kappa^{\vartriangle}(\ell')$, is already constructed for all $\ell' < \ell$, but such that $\tilde{Y}_{\kappa^{\vartriangle}(\ell)}$ is not yet constructed; cf. §I.2.1.2.

If $\kappa^{\vartriangle}(\ell)$ is of the form α/α for some $\alpha \in \bar{\Delta}_n$ with $0 < \alpha$, then choose a pure monomorphism $\tilde{Y}_{\alpha/(\alpha-1)} \longrightarrow\!\!\!\!\!\bullet\;\; \tilde{Y}_{\alpha/\alpha}$ into an object $\tilde{Y}_{\alpha/\alpha}$ of $\mathcal{B}$

We do not necessarily choose $\tilde{Y}_{\alpha/(\alpha-1)}\,\iota$ here.

If $\kappa^{\vartriangle}(\ell) = \alpha^{+1}/\alpha$ for some $\alpha \in \bar{\Delta}_n$ with $0 \leqslant \alpha$, then let $\tilde{Y}_{\alpha^{+1}/\alpha} := 0$.

If $\kappa^{\vartriangle}(\ell)$ is of the form β/α for some $\alpha,\, \beta \in \bar{\Delta}_n$ with $0 < \alpha < \beta < \alpha^{+1}$, then we let

$$(\ \tilde{Y}_{(\beta-1)/(\alpha-1)}\,,\ \tilde{Y}_{(\beta-1)/\alpha}\,,\ \tilde{Y}_{\beta/(\alpha-1)}\,,\ \tilde{Y}_{\beta/\alpha}\ )$$

be a pushout. Recall that by induction assumption, $\tilde{Y}_{(\beta-1)/(\alpha-1)} \to \tilde{Y}_{(\beta-1)/\alpha}$ is purely monomorphic. So $\tilde{Y}_{\beta/(\alpha-1)} \to \tilde{Y}_{\beta/\alpha}$ is purely monomorphic as well.

On $\bar{\Delta}_n^{\triangledown,\cdot}$, we proceed dually, and finally glue along $\dot{\Delta}_n$ to obtain the sought $\tilde{Y}$.

Ad (i). The restriction map ${}_{\mathcal{F}(\bar{\Delta}_n^{\#})}(\tilde{Y}_1, \tilde{Y}_2) \xrightarrow{(-)|_{\dot{\Delta}_n}} {}_{\mathcal{F}(\dot{\Delta}_n)}(Y_1, Y_2)$ is surjective for $Y_1,\, Y_2 \in \operatorname{Ob}\mathcal{F}(\dot{\Delta}_n)$, as we see by induction, using bijectivity to prolong morphisms and universal properties of occurring pushouts and pullbacks.

Ad (ii). Suppose given $X \in \operatorname{Ob}\mathcal{F}^{\square}(\bar{\Delta}_n^{\#})$. Let $X'' := (X|_{\dot{\Delta}_n})^{\tilde{}} \in \operatorname{Ob}\mathcal{F}^{\square}(\bar{\Delta}_n^{\#})$. Let $X' \in \operatorname{Ob}\mathcal{F}^{\square}(\bar{\Delta}_n^{\#})$ be defined by $X'|_{\bar{\Delta}_n^{\vartriangle,\cdot}} = X''|_{\bar{\Delta}_n^{\vartriangle,\cdot}}$ and by $X'|_{\bar{\Delta}_n^{\triangledown,\cdot}} = X|_{\bar{\Delta}_n^{\triangledown,\cdot}}$.

There is a morphism $X' \longrightarrow X$ that restricts to the identity of $X|_{\bar{\Delta}_n^{\triangledown,\cdot}}$ on $\bar{\Delta}_n^{\triangledown,\cdot}$, and hence to the identity of $X|_{\dot{\Delta}_n}$ on $\dot{\Delta}_n$.

There is a morphism $X' \longrightarrow X''$ that restricts to the identity of $X''|_{\bar{\Delta}_n^{\vartriangle,\cdot}}$ on $\bar{\Delta}_n^{\vartriangle,\cdot}$, and hence to the identity of $X|_{\dot{\Delta}_n}$ on $\dot{\Delta}_n$.

Now suppose given $X_1,\, X_2 \in \operatorname{Ob}\mathcal{F}^{\square}(\bar{\Delta}_n^{\#})$ such that $X_1|_{\dot{\Delta}_n} = X_2|_{\dot{\Delta}_n}$. Then there is a sequence of morphisms

$$X_1 \longleftarrow X_1' \longrightarrow X_1'' = X_2'' \longleftarrow X_2' \longrightarrow X_2$$

each of which restricts to the identity of $X_1|_{\dot{\Delta}_n} = X_2|_{\dot{\Delta}_n}$ on $\dot{\Delta}_n$, as required. □

Lemma I.30 *The functors*

$$\mathcal{F}(\dot{\Delta}_n) \xrightarrow{N(\dot{\Delta}_n)} \underline{\mathcal{F}}(\dot{\Delta}_n)$$
$$\mathcal{F}(\dot{\Delta}_n) \xrightarrow{N(\dot{\Delta}_n)N'} \underline{\underline{\mathcal{F}}}(\dot{\Delta}_n)$$

are 1*-epimorphic.*

Proof. Since N' is full and dense, it is 1-epimorphic by Corollary I.77. Therefore, it suffices to show that $N(\dot{\Delta}_n)$ is 1-epimorphic.

We will apply Lemma I.75. Choosing representatives of the occurring morphisms $Z_i \longrightarrow Z_{i+1}$ in an object Z of $\underline{\mathcal{F}}(\dot{\Delta}_n)$, where $i \in [1, n-1]$, we see that $N(\dot{\Delta}_n)$ is dense.

To fulfill condition (C) of loc. cit., we will show that given $X,\, Y \in \operatorname{Ob}\mathcal{F}(\dot{\Delta}_n)$ and a morphism $(X)(N(\dot{\Delta}_n)) \xrightarrow{f} (Y)(N(\dot{\Delta}_n))$, there are morphisms $X' \xrightarrow{h} X$ and $X' \xrightarrow{f'} Y$ in $\mathcal{F}(\dot{\Delta}_n)$ such that $(h)(N(\dot{\Delta}_n))$ is an isomorphism and such that $(h)(N(\dot{\Delta}_n))f = (f')(N(\dot{\Delta}_n))$.

We proceed by induction on $k \in [1, n]$. Suppose given a diagram

$$\begin{array}{ccccccccccc}
\tilde{X}_1 & \xrightarrow{\tilde{x}} & \tilde{X}_2 & \xrightarrow{\tilde{x}} & \tilde{X}_3 & \xrightarrow{\tilde{x}} & \cdots & \xrightarrow{\tilde{x}} & \tilde{X}_{n-1} & \xrightarrow{\tilde{x}} & \tilde{X}_n \\
{\scriptstyle \tilde{f}_1}\downarrow & & {\scriptstyle \tilde{f}_2}\downarrow & & {\scriptstyle \tilde{f}_3}\downarrow & & & & {\scriptstyle \tilde{f}_{n-1}}\downarrow & & {\scriptstyle \tilde{f}_n}\downarrow \\
Y_1 & \xrightarrow{y} & Y_2 & \xrightarrow{y} & Y_3 & \xrightarrow{y} & \cdots & \xrightarrow{y} & Y_{n-1} & \xrightarrow{y} & Y_n
\end{array}$$

in $\mathcal{F}$ such that $\tilde{x}\tilde{f}_{i+1} = \tilde{f}_i y$ for $i \in [1, k-1]$, and such that $\tilde{x}\tilde{f}_{i+1} \equiv_{\mathcal{B}} \tilde{f}_i y$ for $i \in [k, n]$, and a morphism $\tilde{X} \xrightarrow{\tilde{h}} X$ in $\mathcal{F}(\dot{\Delta}_n)$ such that $(\tilde{h})(N(\dot{\Delta}_n))$ is an isomorphism and such that $(\tilde{h})(N(\dot{\Delta}_n))f$ is the morphism in $\underline{\mathcal{F}}(\dot{\Delta}_n)$ represented by $\tilde{f}$.

If $k < n$, we shall construct a morphism $\tilde{X}' \xrightarrow{\tilde{h}'} \tilde{X}$ in $\mathcal{F}(\dot{\Delta}_n)$ with each $\tilde{h}'_i N$ being an isomorphism, and a diagram

$$\begin{array}{ccccccccccc}
\tilde{X}'_1 & \xrightarrow{\tilde{x}'} & \tilde{X}'_2 & \xrightarrow{\tilde{x}'} & \tilde{X}'_3 & \xrightarrow{\tilde{x}'} & \cdots & \xrightarrow{\tilde{x}'} & \tilde{X}'_{n-1} & \xrightarrow{\tilde{x}} & \tilde{X}'_n \\
\downarrow{\scriptstyle \tilde{f}'_1} & & \downarrow{\scriptstyle \tilde{f}'_2} & & \downarrow{\scriptstyle \tilde{f}'_3} & & & & \downarrow{\scriptstyle \tilde{f}'_{n-1}} & & \downarrow{\scriptstyle \tilde{f}'_n} \\
Y_1 & \xrightarrow{y} & Y_2 & \xrightarrow{y} & Y_3 & \xrightarrow{y} & \cdots & \xrightarrow{y} & Y_{n-1} & \xrightarrow{y} & Y_n
\end{array}$$

in $\mathcal{F}$ such that $\tilde{x}'\tilde{f}'_{i+1} = \tilde{f}'_i y$ for $i \in [1, k]$, such that $\tilde{x}'\tilde{f}'_{i+1} - \tilde{f}'_i y \equiv_{\mathcal{B}} 0$ for $i \in [k+1, n]$, and such that $\tilde{h}'_i \tilde{f}_i - \tilde{f}'_i \equiv_{\mathcal{B}} 0$ for all $i \in [1, n]$. For then we obtain a commutative diagram in $\underline{\mathcal{F}}(\dot{\Delta}_n)$

$$\begin{array}{ccc}
\tilde{X}' & & \\
\downarrow{\scriptstyle \tilde{h}'} & \searrow{\scriptstyle \tilde{f}'} & \\
\tilde{X} & & \\
\downarrow{\scriptstyle \tilde{h}} & \searrow{\scriptstyle \tilde{f}} & \\
X & \xrightarrow[f]{} & Y\,,
\end{array}$$

in which we denoted morphisms by their representatives.

Let $\tilde{X}_k \overset{j}{\rightarrowtail} B$ be a pure monomorphism to an object B in $\mathcal{B}$, and let $\tilde{x}\tilde{f}_{k+1} - \tilde{f}_k y = jg$. Let

$$\tilde{X}' \;:=\; (\,\tilde{X}_1 \xrightarrow{\tilde{x}} \cdots \xrightarrow{\tilde{x}} \tilde{X}_k \xrightarrow{(\tilde{x}\ j)} \tilde{X}_{k+1} \oplus B \xrightarrow{\binom{\tilde{x}}{0}} \tilde{X}_{k+2} \xrightarrow{\tilde{x}} \cdots \xrightarrow{\tilde{x}} \tilde{X}_n\,)\,,$$

and let

$$\tilde{f}'_i \;:=\; \begin{cases} \tilde{f}_i & \text{for } i \in [1,n] \smallsetminus \{k+1\} \\ \begin{pmatrix} \tilde{f}_{k+1} \\ -g \end{pmatrix} & \text{for } i = k+1 \end{cases} \quad,\quad \tilde{h}'_i \;:=\; \begin{cases} 1_{\tilde{X}_i} & \text{for } i \in [1,n] \smallsetminus \{k+1\} \\ \begin{pmatrix} 1_{\tilde{X}_{k+1}} \\ 0 \end{pmatrix} & \text{for } i = k+1 \end{cases} \quad.$$

□

Proposition I.31 *The residue class functor* $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) \xrightarrow{M} \underline{\underline{\mathcal{F}}}^{+}(\bar{\Delta}_n^{\#})$ *is* 1*-epimorphic.*

Proof. Consider the commutative quadrangle

$$\begin{array}{ccc}
\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) & \xrightarrow{M} & \underline{\underline{\mathcal{F}}}^{+}(\bar{\Delta}_n^{\#}) \\
\downarrow{\scriptstyle (-)|_{\dot{\Delta}_n}} & & \downarrow{\scriptstyle (-)|_{\dot{\Delta}_n}} \\
\mathcal{F}(\dot{\Delta}_n) & \xrightarrow{N(\dot{\Delta}_n)N'} & \underline{\underline{\mathcal{F}}}(\dot{\Delta}_n)\,.
\end{array}$$

Therein, the functor $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{F}(\dot{\Delta}_n)$ is 1-epimorphic by Lemma I.29. The functor $\mathcal{F}(\dot{\Delta}_n) \xrightarrow{N(\dot{\Delta}_n)\,N'} \underline{\underline{\mathcal{F}}}(\dot{\Delta}_n)$ is 1-epimorphic by Lemma I.30. The functor $\underline{\underline{\mathcal{F}}}^{+}(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\dot{\Delta}_n}} \underline{\underline{\mathcal{F}}}(\dot{\Delta}_n)$ is an equivalence by Proposition I.12. Hence by Remark I.74, the functor $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) \xrightarrow{M} \underline{\underline{\mathcal{F}}}^{+}(\bar{\Delta}_n^{\#})$ is 1-epimorphic. □

We do not claim that the residue class functor $\mathcal{F}^{\square}(\bar{\Delta}_n^{\#}) \xrightarrow{N(\bar{\Delta}_n^{\#})} \underline{\mathcal{F}}^{+}(\bar{\Delta}_n^{\#})$ is 1-epimorphic.

I.4.4 Construction of ϑ

Let $n \geqslant 0$. In the notation of Lemma I.72, we let $C := \bar{\Delta}_n^{\#}$; the role of the category called $\mathcal{E}$ there is played by $\mathcal{F}$ here; we let $\mathcal{G} := \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})$; and finally, we let $\mathcal{H} := \underline{\mathcal{F}}^{+}(\bar{\Delta}_n^{\#})$. Note that $\underline{\mathcal{F}}^{+}(\bar{\Delta}_n^{\#})$ is a characteristic subcategory of $\underline{\mathcal{F}}(\bar{\Delta}_n^{\#})$.

The tuples

$$\begin{array}{rcl} I(n) & = & \Big((I_{X,\,\beta/\alpha})_{\beta/\alpha \in \bar{\Delta}_n^{\#}} \Big)_{X \in \operatorname{Ob} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})} \\ & := & \Big(\Big(X_{\beta/\alpha} \xrightarrow{(x\;x)} X_{\beta/\beta} \oplus X_{\alpha^{+1}/\alpha} \xrightarrow{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} X_{\alpha^{+1}/\beta} \Big)_{\beta/\alpha \in \bar{\Delta}_n^{\#}} \Big)_{X \in \operatorname{Ob} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})} \\ \\ J(n) & = & \Big((J_{X,\,\beta/\alpha})_{\beta/\alpha \in \bar{\Delta}_n^{\#}} \Big)_{X \in \operatorname{Ob} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})} \\ & := & \Big(\Big(X_{\beta/\alpha} \xrightarrow{X_{\beta/\alpha}\,\iota} X_{\beta/\alpha}\,\mathsf{I} = X^{+1}_{\beta/\alpha}\,\mathsf{P} \xrightarrow{X^{+1}_{\beta/\alpha}\,\pi} X^{+1}_{\beta/\alpha} \Big)_{\beta/\alpha \in \bar{\Delta}_n^{\#}} \Big)_{X \in \operatorname{Ob} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#})} \end{array}$$

are $\bar{\Delta}_n^{\#}$-resolving systems, inducing an isomorphism $\mathsf{T}_{I(n)} \xrightarrow[\sim]{\alpha_{I(n),\,J(n)}} \mathsf{T}_{J(n)}$ by Lemma I.72.(2). Recall that $\underline{\mathcal{F}}^{+}(\bar{\Delta}_n^{\#}) \xrightarrow{M''} \underline{\underline{\mathcal{F}^{+}(\bar{\Delta}_n^{\#})}}$ denotes the residue class functor. We have

$$\begin{array}{rcccl} \mathsf{T}_{I(n)}\, M'' & = & M\, \underline{\underline{\mathcal{F}}^{+}(\mathsf{T}_n)} & = & M[-]^{+1} \\ \mathsf{T}_{J(n)}\, M'' & = & M\, \underline{\mathsf{T}^{+}(\bar{\Delta}_n)} & = & M[-^{+1}]\,. \end{array}$$

Since M is 1-epimorphic by Proposition I.31, we obtain

$$\begin{array}{ccc} \mathcal{F}^{\Box}(\bar{\Delta}_n^{\#}) & \overset{\mathsf{T}_{I(n)}}{\underset{\mathsf{T}_{J(n)}}{\rightrightarrows}}\ \Downarrow \alpha_{I(n),\,J(n)} & \underline{\mathcal{F}}^{+}(\bar{\Delta}_n^{\#}) \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle M''} \\ \underline{\underline{\mathcal{F}^{+}(\bar{\Delta}_n^{\#})}} & \overset{[-]^{+1}}{\underset{[-^{+1}]}{\rightrightarrows}}\ \Downarrow \vartheta_n & \underline{\underline{\mathcal{F}^{+}(\bar{\Delta}_n^{\#})}}\ , \end{array}$$

where ϑ_n is characterised by this commutative diagram, i.e. by

$$\alpha_{I(n),\,J(n)} \star M'' \;=\; M \star \vartheta_n\ .$$

Since $\alpha_{I(n),\,J(n)}$ is an isomorphism, so is ϑ_n. Varying n, this defines $\vartheta = (\vartheta_n)_{n \geqslant 0}$.

Theorem I.32 *The tuple $\vartheta = (\vartheta_n)_{n \geqslant 0}$ is a Heller triangulation on $\underline{\mathcal{F}}$.*

Proof. According to Definition I.5.(i), we have to show that the following conditions $(*)$ and $(**)$ hold.

$(*)$ For $m,\, n \geqslant 0$, for a morphism $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$ and for an object $Y \in \operatorname{Ob} \underline{\underline{\mathcal{F}^{+}(\bar{\Delta}_n^{\#})}}$, we have

$$(Y\underline{p}^{\#})\vartheta_m \;=\; (Y\vartheta_n)\underline{p}^{\#}\ .$$

$(**)$ For $n \geqslant 0$ and for an object $Y \in \operatorname{Ob} \underline{\mathcal{F}}^+((2\bar{\Delta}_n)^\#)$, we have

$$(Y\underline{\mathfrak{f}}_n)\vartheta_{n+1} \;=\; (Y\vartheta_{2n+1})\underline{\mathfrak{f}}_n \,.$$

Ad $(*)$. Recall that $\underline{p}^\#$ stands for $\underline{\mathcal{F}}^+(p^\#)$, and that $p^\#$ stands for $\mathcal{F}^\square(p^\#)$. By Proposition I.31, we may assume $Y = XM$ for some $X \in \operatorname{Ob}\mathcal{F}^\square(\bar{\Delta}_n^\#)$. Then

$$\begin{array}{lllll} (XM\underline{p}^\#)\vartheta_m & = & (Xp^\# M)\vartheta_m & = & (Xp^\#\alpha_{I(m),\,J(m)})M'' \\ (XM\vartheta_n)\underline{p}^\# & = & (X\alpha_{I(n),\,J(n)}M'')\underline{p}^\# & = & (X\alpha_{I(n),\,J(n)}p^\#)M''\,, \end{array}$$

so that it suffices to show that $Xp^\#\alpha_{I(m),\,J(m)} = X\alpha_{I(n),\,J(n)}p^\#$.

Starting with $Xp^\#$, the object $Xp^\#\,\mathsf{T}_{I(m)}$ is calculated by means of $(I_{Xp^\#,\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_m^\#}$; whereas $X\,\mathsf{T}_{I(n)}$ is calculated by means of $(I_{X,\,\delta/\gamma})_{\delta/\gamma\in\bar{\Delta}_n^\#}$, so that $X\,\mathsf{T}_{I(n)}\,p^\#$ can be regarded as being calculated by means of $(I_{X,\,\beta p/\alpha p})_{\beta/\alpha\in\bar{\Delta}_m^\#}$. But

$$I_{X,\,\beta p/\alpha p} \;=\; \left(X_{\beta p/\alpha p} \xrightarrow{(x\;x)} X_{\beta p/\beta p}\oplus X_{(\alpha p)^{+1}/\alpha p} \xrightarrow{\binom{x}{-x}} X_{(\alpha p)^{+1}/\beta p}\right) \;=\; I_{Xp^\#,\,\beta/\alpha}\,,$$

whence $Xp^\#\,\mathsf{T}_{I(m)} = X\,\mathsf{T}_{I(n)}\,p^\#$.

Next, starting with $Xp^\#$, the object $Xp^\#\,\mathsf{T}_{J(m)}$ is calculated by means of $(J_{Xp^\#,\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_m^\#}$; whereas $X\,\mathsf{T}_{J(n)}\,p^\#$ can be regarded as being calculated by means of $(J_{X,\,\beta p/\alpha p})_{\beta/\alpha\in\bar{\Delta}_m^\#}$. But

$$J_{X,\,\beta p/\alpha p} \;=\; \left(X_{\beta p/\alpha p} \xrightarrow{X_{\beta p/\alpha p}\iota} X_{\beta p/\alpha p}\,\mathsf{I} = X^{+1}_{\beta p/\alpha p}\,\mathsf{P} \xrightarrow{X^{+1}_{\beta p/\alpha p}\pi} X^{+1}_{\beta p/\alpha p}\right) \;=\; J_{Xp^\#,\,\beta/\alpha}\,,$$

whence $Xp^\#\,\mathsf{T}_{J(m)} = X\,\mathsf{T}_{J(n)}\,p^\#$.

Now

$$Xp^\#\,\mathsf{T}_{I(m)} \xrightarrow[\sim]{Xp^\#\alpha_{I(m),\,J(m)}} Xp^\#\,\mathsf{T}_{J(m)}$$

is induced by $(I_{Xp^\#,\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_m^\#}$ and by $(J_{Xp^\#,\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_m^\#}$; whereas

$$X\,\mathsf{T}_{I(n)}\,p^\# \xrightarrow[\sim]{X\alpha_{I(n),\,J(n)}p^\#} X\,\mathsf{T}_{J(n)}\,p^\#$$

can be regarded as being induced by $(I_{X,\,\beta p/\alpha p})_{\beta/\alpha\in\bar{\Delta}_m^\#}$ and by $(J_{X,\,\beta p/\alpha p})_{\beta/\alpha\in\bar{\Delta}_m^\#}$. We have just seen, however, that these pairs of tuples coincide.

Ad $(**)$. By Proposition I.31, we may assume $Y = XM$ for some $X \in \operatorname{Ob}\mathcal{F}^\square((2\bar{\Delta}_n)^\#)$. By Remark I.27, we have

$$\begin{array}{lllll} (XM\underline{\mathfrak{f}}_n)\vartheta_{n+1} & = & (X\tilde{\mathfrak{f}}_n M)\vartheta_{n+1} & = & (X\tilde{\mathfrak{f}}_n\alpha_{I(n+1),\,J(n+1)})M'' \\ (XM\vartheta_{2n+1})\underline{\mathfrak{f}}_n & = & (X\alpha_{I(2n+1),\,J(2n+1)}M'')\underline{\mathfrak{f}}_n & = & (X\alpha_{I(2n+1),\,J(2n+1)}\mathfrak{f}_n)M''\,, \end{array}$$

so that it suffices to show that $X\tilde{\mathfrak{f}}_n\alpha_{I(n+1),\,J(n+1)} = X\alpha_{I(2n+1),\,J(2n+1)}\mathfrak{f}_n$.

Starting with $X\tilde{\mathfrak{f}}_n$, the object $X\tilde{\mathfrak{f}}_n\,\mathsf{T}_{I(n+1)}$ is calculated by means of $(I_{X\tilde{\mathfrak{f}}_n,\,\beta/\alpha})_{\beta/\alpha\,\in\,\overline{\rho\sqcup\Delta_n}^\#}$; whereas $X\,\mathsf{T}_{I(2n+1)}\,\mathfrak{f}_n$ can be regarded as being calculated by means of the tuple of pure short exact sequences consisting of

$$\begin{cases} 0 & \text{at } (\rho/\rho)^{+z},\ z\in\mathbf{Z} \\ I_{X,\,(\alpha^{+1}/\alpha)^{+z}} & \text{at } (\alpha/\rho)^{+z},\ \alpha\in\Delta_n\,,\ z\in\mathbf{Z} \\ I_{X,\,(\beta^{+1}/\alpha^{+1})^{+z}}\oplus I_{X,\,(\alpha^{+2}/\beta)^{+z}} & \text{at } (\beta/\alpha)^{+z},\ \alpha,\ \beta\,\in\,\Delta_n\,,\ \alpha\leqslant\beta,\ z\in\mathbf{Z}\,. \end{cases}$$

We have $I_{X\tilde{\mathfrak{f}}_n,\,(\rho/\rho)^{+z}} = 0$ for $z \in \mathbf{Z}$. For $\alpha \in \Delta_n$, we have

$$I_{X\tilde{\mathfrak{f}}_n,\,\alpha/\rho} \;=\; \Big(X_{\alpha^{+1}/\alpha} \overset{(x\;x)}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} X_{\alpha^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\alpha} \overset{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)}{\longrightarrow\!\!\!\!\!\!+\;\;} X_{\alpha^{+2}/\alpha^{+1}} \Big) \;=\; I_{X,\,\alpha^{+1}/\alpha}\;,$$

and accordingly at $(\alpha/\rho)^{+z}$ for $z \in \mathbf{Z}$. Moreover, for $\alpha,\ \beta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$, we have

$$I_{X\tilde{\mathfrak{f}}_n,\,\beta/\alpha} \;=\;$$
$$\Big(X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta} \xrightarrow{\left(\begin{smallmatrix} x & 0 & x & 0 \\ 0 & x & 0 & -x \end{smallmatrix}\right)} X_{\beta^{+1}/\beta^{+1}} \oplus X_{\beta^{+2}/\beta} \oplus X_{\alpha^{+3}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\alpha^{+2}} \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & -x \\ -x & 0 \\ 0 & -x \end{smallmatrix}\right)} X_{\alpha^{+3}/\beta^{+1}} \oplus X_{\beta^{+2}/\alpha^{+2}} \Big)$$

and

$$I_{X,\,\beta^{+1}/\alpha^{+1}} \oplus I_{X,\,\alpha^{+2}/\beta} \;=\;$$
$$\Big(X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta} \xrightarrow{\left(\begin{smallmatrix} x & x & 0 & 0 \\ 0 & 0 & x & x \end{smallmatrix}\right)} X_{\beta^{+1}/\beta^{+1}} \oplus X_{\alpha^{+3}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\alpha^{+2}} \oplus X_{\beta^{+2}/\beta} \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ -x & 0 \\ 0 & x \\ 0 & -x \end{smallmatrix}\right)} X_{\alpha^{+3}/\beta^{+1}} \oplus X_{\beta^{+2}/\alpha^{+2}} \Big)$$

Accordingly at $(\beta/\alpha)^{+z}$ for $z \in \mathbf{Z}$.

Since there is an isomorphism from $I_{X\tilde{\mathfrak{f}}_n,\,\alpha/\rho}$ to $I_{X,\,\beta^{+1}/\alpha^{+1}} \oplus I_{X,\,\alpha^{+2}/\beta}$ that has identities on the first and on the third terms of the short exact sequences, completed by

$$X_{\beta^{+1}/\beta^{+1}} \oplus X_{\beta^{+2}/\beta} \oplus X_{\alpha^{+3}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\alpha^{+2}} \xrightarrow[\sim]{\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{smallmatrix}\right)} X_{\beta^{+1}/\beta^{+1}} \oplus X_{\alpha^{+3}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\alpha^{+2}} \oplus X_{\beta^{+2}/\beta}$$

on the second terms, the characterisation in Lemma I.72.(1) shows that we end up altogether with $X\tilde{\mathfrak{f}}_n\,\mathsf{T}_{I(n+1)} = X\,\mathsf{T}_{I(2n+1)}\,\mathfrak{f}_n$.

Starting with $X\tilde{\mathfrak{f}}_n$, the object $X\tilde{\mathfrak{f}}_n\,\mathsf{T}_{J(n+1)}$ is calculated by means of $(J_{X\tilde{\mathfrak{f}}_n,\,\beta/\alpha})_{\beta/\alpha\,\in\,\overline{\rho\sqcup\Delta_n}^{\#}}$; whereas $X\,\mathsf{T}_{J(2n+1)}\,\mathfrak{f}_n$ can be regarded as being calculated by means of the tuple of pure short exact sequences consisting of

$$\begin{cases} 0 & \text{at } (\rho/\rho)^{+z},\ z \in \mathbf{Z} \\ J_{X,\,(\alpha^{+1}/\alpha)^{+z}} & \text{at } (\alpha/\rho)^{+z},\ \alpha \in \Delta_n\,,\ z \in \mathbf{Z} \\ J_{X,\,(\beta^{+1}/\alpha^{+1})^{+z}} \oplus J_{X,\,(\alpha^{+2}/\beta)^{+z}} & \text{at } (\beta/\alpha)^{+z},\ \alpha,\ \beta\ \in\ \Delta_n\,,\ \alpha \leqslant \beta,\ z \in \mathbf{Z}\,. \end{cases}$$

We have $J_{X\tilde{\mathfrak{f}}_n,\,(\rho/\rho)^{+z}} = 0$ for $z \in \mathbf{Z}$. For $\alpha \in \Delta_n$, we have

$$J_{X\tilde{\mathfrak{f}}_n,\,\alpha/\rho} \;=\; \Big(X_{\alpha^{+1}/\alpha} \xrightarrow{X_{\alpha^{+1}/\alpha}\,\iota} X_{\alpha^{+1}/\alpha}\,\mathsf{I} = X^{+1}_{\alpha^{+1}/\alpha}\,\mathsf{P} \xrightarrow{X^{+1}_{\alpha^{+1}/\alpha}\,\pi} X^{+1}_{\alpha^{+1}/\alpha} \Big) \;=\; J_{X,\,\alpha^{+1}/\alpha}\;,$$

and accordingly at $(\alpha/\rho)^{+z}$ for $z \in \mathbf{Z}$.

Moreover, for $\alpha,\ \beta\ \in\ \Delta_n$ with $\alpha \leqslant \beta$, we have

$$\begin{aligned} J_{X\tilde{\mathfrak{f}}_n,\,\beta/\alpha} \;=\; \Big(& X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta} \xrightarrow{(X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta})\,\iota} \\ & (X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta})\,\mathsf{I} = (X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta})^{+1}\,\mathsf{P} \xrightarrow{(X_{\beta^{+1}/\alpha^{+1}} \oplus X_{\alpha^{+2}/\beta})^{+1}\,\pi} \\ & X_{\alpha^{+3}/\beta^{+1}} \oplus X_{\beta^{+2}/\alpha^{+2}} \Big) \;=\; J_{X,\,\beta^{+1}/\alpha^{+1}} \oplus J_{X,\alpha^{+2}/\beta}\;, \end{aligned}$$

and accordingly at $(\beta/\alpha)^{+z}$ for $z \in \mathbf{Z}$.

Hence altogether, we conclude that $X\tilde{\mathfrak{f}}_n\,\mathsf{T}_{J(n+1)} = X\,\mathsf{T}_{J(2n+1)}\,\mathfrak{f}_n$.

Now

$$X\tilde{\mathfrak{f}}_n\,\mathsf{T}_{I(n+1)} \xrightarrow[\sim]{X\tilde{\mathfrak{f}}_n\alpha_{I(n+1),\,J(n+1)}} X\tilde{\mathfrak{f}}_n\,\mathsf{T}_{J(n+1)}$$

is induced by $(I_{X\tilde{\mathfrak{f}}_n,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}^{\#}_{n+1}}$ and by $(J_{X\tilde{\mathfrak{f}}_n,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}^{\#}_{n+1}}$; whereas

$$X\,\mathsf{T}_{I(2n+1)}\,\mathfrak{f}_n \xrightarrow[\sim]{X\alpha_{I(2n+1),\,J(2n+1)}\mathfrak{f}_n} X\,\mathsf{T}_{J(2n+1)}\,\mathfrak{f}_n$$

can be regarded as being induced by the tuple consisting of

$$\begin{cases} 0 & \text{at } (\rho/\rho)^{+z},\ z\in\mathbf{Z} \\ I_{X,(\alpha^{+1}/\alpha)^{+z}} & \text{at } (\alpha/\rho)^{+z},\ \alpha\in\Delta_n\,,\ z\in\mathbf{Z} \\ I_{X,(\beta^{+1}/\alpha^{+1})^{+z}}\oplus I_{X,(\alpha^{+2}/\beta)^{+z}} & \text{at } (\beta/\alpha)^{+z},\ \alpha,\ \beta\ \in\ \Delta_n\,,\ \alpha\leqslant\beta,\ z\in\mathbf{Z}\,. \end{cases}$$

and by $(J_{X\tilde{\mathfrak{f}}_n,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}^{\#}_{n+1}}$.

Since the respective former tuples are isomorphic by a tuple of isomorphisms that has identities on the first and on the third term, and since the respective latter tuples are equal, the characterisation in Lemma I.72.(3) shows that in fact $X\tilde{\mathfrak{f}}_n\alpha_{I(n+1),\,J(n+1)} = X\alpha_{I(2n+1),\,J(2n+1)}\mathfrak{f}_n$. ▫

Corollary I.33 *Let $\mathcal{E}$ be a Frobenius category. There exists a Heller triangulation on $(\underline{\underline{\mathcal{E}}},\mathsf{T})$.*

Concerning the stable category $\underline{\underline{\mathcal{E}}}$, cf. Definition I.47.

Proof. Let $\mathcal{B}\subseteq\mathcal{E}$ be the full subcategory of bijectives. The category $\mathcal{B}^{\mathrm{ac}}$ is functorially Frobenius by Example I.46. Hence $\underline{\underline{\mathcal{E}}} = \underline{\mathcal{B}^{\mathrm{ac}}}$, equipped with the complex shift T, carries a Heller triangulation by virtue of Theorem I.32. ▫

I.4.5 Exact functors induce strictly exact functors

Proposition I.34 *Suppose given an exact functor*

$$\mathcal{F} \xrightarrow{F} \tilde{\mathcal{F}}$$

between functorial Frobenius categories $\mathcal{F} = (\mathcal{F},\mathsf{T},\mathsf{I},\iota,\mathsf{P},\pi)$ and $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}},\tilde{\mathsf{T}},\tilde{\mathsf{I}},\tilde{\iota},\tilde{\mathsf{P}},\tilde{\pi})$ that satisfies

$$\begin{aligned} F\tilde{\mathsf{T}} &= \mathsf{T}F \\ F\tilde{\mathsf{I}} &= \mathsf{I}F \\ F\tilde{\mathsf{P}} &= \mathsf{P}F\,. \end{aligned}$$

Then the induced functor

$$\underline{\mathcal{F}} \xrightarrow{F} \underline{\tilde{\mathcal{F}}}$$

is strictly exact with respect to the Heller triangulations introduced in Theorem I.32.

Proof. Condition (1) of Definition I.5.(iii) is satisfied. Condition (2) of loc. cit. holds since each morphism has a weak kernel that is sent to a weak kernel of its image; and dually. In fact, given a morphism represented by $X \xrightarrow{f} Y$, the residue class of the kernel of $X \oplus Y\,\mathsf{P} \xrightarrow{\binom{f}{Y\pi}} Y$ in $\mathcal{F}$, composed with $X \oplus Y\,\mathsf{P} \xrightarrow{\binom{1}{0}} X$, is a weak kernel of the residue class of $X \xrightarrow{f} Y$ by Lemma I.71 and Remark I.67. Since pure short exact sequences and bijectives are preserved by F, this weak kernel is preserved by $\underline{F}$.

Consider condition (3) of loc. cit. Let ϑ resp. $\tilde{\vartheta}$ be the Heller triangulation on $\underline{\mathcal{F}}$ resp. on $\underline{\tilde{\mathcal{F}}}$ characterised as in Theorem I.32 by

$$\begin{array}{rcl} \alpha_{I(n),\,J(n)}M'' & = & M\vartheta_n \\ \alpha_{\tilde{I}(n),\,\tilde{J}(n)}\tilde{M}'' & = & \tilde{M}\tilde{\vartheta}_n \ , \end{array}$$

where $\tilde{M}$, $\tilde{M}'$, $\tilde{M}''$, $\tilde{I}(n)$ resp. $\tilde{J}(n)$ is defined over $\tilde{\mathcal{F}}$ as M, M', M'', $I(n)$ resp. $J(n)$ is over $\mathcal{F}$. To prove (3), i.e. to show that for $n \geqslant 0$ and $Y \in \operatorname{Ob}\underline{\mathcal{F}^+(\bar{\Delta}_n^\#)}$, we have

$$(Y\vartheta_n)\underline{F^+(\bar{\Delta}_n^\#)} \;=\; (Y\underline{F^+(\bar{\Delta}_n^\#)})\tilde{\vartheta}_n \ ,$$

we may assume by Proposition I.31 that $Y = XM$ for some $X \in \operatorname{Ob}\mathcal{F}^\Box(\bar{\Delta}_n^\#)$. Since

$$\begin{array}{rclcl} (XM\vartheta_n)\underline{F^+(\bar{\Delta}_n^\#)} & = & (X\alpha_{I(n),\,J(n)}M'')\underline{F^+(\bar{\Delta}_n^\#)} & = & (X\alpha_{I(n),\,J(n)}\underline{F}^+(\bar{\Delta}_n^\#))\tilde{M}'' \\ (XM\underline{F^+(\bar{\Delta}_n^\#)})\tilde{\vartheta}_n & = & (XF^\Box(\bar{\Delta}_n^\#)\tilde{M})\tilde{\vartheta}_n & = & (XF^\Box(\bar{\Delta}_n^\#)\alpha_{\tilde{I}(n),\,\tilde{J}(n)})\tilde{M}'' \ , \end{array}$$

it suffices to show that $X\alpha_{I(n),\,J(n)}\underline{F}^+(\bar{\Delta}_n^\#) = XF^\Box(\bar{\Delta}_n^\#)\alpha_{\tilde{I}(n),\,\tilde{J}(n)}$.

Starting with $XF^\Box(\bar{\Delta}_n^\#)$, the object $XF^\Box(\bar{\Delta}_n^\#)\,\mathsf{T}_{\tilde{I}(n)}$ is calculated by means of $(\tilde{I}_{XF^\Box(\bar{\Delta}_n^\#),\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_n^\#}$; whereas $X\,\mathsf{T}_{\tilde{I}(n)}\,\underline{F}^+(\bar{\Delta}_n^\#)$ can be regarded as being calculated by means of $(I_{X,\,\beta/\alpha}F)_{\beta/\alpha\in\bar{\Delta}_n^\#}$, where $I_{X,\,\beta/\alpha}F$ is defined by an application of F to all three terms and both morphisms of the pure short exact sequence $I_{X,\,\beta/\alpha}$. Since F is additive, we get

$$(\tilde{I}_{XF^\Box(\bar{\Delta}_n^\#),\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_n^\#} \;=\; (I_{X,\,\beta/\alpha}F)_{\beta/\alpha\in\bar{\Delta}_n^\#} \ ,$$

whence $XF^\Box(\bar{\Delta}_n^\#)\,\mathsf{T}_{\tilde{I}(n)} = X\,\mathsf{T}_{I(n)}\,\underline{F}^+(\bar{\Delta}_n^\#)$.

Starting with $XF^\Box(\bar{\Delta}_n^\#)$, the object $XF^\Box(\bar{\Delta}_n^\#)\,\mathsf{T}_{\tilde{J}(n)}$ is calculated by means of $(\tilde{J}_{XF^\Box(\bar{\Delta}_n^\#),\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_n^\#}$; whereas $X\,\mathsf{T}_{J(n)}\,\underline{F}^+(\bar{\Delta}_n^\#)$ can be regarded as being calculated by means of $(J_{X,\,\beta/\alpha}F)_{\beta/\alpha\in\bar{\Delta}_n^\#}$, where $J_{X,\,\beta/\alpha}F$ is obtained by entrywise application of F. Since F commutes with P and $\tilde{\mathsf{P}}$, and with I and $\tilde{\mathsf{I}}$, we get

$$(\tilde{J}_{XF^\Box(\bar{\Delta}_n^\#),\,\beta/\alpha})_{\beta/\alpha\in\bar{\Delta}_n^\#} \;=\; (J_{X,\,\beta/\alpha}F)_{\beta/\alpha\in\bar{\Delta}_n^\#} \ ,$$

whence $XF^\Box(\bar{\Delta}_n^\#)\,\mathsf{T}_{\tilde{J}(n)} = X\,\mathsf{T}_{J(n)}\,\underline{F}^+(\bar{\Delta}_n^\#)$.

Moreover, since the defining pairs of tuples coincide, we finally get $XF^\Box(\bar{\Delta}_n^\#)\alpha_{\tilde{I}(n),\,\tilde{J}(n)} = X\alpha_{I(n),\,J(n)}\underline{F}^+(\bar{\Delta}_n^\#)$. □

Suppose given an exact functor

$$\mathcal{E} \xrightarrow{E} \tilde{\mathcal{E}}$$

between Frobenius categories $\mathcal{E}$ and $\tilde{\mathcal{E}}$ that sends all bijective objects in $\mathcal{E}$ to bijective objects in $\tilde{\mathcal{E}}$. Let $\mathcal{B} \subseteq \mathcal{E}$ resp. $\tilde{\mathcal{B}} \subseteq \tilde{\mathcal{E}}$ be the respective subcategories of bijectives. We obtain an induced functor $\mathcal{B}^{\mathrm{ac}} \xrightarrow{E^{\mathrm{ac}}} \tilde{\mathcal{B}}^{\mathrm{ac}}$, inducing in turn a functor

$$\underline{\underline{E}} := \underline{E^{\mathrm{ac}}} \quad : \quad \underline{\underline{\mathcal{E}}} = \underline{\mathcal{B}^{\mathrm{ac}}} \quad \longrightarrow \quad \underline{\tilde{\mathcal{B}}^{\mathrm{ac}}} = \underline{\underline{\tilde{\mathcal{E}}}}$$

modulo split acyclic complexes; cf. Example I.46.(2).

Corollary I.35 *The induced functor*

$$\underline{\underline{\mathcal{E}}} \xrightarrow{\underline{\underline{E}}} \underline{\underline{\tilde{\mathcal{E}}}}$$

is strictly exact with respect to the Heller triangulations on $\underline{\underline{\mathcal{E}}}$ *and on* $\underline{\underline{\tilde{\mathcal{E}}}}$ *introduced in Theorem* I.32 *via the functorial Frobenius categories* $\mathcal{B}^{\mathrm{ac}}$ *and* $\tilde{\mathcal{B}}^{\mathrm{ac}}$*.*

Proof. We may apply Proposition I.34 to $(\mathcal{F} \xrightarrow{F} \tilde{\mathcal{F}}) := (\mathcal{B}^{\mathrm{ac}} \xrightarrow{E^{\mathrm{ac}}} \tilde{\mathcal{B}}^{\mathrm{ac}})$. □

I.5 Some quasicyclic categories

In the definition of a Heller triangulated category, the categories $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$ occur. Replacing this classical stable category by its stable counterpart, these turn out to be Heller triangulated themselves. So we can iterate. Cf. [**5,** Prop. 8.4].

Let $\mathcal{C}$ be a weakly abelian category. Let $n \geqslant 0$.

I.5.1 The category $\mathcal{C}^{+}(\bar{\Delta}_n^{\#})$ is Frobenius

I.5.1.1 The category $\mathcal{A}^{0}(\bar{\Delta}_n^{\#})$ is Frobenius

We proceed in a slightly more general manner than necessary. We generalise the fact that the category of complexes $\mathcal{A}^{0}(\bar{\Delta}_2^{\#})$ over an additive category $\mathcal{A}$ is a Frobenius category, to a category $\mathcal{A}^{0}(\bar{\Delta}_n^{\#})$ for $n \geqslant 0$; cf. Lemma I.37 below. Then we will specialise to our weakly abelian category $\mathcal{C}$ and pass to the full subcategory $\mathcal{C}^{+}(\bar{\Delta}_n^{\#}) \subseteq \mathcal{C}^{0}(\bar{\Delta}_n^{\#})$; cf. Proposition I.40 below.

I.5.1.1.1 Notation

Let $\mathcal{A}$ be an additive category. Let $\mathcal{A}^{0}(\bar{\Delta}_n^{\#})$ be the full subcategory of $\mathcal{A}(\bar{\Delta}_n^{\#})$ defined by

$$\operatorname{Ob}\mathcal{A}^{0}(\bar{\Delta}_n^{\#}) \;:=\; \left\{ X \in \operatorname{Ob}\mathcal{A}(\bar{\Delta}_n^{\#}) \;:\; X_{\alpha/\alpha} = 0 \text{ and } X_{\alpha^{+1}/\alpha} = 0 \text{ for all } \alpha \in \bar{\Delta}_n \right\}.$$

A sequence $X' \xrightarrow{i} X \xrightarrow{p} X''$ in $\mathcal{A}^{0}(\bar{\Delta}_n^{\#})$ is called *pointwise split short exact* if the sequence $X'_\xi \xrightarrow{i_\xi} X_\xi \xrightarrow{p_\xi} X''_\xi$ is split short exact for all $\xi \in \bar{\Delta}_n^{\#}$. A morphism is called *pointwise split monomorphic* (resp. *epimorphic*) if it appears as a kernel (resp. cokernel) in a pointwise split short exact sequence.

The category $\mathcal{A}^{0}(\bar{\Delta}_n^{\#})$ carries an *outer shift* functor $X \longmapsto [X]^{+1}$, where $[X]^{+1}_{\beta/\alpha} := X_{(\beta/\alpha)^{+1}} = X_{\alpha^{+1}/\beta}$ for $\beta/\alpha \in \bar{\Delta}_n^{\#}$.

Recall that $\mathcal{A}$, together with the set of split short exact sequences, is an exact category; cf. Example I.43. So the additive category $\mathcal{A}^0(\bar{\Delta}_n^\#)$, equipped with the set of pointwise split short exact sequences as pure short exact sequences, is an exact category; cf. Example I.44.

Given $\beta/\alpha,\ \delta/\gamma \in \bar{\Delta}_n^\#$, we write $\beta/\alpha \lessdot \delta/\gamma$ if $\alpha < \gamma$ and $\beta < \delta$.

Given $A \in \operatorname{Ob}\mathcal{A}$ and $\beta/\alpha \in \bar{\Delta}_n^\#$, we denote by $A_{]\alpha,\beta]}$ the object in $\mathcal{A}^0(\bar{\Delta}_n^\#)$ consisting of identical morphisms wherever possible and having

$$(A_{]\alpha,\beta]})_{\delta/\gamma} \;:=\; \begin{cases} A & \text{if } \alpha/\beta^{-1} \lessdot \delta/\gamma \leqslant \beta/\alpha \\ 0 & \text{else} \end{cases}$$

for $\delta/\gamma \in \bar{\Delta}_n^\#$. Such an object is called an *extended interval.*

Intuitively, it is a rectangle with upper right corner at β/α, and as large as possible in $\mathcal{A}^0(\bar{\Delta}_n^\#)$.

Let $\mathcal{A}^{+,\,\mathrm{split}}(\bar{\Delta}_n^\#)$ be the full subcategory of $\mathcal{A}^0(\bar{\Delta}_n^\#)$ consisting of objects isomorphic to summands of objects of the form

$$\bigoplus_{\beta/\alpha \,\in\, \bar{\Delta}_n^\#} (A_{\beta/\alpha})_{]\alpha,\beta]}\ ,$$

where $A_{\beta/\alpha} \in \operatorname{Ob}\mathcal{A}$ for $\beta/\alpha \in \bar{\Delta}_n^\#$. This direct sum exists since it is a finite direct sum at each $\delta/\gamma \in \bar{\Delta}_n^\#$. Concerning the notation $\mathcal{A}^{+,\,\mathrm{split}}(\bar{\Delta}_n^\#)$, cf. also Remark I.38 below.

I.5.1.1.2 The periodic case

Let $\mathcal{A}'$ be an additive category, equipped with a *graduation shift* automorphism $X \longmapsto X[+1]$. We write $X \longmapsto X[m]$ for its m-th iteration, where $m \in \mathbf{Z}$.

By entrywise application, there is also a graduation shift on $\mathcal{A}'^0(\bar{\Delta}_n^\#)$, likewise denoted by $X \longmapsto X[+1]$.

As in §I.2.5.3, we define the subcategory $\mathcal{A}'^{0,\mathrm{periodic}}(\bar{\Delta}_n^\#) \subseteq \mathcal{A}'^0(\bar{\Delta}_n^\#)$ to consist of the morphisms $X \xrightarrow{f} Y$ in $\mathcal{A}'^0(\bar{\Delta}_n^\#)$ that satisfy

$$(X[+1] \xrightarrow{f[+1]} Y[+1]) \;=\; ([X]^{+1} \xrightarrow{[f]^{+1}} [Y]^{+1})$$

So the subcategory $\mathcal{A}'^{0,\mathrm{periodic}}(\bar{\Delta}_n^\#) \subseteq \mathcal{A}'^0(\bar{\Delta}_n^\#)$ is not full in general.

Given $A \in \operatorname{Ob}\mathcal{A}'$ and $0 \leqslant i \leqslant j \leqslant n$, we denote by $A_{]i,j]}$ the object in $\mathcal{A}'^0(\bar{\Delta}_n^\#)$ consisting only of zero and identical morphisms and having

$$(A_{]i,j]})_{\delta/\gamma} \;:=\; \begin{cases} A[m] & \text{if } (i/j^{-1})^{+m} \lessdot \delta/\gamma \leqslant (j/i)^{+m} \text{ for some } m \in \mathbf{Z} \\ 0 & \text{else} \end{cases}$$

for $\delta/\gamma \in \bar{\Delta}_n^\#$.

Intuitively, it is a rectangle with upper right corner at j/i, and as large as possible in $\mathcal{A}'^{0,\,\mathrm{periodic}}(\bar{\Delta}_n^\#)$, repeated $\mathbf{Z}$-periodically up to according graduation shift.

Let $\mathcal{A}'^{+,\,\text{split, periodic}}(\bar\Delta_n^\#)$ be the full subcategory of $\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)$ consisting of objects isomorphic to summands of objects of the form

$$\bigoplus_{0\leqslant i\leqslant j\leqslant n} (A_{j,i})_{]i,j]}\ ,$$

where $A_{j,i} \in \operatorname{Ob}\mathcal{A}'$ for $0 \leqslant i \leqslant j \leqslant n$. Such an object is called a *periodic extended interval.*

Lemma I.36 *The category $\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)$, equipped with the pointwise split short exact sequences, is a Frobenius category, having $\mathcal{A}'^{+,\,\text{split, periodic}}(\bar\Delta_n^\#)$ as its subcategory of bijectives.*

Proof. By duality, it suffices to show that the following assertions $(1,2)$ hold.

(1) The object $A_{]i,j]}$ is injective in $\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)$ for any $A \in \operatorname{Ob}\mathcal{A}'$ and any $0 \leqslant i \leqslant j \leqslant n$.

(2) For each object of $\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)$, there exists a pure monomorphism into an object of $\mathcal{A}'^{+,\,\text{split, periodic}}(\bar\Delta_n^\#)$.

Ad (1). Note that we have an adjunction isomorphism

$$\begin{array}{rcl} {}_{\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)}(X, A_{]i,j]}) & \xrightarrow{\sim} & {}_{\mathcal{A}'}(X_{j/i}\,, A) \\ f & \longmapsto & f_{j/i}\ , \end{array}$$

where $X \in \operatorname{Ob}\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)$. Suppose given a pure monomorphism $A_{]i,j]} \rightarrowtail X$ for some $A \in \operatorname{Ob}\mathcal{A}'$. Let $(A \rightarrowtail X_{j/i} \twoheadrightarrow A) = 1_A$. Let $X \longrightarrow A_{]i,j]}$ correspond to $X_{j/i} \twoheadrightarrow A$. The composition $(A_{]i,j]} \rightarrowtail X \longrightarrow A_{]i,j]})$ restricts to 1_A at j/i, hence equals $1_{A_{]i,j]}}$.

Ad (2). Suppose given $X \in \operatorname{Ob}\mathcal{A}'^{0,\,\text{periodic}}(\bar\Delta_n^\#)$. Given $0 \leqslant i \leqslant j \leqslant n$, we let

$$X \xrightarrow{X s_{j/i}} (X_{j/i})_{]i,j]}$$

be the morphism corresponding to $1_{X_{j/i}}$ by adjunction, which is natural in X. Collecting these morphisms yields a morphism

$$X \xrightarrow{Xs} \bigoplus_{0\leqslant i\leqslant j\leqslant n} (X_{j/i})_{]i,j]}\ ,$$

which is pointwise split monomorphic since at j/i, its component $X_{j/i} \longrightarrow X_{j/i}$ is an identity. ▫

I.5.1.1.3 The general case

Lemma I.37 *The category $\mathcal{A}^0(\bar\Delta_n^\#)$, equipped with the pointwise split short exact sequences, is a Frobenius category, having $\mathcal{A}^{+,\,\text{split}}(\bar\Delta_n^\#)$ as its subcategory of bijectives.*

Proof. To prove that $\mathcal{A}^0(\bar\Delta_n^\#)$ is a Frobenius category, we more precisely claim that $\mathcal{A}^{+,\,\text{split}}(\bar\Delta_n^\#)$ is a sufficiently big category of bijective objects in the exact category $\mathcal{A}^0(\bar\Delta_n^\#)$.

Abbreviate $\mathcal{A}^{\mathbf{Z}} := \mathcal{A}(\dot{\mathbf{Z}})$, where $\dot{\mathbf{Z}}$ denotes the discrete category with $\operatorname{Ob}\dot{\mathbf{Z}} = \mathbf{Z}$ and only identical morphisms. The category $\mathcal{A}^{\mathbf{Z}}$ carries the graduation shift automorphism

$$\begin{array}{rcl} \mathcal{A}^{\mathbf{Z}} & \xrightarrow{\sim} & \mathcal{A}^{\mathbf{Z}} \\ (X \xrightarrow{f} Y) & \longmapsto & (X[+1] \xrightarrow{f[+1]} Y[+1]) \ :=\ (X_{i+1} \xrightarrow{f_{i+1}} Y_{i+1})_{i\in\mathbf{Z}}\ . \end{array}$$

We have an isomorphism of categories

$$\begin{array}{rcl} \mathcal{A}^+(\bar{\Delta}_n^\#) & \xrightarrow[\sim]{\Phi} & (\mathcal{A}^{\mathbf{Z}})^{+,\,\text{periodic}}(\bar{\Delta}_n^\#) \\ X & \longmapsto & \big((X_{(\beta/\alpha)^{+i}})_{i\in\mathbf{Z}}\big)_{\beta/\alpha\in\bar{\Delta}_n^\#} \\ \big((Y_{\beta/\alpha})_0\big)_{\beta/\alpha\in\bar{\Delta}_n^\#} & \longleftarrow & Y\,. \end{array}$$

Both categories are exact when equipped with pointwise split short exact sequences, and Φ and Φ^{-1} are exact functors. We have $\mathcal{A}^{+,\,\text{split}}(\bar{\Delta}_n^\#)\Phi = (\mathcal{A}^{\mathbf{Z}})^{+,\,\text{split},\,\text{periodic}}(\bar{\Delta}_n^\#)$.

Putting $\mathcal{A}' := \mathcal{A}^{\mathbf{Z}}$, the result follows by Lemma I.36. ▫

If $\mathcal{A} = \mathcal{C}$ is a weakly abelian category, we have two definitions of $\mathcal{C}^{+,\,\text{split}}(\bar{\Delta}_n^\#)$.

The first one, given in §I.1.2.1.1, defines this category as a full subcategory of $\mathcal{C}^+(\bar{\Delta}_n^\#)$ containing those diagrams in which all morphisms are split in $\hat{\mathcal{C}}$.

The second one, just given, defines this category as a full subcategory of $\mathcal{C}^0(\bar{\Delta}_n^\#)$ containing, up to isomorphism, summands of direct sums of extended intervals.

Remark I.38 *If $\mathcal{A} = \mathcal{C}$ is a weakly abelian category, then the two aforementioned definitions of $\mathcal{C}^{+,\,\text{split}}(\bar{\Delta}_n^\#)$ coincide.*

Proof. First, we notice that an extended interval lies in $\mathcal{C}^+(\bar{\Delta}_n^\#)$, and that all its diagram morphisms are split in $\hat{\mathcal{C}}$.

It remains to be shown that an object in $\mathcal{C}^+(\bar{\Delta}_n^\#)$ all of whose diagram morphisms are split in $\hat{\mathcal{C}}$, is, up to isomorphism, a summand of a direct sum of extended intervals.

Passing to $(\mathcal{C}^{\mathbf{Z}})^{+,\,\text{periodic}}(\bar{\Delta}_n^\#)$, we have to show that an object $X \in \mathrm{Ob}(\mathcal{C}^{\mathbf{Z}})^{+,\,\text{periodic}}(\bar{\Delta}_n^\#)$ all of whose diagram morphisms are split in $\hat{\mathcal{C}}^{\mathbf{Z}}$, is, up to isomorphism, a summand of a direct sum of periodic extended intervals.

By Lemma I.65, applied to the abelian Frobenius category $\hat{\mathcal{C}}^{\mathbf{Z}}$, the object $X|_{\dot{\Delta}_n} \in \mathrm{Ob}\,\mathcal{C}^{\mathbf{Z}}(\dot{\Delta}_n)$ is isomorphic to a summand of a finite direct sum of intervals. Hence, by Lemma I.17, the object X is isomorphic to a summand of a finite direct sum of images of intervals under $\bar{S}$, i.e. of periodic extended intervals, as was to be shown. ▫

I.5.1.2 The subcategory $\mathcal{C}^+(\bar{\Delta}_n^\#) \subseteq \mathcal{C}^0(\bar{\Delta}_n^\#)$

Recall that $\mathcal{C}$ is a weakly abelian category.

Lemma I.39 *Suppose given a pure short exact sequence*

$$X' \rightarrowtail X \twoheadrightarrow X''$$

in $\mathcal{C}^0(\bar{\Delta}_n^\#)$. If two out of the three objects X', X and X'' are in $\mathcal{C}^+(\bar{\Delta}_n^\#)$, so is the third.

Proof. For an object $X \in \operatorname{Ob} \mathcal{C}^0(\bar{\Delta}_n)$ to lie in $\operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n)$, it suffices to know that the complex

$$X(\alpha,\beta,\gamma) := \big(\cdots \longrightarrow X_{\beta/\gamma^{-1}} \longrightarrow X_{\beta/\alpha} \longrightarrow X_{\gamma/\alpha} \longrightarrow X_{\gamma/\beta} \longrightarrow X_{\alpha^{+1}/\beta} \longrightarrow \cdots\big)$$

is acyclic in $\hat{\mathcal{C}}$ for all $\alpha,\ \beta,\ \gamma\ \in\ \bar{\Delta}_n$ with $\alpha \leqslant \beta \leqslant \gamma \leqslant \alpha^{+1}$; which is true, as we take from Lemma I.57; cf. Remark I.67.

Now the long exact homology sequence, applied in $\hat{\mathcal{C}}$ to the short exact sequence $X'(\alpha,\beta,\gamma) \rightarrowtail X(\alpha,\beta,\gamma) \twoheadrightarrow X''(\alpha,\beta,\gamma)$ of complexes, shows that if two of these complexes are acyclic, so is the third. □

Proposition I.40

(1) *The category $\mathcal{C}^+(\bar{\Delta}_n^{\#})$, equipped with the pointwise split short exact sequences, is a Frobenius category, having $\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\#})$ as its subcategory of bijectives.*

Hence its stable category $\underline{\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}}$ is equivalent to its classical stable category $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})} = \mathcal{C}^+(\bar{\Delta}_n^{\#})/\mathcal{C}^{+,\,\mathrm{split}}(\bar{\Delta}_n^{\#})$. So both $\underline{\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}}$ and $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ are weakly abelian.

(2) *Suppose $\mathcal{C}$ to be equipped with an automorphism $X \longmapsto X^{+1}$. The category $\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ (cf. §I.2.5.3), equipped with the pointwise split short exact sequences, is an additively functorial Frobenius category, having $\mathcal{C}^{+,\,\mathrm{split,\,periodic}}(\bar{\Delta}_n^{\#})$ as its subcategory of bijectives.*

We remark that $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ appears in Definition I.5.

Proof. Ad (1). To prove that $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ is an exact category, it remains to be shown, in view of Lemma I.37 and of §I.6.2.2, that a pure short exact sequence in $\mathcal{C}^0(\bar{\Delta}_n^{\#})$ that has the first and the third term in $\operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\#})$, has the second term in $\operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\#})$, too. This follows by Lemma I.39.

To prove that $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ is Frobenius, we may use that $\mathcal{C}^0(\bar{\Delta}_n^{\#})$ is Frobenius, with the bijective objects already lying in $\mathcal{C}^+(\bar{\Delta}_n^{\#})$, thus being a fortiori bijective with respect to $\mathcal{C}^+(\bar{\Delta}_n^{\#})$. By duality, it remains to be shown that the kernel of a pointwise split epimorphism of a bijective object to a given object in $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ is again in $\mathcal{C}^+(\bar{\Delta}_n^{\#})$, thus showing that this epimorphism is actually pure in $\mathcal{C}^+(\bar{\Delta}_n^{\#})$. This follows by Lemma I.39.

Ad (2). In view of Lemma I.36, this follows as (1). □

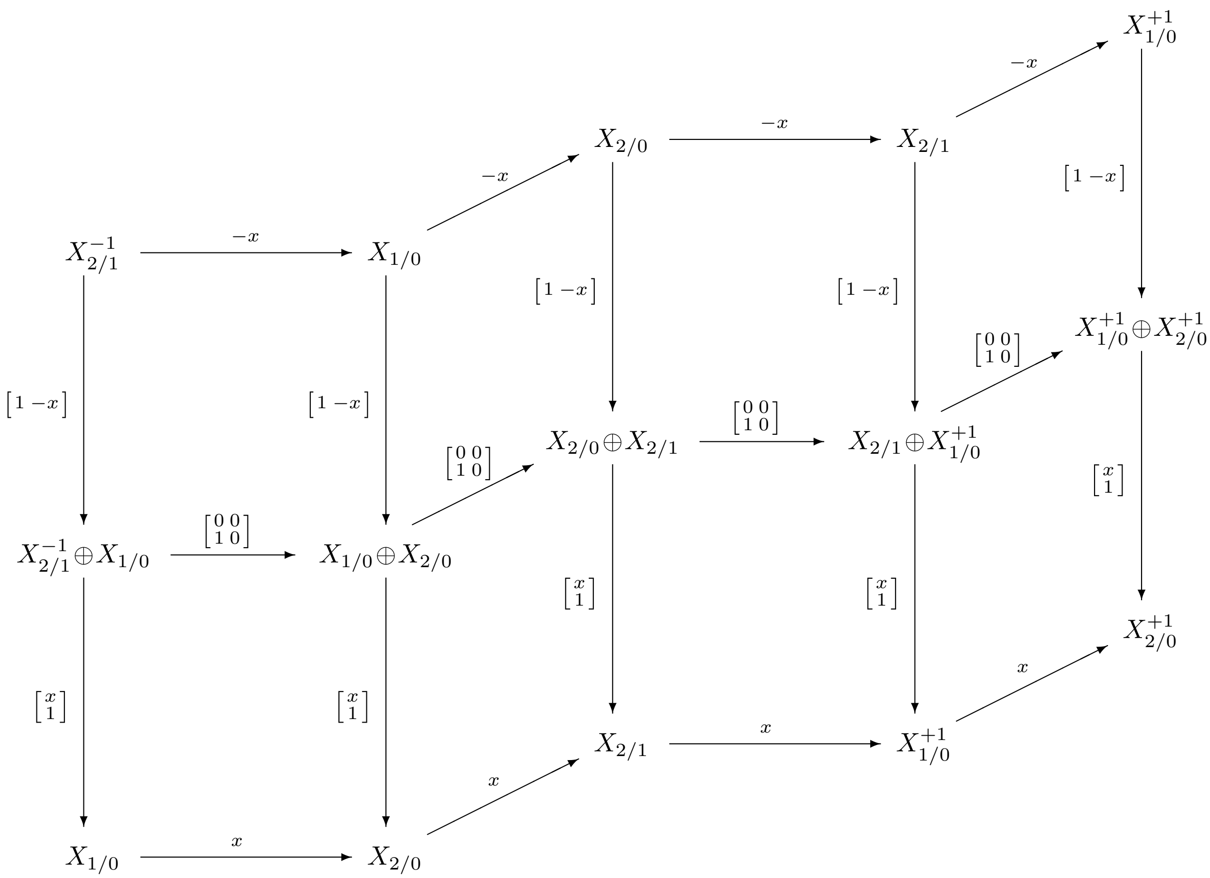
I do not know whether $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ is Frobenius. It seems doubtful, since this question is reminiscent of the example of A. Neeman that shows that the mapping cone of a morphism of distinguished triangles in the sense of Verdier need not be distinguished [**49,** p. 234].

I.5.1.3 Two examples

Suppose $\mathcal{C}$ to be equipped with an automorphism $X \longmapsto X^{+1}$.

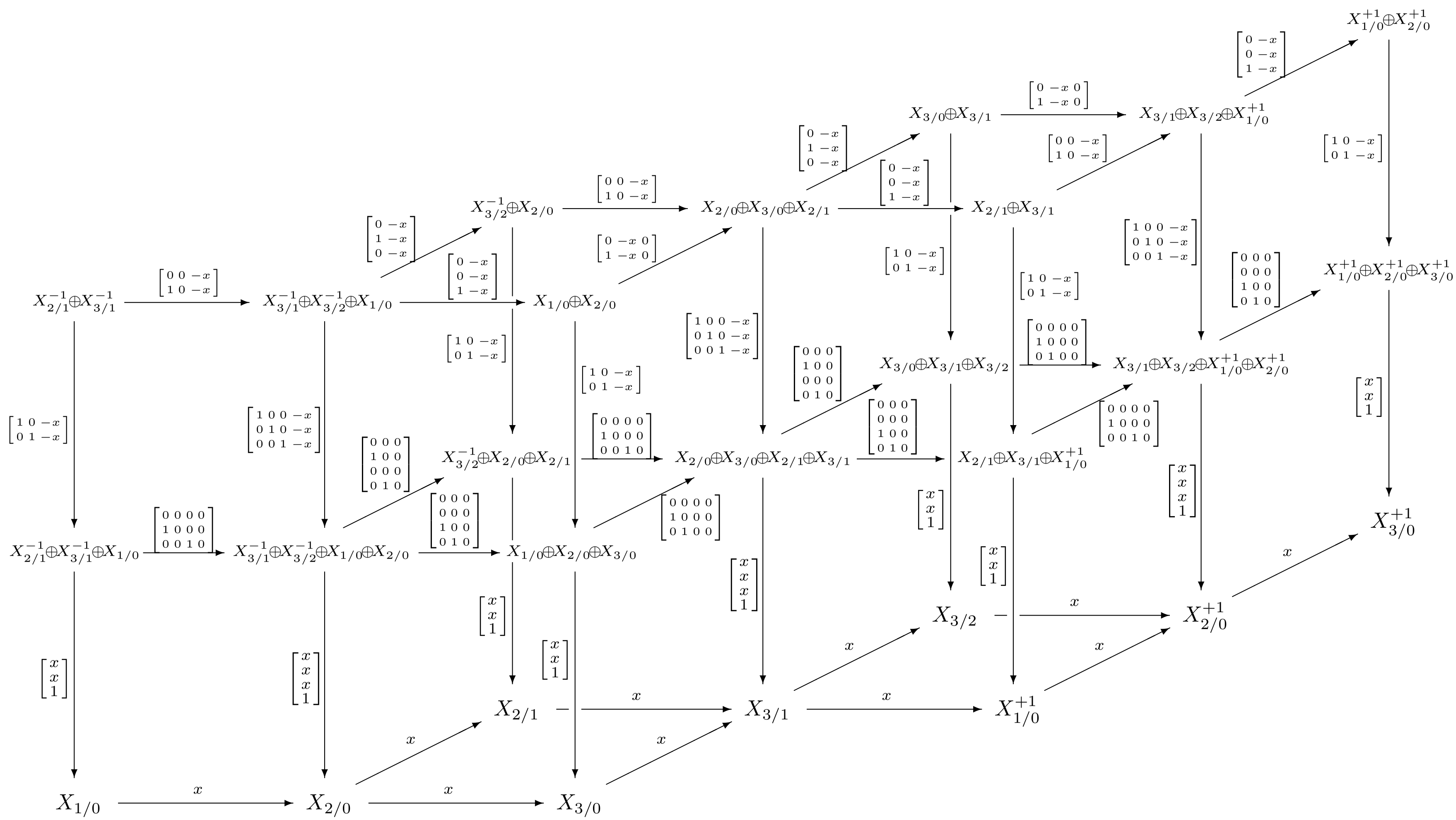
The category $\mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ being a Frobenius category by Proposition I.40.(2), its classical stable category $\underline{\mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})}$ carries a Heller operator, defined on $X \in \operatorname{Ob} \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ as the kernel of $B \twoheadrightarrow X$, where $B \in \operatorname{Ob} \mathcal{C}^{+,\,\text{split},\,\text{periodic}}(\bar{\Delta}_n^{\#})$. As examples, we calculate the Heller operator for $n \in \{2, 3\}$ on periodic n-pretriangles.

Suppose $n = 2$. Let $X \in \operatorname{Ob} \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_2^{\#})$ be a periodic 2-pretriangle. We obtain



In particular, if X is a 2-triangle, i.e. an object of $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_2^{\#})$, then this Heller shift of X is also a 2-triangle; cf. Lemma I.22.

Suppose $n = 3$. Let $X \in \mathrm{Ob}\ \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_3^{\#})$ be a periodic 3-pretriangle. We obtain



If X is a 3-triangle, i.e. an object $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_3^{\#})$, I do not know whether this Heller shift of X is again a 3-triangle.

I.5.2 A quasicyclic category

The category of *quasicyclic categories* is defined to be the category of contravariant functors from $\bar{\mathbf{\Delta}}^\circ$ to the (1-)category (Cat) of categories. Recall that we have a functor $\mathbf{\Delta} \longrightarrow \bar{\mathbf{\Delta}}$, $\Delta_n \longmapsto \bar{\Delta}_n$ that allows to restrict a quasicyclic category to its *underlying simplicial category.*

Given a category $\mathcal{U}$, we denote by $\operatorname{Iso}\mathcal{U} \subseteq \mathcal{U}$ its subcategory consisting of isomorphisms. Given a functor $\mathcal{U} \xrightarrow{U} \mathcal{U}'$, we denote by $\operatorname{Iso} F : \operatorname{Iso}\mathcal{U} \longrightarrow \operatorname{Iso}\mathcal{U}'$ the induced functor.

We define

$$\begin{array}{ccc} \bar{\mathbf{\Delta}}^\circ & \xrightarrow{\operatorname{qcyc}_\bullet \mathcal{C}} & (\mathrm{Cat}) \\ (\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m) & \longmapsto & \Big(\operatorname{Iso}\mathcal{C}^+(\bar{\Delta}_n^\#) \xrightarrow{\operatorname{Iso}\mathcal{C}^+(p^\#)} \operatorname{Iso}\mathcal{C}^+(\bar{\Delta}_m^\#)\Big) \,. \end{array}$$

More intuitively written, $\operatorname{qcyc}_\bullet \mathcal{C} := \operatorname{Iso}\mathcal{C}^+(\bar{\Delta}_\bullet^\#)$. Note that $\operatorname{qcyc}_0 \mathcal{C}$ consists only of zero-objects.

A strictly exact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ induces a functor $\mathcal{C}^+(\bar{\Delta}_n^\#) \xrightarrow{F^+(\bar{\Delta}_n^\#)} \mathcal{C}'^+(\bar{\Delta}_n^\#)$ for $n \geqslant 0$, and thus a morphism

$$\operatorname{qcyc}_\bullet \mathcal{C} \xrightarrow{\operatorname{qcyc}_\bullet F} \operatorname{qcyc}_\bullet \mathcal{C}'$$

of quasicyclic categories.

As variants, we mention

$$\begin{array}{rcl} \operatorname{qcyc}_\bullet^{\mathrm{periodic}} \mathcal{C} & := & \operatorname{Iso}\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_\bullet^\#) \\ \operatorname{qcyc}_\bullet^{\vartheta=1} \mathcal{C} & := & \operatorname{Iso}\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_\bullet^\#) \,. \end{array}$$

I.5.3 A biquasicyclic category

> As an attempt in the direction described in [**58,** p. 330], we define a first step of an "iteration" of the construction $\mathcal{C} \mapsto \operatorname{qcyc}_\bullet \mathcal{C}$.

By Proposition I.40, we may form the category $\underline{\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}}^+(\bar{\Delta}_m^\#)$. Note that a morphism $\bar{\Delta}_m \xleftarrow{f} \bar{\Delta}_{m'}$ of periodic linearly ordered sets induces a functor $\underline{\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}}^+(f^\#)$ in the second variable.

By Lemma I.34, a morphism $\bar{\Delta}_n \xleftarrow{g} \bar{\Delta}_{n'}$ of periodic linearly ordered sets induces a strictly exact functor $\underline{\underline{\mathcal{C}^+(g^\#)}}$, and so a functor $\underline{\underline{\mathcal{C}^+(g^\#)}}^+(\bar{\Delta}_m^\#)$ in the first variable for $m \geqslant 0$.

The functors induced by f and by g commute.

We may define

$$\operatorname{qcyc}_{\bullet\bullet} \mathcal{C} \;:=\; \operatorname{Iso}\Big(\underline{\underline{\mathcal{C}^+(\bar{\Delta}_\bullet^\#)}}^+(\bar{\Delta}_\bullet^\#)\Big) \,,$$

which yields a *biquasicyclic category,* i.e. a functor from $\bar{\mathbf{\Delta}}^\circ \times \bar{\mathbf{\Delta}}^\circ$ to the (1-)category (Cat) of categories.

By Lemma I.34, a strictly exact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ induces a functor $\underline{\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}} \xrightarrow{\underline{\underline{F^+(\bar{\Delta}_n^\#)}}} \underline{\underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}}$ for $n \geqslant 0$, and thus a morphism

$$\operatorname{qcyc}_{\bullet\bullet} \mathcal{C} \xrightarrow{\operatorname{qcyc}_{\bullet\bullet} F} \operatorname{qcyc}_{\bullet\bullet} \mathcal{C}'$$

of quasicyclic categories.

As variants, we mention

$$\begin{array}{rcl} \mathrm{qcyc}^{\mathrm{periodic}}_{\bullet\bullet}\mathcal{C} & := & \mathrm{Iso}\left(\underline{\underline{\mathcal{C}^+(\bar{\Delta}^{\#}_{\bullet})}}^{+,\,\mathrm{periodic}}(\bar{\Delta}^{\#}_{\bullet})\right) \\ \mathrm{qcyc}^{\vartheta=1}_{\bullet\bullet}\mathcal{C} & := & \mathrm{Iso}\left(\underline{\underline{\mathcal{C}^+(\bar{\Delta}^{\#}_{\bullet})}}^{+,\,\vartheta=1}(\bar{\Delta}^{\#}_{\bullet})\right) . \end{array}$$

Cf. Remark I.20.

This procedure can be iterated to obtain triquasicyclic categories etc.

I.6 Some general lemmata

This appendix is a tool kit consisting of folklore lemmata (with proof) and known results (mainly without proof). We do not claim originality.

I.6.1 An additive lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, and let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a full and dense additive functor. Let $\mathcal{N} \subseteq \mathcal{B}$ be a full additive subcategory. Let $\mathcal{M} \subseteq \mathcal{A}$ be the full subcategory determined by

$$\mathrm{Ob}\,\mathcal{M} \; := \; \{A \in \mathrm{Ob}\,\mathcal{A} \; : \; AF \text{ is isomorphic to an object of } \mathcal{N}\} \; .$$

Lemma I.41 *Suppose that for each morphism $A \xrightarrow{a_0} A'$ in $\mathcal{A}$ such that $a_0 F = 0$, there exists a factorisation*

$$(A \xrightarrow{a_0} A') \; = \; (A \xrightarrow{a_0'} M_0 \xrightarrow{a_0''} A')$$

with $M_0 \in \mathrm{Ob}\,\mathcal{M}$. Then the induced functor

$$\begin{array}{rcl} \mathcal{A}/\mathcal{M} & \xrightarrow{\underline{F}} & \mathcal{B}/\mathcal{N} \\ (A \xrightarrow{a} A') & \longmapsto & (AF \xrightarrow{aF} A'F) \end{array}$$

is an equivalence.

Proof. We have to show that $\underline{F}$ is faithful. Suppose given $A \xrightarrow{a} A'$ in $\mathcal{A}$ such that

$$(AF \xrightarrow{aF} A'F) \; = \; (AF \xrightarrow{b'} N \xrightarrow{b''} A'F) \; ,$$

where $N \in \mathrm{Ob}\,\mathcal{N}$. Since F is dense, we may assume $N = MF$ for some $M \in \mathrm{Ob}\,\mathcal{M}$. Since F is full, there exist a' and a'' in $\mathcal{A}$ with $a'F = b'$ and $a''F = b''$. Then

$$(A \xrightarrow{a} A') \; = \; (A \xrightarrow{a'a''} A') + (A \xrightarrow{a_0} A')$$

with $a_0 F = 0$. Since $a'a''$ factors over $M \in \mathrm{Ob}\,\mathcal{M}$, and since a_0 factors over an object of $\mathcal{M}$ by assumption on F, we conclude that a factors over an object of $\mathcal{M}$. □

I.6.2 Exact categories

I.6.2.1 Definition

The concept of exact categories is due to Quillen **[52]**, who uses a different, but equivalent set of axioms. In **[31,** App. A], Keller has cut down redundancies in this set of axioms. We give still another equivalent reformulation.

An *additive category* $\mathcal{A}$ is a category with zero object, binary products and binary coproducts such that the natural map from the coproduct to the product is an isomorphism; which allows to define a commutative and associative addition $(+)$ on ${}_{\mathcal{A}}(X,Y)$, where X, $Y \in \operatorname{Ob}\mathcal{A}$; and such that there exists an endomorphism -1_X for each $X \in \operatorname{Ob}\mathcal{A}$ that is characterised by $1_X + (-1_X) = 0_X$.

A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called *short exact* if f is a kernel of g and g is a cokernel of f.

A short exact sequence isomorphic to a short exact sequence of the form

$$X \xrightarrow{(1\;0)} X \oplus Y \xrightarrow{\binom{0}{1}} Y\,,$$

where X, $Y \in \operatorname{Ob}\mathcal{A}$, is called *split short exact.* A morphism appearing as a kernel in a split short exact sequence is *split monomorphic*, a morphism appearing as a cokernel in a split short exact sequence is called *split epimorphic.* A split short exact sequence is isomorphic to a sequence of the form just displayed by an isomorphism having an identity on the first and on the third term.

An *exact category* $(\mathcal{E},\mathcal{S})$ consists of an additive category $\mathcal{E}$ and an isomorphism closed set $\mathcal{S}$ of short exact sequences in $\mathcal{E}$, called *pure short exact sequences* ([8]), such that the following axioms (Ex 1, 2, 3, 1°, 2°, 3°) are satisfied.

A monomorphism fitting into a pure short exact sequence is called a *pure monomorphism*, denoted by $\overset{\bullet}{\longrightarrow}$; an epimorphism fitting into a pure short exact sequence is called a *pure epimorphism*, denoted by $\overset{|}{\longrightarrow}$. A morphism which can be written as a composition of a pure epimorphism followed by a pure monomorphism is called *pure.*

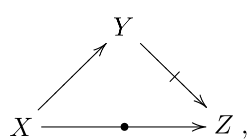
(Ex 1) Split monomorphisms are pure monomorphisms.

(Ex 1°) Split epimorphisms are pure epimorphisms.

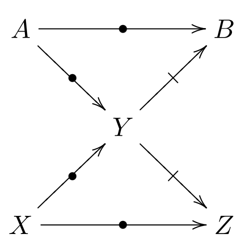
(Ex 2) The composite of two pure monomorphisms is purely monomorphic.

(Ex 2°) The composite of two pure epimorphisms is purely epimorphic.

(Ex 3) Given a commutative diagram



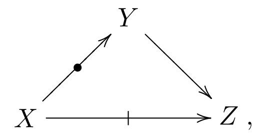
we may insert it into a commutative diagram



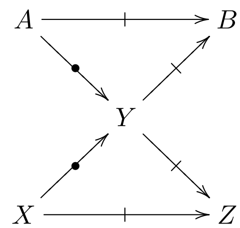
with (X,Y,B) and (A,Y,Z) pure short exact sequences.

[8] This notion is borrowed from the particular cases of pure short exact sequences of lattices over orders and of $\otimes$-pure short exact sequences of modules. Other frequently used names are *admissible short exact sequence*, consisting of an *admissible monomorphism* and an *admissible epimorphism*; and *conflation,* consisting of an *inflation* and a *deflation.*

(Ex 3°) Given a commutative diagram



we may insert it into a commutative diagram



with (X, Y, B) and (A, Y, Z) pure short exact sequences.

An *exact functor* from an exact category $(\mathcal{E}, \mathcal{S})$ to an exact category $(\mathcal{F}, \mathcal{T})$ is given by an additive functor $\mathcal{E} \xrightarrow{F} \mathcal{F}$ such that $\mathcal{S}F \subseteq \mathcal{T}$, where, by abuse of notation, F also denotes the functor induced by F on diagrams of shape $\bullet \longrightarrow \bullet \longrightarrow \bullet$.

Frequently, the exact category $(\mathcal{E}, \mathcal{S})$ is simply referred to by $\mathcal{E}$.

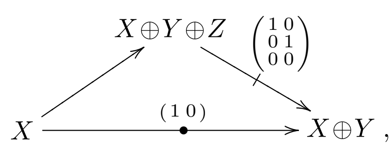
Example I.42

(1) An abelian category, equipped with the set of all short exact sequences as pure short exact sequences, is an exact category.

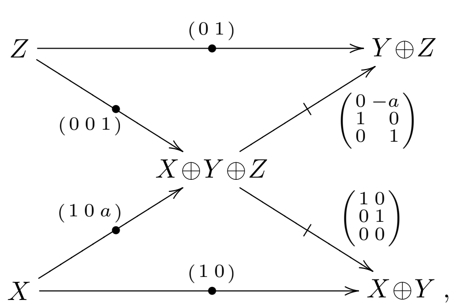
(2) If $\mathcal{E}$ is an exact category, so is $\mathcal{E}^\circ$, equipped with the pure short exact sequences of $\mathcal{E}$ considered as short exact sequences in $\mathcal{E}^\circ$, with the roles of kernel and cokernel interchanged.

Example I.43 An additive category $\mathcal{A}$, equipped with the set of split short exact sequences as pure short exact sequences, is an exact category.

In fact, (Ex 1, 2) are fulfilled, and it remains to prove (Ex 3); then the dual axioms ensue by duality. Given



we get



where $X \xrightarrow{a} Z$ is the third component of the given morphism $X \longrightarrow X \oplus Y \oplus Z$.

Example I.44 Suppose given an exact category $\mathcal{E}$ and a category D. Let a short exact sequence (X, Y, Z) in $\mathcal{E}(D)$ be pure if the sequence (X_d, Y_d, Z_d) is a pure short exact sequence in $\mathcal{E}$ for all $d \in \operatorname{Ob} D$. Then $\mathcal{E}(D)$ is an exact category.

I.6.2.2 Embedding exact categories

By a theorem

- stated by QUILLEN [**52,** p. 100],
- proven by LAUMON [**41,** Th. 1.0.3],
- re-proven by KELLER [**31,** Prop. A.2],
- where QUILLEN resp. KELLER refer to [**16**] for a similar resp. an auxiliary technique,

for any exact category $\mathcal{E}$, there exists an abelian category $\tilde{\mathcal{E}}$ containing $\mathcal{E}$ as a full subcategory closed under extensions, the pure short exact sequences in $\mathcal{E}$ being the short exact sequences in $\tilde{\mathcal{E}}$ with all three objects in $\operatorname{Ob}\mathcal{E}$.

Conversely, suppose given an exact category $\mathcal{E}$ and a full subcategory $\mathcal{E}' \subseteq \mathcal{E}$ such that whenever (X, Y, Z) is a pure short exact sequence in $\mathcal{E}$ with $X,\ Z \in \operatorname{Ob}\mathcal{E}'$, then also $Y \in \operatorname{Ob}\mathcal{E}'$. Then the subcategory $\mathcal{E}'$, equipped with the pure short exact sequences in $\mathcal{E}$ with all three terms in $\operatorname{Ob}\mathcal{E}'$ as pure short exact sequences in $\mathcal{E}'$, is an exact category.

I.6.2.3 Frobenius categories: definitions

Definition I.45

(1) A *bijective object* in an exact category $\mathcal{E}$ is an object B for which ${}_{\mathcal{E}}(B, -)$ and ${}_{\mathcal{E}}(-, B)$ are exact functors from $\mathcal{E}$ resp. from $\mathcal{E}^\circ$ to $\mathbf{Z}$-Mod.

(2) A *Frobenius category* is an exact category for which each object X allows for a diagram $B \twoheadrightarrow X \rightarrowtail B'$ with B and B' bijective.

(3) Suppose given an exact category $\mathcal{F}$ carrying a shift automorphism $\mathsf{T} : X \longmapsto X\,\mathsf{T} = X^{+1}$ and two additive endofunctors I and P together with natural transformations $1_{\mathcal{F}} \xrightarrow{\iota} \mathsf{I}$ and $\mathsf{P} \xrightarrow{\pi} 1_{\mathcal{F}}$ such that $\mathsf{T}\,\mathsf{P} = \mathsf{I}$ and such that

$$X \xrightarrow{X\iota} X\,\mathsf{I} = X^{+1}\,\mathsf{P} \xrightarrow{X^{+1}\pi} X^{+1}$$

is a pure short exact sequence with bijective middle term for all $X \in \operatorname{Ob}\mathcal{C}$. Then $(\mathcal{F}, \mathsf{T}, \mathsf{I}, \iota, \mathsf{P}, \pi)$ is called a *functorial Frobenius category.* Often we write just $\mathcal{F}$ for $(\mathcal{F}, \mathsf{T}, \mathsf{I}, \iota, \mathsf{P}, \pi)$.

Example I.46

(1) Let $\mathcal{A}$ be an additive category. Let $\dot{\mathbf{Z}}$ denote the discrete category with $\operatorname{Ob}\dot{\mathbf{Z}} = \mathbf{Z}$ and only identical morphisms. The category $\mathcal{A}(\dot{\mathbf{Z}})$ carries the shift functor $X^\bullet \longmapsto X^{\bullet+1}$, where $(X^{\bullet+1})^i = X^{i+1}$. An object in the category $\mathrm{C}(\mathcal{A})$ of complexes with entries in $\mathcal{A}$ is written $(X^\bullet, d^\bullet)$, where X is an object of $\mathcal{A}(\dot{\mathbf{Z}})$ and where $X^\bullet \xrightarrow{d^\bullet} X^{\bullet+1}$ with $d^\bullet d^{\bullet+1} = 0$. The category $\mathrm{C}(\mathcal{A})$, equipped with pointwise split short exact sequences, is an exact category; cf. Examples I.43, I.44. Given a complex $(X^\bullet, d^\bullet)$, we let $(X^\bullet, d^\bullet)\,\mathsf{T} = (X^\bullet, d^\bullet)^{+1} := (X^{\bullet+1}, -d^{\bullet+1})$ and

$$\Big((X^\bullet, d^\bullet) \xrightarrow{(X^\bullet, d^\bullet)\iota} (X^\bullet, d^\bullet)\,\mathsf{I} = (X^\bullet, d^\bullet)^{+1}\,\mathsf{P} \xrightarrow{(X^\bullet, d^\bullet)^{+1}\pi} (X^\bullet, d^\bullet)^{+1}\Big)$$

$$:= \Big((X^\bullet, d^\bullet) \xrightarrow{(1\ d^\bullet)} \big(X^\bullet \oplus X^{\bullet+1}, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\big) \xrightarrow{\left(\begin{smallmatrix} -d^\bullet \\ 1 \end{smallmatrix}\right)} (X^{\bullet+1}, -d^{\bullet+1})\Big)\,.$$

Then $(\mathrm{C}(\mathcal{A}), \mathsf{T}, \mathsf{I}, \iota, \mathsf{P}, \pi)$ is a functorial Frobenius category.

(2) Suppose $\mathcal{E}$ to be a Frobenius category. Let $\mathcal{B} \subseteq \mathcal{E}$ be a *sufficiently big full subcategory of bijective objects,* i.e. each object of $\mathcal{B}$ is bijective in $\mathcal{E}$, and each object X of $\mathcal{E}$ admits $B \twoheadrightarrow X \rightarrowtail B'$ with $B,\ B' \in \operatorname{Ob}\mathcal{B}$. In other words, each bijective object of $\mathcal{E}$ is isomorphic to a direct summand of an object of $\mathcal{B}$.

Let $\mathcal{B}^{\mathrm{ac}} \subseteq \mathrm{C}(\mathcal{B})$ denote the full subcategory of *purely acyclic* complexes, i.e. complexes $(X^\bullet, d^\bullet)$ such that all differentials $X^i \xrightarrow{d} X^{i+1}$ are pure, factoring in $\mathcal{E}$ as $d = \bar{d}\dot{d}$ with $\bar{d}$ purely epi- and $\dot{d}$ purely monomorphic, and such that all resulting sequences $(\dot{d}, \bar{d})$ are purely short exact. For short, a complex is purely acyclic if it decomposes into pure short exact sequences.

Then $\mathcal{B}^{\mathrm{ac}}$ is a functorial Frobenius category, equipped with the restricted functors and transformations of $\mathrm{C}(\mathcal{B})$ as defined in (1); cf. [**48,** Lem. 1.1]. Let $\mathcal{B}^{\mathrm{sp\,ac}} \subseteq \mathcal{B}^{\mathrm{ac}}$ be the full subcategory of split acyclic complexes, i.e. of complexes isomorphic to a complex of the form $(T^\bullet \oplus T^{\bullet+1}, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$ for some $T^\bullet \in \mathrm{Ob}\,\mathcal{B}(\dot{\mathbf{Z}})$. Then $\mathcal{B}^{\mathrm{sp\,ac}}$ is a sufficiently big full subcategory of bijective objects in $\mathcal{B}^{\mathrm{ac}}$.

Definition I.47 Suppose given a Frobenius category $\mathcal{E}$, and a sufficiently big full subcategory $\mathcal{B} \subseteq \mathcal{E}$ of bijectives. Let

$$\begin{array}{rcll} \underline{\mathcal{E}} & := & \mathcal{E}/\mathcal{B} & \text{be the } \textit{classical stable category} \text{ of } \mathcal{E}\,; \\ \underline{\underline{\mathcal{E}}} & := & \mathcal{B}^{\mathrm{ac}}/\mathcal{B}^{\mathrm{sp\,ac}} & \text{be the } \textit{stable category} \text{ of } \mathcal{E}\,. \end{array}$$

In other words, the stable category $\underline{\underline{\mathcal{E}}}$ of $\mathcal{E}$ is defined to be the classical stable category $\underline{\mathcal{B}^{\mathrm{ac}}}$ of $\mathcal{B}^{\mathrm{ac}}$. The shift functor induced by the automorphism T of $\mathcal{B}^{\mathrm{ac}}$ on $\underline{\underline{\mathcal{E}}}$ is also denoted by T.

Lemma I.48 *The functor*

$$\begin{array}{rcl} \mathcal{B}^{\mathrm{ac}} & \xrightarrow{I} & \mathcal{E} \\ (X, d) & \longmapsto & \mathrm{Im}(X^0 \xrightarrow{d} X^1) \end{array}$$

induces an equivalence

$$\underline{\underline{\mathcal{E}}} = \underline{\mathcal{B}^{\mathrm{ac}}} \xrightarrow[\sim]{\underline{I}} \underline{\mathcal{E}}\,.$$

Cf. [**33,** Sec. 4.3].

Proof. This is an application of Lemma I.41. □

We choose an inverse equivalence R to $\underline{I}$. We have the residue class functor $\mathcal{E} \xrightarrow{N} \underline{\mathcal{E}}$, and, by abuse of notation, a second residue class functor $(\mathcal{E} \xrightarrow{N} \underline{\underline{\mathcal{E}}}) := (\mathcal{E} \xrightarrow{N} \underline{\mathcal{E}} \xrightarrow[\sim]{R} \underline{\underline{\mathcal{E}}})$.

A morphism $X \xrightarrow{f} Y$ is zero in $\underline{\mathcal{E}}$ if and only if for any monomorphism $X \xrightarrow{i} X'$ and any epimorphism $Y' \xrightarrow{p} Y$, there is a factorisation $f = if'p$. This defines $\underline{\mathcal{E}}$ without mentioning bijective objects in $\mathcal{E}$. So one might speculate whether the class of Frobenius categories within the class of exact categories could be extended still without losing essential properties of Frobenius categories.

I.6.3 Kernel-cokernel-criteria

Let $\mathcal{A}$ be an abelian category. The *circumference lemma* states that given a commutative triangle in $\mathcal{A}$, the induced sequence on kernels and cokernels, with zeroes attached to the ends, is long exact.

Definition I.49 A *weak square* in $\mathcal{A}$ is a commutative quadrangle (A, B, C, D) in $\mathcal{A}$ whose diagonal sequence $(A, B \oplus C, D)$ is exact at $B \oplus C$. It is denoted by a "+"-sign in the commutative diagram,

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & + & \uparrow \\ A & \longrightarrow & B\,. \end{array}$$

A *pullback* is a weak square with first morphism in the diagonal sequence being monomorphic. It is denoted

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \llcorner & \uparrow \\ A & \longrightarrow & B\,. \end{array}$$

A *pushout* is a weak square with second morphism in the diagonal sequence being epimorphic. It is denoted

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \urcorner & \uparrow \\ A & \longrightarrow & B\,. \end{array}$$

A *square* is a commutative quadrangle that is a pullback and a pushout, i.e. that has a short exact diagonal sequence. It is denoted

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \square & \uparrow \\ A & \longrightarrow & B\,. \end{array}$$

Remark I.50 *If a commutative quadrangle in* $\mathcal{A}$

$$\begin{array}{ccc} C & \xrightarrow{c} & D \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle d} \\ A & \xrightarrow[b]{} & B \end{array}$$

is a square, then the induced morphism from the kernel of $A \xrightarrow{a} C$ *to the kernel of* $B \xrightarrow{d} D$ *is an isomorphism and the induced morphism from the cokernel of* $A \xrightarrow{a} C$ *to the cokernel of* $B \xrightarrow{d} D$ *is an isomorphism.*

Proof. If (A, B, C, D) is a square, then the circumference lemma, applied to the commutative triangle

$$\begin{array}{ccc} C & & \\ & \nwarrow^{\binom{0}{1}} & \\ {\scriptstyle a}\uparrow & & B \oplus C \\ & \nearrow_{(b\ a)} & \\ A & & \end{array},$$

yields a long exact sequence

$$0 \longrightarrow \mathrm{K}_a \xrightarrow{j} B \xrightarrow{-d} D \xrightarrow{q} \mathrm{C}_a \longrightarrow 0\,,$$

where $\mathrm{K}_a \xrightarrow{i} A$ is the kernel of a, and where $C \xrightarrow{p} \mathrm{C}_a$ is the cokernel of a. Since $ib = j$ and $cb = p$, the induced morphisms on the kernels and on the cokernels of a and d are isomorphisms. ◻

Lemma I.51 *A commutative quadrangle in* $\mathcal{A}$

$$\begin{array}{ccc} C & \xrightarrow{c} & D \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle d} \\ A & \xrightarrow[b]{} & B \end{array}$$

is a weak square if and only if the induced morphism $\mathrm{K}_a \longrightarrow \mathrm{K}_d$ *from the kernel of* $A \xrightarrow{a} C$ *to the kernel of* $B \xrightarrow{d} D$ *is an epimorphism and the induced morphism* $\mathrm{C}_a \longrightarrow \mathrm{C}_d$ *from the cokernel of* $A \xrightarrow{a} C$ *to the cokernel of* $B \xrightarrow{d} D$ *is a monomorphism.*

It is a pullback if and only if $\mathrm{K}_a \xrightarrow{\sim} \mathrm{K}_d$ *and* $\mathrm{C}_a \rightarrowtail \mathrm{C}_d$.

It is a pushout if and only if $\mathrm{K}_a \twoheadrightarrow \mathrm{K}_d$ *and* $\mathrm{C}_a \xrightarrow{\sim} \mathrm{C}_d$.

It is a square if and only if $\mathrm{K}_a \xrightarrow{\sim} \mathrm{K}_d$ *and* $\mathrm{C}_a \xrightarrow{\sim} \mathrm{C}_d$.

Proof. Let A' be the pullback of (C, B, D), and let D' be the pushout of (A', C, B). We obtain induced morphisms $A \longrightarrow A'$ and $D' \twoheadrightarrow D$. The circumference lemma, applied to (B, D', D), shows $\mathrm{C}_{B\to D'} \twoheadrightarrow \mathrm{C}_{B\to D}$.

The quadrangle (A, B, C, D) is a weak square if and only if $A \rightarrowtail A'$; which in turn, by the circumference lemma applied to (A, A', C), is equivalent to $\mathrm{K}_{A\to C} \rightarrowtail \mathrm{K}_{A'\to C}$ and $\mathrm{C}_{A\to C} \xrightarrow{\sim} \mathrm{C}_{A'\to C}$; which, by composition and by Remark I.50, applied to the square (A', B, C, D'), is equivalent to $\mathrm{K}_{A\to C} \rightarrowtail \mathrm{K}_{B\to D}$ and $\mathrm{C}_{A\to C} \twoheadrightarrow \mathrm{C}_{B\to D}$.

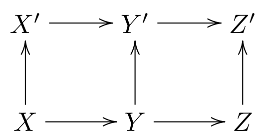
The quadrangle (A, B, C, D) is a pullback if and only if $A \xrightarrow{\sim} A'$; which in turn, by the circumference lemma applied to (A, A', C), is equivalent to $\mathrm{K}_{A\to C} \xrightarrow{\sim} \mathrm{K}_{A'\to C}$ and $\mathrm{C}_{A\to C} \xrightarrow{\sim} \mathrm{C}_{A'\to C}$; which, by composition and by Remark I.50, applied to the square (A', B, C, D'), is equivalent to $\mathrm{K}_{A\to C} \xrightarrow{\sim} \mathrm{K}_{B\to D}$ and $\mathrm{C}_{A\to C} \twoheadrightarrow \mathrm{C}_{B\to D}$.

The quadrangle (A, B, C, D) is a square if and only if $A \xrightarrow{\sim} A'$ and $D' \xrightarrow{\sim} D$; which in turn, by the circumference lemma applied to (A, A', C), is equivalent to $\mathrm{K}_{A\to C} \xrightarrow{\sim} \mathrm{K}_{A'\to C}$, $\mathrm{C}_{A\to C} \xrightarrow{\sim} \mathrm{C}_{A'\to C}$ and $\mathrm{C}_{B\to D'} \xrightarrow{\sim} \mathrm{C}_{B\to D}$; which, by composition and by Remark I.50, applied to the square (A', B, C, D'), is equivalent to $\mathrm{K}_{A\to C} \xrightarrow{\sim} \mathrm{K}_{B\to D}$ and $\mathrm{C}_{A\to C} \xrightarrow{\sim} \mathrm{C}_{B\to D}$. ▫

I.6.4 An exact lemma

Let $\mathcal{E}$ be an exact category. A *pure square* in $\mathcal{E}$ is a commutative quadrangle (A, B, C, D) in $\mathcal{E}$ that has a pure short exact diagonal sequence $(A, B \oplus C, D)$. Just as a square in abelian categories, a pure square is denoted by a box "□".

Lemma I.52 *Suppose given a composition*



of commutative quadrangles in $\mathcal{E}$*. If two out of the three quadrangles* (X, Y, X', Y')*,* (Y, Z, Y', Z')*,* (X, Z, X', Z') *are pure squares, so is the third.*

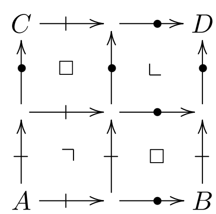
Proof. In an abelian category, this follows from Lemma I.51.

As explained in §I.6.2.2, we may embed $\mathcal{E}$ fully, faithfully and additively into an abelian category $\tilde{\mathcal{E}}$ such that the pure short exact sequences in $\mathcal{E}$ are precisely the short exact sequences in $\tilde{\mathcal{E}}$ with all three objects in $\operatorname{Ob}\mathcal{E}$. In particular, the pure squares in $\mathcal{E}$ are precisely the squares in $\tilde{\mathcal{E}}$ with all four objects in $\operatorname{Ob}\mathcal{E}$, and the assertion in $\mathcal{E}$ follows from the assertion in $\tilde{\mathcal{E}}$. ▫

I.6.5 Some abelian lemmata

Let $\mathcal{A}$ be an abelian category.

Lemma I.53 *Inserting images, a weak square* (A, B, C, D) *in* $\mathcal{A}$ *decomposes into*



Proof. The assertion follows using the characterisation of weak squares, pullbacks and pushouts given in Lemma I.51. ▫

Lemma I.54 *If, in a commutative diagram*

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \uparrow & + & \uparrow & + & \uparrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in $\mathcal{A}$, the quadrangles (X, Y, X', Y') and (Y, Z, Y', Z') are weak squares, then the composite quadrangle (X, Z, X', Z') is also a weak square.

Proof. The assertion follows using the characterisation of weak squares given in Lemma I.51. ▫

Lemma I.55 *If, in a commutative diagram*

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \uparrow & \urcorner & \uparrow & & \uparrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in $\mathcal{A}$, the left hand side quadrangle (X, Y, X', Y') is a pushout, as indicated, and the outer quadrangle (X, Z, X', Z') is a weak square, then the right hand side quadrangle (Y, Z, Y', Z') is also a weak square.

If the left hand side quadrangle (X, Y, X', Y') and the outer quadrangle (X, Z, X', Z') are pushouts, then the right hand side quadrangle (Y, Z, Y', Z') is also a pushout.

Proof. This follows using Lemma I.51. ▫

Lemma I.56 *If, in a commutative quadrangle in $\mathcal{A}$*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & Y \end{array},$$

the morphism $X \longrightarrow Y$ is an epimorphism and the morphism $X' \longrightarrow Y'$ is a monomorphism, then the quadrangle is a weak square.

Proof. This follows using Lemma I.51, applied horizontally. ▫

Lemma I.57 *Given a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & W' \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$ such that $(X, Z, 0, Z')$ and $(Y, 0, Y', W')$ are weak squares, then (Y, Z, Y', Z') is a weak square.

Proof. This follows using Lemma I.51. ▫

Lemma I.58 *Given a diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & Y'' & \xrightarrow{v''} & Z'' \\ \uparrow & + & \uparrow{\scriptstyle y'} & + & \uparrow{\scriptstyle z'} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \\ {\scriptstyle x}\uparrow & + & {\scriptstyle y}\uparrow & + & \uparrow \\ X & \xrightarrow{u} & Y & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$ consisting of weak squares, as indicated by $+$, the sequence

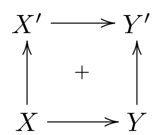
$$X \xrightarrow{xu'} Y' \xrightarrow{(y'\ v')} Y'' \oplus Z' \xrightarrow{\binom{v''}{-z'}} Z''$$

is exact at Y' and at $Y'' \oplus Z'$.

Proof. At Y', we reduce to the case u, u', x and y monomorphic and (X, Y, X', Y') being a pullback via Lemma I.53. Suppose given $T \xrightarrow{t} Y'$ with $ty' = 0$ and $tv' = 0$. First of all, there exist $T \xrightarrow{a} X'$ and $T \xrightarrow{b} Y$ such that $au' = t = by$. Thus there exists $T \xrightarrow{c} X$ such that $cx = a$ and $cu = b$. In particular, $cxu' = au' = t$. Hence a factorisation of t over xu' exists. Uniqueness follows by monomorphy of xu'. ▫

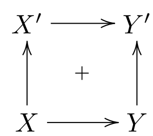
Lemma I.59

(1) *Suppose given a weak square in $\mathcal{A}$*



with X' bijective. If the images of $X \longrightarrow Y$, of $X \longrightarrow X'$ and of $Y \longrightarrow Y'$ are bijective, then the images of $X' \longrightarrow Y'$ and of $X \longrightarrow Y'$ are bijective, too.

(2) *Suppose given a weak square in $\mathcal{A}$*



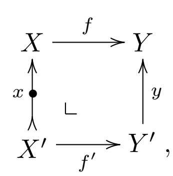
with Y bijective. If the images of $X' \longrightarrow Y'$, of $X \longrightarrow X'$ and of $Y \longrightarrow Y'$ are bijective, then the images of $X \longrightarrow Y$ and of $X \longrightarrow Y'$ are bijective, too.

Proof. Ad (1). We decompose (X, Y, X', Y') according to Lemma I.53 and denote the image of $X \longrightarrow Y$ by $\mathrm{Im}_{X,Y}$, etc.

The diagonal sequence of the square $(\mathrm{Im}_{X,Y},\ Y,\ \mathrm{Im}_{X,Y'},\ \mathrm{Im}_{Y,Y'})$ shows that $\mathrm{Im}_{X,Y'}$ is bijective.

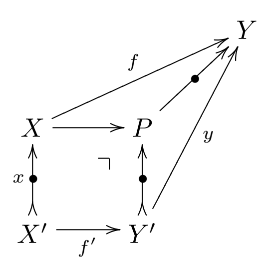
The diagonal sequence of the square $(\mathrm{Im}_{X,X'},\ \mathrm{Im}_{X,Y'},\ X',\ \mathrm{Im}_{X',Y'})$ shows that $\mathrm{Im}_{X',Y'}$ is bijective. ▫

Lemma I.60 *Given a pullback*



in $\mathcal{A}$ with Y' injective, the morphism $(X', Y') \xrightarrow{(x,y)} (X, Y)$ is split monomorphic in $\mathcal{A}(\Delta_1)$. More precisely, any retraction for x may be extended to a retraction for (x, y).

Proof. Let $xx' = 1_{X'}$. We form the pushout.



There is an induced morphism $P \longrightarrow Y'$ such that $(X \longrightarrow P \longrightarrow Y') = (X \xrightarrow{x'f'} Y')$ and such that $(Y' \longrightarrow P \longrightarrow Y') = (Y' \xrightarrow{1_{Y'}} Y')$. Since Y' is injective, we obtain a factorisation $(P \longrightarrow Y') = (P \twoheadrightarrow Y \longrightarrow Y')$. ▫

Lemma I.61 *Suppose given a morphism $X \longrightarrow Y$ of commutative quadrangles in $\mathcal{A}$, i.e. a morphism in $\mathcal{A}(\Delta_1 \times \Delta_1)$.*

(1) *If X is a pushout and Y is a weak square, then the cokernel of $X \longrightarrow Y$ is a weak square.*

(2) *If X is a weak square and Y is a pullback, then the kernel of $X \longrightarrow Y$ is a weak square.*

Proof. Ad (1). A morphism of commutative quadrangles gives rise to a morphism of the diagonal sequences; namely from a sequence that is exact in the middle and has an epimorphic second morphism, stemming from X, to a sequence that is exact in the middle, stemming from Y. In order to prove that the cokernel sequence is exact in the middle, we reduce by insertion of the image of the first morphism of the diagonal sequence and by an application of the circumference lemma to the case in which the sequence stemming from Y has a monomorphic first morphism. Then the snake lemma yields the result. ▫

Lemma I.62 *Suppose given a diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & Y'' & \longrightarrow & Z'' \\ \uparrow & + & \uparrow & + & \uparrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \uparrow & + & \uparrow & + & \uparrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in $\mathcal{A}$, consisting of weak squares. The induced morphisms furnish a short exact sequence

$$\mathrm{Im}(X \longrightarrow Z') \longrightarrow \mathrm{Im}(Y \longrightarrow Z') \longrightarrow \mathrm{Im}(Y \longrightarrow Z'')\,.$$

Proof. Abbreviate $\mathrm{Im}(X \longrightarrow Z')$ by $\mathrm{Im}_{X,Z'}$ etc. The morphism $\mathrm{Im}_{X,Z'} \longrightarrow \mathrm{Im}_{Y,Z'}$ is monomorphic by composition, and, dually, the morphism $\mathrm{Im}_{Y,Z'} \longrightarrow \mathrm{Im}_{Y,Z''}$ is epimorphic. Now since $\mathrm{Im}_{X,X'} \longrightarrow \mathrm{Im}_{X,Z'}$ is epimorphic and $\mathrm{Im}_{Y,Z''} \longrightarrow \mathrm{Im}_{Y'',Z''}$ is monomorphic, it suffices to show that

$$\mathrm{Im}_{X,X'} \longrightarrow \mathrm{Im}_{Y,Z'} \longrightarrow \mathrm{Im}_{Y'',Z''}$$

is exact at $\mathrm{Im}_{Y,Z'}$. This follows from the diagram obtained by Lemma I.53

$$\begin{array}{ccccc} 0 & \longrightarrow & Y'' & \rightarrowtail & \mathrm{Im}_{Y'',Z''} \\ \uparrow & + & \uparrow & \urcorner & \uparrow \\ X' & \longrightarrow & Y' & \rightarrowtail & \mathrm{Im}_{Y',Z'} \\ \uparrow\bullet & \llcorner & \uparrow\bullet & \Box & \uparrow\bullet \\ \mathrm{Im}_{X,X'} & \longrightarrow & \mathrm{Im}_{Y,Y'} & \rightarrowtail & \mathrm{Im}_{Y,Z'}\,, \end{array}$$

since by Lemma I.54, weak squares are stable under composition. ▫

I.6.6 On Frobenius categories

I.6.6.1 Some Frobenius-abelian lemmata

Suppose given an abelian Frobenius category $\mathcal{A}$; cf. Definition I.45. Let $\mathcal{B}$ be its full subcategory of bijective objects. Recall that the classical stable category of $\mathcal{A}$ is defined as $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{B}$; cf. Definition I.47. A morphism in $\mathcal{A}$ whose residue class in $\underline{\mathcal{A}}$ is an isomorphism is called a *homotopism.* A morphism in $\mathcal{A}$ whose residue class in $\underline{\mathcal{A}}$ is a retraction is called a *retraction up to homotopy.*

Lemma I.63 *Given a retraction up to homotopy $X \xrightarrow{f} Y$ and an epimorphism $Y' \xrightarrow{y} Y$ in $\mathcal{A}$, in the pullback*

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array},$$

the morphism $X' \xrightarrow{f'} Y'$ is a retraction up to homotopy, too. More precisely, if $gf \equiv_{\mathcal{B}} 1_Y$, then we may find a morphism g' with $g'f' \equiv_{\mathcal{B}} 1_{Y'}$ as a pullback of g along x.

Proof. Let $Y \xrightarrow{g} X$ be such that $gf = 1_Y + h$, where

$$(Y \xrightarrow{h} Y) \; = \; (Y \xrightarrow{h_1} B \xrightarrow{h_2} Y)$$

for some $B \in \operatorname{Ob}\mathcal{B}$ and some morphisms h_1 and h_2 in $\mathcal{A}$. Let $B \xrightarrow{h_2'} Y'$ be a morphism such that

$$(B \xrightarrow{h_2'} Y' \xrightarrow{y} Y) \; = \; (B \xrightarrow{h_2} Y)\,,$$

which exists since B is projective and y is epimorphic. The commutative quadrangle

$$\begin{array}{ccc} Y' & \xrightarrow{y} & Y \\ {\scriptstyle 1_{Y'}+yh_1h_2'}\downarrow & & \downarrow{\scriptstyle 1_Y+h} \\ Y' & \xrightarrow{y} & Y \end{array}$$

is a pullback since the induced morphism on the horizontal kernels is an identity; cf. Lemma I.51. So we may form the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{y} & Y \\ {\scriptstyle g'}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array},$$

in which g' with $g'x = yg$ and $g'f' = 1_{Y'} + yh_1h_2'$ is induced by the universal property of the lower pullback (X', X, Y', Y), and in which the resulting upper quadrangle (Y', Y, X', X) is a pullback by Lemma I.51. □

Lemma I.64 *Given a homotopism $X \xrightarrow{f} Y$ and an epimorphism $Y' \xrightarrow{y} Y$ in $\mathcal{A}$, in the pullback*

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array},$$

the morphism $X' \xrightarrow{f'} Y'$ is a homotopism, too.

Proof. Let $gf \equiv_{\mathcal{B}} 1_Y$ and $fg \equiv_{\mathcal{B}} 1_X$. By Lemma I.63, we may form the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{y} & Y \\ {\scriptstyle g'}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array},$$

in which $g'f' \equiv_{\mathcal{B}} 1_{Y'}$. Since g is a retraction up to homotopy, so is g' by Lemma I.63. Therefore g' is a homotopism. Hence also f' is a homotopism. ▫

I.6.6.2 Decomposing split diagrams in intervals

Let $\mathcal{A}$ be an abelian Frobenius category, and let $\mathcal{B}$ be its full subcategory of bijective objects. Suppose given $n \geqslant 1$. Write $\dot{\Delta}_n := \Delta_n \smallsetminus \{0\}$. An object X in $\mathcal{A}(\dot{\Delta}_n)$ is called *split* if $X_k \xrightarrow{x} X_l$ is split for all $k,\, l \in [1,n]$ with $k \leqslant l$.

Given $C \in \operatorname{Ob}\mathcal{A}$ and $k,\, l \in [1,n]$ with $k \leqslant l$, we denote by $C_{[k,l]}$ the object of $\mathcal{A}(\dot{\Delta}_n)$ given by $(C_{[k,l]})_j = 0$ for $j \in [1,n] \smallsetminus [k,l]$, by $(C_{[k,l]})_j = C$ for $j \in [k,l]$, and by $\big((C_{[k,l]})_j \xrightarrow{c} (C_{[k,l]})_{j'}\big) = (C \xrightarrow{1_C} C)$ for $j,\, j' \in [k,l]$ with $j \leqslant j'$. An object in $\mathcal{A}(\dot{\Delta}_n)$ of the form $C_{[k,l]}$ for some $C \in \operatorname{Ob}\mathcal{A}$ and some $k,\, l \in [1,n]$ with $k \leqslant l$, is called an *interval.*

Lemma I.65 *Any split object in $\mathcal{B}(\dot{\Delta}_n)$ is isomorphic to a finite direct sum of intervals.*

Proof. We proceed by induction on n. Suppose given a split object X in $\mathcal{B}(\dot{\Delta}_n)$. Let $X' := X{\uparrow_0}$ be defined as a pointwise pullback at n, using $0 \xrightarrow{0} X_n$ (cf. §I.6.7 below). We have $X' \in \operatorname{Ob}\mathcal{B}(\dot{\Delta}_n)$ with $X'_n = 0$. Hence, by induction, X' is isomorphic to a finite direct sum of intervals. There is a pure monomorphism $X' \to X$ whose cokernel is a diagram in $\operatorname{Ob}\mathcal{B}(\dot{\Delta}_n)$ consisting of split monomorphisms; cf. Lemma I.51. Moreover, by an iterated application of Lemma I.60, starting at position 1, this pure monomorphism $X' \to X$ is split as a morphism of $\mathcal{A}(\dot{\Delta}_n)$ ([9]). Thus X is isomorphic to the direct sum of X' and the cokernel of $X' \to X$, and it remains to be shown that this cokernel is isomorphic to a finite direct sum of intervals.

Therefore, we may assume that X consists of split monomorphisms $X_k \xrightarrow{x} X_l$ for $k,\, l \in [1,n]$. We have a monomorphism $(X_1)_{[1,n]} \xrightarrow{i} X$. Choosing a retraction to $X_1 \xrightarrow{x} X_n$ and composing, we obtain a coretraction to i, so that X is isomorphic to the direct sum of the interval $(X_1)_{[1,n]}$ and the cokernel of i. Since the cokernel of i has a zero term at position 1, we are done by induction. ▫

I.6.6.3 A Freyd category reminder

The construction of the Freyd category and its properties are due to FREYD [**15,** Th. 3.1].

Definition I.66 Suppose given an additive category $\mathcal{C}$ and a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$.

(1) A morphism $K \xrightarrow{i} X$ is a *weak kernel* of $X \xrightarrow{f} Y$ if the sequence of abelian groups

$$(T,K) \xrightarrow{(-)i} (T,X) \xrightarrow{(-)f} (T,Y)$$

is exact at (T,X) for every $T \in \operatorname{Ob}\mathcal{C}$.

(2) A morphism $Y \xrightarrow{p} C$ is a *weak cokernel* of $X \xrightarrow{f} Y$ if the sequence of abelian groups

$$(X,T) \xleftarrow{f(-)} (Y,T) \xleftarrow{p(-)} (C,T)$$

is exact at (Y,T) for every $T \in \operatorname{Ob}\mathcal{C}$.

(3) The category $\mathcal{C}$ is called *weakly abelian* if every morphism has a weak kernel and a weak cokernel, and if every morphism is a weak kernel (of some morphism) and a weak cokernel (of some morphism).

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[9] At this point, we use that $\dot{\Delta}_n$ is linearly ordered.

Let $\mathcal{C}$ be a weakly abelian category. Let $\mathcal{C}^0(\Delta_1)$ be the full subcategory of $\mathcal{C}(\Delta_1)$ whose objects are zero morphisms. The *Freyd category* $\hat{\mathcal{C}}$ of $\mathcal{C}$ is defined to be the quotient category

$$\hat{\mathcal{C}} \;:=\; \mathcal{C}(\Delta_1)/\mathcal{C}^0(\Delta_1)\ .$$

We collect some elementary facts and constructions and mention some conventions.

(1) The category $\hat{\mathcal{C}}$ is abelian. The kernel and the cokernel of a morphism $X \xrightarrow{f} Y$ represented by (f', f'') are constructed as

$$\begin{array}{ccccccc} K & \xrightarrow{i} & X' & \xrightarrow{f'} & Y' & \xrightarrow{1_{Y'}} & Y' \\ {\scriptstyle ix}\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle yp} \\ X'' & \xrightarrow{1_{X''}} & X'' & \xrightarrow{f''} & Y'' & \xrightarrow{p} & C\ , \end{array}$$

where i is a chosen weak kernel and p a chosen weak cokernel of the diagonal morphism $f'y = xf''$. If $f'y = xf'' = 0$, we choose $X' \xrightarrow{1_{X'}} X'$ as weak kernel and $Y'' \xrightarrow{1_{Y''}} Y''$ as weak cokernel.

Choosing a kernel and a cokernel for each object in $\hat{\mathcal{C}}(\Delta_1)$, we obtain a kernel and a cokernel functor $\hat{\mathcal{C}}(\Delta_1) \rightrightarrows \hat{\mathcal{C}}$, as for any abelian category.

(2) We stipulate that the pullback resp. the pushout of an identity morphism along a morphism is chosen to be an identity morphism.

(3) We have a full and faithful functor $\mathcal{C} \longrightarrow \hat{\mathcal{C}}$, $X \longmapsto (X \xrightarrow{1_X} X)$. Its image, identified with $\mathcal{C}$, consists of bijective objects.

(4) For each $X = (X' \xrightarrow{x} X'') \in \operatorname{Ob}\hat{\mathcal{C}}$, we may define objects and morphisms

$$X\,\mathsf{P} \xrightarrow{X\pi} X \xrightarrow{X\iota} X\,\mathsf{I}$$

by

$$\begin{array}{ccccc} X' & \xrightarrow{1_{X'}} & X' & \xrightarrow{x} & X'' \\ {\scriptstyle 1_{X'}}\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1_{X''}} \\ X' & \xrightarrow{x} & X'' & \xrightarrow{1_{X''}} & X''\ . \end{array}$$

As already mentioned in (3), the objects $X\,\mathsf{P}$ and $X\,\mathsf{I}$ are bijective, and thus $\hat{\mathcal{C}}$ is Frobenius.

Sometimes, we write just ι for $X\iota$ and π for $X\pi$. Note that $X\pi = 1_X$ and $X\iota = 1_X$ if $X \in \operatorname{Ob}\mathcal{C}$.

This construction $X\,\mathsf{P} \xrightarrow{X\pi} X \xrightarrow{X\iota} X\,\mathsf{I}$ is not meant to be functorial in $(X' \xrightarrow{x} X'')$, however.

Remark I.67 *Suppose given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. The following assertions are equivalent.*

(i) *The morphism f is a weak kernel of g.*

(ii) *The morphism g is a weak cokernel of f.*

(iii) *The sequence (f, g) is exact at Y when considered in $\hat{\mathcal{C}}$.*

Proof. Ad (i) $\Longrightarrow$ (iii). Suppose that f is a weak kernel of g. Let $K \xrightarrow{i} Y$ be the kernel of g in $\hat{\mathcal{C}}$. Factor $f = f'i$. Since f is a weak kernel of g in $\mathcal{C}$, we may factor $(K\pi)i = uf$, whence $K\pi = uf'$. Hence f' is epimorphic.

Ad (iii) $\Longrightarrow$ (i). Suppose (f, g) to be exact at Y. Let $T \xrightarrow{t} Y$ in $\mathcal{C}$ be such that $tg = 0$. Then t factors over the kernel of g, taken in $\hat{\mathcal{C}}$, and therefore, by projectivity of T in $\hat{\mathcal{C}}$, also over X. □

Remark I.68 *A morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ is monomorphic if and only if it is a coretraction. Dually, it is epimorphic if and only if it is a retraction.*

Proof. Suppose f to be monomorphic in $\mathcal{C}$. It suffices to show that f is monomorphic in $\hat{\mathcal{C}}$, for then f is a coretraction since X is injective in $\hat{\mathcal{C}}$. Let $K \xrightarrow{i} X$ be the kernel of f in $\hat{\mathcal{C}}$. From $(K\pi)if = 0$, we conclude $(K\pi)i = 0$ since f is monomorphic in $\mathcal{C}$, and thus $K \simeq 0$ since $K\pi$ is epimorphic and i is monomorphic in $\hat{\mathcal{C}}$. □

In particular, an abelian category is weakly abelian if and only if it is semisimple, i.e. if and only if every morphism in $\mathcal{A}$ splits. Hence the notion "weakly abelian" is slightly abusive.

Let $\mathcal{E}$ be a Frobenius category; cf. §I.6.2.3.

Lemma I.69 *Suppose given a pure short exact sequence $X' \overset{i}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} X''$ in $\mathcal{E}$. In $\underline{\mathcal{E}}$, the residue class iN is a weak kernel of pN, and the residue class pN is a weak cokernel of iN.*

Proof. By duality, it suffices to show that iN is a weak kernel of pN. So suppose given $T \xrightarrow{t} X$ in $\mathcal{E}$ with $tp \equiv_{\mathcal{B}} 0$. We have to show that there exists a morphism $T \xrightarrow{t'} X'$ such that $t'i \equiv_{\mathcal{B}} t$. Let $(T \xrightarrow{t} X \xrightarrow{p} X'') = (T \xrightarrow{u} B \xrightarrow{q} X'')$, where B is bijective. Let $P \overset{\tilde{p}}{\twoheadrightarrow} B$ be the pullback of p along q. We have a factorisation $(T \xrightarrow{t} X) = (T \xrightarrow{v} P \xrightarrow{w} X)$. We have a factorisation $(X' \overset{i}{\rightarrowtail} X) = (X' \overset{\tilde{i}}{\rightarrowtail} P \xrightarrow{w} X)$; moreover, $(\tilde{i}, \tilde{p})$ is a pure short exact sequence, hence split by projectivity of B; cf. Lemma I.51 and §I.6.2.2. Let $\tilde{i}r = 1$. Then $r\tilde{i} - 1 \equiv_{\mathcal{B}} 0$, since it factors over B. We obtain $(vr)i = vr\tilde{i}w \equiv_{\mathcal{B}} vw = t$. □

Remark I.70 *The stable category $\underline{\underline{\mathcal{E}}}$ and the classical stable category $\underline{\mathcal{E}}$ of the Frobenius category $\mathcal{E}$ are weakly abelian. The stable category $\underline{\underline{\mathcal{E}}}$ carries an automorphism T, induced by shifting an acyclic complex to the left by one position and negating the differentials.*

Cf. Definition I.47.

Proof. By Lemma I.48, it remains to prove that $\underline{\mathcal{E}}$ is weakly abelian. Suppose given a morphism $X \xrightarrow{f} Y$ in $\mathcal{E}$. By duality, it suffices to show that the residue class of $X \xrightarrow{f} Y$ in $\underline{\mathcal{E}}$ is a weak cokernel and has a weak kernel. Substituting isomorphically in $\underline{\mathcal{E}}$ by adding a bijective object to X, we may assume f to be a pure epimorphism in $\mathcal{E}$. So we may complete to a pure short exact sequence and apply Lemma I.69. □

Lemma I.71 *A pure short exact sequence $X' \overset{i}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} X''$ in $\mathcal{E}$ is mapped via the residue class functor N to a sequence in $\underline{\mathcal{E}}$ that is exact at X when considered in the Freyd category $\hat{\underline{\mathcal{E}}}$ of $\underline{\mathcal{E}}$. In particular, a pure square in $\mathcal{E}$ is mapped to a weak square in $\underline{\mathcal{E}}$.*

Proof. By Remark I.67, we may apply Lemma I.69. □

I.6.6.4 Heller operators for diagrams

In Definition I.5, the central role is attributed to the tuple $\vartheta = (\vartheta_n)_{n \geqslant 0}$ of isomorphisms. In the case of $\mathcal{C}$ being the stable category of a Frobenius category, such an isomorphism ϑ_n arises from different choices of pure monomorphisms into bijective objects. To that end, we provide a comparison lemma, which suitably organises wellknown facts.

Let C be a category.

Given a category $\mathcal{D}$ and a full subcategory $\mathcal{U} \subseteq \mathcal{D}(C)$, we say that $\mathcal{U}$ is *characteristic* in $\mathcal{D}(C)$ if the image of $\mathcal{U}$ under $A(C)$ is contained in $\mathcal{U}$ for any autoequivalence $\mathcal{D} \overset{A}{\underset{\sim}{\longrightarrow}} \mathcal{D}$, and if $\mathcal{U}$ is closed under isomorphy in $\mathcal{D}(C)$, i.e. $X \simeq X'$ in $\mathcal{D}(C)$ and $X' \in \operatorname{Ob} \mathcal{U}$ implies $X \in \operatorname{Ob} \mathcal{U}$.

Let $\mathcal{E}$ be a Frobenius category. Denote by $\underline{\mathcal{E}}$ its classical stable category, and denote by $\mathcal{E} \xrightarrow{N} \underline{\mathcal{E}}$ the residue class functor. Let $\mathcal{G} \subseteq \mathcal{E}(C)$ be a full additive subcategory. Let $\mathcal{H} \subseteq \underline{\mathcal{E}}(C)$ be a full additive characteristic subcategory such that $(\mathcal{G})(N(C)) \subseteq \mathcal{H}$.

$$\begin{array}{ccc} \mathcal{G} & \hookrightarrow & \mathcal{E}(C) \\ \downarrow & & \downarrow{\scriptstyle N(C)} \\ \mathcal{H} & \hookrightarrow & \underline{\mathcal{E}}(C) \end{array}$$

A C*-resolving system* I consists of pure short exact sequences

$$I \;=\; \Big(\Big(X_c \xrightarrow{i_{X,c}} I_{X,c} \xrightarrow{p_{X,c}} \tilde{X}_c\Big)_{c\in \operatorname{Ob} C}\Big)_{X\in \operatorname{Ob}\mathcal{G}} \;,$$

with bijective objects $I_{X,c}$ in $\mathcal{E}$ as middle terms.

Lemma I.72

(1) *Given a C-resolving system*

$$I \;=\; \Big(\Big(X_c \xrightarrow{i_{X,c}} I_{X,c} \xrightarrow{p_{X,c}} \tilde{X}_c\Big)_{c\in \operatorname{Ob} C}\Big)_{X\in \operatorname{Ob}\mathcal{G}} \;,$$

there exists a functor

$$\mathcal{G} \xrightarrow{\mathsf{T}_I} \mathcal{H}$$

that is uniquely characterised by the following properties.
On objects $X \in \operatorname{Ob}\mathcal{G} \subseteq \operatorname{Ob}\mathcal{E}(C)$, the image $X\,\mathsf{T}_I \in \operatorname{Ob}\mathcal{H} \subseteq \operatorname{Ob}\underline{\mathcal{E}}(C)$ is characterised as follows.

(∗) *For any $(c \xrightarrow{\gamma} d) \in C$, there exist*

- *a representative $(X\,\mathsf{T}_I)^{\sim}_{\gamma}$ in $\mathcal{E}$ of the evaluation $(X\,\mathsf{T}_I)_c \xrightarrow{(X\,\mathsf{T}_I)_\gamma} (X\,\mathsf{T}_I)_d$ in $\underline{\mathcal{E}}$ at $c \xrightarrow{\gamma} d$ of the diagram $X\,\mathsf{T}_I \in \operatorname{Ob}\mathcal{H} \subseteq \operatorname{Ob}\underline{\mathcal{E}}(C)$, and*
- *a morphism $I_{X,c} \longrightarrow I_{X,d}$ in $\mathcal{E}$*

such that

$$\begin{array}{ccccc} X_c & \xrightarrow{i_{X,c}} & I_{X,c} & \xrightarrow{p_{X,c}} & (X\,\mathsf{T}_I)_c \\ {\scriptstyle X_\gamma}\downarrow & & \downarrow & & \downarrow{\scriptstyle (X\,\mathsf{T}_I)^{\sim}_{\gamma}} \\ X_d & \xrightarrow{i_{X,d}} & I_{X,d} & \xrightarrow{p_{X,d}} & (X\,\mathsf{T}_I)_d \end{array}$$

is a morphism of pure short exact sequences.

On morphisms $(X \xrightarrow{f} Y) \in \mathcal{G} \subseteq \mathcal{E}(C)$, the image $(X\,\mathsf{T}_I \xrightarrow{f\,\mathsf{T}_I} Y\,\mathsf{T}_I) \in \mathcal{H} \subseteq \underline{\mathcal{E}}(C)$ is characterised as follows.

(∗∗) *For any $c \in \operatorname{Ob} C$, there exist*

- *a representative $(f\,\mathsf{T}_I)^{\sim}_{c}$ in $\mathcal{E}$ of the evaluation $(X\,\mathsf{T}_I)_c \xrightarrow{(f\,\mathsf{T}_I)_c} (Y\,\mathsf{T}_I)_c$ in $\underline{\mathcal{E}}$ at c of the diagram morphism $(X\,\mathsf{T}_I \xrightarrow{f\,\mathsf{T}_I} Y\,\mathsf{T}_I) \in \mathcal{H} \subseteq \underline{\mathcal{E}}(C)$, and*
- *a morphism $I_{X,c} \longrightarrow I_{Y,c}$ in $\mathcal{E}$*

such that

$$\begin{array}{ccccc} X_c & \xrightarrow{i_{X,c}} & I_{X,c} & \xrightarrow{p_{X,c}} & (X\,\mathsf{T}_I)_c \\ {\scriptstyle f_c}\downarrow & & \downarrow & & \downarrow{\scriptstyle (f\,\mathsf{T}_I)^{\sim}_{c}} \\ Y_c & \xrightarrow{i_{Y,c}} & I_{Y,c} & \xrightarrow{p_{Y,c}} & (Y\,\mathsf{T}_I)_c \end{array}$$

is a morphism of pure short exact sequences.

(2) *Given C-resolving systems*

$$I = \Big(\big(X_c \xrightarrow{i_{X,c}} I_{X,c} \xrightarrow{p_{X,c}} \tilde{X}_c\big)_{c\in \operatorname{Ob} C}\Big)_{X\in\operatorname{Ob}\mathcal{G}} ,$$

$$I' = \Big(\big(X_c \xrightarrow{i'_{X,c}} I'_{X,c} \xrightarrow{p'_{X,c}} \tilde{X}'_c\big)_{c\in \operatorname{Ob} C}\Big)_{X\in\operatorname{Ob}\mathcal{G}} ,$$

there exists an isomorphism

$$\mathsf{T}_I \xrightarrow[\sim]{\alpha_{I,I'}} \mathsf{T}_{I'}$$

that is uniquely characterised by the following property.

(∗∗∗) *For any $X \in \operatorname{Ob}\mathcal{G} \subseteq \operatorname{Ob}\underline{\mathcal{E}}(C)$ and for any $c \in \operatorname{Ob} C$, there exist*

- *a representative $(X\alpha_{I,I'})^\sim$ in $\mathcal{E}$ of the evaluation $(X\,\mathsf{T}_I)_c \xrightarrow{(X\alpha_{I,I'})_c} (X\,\mathsf{T}_{I'})_c$ in $\underline{\mathcal{E}}$ at c of the evaluation $X\,\mathsf{T}_I \xrightarrow{X\alpha_{I,I'}} X\,\mathsf{T}_{I'}$ in $\mathcal{H} \subseteq \underline{\mathcal{E}}(C)$ of $\alpha_{I,I'}$ at X, and*
- *a morphism $I_{X,c} \longrightarrow I'_{X,c}$ in $\mathcal{E}$*

such that

$$\begin{array}{ccccc} X_c & \xrightarrow{i_{X,c}} & I_{X,c} & \xrightarrow{p_{X,c}} & (X\,\mathsf{T}_I)_c \\ \| & & \downarrow & & \downarrow{\scriptstyle (X\alpha_{I,I'})^\sim} \\ X_c & \xrightarrow{i'_{X,c}} & I'_{X,c} & \xrightarrow{p'_{X,c}} & (X\,\mathsf{T}_{I'})_c \end{array}$$

is a morphism of pure short exact sequences.

Proof. Let us first assume that $\mathcal{H} = \underline{\mathcal{E}}(C)$. Having proven all assertions in this case, it then finally will remain to be shown that given $\mathcal{H} \subseteq \underline{\mathcal{E}}(C)$ and a C-resolving system I, we have $X\,\mathsf{T}_I \in \operatorname{Ob}\mathcal{H} \subseteq \operatorname{Ob}\underline{\mathcal{E}}(C)$ for $X \in \operatorname{Ob}\mathcal{G}$.

We remark that starting from a morphism $U \xrightarrow{u} U'$ in $\mathcal{E}$ and from chosen pure short exact sequences (U, B, V) and (U', B', V') with bijective middle terms B resp. B', we may define a morphism $V \xrightarrow{v} V'$ in $\underline{\mathcal{E}}$ by the existence of a morphism

$$\begin{array}{ccccc} U & \xrightarrow{i} & B & \xrightarrow{p} & V \\ {\scriptstyle u}\downarrow & & \downarrow & & \downarrow{\scriptstyle v^\sim} \\ U' & \xrightarrow{i'} & B' & \xrightarrow{p'} & V' \end{array}$$

of pure short exact sequences in $\mathcal{E}$, where $V \xrightarrow{v} V'$ is the image in $\underline{\mathcal{E}}$ of the morphism $V \xrightarrow{v^\sim} V'$ in $\mathcal{E}$.

Ad (1). Given $X \in \operatorname{Ob}\mathcal{G}$, we define $X\,\mathsf{T}_I \in \underline{\mathcal{E}}(C)$ at the morphism $c \xrightarrow{\gamma} d$ of C by the diagram in (∗). The characterisation (∗) shows that $X\,\mathsf{T}_I$ is in fact in $\operatorname{Ob}\underline{\mathcal{E}}(C)$.

Given a morphism $X \xrightarrow{f} Y$ in $\mathcal{G}$, we define the morphism $X\,\mathsf{T}_I \xrightarrow{f\,\mathsf{T}_I} Y\,\mathsf{T}_I$ in $\underline{\mathcal{E}}(C)$ at $c \in \operatorname{Ob} C$ by the diagram in (∗∗). Combining (∗) and (∗∗), we see that $f\,\mathsf{T}_I$ is in fact in $\underline{\mathcal{E}}(C)$. From (∗∗) we conclude that T_I is indeed a functor.

Ad (2). Given $X \in \operatorname{Ob}\mathcal{G}$, we define $X\,\mathsf{T}_I \xrightarrow{X\alpha_{I,I'}} X\,\mathsf{T}_{I'}$ at $c \in \operatorname{Ob} C$ by the diagram in (∗∗∗).

Combining (∗∗∗) and (∗), we see that $X\,\mathsf{T}_I \xrightarrow{X\alpha_{I,I'}} X\,\mathsf{T}_{I'}$ is indeed in $\underline{\mathcal{E}}(C)$. Combining (∗∗∗) and (∗∗), we see that $\alpha_{I,I'}$ is indeed a transformation.

Suppose given resolving systems I, I' and I''. The characterisation of $\alpha_{I,I'}$ etc. implies that $\alpha_{I,I'}\alpha_{I',I''} = \alpha_{I,I''}$ and that $\alpha_{I,I} = 1_{\mathsf{T}_I}$. Hence in particular, $\alpha_{I,I'}\alpha_{I',I} = 1_{\mathsf{T}_I}$ and $\alpha_{I',I}\alpha_{I,I'} = 1_{\mathsf{T}_{I'}}$, and so $\alpha_{I,I'}$ is an isomorphism from T_I to $\mathsf{T}_{I'}$.

Consider the case $C = \Delta_0$, i.e. the terminal category, let $\mathcal{G} = \mathcal{E}(\Delta_0) = \mathcal{E}$ and let $\mathcal{H} = \underline{\mathcal{E}}(\Delta_0) = \underline{\mathcal{E}}$. For a Δ_0-resolving system J, we obtain a functor $\mathcal{E} \xrightarrow{\mathsf{T}_J} \underline{\mathcal{E}}$ that factors as

$$(\mathcal{E} \xrightarrow{\mathsf{T}_J} \underline{\mathcal{E}}) \;=\; (\mathcal{E} \xrightarrow{N} \underline{\mathcal{E}} \underset{\sim}{\xrightarrow{\bar{\mathsf{T}}_J}} \underline{\mathcal{E}}) \,.$$

In fact, for a morphism b that factors over a bijective object B, we can choose 0 as a representative of $b\,\mathsf{T}_J$, inserting the pure short exact sequence $(B, B, 0)$. Moreover, $\bar{\mathsf{T}}_J$ is an equivalence, for it is full; faithful, using the dual of the argument just given; and dense, since given a morphism of short exact sequences in $\mathcal{E}$ with bijective middle terms and an identity on the kernels, the morphism on the cokernels is a homotopism.

Now return to the general case $\mathcal{H} \subseteq \underline{\mathcal{E}}(C)$. Let J' be a C-resolving system consisting of pure short exact sequences with bijective middle term that already occur in the chosen Δ_0-resolving system J. Then, for $X \in \operatorname{Ob}\mathcal{G}$, we have $X\,\mathsf{T}_{J'} = X(N(C))(\bar{\mathsf{T}}_J(C))$. Since $X(N(C)) \in \operatorname{Ob}\mathcal{H}$ by assumption, and since, moreover, $\mathcal{H}$ is assumed to be a characteristic subcategory of $\underline{\mathcal{E}}(C)$, we conclude that $X(N(C))(\bar{\mathsf{T}}_J(C)) = X\,\mathsf{T}_{J'}$ is in $\operatorname{Ob}\mathcal{H}$. Finally, let I be an arbitrary C-resolving system. We have $X\,\mathsf{T}_I \underset{\sim}{\xrightarrow{\alpha_{I,J'}}} X\,\mathsf{T}_{J'}$ in $\underline{\mathcal{E}}(C)$, and thus $X\,\mathsf{T}_{J'} \in \operatorname{Ob}\mathcal{H}$ implies $X\,\mathsf{T}_I \in \operatorname{Ob}\mathcal{H}$, since a characteristic subcategory of $\underline{\mathcal{E}}(C)$ is, by definition, closed under isomorphy. ▫

I.6.7 Pointwise pullback and pushout

Suppose given an abelian category $\mathcal{A}$, a poset E and an element $\varepsilon \in E$. Let $E^\varepsilon := E \sqcup \{\varepsilon'\}$ be the poset defined by requiring that $\varepsilon \leqslant \varepsilon'$, that $\alpha \not\leqslant \varepsilon'$ whenever $\alpha \not\leqslant \varepsilon$ and that $\varepsilon' \not\leqslant \alpha$ for all $\alpha \in E$; and the remaining relations within $E \subseteq E^\varepsilon$ inherited from E. We define the *pushout at* ε

$$\begin{array}{ccc} \mathcal{A}(E^\varepsilon) & \xrightarrow{(-)\uparrow^{(=)}} & \mathcal{A}(E) \\ X' & \longmapsto & X{\uparrow}^{x'} \,, \end{array}$$

where $X := X'|_E$, and $(X'_\varepsilon \xrightarrow{x'} X'_{\varepsilon'}) = (X'_\varepsilon \xrightarrow{X'_{\varepsilon'/\varepsilon}} X'_{\varepsilon'})$; and a transformation

$$X'|_E = X \xrightarrow{\;i \;=\; iX'\;} X{\uparrow}^{x'} \,,$$

natural in X', by the following construction. Abbreviating $X{\uparrow}^{x'}$ by $\tilde{X}$, we let

$$\begin{array}{ccc} X'_{\varepsilon'} = \tilde{X}_\varepsilon & \xrightarrow{\tilde{X}_{\alpha/\varepsilon}} & \tilde{X}_\alpha \\ {\scriptstyle x'}\uparrow & \lrcorner & \uparrow{\scriptstyle i_\alpha} \\ X_\varepsilon & \xrightarrow{X_{\alpha/\varepsilon}} & X_\alpha \end{array}$$

for $\alpha \in E$ with $\varepsilon \leqslant \alpha$. If $\varepsilon \not\leqslant \alpha$, we let $\tilde{X}_\alpha = X_\alpha$ and $i_\alpha = 1_{X_\alpha}$.

Given $\alpha \leqslant \beta$ in E, we let

$$\begin{array}{llll} (\tilde{X}_\alpha \xrightarrow{\tilde{X}_{\beta/\alpha}} \tilde{X}_\beta) & \text{be induced by pushout} & & \text{if } \varepsilon \leqslant \alpha \leqslant \beta \,, \\ (\tilde{X}_\alpha \xrightarrow{\tilde{X}_{\beta/\alpha}} \tilde{X}_\beta) & := & (X_\alpha \xrightarrow{X_{\beta/\alpha}} X_\beta \xrightarrow{i_\beta} \tilde{X}_\beta) & \text{if } \varepsilon \not\leqslant \alpha, \text{ but } \varepsilon \leqslant \beta \,, \\ (\tilde{X}_\alpha \xrightarrow{\tilde{X}_{\beta/\alpha}} \tilde{X}_\beta) & := & (X_\alpha \xrightarrow{X_{\beta/\alpha}} X_\beta) & \text{if } \varepsilon \not\leqslant \beta \,. \end{array}$$

The morphism $X \xrightarrow{i} X{\uparrow}^{x'}$ is the solution to the following universal problem. Suppose given a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}(E)$ such that at $\varepsilon \in E$ we have a factorisation

$$(X_\varepsilon \xrightarrow{f_\varepsilon} Y_\varepsilon) \;=\; (X_\varepsilon \xrightarrow{x'} X'_{\varepsilon'} \longrightarrow Y_\varepsilon) \,.$$

Then there is a unique morphism $X{\uparrow}^{x'} \xrightarrow{g} Y$ such that

$$(X \xrightarrow{f} Y) \;=\; (X \xrightarrow{i} X{\uparrow}^{x'} \xrightarrow{g} Y) \,.$$

Dually, let $E_\varepsilon := E \sqcup \{\varepsilon'\}$ be the poset defined by requiring that $\varepsilon \geqslant \varepsilon'$, that $\alpha \not\geqslant \varepsilon'$ whenever $\alpha \not\geqslant \varepsilon$ and that $\varepsilon' \not\geqslant \alpha$ for all $\alpha \in E$; and the remaining relations within $E \subseteq E_\varepsilon$ inherited from E. We define the *pullback at* ε

$$\begin{array}{ccc} \mathcal{A}(E_\varepsilon) & \xrightarrow{(-)\uparrow_{(=)}} & \mathcal{A}(E) \\ X' & \longmapsto & X\uparrow_{x'} \,, \end{array}$$

where $X := X'|_E$, and $(X'_{\varepsilon'} \xrightarrow{x'} X'_\varepsilon) = (X'_{\varepsilon'} \xrightarrow{X'_{\varepsilon/\varepsilon'}} X'_\varepsilon)$; and a transformation

$$X'|_E = X \xleftarrow{\;p\;=\;pX'\;} X\uparrow_{x'} \,,$$

natural in X', being the solution to the universal problem dual to the one described above.

I.6.8 1-epimorphic functors

Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between categories $\mathcal{C}$ and $\mathcal{D}$.

Definition I.73 The functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is 1-*epimorphic* if the induced functor "restriction along F"

$$[\![\mathcal{C}, \mathcal{E}]\!] \xleftarrow{F(-)} [\![\mathcal{D}, \mathcal{E}]\!]$$

is full and faithful for any category $\mathcal{E}$. In particular, given functors $\mathcal{D} \overset{G}{\underset{H}{\rightrightarrows}} \mathcal{E}$ with $FG \simeq FH$, we can conclude that $G \simeq H$; whence the notion of 1-epimorphy.

Remark I.74 *Suppose given a diagram of categories and functors*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle S}\downarrow\wr & & \wr\downarrow{\scriptstyle T} \\ \mathcal{C}' & \xrightarrow{F'} & \mathcal{D}' \end{array}$$

with equivalences S and T, and with $FT \simeq SF'$. Then F is 1*-epimorphic if and only if F' is* 1*-epimorphic.*

Let $C, C' \in \operatorname{Ob}\mathcal{C}$. An F-*epizigzag* (resp. an F-*monozigzag*) $C \overset{u}{\rightsquigarrow} C'$ is a finite sequence of morphisms

$$C \;=\; C_0 \xrightarrow{u_0} Z_0 \xleftarrow{u'_0} C_1 \xrightarrow{u_1} Z_1 \xleftarrow{u'_1} C_1 \xrightarrow{u_2} \cdots \xleftarrow{u'_{k-2}} C_{k-1} \xrightarrow{u_{k-1}} Z_{k-1} \xleftarrow{u'_{k-1}} C_k \;=\; C'$$

in $\mathcal{C}$ of *length* $k \geqslant 0$ such that $u'_i F$ is an isomorphism for all $i \in [0, k]$, and such that

$$uF \;:=\; (u_0F)(u'_0F)^-(u_1F)(u'_1F)^- \cdots (u_{k-1}F)(u'_{k-1}F)^- \;:\; CF \longrightarrow C'F$$

is a retraction (resp. a coretraction) in $\mathcal{D}$.

Lemma I.75 *Suppose the functor*

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

to be dense, and to satisfy the following condition (C).

(C) $\left\{\begin{array}{l} \text{Given objects } C, C' \in \operatorname{Ob}\mathcal{C} \text{ and a morphism } CF \xrightarrow{d} C'F \text{ in } \mathcal{D}\text{, there exists} \\ \text{an } F\text{-epizigzag } C_{\mathrm{s}} \overset{c_{\mathrm{s}}}{\rightsquigarrow} C\text{, an } F\text{-monozigzag } C' \overset{c'_{\mathrm{t}}}{\rightsquigarrow} C'_{\mathrm{t}} \text{ and a morphism } C_{\mathrm{s}} \xrightarrow{c} C'_{\mathrm{t}} \\ \text{such that} \\ \qquad (C_{\mathrm{s}}F \xrightarrow{c_{\mathrm{s}}F} CF \xrightarrow{d} C'F \xrightarrow{c'_{\mathrm{t}}F} C'_{\mathrm{t}}F) \;=\; (C_{\mathrm{s}}F \xrightarrow{cF} C'_{\mathrm{t}}F)\,. \end{array}\right.$

Then F is 1-epimorphic.

Proof. Since F is dense, Remark I.74 allows to assume that F is surjective on objects, i.e. $(\mathrm{Ob}\,\mathcal{C})F = \mathrm{Ob}\,\mathcal{D}$.

Let us prove that $\mathcal{E}(\mathcal{C}) \xleftarrow{F(-)} \mathcal{E}(\mathcal{D})$ is faithful. Suppose given functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \overset{G}{\underset{H}{\rightrightarrows}} \mathcal{E}$ and morphisms $G \xrightarrow{\gamma} H$ and $G \xrightarrow{\gamma'} H$ such that $F\gamma = F\gamma'$. Given $D \in \mathrm{Ob}\,\mathcal{D}$, we have to show that $D\gamma = D\gamma'$. Writing $D = CF$ for some $C \in \mathrm{Ob}\,\mathcal{C}$, this follows from $D\gamma = CF\gamma = CF\gamma' = D\gamma'$.

Let us prove that $\mathcal{E}(\mathcal{C}) \xleftarrow{F(-)} \mathcal{E}(\mathcal{D})$ is full. Suppose given functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \overset{G}{\underset{H}{\rightrightarrows}} \mathcal{E}$ and a morphism $FG \xrightarrow{\delta} FH$. Define $G \xrightarrow{\hat{\delta}} H$ by $(CF)\hat{\delta} := C\delta$ for $C \in \mathrm{Ob}\,\mathcal{C}$.

We have to prove that $D\hat{\delta}$ is a welldefined morphism for $D \in \mathrm{Ob}\,\mathcal{D}$. So suppose that $D = CF = C'F$. We have to show that $C\delta = C'\delta$. By assumption (C), applied to $d = 1_D = 1_{CF} = 1_{C'F}$, there exist an F-epizigzag $C_\mathrm{s} \overset{c_\mathrm{s}}{\rightsquigarrow} C$, an F-monozigzag $C' \overset{c'_\mathrm{t}}{\rightsquigarrow} C'_\mathrm{t}$ and a morphism $C_\mathrm{s} \xrightarrow{c} C'_\mathrm{t}$ such that $(c_\mathrm{s}F)(c'_\mathrm{t}F) = cF$. We obtain

$$\begin{aligned} (c_\mathrm{s}FG)(C\delta)(c'_\mathrm{t}FH) &= (C_\mathrm{s}\delta)(c_\mathrm{s}FH)(c'_\mathrm{t}FH) \\ &= (C_\mathrm{s}\delta)(cFH) \\ &= (cFG)(C'_\mathrm{t}\delta) \\ &= (c_\mathrm{s}FG)(c'_\mathrm{t}FG)(C'_\mathrm{t}\delta) \\ &= (c_\mathrm{s}FG)(C'\delta)(c'_\mathrm{t}FH)\,, \end{aligned}$$

whence $C\delta = C'\delta$ by epimorphy of $c_\mathrm{s}FG$ and by monomorphy of $c'_\mathrm{t}FH$.

We have to prove that $\hat{\delta}$ is natural. Suppose given $CF \xrightarrow{d} C'F$ in $\mathcal{D}$ for some $C, C' \in \mathrm{Ob}\,\mathcal{C}$. We have to show that $(dG)((C'F)\hat{\delta}) = ((CF)\hat{\delta})(dH)$, i.e. that $(dG)(C'\delta) = (C\delta)(dH)$. By assumption (C), there exist an F-epizigzag $C_\mathrm{s} \overset{c_\mathrm{s}}{\rightsquigarrow} C$, an F-monozigzag $C' \overset{c'_\mathrm{t}}{\rightsquigarrow} C'_\mathrm{t}$ and a morphism $C_\mathrm{s} \xrightarrow{c} C'_\mathrm{t}$ such that $(c_\mathrm{s}F)d(c'_\mathrm{t}F) = cF$. We obtain

$$\begin{aligned} (c_\mathrm{s}FG)(dG)(C'\delta)(c'_\mathrm{t}FH) &= (c_\mathrm{s}FG)(dG)(c'_\mathrm{t}FG)(C'_\mathrm{t}\delta) \\ &= (cFG)(C'_\mathrm{t}\delta) \\ &= (C_\mathrm{s}\delta)(cFH) \\ &= (C_\mathrm{s}\delta)(c_\mathrm{s}FH)(dH)(c'_\mathrm{t}FH) \\ &= (c_\mathrm{s}FG)(C\delta)(dH)(c'_\mathrm{t}FH)\,, \end{aligned}$$

whence $(dG)(C'\delta) = (C\delta)(dH)$ by epimorphy of $c_\mathrm{s}FG$ and by monomorphy of $c'_\mathrm{t}FH$. ▫

Corollary I.76 *If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a functor such that* (i, ii) *hold, then F is 1-epimorphic.*

(i) *For all morphisms $D \xrightarrow{d} D'$ in $\mathcal{D}$, there is a morphism $C \xrightarrow{c} C'$ in $\mathcal{C}$ such that*

$$(C \xrightarrow{c} C')F \;=\; (D \xrightarrow{d} D')\,.$$

(ii) *For any $C, C' \in \mathrm{Ob}\,\mathcal{C}$ such that $CF = C'F$, there exists a finite sequence of morphisms*

$$C \;=\; C_0 \xrightarrow{u_0} Z_0 \xleftarrow{u'_0} C_1 \xrightarrow{u_1} Z_1 \xleftarrow{u'_1} C_2 \xrightarrow{u_2} \cdots \xleftarrow{u'_{k-2}} C_{k-1} \xrightarrow{u_{k-1}} Z_{k-1} \xleftarrow{u'_{k-1}} C_k \;=\; C'$$

from C to C' such that $u_iF = u'_iF = 1_{CF} = 1_{C'F}$ for all $i \in [0, k]$.

Proof. The functor F is dense, even surjective on objects, because identities have inverse images under F. To fulfill condition (C) of Lemma I.75, given objects $C, C' \in \mathrm{Ob}\,\mathcal{C}$ and a morphism $CF \xrightarrow{d} C'F$ in $\mathcal{D}$, we may take some morphism $C_\mathrm{s} \xrightarrow{c} C'_\mathrm{t}$ in $\mathcal{C}$ such that $(C_\mathrm{s} \xrightarrow{c} C'_\mathrm{t})F = (CF \xrightarrow{d} C'F)$, we may take for c_s a sequence as given by assumption because of $C_\mathrm{s}F = CF$, and we may take for c_t a sequence as given by assumption because of $C'_\mathrm{t}F = C'F$. ▫

Corollary I.77 *If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a full and dense functor, then F is 1-epimorphic.*

Proof. In fact, in condition (C) of Lemma I.75, we may take an F-monozigzag and an F-epizigzag of length 0. ▫

Chapter II

On exact functors for Heller triangulated categories

II.0 Introduction

II.0.1 Extending from Verdier to Heller

The following facts are part of the classical theory that Verdier triangulated categories.

- Verdier triangulated categories are stable under formation of the Karoubi hull [**2**].
- The Karoubi hull construction is functorial within Verdier triangulated categories and exact functors [**2**].
- Verdier triangulated categories are stable under localisation at a thick subcategory [**56**].
- Such a localisation has a universal property within Verdier triangulated categories and exact functors [**56**].
- An adjoint functor of an exact functor is exact [**44,** App. 2, Prop. 11], [**34,** 1.6].

We extend these assertions somewhat to fit into the Heller triangulated setting.

- Heller triangulated categories are stable under formation of the Karoubi hull; cf. Proposition II.12.
- The Karoubi hull construction is functorial within Heller triangulated categories and exact functors; cf. Proposition II.13.
- Closed Heller triangulated categories are stable under localisation at a thick subcategory; cf. Proposition II.36. (Concerning closedness, see remark below.)
- Such a localisation has a universal property within closed Heller triangulated categories and exact functors; cf. Proposition II.38.

- An adjoint functor of an exact functor is exact; cf. Proposition II.28.

In a general Heller triangulated category, it is unknown whether there exists a cone on a given morphism. This however is true if all idempotents split; cf. Lem. I.18. It is technically convenient to extend this assertion in the following manner. Define a Heller triangulated category to be closed if this property holds; cf. Def. III.13, Definition II.14, Remark II.15, Lemma II.20. Prove that certain constructions yield closed Heller triangulated categories or preserve closedness; cf. Cor. III.21, Proposition II.36.

An exact functor between Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$ is a pair (F, a) consisting of a subexact functor F and an isotransformation $a : \mathsf{T}\, F \longrightarrow F\, \mathsf{T}'$ such that ϑ, ϑ' and a are compatible; cf. [**34,** Def. 1.4], Definition II.1. Exactness of such a pair can also be characterised via n-triangles; cf. Proposition II.25. The deeper reason behind that fact is that closed Heller triangulated categories can, alternatively, be defined via sets of n-triangles for $n \geqslant 0$ with suitable properties with respect to quasicyclic and folding operations, as S. Thomas informed me.

The proof of the exactness of an adjoint of an exact functor does not have to make recourse to n-triangles. Neither does the construction of the Heller triangulation on the Karoubi hull. This shows the convenience of the definition of a triangulation via a tuple $\vartheta = (\vartheta_n)_{n \geqslant 0}$ of isomorphisms between certain shift functors, and to view the n-triangles as accessory, if useful; which is no longer the point of view taken in [**35**].

II.0.2 Desirables

Still missing is a precise formulation in which sense the dual of a Heller triangulated category is again a Heller triangulated category, and also in which sense the constructions above are compatible with duality. Moreover, we do not treat exactness of derived functors, except implicitly, in those cases where a derived functor can be written as a composite of an adjoint of a localisation functor, an exact functor and another localisation functor. Still missing, in the Heller triangulated context, is furthermore the exactness of the lift of the inclusion of the heart to a functor on the bounded derived category [**8,** Prop. 3.1.10], or more generally, the functor Z appearing in the construction of [**34,** Ex. 2.3] ; cf. [**34,** Th. 3.2].

II.0.3 Notations and conventions

We use the notations and conventions from Chapter I. In particular, we write composition of morphisms and functors in the natural order; viz. morphisms as $\xrightarrow{f}\xrightarrow{g} = \xrightarrow{fg} = \xrightarrow{f\cdot g}$ and functors as $\xrightarrow{F}\xrightarrow{G} = \xrightarrow{FG} = \xrightarrow{F\star G}$. Similarly for transformations.

Epic and *epimorphic* are synonymous, and so are *monic* and *monomorphic.*

II.1 Exact functors

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$, $(\mathcal{C}', \mathsf{T}', \vartheta')$ and $(\mathcal{C}'', \mathsf{T}'', \vartheta'')$ be Heller triangulated categories; cf. Def. I.5.(ii).

In Def. I.5.(iii), we required a strictly exact functor $\mathcal{C} \longrightarrow \mathcal{C}'$ to satisfy $F\,\mathsf{T}' = \mathsf{T}\,F$. The adjoint functor of a strictly exact functor does not always seem to be strictly exact. Following Keller and Vossieck, we shall prove below that if we call a functor *exact,* if it satisfies the weakened condition $F\,\mathsf{T}' \simeq \mathsf{T}\,F$ instead (and an accordingly modified compatibility with the Heller triangulations), then an adjoint of an exact functor is exact; cf. [**34,** 1.4].

Nonetheless, generally speaking, usually one deals with strictly exact functors. Hence we shall also state an extra condition of shiftcompatibility on the adjunction that ensures a shiftcompatibly adjoint functor of a strictly exact functor to be strictly exact.

Given $n \geqslant 0$ and a transformation $G \xrightarrow{a} G'$ between subexact additive functors $\mathcal{C} \overset{G}{\underset{G'}{\rightrightarrows}} \mathcal{C}'$, we denote by $G^+(\bar{\Delta}_n^\#) \xrightarrow{a^+(\bar{\Delta}_n^\#)} G'^+(\bar{\Delta}_n^\#)$ the transformation given by

$$\big(X(a^+(\bar{\Delta}_n^\#))\big)_{\beta/\alpha} \;:=\; X_{\beta/\alpha}a \;:\; X_{\beta/\alpha}G \longrightarrow X_{\beta/\alpha}G'$$

for $X \in \operatorname{Ob}\mathcal{C}(\bar{\Delta}_n^\#)$, and for $\beta/\alpha \in \bar{\Delta}_n^\#$, i.e. for $\alpha,\ \beta \in \bar{\Delta}_n$ with $\beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}$. Moreover, we denote by $\underline{G^+(\bar{\Delta}_n^\#)} \xrightarrow{\underline{a^+(\bar{\Delta}_n^\#)}} \underline{G'^+(\bar{\Delta}_n^\#)}$ the induced transformation between the induced functors on the stable categories.

Sometimes, we abbreviate $(\underline{G} \xrightarrow{\underline{a}} \underline{G}') := \big(\underline{G^+(\bar{\Delta}_n^\#)} \xrightarrow{\underline{a^+(\bar{\Delta}_n^\#)}} \underline{G'^+(\bar{\Delta}_n^\#)}\big)$.

Definition II.1

A pair (F, a), consisting of an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and a transformation $\mathsf{T}\,F \xrightarrow{a} F\,\mathsf{T}'$, is called an *exact pair,* or an *exact functor,* if the following conditions hold.

(1) a is an isotransformation.

(2) F is subexact, i.e. its induced functor $\hat{\mathcal{C}} \xrightarrow{\hat{F}} \hat{\mathcal{C}}'$ on the Freyd categories is exact.

(3) We have

$$(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^\#)}) \cdot \underline{a^+(\bar{\Delta}_n^\#)} \;=\; \underline{F^+(\bar{\Delta}_n^\#)} \star \vartheta'_n$$

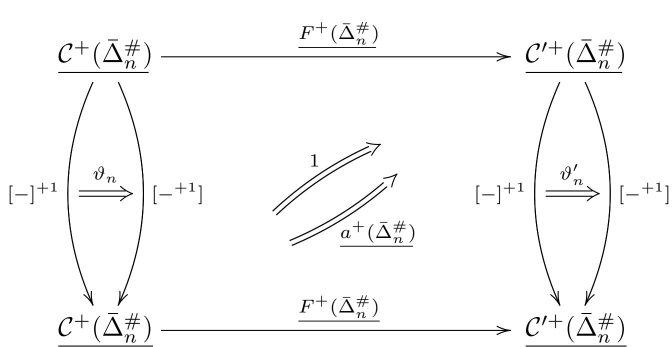
for all $n \geqslant 0$.

In particular, provided $\mathsf{T}\,F = F\,\mathsf{T}'$, then $(F, 1)$ is exact if and only if F is strictly exact; cf. Def. I.5.(iii). In this case, we sometimes identify F and $(F, 1)$.

Calling a pair (F, a) an exact functor instead of an exact pair is an abuse of notation.

We shall not discuss whether condition (1) is redundant; we need it for the construction of $\tilde{Y}$ in §II.4, but that may be due to the order of our arguments.

Condition (3) asserts that the following cylindrical diagram commutes for all $n \geqslant 0$.



I.e., using the abbreviation just introduced, we require $X\vartheta_n\underline{F}\cdot X\underline{a} = X\underline{F}\vartheta'_n$ to hold in $\underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$ for all $X \in \operatorname{Ob}\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} = \operatorname{Ob}\mathcal{C}^+(\bar{\Delta}_n^\#)$.

Definition II.2 Suppose given exact functors (F,a) from $\mathcal{C}$ to $\mathcal{C}'$, and (F',a') from $\mathcal{C}'$ to $\mathcal{C}''$. The composite of (F,a) and (F',a') is defined to be

$$(F,a)\star(F',a') \;=\; (F,a)(F',a') \;:=\; (FF',\,(aF')(Fa')) \;=\; (F\star F',\,(a\star F')\cdot(F\star a'))\;.$$

Composition is associative.

Remark II.3 *If (F,a) and (F',a') are exact, then so is their composite $(F,a)(F',a')$.*

Proof. To be able to distinguish more easily, we shall make use, from the second to the last but first step, of the notation $a\star F = aF$, $F\star F' = FF'$ etc. Given $n \geqslant 0$, we obtain

$$\begin{array}{cl} & \Big(\vartheta_n\star\underline{(F\star F')^+(\bar{\Delta}_n^\#)}\Big)\cdot\underline{\big((a\star F')\cdot(F\star a')\big)^+(\bar{\Delta}_n^\#)} \\ = & \Big(\vartheta_n\star\underline{F^+(\bar{\Delta}_n^\#)}\star\underline{F'^+(\bar{\Delta}_n^\#)}\Big)\cdot\Big(\underline{a^+(\bar{\Delta}_n^\#)}\star\underline{F'^+(\bar{\Delta}_n^\#)}\Big)\cdot\Big(\underline{F^+(\bar{\Delta}_n^\#)}\star\underline{a'^+(\bar{\Delta}_n^\#)}\Big) \\ = & \Big(\underline{F^+(\bar{\Delta}_n^\#)}\star\vartheta'_n\star\underline{F'^+(\bar{\Delta}_n^\#)}\Big)\cdot\Big(\underline{F^+(\bar{\Delta}_n^\#)}\star\underline{a'^+(\bar{\Delta}_n^\#)}\Big) \\ = & \underline{F^+(\bar{\Delta}_n^\#)}\star\underline{F'^+(\bar{\Delta}_n^\#)}\star\vartheta''_n \\ = & \underline{(F\star F')^+(\bar{\Delta}_n^\#)}\star\vartheta''_n\;. \end{array}$$

□

Definition II.4 Suppose given exact functors (F,a) and (G,b) from $(\mathcal{C},\mathsf{T},\vartheta)$ to $(\mathcal{C}',\mathsf{T}',\vartheta')$. A transformation $F\xrightarrow{s}G$ such that $(\mathsf{T}\star s)\cdot b = a\cdot(s\star\mathsf{T}')$ holds, is called *periodic.*

The periodicity condition requires that

$$\begin{array}{ccc} X^{+1}F & \xrightarrow{X^{+1}s} & X^{+1}G \\ {\scriptstyle Xa}\downarrow{\scriptstyle\wr} & & {\scriptstyle\wr}\downarrow{\scriptstyle Xb} \\ (XF)^{+1} & \xrightarrow{(Xs)^{+1}} & (XG)^{+1} \end{array}$$

commute for all $X \in \operatorname{Ob}\mathcal{C}$.

Remark II.5 Suppose given exact functors (F,a), (G,b) and (H,c) from $\mathcal{C}$ to $\mathcal{C}'$, and periodic transformations $F \overset{s}{\underset{s'}{\rightrightarrows}} G \xrightarrow{t} H$.

(1) The composite $F\xrightarrow{s\cdot t}H$ is periodic.

(2) The identity $F\xrightarrow{1}F$ is periodic.

(3) If s is a periodic isotransformation from (F,a) to (G,b), then s^- is a periodic isotransformation from (G,b) to (F,a).

(4) The difference $F \xrightarrow{s-s'} G$ of two periodic transformations is periodic.

(5) The direct sum $(F,a) \oplus (G,b) := (F \oplus G, a \oplus b) = (F \oplus G, \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right))$ is exact, with periodic inclusions from and periodic projections to (F,a) and (G,b).

Definition II.6 Write $[\![\mathcal{C},\mathcal{C}']\!]_{\text{ex}}$ for the category of the exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$; cf. Definitions II.1, II.4, Remark II.5.

Write $[\![\mathcal{C},\mathcal{C}']\!]_{\text{st ex}}$ for the full subcategory of $[\![\mathcal{C},\mathcal{C}']\!]_{\text{ex}}$ of the strictly exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$.

II.2 Idempotents and cones

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a Heller triangulated category; cf. Def. I.5.

II.2.1 A general remark on residue classes

Concerning Frobenius categories, cf. e.g. §I.6.2.3.

Remark II.7 *Given a full and faithful exact functor $G : \mathcal{F} \longrightarrow \mathcal{F}'$ of Frobenius categories that sends all bijective objects to bijective objects. Then the induced functor $\underline{G} : \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{F}}'$ on the classical stable categories is full and faithful.*

Proof. By construction, it is full. We claim that it is faithful. Suppose given $X \longrightarrow Y$ in $\mathcal{F}$ such that

$$(XG \longrightarrow YG) \;=\; (XG \longrightarrow B' \longrightarrow YG)$$

in $\mathcal{C}'$ for some bijective object B' of $\mathcal{C}'$. Choose $X \rightarrowtail B$ in $\mathcal{C}$ with B bijective in $\mathcal{C}$. Since G preserves pure monomorphy, $XG \longrightarrow B'$ factors over $XG \rightarrowtail BG$, whence $XG \longrightarrow YG$ factors over $XG \rightarrowtail BG$, whence $X \longrightarrow Y$ factors over $X \rightarrowtail B$. □

Suppose given weakly abelian categories $\mathcal{A}$ and $\mathcal{A}'$. Suppose given a subexact functor $\mathcal{A} \xrightarrow{F} \mathcal{A}'$. Suppose given $n \geqslant 0$. We obtain an induced functor

$$\underline{F^+(\bar\Delta_n^\#)} \;:\; \underline{\mathcal{A}^+(\bar\Delta_n^\#)} \;\longrightarrow\; \underline{\mathcal{A}'^+(\bar\Delta_n^\#)}$$

on the respective stable categories of n-pretriangles. Cf. §I.1.2.1.3, §I.6.6.3.

Remark II.8 *If F is full and faithful, so is $\underline{F^+(\bar\Delta_n^\#)}$.*

In particular, if F is the embedding of a full subcategory, we may and will also consider $\underline{F^+(\bar\Delta_n^\#)}$ to be the embedding of a full subcategory.

Proof. By Prop. I.40, both $\mathcal{A}^+(\bar\Delta_n^\#)$ and $\mathcal{A}'^+(\bar\Delta_n^\#)$ are Frobenius categories; and the full and faithful functor $F^+(\bar\Delta_n^\#) : \mathcal{A}^+(\bar\Delta_n^\#) \longrightarrow \mathcal{A}'^+(\bar\Delta_n^\#)$ induced by F preserves bijective objects, viz. split objects, and pure short exact sequences, viz. pointwise split short exact sequences. So by Remark II.7, the assertion follows. □

II.2.2 A Heller triangulation on the Karoubi hull

Let $\hat{\mathcal{C}}$ denote the Freyd category of $\mathcal{C}$; cf. e.g. § I.6.6.3. We consider the full and faithful functor $\mathcal{C} \longrightarrow \hat{\mathcal{C}}$ as an embedding of a full subcategory. Let $\tilde{\mathcal{C}}$ denote the full subcategory of bijectives in the abelian Frobenius category $\hat{\mathcal{C}}$. So we have full subcategories

$$\mathcal{C} \subseteq \tilde{\mathcal{C}} \subseteq \hat{\mathcal{C}}$$

Since the image of $\mathcal{C}$ in $\hat{\mathcal{C}}$ is a big enough subcategory of bijectives, the embedding $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is a Karoubi hull of $\mathcal{C}$; cf. [**30,** III.II]. Cf. also Remark II.43, Lemma II.44 – which we will not use and argue directly instead.

We shall give a Heller triangulation on this Karoubi hull $\tilde{\mathcal{C}}$ of $\mathcal{C}$. The Verdier triangulated version of this construction is due to Balmer and Schlichting; cf. [**2,** Th. 1.12].

As a full subcategory of bijective objects in abelian Frobenius category, the category $\tilde{\mathcal{C}}$ is weakly abelian.

The shift T on $\mathcal{C}$ induces a shift $\hat{\mathsf{T}}$ on $\hat{\mathcal{C}}$, which restricts to a shift $\tilde{\mathsf{T}}$ on $\tilde{\mathcal{C}}$.

Remark II.9 *Suppose given $n \geqslant 0$ and $X \in \operatorname{Ob} \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$. There exists $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ such that X is isomorphic to a direct summand of Z in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$. In other words, there exists $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ and a split monomorphism $X \xrightarrow{i} Z$ in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$.*

Proof. By Prop. I.12, it suffices to prove that given $X \in \operatorname{Ob} \underline{\tilde{\mathcal{C}}(\dot{\Delta}_n)}$, there exists $Z \in \operatorname{Ob} \underline{\mathcal{C}(\dot{\Delta}_n)}$ such that X is isomorphic, in $\underline{\tilde{\mathcal{C}}(\dot{\Delta}_n)}$, to a direct summand of Z.

It suffices to prove the existence of a split monomorphism $X \longrightarrow Z$ in $\tilde{\mathcal{C}}(\dot{\Delta}_n)$ with $Z \in \operatorname{Ob} \mathcal{C}(\dot{\Delta}_n)$.

For $i \in [1, n]$, let $Y_i \in \operatorname{Ob} \tilde{\mathcal{C}}$ be such that $X_i \oplus Y_i$ is isomorphic to an object in $\mathcal{C}$. Let $Y \in \operatorname{Ob} \tilde{\mathcal{C}}(\dot{\Delta}_n)$ have entry Y_i at position i for $1 \leqslant i \leqslant n$ and the morphism from position i to position j be zero for $1 \leqslant i < j \leqslant n$. The diagram $X \oplus Y$ has X as a summand and is isomorphic to an object in $\mathcal{C}(\dot{\Delta}_n)$. ▫

Remark II.10 *Given $n \geqslant 0$, a diagram $X \in \operatorname{Ob} \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$, a split monomorphism $X \longrightarrow Z$ with $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ and a morphism $X \xrightarrow{x} X'$, then there exists a commutative quadrangle*

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z' \end{array}$$

in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$ with $Z' \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$.

Moreover, if $X \longrightarrow X'$ is a split monomorphism, we may choose $Z \longrightarrow Z'$ to be a split monomorphism.

Proof. We form

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ {\scriptstyle (1\,0)}\downarrow & & \downarrow{\scriptstyle (1\,0)} \\ X \oplus Y & \xrightarrow[\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)]{} & X' \oplus Y \end{array},$$

where $X \oplus Y \simeq Z$. By Remark II.9, there is a split monomorphism from $X' \oplus Y$ to an object Z' of Ob $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Moreover, if $X \xrightarrow{x} X'$ is split monic, so is the composite $(X \oplus Y \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)} X' \oplus Y \longrightarrow Z')$. ▫

Construction II.11 Given $n \geqslant 0$, we define $[-]^{+1} \xrightarrow{\tilde{\vartheta}_n} [-^{+1}]$ on $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ as follows.

Given $X \in$ Ob $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$, choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in$ Ob $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$, existent by Remark II.9, and choose a retraction p to i. Define

$$([X]^{+1} \xrightarrow{X\tilde{\vartheta}_n} [X^{+1}]) \;:=\; ([X]^{+1} \xrightarrow{[i]^{+1}} [Z]^{+1} \xrightarrow{Z\vartheta_n} [Z^{+1}] \xrightarrow{[p^{+1}]} [X^{+1}]) \,.$$

To prove that $X\tilde{\vartheta}_n$ is welldefined, we shall first show that it is independent of the choice of the retraction p. Given $d : Z \longrightarrow X$ with $id = 0$, we have to show that $[i]^{+1} Z\vartheta_n [d^{+1}] = 0$. Since $[i^{+1}]$ is monic, it suffices to show that $[i]^{+1} Z\vartheta_n [d^{+1}][i^{+1}] = 0$. In fact,

$$[i]^{+1} Z\vartheta_n [d^{+1}][i^{+1}] \;=\; [i]^{+1} Z\vartheta_n [(di)^{+1}] \;=\; [i]^{+1} [di]^{+1} Z\vartheta_n \;=\; [id]^{+1}[i^{+1}] Z\vartheta_n \;=\; 0\,,$$

since di is in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Now assume given another split monomorphism $X \longrightarrow Z'$ with $Z' \in$ Ob $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$. By Remark II.10, we may assume that this split monomorphism factors into two split monomorphisms $X \xrightarrow{i} Z \xrightarrow{i'} Z'$. Let $ip = 1$ and $i'p' = 1$. Then $(ii')(p'p) = 1$, and we may conclude

$$[ii']^{+1}(Z'\vartheta_n)[(p'p)^{+1}] \;=\; [i]^{+1}[i']^{+1}\Big(Z'\vartheta_n[p'^{+1}]\Big)[p^{+1}] \;=\; [i]^{+1}[i']^{+1}\Big([p']^{+1}Z\vartheta_n\Big)[p^{+1}] \;=\; [i]^{+1}Z\vartheta_n[p^{+1}]\,,$$

since p' is in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

To show that ϑ_n is a transformation, we suppose given a morphism $X \xrightarrow{f} X'$ in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ and have to show that $X\tilde{\vartheta}_n[f^{+1}] \overset{!}{=} [f]^{+1}X'\tilde{\vartheta}_n$. By Remarks II.9 and II.10, we find a commutative quadrangle

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Z & \xrightarrow{g} & Z' \end{array}$$

in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ with $Z,\, Z' \in$ Ob $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$. Choose p and p' such that $ip = 1$ and $i'p' = 1$. It suffices to show that $X\tilde{\vartheta}_n[f^{+1}][i'^{+1}] \overset{!}{=} [f]^{+1}X'\tilde{\vartheta}_n[i'^{+1}]$ by monomorphy of $[i'^{+1}]$. In fact,

$$\begin{array}{rlll} & X\tilde{\vartheta}_n[f^{+1}][i'^{+1}] & = X\tilde{\vartheta}_n[i^{+1}][g^{+1}] & = [i]^{+1}Z\vartheta_n[p^{+1}][i^{+1}][g^{+1}] \\ = & [i]^{+1}Z\vartheta_n[(pig)^{+1}] & = [i]^{+1}[pig]^{+1}Z'\vartheta_n & = [ig]^{+1}Z'\vartheta_n \\ = & [fi']^{+1}Z'\vartheta_n & = [fi']^{+1}[p'i']^{+1}Z'\vartheta_n & = [fi']^{+1}Z'\vartheta_n[(p'i')^{+1}] \\ = & [f]^{+1}[i']^{+1}Z'\vartheta_n[p'^{+1}][i'^{+1}] & = [f]^{+1}X'\tilde{\vartheta}_n[i'^{+1}]\,. & \end{array}$$

Note that $Z\tilde{\vartheta}_n = Z\vartheta_n$ for $Z \in$ Ob $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

End of construction.

Proposition II.12

(1) *The tuple* $\tilde{\vartheta} := (\tilde{\vartheta}_n)_{n\geqslant 0}$ *is the unique Heller triangulation on* $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}})$ *such that the full and faithful inclusion functor* $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ *is strictly exact; cf. Def. I.5.(i, ii).*

(2) *An* n*-pretriangle* $U \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ *is an* n*-triangle with respect to* $(\mathcal{C}, \mathsf{T}, \vartheta)$ *if and only if it is an* n*-triangle with respect to* $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}}, \tilde{\vartheta})$*.*

Proof. Ad (1). We have to show that given $m,\, n \geqslant 0$ and a periodic monotone map $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$, we have $\underline{p}^\# \star \tilde{\vartheta}_m \overset{!}{=} \tilde{\vartheta}_n \star \underline{p}^\#$. Let us verify this at $X \in \operatorname{Ob} \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$. Choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ by Remark II.9. It suffices to show that $(X\underline{p}^\#)\tilde{\vartheta}_m[(i\underline{p}^\#)^{+1}] \overset{!}{=} (X\tilde{\vartheta}_n)\underline{p}^\#[(i\underline{p}^\#)^{+1}]$. In fact, we obtain

$$\begin{aligned}
(X\underline{p}^\#)\tilde{\vartheta}_m[(i\underline{p}^\#)^{+1}] &= [i\underline{p}^\#]^{+1}(Z\underline{p}^\#)\vartheta_m &&= [i\underline{p}^\#]^{+1}(Z\vartheta_n)\underline{p}^\# \\
&= ([i]^{+1}Z\vartheta_n)\underline{p}^\# &&= (X\tilde{\vartheta}_n[i^{+1}])\underline{p}^\# \quad = (X\tilde{\vartheta}_n)\underline{p}^\#[(i\underline{p}^\#)^{+1}]\,.
\end{aligned}$$

We have to show that given $n \geqslant 0$, we have $\underline{\mathfrak{f}}_n \star \tilde{\vartheta}_{n+1} \overset{!}{=} \tilde{\vartheta}_{2n+1} \star \underline{\mathfrak{f}}_n$. Let us verify this at $X \in \operatorname{Ob} \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_{2n+1}^\#)}$. Choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_{2n+1}^\#)}$ by Remark II.9. It suffices to show that $(X\underline{\mathfrak{f}}_n)\tilde{\vartheta}_{n+1}[(i\underline{\mathfrak{f}}_n)^{+1}] \overset{!}{=} (X\tilde{\vartheta}_{2n+1})\underline{\mathfrak{f}}_n[(i\underline{\mathfrak{f}}_n)^{+1}]$. In fact, we obtain

$$\begin{aligned}
(X\underline{\mathfrak{f}}_n)\tilde{\vartheta}_{n+1}[(i\underline{\mathfrak{f}}_n)^{+1}] &= [i\underline{\mathfrak{f}}_n]^{+1}(Z\underline{\mathfrak{f}}_n)\vartheta_{n+1} &&= [i\underline{\mathfrak{f}}_n]^{+1}(Z\vartheta_{2n+1})\underline{\mathfrak{f}}_n \\
&= ([i]^{+1}Z\vartheta_{2n+1})\underline{\mathfrak{f}}_n &&= (X\tilde{\vartheta}_{2n+1}[i^{+1}])\underline{\mathfrak{f}}_n \quad = (X\tilde{\vartheta}_{2n+1})\underline{\mathfrak{f}}_n[(i\underline{\mathfrak{f}}_n)^{+1}]\,.
\end{aligned}$$

The inclusion functor $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is strictly exact since it strictly commutes with shift by construction, since it is subexact because the induced functor on the Freyd categories is an equivalence, and since $Z\tilde{\vartheta}_n = Z\vartheta_n$ for $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Now suppose that both $\tilde{\vartheta}$ and $\tilde{\vartheta}'$ are Heller triangulations on $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}})$ such that $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is strictly exact. Suppose given $n \geqslant 0$ and $X \in \operatorname{Ob} \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$. We have to show that $X\tilde{\vartheta}_n \overset{!}{=} X\tilde{\vartheta}'_n$. Choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$; cf. Remark II.9. It suffices to show that $X\tilde{\vartheta}_n[i^{+1}] \overset{!}{=} X\tilde{\vartheta}'_n[i^{+1}]$. In fact,

$$X\tilde{\vartheta}_n[i^{+1}] \;=\; [i]^{+1}Z\tilde{\vartheta}_n \;=\; [i]^{+1}Z\vartheta_n \;=\; [i]^{+1}Z\tilde{\vartheta}'_n \;=\; X\tilde{\vartheta}'_n[i^{+1}]\,.$$

Ad (2). Suppose given an n-pretriangle $U \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$. Now U is an n-triangle with respect to $(\mathcal{C}, \mathsf{T}, \vartheta)$ if and only if $U\vartheta_n = 1$, and with respect to $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}}, \tilde{\vartheta})$ if and only if $U\tilde{\vartheta}_n = 1$; cf. Def. I.5.(ii). Since $U\vartheta_n = U\tilde{\vartheta}_n$, these assertions are equivalent. Cf. also Lem. I.25. ▫

II.2.3 Functoriality of the Karoubi hull

We shall prove the universal property of the Karoubi hull directly, without making recourse to Remark II.43 and Lemma II.44. We will make use of the universal property and the abelianness of the Freyd category, however.

Proposition II.13 *Suppose given Heller triangulated categories* $(\mathcal{C}, \mathsf{T}, \vartheta)$, $(\mathcal{C}', \mathsf{T}', \vartheta')$. *Call the strictly exact inclusion functors* $\mathsf{K} : \mathcal{C} \longrightarrow \tilde{\mathcal{C}}$ *and* $\mathsf{K}' : \mathcal{C}' \longrightarrow \tilde{\mathcal{C}}'$.

(1) *Suppose given an exact functor* (F, a) *from* $\mathcal{C}$ *to* $\mathcal{C}'$.

We may construct an exact functor $(\tilde{F}, \tilde{a})$ *from* $\tilde{\mathcal{C}}$ *to* $\tilde{\mathcal{C}}'$ *such that*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(F,a)} & \mathcal{C}' \\ {\scriptstyle \mathsf{K}}\downarrow & & \downarrow{\scriptstyle \mathsf{K}'} \\ \tilde{\mathcal{C}} & \xrightarrow{(\tilde{F},\tilde{a})} & \tilde{\mathcal{C}}' \end{array}$$

commutes, i.e. such that $(F, a)(\mathsf{K}', 1) = (\mathsf{K}, 1)(\tilde{F}, \tilde{a})$, *i.e. such that* $F \star \mathsf{K}' = \mathsf{K} \star \tilde{F}$ *and* $a \star \mathsf{K}' = \mathsf{K} \star \tilde{a}$, *i.e. such that* $u\tilde{F} = uF$ *for* $u \in \operatorname{Mor} \mathcal{C}$ *and* $Z\tilde{a} = Za$ *for* $Z \in \operatorname{Ob} \mathcal{C}$.

The functor $\tilde{F}$ *and the condition* $a \star \mathsf{K}' = \mathsf{K} \star \tilde{a}$ *uniquely determines* $\tilde{a}$. *If* $a = 1$, *then* $\tilde{a} = 1$.

(2) *Given two exact functors* $(\tilde{F}_1, \tilde{a}_1)$ *and* $(\tilde{F}_2, \tilde{a}_2)$ *such that* $(F, a)(\mathsf{K}', 1) = (\mathsf{K}, 1)(\tilde{F}_1, \tilde{a}_1) = (\mathsf{K}, 1)(\tilde{F}_2, \tilde{a}_2)$, *there exists a unique isotransformation* $\tilde{F}_1 \overset{\varphi}{\underset{\sim}{\longrightarrow}} \tilde{F}_2$ *such that* $\mathsf{K} \star \varphi = 1$, *i.e. such that* $Z\varphi = 1$ *for* $Z \in \operatorname{Ob} \mathcal{C}$. *This isotransformation* φ *is periodic.*

(3) *Suppose given exact functors* (F, a) *and* (G, b) *from* $\mathcal{C}$ *to* $\mathcal{C}'$. *Suppose given a periodic transformation* s *from* F *to* G.

Construct $(\tilde{F}, \tilde{a})$ *and* $(\tilde{G}, \tilde{b})$ *as in* (1).

There exists a unique periodic transformation $\tilde{s}$ *from* $\tilde{F}$ *to* $\tilde{G}$ *such that* $\mathsf{K} \star \tilde{s} = s \star \mathsf{K}'$, *i.e. such that* $Z\tilde{s} = Zs$ *for* $Z \in \operatorname{Ob} \mathcal{C}$.

Proof. Given $X \in \operatorname{Ob} \tilde{\mathcal{C}}$, we choose $X \xrightarrow{i_X} Z_X \xrightarrow{p_X} X$ in $\tilde{\mathcal{C}}$ such that $i_X \cdot p_X = 1_X$ and such that $Z_X \in \operatorname{Ob} \mathcal{C}$.

Moreover, choose these objects and morphisms in such a way that $Z_{X\tilde{\mathsf{T}}} = Z_X \mathsf{T}$, $i_{X\tilde{\mathsf{T}}} = i_X \tilde{\mathsf{T}}$ and $p_{X\tilde{\mathsf{T}}} = p_X \tilde{\mathsf{T}}$ for $X \in \operatorname{Ob} \tilde{\mathcal{C}}$.

Furthermore, if $X \in \operatorname{Ob} \mathcal{C}$, then choose $Z_X = X$ and $i_X = 1_X$ and $p_X = 1_X$.

Given $X \xrightarrow{u} Y$ in $\tilde{\mathcal{C}}$, we let $Z_X \xrightarrow{z_u} Z_Y$ be defined by $z_u := p_X \cdot u \cdot i_Y$; cf. Remark II.39.

Ad (1). Since F is subexact, $\hat{F}$ is exact. Since W is a summand of an object in $\mathcal{C}$, also $W\hat{F}$ is a summand of an object in $\mathcal{C}'$, hence bijective. So $\tilde{F} := \hat{F}|_{\tilde{\mathcal{C}}}^{\tilde{\mathcal{C}}'}$ is welldefined.

We want to show that the functor $\tilde{F}$ preserves weak kernels and is therefore subexact; cf. Lemma II.41. In fact, given $W \xrightarrow{w} B \xrightarrow{f} C$ in $\tilde{\mathcal{C}}$ such that w is a weak kernel of f, we get a factorisation $w = w'i$, where $K \overset{i}{\rightarrowtail} B$ is a kernel of f in $\hat{\mathcal{C}}$. Considering an epimorphism $P \overset{p}{\twoheadrightarrow} K$ in $\hat{\mathcal{C}}$ with $P \in \operatorname{Ob} \tilde{\mathcal{C}}$, we obtain a factorisation $pi = p'w = p'w'i$, whence $p = p'w'$, whence w' is epic. Since $w'\hat{F}$ is epic and $i\hat{F}$ is a kernel of $f\hat{F}$, we obtain that $w\hat{F} = w\tilde{F}$ is a weak kernel of $f\hat{F} = f\tilde{F}$.

The universal property of the Freyd construction yields a transformation $\hat{a} : \hat{\mathsf{T}}\hat{F} \longrightarrow \hat{F}\,\hat{\mathsf{T}}'$. We let the transformation $\tilde{a} : \tilde{\mathsf{T}}\tilde{F} \longrightarrow \tilde{F}\,\tilde{\mathsf{T}}'$ be defined on $X \in \operatorname{Ob} \tilde{\mathcal{C}} \subseteq \operatorname{Ob} \hat{\mathcal{C}}$ as $X\tilde{a} := X\hat{a}$. In particular, $Z\tilde{a} = Za$ for $Z \in \operatorname{Ob} \mathcal{C}$.

Given $n \geqslant 0$, it remains to be shown that $\underline{\tilde F^+(\bar\Delta_n^\#)} \star \tilde\vartheta'_n \overset{!}{=} (\tilde\vartheta_n \star \underline{\tilde F^+(\bar\Delta_n^\#)}) \cdot \underline{\tilde a^+(\bar\Delta_n^\#)}$; cf. Definition II.1. Let us verify this at $X \in \operatorname{Ob} \underline{\tilde{\mathcal C}^+(\bar\Delta_n^\#)}$. Let $X \xrightarrow{i} Z$ be a split monomorphism with $Z \in \operatorname{Ob} \underline{\mathcal C^+(\bar\Delta_n^\#)}$, existent by Remark II.9. It suffices to show that

$$(X\underline{\tilde F^+(\bar\Delta_n^\#)})\tilde\vartheta'_n \cdot [(i\underline{\tilde F^+(\bar\Delta_n^\#)})^{+1}] \overset{!}{=} (X\tilde\vartheta_n)\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot X\underline{\tilde a^+(\bar\Delta_n^\#)} \cdot [(i\underline{\tilde F^+(\bar\Delta_n^\#)})^{+1}]\,.$$

In fact, we obtain

$$\begin{aligned}
(X\underline{\tilde F^+(\bar\Delta_n^\#)})\tilde\vartheta'_n \cdot [(i\underline{\tilde F^+(\bar\Delta_n^\#)})^{+1}] &= [i\underline{\tilde F^+(\bar\Delta_n^\#)}]^{+1} \cdot (Z\underline{\tilde F^+(\bar\Delta_n^\#)})\tilde\vartheta'_n \\
&= [i\underline{\tilde F^+(\bar\Delta_n^\#)}]^{+1} \cdot (Z\underline{F^+(\bar\Delta_n^\#)})\vartheta'_n \\
&\overset{(F,a)\text{ ex.}}{=} [i\underline{\tilde F^+(\bar\Delta_n^\#)}]^{+1} \cdot (Z\vartheta_n)\underline{F^+(\bar\Delta_n^\#)} \cdot Z\underline{a^+(\bar\Delta_n^\#)} \\
&= [i\underline{\tilde F^+(\bar\Delta_n^\#)}]^{+1} \cdot (Z\tilde\vartheta_n)\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot Z\underline{\tilde a^+(\bar\Delta_n^\#)} \\
&= [i]^{+1}\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot (Z\tilde\vartheta_n)\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot Z\underline{\tilde a^+(\bar\Delta_n^\#)} \\
&= (X\tilde\vartheta_n)\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot [i^{+1}]\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot Z\underline{\tilde a^+(\bar\Delta_n^\#)} \\
&= (X\tilde\vartheta_n)\underline{\tilde F^+(\bar\Delta_n^\#)} \cdot X\underline{\tilde a^+(\bar\Delta_n^\#)} \cdot [(i\underline{\tilde F^+(\bar\Delta_n^\#)})^{+1}]\,.
\end{aligned}$$

If $a = 1$, then $\hat a = 1$, so $\tilde a = 1$.

It remains to show that $\tilde a$ is uniquely determined by $\tilde F$ and the condition $a \star \mathsf{K}' = \mathsf{K} \star \tilde a$. In fact, given $X \in \operatorname{Ob}\tilde{\mathcal C}$, we have

$$X\tilde a \cdot i_X \tilde F \tilde{\mathsf T}' = i_X \tilde{\mathsf T}\tilde F \cdot Z_X \tilde a = i_X \tilde{\mathsf T}\tilde F \cdot Z_X a\,,$$

and $i_X \tilde F \tilde{\mathsf T}'$ is monic.

Ad (2). Define $\varphi : \tilde F_1 \xrightarrow{\sim} \tilde F_2$ at $X \in \operatorname{Ob}\tilde{\mathcal C}$ by

$$\begin{array}{ccccc}
Z_X F & \xrightarrow{p_X \tilde F_1} & X\tilde F_1 & \xrightarrow{i_X \tilde F_1} & Z_X F \\
{\scriptstyle 1}\downarrow & & {\scriptstyle X\varphi}\downarrow\wr & & \downarrow{\scriptstyle 1} \\
Z_X F & \xrightarrow{p_X \tilde F_2} & X\tilde F_2 & \xrightarrow{i_X \tilde F_2} & Z_X F\,;
\end{array}$$

cf. Remark II.40.

The tuple $\varphi = (X\varphi)_{X\in\operatorname{Ob}\tilde{\mathcal C}}$ is actually a transformation from $\tilde F_1$ to $\tilde F_2$, for given $X \xrightarrow{u} Y$ in $\tilde{\mathcal C}$, we obtain

$$\begin{array}{rlrl}
 & p_X\tilde F_1 \cdot u\tilde F_1 \cdot Y\varphi \cdot i_Y \tilde F_2 & = & p_X \tilde F_1 \cdot u\tilde F_1 \cdot i_Y \tilde F_1 \\
= & p_X \tilde F_1 \cdot (u \cdot i_Y)\tilde F_1 & = & p_X \tilde F_1 \cdot (i_X \cdot z_u)\tilde F_1 \\
= & p_X \tilde F_1 \cdot i_X \tilde F_1 \cdot z_u \tilde F_1 & = & ((p_X \cdot i_X) \cdot z_u)F \\
= & (z_u \cdot (p_Y \cdot i_Y))F & = & z_u \tilde F_2 \cdot p_Y \tilde F_2 \cdot i_Y \tilde F_2 \\
= & (z_u \cdot p_Y)\tilde F_2 \cdot i_Y \tilde F_2 & = & (p_X \cdot u)\tilde F_2 \cdot i_Y \tilde F_2 \\
= & p_X \tilde F_2 \cdot u \tilde F_2 \cdot i_Y \tilde F_2 & = & p_X \tilde F_1 \cdot X\varphi \cdot u\tilde F_2 \cdot i_Y \tilde F_2\,,
\end{array}$$

and $p_X \tilde F_1$ is epic and $i_Y \tilde F_2$ is monic.

Note that commutativity of the diagram above is also necessary, for we require $\mathsf{K}\star\varphi = 1$. This ensures uniqueness of φ.

It remains to show that φ is a periodic transformation from $(\tilde{F}_1\,,\,\tilde{a}_1)$ to $(\tilde{F}_2\,,\,\tilde{a}_2)$. In fact, given $X \in \operatorname{Ob}\tilde{\mathcal{C}}$, we get

$$
\begin{array}{rlcl}
 & X\tilde{a}_1\cdot X\varphi\tilde{\mathsf{T}}'\cdot i_X\tilde{F}_2\tilde{\mathsf{T}}' & = & X\tilde{a}_1\cdot i_X\tilde{F}_1\tilde{\mathsf{T}}' \\
= & i_X\tilde{\mathsf{T}}\tilde{F}_1\cdot Z_X\tilde{a}_1 & = & i_{X\tilde{\mathsf{T}}}\tilde{F}_1\cdot Z_X\tilde{a}_1 \\
= & i_{X\tilde{\mathsf{T}}}\tilde{F}_1\cdot Z_X a & = & i_{X\tilde{\mathsf{T}}}\tilde{F}_1\cdot Z_X\tilde{a}_2 \\
= & X\tilde{\mathsf{T}}\varphi\cdot i_{X\tilde{\mathsf{T}}}\tilde{F}_2\cdot Z_X\tilde{a}_2 & = & X\tilde{\mathsf{T}}\varphi\cdot i_X\tilde{\mathsf{T}}\tilde{F}_2\cdot Z_X\tilde{a}_2 \\
= & X\tilde{\mathsf{T}}\varphi\cdot X\tilde{a}_2\cdot i_X\tilde{F}_2\tilde{\mathsf{T}}'\,, & &
\end{array}
$$

and $i_X\tilde{F}_2\tilde{\mathsf{T}}'$ is monic.

Ad (3). Define $\tilde{s} : \tilde{F} \longrightarrow \tilde{G}$ at $X \in \operatorname{Ob}\tilde{\mathcal{C}}$ by

$$
\begin{array}{ccccc}
Z_X F & \xrightarrow{p_X\tilde{F}} & X\tilde{F} & \xrightarrow{i_X\tilde{F}} & Z_X F \\
{\scriptstyle Z_X s}\downarrow & & {\scriptstyle X\tilde{s}}\downarrow\wr & & \downarrow{\scriptstyle Z_X s} \\
Z_X G & \xrightarrow{p_X\tilde{G}} & X\tilde{G} & \xrightarrow{i_X\tilde{G}} & Z_X G\,;
\end{array}
$$

cf. Remark II.40.

The tuple $s = (Xs)_{X\in\operatorname{Ob}\tilde{\mathcal{C}}}$ is actually a transformation from $\tilde{F}$ to $\tilde{G}$, for given $X \xrightarrow{u} Y$ in $\tilde{\mathcal{C}}$, we obtain

$$
\begin{array}{rlcl}
 & p_X\tilde{F}\cdot u\tilde{F}\cdot Y\tilde{s}\cdot i_Y\tilde{G} & = & p_X\tilde{F}\cdot u\tilde{F}\cdot i_Y\tilde{F}\cdot Z_Y s \\
= & p_X\tilde{F}\cdot(u\cdot i_Y)\tilde{F}\cdot Z_Y s & = & p_X\tilde{F}\cdot(i_X\cdot z_u)\tilde{F}\cdot Z_Y s \\
= & p_X\tilde{F}\cdot i_X\tilde{F}\cdot z_u\tilde{F}\cdot Z_Y s & = & ((p_X\cdot i_X)\cdot z_u)F\cdot Z_Y s \\
= & (z_u\cdot(p_Y\cdot i_Y))F\cdot Z_Y s & = & Z_X s\cdot(z_u\cdot(p_Y\cdot i_Y))G \\
= & Z_X s\cdot z_u\tilde{G}\cdot p_Y\tilde{G}\cdot i_Y\tilde{G} & = & Z_X s\cdot(z_u\cdot p_Y)\tilde{G}\cdot i_Y\tilde{G} \\
= & Z_X s\cdot(p_X\cdot u)\tilde{G}\cdot i_Y\tilde{G} & = & Z_X s\cdot p_X\tilde{G}\cdot u\tilde{G}\cdot i_Y\tilde{G} \\
= & p_X\tilde{F}\cdot X\tilde{s}\cdot u\tilde{G}\cdot i_Y\tilde{G}\,, & &
\end{array}
$$

and $p_X\tilde{F}$ is epic and $i_Y\tilde{G}$ is monic.

Note that commutativity of the diagram above is also necessary, for we require $\mathsf{K}\star\tilde{s} = s\star\mathsf{K}'$. This ensures uniqueness of s.

It remains to show that $\tilde{s}$ is a periodic transformation from $(\tilde{F},\tilde{a})$ to $(\tilde{G},\tilde{b})$. In fact, given $X \in \operatorname{Ob}\tilde{\mathcal{C}}$, we get

$$
\begin{array}{rlcl}
 & X\tilde{a}\cdot X\tilde{s}\tilde{\mathsf{T}}'\cdot i_X\tilde{G}\,\tilde{\mathsf{T}}' & = & X\tilde{a}\cdot i_X\tilde{F}\,\tilde{\mathsf{T}}'\cdot Z_X s\tilde{\mathsf{T}}' \\
= & i_X\tilde{\mathsf{T}}\tilde{F}\cdot Z_X\tilde{a}\cdot Z_X s\tilde{\mathsf{T}}' & = & i_{X\tilde{\mathsf{T}}}\tilde{F}\cdot Z_X\tilde{a}\cdot Z_X s\tilde{\mathsf{T}}' \\
= & i_{X\tilde{\mathsf{T}}}\tilde{F}\cdot Z_X a\cdot Z_X s\,\mathsf{T}' & = & i_{X\tilde{\mathsf{T}}}\tilde{F}\cdot Z_X\,\mathsf{T}\,s\cdot Z_X b \\
= & i_{X\tilde{\mathsf{T}}}\tilde{F}\cdot Z_X\tilde{\mathsf{T}}s\cdot Z_X\tilde{b} & = & i_{X\tilde{\mathsf{T}}}\tilde{F}\cdot Z_{X\tilde{\mathsf{T}}}s\cdot Z_X\tilde{b} \\
= & X\tilde{\mathsf{T}}\tilde{s}\cdot i_{X\tilde{\mathsf{T}}}\tilde{G}\cdot Z_X\tilde{b} & = & X\tilde{\mathsf{T}}\tilde{s}\cdot i_X\tilde{\mathsf{T}}\tilde{G}\cdot Z_X\tilde{b} \\
= & X\tilde{\mathsf{T}}\tilde{s}\cdot X\tilde{b}\cdot i_X\tilde{G}\,\tilde{\mathsf{T}}'\,, & &
\end{array}
$$

and $i_X \tilde{G} \tilde{\mathsf{T}}'$ is monic. ▫

II.2.4 Closed Heller triangulated categories

Recall that given a Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$, its Karoubi hull $\tilde{\mathcal{C}}$ is Heller triangulated, too; cf. Proposition II.12.(1). More precisely, $(\tilde{\mathcal{C}}, \tilde{T}, \tilde{\vartheta})$ is Heller triangulated, where $\tilde{T}$ and $\tilde{\vartheta}$ are as in §II.2.2.

Definition II.14

> A Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is called *closed* if whenever $(X, Y, \tilde{Z})$ is a 2-triangle in $\tilde{\mathcal{C}}$ and $X,\, Y \in \operatorname{Ob} \mathcal{C}$, then $\tilde{Z}$ is isomorphic to an object of $\mathcal{C}$.

Cf. Def. I.5.(i, iii).

I do not know an example of a non-closed Heller triangulated category.

As usual, we will call $\tilde{Z}$ the *cone* of $X \longrightarrow Y$, being unique up to isomorphism. Thus we may rephrase that by definition, $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed if it is closed under taking cones in the Karoubi hull $\tilde{\mathcal{C}}$.

Remark II.15 *The Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed if and only if given $X \xrightarrow{f} Y$ in $\mathcal{C}$, there exists a 2-triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X^{+1}$ in $\mathcal{C}$.*

Cf. Def. III.13.

Proof. If $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed, then given $X \xrightarrow{f} Y$ in $\mathcal{C}$, there exists a 2-triangle $X \xrightarrow{f} Y \longrightarrow \tilde{Z} \longrightarrow X^{+1}$ in $\tilde{\mathcal{C}}$ by Lem. I.18, and we may substitute $\tilde{Z}$ isomorphically by an object Z in $\operatorname{Ob} \mathcal{C}$, so we are done by Lem. I.21.(4).

Conversely, if we dispose of this existence property, and if we are given a 2-triangle $(X, Y, \tilde{Z})$ in $\tilde{\mathcal{C}}$ with $X,\, Y \in \operatorname{Ob} \mathcal{C}$, then there exists a 2-triangle (X, Y, Z) with $Z \in \operatorname{Ob} \mathcal{C}$, too, and we may apply Lem. I.21.(6) to conclude that $Z \simeq \tilde{Z}$. So $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed. ▫

Remark II.16 *If idempotents split in $\mathcal{C}$, then $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed.*

Proof. If idempotents split in $\mathcal{C}$, then $\mathcal{C} = \tilde{\mathcal{C}}$. ▫

Remark II.17 *Suppose given Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$, $(\mathcal{C}', \mathsf{T}', \vartheta')$ and a full and faithful strictly exact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$. Furthermore, suppose that whenever given a 2-triangle (XF, YF, Z') in $\mathcal{C}'$, where $X,\, Y \in \operatorname{Ob} \mathcal{C}$, then there exists $Z \in \operatorname{Ob} \mathcal{C}$ such that $Z' \simeq ZF$.*

Suppose that $\mathcal{C}'$ is closed. Then $\mathcal{C}$ is closed.

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$. There exists a 2-triangle $XF \xrightarrow{fF} YF \longrightarrow Z' \longrightarrow XF^{+1}$ in $\mathcal{C}'$. By assumption, there exists $Z \in \operatorname{Ob}\mathcal{C}$ such that $ZF \simeq Z'$. By isomorphic substitution and fullness of F, we obtain a 2-triangle $XF \xrightarrow{fF} YF \xrightarrow{gF} ZF \xrightarrow{hF} XF^{+1}$ in $\mathcal{C}'$; cf. Lem. I.21.(4). Since

$$(X,Y,Z)\vartheta_2 \underline{F^+(\bar{\Delta}_2^\#)} \;=\; (X,Y,Z)\underline{F^+(\bar{\Delta}_2^\#)}\vartheta_2' \;=\; (XF,YF,ZF)\vartheta_2' \;=\; 1\,,$$

we conclude by faithfulness of $\underline{F^+(\bar{\Delta}_2^\#)}$ that $(X,Y,Z)\vartheta_2 = 1$; cf. Remark II.8, Def. I.5.(ii). So we are done by Remark II.15. ▫

Remark II.18 *A closed Heller triangulated category is Verdier triangulated.*

Proof. Its Karoubian hull is Verdier triangulated by Prop. I.23. An additive shift-closed subcategory of a Verdier triangulated category that is closed under forming cones is Verdier triangulated. ▫

Definition II.19 Suppose given a closed Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$.

Suppose given $n \geqslant 0$ and $Y \in \operatorname{Ob}\mathcal{C}(\dot{\Delta}_n)$ and $X \in \operatorname{Ob}\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^\#)$ such that $X|_{\dot{\Delta}_n} = Y$.

Then Y is called the *base* of the n-triangle X.

Lemma II.20 *Suppose given a closed Heller triangulated category* $(\mathcal{C}, \mathsf{T}, \vartheta)$ *and* $n \geqslant 0$. *The restriction functor* $\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^\#) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ *is strictly dense, i.e. surjective on objects. In other words, each object* $Y \in \operatorname{Ob}\mathcal{C}(\dot{\Delta}_n)$ *is the base of an* n*-triangle.*

> So weakening the assumption in Lem. I.18 that idempotents be split in $\mathcal{C}$ to the assumption that $\mathcal{C}$ be closed, we nonetheless obtain the conclusion of loc. cit.

Proof. Suppose given $Y \in \operatorname{Ob}\mathcal{C}(\dot{\Delta}_n)$. By Lem. I.18, we obtain an n-triangle $\tilde{X} \in \operatorname{Ob}\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^\#)$ such that $\tilde{X}|_{\dot{\Delta}_n} = Y$.

By Lem. I.21.(1), we have a triangle $(\tilde{X}_{\alpha/0}\,, \tilde{X}_{\beta/0}\,, \tilde{X}_{\beta/\alpha})$ whenever $0 < \alpha < \beta < 0^{+1}$. Since $\mathcal{C}$ is closed, $\tilde{X}_{\beta/\alpha}$ is isomorphic to an object of $\mathcal{C}$. Isomorphic substitution, which is permitted without leaving $\tilde{\mathcal{C}}^{+,\vartheta=1}(\bar{\Delta}_n^\#)$ by Lem. I.21.(4), yields an n-triangle in $\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^\#)$ that restricts to Y on $\dot{\Delta}_n$; cf. Proposition II.12.(2). ▫

II.3 Heller triangulated subcategories

Definition II.21 Given a Heller triangulated category $(\mathcal{C}', \mathsf{T}', \vartheta')$, a full subcategory $\mathcal{C} \subseteq \mathcal{C}'$ is called a full *Heller triangulated subcategory* of $\mathcal{C}'$ if there exist T and ϑ such that $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a Heller triangulated category and such that the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}'$ is strictly exact.

We remark that in this case, the automorphism T and the tuple of transformations ϑ are uniquely determined by $(\mathcal{C}', \mathsf{T}', \vartheta')$ as respective restrictions; cf. Def. I.5.(iii), Remark II.8.

Example II.22 Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a Heller triangulated category. Let $\tilde{\mathcal{C}}$ be the Karoubi hull of $\mathcal{C}$, and let $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}}, \tilde{\vartheta}_n)$ be the Heller triangulated category from Construction II.11. By Proposition II.12.(1), $\mathcal{C}$ is a Heller triangulated subcategory of $\tilde{\mathcal{C}}$.

Lemma II.23 *Suppose given a closed Heller triangulated category $(\mathcal{C}', \mathsf{T}', \vartheta')$, and a full subcategory $\mathcal{C} \subseteq \mathcal{C}'$ such that the following conditions $(1, 2)$ hold.*

(1) $\mathcal{C}\,\mathsf{T}' = \mathcal{C}$.

(2) *Given a 2-triangle (X, Y, Z') in $\mathcal{C}'$ with $X,\, Y \in \operatorname{Ob}\mathcal{C}$, then Z' is isomorphic to an object of $\mathcal{C}$.*

Then $\mathcal{C}$, equipped with the shift T and the tuple ϑ obtained by restriction from T' and ϑ', respectively, is a Heller triangulated subcategory of $\mathcal{C}'$. Moreover, $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed.

Proof. Let T denote the restriction of T' to an automorphism of $\mathcal{C}$, which exists by assumption (1). Write $\mathcal{C} \xrightarrow{i} \mathcal{C}'$ for the inclusion functor.

Since $\mathcal{C}'$ is closed, assumption (2) allows to conclude that $\mathcal{C}$ is a full additive subcategory of $\mathcal{C}'$, and moreover, that $\mathcal{C}$ is weakly abelian such that i is subexact; cf. Lemma II.41.

Given $n \geqslant 0$ and $X \in \operatorname{Ob}\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$, we define, by restriction, $([X]^{+1} \xrightarrow{X\vartheta_n} [X^{+1}]) := ([X]^{+1} \xrightarrow{X\vartheta'_n} [X^{+1}])$. Since $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} \xrightarrow{i} \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$ is full and faithful by Remark II.8, this is a welldefined transformation satisfying $\vartheta_n \star \underline{i} = \underline{i} \star \vartheta'_n$.

Given $m,\, n \geqslant 0$ and a periodic monotone map $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$, we have $\underline{p}^\# \star \underline{i} = \underline{i} \star \underline{p}^\#$, whence

$$\underline{p}^\# \star \vartheta_m \star \underline{i} \;=\; \underline{p}^\# \star \underline{i} \star \vartheta'_m \;=\; \underline{i} \star \underline{p}^\# \star \vartheta'_m \;=\; \underline{i} \star \vartheta'_n \star \underline{p}^\# \;=\; \vartheta_n \star \underline{i} \star \underline{p}^\# \;=\; \vartheta_n \star \underline{p}^\# \star \underline{i}\,,$$

so that we may conclude that $\underline{p}^\# \star \vartheta_m = \vartheta_n \star \underline{p}^\#$, for $\underline{i}$ is faithful.

Given $n \geqslant 0$, we have $\underline{\mathfrak{f}}_n \star \underline{i} = \underline{i} \star \underline{\mathfrak{f}}_n$, whence

$$\underline{\mathfrak{f}}_n \star \vartheta_{n+1} \star \underline{i} \;=\; \underline{\mathfrak{f}}_n \star \underline{i} \star \vartheta'_{n+1} \;=\; \underline{i} \star \underline{\mathfrak{f}}_n \star \vartheta'_{n+1} \;=\; \underline{i} \star \vartheta'_{2n+1} \star \underline{\mathfrak{f}}_n \;=\; \vartheta_{2n+1} \star \underline{i} \star \underline{\mathfrak{f}}_n \;=\; \vartheta_{2n+1} \star \underline{\mathfrak{f}}_n \star \underline{i}\,,$$

so that we may conclude that $\underline{\mathfrak{f}}_n \star \vartheta_{n+1} = \vartheta_{2n+1} \star \underline{\mathfrak{f}}_n$, for $\underline{i}$ is faithful.

Hence ϑ is a Heller triangulation on $(\mathcal{C}, \mathsf{T})$; cf. Def. I.5.(i). By construction, $\mathcal{C} \xrightarrow{i} \mathcal{C}'$ is strictly exact.

By (2) and Remark II.17, the Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed. □

II.4 Functors are exact if and only if they are compatible with n-triangles

Suppose given Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$.

Concerning the notion of n-triangles in a Heller triangulated category, cf. Def. I.5.(ii).

For $n \geqslant 0$, an object Y in $\mathcal{C}(\bar{\Delta}_n^\#)$ is called *periodic* if $[Y]^{+1} = [Y^{+1}]$.

Suppose given an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and an isomorphism $\mathsf{T}\,F \xrightarrow[\sim]{a} F\,\mathsf{T}'$.

For $z \in \mathbf{Z}$, we let $\mathsf{T}^z\,F \xrightarrow[\sim]{a^{(z)}} F\,\mathsf{T}'^z$ be defined by

$$\begin{array}{lcll} a^{(0)} & := & 1_F & \\ a^{(z+1)} & := & (\mathsf{T} \star a^{(z)}) \cdot (a \star \mathsf{T}'^z) & \text{for } z \geqslant 0 \\ a^{(z-1)} & := & (\mathsf{T}^- \star a^{(z)}) \cdot (\mathsf{T}^- \star a^- \star \mathsf{T}'^{z-1}) & \text{for } z \leqslant 0 \end{array}$$

Then $(\mathsf{T}^z \star a^{(w)}) \cdot (a^{(z)} \star \mathsf{T}'^w) = a^{(z+w)} : \mathsf{T}^{z+w}\,F \xrightarrow{\sim} F\,\mathsf{T}'^{z+w}$ for $z,\, w \in \mathbf{Z}$.

Given a periodic n-pretriangle $X \in \operatorname{Ob} \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^\#)$, for sake of brevity we denote in this chapter by

$$Y \;:=\; X(F(\bar{\Delta}_n^\#)) \;\in\; \operatorname{Ob} \mathcal{C}'(\bar{\Delta}_n^\#)$$

the diagram obtained by pointwise application of F to X. We have

$$Y|_{\dot{\Delta}_n^{+1}} \;=\; X|_{\dot{\Delta}_n}((\mathsf{T}\,F)(\dot{\Delta}_n)) \;\xrightarrow[\sim]{X|_{\dot{\Delta}_n}(a(\dot{\Delta}_n))}\; X|_{\dot{\Delta}_n}((F\,\mathsf{T}')(\dot{\Delta}_n)) \;=\; (Y|_{\dot{\Delta}_n})^{+1}\,.$$

Isomorphic substitution along this isomorphism turns $Y|_{\bar{\Delta}_n^{\Delta\nabla}}$ into a diagram $\breve{Y}|_{\bar{\Delta}_n^{\Delta\nabla}}$ for a periodic object $\breve{Y} \in \operatorname{Ob} \mathcal{C}(\bar{\Delta}_n^\#)$ thus defined. We have an isomorphism $Y \xrightarrow[\sim]{\breve{a}} \breve{Y}$ in $\mathcal{C}'(\bar{\Delta}_n^\#)$ that at $(\beta/\alpha)^{+z}$ for $0 \leqslant \alpha \leqslant \beta \leqslant n$ and $z \in \mathbf{Z}$ is given by

$$\Big(Y_{(\beta/\alpha)^{+z}} \xrightarrow[\sim]{\breve{a}_{(\beta/\alpha)^{+z}}} \breve{Y}_{(\beta/\alpha)^{+z}}\Big) \;:=\; \Big(X_{\beta/\alpha}\,\mathsf{T}^z\,F \xrightarrow[\sim]{X_{\beta/\alpha}\,a^{(z)}} X_{\beta/\alpha}F\,\mathsf{T}'^z\Big)\,.$$

In fact, given $0 \leqslant \alpha \leqslant n$ and $z \in \mathbf{Z}$, we obtain a commutative quadrangle

$$\begin{array}{ccc} X_{n/\alpha}\,\mathsf{T}^z\,F & \xrightarrow{x\,\mathsf{T}^z\,F} & X_{\alpha/0}\,\mathsf{T}^{z+1}\,F \\ {\scriptstyle X_{n/\alpha}\,a^{(z)}}\downarrow\wr & & \wr\downarrow{\scriptstyle X_{\alpha/0}\,a^{(z+1)}} \\ X_{n/\alpha}F\,\mathsf{T}'^z & \xrightarrow{(xF\,\mathsf{T}'^z)(X_{\alpha/0}\,a\,\mathsf{T}'^z)} & X_{\alpha/0}F\,\mathsf{T}'^{z+1} \end{array},$$

for

$$(X_{n/\alpha}\,a^{(z)})(xF\,\mathsf{T}'^z)(X_{\alpha/0}\,a\,\mathsf{T}'^z) \;=\; (x\,\mathsf{T}^z\,F)(X_{\alpha/0}\,\mathsf{T}\,a^{(z)})(X_{\alpha/0}\,a\,\mathsf{T}'^z) \;=\; (x\,\mathsf{T}^z\,F)(X_{\alpha/0}\,a^{(z+1)})\,.$$

The remaining commutativities required for the naturality of $Y \xrightarrow[\sim]{\breve{a}} \breve{Y}$ follow by naturality of $a^{(z)}$.

We remark that $\breve{a}|_{\dot{\Delta}_n} = 1_{F(\dot{\Delta}_n)}$.

If F is subexact, then Y is an n-pretriangle and $\breve{Y}$ is a periodic n-pretriangle.

Lemma II.24 *Suppose given an exact functor (F, a).*

Then for each n-triangle X of $\mathcal{C}$, i.e. $X \in \operatorname{Ob}\mathcal{C}^{\vartheta=1,+}(\bar{\Delta}_n^\#)$, the object $\breve{Y}$ of $\mathcal{C}'(\bar{\Delta}_n^\#)$ defined by (1) and (2) is an n-triangle of $\mathcal{C}'$, i.e. $\breve{Y} \in \operatorname{Ob}\mathcal{C}'^{\vartheta=1,+}(\bar{\Delta}_n^\#)$.

(1) *We have $[\breve{Y}]^{+1} = [\breve{Y}^{+1}]$.*

(2) *On* $\bar{\Delta}_n^{\vartriangle\triangledown}$, *the object* $\breve{Y}|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$ *arises from* $Y := X(F(\bar{\Delta}_n^{\#}))|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$ *by isomorphic substitution along* $Y|_{\dot{\Delta}_n^{+1}} = X|_{\dot{\Delta}_n}(\mathsf{T}(\dot{\Delta}_n))(F(\dot{\Delta}_n)) \overset{a(\dot{\Delta}_n)}{\underset{\sim}{\longrightarrow}} X|_{\dot{\Delta}_n}(F(\dot{\Delta}_n))(\mathsf{T}'(\dot{\Delta}_n)) = (Y|_{\dot{\Delta}_n})^{+1}$.

Cf. Lem. I.25 for the case of a strictly exact functor.

Proof. Suppose given $n \geqslant 0$ and an n-triangle $X \in \operatorname{Ob} \mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^{\#})$. By construction, $\breve{Y}$ is periodic. We have to show that $\breve{Y}\vartheta'_n \overset{!}{=} 1_{[\breve{Y}]^{+1}}$. We obtain

$$\begin{aligned} Y\vartheta'_n &= X\left(\underline{F^+(\bar{\Delta}_n^{\#})} \star \vartheta'_n\right) \\ &= X\left(\left(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})}\right) \cdot \underline{a^+(\bar{\Delta}_n^{\#})}\right) \\ &= X\left(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})}\right) \cdot X\underline{a^+(\bar{\Delta}_n^{\#})} \\ &= X\underline{a^+(\bar{\Delta}_n^{\#})}\,. \end{aligned}$$

In particular, $Y\vartheta'|_{\dot{\Delta}_n} = X|_{\dot{\Delta}_n}\underline{a(\dot{\Delta}_n)}$. Hence, restricting the stably commutative quadrangle

$$\begin{array}{ccc} [Y]^{+1} & \xrightarrow{[\breve{a}]^{+1}} & [\breve{Y}]^{+1} \\ {\scriptstyle Y\vartheta'_n}\downarrow & & \downarrow{\scriptstyle \breve{Y}\vartheta'_n} \\ [Y^{+1}] & \xrightarrow{[\breve{a}^{+1}]} & [\breve{Y}^{+1}] \end{array}$$

to $\dot{\Delta}_n$, we obtain the stably commutative quadrangle

$$\begin{array}{ccc} Y|_{\dot{\Delta}_n^{+1}} & \xrightarrow{X|_{\dot{\Delta}_n}\underline{a(\dot{\Delta}_n)}} & (Y|_{\dot{\Delta}_n})^{+1} \\ {\scriptstyle X|_{\dot{\Delta}_n}\underline{a(\dot{\Delta}_n)}}\downarrow & & \downarrow{\scriptstyle \breve{Y}\vartheta'_n|_{\dot{\Delta}_n}} \\ (Y|_{\dot{\Delta}_n})^{+1} & \xrightarrow{1} & (Y|_{\dot{\Delta}_n})^{+1}\,. \end{array}$$

whence $\breve{Y}\vartheta'_n|_{\dot{\Delta}_n} = 1_{(Y|_{\dot{\Delta}_n})^{+1}}$. Since the functor from $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ to $\underline{\mathcal{C}(\dot{\Delta}_n)}$ induced by restriction is an equivalence by Prop. I.12, this implies that $\breve{Y}\vartheta'_n = 1_{[\breve{Y}]^{+1}}$. ▫

Proposition II.25 *Suppose* $\mathcal{C}$ *to be closed.*

The pair (F, a) *is an exact functor if and only if for each* n*-triangle* X *of* $\mathcal{C}$*, the object* $\breve{Y}$ *of* $\mathcal{C}'(\bar{\Delta}_n^{\#})$ *defined by* $(1, 2)$ *is an* n*-triangle of* $\mathcal{C}'$*.*

(1) *We have* $[\breve{Y}]^{+1} = [\breve{Y}^{+1}]$.

(2) *On* $\bar{\Delta}_n^{\vartriangle\triangledown}$, *the object* $\breve{Y}|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$ *arises from* $Y := X(F(\bar{\Delta}_n^{\#}))|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$ *by isomorphic substitution along* $Y|_{\dot{\Delta}_n^{+1}} = X|_{\dot{\Delta}_n}(\mathsf{T}(\dot{\Delta}_n))(F(\dot{\Delta}_n)) \overset{a(\dot{\Delta}_n)}{\underset{\sim}{\longrightarrow}} X|_{\dot{\Delta}_n}(F(\dot{\Delta}_n))(\mathsf{T}'(\dot{\Delta}_n)) = (Y|_{\dot{\Delta}_n})^{+1}$.

Proof. In view of Lemma II.24, it suffices to show that if each n-triangle X in $\mathcal{C}$ yields an n-triangle $\breve{Y}$ in $\mathcal{C}'$ by $(1, 2)$, then (F, a) is exact.

We *claim* that F is subexact. By Lemma II.41, it suffices to show that given a morphism $S \xrightarrow{p} T$ in $\mathcal{C}$, there exists a weak cokernel of p that is mapped by F to a weak cokernel. Since $\mathcal{C}$ is a closed Heller triangulated category, a weak cokernel of p is contained in the the completion of $S \xrightarrow{p} T$ to a 2-triangle X by Lemma II.20. We form the corresponding 2-triangle $\breve{Y}$ defined by (1, 2). Since it contains a weak cokernel of $SF \xrightarrow{pF} TF$, and since $\breve{Y}$ is isomorphic, in $\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})$, to $X(F^+(\bar{\Delta}_2^{\#}))$, the image under F of the weak cokernel of p that is contained in the 2-triangle X is in fact a weak cokernel of pF. This proves the *claim*.

We *claim* that

$$(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})}) \cdot \underline{a^+(\bar{\Delta}_n^{\#})} \;=\; \underline{F^+(\bar{\Delta}_n^{\#})} \star \vartheta'_n$$

for all $n \geqslant 0$. Suppose given $X \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^{\#})$. Since $\mathcal{C}$ is a closed Heller triangulated category, there exists an n-triangle X' such that $X'|_{\dot{\Delta}_n} = X|_{\dot{\Delta}_n}$; cf. Lemma II.20. By Prop. I.12, we have an isomorphism $X \xrightarrow[\sim]{f} X'$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ that restricts to the identity on $\dot{\Delta}_n$. We dispose of a commutative diagram

$$\begin{array}{ccc}
[X]^{+1} & \xrightarrow{X\vartheta_n} & [X^{+1}] \\
{\scriptstyle [f]^{+1}}\downarrow & & \downarrow{\scriptstyle [f^{+1}]} \\
[X']^{+1} & \xrightarrow{X'\vartheta_n} & [X'^{+1}]
\end{array}$$

in $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. Since, by construction, $X'\vartheta_n = 1$, we have $X\vartheta_n = [f]^{+1} \cdot [f^{+1}]^{-}$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$.

Likewise, we have a commutative quadrangle

$$\begin{array}{ccc}
[X\underline{F^+(\bar{\Delta}_n)}]^{+1} & \xrightarrow{X\underline{F^+(\bar{\Delta}_n)}\vartheta'_n} & [(X\underline{F^+(\bar{\Delta}_n)})^{+1}] \\
{\scriptstyle [f\underline{F^+(\bar{\Delta}_n)}]^{+1}}\downarrow & & \downarrow{\scriptstyle [(f\underline{F^+(\bar{\Delta}_n)})^{+1}]} \\
[X'\underline{F^+(\bar{\Delta}_n)}]^{+1} & \xrightarrow{X'\underline{F^+(\bar{\Delta}_n)}\vartheta'_n} & [(X'\underline{F^+(\bar{\Delta}_n)})^{+1}]\,,
\end{array}$$

in $\underline{\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})}$. We want to calculate its lower arrow. Since X' is an n-triangle, we have an isomorphism $Y' \xrightarrow[\sim]{\breve{a}'} \breve{Y}'$ formed as above, where $\breve{Y}'\vartheta'_n = 1$. The stably commutative quadrangle

$$\begin{array}{ccc}
[Y']^{+1} & \xrightarrow{[\breve{a}']^{+1}} & [\breve{Y}']^{+1} \\
{\scriptstyle Y'\vartheta'_n}\downarrow & & \downarrow{\scriptstyle \breve{Y}'\vartheta'_n = 1} \\
[Y'^{+1}] & \xrightarrow{[\breve{a}'^{+1}]} & [\breve{Y}'^{+1}]
\end{array}$$

yields by restriction to $\dot{\Delta}_n$ the commutative diagram

$$\begin{array}{ccc}
X'|_{\dot{\Delta}_n}\,\underline{(\mathsf{T}\,F)(\dot{\Delta}_n)} & \xrightarrow{X'|_{\dot{\Delta}_n}\,\underline{a(\dot{\Delta}_n)}} & X'|_{\dot{\Delta}_n}\,\underline{(F\;\mathsf{T}')(\dot{\Delta}_n)} \\
\| & & \| \\
[Y']^{+1}|_{\dot{\Delta}_n} & \xrightarrow{[\breve{a}]^{+1}|_{\dot{\Delta}_n}} & [\breve{Y}']^{+1}|_{\dot{\Delta}_n} \\
{\scriptstyle Y'\vartheta'_n|_{\dot{\Delta}_n}}\downarrow & & \downarrow{\scriptstyle 1} \\
[Y'^{+1}]|_{\dot{\Delta}_n} & \xrightarrow{1} & [\breve{Y}'^{+1}]|_{\dot{\Delta}_n}\,,
\end{array}$$

whence $Y'\vartheta'_n|_{\dot{\Delta}_n} = X'|_{\dot{\Delta}_n}\,\underline{a(\dot{\Delta}_n)} = X'\underline{a^+(\bar{\Delta}_n^{\#})}|_{\dot{\Delta}_n}$. Since the functor from $\underline{\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})}$ to $\underline{\mathcal{C}'(\dot{\Delta}_n)}$ induced by restriction is an equivalence by Prop. I.12, this implies that

$$X'\underline{F^+(\bar{\Delta}_n)}\vartheta'_n \;=\; Y'\vartheta'_n \;=\; X'\underline{a^+(\bar{\Delta}_n^{\#})}\,.$$

So we can conclude that

$$\begin{aligned}
X\underline{F^+(\bar\Delta_n)}\vartheta'_n &= [f\underline{F^+(\bar\Delta_n)}]^{+1}\cdot X'\underline{F^+(\bar\Delta_n)}\vartheta'_n\cdot[(f\underline{F^+(\bar\Delta_n)})^{+1}]^- \\
&= [f\underline{F^+(\bar\Delta_n)}]^{+1}\cdot X'\underline{a^+(\bar\Delta_n^\#)}\cdot[(f\underline{F^+(\bar\Delta_n)})^{+1}]^- \\
&= [f\underline{F^+(\bar\Delta_n^\#)}]^{+1}\cdot X'\underline{a^+(\bar\Delta_n^\#)}\cdot(f\underline{(F\,\mathsf{T}')^+(\bar\Delta_n^\#)})^- \\
&= [f\underline{F^+(\bar\Delta_n^\#)}]^{+1}\cdot(f\underline{(\mathsf{T}\,F)^+(\bar\Delta_n^\#)})^-\cdot X\underline{a^+(\bar\Delta_n^\#)} \\
&= ([f]^{+1}\cdot[f^{+1}]^-)\underline{F^+(\bar\Delta_n^\#)}\cdot X\underline{a^+(\bar\Delta_n^\#)} \\
&= X(\vartheta_n\star\underline{F^+(\bar\Delta_n^\#)})\cdot X\underline{a^+(\bar\Delta_n^\#)}\ .
\end{aligned}$$

This proves the *claim.* □

Corollary II.26 *Suppose* $(\mathcal{C},\mathsf{T},\vartheta)$ *to be closed.*

Suppose given an additive functor $\mathcal{C}\xrightarrow{F}\mathcal{C}'$ *such that* $\mathsf{T}\,F = F\,\mathsf{T}'$.

Then F *is strictly exact if and only if for each* $n\geqslant 0$ *and each* n*-triangle* $X\in\operatorname{Ob}\mathcal{C}^{+,\vartheta=1}(\bar\Delta_n^\#)$, *the diagram* $X(F(\bar\Delta_n^\#))\in\operatorname{Ob}\mathcal{C}'(\bar\Delta_n^\#)$, *obtained by pointwise application of* F, *is an* n*-triangle.*

Proof. In this case, we have $a = 1_{\mathsf{T}\,F} = 1_{F\,\mathsf{T}'}$ and $\check{Y} = Y = X(F(\bar\Delta_n^\#))$. Since $(F,1)$ is exact if and only if F is strictly exact, the assertion follows by Proposition II.25. □

II.5 Adjoints

II.5.1 Adjoints and shifts

Suppose given categories $\mathcal{A}$ and $\mathcal{A}'$. Suppose given an endofunctor T of $\mathcal{A}$. Suppose given an endofunctor T' of $\mathcal{A}'$.

Suppose given functors $\mathcal{A}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{A}'$ such that $F\dashv G$ via unit $1\xrightarrow{\varepsilon}FG$ and counit $GF\xrightarrow{\eta}1$, i.e. $(G\varepsilon)(\eta G) = 1_G$ and $(\varepsilon F)(F\eta) = 1_F$.

Suppose given $TF\xrightarrow{\alpha}FT'$.

Let

$$(GT\xrightarrow{\beta}T'G)\ :=\ \big(GT\xrightarrow{GT\varepsilon}GTFG\xrightarrow{G\alpha G}GFT'G\xrightarrow{\eta T'G}T'G\big)\ .$$

So we have the commutative diagram

$$\begin{array}{ccc}
GT & \xrightarrow{\beta} & T'G \\
{\scriptstyle GT\varepsilon}\downarrow & & \uparrow{\scriptstyle \eta T'G} \\
GTFG & \xrightarrow{G\alpha G} & GFT'G\ .
\end{array}$$

Lemma II.27

(1) *We have the commutative diagram*

$$\begin{array}{ccc} TF & \xrightarrow{\alpha} & FT' \\ {\scriptstyle \varepsilon TF}\downarrow & & \uparrow{\scriptstyle FT'\eta} \\ FGTF & \xrightarrow{F\beta F} & FT'GF \end{array} .$$

(2) *We have the commutative quadrangle*

$$\begin{array}{ccc} T & \xrightarrow{\varepsilon T} & FGT \\ {\scriptstyle T\varepsilon}\downarrow & & \downarrow{\scriptstyle F\beta} \\ TFG & \xrightarrow[\alpha G]{} & FT'G \end{array} .$$

(2°) *We have the commutative quadrangle*

$$\begin{array}{ccc} GTF & \xrightarrow{G\alpha} & GFT' \\ {\scriptstyle \beta F}\downarrow & & \downarrow{\scriptstyle \eta T'} \\ T'GF & \xrightarrow[T'\eta]{} & T' \end{array} .$$

(3) *Suppose that T and T' are autofunctors. Write $G' = T'GT^-$. If α is an isotransformation, then so is β, where*

$$\begin{array}{l} (T'G \xrightarrow{\beta^-} GT) \;= \\ \big(T'G = T'GT^-T \xrightarrow{T'GT^-\varepsilon T} T'GT^-FGT = \\ T'GT^-FT'T'^-GT \xrightarrow{T'GT^-\alpha^-T'^-GT} T'GT^-TFT'^-GT = \\ T'GFT'^-GT \xrightarrow{T'\eta T'^-GT} T'T'^-GT = GT\big) . \end{array}$$

(3°) *Suppose that T and T' are autofunctors. If β is an isotransformation, then so is α, where*

$$\begin{array}{l} (FT' \xrightarrow{\alpha^-} TF) \;= \\ \big(FT' = TT^-FT' \xrightarrow{T\varepsilon T^-FT'} TFGT^-FT' = \\ TFT'^-T'GT^-FT' \xrightarrow{TFT'^-\beta^-T^-FT'} TFT'^-GTT^-FT' = \\ TFT'^-GFT' \xrightarrow{TFT'^-\eta T'} TFT'^-T' = TF\big) . \end{array}$$

Proof. Ad (2). We have

$$\begin{array}{rcl} (\varepsilon T)(F\beta) & = & (\varepsilon T)(FGT\varepsilon)(FG\alpha G)(F\eta T'G) \\ & = & (T\varepsilon)(\varepsilon TFG)(FG\alpha G)(F\eta T'G) \\ & = & (T\varepsilon)(\alpha G)(\varepsilon FT'G)(F\eta T'G) \\ & = & (T\varepsilon)(\alpha G) . \end{array}$$

Ad (1). We have

$$\begin{aligned} (\varepsilon T F)(F\beta F)(FT'\eta) &\overset{(2)}{=} (T\varepsilon F)(\alpha G F)(FT'\eta) \\ &= (T\varepsilon F)(TF\eta)\alpha \\ &= \alpha\,. \end{aligned}$$

□

Ad (3). We have

$$\begin{aligned} & \beta\cdot(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT) \\ =\; & (\beta T^-T)(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT) \\ =\; & (GTT^-\varepsilon T)(\beta T^-FGT)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT) \\ =\; & (G\varepsilon T)(\beta T^-FT'T'^-GT)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT) \\ =\; & (G\varepsilon T)(GTT^-\alpha^-T'^-GT)(\beta T^-TFT'^-GT)(T'\eta T'^-GT) \\ =\; & (G\varepsilon T)(G\alpha^-T'^-GT)(\beta FT'^-GT)(T'\eta T'^-GT) \\ \overset{(2^\circ)}{=}\; & (G\varepsilon T)(G\alpha^-T'^-GT)(G\alpha T'^-GT)(\eta T'T'^-GT) \\ =\; & (G\varepsilon T)(\eta GT) \\ =\; & 1 \end{aligned}$$

and

$$\begin{aligned} & (T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)\cdot\beta \\ =\; & (T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)(T'T'^-\beta) \\ =\; & (T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'GFT'^-\beta)(T'\eta T'^-T'G) \\ =\; & (T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'GT^-TFT'^-\beta)(T'\eta G) \\ =\; & (T'GT^-\varepsilon T)(T'GT^-FT'T'^-\beta)(T'GT^-\alpha^-T'^-T'G)(T'\eta G) \\ =\; & (T'GT^-\varepsilon T)(T'GT^-F\beta)(T'GT^-\alpha^-G)(T'\eta G) \\ \overset{(2)}{=}\; & (T'GT^-T\varepsilon)(T'GT^-\alpha G)(T'GT^-\alpha^-G)(T'\eta G) \\ =\; & (T'G\varepsilon)(T'\eta G) \\ =\; & 1\,. \end{aligned}$$

II.5.2 An adjoint of an exact functor is exact

The Verdier triangulated version of the following proposition is due to Margolis [**44,** App. 2, Prop. 11], and, in a more general form, to Keller and Vossieck [**34,** 1.6].

Proposition II.28 *Suppose given Heller triangulated categories* $(\mathcal{C}, \mathsf{T}, \vartheta)$ *and* $(\mathcal{C}', \mathsf{T}', \vartheta')$.

Suppose given an exact functor (F, a) *from* $\mathcal{C}$ *to* $\mathcal{C}'$ *; cf. Definition* II.1.

Suppose given a functor $\mathcal{C} \xleftarrow{G} \mathcal{C}'$.

So $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}'$ *and* $\mathsf{T}\,F \xrightarrow[\sim]{a} F\,\mathsf{T}'$.

(1) *If* $F \dashv G$*, then there exists an isomorphism* $\mathsf{T}'\,G \xrightarrow[\sim]{b} G\,\mathsf{T}$ *such that* (G, b) *is an exact functor from* $\mathcal{C}'$ *to* $\mathcal{C}$.

Choose a unit $1_{\mathcal{C}'} \xrightarrow{\varepsilon} FG$ *and a counit* $GF \xrightarrow{\eta} 1_{\mathcal{C}}$. *Then, more precisely, we may choose*

$$(\mathsf{T}'G \xrightarrow{b} G\mathsf{T}) \;:=\; (G\mathsf{T} \xrightarrow{G\mathsf{T}\varepsilon} G\mathsf{T}FG \xrightarrow{GaG} GF\mathsf{T}'G \xrightarrow{\eta\mathsf{T}'G} \mathsf{T}'G)^- .$$

(1°) *If* $G \dashv F$, *then there exists an isomorphism* $\mathsf{T}'G \xrightarrow[\sim]{b} G\mathsf{T}$ *such that* (G,b) *is an exact functor from* $\mathcal{C}'$ *to* $\mathcal{C}$.

Choose a unit $1_{\mathcal{C}'} \xrightarrow{\varepsilon} GF$ *and a counit* $FG \xrightarrow{\eta} 1_{\mathcal{C}}$. *Then, more precisely, we may choose*

$$(\mathsf{T}'G \xrightarrow{b} G\mathsf{T}) \;:=\; (\mathsf{T}'G \xrightarrow{\varepsilon\mathsf{T}'G} GF\mathsf{T}'G \xrightarrow{Ga^-G} G\mathsf{T}FG \xrightarrow{G\mathsf{T}\eta} G\mathsf{T}) .$$

Proof. Ad (1). By Lemma II.42.(1°), G is subexact.

Lemma II.27.(3) yields the isotransformation $b^- := (G\mathsf{T}\varepsilon)(GaG)(\eta\mathsf{T}'G)$.

Suppose given $n \geqslant 0$. We shall make use of the abbreviation $\underline{G} = \underline{G^+(\bar\Delta_n^\#)}$, etc. We have to show that

$$(\vartheta'_n \star \underline{G}) \cdot \underline{b} \overset{!}{=} \underline{G} \star \vartheta_n \,,$$

i.e. that

$$(\underline{G} \star \vartheta_n) \cdot \underline{b}^- \overset{!}{=} \vartheta'_n \star \underline{G} \,,$$

i.e. that

$$(\underline{G} \star \vartheta_n) \cdot (\underline{G} \star \underline{\mathsf{T}} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}}' \star \underline{G}) \overset{!}{=} \vartheta'_n \star \underline{G}$$

Recall that $[-]^{+1}$ denotes the outer shift, that $[-^{+1}]$ denotes the inner shift and that $\vartheta_n : [-]^{+1} \xrightarrow{\sim} [-^{+1}]$ on $\underline{\mathcal{C}^+(\bar\Delta_n^\#)}$; similarly on $\underline{\mathcal{C}'^+(\bar\Delta_n^\#)}$.

We obtain

$$\begin{array}{cl}
 & (\underline{G} \star \vartheta_n) \cdot (\underline{G} \star \underline{\mathsf{T}} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}}' \star \underline{G}) \\
= & (\underline{G} \star \vartheta_n) \cdot (\underline{G} \star [-^{+1}] \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}}' \star \underline{G}) \\
= & (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star \vartheta_n \star \underline{F} \star \underline{G}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}}' \star \underline{G}) \\
= & (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star ((\vartheta_n \star \underline{F}) \cdot \underline{a}) \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}}' \star \underline{G}) \\
\overset{(F,a)\text{ ex.}}{=} & (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{F} \star \vartheta'_n \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}}' \star \underline{G}) \\
= & (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{F} \star \vartheta'_n \star \underline{G}) \cdot (\underline{\eta} \star [-^{+1}] \star \underline{G}) \\
= & (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{\eta} \star [-]^{+1} \star \underline{G}) \cdot (\vartheta'_n \star \underline{G}) \\
= & (\underline{G} \star \underline{\varepsilon} \star [-]^{+1}) \cdot (\underline{\eta} \star \underline{G} \star [-]^{+1}) \cdot (\vartheta'_n \star \underline{G}) \\
= & \vartheta'_n \star \underline{G} \,.
\end{array}$$

Ad (1°). Cf. Lemma II.27.(1). ▫

Example II.29 Suppose we are in the situation of Proposition II.28.(1). Then ε and η are periodic; cf. Definition II.4.

Ad $\varepsilon : 1 \longrightarrow FG$. The functor $(F,a)(G,b) = (FG\,,\, aG \cdot Fb)$ is exact; cf. Remark II.3. The functor $(1_{\mathcal{C}}\,,\,1)$ is exact. The quadrangle

$$\begin{array}{ccc}
\mathsf{T} & \xrightarrow{T\varepsilon} & \mathsf{T}FG \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle aG\cdot Fb} \\
\mathsf{T} & \xrightarrow{\varepsilon T} & FG\mathsf{T}
\end{array}$$

commutes by Lemma II.27.(2).

Ad $\eta : GF \longrightarrow 1$. The functor $(G, b)(F, a) = (GF\,,\, bF \cdot Ga)$ is exact; cf. Remark II.3. The functor $(1_{\mathcal{C}'}\,,\, 1)$ is exact. The quadrangle

$$\begin{array}{ccc} \mathsf{T}'\,GF & \xrightarrow{\mathsf{T}'\,\eta} & \mathsf{T}' \\ {\scriptstyle bF\cdot Ga}\downarrow & & \downarrow{\scriptstyle 1} \\ GF\,\mathsf{T}' & \xrightarrow{\eta\,\mathsf{T}'} & \mathsf{T}' \end{array}$$

commutes by Lemma II.27.(2°).

II.5.3 A functor shiftcompatibly adjoint to a strictly exact functor is strictly exact

Suppose given closed Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$

Recall that an additive functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is strictly exact if and only if $(F, 1)$ is exact; cf. Def. I.5.(iii), Definition II.1.

Corollary II.30

Suppose given a strictly exact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ *.*

Suppose given a functor $\mathcal{C} \xleftarrow{G} \mathcal{C}'$ *.*

(1) *If* $F \dashv G$*, with unit* $\varepsilon : 1 \longrightarrow FG$ *and counit* $\eta : GF \longrightarrow 1$ *such that* $(G\,\mathsf{T}\,\varepsilon)(\eta\,\mathsf{T}'\,G) = 1$*, then* G *is strictly exact.*

(1°) *If* $G \dashv F$*, with unit* $\varepsilon : 1 \longrightarrow FG$ *and counit* $\eta : GF \longrightarrow 1$ *such that* $(\varepsilon\,\mathsf{T}'\,G)(G\,\mathsf{T}\,\eta) = 1$*, then* G *is strictly exact.*

Proof. Ad (1). In the notation of Proposition II.28.(1), we have $a = 1$, and, consequently, $b = (G\,\mathsf{T}\,\varepsilon)(\eta\,\mathsf{T}'\,G) = 1$. Hence by loc. cit., $(G, 1)$ is exact, i.e. G is strictly exact. □

II.6 Localisation

We prove that the localisation $\mathcal{C}/\!\!/\mathcal{N}$ of a Heller triangulated category $\mathcal{C}$ at a thick subcategory $\mathcal{N}$ is Heller triangulated in such a way that the localisation functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is strictly exact; cf. Def. I.5. There is considerable overlap with the classical localisation theory of Verdier triangulated categories, due to Verdier [**56**], which we include for sake of self-containedness.

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a closed Heller triangulated category; cf. Definition II.14.

Definition II.31 A full additive subcategory $\mathcal{N} \subseteq \mathcal{C}$ is called *thick* if the conditions $(1, 2, 3)$ are satisfied; cf. [**53,** Prop. 1.3]

(1) We have $\mathcal{N}^{+1} = \mathcal{N}$ (closed under shift).

(2) Given a 2-triangle (X, Y, Z) in $\mathcal{C}$ with X and Y in $\operatorname{Ob}\mathcal{N}$, then $Z \in \operatorname{Ob}\mathcal{N}$ (closed under taking cones).

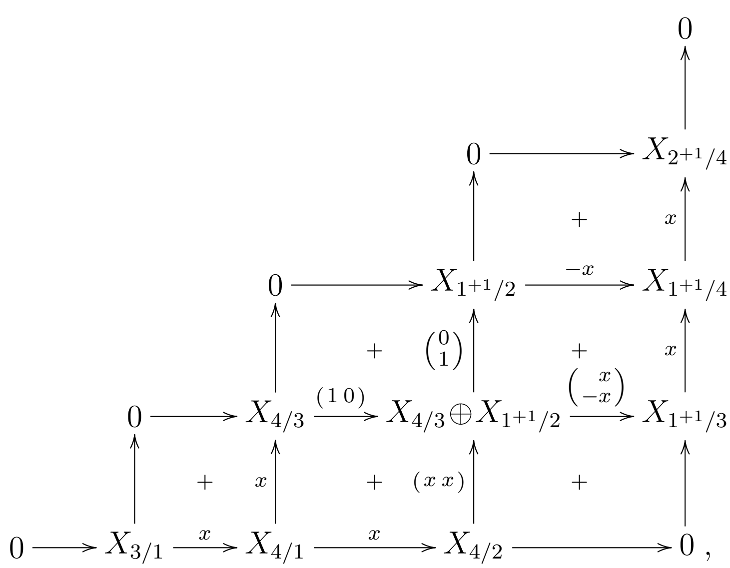
(3) Given $X, Y \in \operatorname{Ob}\mathcal{C}$ with $X \oplus Y$ in $\operatorname{Ob}\mathcal{N}$, then $X \in \operatorname{Ob}\mathcal{N}$ (closed under taking summands).

Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$. By Lemma II.23, conditions (1) and (2) of Definition II.31 yield that $\mathcal{N}$ is a Heller triangulated subcategory of $\mathcal{C}$.

Let $M(\mathcal{N}) := \{(X \xrightarrow{f} Y) \in \mathcal{C} \;:\; \text{the cone of } f \text{ is in } \operatorname{Ob}\mathcal{N}\}$. An element of $M(\mathcal{N})$ is called an *$M(\mathcal{N})$-isomorphism* or often just an *$\mathcal{N}$-isomorphism* (not to be confused with "an isomorphism in $\mathcal{N}$"). If $\mathcal{N}$ is unambiguous, then an $\mathcal{N}$-isomorphism is denoted by $X \Longrightarrow Y$. For instance, $X \Longrightarrow 0$ if and only if $0 \Longrightarrow X$ if and only if $X \in \operatorname{Ob}\mathcal{N}$.

Lemma II.32 *The subset $M(\mathcal{N})$ of $\mathcal{N}$-isomorphisms in $\mathcal{C}$ is a multiplicative system in $\mathcal{C}$ in the sense of Definition* II.45.

Proof. Ad (Fr 2). Suppose given $X_{1/0} \xrightarrow{x} X_{2/0} \xrightarrow{x} X_{3/0} \xrightarrow{x} X_{4/0}$ such that $X_{1/0} \xrightarrow{x} X_{3/0}$ and $X_{2/0} \xrightarrow{x} X_{4/0}$ are M-isomorphisms. We complete to a 4-triangle $X \in \operatorname{Ob}\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^{\#})$ using closedness of $\mathcal{C}$; cf. Lemma II.20. By Lem. I.21.(1, 6), we have $X_{3/1}, X_{4/2} \in \operatorname{Ob}\mathcal{N}$. We have to show that $X_{2/1}, X_{3/2}, X_{4/3}, X_{4/1} \in \operatorname{Ob}\mathcal{N}$. Let the periodic monotone map $\bar{\Delta}_5 \xrightarrow{p} \bar{\Delta}_4$ be defined by $0p := 1$, $1p := 1$, $2p := 2$, $3p := 3$, $4p := 4$ and $5p := 4$. The 2-triangle $Xp^{\#}\mathfrak{f}_2 \in \operatorname{Ob}\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_2^{\#})$ is given by



cf. Lem. I.21.(1, 2), §I.1.2.1.2, §I.1.2.2.2. Since $X_{3/1}, X_{4/2} \in \operatorname{Ob}\mathcal{N}$, and since $\mathcal{N}$ is closed under cones, we have $X_{4/3} \oplus X_{1^{+1}/2} \in \operatorname{Ob}\mathcal{N}$. Since $\mathcal{N}$ is closed under summands and under shift, we obtain $X_{4/3}, X_{2/1} \in \operatorname{Ob}\mathcal{N}$. Since $\mathcal{N}$ is closed under cones and under shift, $X_{4/1} \in \operatorname{Ob}\mathcal{N}$ ensues. Considering X again, since $\mathcal{N}$ is closed under cones, we finally obtain $X_{3/2} \in \operatorname{Ob}\mathcal{N}$.

Ad (Fr 3). Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$ such that there exists $Y \overset{s}{\Longrightarrow} Z$ with $fs = 0$. We obtain a factorisation $(X \xrightarrow{f} Y) = (X \xrightarrow{u} N \xrightarrow{v} Y)$ with $N \in \operatorname{Ob}\mathcal{N}$. Completing

$(T_{1/0} \xrightarrow{t} T_{2/0} \xrightarrow{t} T_{3/0}) := (X \xrightarrow{u} N \xrightarrow{v} Y)$ to a 3-triangle by Lemma II.20, we obtain $T_{1/2^{-1}} \overset{t}{\Longrightarrow} T_{1/0}$, which composes to zero with $(T_{1/0} \xrightarrow{t} T_{3/0}) = (X \xrightarrow{f} Y)$.

Ad (Fr 4). Suppose given

$$\begin{array}{c} X' \\ \uparrow \\ X \Longrightarrow Y \end{array}$$

in $\mathcal{C}$. Prolonging $X \longrightarrow X'$ to a 2-triangle (X'', X, X'), then completing $X'' \longrightarrow X \Longrightarrow Y$ to a 3-triangle using Lemma II.20, we obtain, by Lem. I.21.(6), a 3-triangle T with $(T_{2/0} \xrightarrow{t} T_{3/0}) = (X \Longrightarrow Y)$ and $(T_{2/0} \xrightarrow{t} T_{2/1}) = (X \longrightarrow X')$. Then $T_{3/2} \in \operatorname{Ob}\mathcal{N}$, whence $T_{2/1} \Longrightarrow T_{3/1}$. The weak square $(T_{2/0}, T_{3/0}, T_{2/1}, T_{3/1})$ is a completion as sought. ▫

Note that if $X \in (\operatorname{Ob}\mathcal{C}) \smallsetminus (\operatorname{Ob}\mathcal{N})$, then $(0,0,0,X)$ is a weak square in which $0 \longrightarrow 0$ is an $\mathcal{N}$-isomorphism, but $0 \longrightarrow X$ is not.

The localisation of $\mathcal{C}$ at $M(\mathcal{N})$, defined as in §II.7.4, is also called the *localisation of* $\mathcal{C}$ *at* $\mathcal{N}$, and also written $\mathcal{C}/\!\!/\mathcal{N} := \mathcal{C}_{M(\mathcal{N})}$. Concerning the *localisation functor* $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$, we refer to §II.7.4.

Recall that an additive functor between weakly abelian categories is called subexact if it induces an exact functor on the Freyd categories; cf. §I.1.2.1.3; cf. also Lemma II.41.

Lemma II.33 *The category* $\mathcal{C}/\!\!/\mathcal{N}$ *is weakly abelian. The functor* $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ *is subexact.*

Proof. By Remark II.51, the category $\mathcal{C}/\!\!/\mathcal{N}$ is additive, and the localisation functor $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}/\!\!/\mathcal{N}$ is additive. We claim that L maps weak kernels to weak kernels. Let $X \xrightarrow{f} Y$ be a weak kernel of $Y \xrightarrow{g} Z$ in $\mathcal{C}$. We claim that it remains a weak kernel in $\mathcal{C}/\!\!/\mathcal{N}$. Suppose given a morphism $T \xrightarrow{t} Y$ in $\mathcal{C}$ such that $tg = 0$ in $\mathcal{C}/\!\!/\mathcal{N}$, which we, by isomorphic replacement, may assume given. Let $T' \overset{s}{\Longrightarrow} T$ be such that $stg = 0$ in $\mathcal{C}$; cf. Remark II.47. Since f is a weak kernel of g in $\mathcal{C}$, we have a factorisation $st = uf$. Hence $t = (s^{-1}u)f$ is a factorisation of t over f in $\mathcal{C}/\!\!/\mathcal{N}$.

Substituting isomorphically in $\mathcal{C}/\!\!/\mathcal{N}$ and using duality, for $\mathcal{C}/\!\!/\mathcal{N}$ to be weakly abelian, it suffices to show that each morphism $X \xrightarrow{f} Y$ has a weak kernel resp. is a weak kernel in $\mathcal{C}/\!\!/\mathcal{N}$. But by the property of L just shown, we may use a weak kernel of f in $\mathcal{C}$ resp. a morphism f is a weak kernel of in $\mathcal{C}$. ▫

Remark II.34 *The category* $\mathcal{C}/\!\!/\mathcal{N}$ *carries a shift automorphism* $\mathcal{C}/\!\!/\mathcal{N} \xrightarrow{\mathsf{T}} \mathcal{C}/\!\!/\mathcal{N}$, $f/t \longmapsto f^{+1}/t^{+1}$. *We have* $\mathsf{L}\,\mathsf{T} = \mathsf{T}\,\mathsf{L}$.

Proof. This functor is welldefined since $\mathcal{N}$, and hence $M(\mathcal{N})$, is closed under shift in $\mathcal{C}$. Likewise, its inverse $f/t \longmapsto f^{-1}/t^{-1}$ is welldefined. ▫

Lemma II.35 *Suppose given a Heller triangulated category* $(\mathcal{D}, \mathsf{T}, \theta)$.

Suppose given a weakly abelian category $\mathcal{D}'$ *and an automorphism* $\mathcal{D}' \xrightarrow{\mathsf{T}'} \mathcal{D}'$. *Suppose given a subexact additive functor* $\mathcal{D} \xrightarrow{G} \mathcal{D}'$ *strictly compatible with shift, i.e.* $G\,\mathsf{T}' = \mathsf{T}\,G$. *Suppose that* $\mathcal{D}(\dot{\Delta}_n) \xrightarrow{G(\dot{\Delta}_n)} \mathcal{D}'(\dot{\Delta}_n)$ *is* 1*-epimorphic for* $n \geqslant 0$.

Then the functor $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)}$ *is* 1*-epimorphic.*

Moreover, there exists a unique Heller triangulation θ' *on* $(\mathcal{D}', \mathsf{T}')$ *such that* $\mathcal{D} \xrightarrow{G} \mathcal{D}'$ *is strictly exact; cf. Def. I.5.*

Proof. Given $n \geqslant 0$. Since the residue class functors $\mathcal{D}(\dot{\Delta}_n) \longrightarrow \underline{\mathcal{D}(\dot{\Delta}_n)}$ and $\mathcal{D}'(\dot{\Delta}_n) \longrightarrow \underline{\mathcal{D}'(\dot{\Delta}_n)}$ are full and dense, they are 1-epimorphic by Cor. I.77; concerning notation, cf. §I.2.4. The commutative quadrangle

$$\begin{array}{ccc} \mathcal{D}(\dot{\Delta}_n) & \xrightarrow{G(\dot{\Delta}_n)} & \mathcal{D}'(\dot{\Delta}_n) \\ \downarrow & & \downarrow \\ \underline{\mathcal{D}(\dot{\Delta}_n)} & \xrightarrow{\underline{G(\dot{\Delta}_n)}} & \underline{\mathcal{D}'(\dot{\Delta}_n)} \end{array}$$

shows that $\underline{G(\dot{\Delta}_n)}$ is 1-epimorphic. Restriction induces equivalences $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} \xrightarrow{(-)|_{\dot{\Delta}_n}} \underline{\mathcal{D}(\dot{\Delta}_n)}$ and $\underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \xrightarrow{(-)|_{\dot{\Delta}_n}} \underline{\mathcal{D}'(\dot{\Delta}_n)}$ by Prop. I.12. Therefore, the commutative quadrangle

$$\begin{array}{ccc} \underline{\mathcal{D}(\dot{\Delta}_n)} & \xrightarrow{\underline{G(\dot{\Delta}_n)}} & \underline{\mathcal{D}'(\dot{\Delta}_n)} \\ {\scriptstyle (-)|_{\dot{\Delta}_n}}\uparrow & & \uparrow{\scriptstyle (-)|_{\dot{\Delta}_n}} \\ \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \end{array}$$

shows that $\underline{G^+(\bar{\Delta}_n^\#)}$ is 1-epimorphic; concerning notation, cf. §I.1.2.1.1, §I.1.2.1.3. Therefore, we may define a transformation θ'_n for $\mathcal{D}'$ by the requirement that

$$\begin{array}{ccc} \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \\ {\scriptstyle [-]^{+1}} \Big\downarrow \overset{\theta_n}{\Longrightarrow} \Big\downarrow {\scriptstyle [-^{+1}]} & & {\scriptstyle [-]^{+1}} \Big\downarrow \overset{\theta'_n}{\Longrightarrow} \Big\downarrow {\scriptstyle [-^{+1}]} \\ \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \end{array}$$

be commutative, i.e. that $\theta_n \star \underline{G^+(\bar{\Delta}_n^\#)} = \underline{G^+(\bar{\Delta}_n^\#)} \star \theta'_n$. In other words, there exists a unique θ'_n making this diagram commutative.

Let $\theta' := (\theta'_n)_{n \geqslant 0}$, where for $n = 0$, we make use of $\underline{\mathcal{D}'^+(\bar{\Delta}_0^\#)} = 0$. We claim that θ' is a Heller triangulation on $(\mathcal{D}', \mathsf{T}')$, i.e. that $(\mathcal{D}', \mathsf{T}', \theta')$ is a Heller triangulated category. Once this is proven, we see that by construction, $\mathcal{D} \xrightarrow{G} \mathcal{D}'$ is strictly exact; cf. Def. I.5.(iii).

Suppose given $m, n \geqslant 0$ and a periodic monotone map $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$. To prove that $\underline{p}^\# \star \theta'_m \overset{!}{=} \theta'_n \star \underline{p}^\#$, we may precompose with the 1-epimorphic functor $\underline{G^+(\bar{\Delta}_n^\#)}$ to obtain

$$\begin{array}{rcl} \underline{G^+(\bar{\Delta}_n^\#)} \star \underline{p}^\# \star \theta'_m & = & \underline{p}^\# \star \underline{G^+(\bar{\Delta}_m^\#)} \star \theta'_m \;=\; \underline{p}^\# \star \theta_m \star \underline{G^+(\bar{\Delta}_m^\#)} \\ & \underset{\text{Heller triangulated}}{\overset{(\mathcal{D},\mathsf{T},\theta)}{=}} & \theta_n \star \underline{p}^\# \star \underline{G^+(\bar{\Delta}_m^\#)} \;=\; \theta_n \star \underline{G^+(\bar{\Delta}_n^\#)} \star \underline{p}^\# \;=\; \underline{G^+(\bar{\Delta}_n^\#)} \star \theta'_n \star \underline{p}^\# . \end{array}$$

Suppose given $n \geqslant 0$. To prove that $\mathfrak{f}_n \star \theta'_{n+1} \overset{!}{=} \theta'_{2n+1} \star \mathfrak{f}_n$, we may precompose with the 1-epimorphic functor $\underline{G^+(\bar{\Delta}^\#_{2n+1})}$ to obtain

$$\begin{array}{rcl} \underline{G^+(\bar{\Delta}^\#_{2n+1})} \star \mathfrak{f}_n \star \theta'_{n+1} & = & \mathfrak{f}_n \star \underline{G^+(\bar{\Delta}^\#_{n+1})} \star \theta'_{n+1} \\ & = & \mathfrak{f}_n \star \theta_{n+1} \star \underline{G^+(\bar{\Delta}^\#_{n+1})} \\ & \underset{\text{Heller triangulated}}{\overset{(\mathcal{D},\mathsf{T},\theta)}{=}} & \theta_{2n+1} \star \mathfrak{f}_n \star \underline{G^+(\bar{\Delta}^\#_{n+1})} \\ & = & \theta_{2n+1} \star \underline{G^+(\bar{\Delta}^\#_{2n+1})} \star \mathfrak{f}_n \;=\; \underline{G^+(\bar{\Delta}^\#_{2n+1})} \star \theta'_{2n+1} \star \mathfrak{f}_n \,. \end{array}$$

□

Proposition II.36 *Recall that $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a closed Heller triangulated category, and that $\mathcal{N}$ is a thick subcategory of $\mathcal{C}$.*

There exists a unique Heller triangulation θ on $(\mathcal{C}/\!\!/\mathcal{N}, \mathsf{T})$ such that $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is strictly exact; cf. Def. I.5.

Then $(\mathcal{C}/\!\!/\mathcal{N}, \mathsf{T}, \theta)$ is a closed Heller triangulated category; cf. Definition II.14.

Proof. By Lemma II.33, the category $\mathcal{C}/\!\!/\mathcal{N}$ is weakly abelian, and $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}/\!\!/\mathcal{N}$ is subexact. By Remark II.34, $\mathcal{C}/\!\!/\mathcal{N}$ carries a shift automorphism, and L is compatible with the shift automorphisms on $\mathcal{C}$ and on $\mathcal{C}/\!\!/\mathcal{N}$. By Lemma II.50, the functor $\mathcal{C}(\dot{\Delta}_n) \xrightarrow{\mathsf{L}(\dot{\Delta}_n)} (\mathcal{C}/\!\!/\mathcal{N})(\dot{\Delta}_n)$ is 1-epimorphic for $n \geqslant 0$. Therefore, existence and uniqueness of θ follow by Lemma II.35.

It remains to be shown that $\mathcal{C}/\!\!/\mathcal{N}$ is closed. By isomorphic substitution, it suffices to show that each morphism in the image of L has a cone in $\mathcal{C}/\!\!/\mathcal{N}$; cf. Lem. I.21.(6). But this follows from $\mathcal{C}$ being closed and from L being strictly exact. □

An object $(X \xrightarrow{x} X')$ of the Freyd category $\hat{\mathcal{C}}$ is called *$\mathcal{N}$-zero* if x factors over an object of $\mathcal{N}$; concerning $\hat{\mathcal{C}}$, cf. §I.6.6.3. Note that an object of $\hat{\mathcal{C}}$ that is isomorphic to a summand of an $\mathcal{N}$-zero object is itself $\mathcal{N}$-zero.

Remark II.37 *A morphism in $\mathcal{C}$ is an $\mathcal{N}$-isomorphism if and only if its kernel and its cokernel, taken in $\hat{\mathcal{C}}$, are $\mathcal{N}$-zero.*

Note that this criterion does not make reference to the Heller triangulated structure on $\mathcal{C}$, but only to the fact that $\mathcal{C}$ is weakly abelian. One might ask for conditions on $\mathcal{N}$ that only use weak abelianess of $\mathcal{C}$, and that nonetheless suffice to turn $\mathcal{C}_{M(\mathcal{N})}$ into a weakly abelian category – where now $M(\mathcal{N})$ is the subset of morphisms of $\mathcal{C}$ defined by the criterion given in Remark II.37.

Proof of Remark II.37. Suppose that $X \xrightarrow{f} Y$ is an $\mathcal{N}$-isomorphism in $\mathcal{C}$. Then it has a weak kernel N and a weak cokernel M in $\operatorname{Ob}\mathcal{N}$. By construction of the kernel in $\hat{\mathcal{C}}$, it is of the form $(N \longrightarrow X)$. Dually, the cokernel is of the form $(Y \longrightarrow M)$; cf. §I.6.6.3.

Conversely, suppose that the kernel and the cokernel of the morphism $X \xrightarrow{f} Y$, taken in $\hat{\mathcal{C}}$, are $\mathcal{N}$-zero. Consider the exact functor $\hat{\mathcal{C}} \xrightarrow{\hat{\mathsf{L}}} (\mathcal{C}/\!\!/\mathcal{N})\hat{}$ that prolongs L on the level of Freyd categories. It maps f to an isomorphism, since in the abelian category $(\mathcal{C}/\!\!/\mathcal{N})\hat{}$, the image of f has zero kernel and zero cokernel. Since $\mathcal{C}/\!\!/\mathcal{N} \longrightarrow (\mathcal{C}/\!\!/\mathcal{N})\hat{}$ is full and faithful, the image of f under L in $\mathcal{C}/\!\!/\mathcal{N}$ is an isomorphism, too. Hence f is an $\mathcal{N}$-isomorphism in $\mathcal{C}$; cf. Remark II.46. □

Proposition II.38 (universal property) *Recall that $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a closed Heller triangulated category, and that $\mathcal{N}$ is a thick subcategory of $\mathcal{C}$.*

Let θ be the unique Heller triangulation on $(\mathcal{C}/\!\!/\mathcal{N}, \mathsf{T})$ such that the localisation functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is strictly exact; cf. Proposition II.36. *Suppose given a Heller triangulated category $(\mathcal{C}', \mathsf{T}', \vartheta')$.*

Recall that we write $[\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex}}$ for the category of exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$; cf. Definition II.6.

Write $[\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex},\,\mathcal{N}} \subseteq [\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex}}$ for the full subcategory consisting of exact functors (F, a) such that $NF \simeq 0$ for all $N \in \mathrm{Ob}\,\mathcal{N}$.

Recall that we write $[\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex}}$ for the category of strictly exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$; cf. Definition II.6.

Write $[\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex},\,\mathcal{N}} \subseteq [\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex}}$ for the full subcategory consisting of strictly exact functors F such that $NF \simeq 0$ for all $N \in \mathrm{Ob}\,\mathcal{N}$.

(1) *We have a strictly dense equivalence*

$$\begin{array}{rcl} [\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex},\,\mathcal{N}} & \xleftarrow{\mathsf{L}\star(-)} & [\![\, \mathcal{C}/\!\!/\mathcal{N}, \mathcal{C}']\!]_{\mathrm{ex}} \\ (\mathsf{L}\star G\,,\, \mathsf{L}\star b) \;=\; (\mathsf{L}, 1) \star (G, b) & \longleftarrow\!\shortmid & (G, b)\;. \end{array}$$

(2) *We have a strictly dense equivalence*

$$\begin{array}{rcl} [\![\, \mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex},\,\mathcal{N}} & \xleftarrow{\mathsf{L}\star(-)} & [\![\, \mathcal{C}/\!\!/\mathcal{N}, \mathcal{C}']\!]_{\mathrm{st\,ex}} \\ \mathsf{L}\star G & \longleftarrow\!\shortmid & G\;. \end{array}$$

Proof.

Ad (1). Welldefinedness of the functor $\mathsf{L}\star(-)$ follows from L being strictly exact and exact functors being stable under composition; cf. Proposition II.36, Remark II.3.

We make use of the universal property of the localisation to the extent stated in Remark II.51.

Suppose given exact functors $\mathcal{C} \overset{(F,a)}{\underset{(G,b)}{\rightrightarrows}} \mathcal{C}'$ and a periodic transformation $F \xrightarrow{u} G$.

Let $\breve{F} : \mathcal{C}/\!\!/\mathcal{N} \longrightarrow \mathcal{C}'$ be defined by $\mathsf{L}\star\breve{F} := F$. Let $\breve{G} : \mathcal{C}/\!\!/\mathcal{N} \longrightarrow \mathcal{C}'$ be defined by $\mathsf{L}\star\breve{G} := G$.

Recall that the shift on $\mathcal{C}/\!\!/\mathcal{N}$ is, abusively, also denoted by T, so that $\mathsf{T}\star\mathsf{L} = \mathsf{L}\star\mathsf{T}$. Let the transformations $\breve{a}$ and $\breve{b}$ be defined by

$$\begin{array}{rcl} \mathsf{L}\star(\mathsf{T}\star\breve{F} \xrightarrow{\breve{a}} \breve{F}\star\mathsf{T}') & := & (\mathsf{T}\star F \xrightarrow{a} F\star\mathsf{T}') \\ \mathsf{L}\star(\mathsf{T}\star\breve{G} \xrightarrow{\breve{b}} \breve{G}\star\mathsf{T}') & := & (\mathsf{T}\star G \xrightarrow{b} G\star\mathsf{T}')\;. \end{array}$$

Let the transformation $\breve{F} \xrightarrow{\breve{u}} \breve{G}$ be defined by

$$\mathsf{L}\star(\breve{F} \xrightarrow{\breve{u}} \breve{G}) \;:=\; (F \xrightarrow{u} G)\;.$$

We have to show that $(\breve{F}, \breve{a})$ is exact and that $\breve{u}$ is periodic.

Ad $\breve{F}$ exact. Since $X\breve{a} = X\,\mathsf{L}\,\breve{a} = Xa$ is an isomorphism for $X \in \operatorname{Ob}\mathcal{C}/\!\!/\mathcal{N} = \operatorname{Ob}\mathcal{C}$, the transformation a is an isotransformation.

To show that $\breve{F}$ is subexact, by Lemma II.41, it suffices to show that given a morphism f in $\mathcal{C}/\!\!/\mathcal{N}$, it has a weak cokernel that is preserved by $\breve{F}$. By isomorphic substitution, we may assume that $f = f'\,\mathsf{L}$ for some morphism f' in $\mathcal{C}$. Let (f', g', h') be a 2-triangle in $\mathcal{C}$; cf. Lemma II.20. Since L is strictly exact, the 2-triangle $(f, g'\,\mathsf{L}, h'\,\mathsf{L})$ results. In particular, $g'\,\mathsf{L}$ is a weak cokernel of f. Since F is subexact, $g'F = g'\,\mathsf{L}\,\breve{F}$ is a weak cokernel of $f'F = f'\,\mathsf{L}\,\breve{F} = f\breve{F}$.

Suppose given $n \geqslant 0$. We shall make use of the abbreviation $\underline{F} = \underline{F^+(\bar{\Delta}_n^\#)}$, etc. It remains to show that

$$(\theta_n \star \underline{\breve{F}}) \cdot \underline{\breve{a}} \overset{!}{=} \underline{\breve{F}} \star \vartheta'_n \,.$$

Since $\underline{\mathsf{L}} = \underline{\mathsf{L}^+(\bar{\Delta}_n^\#)}$ is 1-epimorphic by Lemmata II.50 and II.35, it suffices to show that

$$\underline{\mathsf{L}} \star ((\theta_n \star \underline{\breve{F}}) \cdot \underline{\breve{a}}) \overset{!}{=} \underline{\mathsf{L}} \star \underline{\breve{F}} \star \vartheta'_n \,.$$

In fact,

$$\begin{array}{rcl} \underline{\mathsf{L}} \star ((\theta_n \star \underline{\breve{F}}) \cdot \underline{\breve{a}}) & = & (\underline{\mathsf{L}} \star \theta_n \star \underline{\breve{F}}) \cdot (\underline{\mathsf{L}} \star \underline{\breve{a}}) \\ & \overset{\mathsf{L}\text{ ex.}}{=} & (\vartheta_n \star \underline{\mathsf{L}} \star \underline{\breve{F}}) \cdot (\underline{\mathsf{L}} \star \underline{\breve{a}}) \\ & = & (\vartheta_n \star \underline{F}) \cdot \underline{a} \\ & \overset{(F,a)\text{ ex.}}{=} & \underline{F} \star \vartheta'_n \\ & = & \underline{\mathsf{L}} \star \underline{\breve{F}} \star \vartheta'_n \,. \end{array}$$

Ad $\breve{u}$ periodic. We have to show that

$$(\mathsf{T} \star \breve{u}) \cdot \breve{b} \overset{!}{=} \breve{a} \cdot (\breve{u} \star \mathsf{T}')$$

as transformations from $\mathsf{T} \star \breve{F}$ to $\breve{G} \star \mathsf{T}'$. By Remark II.51, it suffices to show that

$$\mathsf{L} \star ((\mathsf{T} \star \breve{u}) \cdot \breve{b}) \overset{!}{=} \mathsf{L} \star (\breve{a} \cdot (\breve{u} \star \mathsf{T}')) \,.$$

In fact,

$$\begin{array}{rcl} \mathsf{L} \star ((\mathsf{T} \star \breve{u}) \cdot \breve{b}) & = & (\mathsf{L} \star \mathsf{T} \star \breve{u}) \cdot (\mathsf{L} \star \breve{b}) \\ & = & (\mathsf{T} \star \mathsf{L} \star \breve{u}) \cdot (\mathsf{L} \star \breve{b}) \\ & = & (\mathsf{T} \star u) \cdot b \\ & \overset{u\text{ per.}}{=} & a \cdot (u \star \mathsf{T}') \\ & = & (\mathsf{L} \star \breve{a}) \cdot (\mathsf{L} \star \breve{u} \star \mathsf{T}') \\ & = & \mathsf{L} \star (\breve{a} \cdot (\breve{u} \star \mathsf{T}')) \,. \end{array}$$

Ad (2). Welldefinedness of the functor $\mathsf{L} \star (-)$ follows from L being strictly exact and strictly exact functors being stable under composition; cf. Proposition II.36, Remark II.3.

Keep the notation of the proof of (1). Given an exact functor (F, a) from $\mathcal{C}$ to $\mathcal{C}'$, we infer from $a = 1$, using $\mathsf{L} \star \breve{a} = a$, that $\breve{a} = 1$. □

II.7 Some general assertions

This appendix serves as a tool kit consisting of known results and folklore lemmata. We do not claim originality.

II.7.1 Remarks on coretractions and retractions

Remark II.39 *Let $\mathcal{A}$ be a category.*

Suppose given X, Z in $\operatorname{Ob}\mathcal{A}$, and morphisms $X \xrightarrow{i} Z \xrightarrow{p} X$ such that $ip = 1_X$.

Suppose given Y, W in $\operatorname{Ob}\mathcal{A}$, and morphisms $Y \xrightarrow{j} W \xrightarrow{q} Y$ such that $jq = 1_Y$.

Suppose given $X \xrightarrow{u} Y$ in $\mathcal{A}$. Let $Z \xrightarrow{v} W$ be defined by $v := puj$. Then $vq = pu$ and $iv = uj$.

$$\begin{array}{ccccc} Z & \xrightarrow{p} & X & \xrightarrow{i} & Z \\ {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ W & \xrightarrow{q} & Y & \xrightarrow{j} & W \end{array}$$

Proof. We have $vq = pujq = pu$ and $iv = ipuj = uj$. ▫

Remark II.40 *Let $\mathcal{A}$ be a category. Suppose given Z, X, Z', W, Y, $W' \in \operatorname{Ob}\mathcal{A}$.*

Suppose given morphisms $X \xrightarrow{i} Z \xrightarrow{p} X$ such that $ip = 1_X$.

Suppose given morphisms $X \xrightarrow{i'} Z' \xrightarrow{p'} X$ such that $i'p' = 1_X$.

Suppose given morphisms $Y \xrightarrow{j} W \xrightarrow{q} Y$ such that $jq = 1_Y$.

Suppose given morphisms $Y \xrightarrow{j'} W' \xrightarrow{q'} Y$ such that $j'q' = 1_Y$.

Suppose given $Z \xrightarrow{v} W$ and $Z' \xrightarrow{v'} W'$ such that $pi'v' = vqj'$.

Then there exists a unique morphism $X \xrightarrow{u} Y$ in $\mathcal{A}$ such that $vq = pu$ and $i'v' = uj'$.

$$\begin{array}{ccccc} Z & \xrightarrow{p} & X & \xrightarrow{i'} & Z' \\ {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v'} \\ W & \xrightarrow{q} & Y & \xrightarrow{j'} & W' \end{array}$$

If v and v' are isomorphisms, so is u.

Proof. Uniqueness follows from p being epic and j' being monic.

For existence, we let $u := ivq = i'v'q'$, the latter equality holding because of $pivqj' = pipi'v' = pi'v' = vqj' = vqj'q'j' = pi'v'q'j'$, using p epic and j' monic. Then $pu = pi'v'q' = vqj'q' = vq$ and $uj' = ivqj' = ipi'v' = i'v'$.

If v and v' are isomorphisms, then let $u' := jv^{-}p = j'v'^{-}p'$ to get $uu' = ivqj'v'^{-}p' = ipi'v'v'^{-}p' = 1$ and $u'u = jv^{-}p\,i'v'q' = jv^{-}vqj'q' = 1$, so that $u' = u^{-}$. In particular, u is an isomorphism. ▫

II.7.2 Two lemmata on subexact functors

Suppose given weakly abelian categories $\mathcal{A}$ and $\mathcal{A}'$; cf. e.g. Def. I.66.(3). Suppose given an additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$. Recall that F is called subexact if the induced functor $\hat{F} : \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{A}}'$ on the Freyd categories is exact; cf. §I.1.2.1.3.

Lemma II.41 *The following assertions* $(1, 2, 3, 3^\circ, 4, 4^\circ)$ *are equivalent.*

(1) *The functor* F *is subexact.*

(2) *The functor* F *preserves weak kernels and weak cokernels.*

(3) *The functor* F *preserves weak kernels.*

(3°) *The functor* F *preserves weak cokernels.*

(4) *For each morphism* $X \xrightarrow{t} Y$ *in* $\mathcal{A}$*, there exists a weak kernel* $W \xrightarrow{w} X$ *such that* wF *is a weak kernel of* tF*.*

(4°) *For each morphism* $X \xrightarrow{t} Y$ *in* $\mathcal{A}$*, there exists a weak cokernel* $Y \xrightarrow{w'} W'$ *such that* $w'F$ *is a weak cokernel of* tF*.*

Proof. Ad (1) $\Rightarrow$ (4). Suppose given a morphism $X \xrightarrow{t} Y$ in $\mathcal{A}$. Let $K \xrightarrow{i} X$ be a kernel of t in $\hat{\mathcal{A}}$. Choose $A \xrightarrow{b} K$ with $A \in \operatorname{Ob}\mathcal{A}$. Since $\hat{F}$ is exact, $A\hat{F} \xrightarrow{(bi)\hat{F}} X\hat{F} \xrightarrow{t\hat{F}} Y\hat{F}$ is exact at $X\hat{F}$. So $(bi)\hat{F} = (bi)F$ is a weak kernel of $t\hat{F} = tF$ in $\mathcal{A}'$.

Ad (4) $\Rightarrow$ (3). Given a morphism $X \xrightarrow{t} Y$ in $\mathcal{A}$ and a weak kernel $W \xrightarrow{w} X$, a morphism $V \xrightarrow{v} X$ is a weak kernel of t if and only if both (w factors over v) and (v factors over w). So if wF is a weak kernel of tF, so is vF. Consequently, if F preserves a single weak kernel of t, it preserves all of them.

Ad (3) $\Rightarrow$ (2). This follows by Rem. I.67.

Ad (2) $\Rightarrow$ (1). Using duality and uniqueness of the kernel up to isomorphism, it suffices to show that $\hat{F}$ maps a chosen kernel of a given morphism to a kernel of its image under $\hat{F}$. Since F preserves weak kernels, this follows by construction of a kernel; cf. e.g. §I.6.6.3, item (1) before Rem. I.67. ▫

Lemma II.42 *Suppose that* $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ *is subexact. Suppose given a functor* $\mathcal{C} \xleftarrow{G} \mathcal{C}'$*.*

(1) *If* $G \dashv F$*, then* G *is subexact.*

(1°) *If* $F \dashv G$*, then* G *is subexact.*

Proof. Ad (1). As an adjoint functor between additive categories, G is additive.

Let $1 \xrightarrow{\varepsilon} GF$ be a unit and $FG \xrightarrow{\eta} 1$ a counit of the adjunction $G \dashv F$.

By Lemma II.41, it suffices to show that G preserves weak cokernels. Suppose given $X' \xrightarrow{u} X \xrightarrow{v} X''$ such that v is a weak cokernel of u. We have to show that Gv is a weak cokernel of Gu. Suppose given $t : XG \longrightarrow T$ such that $uG \cdot t = 0$. Then

$$u \cdot X\varepsilon \cdot tF \;=\; X'\varepsilon \cdot uGF \cdot tF \;=\; X'\varepsilon \cdot (uG \cdot t)F \;=\; 0\,.$$

Since v is a weak cokernel of u, we obtain a morphism $s : X'' \longrightarrow TF$ such that $v \cdot s = X\varepsilon \cdot tF$. Then

$$vG \cdot (sG \cdot T\eta) \;=\; X\varepsilon G \cdot tFG \cdot T\eta \;=\; X\varepsilon G \cdot XG\eta \cdot t \;=\; t\,.$$

▫

II.7.3 Karoubi hull

The construction of the Karoubi hull is due to Karoubi; cf. [**30,** III.II].

Suppose given an additive category $\mathcal{A}$. The *Karoubi hull* $\tilde{\mathcal{A}}$ has

$$\operatorname{Ob}\tilde{\mathcal{A}} \;:=\; \{\,(A, e) \,:\, A \in \operatorname{Ob}\mathcal{A},\, e \in {}_{\mathcal{A}}(A, A) \text{ with } e^2 = e\,\}$$

and, given (A, e), $(B, f) \in \operatorname{Ob} \tilde{\mathcal{A}}$,

$$ {}_{\mathcal{A}}\big((A,e),\,(B,f)\big) \;:=\; \{\, u \in {}_{\mathcal{A}}(A,B) \,:\, e \cdot u \cdot f = u \,\}\,. $$

Then $\tilde{\mathcal{A}}$ is an additive category, in which all idempotents are split.

Composition is inherited from $\mathcal{A}$. We have a full and faithful additive functor

$$ \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathsf{K}} & \tilde{\mathcal{A}} \\ (X \xrightarrow{u} Y) & \longmapsto & \big((X,1) \xrightarrow{u} (Y,1)\big)\,, \end{array} $$

which we often consider as an inclusion of a full subcategory.

Suppose given an additive category $\mathcal{B}$ in which all idempotents are split.

Remark II.43 Write $[\![\mathcal{A},\mathcal{B}]\!]_{\text{add}}$ for the category of additive functors and transformations between such from $\mathcal{A}$ to $\mathcal{B}$. The induced functor $[\![\mathcal{A},\mathcal{B}]\!] \xleftarrow{\mathsf{K}\star(-)} [\![\tilde{\mathcal{A}},\mathcal{B}]\!]$ restricts to a strictly dense equivalence

$$ [\![\mathcal{A},\mathcal{B}]\!]_{\text{add}} \xleftarrow{\mathsf{K}\star(-)} [\![\tilde{\mathcal{A}},\mathcal{B}]\!]_{\text{add}}\,. $$

Lemma II.44 Suppose given an additive functor $\mathcal{A} \xrightarrow{I} \mathcal{A}'$ to an additive category $\mathcal{A}'$ in which all idempotents split. By Remark II.43, we obtain a functor $J : \tilde{\mathcal{A}} \longrightarrow \mathcal{A}'$, unique up to isomorphism, such that the following triangle of functors commutes.

$$ \begin{array}{ccc} \mathcal{A} & \xrightarrow{I} & \mathcal{A}' \\ {\scriptstyle \mathsf{K}}\downarrow & \nearrow{\scriptstyle J} & \\ \tilde{\mathcal{A}} & & \end{array} $$

If I is full and faithful, and if every object of $\mathcal{A}'$ is a direct summand of an object in the image of I, then J is an equivalence.

By abuse of notation, in the situation of Lemma II.44, we also write $\tilde{\mathcal{A}} = \mathcal{A}'$ and consider I to be an inclusion of a full subcategory.

II.7.4 Multiplicative systems

The construction of the quotient category of a Verdier triangulated category is due to VERDIER; cf. **[56]**.

Suppose given a category $\mathcal{C}$.

Definition II.45 A set M of morphisms of $\mathcal{C}$ is called a *multiplicative system* in $\mathcal{C}$ if (Fr 1-4) are satisfied. An element of M is called an M-isomorphism and denoted by $X \Longrightarrow Y$.

(Fr 1) Each identity in $\mathcal{C}$ is an M-isomorphism.

(Fr 2) Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{C}$ such that fg and gh are M-isomorphisms.
Then f, g, h and $f \cdot g \cdot h$ are M-isomorphisms.

(Fr 3) Suppose given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{C}$. There exists an M-isomorphism s such that $sf = sg$ if and only if there exists an M-isomorphism t such that $ft = gt$.

(Fr 4) Given $\begin{array}{ccc} X & \Longrightarrow & Y \\ \big\downarrow & & \\ X' & & \end{array}$ in $\mathcal{C}$, there exists a completion to a commutative quadrangle $\begin{array}{ccc} X & \Longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ X' & \Longrightarrow & Y' \end{array}$.

Dually, given $\begin{array}{ccc} & & Y \\ & & \big\downarrow \\ X' & \Longrightarrow & Y' \end{array}$ in $\mathcal{C}$, there exists a completion to a commutative quadrangle $\begin{array}{ccc} X & \Longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ X' & \Longrightarrow & Y' \end{array}$.

Cf. [**56,** §2, no. 1].

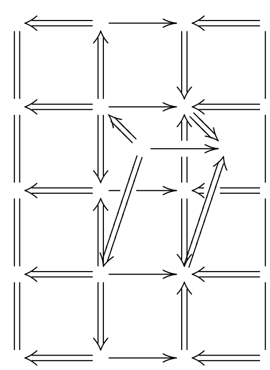
Suppose given a multiplicative system M in $\mathcal{C}$. Using (Fr 2), we note that in the first assertion of (Fr 4), if $X \longrightarrow X'$ is an M-isomorphism, then there exists a commutative completion with $Y \longrightarrow Y'$ being an M-isomorphism. And dually.

The category $\mathcal{C}_M$, called *localisation of $\mathcal{C}$ at M,* is defined as follows. Let $\mathrm{Ob}\,\mathcal{C}_M := \mathrm{Ob}\,\mathcal{C}$. A morphism from X to Y is a *double fraction,* which is an equivalence class of diagrams of the following form.

$$X \overset{s}{\Longleftarrow} X' \overset{f}{\longrightarrow} Y' \overset{t}{\Longleftarrow} Y$$

The diagrams (s, f, t) and $(s's, s'ft', tt')$ are declared to be *elementarily equivalent*, provided s' and t' are M-isomorphisms. To form double fractions, we take the equivalence relation generated by elementary equivalence.

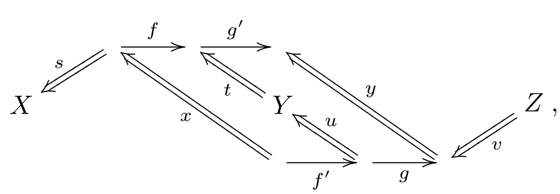
The equivalence class of the diagram (s, f, t) is written $s\backslash f/t$. So $s\backslash f/t = \tilde{s}\backslash \tilde{f}/\tilde{t}$ if and only if there exist M-isomorphisms u, $\tilde{u}$, v, $\tilde{v}$ such that $us = \tilde{u}\tilde{s}$ and $tv = \tilde{t}\tilde{v}$ and $ufv = \tilde{u}\tilde{f}\tilde{v}$.



Write $f/t := 1\backslash f/t$, called a *right fraction*, and $s\backslash f := s\backslash f/1$, called a *left fraction.* Using (Fr 4), each morphism in $\mathcal{C}_M$ can be represented both by a left fraction and by a right fraction. Given right fractions f/t and $\tilde{f}/\tilde{t}$, they are equal if there exist M-isomorphisms u, v and $\tilde{v}$ such that $ufv = u\tilde{f}\tilde{v}$ and $tv = \tilde{t}\tilde{v}$. By (Fr 3), this implies the existence of M-isomorphisms v, $\tilde{v}$ and u' such that $f(vu') = \tilde{f}(\tilde{v}u')$ and $t(vu') = \tilde{t}(\tilde{v}u')$. Dually for left fractions.

So double fractions are a self-dual way to represent morphisms in $\mathcal{C}_M$. Right or left fractions are more efficient in many arguments.

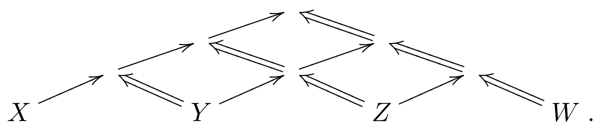
The composite of two double fractions $s\backslash f/t$ and $u\backslash g/v$ is defined, using (Fr 4) for the commutative diagram



to be equivalently $s\backslash fg'/vy$ or $xs\backslash f'g/v$. By (Fr 4, 2, 3), this definition is independent of the chosen completion with g' and y, and, likewise, of the chosen completion with x and f'.

Independence of the choice of the representative $s\backslash f/t$ is seen considering an elementary equivalence and using (Fr 4, 2), thus obtaining an elementary equivalence of the two possible representatives of the composite. Likewise independence of the representative $u\backslash g/v$.

Associativity follows using right fractions and a commutative diagram constructed by means of (Fr 4),



Given $f \in \operatorname{Mor}\mathcal{C}$, we also write $1\backslash f/1 =: f$ in $\mathcal{C}_M$, by abuse of notation. Note that in $\mathcal{C}_M$, we have $s\backslash f/t = s^- f\, t^-$.

Remark II.46 *A double fraction $s\backslash f/t$ represents an isomorphism in $\mathcal{C}_M$ if and only if f is an M-isomorphism.*

Sketch. First, using (Fr 2), we reduce to the case of a right fraction g/u. For a right fraction in turn, the assertion follows applying (Fr 2) to an associativity diagram as above. ▫

Remark II.47 Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$, we have $f = g$ in $\mathcal{C}_M$ if and only if there exists an M-isomorphism t such that $ft = gt$ in $\mathcal{C}$, or, equivalently, if and only if there exists an M-isomorphism s such that $sf = sg$ in $\mathcal{C}$.

Remark II.48 *We have a functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}_M$, $f \longmapsto 1\backslash f/1 = f$, called localisation functor.*

Given a category $\mathcal{T}$, we let $[\![\, \mathcal{C}, \mathcal{T} \,]\!]_M$ be the full subcategory of $[\![\, \mathcal{C}, \mathcal{T} \,]\!]$ consisting of functors that send all M-isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{T}$. The induced functor

$$[\![\, \mathcal{C}\, , \mathcal{T} \,]\!]_M \xleftarrow{\mathsf{L}\star(-)} [\![\, \mathcal{C}_M\, , \mathcal{T} \,]\!]$$

is a strictly dense equivalence, i.e. it is surjective on objects, full and faithful.

Sketch. Given a functor $F \in \operatorname{Ob}[\![\, \mathcal{C}\, , \mathcal{T} \,]\!]_M$, we may define $\breve{F}$ on $\mathcal{C}_M$ by letting $X\breve{F} := XF$ for $X \in \operatorname{Ob}\mathcal{C}_M = \operatorname{Ob}\mathcal{C}$ and by $(s\backslash f/t)\breve{F} := (sF)^- \cdot (fF) \cdot (tF)^-$. Then $\mathsf{L}\star\breve{F} = F$.

Given a transformation $(F \xrightarrow{u} G) \in \operatorname{Mor}[\![\, \mathcal{C}\, , \mathcal{T} \,]\!]_M$, we may define $\breve{F} \xrightarrow{\breve{u}} \breve{G}$ by setting $X\breve{u} := Xu$ for $X \in \operatorname{Ob}\mathcal{C}_M = \operatorname{Ob}\mathcal{C}$. Then $\mathsf{L}\star\breve{u} = u$. ▫

Lemma II.49 *Given $n \geqslant 0$, the functor*

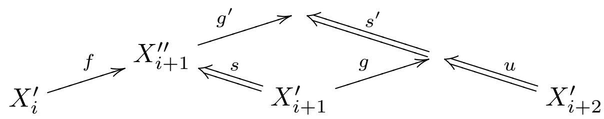
$$\mathcal{C}(\dot{\Delta}_n) \xrightarrow{\mathsf{L}(\dot{\Delta}_n)} \mathcal{C}_M(\dot{\Delta}_n)\ ,$$

given by pointwise application of L, is dense.

Proof. We may assume $n \geqslant 1$.

Suppose given $X \in \operatorname{Ob}\mathcal{C}_M(\dot{\Delta}_n)$. To prove that for $i \in [1, n-1]$ there exists an $X' \in \operatorname{Ob}\mathcal{C}_M(\dot{\Delta}_n)$ isomorphic to X such that $X'_j \xrightarrow{x'} X'_{j+1}$ is in the image of L for $j \in [1, i-1]$, we proceed by induction on $i \geqslant 1$. Suppose the assertion to be true for i. Let us prove the assertion for $i+1$. Write $X'_i \xrightarrow{x'} X'_{i+1}$ as a right fraction f/s. If

$i = n-1$, we replace X'_{i+1} by the target of f, and f/s by f. If $i \leqslant n-2$, we write $X'_{i+1} \xrightarrow{x'} X'_{i+2}$ as a right fraction g/u and construct the following commutative diagram using (Fr 4).



We replacing the object X'_{i+1} by X''_{i+1}, the morphism f/s by f and the morphism g/u by g'/us'.

In both cases, we obtain a diagram isomorphic to X' that conincides with X' on $[1, i]$ and whose morphism from i to $i+1$ is in the image of L. ▫

Lemma II.50 *Given $n \geqslant 0$, the functor*

$$\mathcal{C}(\dot{\Delta}_n) \xrightarrow{\mathsf{L}(\dot{\Delta}_n)} \mathcal{C}_M(\dot{\Delta}_n)$$

is 1*-epimorphic.*

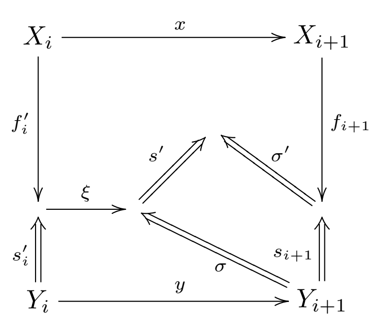
Proof. We shall apply Lem. I.75. By Lemma II.49, $\mathsf{L}(\dot{\Delta}_n)$ is dense.

Suppose given $X, Y \in \operatorname{Ob} \mathcal{C}(\dot{\Delta}_n)$ and a morphism $X\,\mathsf{L}(\dot{\Delta}_n) \xrightarrow{g} Y\,\mathsf{L}(\dot{\Delta}_n)$ in $\mathcal{C}_M(\dot{\Delta}_n)$. Let g_i be represented by a right fraction f_i/s_i for $i \in [1, n]$.

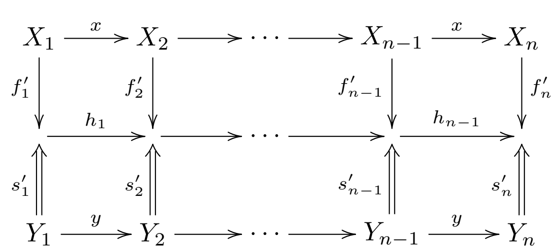
We *claim* that for $i \in [1, n]$, we can find representatives f'_j/s'_j for $j \in [1, i]$ such that there exist h_j with $s'_j h_j = y s'_{j+1}$ and $f'_j h_j = x f'_{j+1}$ in $\mathcal{C}$ for $j \in [1, i-1]$. Let $f'_1 := f_1$ and $s'_1 := s_1$. Proceeding by induction on i, we have to write the right fraction f_{i+1}/s_{i+1} suitably as f'_{i+1}/s'_{i+1}. First of all, by (Fr 4), we find an M-isomorphism σ and a morphism ξ such that $y\sigma = s'_i \xi$ in $\mathcal{C}$. We have $f'_i \xi \sigma^- = x f_{i+1} s^-_{i+1}$ in $\mathcal{C}_M$. Using (Fr 4) and (Fr 2), we find M-isomorphisms s' and σ' such that $\sigma s' = s_{i+1}\sigma'$ in $\mathcal{C}$. Hence

$$f'_i \xi s' \;=\; f'_i \xi s' s'^- \sigma^- s_{i+1} \sigma' \;=\; f'_i \xi \sigma^- s_{i+1} \sigma' \;=\; x f_{i+1} s^-_{i+1} s_{i+1} \sigma' \;=\; x f_{i+1} \sigma'$$

in $\mathcal{C}_M$. Composing with a further M-isomorphism, we may assume that $f'_i \xi s' = x f_{i+1} \sigma'$ in $\mathcal{C}$; cf. Remark II.47. We take $h_i := \xi s'$ and $s'_{i+1} := s_{i+1}\sigma'$ and $f'_{i+1} := f_{i+1}\sigma'$.



This proves the *claim,* in particular for $i = n$.



Condition (C) of loc. cit. is satisfied letting the epizigzag have length 0, letting the monozigzag be the single backwards diagram morphism consisting of the morphisms s'_i, and letting the required diagram morphism in the image of $L(\dot{\Delta}_n)$ consist of the morphisms f'_i. ▫

Remark II.51 *Suppose the category $\mathcal{C}$ to be additive.*

(1) *An object X is isomorphic to 0 in $\mathcal{C}_M$ if and only if $X \Longrightarrow 0$, or, equivalently, if and only if $0 \Longrightarrow X$.*

(2) *The category $\mathcal{C}_M$ is additive, and the functor $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}_M$ is additive.*

(3) *Given an additive category $\mathcal{T}$, the strictly dense equivalence*

$$[\![\, \mathcal{C}\, ,\, \mathcal{T}\,]\!]_M \xleftarrow{\mathsf{L}\star(-)} [\![\, \mathcal{C}_M\, ,\, \mathcal{T}\,]\!]$$

restricts to a strictly dense equivalence from the category of additive functors from $\mathcal{C}_M$ to $\mathcal{T}$ to the category of additive functors from $\mathcal{C}$ to $\mathcal{T}$ that sends all M-isomorphisms to isomorphisms, written

$$[\![\, \mathcal{C}\, ,\, \mathcal{T}\,]\!]_{\mathrm{add},\, M} \xleftarrow{\mathsf{L}\star(-)} [\![\, \mathcal{C}_M\, ,\, \mathcal{T}\,]\!]_{\mathrm{add}} \; .$$

Sketch.

Ad (1). If X is isomorphic to 0, then $X \Longrightarrow X' \Longleftarrow 0$; cf. Remark II.46. By (Fr 4), we conclude that $0 \Longrightarrow X$.

Ad (2). Given $X, Y \in \mathrm{Ob}\,\mathcal{C}$, the direct sum $X \oplus Y$, together with $X \xrightarrow{(1\,0)} X \oplus Y$ and $Y \xrightarrow{(0\,1)} X \oplus Y$, remains a coproduct in $\mathcal{C}_M$.

For existence of an induced morphism from the coproduct, we use (Fr 4, 2) to produce a common denominator of two right fractions.

To prove uniqueness of the induced morphism, we suppose given $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)/s$ and $\left(\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right)/s$, without loss of generality with common denominator, such that $f/s = (1\,0) \cdot \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)/s = (1\,0) \cdot \left(\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right)/s = f'/s$ and $g/s = (0\,1) \cdot \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)/s = (0\,1) \cdot \left(\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right)/s = g'/s$ in $\mathcal{C}_M$. So there exists an M-isomorphism u such that $fu = f'u$, and an M-isomorphism v such that $gv = g'v$, both in $\mathcal{C}$. By (Fr 4, 2), we obtain a common M-isomorphism w such that $fw = f'w$ and $gw = g'w$ in $\mathcal{C}$. Hence $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) w = \left(\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right) w$ in $\mathcal{C}$. Therefore $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)/s = \left(\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right)/s$ in $\mathcal{C}_M$.

Moreover, the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of $X \oplus X$ remains an automorphism in $\mathcal{C}_M$.

Ad (3). Since L is additive, $\mathsf{L}\star(-)$ sends additive functors to additive functors. Conversely, given an additive functor $F : \mathcal{C} \longrightarrow \mathcal{T}$, the functor $\breve{F}$ as constructed in the proof of Remark II.48 is additive. ▫

Chapter III

Nonisomorphic Verdier octahedra on the same base

III.0 Introduction

III.0.1 Is being a 3-triangle characterised by 2-triangles?

Verdier (implicitly) defined a Verdier octahedron to be a diagram in a triangulated category in the shape of an octahedron, four of whose triangles are distinguished, the four others commutative [**56,** Def. 1-1]; cf. also [**8,** 1.1.6]. It arises as follows.

To a morphism in a triangulated category, we can attach an object, called its *cone*. The morphism we start with and its cone are contained in a distinguished triangle. To the morphism we started with, we refer as the *base* of this distinguished triangle.

Now given a commutative triangle, we can form the cone on the first morphism, on the second morphism and on their composite, yielding three distinguished triangles. These three cones in turn are contained in a fourth distinguished triangle. The whole diagram obtained by this construction is a Verdier octahedron. We shall refer to the commutative triangle we started with as the *base* of this Verdier octahedron.

A distinguished triangle has the property of being determined up to isomorphism by its base. Moreover, any morphism between the bases of two distinguished triangles can be extended to a morphism between the whole distinguished triangles.

We shall show that the analogous assertion is not true for Verdier octahedra. In §III.3, we give an example of two nonisomorphic Verdier octahedra on the same base. In particular, the identity morphism between the bases cannot be prolonged to a morphism between the whole Verdier octahedra.

The reader particularly interested in Verdier octahedra can read §III.1.1, §III.1.2, §III.1.4 and §III.3.

In the terminology of Heller triangulated categories, a Verdier octahedron is a periodic 3-pretriangle X such that $Xd^{\#}$ is a 2-triangle (i.e. a distinguished triangle) for all injective periodic monotone maps $\bar{\Delta}_3 \overset{d}{\longleftarrow} \bar{\Delta}_2$.

One of the two Verdier octahedra in our example will be a 3-triangle in the sense of Def. I.5, i.e. a "distinguished octahedron", whereas the other will not.

Note that unlike a Verdier octahedron, a 3-triangle is uniquely determined up to isomorphism by its base in the Heller triangulated context; cf. Lem. I.21.(6).

III.0.2 Is being an n-triangle characterised by $(n-1)$-triangles?

The situation of §III.0.1 can be generalised in the following manner.

Suppose given a closed Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$; cf. Def. I.5, Definition III.13.

The Heller triangulation $\vartheta = (\vartheta_n)_{n \geqslant 0}$ on $(\mathcal{C}, \mathsf{T})$ can be viewed as a means to distinguish certain periodic n-pretriangles as n-triangles. Namely, a periodic n-pretriangle X is, by definition, an n-triangle if $X\vartheta_n = 1$; cf. Def. I.5.(ii.2). For instance, 2-triangles are distinguished triangles in the sense of Verdier; 3-triangles are particular, "distinguished" Verdier octahedra.

III.0.2.1 The example

Let $n \geqslant 3$. Let X be a periodic n-pretriangle. Suppose that $Xd^{\#}$ is an $(n-1)$-triangle for all injective periodic monotone maps $\bar{\Delta}_n \overset{d}{\longleftarrow} \bar{\Delta}_{n-1}$. One might ask whether X is an n-triangle.

We shall show in §III.2 by an example that this is, in general, not the case.

III.0.2.2 Consequences

Suppose given $n \geqslant 3$ and a subset of the set of periodic n-pretriangles. We shall say for the moment that *determination* holds for this subset if for X and $\tilde{X}$ out of this subset, $X|_{\dot{\Delta}_n} \simeq \tilde{X}|_{\dot{\Delta}_n}$ implies that there is a periodic isomorphism $X \simeq \tilde{X}$. We shall say that *prolongation* holds for this subset, if for X and $\tilde{X}$ out of this subset and a morphism $X|_{\dot{\Delta}_n} \longrightarrow \tilde{X}|_{\dot{\Delta}_n}$, there exists a periodic morphism $X \longrightarrow \tilde{X}$ that restricts on $\dot{\Delta}_n$ to that given morphism. If prolongation holds, then determination holds.

- Consider the subset of periodic n-pretriangles X such that $Xd^{\#}$ is an $(n-1)$-triangle for all injective periodic monotone maps $\bar{\Delta}_n \overset{d}{\longleftarrow} \bar{\Delta}_{n-1}$. Our example shows that in general, determination and prolongation do not hold for this subset. In fact, if X is such an n-pretriangle, but not an n-triangle, then the n-triangle on the base $X|_{\dot{\Delta}_n}$ is not isomorphic to X; cf. Lem. I.21.(1, 4).

- Bernstein, Beilinson and Deligne considered the subset of periodic n-pretriangles X such that $Xd^{\#}$ is a 2-triangle (i.e. a distinguished triangle) for all injective periodic monotone maps $\bar{\Delta}_n \overset{d}{\longleftarrow} \bar{\Delta}_2$ [**8**, 1.1.14]. Our example shows that in general, determination and prolongation do not hold for this subset. In fact, this subset contains the previously described subset.

In both of the cases above, if $n = 3$, then the condition singles out the subset of Verdier octahedra.

- By Lem. I.21.(6), Lem. I.19, determination and prolongation hold for the set of n-triangles.

So morally, our example shows that it makes sense to let the Heller triangulation ϑ distinguish n-triangles for all $n \geqslant 0$. There is no "sufficiently large" n we could be content with.

III.0.3 An appendix on transport of structure

Suppose given a Frobenius category $\mathcal{E}$; that is, an exact category with enough bijective objects (relative to pure short exact sequences). Let $\mathcal{B} \subseteq \mathcal{E}$ denote the full subcategory of bijective objects.

There are two variants of the stable category of $\mathcal{E}$. First, there is the *classical stable category* $\underline{\mathcal{E}}$, defined as the quotient of $\mathcal{E}$ modulo $\mathcal{B}$. Second, there is the *stable category* $\underline{\underline{\mathcal{E}}}$, defined as the quotient of the category of purely acyclic complexes with entries in $\mathcal{B}$ modulo the category of split acyclic complexes with entries in $\mathcal{B}$. The categories $\underline{\mathcal{E}}$ and $\underline{\underline{\mathcal{E}}}$ are equivalent. The advantage of the variant $\underline{\underline{\mathcal{E}}}$ is that it carries a shift automorphism, whereas $\underline{\mathcal{E}}$ carries a shift autoequivalence.

In Cor. I.33, we have endowed $\underline{\underline{\mathcal{E}}}$ with a Heller triangulation. Now in our particular situation, also $\underline{\mathcal{E}}$ carries a shift automorphism. Since $\underline{\mathcal{E}}$ is better suited for calculations within that category, the question arises whether the equivalence $\underline{\mathcal{E}} \simeq \underline{\underline{\mathcal{E}}}$ can be used to transport the structure of a Heller triangulated category from $\underline{\underline{\mathcal{E}}}$ to $\underline{\mathcal{E}}$. This is indeed the case; cf. Proposition III.22.(1). Moreover, we give recipes how to detect and how to construct n-triangles in $\underline{\mathcal{E}}$; cf. Propositions III.22.(2, 3), III.25.

Roughly put, the variant $\underline{\underline{\mathcal{E}}}$ is rather suited for theoretical purposes, the variant $\underline{\mathcal{E}}$ is rather suited for practical purposes, and we had to pass a result from $\underline{\underline{\mathcal{E}}}$ to $\underline{\mathcal{E}}$. Not surprisingly, to do so, we had to grapple with the various equivalences and isomorphisms involved.

III.0.4 Acknowledgements

I thank Amnon Neeman for pointing out, years ago, why a counterexample as in §III.2 should exist, contrary to what I had believed.

This example has been found using the computer algebra system Magma [**9**]. I thank Markus Kirschmer for help with a Magma program.

I thank the referee for helpful comments.

III.0.5 Notations and conventions

We use the conventions listed in §I.0.7. In addition, we use the following conventions.

(i) If x and y are elements of a set, we let $\partial_{x,y} := 1$ if $x = y$, and we let $\partial_{x,y} := 0$ if $x \neq y$.

(ii) Given $a \in \mathbf{Z}$, we write $\mathbf{Z}/a := \mathbf{Z}/a\mathbf{Z}$.

(iii) Given a ring R and R-modules X and Y, we write, by choice, ${}_R(X,Y) = {}_{R\text{-Mod}}(X,Y) = \mathrm{Hom}_R(X,Y)$. Moreover, given $k \geqslant 0$, we write $X^{\oplus k} := \bigoplus_{i \in [1,k]} X$.

(iv) An *automorphism* T of a category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ for which there exists an endofunctor S such that $ST = 1_{\mathcal{C}}$ and $TS = 1_{\mathcal{C}}$. An *autoequivalence* T of a category $\mathcal{C}$ is an endofunctor on $\mathcal{C}$ for which there exists an endofunctor S such that $ST \simeq 1_{\mathcal{C}}$ and $TS \simeq 1_{\mathcal{C}}$.

(v) Let $n \geqslant 0$. Recall that $\bar{\Delta}_n^{\Delta\nabla} = \{\beta/\alpha \in \bar{\Delta}_n^{\#} : 0 \leqslant \alpha \leqslant \beta \leqslant 0^{+1}\} \subseteq \bar{\Delta}_n^{\#}$. We will often display an n-triangle or a periodic n-pretriangle in a Heller triangulated category $\mathcal{C}$ by showing its restriction to $\bar{\Delta}_n^{\Delta\nabla} \smallsetminus (\{\alpha/\alpha \; : \; 0 \leqslant \alpha \leqslant 0^{+1}\} \cup \{0^{+1}/0\})$. This is possible without loss of information, for we can reconstruct the whole diagram by adding zeroes on α/α for $0 \leqslant \alpha \leqslant 0^{+1}$ and on $0^{+1}/0$, and then by periodic prolongation.

(vi) Suppose given a Heller triangulated category $\mathcal{C}$. A *Verdier octahedron* in $\mathcal{C}$ is a periodic 3-pretriangle $X \in \mathrm{Ob}\, \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_3^{\#})$ such that $Xd^{\#} \in \mathrm{Ob}\, \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_2^{\#})$ is a 2-triangle for all injective periodic monotone maps $\bar{\Delta}_3 \xleftarrow{d} \bar{\Delta}_2$.

Henceforth, let $p \geqslant 2$ be a prime.

III.1 The classical stable category of $(\mathbf{Z}/p^m)$-mod

III.1.1 The category $(\mathbf{Z}/p^m)$-mod

Let $m \geqslant 0$. By $\mathcal{E} := (\mathbf{Z}/p^m)$-mod we understand the following category.

The objects are indexed by tuples $(a_i)_{i \in [0,m]}$ with $a_i \in \mathbf{Z}_{\geqslant 0}$. To such an index, we attach the object

$$\bigoplus_{i \in [0,m]} (\mathbf{Z}/p^i)^{\oplus a_i} \; .$$

As morphisms, we take $\mathbf{Z}/p^m$-linear maps.

Note that we have *not* chosen a skeleton. The trick here is to pick *several* zero objects.

The duality contrafunctor ${}_{\mathbf{Z}/p^m}(-, \mathbf{Z}/p^m)$ on $\mathcal{E}$, which sends $\mathbf{Z}/p^i$ to $\mathbf{Z}/p^i$ for $i \in [1,m]$, shows that an object in this category is injective if and only if it is projective. An object of $\mathcal{E}$ is bijective if and only if it is isomorphic to a finite direct sum of copies of $\mathbf{Z}/p^m$. The category $\mathcal{E}$ is an abelian Frobenius category, with all short exact sequences stipulated to be pure; cf. e.g. Def. I.45.

III.1.2 The shift on $\underline{(\mathbf{Z}/p^m)\text{-mod}}$

To define a shift automorphism on the classical stable category $\underline{\mathcal{E}} = \underline{(\mathbf{Z}/p^m)\text{-mod}}$, we shall distinguish certain (pure) short exact sequences in $\mathcal{E}$; cf. §III.4.4.2.1, Def. I.47.

Let $\mathrm{E}_k := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ denote the unit matrix of size $k \times k$; let $\mathrm{E}'_k := \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ denote the reversed unit matrix of size $k \times k$.

As distinguished (pure) short exact sequences we take those of the form

$$\bigoplus_{i\in[0,m]} (\mathbf{Z}/p^i)^{\oplus a_i} \xrightarrow{\begin{pmatrix} p^m \mathrm{E}_{a_0} & & & \\ & p^{m-1}\mathrm{E}_{a_1} & & \\ & & \ddots & \\ & & & p^0 \mathrm{E}_{a_m} \end{pmatrix}} (\mathbf{Z}/p^m)^{\oplus \sum_{i\in[0,m]} a_i} \xrightarrow{\begin{pmatrix} & & & \mathrm{E}'_{a_0} \\ & & \mathrm{E}'_{a_1} & \\ & \cdot^{\cdot^{\cdot}} & & \\ \mathrm{E}'_{a_m} & & & \end{pmatrix}} \bigoplus_{i\in[0,m]} (\mathbf{Z}/p^i)^{\oplus a_{m-i}}$$

So roughly speaking, distinguished short exact sequences are direct sums of those of the form

$$\mathbf{Z}/p^i \xrightarrow{p^{m-i}} \mathbf{Z}/p^m \xrightarrow{1} \mathbf{Z}/p^{m-i} ,$$

where $i \in [0, m]$; we reorder the summands the cokernel term consists of.

With this choice, conditions (i, ii, iii) of §III.4.4.2.1 are satisfied.

On indecomposable objects and morphisms between them, the shift automorphism induced on $\underline{\mathcal{E}}$ by our set of distinguished short exact sequences is given by

$$(\mathbf{Z}/p^i \xrightarrow{a} \mathbf{Z}/p^j)^{+1} \;=\; (\mathbf{Z}/p^{m-i} \xrightarrow{p^{i-j}a} \mathbf{Z}/p^{m-j}) ,$$

where $i,\, j \in [0, m]$, and where a is a representative in $\mathbf{Z}$. Note that if $i < j$, then a is divisible by p^{j-i}.

Note that $\mathbf{Z}/p^i \xrightarrow{a} \mathbf{Z}/p^j$ represents zero in $\underline{\mathcal{E}}$ if and only if a is divisible by $p^{\min(m-i,\, j)}$.

III.1.3 A Heller triangulation on $\underline{(\mathbf{Z}/p^m)\text{-mod}}$

Concerning the notation $\mathcal{E}^{\Box}(\bar{\Delta}_n^{\vartriangle\triangledown})$, cf. §III.4.3. Given $n \geqslant 0$ and $X \in \operatorname{Ob} \mathcal{E}^{\Box}(\bar{\Delta}_n^{\vartriangle\triangledown})$, we form $X^\tau \in \operatorname{Ob} \underline{\mathcal{E}}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\vartriangle\triangledown})$ with respect to the set of distinguished short exact sequences of §III.1.2 as described in §III.4.4.2.3. That is, we replace the rightmost column of X by the column obtained using distinguished short exact sequences, so that $(X^\tau)_{0^{+1}/*} = ((X^\tau)_{*/0})^{+1} = (X_{*/0})^{+1}$; cf. §III.4.4.2.3.

Remark III.1 *If the short exact sequences*

$$X_{\alpha/0} \xrightarrow{(x\; x)} X_{\alpha/\alpha} \oplus X_{0^{+1}/0} \xrightarrow{\binom{x}{-x}} X_{0^{+1}/\alpha}$$

appearing in the diagram X for $1 \leqslant \alpha \leqslant n$ already are distinguished, then the image of X in $\operatorname{Ob} \underline{\mathcal{E}}^{+}(\bar{\Delta}_n^{\vartriangle\triangledown})$ equals X^τ.

Concerning the notion of a closed Heller triangulated category, cf. Definition III.13 in §III.4.2.

Remark III.2 *The classical stable category* $\underline{\mathcal{E}} = \underline{(\mathbf{Z}/p^m)\text{-mod}}$ *carries a closed Heller triangulation such that given* $n \geqslant 0$ *and* $X \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\vartriangle\triangledown})$, *the periodic prolongation of* X^τ *to an object of* $\underline{\mathcal{E}}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ *is an* n*-triangle.*

Proof. The assertion follows by Proposition III.22.(1) in §III.4.4.2.1 and Proposition III.25 in §III.4.4.2.4. ▫

III.1.4 A Verdier triangulation on $\underline{(\mathbf{Z}/p^m)\text{-mod}}$

By [**22,** Th. 2.6], $\underline{\mathcal{E}} = \underline{(\mathbf{Z}/p^m)\text{-mod}}$ is a Verdier triangulated category, i.e. a triangulated category in the sense of Verdier [**56,** Def. 1-1].

> This also follows by Remark III.2 and by Prop. I.23, which says that any Heller triangulated category in which idempotents split is also Verdier triangulated. The 2-triangles in the Heller context are the distinguished triangles in the Verdier context.

Given a morphism $X \xrightarrow{f} Y$ in $\mathcal{E}$, using the distinguished short exact sequence $X \rightarrowtail B \twoheadrightarrow X^{+1}$, where B is bijective, we can form the morphism

$$\begin{array}{ccccc} Y & \rightarrowtail & Z & \twoheadrightarrow & X^{+1} \\ {\scriptstyle f}\uparrow & & \uparrow & & \| \\ X & \rightarrowtail & B & \twoheadrightarrow & X^{+1} \end{array}$$

of short exact sequences, from which the sequence

$$X \xrightarrow{f} Y \rightarrowtail Z \twoheadrightarrow X^{+1}$$

represents a distinguished triangle in the Verdier triangulated category $\underline{\mathcal{E}}$.

III.2 Nonisomorphic periodic n-pretriangles

> Nonisomorphic periodic n-pretriangles whose periodic $(n-1)$-pretriangles are all $(n-1)$-triangles, to be specific.

Let $n \geqslant 3$. Let $\mathcal{C} := \underline{(\mathbf{Z}/p^{2n})\text{-mod}}$, and let it be endowed with a shift automorphism as in §III.1.2 and a Heller triangulation as in §III.1.3.

III.2.1 A $(2n-1)$-triangle

Let Y be the following $(2n-1)$-triangle in $\mathcal{C}$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & \mathbf{Z}/p^1 \\
 & & & & & & & & & & & & \uparrow{\scriptstyle 1} \\
 & & & & & & & & & & \mathbf{Z}/p^1 & \xrightarrow{-p} & \mathbf{Z}/p^2 \\
 & & & & & & & & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} \\
 & & & & & & & & \mathbf{Z}/p^1 & \xrightarrow{p} & \mathbf{Z}/p^2 & \xrightarrow{-p} & \mathbf{Z}/p^3 \\
 & & & & & & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} \\
 & & & & & & \cdot^{\cdot^{\cdot}} & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} \\
 & & & & \mathbf{Z}/p^1 & \xrightarrow{p} & \cdots & \xrightarrow{p} & \mathbf{Z}/p^{2n-4} & \xrightarrow{p} & \mathbf{Z}/p^{2n-3} & \xrightarrow{-p} & \mathbf{Z}/p^{2n-2} \\
 & & & & \uparrow{\scriptstyle 1} & & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} \\
 & & \mathbf{Z}/p^1 & \xrightarrow{p} & \mathbf{Z}/p^2 & \xrightarrow{p} & \cdots & \xrightarrow{p} & \mathbf{Z}/p^{2n-3} & \xrightarrow{p} & \mathbf{Z}/p^{2n-2} & \xrightarrow{-p} & \mathbf{Z}/p^{2n-1} \\
 & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \\
\mathbf{Z}/p^1 & \xrightarrow{p} & \mathbf{Z}/p^2 & \xrightarrow{p} & \mathbf{Z}/p^3 & \xrightarrow{p} & \cdots & \xrightarrow{p} & \mathbf{Z}/p^{2n-2} & \xrightarrow{p} & \mathbf{Z}/p^{2n-1} & &
\end{array}
$$

Here we have made use of the convention from §III.0.5 that we display of Y only its restriction to the subposet $\{\beta/\alpha \in \bar{\Delta}^{\#}_{2n-1} \;:\; 0 \leqslant \alpha < \beta \leqslant 0^{+1},\ \beta/\alpha \neq 0^{+1}/0\}$, which is possible without loss of information. Similarly below.

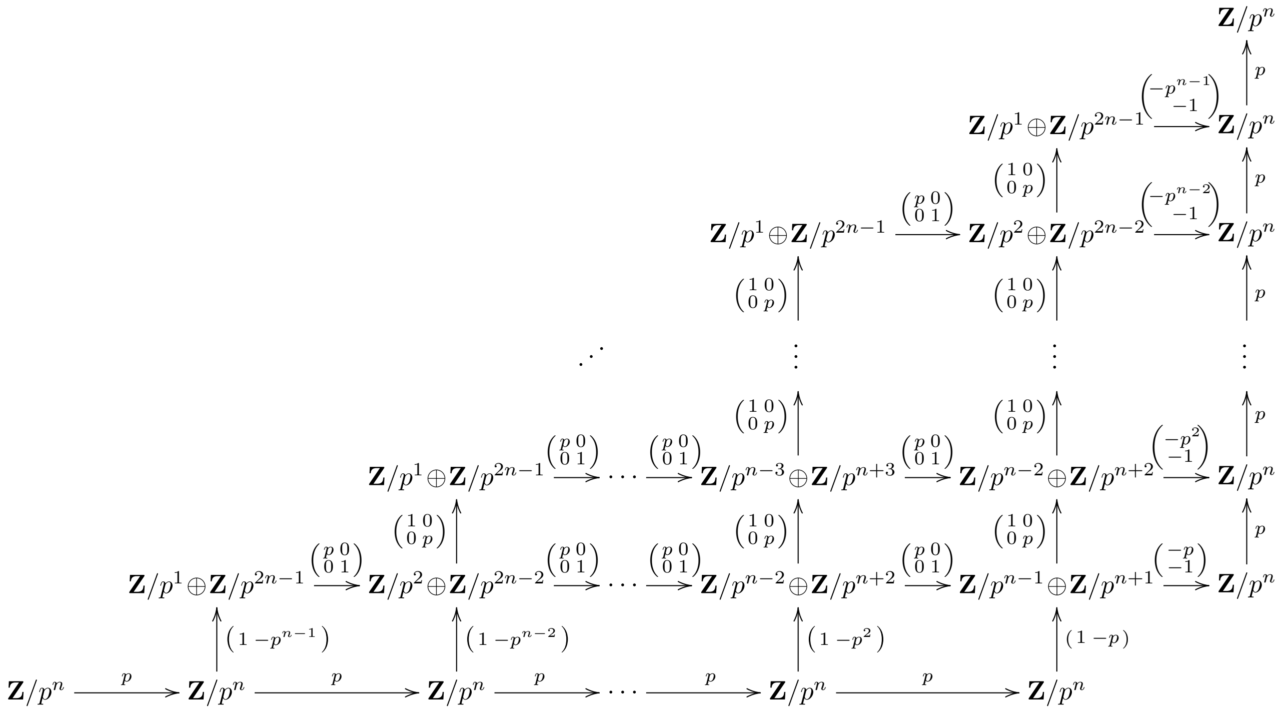
It arises from a diagram on $\bar{\Delta}_n^{\vartriangle\triangledown}$ with values in $(\mathbf{Z}/p^{2n})$-mod that consists of squares, has entry $\mathbf{Z}/p^{2n}$ at position $0^{+1}/0$, and has the quadrangle

$$
\begin{array}{ccc}
\mathbf{Z}/p^{2n-2} & \xrightarrow{-p} & \mathbf{Z}/p^{2n-1} \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle -1} \\
\mathbf{Z}/p^{2n-1} & \xrightarrow{p} & \mathbf{Z}/p^{2n}
\end{array}
$$

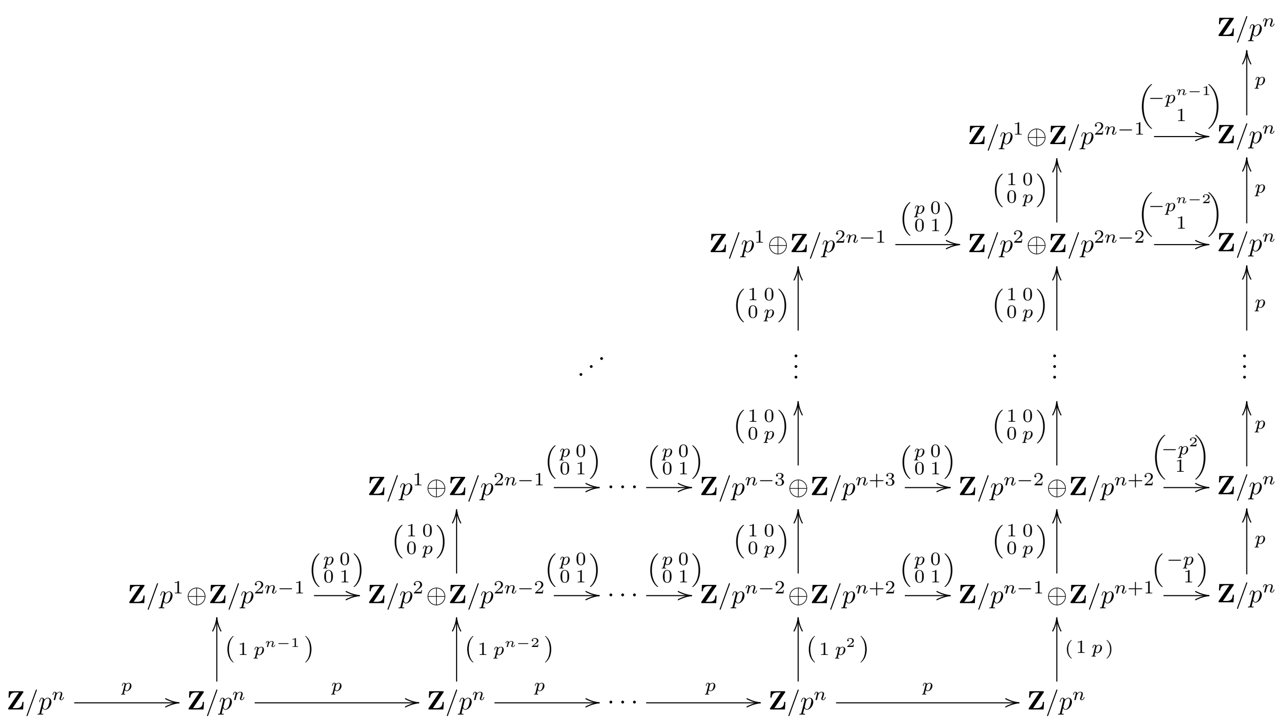
in its lower right corner. This diagram contains the necessary distinguished short exact sequences with the necessary signs inserted for Y to be in fact a $(2n-1)$-triangle; cf. Remarks III.1, III.2.

III.2.2 An n-triangle and a periodic n-pretriangle

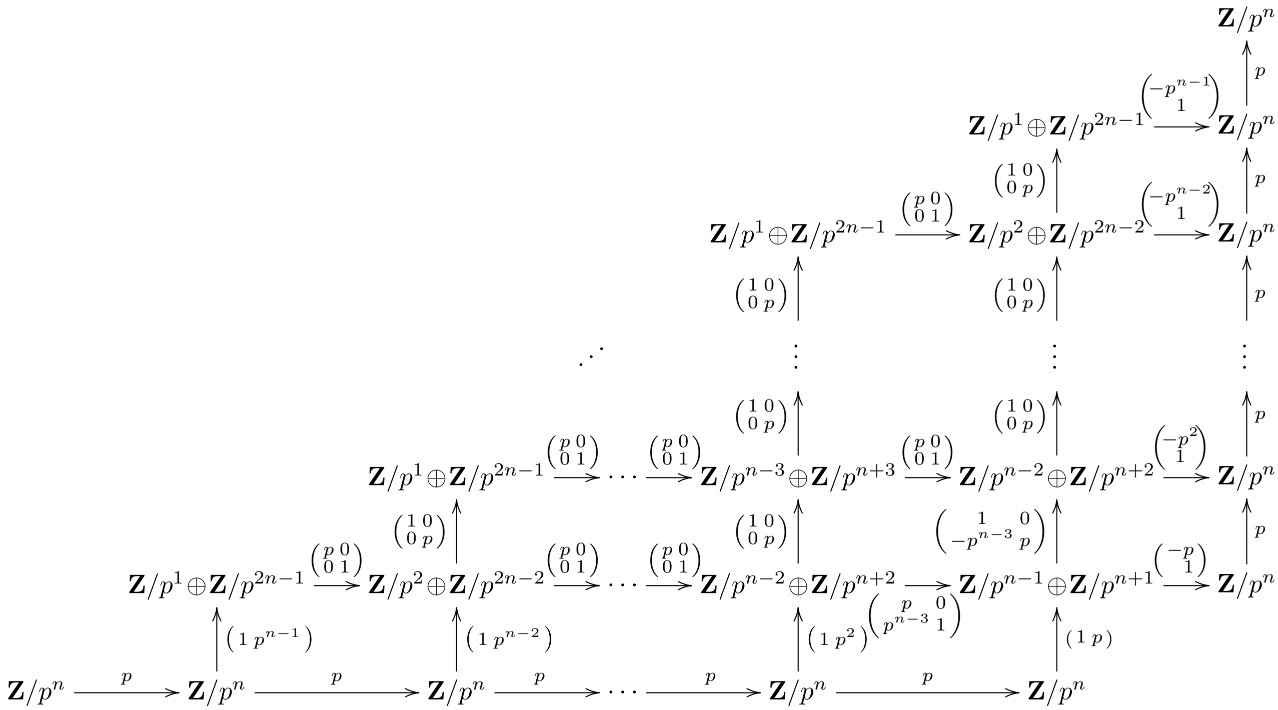
We apply the folding operator $\mathfrak{f}_{n-1}$ to the $(2n-1)$-triangle Y obtained in §III.2.1, yielding the n-triangle $Y\mathfrak{f}_{n-1}$, which we shall display now; cf. Lem. I.21.(2), §I.1.2.2.3.



Let X be the n-triangle obtained from $Y\mathfrak{f}_{n-1}$ by isomorphic substitution along $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ on all terms consisting of two summands; cf. Lem. I.21.(4). So X can be displayed as follows.



Let $\tilde{X}$ be the following periodic n-pretriangle.



To verify that $\tilde{X}$ actually is an n-pretriangle, a comparison with X reduces us to show that the three quadrangles depicted in full in the lower right corner of $\tilde{X}$ are weak squares. Of these three, the middle quadrangle arises from the corresponding one of X by an isomorphic substitution along $\mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1} \xrightarrow[\sim]{\left(\begin{smallmatrix} 1 & 0 \\ p^{n-3} & 1 \end{smallmatrix}\right)} \mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1}$, and thus is a weak square. For the lower one, we may apply Lem. I.57 to the diagram $(\tilde{X}_{1/0},\, \tilde{X}_{n-1/0},\, \tilde{X}_{n/0},\, \tilde{X}_{0^{+1}/0},\, \tilde{X}_{1/1},\, \tilde{X}_{n-1/1},\, \tilde{X}_{n/1},\, \tilde{X}_{0^{+1}/1})$ and compare with X to show that it is a weak square. For the right hand side one, we may apply Lem. I.57 to the diagram $(\tilde{X}_{n/0},\, \tilde{X}_{n/1},\, \tilde{X}_{n/2},\, \tilde{X}_{n/n},\, \tilde{X}_{0^{+1}/0},\, \tilde{X}_{0^{+1}/1},\, \tilde{X}_{0^{+1}/2},\, \tilde{X}_{0^{+1}/n})$ and compare with X to show that it is a weak square.

Given $k \in [0,n]$, we let $\bar{\Delta}_n \xleftarrow{\mathrm{d}_k} \bar{\Delta}_{n-1}$ be the periodic monotone map determined by $[0,n-1]\mathrm{d}_k = [0,n] \smallsetminus \{k\}$.

Lemma III.3 *Suppose given $k \in [0,n]$.*

(1) *The diagram $\tilde{X}\mathrm{d}_k^{\#}$ is an $(n-1)$-triangle.*

(2) *We have $X\mathrm{d}_k^{\#} \simeq \tilde{X}\mathrm{d}_k^{\#}$ in $\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_{n-1}^{\#})$.*

Proof. Since X is an n-triangle, $X\mathrm{d}_k^{\#}$ is an $(n-1)$-triangle; cf. Lem. I.21.(1). Since $X\mathrm{d}_k^{\#}|_{\dot{\Delta}_{n-1}} = \tilde{X}\mathrm{d}_k^{\#}|_{\dot{\Delta}_{n-1}}$, the diagram $\tilde{X}\mathrm{d}_k^{\#}$ is an $(n-1)$-triangle if and only if it is isomorphic to $X\mathrm{d}_k^{\#}$ in $\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_{n-1}^{\#})$; cf. Lem. I.21.(4, 6). So assertions (1) and (2) are equivalent. We will prove (2).

When referring to an object on a certain position in the diagram $X\mathrm{d}_k^{\#}$ resp. $\tilde{X}\mathrm{d}_k^{\#}$, we shall also mention in parentheses its position as an object in the diagram X resp. $\tilde{X}$ for ease of orientation.

When constructing a morphism in $\mathcal{C}^{+,\,\text{periodic}}(\bar\Delta^\#_{n-1})$, we will give its components on $\{j/i \,:\, 0 \leqslant i \leqslant j \leqslant n-1\} \subseteq \bar\Delta^\#_{n-1}$; the remaining components result thereof by periodic repetition.

Case $k \in \{1,\, n\}$. We have $X\mathrm{d}_1^\# = \tilde X\mathrm{d}_1^\#$ and $X\mathrm{d}_n^\# = \tilde X\mathrm{d}_n^\#$.

Case $k = 0$. We *claim* that $X\mathrm{d}_0^\#$ is isomorphic to $\tilde X\mathrm{d}_0^\#$ in $\mathcal{C}^{+,\,\text{periodic}}(\bar\Delta^\#_{n-1})$. In fact, an isomorphism $X\mathrm{d}_0^\# \xrightarrow{\sim} \tilde X\mathrm{d}_0^\#$ is given by

$$\mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1} \xrightarrow[\sim]{\left(\begin{smallmatrix} 1 & 0 \\ p^{n-3} & 1 \end{smallmatrix}\right)} \mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1}$$

at position $(n-1)/0$ (position $n/1$ in X resp. $\tilde X$), and by the identity elsewhere. This proves the *claim.*

Case $k \in [2, n-1]$. We *claim* that $X\mathrm{d}_k^\#$ is isomorphic to $\tilde X\mathrm{d}_k^\#$ in $\mathcal{C}^{+,\,\text{periodic}}(\bar\Delta^\#_{n-1})$. In fact, an isomorphism $X\mathrm{d}_k^\# \xrightarrow{\sim} \tilde X\mathrm{d}_k^\#$ is given as follows.

At position $j/0$ for $j \in [1, n-1]$ (position $j/0$ if $j \leqslant k-1$ and $(j+1)/0$ if $j \geqslant k$ in X resp. $\tilde X$), it is given by the identity on $\mathbf{Z}/p^n$.

At position j/i for $i,\, j \in [1, k-1]$ such that $i < j$ (position j/i in X resp. $\tilde X$), it is given by the identity on $\mathbf{Z}/p^{j-i} \oplus \mathbf{Z}/p^{2n-j+i}$.

At position j/i for $i,\, j \in [k, n-1]$ such that $i < j$ (position $(j+1)/(i+1)$ in X resp. $\tilde X$), it is given by the identity on $\mathbf{Z}/p^{j-i} \oplus \mathbf{Z}/p^{2n-j+i}$.

At position j/i for $i \in [1, k-1]$ and $j \in [k, n-1]$ such that $j/i \neq (n-1)/1$ (position $(j+1)/i$ in X resp. $\tilde X$), it is given by

$$\mathbf{Z}/p^{j+1-i} \oplus \mathbf{Z}/p^{2n-j-1+i} \xrightarrow[\sim]{\left(\begin{smallmatrix} 1 & 0 \\ -p^{j-1-i} & 1 \end{smallmatrix}\right)} \mathbf{Z}/p^{j+1-i} \oplus \mathbf{Z}/p^{2n-j-1+i}$$

At position $(n-1)/1$ (position $n/1$ in X resp. $\tilde X$), it is given by the identity on $\mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1}$.

This proves the *claim.* ▫

Lemma III.4 *X is not isomorphic to $\tilde X$ in $\mathcal{C}^{+,\,\text{periodic}}(\bar\Delta^\#_n)$.*

In particular, $\tilde X$ is not an n-triangle; cf. Lem. I.21.(6).

Proof. We *assume* the contrary. By Lem. I.21.(4), X and $\tilde X$ are n-triangles. Thus, by Lem. I.21.(6), there is an isomorphism $X \xrightarrow{\sim} \tilde X$ that is identical at $i/0$ and at $0^{+1}/i$ for $i \in [1, n]$. Let

$$\mathbf{Z}/p^{\ell-k} \oplus \mathbf{Z}/p^{2n-\ell+k} \xrightarrow{\left(\begin{smallmatrix} a_{\ell/k} & p^{2n-2\ell+2k} b_{\ell/k} \\ c_{\ell/k} & d_{\ell/k} \end{smallmatrix}\right)} \mathbf{Z}/p^{\ell-k} \oplus \mathbf{Z}/p^{2n-\ell+k}$$

denote the entry of this isomorphism at ℓ/k, where $1 \leqslant k < \ell \leqslant n$.

If $\ell - k \geqslant 2$, we have the following commutative quadrangle in $\mathcal{C}$ on $\ell/k \longrightarrow \ell/(k+1)$.

$$\begin{array}{ccc}
\mathbf{Z}/p^{\ell-k} \oplus \mathbf{Z}/p^{2n-\ell+k} & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)} & \mathbf{Z}/p^{\ell-k-1} \oplus \mathbf{Z}/p^{2n-\ell+k+1} \\
{\scriptstyle \left(\begin{smallmatrix} a_{\ell/k} & p^{2n-2\ell+2k} b_{\ell/k} \\ c_{\ell/k} & d_{\ell/k} \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} a_{\ell/(k+1)} & p^{2n-2\ell+2k+2} b_{\ell/(k+1)} \\ c_{\ell/(k+1)} & d_{\ell/(k+1)} \end{smallmatrix}\right)} \\
\mathbf{Z}/p^{\ell-k} \oplus \mathbf{Z}/p^{2n-\ell+k} & \xrightarrow[\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) - \partial_{\ell/k,\, n/1} \left(\begin{smallmatrix} 0 & 0 \\ p^{n-3} & 0 \end{smallmatrix}\right)]{} & \mathbf{Z}/p^{\ell-k-1} \oplus \mathbf{Z}/p^{2n-\ell+k+1}
\end{array}$$

We read off the congruences

$$\text{(i)} \qquad c_{\ell/k} - \partial_{\ell/k,\,n/1}\, p^{n-3} d_{\ell/k} \;\equiv_{p^{\ell-k-1}}\; p c_{\ell/(k+1)}$$
$$\text{(ii)} \qquad b_{\ell/k} \;\equiv_{p^{\ell-k-1}}\; p b_{\ell/(k+1)}\ .$$

From (i) we infer

$$\text{(iii)} \qquad c_{n/1} - p^{n-3} d_{n/1} \;\equiv_{p^{n-2}}\; p^1 c_{n/2} \;\equiv_{p^{n-2}}\; p^2 c_{n/3} \;\equiv_{p^{n-2}}\; \dots \;\equiv_{p^{n-2}}\; p^{n-2} c_{n/(n-1)} \;\equiv_{p^{n-2}}\; 0\ .$$

From (ii) we infer

$$\text{(iv)} \qquad b_{n/1} \;\equiv_{p^{n-2}}\; p b_{n/2} \;\equiv_{p^{n-2}}\; p^2 b_{n/3} \;\equiv_{p^{n-2}}\; \dots \;\equiv_{p^{n-2}}\; p^{n-2} b_{n/(n-1)} \;\equiv_{p^{n-2}}\; 0\ .$$

On $n/1 \longrightarrow 0^{+1}/1$, we have the following commutative quadrangle in $\mathcal{C}$.

$$\begin{array}{ccc}
\mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1} & \xrightarrow{\left(\begin{smallmatrix} -p \\ 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^n \\
{\scriptstyle \left(\begin{smallmatrix} a_{n/1} & p^2 b_{n/1} \\ c_{n/1} & d_{n/1} \end{smallmatrix}\right)} \downarrow & & \| \\
\mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1} & \xrightarrow[\left(\begin{smallmatrix} -p \\ 1 \end{smallmatrix}\right)]{} & \mathbf{Z}/p^n
\end{array}$$

We read off the congruence

$$\text{(v)} \qquad -p c_{n/1} + d_{n/1} \;\equiv_{p^{n-1}}\; 1\ .$$

On $n/0 \longrightarrow n/1$, we have the following commutative quadrangle in $\mathcal{C}$.

$$\begin{array}{ccc}
\mathbf{Z}/p^n & \xrightarrow{(1\ p)} & \mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1} \\
\| & & \downarrow {\scriptstyle \left(\begin{smallmatrix} a_{n/1} & p^2 b_{n/1} \\ c_{n/1} & d_{n/1} \end{smallmatrix}\right)} \\
\mathbf{Z}/p^n & \xrightarrow[(1\ p)]{} & \mathbf{Z}/p^{n-1} \oplus \mathbf{Z}/p^{n+1}
\end{array}$$

We read off the congruence

$$\text{(vi)} \qquad p b_{n/1} + d_{n/1} \;\equiv_{p^{n-1}}\; 1\ .$$

By (iii) resp. (iv) we conclude from (v) resp. (vi) that

$$\text{(v}'\text{)} \qquad (1 - p^{n-2}) d_{n/1} \;\equiv_{p^{n-1}}\; 1$$
$$\text{(vi}'\text{)} \qquad d_{n/1} \;\equiv_{p^{n-1}}\; 1\ .$$

Substituting (vi′) into (v′), we obtain

$$1 - p^{n-2} \;\equiv_{p^{n-1}}\; 1\ ,$$

which is *absurd.* □

III.3 Nonisomorphic Verdier octahedra

Since in §III.2, the category $\mathcal{C}$ is also a Verdier triangulated category, specialising to $n = 3$ yields two nonisomorphic Verdier octahedra on the same base. In this particular case, we shall now give a somewhat longer argument alternative to that given in §III.2 that is independent of Chapter I, whose techniques might not be familiar to all readers. Nonetheless, §III.3 is a particular case of §III.2.

Let $\mathcal{C} := \underline{(\mathbf{Z}/p^6)\text{-mod}}$, and let it be endowed with a shift automorphism as in §III.1.2 and a Verdier triangulation as in §III.1.4.

Let the diagram X be given by

$$
\begin{array}{ccccccc}
 & & & & & & \mathbf{Z}/p^3 \\
 & & & & & & \uparrow{\scriptstyle p} \\
 & & & & \mathbf{Z}/p^1 \oplus \mathbf{Z}/p^5 & \xrightarrow{\left(\begin{smallmatrix} -p^2 \\ 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^3 \\
 & & & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)} & & \uparrow{\scriptstyle p} \\
 & & \mathbf{Z}/p^1 \oplus \mathbf{Z}/p^5 & \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^2 \oplus \mathbf{Z}/p^4 & \xrightarrow{\left(\begin{smallmatrix} -p \\ 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^3 \\
 & & \uparrow{\scriptstyle (1\ p^2)} & & \uparrow{\scriptstyle (1\ p)} & & \\
\mathbf{Z}/p^3 & \xrightarrow{p} & \mathbf{Z}/p^3 & \xrightarrow{p} & \mathbf{Z}/p^3 & & .
\end{array}
$$

Let the diagram $\tilde{X}$ be given by

$$
\begin{array}{ccccccc}
 & & & & & & \mathbf{Z}/p^3 \\
 & & & & & & \uparrow{\scriptstyle p} \\
 & & & & \mathbf{Z}/p^1 \oplus \mathbf{Z}/p^5 & \xrightarrow{\left(\begin{smallmatrix} -p^2 \\ 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^3 \\
 & & & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ -1 & p \end{smallmatrix}\right)} & & \uparrow{\scriptstyle p} \\
 & & \mathbf{Z}/p^1 \oplus \mathbf{Z}/p^5 & \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 1 & 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^2 \oplus \mathbf{Z}/p^4 & \xrightarrow{\left(\begin{smallmatrix} -p \\ 1 \end{smallmatrix}\right)} & \mathbf{Z}/p^3 \\
 & & \uparrow{\scriptstyle (1\ p^2)} & & \uparrow{\scriptstyle (1\ p)} & & \\
\mathbf{Z}/p^3 & \xrightarrow{p} & \mathbf{Z}/p^3 & \xrightarrow{p} & \mathbf{Z}/p^3 & & .
\end{array}
$$

Lemma III.5 *Both X and $\tilde{X}$ are Verdier octahedra.*

In contrast to the procedure in §III.2, to prove this, we will not make use of the folding operation.

Proof. For the periodic monotone map $\bar{\Delta}_3 \xleftarrow{\mathrm{d}_3} \bar{\Delta}_2$ that maps $0 \leftarrow\!\shortmid 0$, $1 \leftarrow\!\shortmid 1$ and $2 \leftarrow\!\shortmid 2$, we obtain $X\mathrm{d}_3^\# = \tilde{X}\mathrm{d}_3^\#$, horizontally displayed as

$$
\mathbf{Z}/p^3 \xrightarrow{p} \mathbf{Z}/p^3 \xrightarrow{(1\ p^2)} \mathbf{Z}/p \oplus \mathbf{Z}/p^5 \xrightarrow{\left(\begin{smallmatrix} -p^2 \\ 1 \end{smallmatrix}\right)} \mathbf{Z}/p^3 \ .
$$

The following morphism of short exact sequences in $(\mathbf{Z}/p^6)$-mod shows $X\mathrm{d}_3^{\#}$ to be a distinguished triangle.

$$\begin{array}{ccccc}
\mathbf{Z}/p^3 & \xrightarrow{(1\ p^2)} & \mathbf{Z}/p\oplus\mathbf{Z}/p^5 & \xrightarrow{\binom{-p^2}{1}} & \mathbf{Z}/p^3 \\
\uparrow{\scriptstyle p} & & \uparrow{\scriptstyle (0\ 1)} & & \| \\
\mathbf{Z}/p^3 & \xrightarrow{p^3} & \mathbf{Z}/p^6 & \xrightarrow{1} & \mathbf{Z}/p^3
\end{array}$$

For the periodic monotone map $\bar{\Delta}_3 \xleftarrow{\mathrm{d}_1} \bar{\Delta}_2$ that maps $0 \leftarrow\!\shortmid 0$, $2 \leftarrow\!\shortmid 1$ and $3 \leftarrow\!\shortmid 2$, we obtain the distinguished triangle $X\mathrm{d}_1^{\#} = \tilde{X}\mathrm{d}_1^{\#} = X\mathrm{d}_3^{\#}$ again.

For the periodic monotone map $\bar{\Delta}_3 \xleftarrow{\mathrm{d}_2} \bar{\Delta}_2$ that maps $0 \leftarrow\!\shortmid 0$, $1 \leftarrow\!\shortmid 1$ and $3 \leftarrow\!\shortmid 2$, we obtain the diagram $X\mathrm{d}_2^{\#} = \tilde{X}\mathrm{d}_2^{\#}$, horizontally displayed as

$$\mathbf{Z}/p^3 \xrightarrow{p^2} \mathbf{Z}/p^3 \xrightarrow{(1\ p)} \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 \xrightarrow{\binom{-p}{1}} \mathbf{Z}/p^3 \ .$$

The following morphism of short exact sequences in $(\mathbf{Z}/p^6)$-mod shows $X\mathrm{d}_2^{\#}$ to be a distinguished triangle.

$$\begin{array}{ccccc}
\mathbf{Z}/p^3 & \xrightarrow{(1\ p)} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\binom{-p}{1}} & \mathbf{Z}/p^3 \\
\uparrow{\scriptstyle p^2} & & \uparrow{\scriptstyle (0\ 1)} & & \| \\
\mathbf{Z}/p^3 & \xrightarrow{p^3} & \mathbf{Z}/p^6 & \xrightarrow{1} & \mathbf{Z}/p^3
\end{array}$$

For the periodic monotone map $\bar{\Delta}_3 \xleftarrow{\mathrm{d}_0} \bar{\Delta}_2$ that maps $1 \leftarrow\!\shortmid 0$, $2 \leftarrow\!\shortmid 1$ and $3 \leftarrow\!\shortmid 2$, we obtain the periodic isomorphism $X\mathrm{d}_0^{\#} \xrightarrow{\sim} \tilde{X}\mathrm{d}_0^{\#}$, horizontally displayed as

$$\begin{array}{ccccccc}
\mathbf{Z}/p\oplus\mathbf{Z}/p^5 & \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} & \mathbf{Z}/p\oplus\mathbf{Z}/p^5 & \xrightarrow{\begin{pmatrix} 0 & -p^4 \\ 1 & 0 \end{pmatrix}} & \mathbf{Z}/p\oplus\mathbf{Z}/p^5 \\
\| & & \wr\downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} & & \| & & \| \\
\mathbf{Z}/p\oplus\mathbf{Z}/p^5 & \xrightarrow{\begin{pmatrix} p & 0 \\ 1 & 1 \end{pmatrix}} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ -1 & p \end{pmatrix}} & \mathbf{Z}/p\oplus\mathbf{Z}/p^5 & \xrightarrow{\begin{pmatrix} 0 & -p^4 \\ 1 & 0 \end{pmatrix}} & \mathbf{Z}/p\oplus\mathbf{Z}/p^5 \ .
\end{array}$$

So we are reduced to show that $X\mathrm{d}_0^{\#}$ is a distinguished triangle, which it is as a direct sum of two distinguished triangles, as the following morphisms of short exact sequences in $(\mathbf{Z}/p^6)$-mod show.

$$\begin{array}{ccccc}
\mathbf{Z}/p & \xrightarrow{p} & \mathbf{Z}/p^2 & \xrightarrow{1} & \mathbf{Z}/p \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \| \\
\mathbf{Z}/p^5 & \xrightarrow{p} & \mathbf{Z}/p^6 & \xrightarrow{1} & \mathbf{Z}/p
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\mathbf{Z}/p^4 & \xrightarrow{p} & \mathbf{Z}/p^5 & \xrightarrow{1} & \mathbf{Z}/p \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \| \\
\mathbf{Z}/p^5 & \xrightarrow{p} & \mathbf{Z}/p^6 & \xrightarrow{1} & \mathbf{Z}/p
\end{array}$$

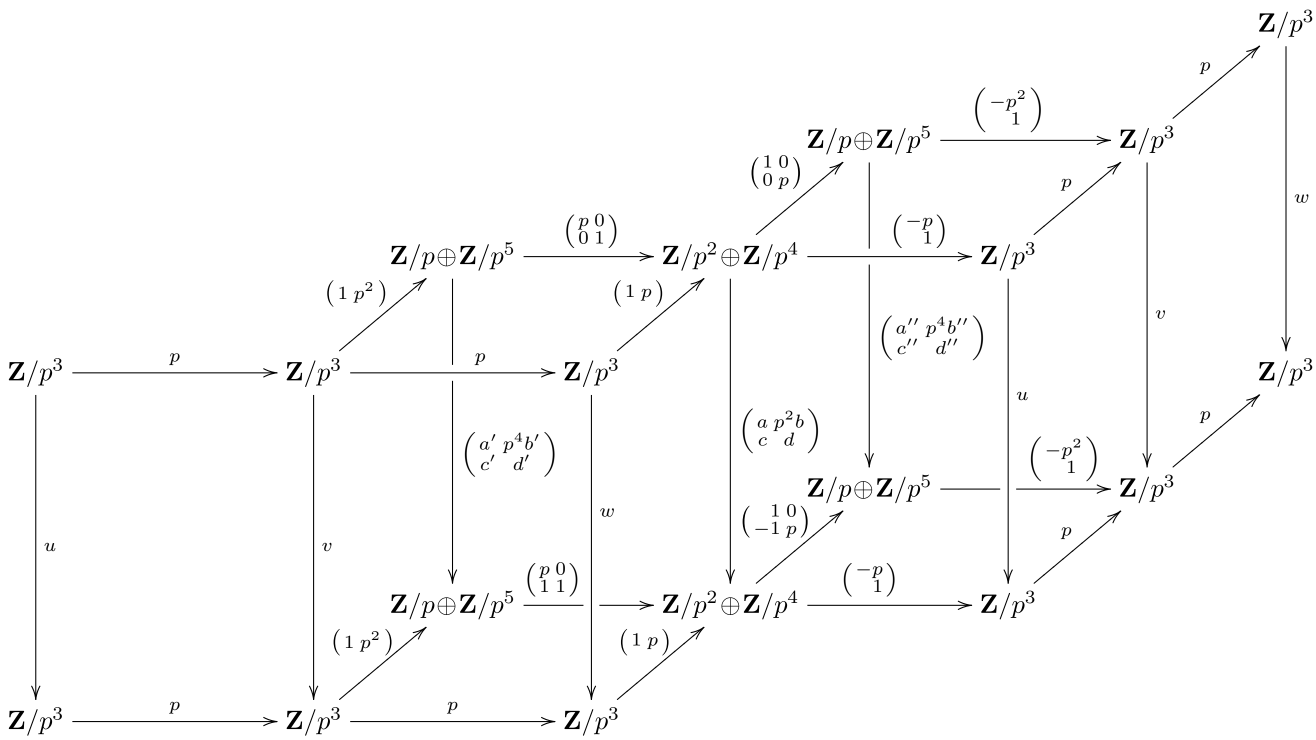
□

Lemma III.6 *The Verdier octahedra X and $\tilde{X}$ are not isomorphic in $\mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_3^{\#})$.*

That is, there is no isomorphism between the displayed parts of X resp. of $\tilde{X}$ such that its entries on the rightmost vertical column arise by an application of the shift functor of $\mathcal{C}$ to its entries on the lower row.

We will not use the fact that X is a 3-triangle, which in conjunction with Lem. I.21.(4,6) would permit us to restrict ourselves to consider isomorphisms that are identical on the lower row and the rightmost vertical column, as we did in Lemma III.4.

Proof. We *assume* the contrary and depict an isomorphism $X \xrightarrow{\sim} \tilde{X}$ as follows.



Note that all vertical quadrangles commute in $\mathcal{C}$.

The commutative quadrangles on $1/0 \longrightarrow 2/0 \longrightarrow 3/0$ yield $u \equiv_{p^2} v \equiv_{p^2} w$.

The commutative quadrangle on $3/0 \longrightarrow 3/1$ yields $pb + d \equiv_{p^2} w$.

The commutative quadrangle on $3/1 \longrightarrow 0^{+1}/1$ yields $-pc + d \equiv_{p^2} u$.

The commutative quadrangle on $3/1 \longrightarrow 3/2$ yields $b \equiv_p 0$ and $c \equiv_p d$.

Altogether, we have

$$u \;\equiv_{p^2}\; w \;\equiv_{p^2}\; pb + d \;\equiv_{p^2}\; d \;\equiv_{p^2}\; u + pc \;\equiv_{p^2}\; u + pd\;,$$

whence

$$0 \;\equiv_p\; d \;\equiv_p\; w\;.$$

Since $\mathbf{Z}/p^3 \xrightarrow{w} \mathbf{Z}/p^3$ is an isomorphism in $\mathcal{C}$, we have $w \not\equiv_p 0$. This is *absurd.* ▫

In [**8,** 1.1.13], it is described how an octahedron gives rise to two "extra" triangles. As cone of the diagonal of a quadrangle appearing in that octahedron, we take the direct sum of the non-diagonal terms of the subsequent quadrangle, the morphisms being taken from the octahedron, with one minus sign inserted to ensure that the composition of two morphisms in the constructed triangle vanishes.

Remark III.7 *The triangles arising from X and from $\tilde{X}$ as described in* [**8,** 1.1.13] *are distinguished.*

Proof. ([1]) The morphism of short exact sequences in $(\mathbf{Z}/p^6)$-mod

$$\begin{array}{ccccc}
\mathbf{Z}/p^3 & \xrightarrow[\bullet]{\begin{pmatrix}1 & p^2 & -p\end{pmatrix}} & \mathbf{Z}/p \oplus \mathbf{Z}/p^5 \oplus \mathbf{Z}/p^3 & \xrightarrow[|]{\begin{pmatrix}p & 0\\ 0 & 1\\ 1 & p\end{pmatrix}} & \mathbf{Z}/p^2 \oplus \mathbf{Z}/p^4 \\
\uparrow {\scriptstyle \begin{pmatrix}-p\\ p^2\end{pmatrix}} & & \uparrow {\scriptstyle \begin{pmatrix}0 & -p & 1\\ 0 & 1 & 0\end{pmatrix}} & & \| \\
\mathbf{Z}/p^4 \oplus \mathbf{Z}/p^2 & \xrightarrow[\bullet]{\begin{pmatrix}p^2 & 0\\ 0 & p^4\end{pmatrix}} & \mathbf{Z}/p^6 \oplus \mathbf{Z}/p^6 & \xrightarrow[|]{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} & \mathbf{Z}/p^2 \oplus \mathbf{Z}/p^4
\end{array}$$

and the isomorphism of diagrams with coefficients in $\mathcal{C}$

$$\begin{array}{ccccccc}
\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}1 & p^2 & -p\end{pmatrix}} & \mathbf{Z}/p\oplus\mathbf{Z}/p^5\oplus\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}p & 0\\ 0 & 1\\ 1 & p\end{pmatrix}} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\begin{pmatrix}-p^2\\ p\end{pmatrix}} & \mathbf{Z}/p^3 \\
\| & & \| & & \wr\downarrow {\scriptstyle \begin{pmatrix}1-p & 0\\ 1 & 1\end{pmatrix}} & & \| \\
\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}1 & p^2 & -p\end{pmatrix}} & (\mathbf{Z}/p\oplus\mathbf{Z}/p^5)\oplus\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}p & 0\\ 1 & 1\\ 1 & p\end{pmatrix}} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\begin{pmatrix}-p^2\\ p\end{pmatrix}} & \mathbf{Z}/p^3
\end{array}$$

show one of the triangles mentioned in loc. cit. to be distinguished in X and in $\tilde{X}$.

The morphism of short exact sequences in $(\mathbf{Z}/p^6)$-mod

$$\begin{array}{ccccc}
\mathbf{Z}/p^2 \oplus \mathbf{Z}/p^4 & \xrightarrow[\bullet]{\begin{pmatrix}1 & 0 & p\\ 0 & p & -1\end{pmatrix}} & \mathbf{Z}/p \oplus \mathbf{Z}/p^5 \oplus \mathbf{Z}/p^3 & \xrightarrow[|]{\begin{pmatrix}-p^2\\ 1\\ p\end{pmatrix}} & \mathbf{Z}/p^3 \\
\uparrow {\scriptstyle \begin{pmatrix}p & p^2\end{pmatrix}} & & \uparrow {\scriptstyle \begin{pmatrix}0 & 1 & 0\end{pmatrix}} & & \| \\
\mathbf{Z}/p^3 & \xrightarrow[\bullet]{p^3} & \mathbf{Z}/p^6 & \xrightarrow[|]{1} & \mathbf{Z}/p^3
\end{array}$$

and the isomorphism of diagrams with coefficients in $\mathcal{C}$

$$\begin{array}{ccccccc}
\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}p & p^2\end{pmatrix}} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\begin{pmatrix}1 & 0 & p\\ 0 & p & -1\end{pmatrix}} & (\mathbf{Z}/p\oplus\mathbf{Z}/p^5)\oplus\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}-p^2\\ 1\\ p\end{pmatrix}} & \mathbf{Z}/p^3 \\
\| & & \| & & \wr\downarrow {\scriptstyle \begin{pmatrix}1 & 0 & -p^2\\ 0 & 1 & 1\\ 1 & 0 & 1+p\end{pmatrix}} & & \| \\
\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}p & p^2\end{pmatrix}} & \mathbf{Z}/p^2\oplus\mathbf{Z}/p^4 & \xrightarrow{\begin{pmatrix}1 & 0 & p\\ -1 & p & -1\end{pmatrix}} & (\mathbf{Z}/p\oplus\mathbf{Z}/p^5)\oplus\mathbf{Z}/p^3 & \xrightarrow{\begin{pmatrix}-p^2\\ 1\\ p\end{pmatrix}} & \mathbf{Z}/p^3
\end{array}$$

show the other of the triangles mentioned in loc. cit. to be distinguished in X and in $\tilde{X}$. □

III.4 Transport of structure

We use the notation of §I.1, §I.2.

III.4.1 Transport of a Heller triangulation

Concerning weakly abelian categories, see e.g. §I.6.6.3. Recall that an additive functor between weakly abelian categories is called subexact if it induces an exact functor on the Freyd categories; cf. §I.1.2.1.3. For instance, an equivalence is subexact.

[1]Strictly speaking, we should reorder summands in the diagrams that follow; cf. §III.1.1. But then the proof would be more difficult to read.

Setup III.8

> Suppose given a Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$; cf. Def. I.5. Suppose given a weakly abelian category $\mathcal{C}'$ and an automorphism T' on $\mathcal{C}'$, called *shift*; cf. Def. I.66.
>
> Assume given subexact functors $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and $\mathcal{C}' \xrightarrow{G} \mathcal{C}$, and isotransformations $1_{\mathcal{C}'} \underset{\sim}{\xrightarrow{\varepsilon}} GF$ and $\mathsf{T}' G \underset{\sim}{\xrightarrow{\sigma}} G\mathsf{T}$.

Suppose given $n \geqslant 0$. By abuse of notation, we write $F := \underline{F^+(\bar{\Delta}_n^\#)} : \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)} \longrightarrow \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$ for the functor obtained by pointwise application of F.

Similarly, we write $\varepsilon := \underline{\varepsilon^+(\bar{\Delta}_n^\#)} : \underline{(1_{\mathcal{C}'})^+(\bar{\Delta}_n^\#)} \xrightarrow{\sim} \underline{(GF)^+(\bar{\Delta}_n^\#)}$ for the isotransformation obtained by pointwise application of ε.

More generally speaking, for notational convenience, induced functors of type $\underline{A^+(\bar{\Delta}_n^\#)}$ will often be abbreviated by A, and induced transformations of type $\underline{\alpha^+(\bar{\Delta}_n^\#)}$ will often be abbreviated by α. For instance, given $X \in \operatorname{Ob} \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$, we will allow ourselves to write $X\,\mathsf{T} = X\underline{\mathsf{T}^+(\bar{\Delta}_n^\#)}\ (= [X^{+1}])$.

Given $X' \in \operatorname{Ob} \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)} = \operatorname{Ob} \mathcal{C}'^+(\bar{\Delta}_n^\#)$, we define the isomorphism $[X']^{+1} \underset{\sim}{\xrightarrow{X'\vartheta'_n}} [X'^{+1}]$ in $\underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$ by the following commutative diagram.

$$
\begin{array}{ccc}
[X']^{+1} & \underset{\sim}{\xrightarrow{X'\vartheta'_n}} & [X'^{+1}] \\
{\scriptstyle [X'\varepsilon]^{+1}} \downarrow \wr & & \wr \downarrow {\scriptstyle [X'^{+1}]\varepsilon} \\
[X'GF]^{+1} & & [X'^{+1}]GF \\
\Vert & & {\scriptstyle X'\sigma F} \downarrow \wr \\
[X'G]^{+1}F & \underset{\sim}{\xrightarrow{X'G\vartheta_n F}} & [(X'G)^{+1}]F
\end{array}
$$

In other words, we let

$$X'\vartheta'_n \ := \ ([X'\varepsilon]^{+1})(X'G\vartheta_n F)(X'\sigma^- F)([X'^{+1}]\varepsilon^-) \, .$$

As a composite of isotransformations, $(X'\vartheta'_n)_{X' \in \operatorname{Ob} \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}}$ is an isotransformation. Let $\vartheta' := (\vartheta'_n)_{n \geqslant 0}$.

Lemma III.9 *The triple $(\mathcal{C}', \mathsf{T}', \vartheta')$ is a Heller triangulated category.*

Cf. Def. I.5. We will say that ϑ' is *transported from $(\mathcal{C}, \mathsf{T}, \vartheta)$ via F and G.* Strictly speaking, we should mention ε and σ here as well.

Proof. Suppose given $m, n \geqslant 0$, a periodic monotone map $\bar{\Delta}_n \xleftarrow{q} \bar{\Delta}_m$ and $X' \in \operatorname{Ob} \underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$. We *claim* that $X'\underline{q}^\#\vartheta'_m = X'\vartheta'_n\underline{q}^\#$. We have

$$
\begin{array}{rcl}
X'\underline{q}^\#\vartheta'_m & = & ([X'\underline{q}^\#\varepsilon]^{+1})(X'\underline{q}^\# G\vartheta_m F)(X'\underline{q}^\#\sigma^- F)([X'\underline{q}^{\#}{}^{+1}]\varepsilon^-) \, , \\
X'\vartheta'_n\underline{q}^\# & = & ([X'\varepsilon]^{+1}\underline{q}^\#)(X'G\vartheta_n F\underline{q}^\#)(X'\sigma^- F\underline{q}^\#)([X'^{+1}]\varepsilon^-\underline{q}^\#) \, .
\end{array}
$$

By respective pointwise definition, we have $[X'\underline{q}^\#\varepsilon]^{+1} = [X'\varepsilon]^{+1}\underline{q}^\#$ (using that q is periodic), $X'\underline{q}^\#\sigma^- F = X'\sigma^- F\underline{q}^\#$ and $[X'\underline{q}^{\#}{}^{+1}]\varepsilon^- = [X'^{+1}]\varepsilon^-\underline{q}^\#$. Moreover, since $(\mathcal{C}, \mathsf{T}, \vartheta)$ is Heller triangulated, we get

$$X'\underline{q}^\# G\vartheta_m F \ = \ X'G\underline{q}^\#\vartheta_m F \ = \ X'G\vartheta_n\underline{q}^\# F \ = \ X'G\vartheta_n F\underline{q}^\# \ .$$

This proves the *claim.*

Suppose given $n \geqslant 0$ and $X' \in \operatorname{Ob} \underline{\mathcal{C}'^+(\bar{\Delta}_{2n+1}^\#)}$. We *claim* that $X'\underline{\mathfrak{f}}_n\vartheta'_{n+1} = X'\vartheta'_{2n+1}\underline{\mathfrak{f}}_n$. We have

$$
\begin{array}{rcl}
X'\underline{\mathfrak{f}}_n\vartheta'_{n+1} & = & ([X'\underline{\mathfrak{f}}_n\varepsilon]^{+1})(X'\underline{\mathfrak{f}}_n G\vartheta_{n+1} F)(X'\underline{\mathfrak{f}}_n\sigma^- F)([X'\underline{\mathfrak{f}}_n{}^{+1}]\varepsilon^-) \, , \\
X'\vartheta'_{2n+1}\underline{\mathfrak{f}}_n & = & ([X'\varepsilon]^{+1}\underline{\mathfrak{f}}_n)(X'G\vartheta_{2n+1}F\underline{\mathfrak{f}}_n)(X'\sigma^- F\underline{\mathfrak{f}}_n)([X'^{+1}]\varepsilon^-\underline{\mathfrak{f}}_n) \, .
\end{array}
$$

By additivity of F, G and T' and by respective pointwise definition, we have $[X'\mathfrak{f}_{\underline{n}}\varepsilon]^{+1} = [X'\varepsilon]^{+1}\mathfrak{f}_{\underline{n}}$ (using shiftcompatibility of $\mathfrak{f}_{\underline{n}}$), $X'\mathfrak{f}_{\underline{n}}\sigma^- F = X'\sigma^- F\mathfrak{f}_{\underline{n}}$ and $[X'\mathfrak{f}_{\underline{n}}{}^{+1}]\varepsilon^- = [X'^{+1}]\varepsilon^-\mathfrak{f}_{\underline{n}}$. Moreover, since $(\mathcal{C}, \mathsf{T}, \vartheta)$ is Heller triangulated, we get

$$X'\mathfrak{f}_{\underline{n}} G\vartheta_{n+1}F \;=\; X'G\mathfrak{f}_{\underline{n}}\vartheta_{n+1}F \;=\; X'G\vartheta_{2n+1}\mathfrak{f}_{\underline{n}}F \;=\; X'G\vartheta_{2n+1}F\mathfrak{f}_{\underline{n}} \,.$$

This proves the *claim.* ▫

III.4.2 Detecting n-triangles

Setup III.10

Suppose given a Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$; cf. Def. I.5. Suppose given an additive category $\mathcal{C}'$ and an automorphism T' on $\mathcal{C}'$, called *shift.*

Suppose given mutually inverse equivalences $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and $\mathcal{C}' \xrightarrow{G} \mathcal{C}$. Note that $G \dashv F$, whence there exist isotransformations $1_{\mathcal{C}'} \xrightarrow[\sim]{\varepsilon} GF$ and $FG \xrightarrow[\sim]{\eta} 1_{\mathcal{C}}$ such that both $(F\varepsilon)(\eta F) = 1_F$ and $(\varepsilon G)(G\eta) = 1_G$ hold. We fix such ε and η.

Suppose given an isotransformation $\mathsf{T}'\, G \xrightarrow[\sim]{\sigma} G\,\mathsf{T}$.

Note that $\mathcal{C}'$ is weakly abelian, being equivalent to the weakly abelian category $\mathcal{C}$.

Let ϑ' be transported from $(\mathcal{C}, \mathsf{T}, \vartheta)$ via F and G.

That is, we let $X'\vartheta'_n \;:=\; ([X'\varepsilon]^{+1})(X'G\vartheta_n F)(X'\sigma^- F)([X'^{+1}]\varepsilon^-)$ for $n \geqslant 0$ and $X' \in \operatorname{Ob} \underline{\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})}$, defining $\vartheta' := (\vartheta'_n)_{n\geqslant 0}$.

By Lemma III.9, the triple $(\mathcal{C}', \mathsf{T}', \vartheta')$ is a Heller triangulated category.

Moreover, let

$$(\mathsf{T}\,F \xrightarrow[\sim]{\rho} F\,\mathsf{T}') \;=\; (\mathsf{T}\,F \xrightarrow[\sim]{\eta^-\,\mathsf{T}\,F} FG\,\mathsf{T}\,F \xrightarrow[\sim]{F\sigma^- F} F\,\mathsf{T}'\,GF \xrightarrow[\sim]{F\,\mathsf{T}'\,\varepsilon^-} F\,\mathsf{T}')\,.$$

Notation III.11 Suppose given $n \geqslant 0$. Concerning the full subposet

$$\bar{\Delta}_n^{\vartriangle\triangledown} \;=\; \{\beta/\alpha \in \Delta_n^{\#} : 0 \leqslant \alpha \leqslant \beta \leqslant 0^{+1}\} \;\subseteq\; \bar{\Delta}_n^{\#}\,,$$

cf. §I.2.5.1.

(1) Suppose given $X' \in \operatorname{Ob} \mathcal{C}'^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$, where periodic means $[X']^{+1} = [X'^{+1}]$; cf. §I.2.5.1. Consider the diagram $X'G|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$. Denote by $X'G|^{\sigma}_{\bar{\Delta}_n^{\vartriangle\triangledown}} \in \operatorname{Ob} \mathcal{C}'^{+}(\bar{\Delta}_n^{\#})$ the diagram $X'G|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$ with $(X'G)_{0^{+1}/i} = X'_{0^{+1}/i}G = X'_{i/0}\,\mathsf{T}'\,G$ isomorphically replaced via $X'_{i/0}\sigma$ by $X'_{i/0}G\,\mathsf{T}$. Denote by $X'G^{\sigma} \in \operatorname{Ob} \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ its periodic prolongation, characterised by $X'G^{\sigma}|_{\bar{\Delta}_n^{\vartriangle\triangledown}} = X'G|^{\sigma}_{\bar{\Delta}_n^{\vartriangle\triangledown}}$; cf. §I.2.5.1. Using, for $k \geqslant 0$,

$$\Big([X'G]^{+k} \xrightarrow{\sim} [X'G^{+1}]^{+(k-1)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} [X'G^{+k}]\Big)\Big|_{\bar{\Delta}_n^{\vartriangle\triangledown}}\,,$$

given by

$$X'_{(j/i)^{+k}}G \;=\; X'_{j/i}\,\mathsf{T}'^{k}\,G\,\mathsf{T}^0 \xrightarrow[\sim]{\mathsf{T}'^{k-1}\,\sigma\,\mathsf{T}^0} X'_{j/i}\,\mathsf{T}'^{k-1}\,G\,\mathsf{T}^1 \xrightarrow[\sim]{\mathsf{T}'^{k-2}\,\sigma\,\mathsf{T}^1} \cdots \xrightarrow[\sim]{\mathsf{T}'^{0}\,\sigma\,\mathsf{T}^{k-1}} X'_{j/i}\,\mathsf{T}'^{0}\,G\,\mathsf{T}^k$$

at j/i for $0 \leqslant i \leqslant j \leqslant 0^{+1}$, and similarly for $k \leqslant 0$, using $\mathsf{T}'^{-}\,G \xrightarrow[\sim]{\mathsf{T}'^{-}\,\sigma^-\,\mathsf{T}^-} G\,\mathsf{T}^-$, we obtain an isomorphism $X'G \xrightarrow[\sim]{\varphi} X'G^{\sigma}$ in $\mathcal{C}^{+}(\bar{\Delta}_n^{\#})$ such that $\varphi_{i/0} = 1_{X'_{i/0}G}$ and $\varphi_{0^{+1}/i} = X'_{i/0}\,\sigma$ for $i \in [1, n]$.

(2) Suppose given $X \in \operatorname{Ob} \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$. Consider the diagram $XF|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$. Denote by $XF|^{\rho}_{\bar{\Delta}_n^{\vartriangle\triangledown}} \in \operatorname{Ob} \mathcal{C}^{+}(\bar{\Delta}_n^{\#})$ the diagram $XF|_{\bar{\Delta}_n^{\vartriangle\triangledown}}$ with $(XF)_{0^{+1}/i} = X_{0^{+1}/i}F = X_{i/0}\,\mathsf{T}\,F$ isomorphically replaced via $X_{i/0}\,\rho$ by $X_{i/0}F\,\mathsf{T}'$. Denote by $XF^{\rho} \in \operatorname{Ob} \mathcal{C}'^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$ its periodic prolongation, characterised by $XF^{\rho}|_{\bar{\Delta}_n^{\vartriangle\triangledown}} = XF|^{\rho}_{\bar{\Delta}_n^{\vartriangle\triangledown}}$; cf. §I.2.5.1.

Similarly as in (1), we have an isomorphism $XF \xrightarrow[\sim]{\psi} XF^{\rho}$ in $\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})$ such that $\psi_{i/0} = 1_{X_{i/0}F}$ and $\psi_{0^{+1}/i} = X_{i/0}\,\rho$ for $i \in [1, n]$.

Lemma III.12 *Suppose given $n \geqslant 0$.*

(1) *Suppose given $X' \in \operatorname{Ob} \mathcal{C}'^{+,\,\text{periodic}}(\bar{\Delta}_n^\#)$. Then X' is an n-triangle if and only if $X'G^\sigma$ is an n-triangle.*

(2) *Suppose given $X \in \operatorname{Ob} \mathcal{C}^{+,\,\text{periodic}}(\bar{\Delta}_n^\#)$. Then X is an n-triangle if and only if XF^ρ is an n-triangle.*

Cf. Def. I.5.(ii.2).

Proof. Ad (1). Since ϑ_n is a transformation, there exists a commutative quadrangle

$$\begin{array}{ccc} [X'G^\sigma]^{+1} & \xrightarrow[\sim]{X'G^\sigma\vartheta_n} & [(X'G^\sigma)^{+1}] \\ \wr\uparrow{\scriptstyle [\varphi]^{+1}} & & \wr\uparrow{\scriptstyle [\varphi^{+1}]} \\ [X'G]^{+1} & \xrightarrow[\sim]{X'G\vartheta_n} & [(X'G)^{+1}] \end{array}$$

in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$. Therefore, $X'G^\sigma$ is an n-triangle if and only if $[\varphi]^{+1} = (X'G\vartheta_n)[\varphi^{+1}]$. By Prop. I.12, this equation is equivalent to $[\varphi]^{+1}|_{\dot{\Delta}_n} = (X'G\vartheta_n)|_{\dot{\Delta}_n}[\varphi^{+1}]|_{\dot{\Delta}_n}$; cf. §I.2.1.1; in other words, to

$$X'\sigma|_{\dot{\Delta}_n} \;=\; X'G\vartheta_n|_{\dot{\Delta}_n}$$

as morphisms from $X'\mathsf{T}'G|_{\dot{\Delta}_n}$ to $X'G\mathsf{T}|_{\dot{\Delta}_n}$ in $\underline{\mathcal{C}(\dot{\Delta}_n)}$. This, in turn, is equivalent to $X'\sigma \;=\; X'G\vartheta_n$ as morphisms from $X'\mathsf{T}'G$ to $X'G\mathsf{T}$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ by Prop. I.12.

Now X' being an n-triangle is equivalent to $X'\vartheta'_n = 1$; i.e. to

$$([X'\varepsilon]^{+1})(X'G\vartheta_n F)(X'\sigma^- F)([X'^{+1}]\varepsilon^-) \;=\; 1\,.$$

Since $[X']^{+1} \;=\; [X'^{+1}]$, we have $[X'\varepsilon]^{+1} \;=\; [X']^{+1}\varepsilon \;=\; [X'^{+1}]\varepsilon$, whence this equation is equivalent to $(X'G\vartheta_n F)(X'\sigma^- F) = 1$. Since F is an equivalence, this amounts to $(X'G\vartheta_n)(X'\sigma^-) = 1$, as was to be shown.

Ad (2). Since ϑ'_n is a transformation, there exists a commutative quadrangle

$$\begin{array}{ccc} [XF^\rho]^{+1} & \xrightarrow[\sim]{XF^\rho\vartheta'_n} & [(XF^\rho)^{+1}] \\ \wr\uparrow{\scriptstyle [\psi]^{+1}} & & \wr\uparrow{\scriptstyle [\psi^{+1}]} \\ [XF]^{+1} & \xrightarrow[\sim]{XF\vartheta'_n} & [(XF)^{+1}] \end{array}$$

in $\underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$. Therefore, XF^ρ is an n-triangle if and only if $[\psi]^{+1} = (XF\vartheta'_n)[\psi^{+1}]$. By Prop. I.12, this equation is equivalent to $[\psi]^{+1}|_{\dot{\Delta}_n} = (XF\vartheta'_n)|_{\dot{\Delta}_n}[\psi^{+1}]|_{\dot{\Delta}_n}$; in other words, to

$$X\rho|_{\dot{\Delta}_n} \;=\; XF\vartheta'_n|_{\dot{\Delta}_n}$$

as morphisms from $X\mathsf{T}F|_{\dot{\Delta}_n}$ to $XF\mathsf{T}'|_{\dot{\Delta}_n}$ in $\underline{\mathcal{C}'(\dot{\Delta}_n)}$. This, in turn, is equivalent to $X\rho \;=\; XF\vartheta'_n$ as morphisms from $X\mathsf{T}F$ to $XF\mathsf{T}'$ in $\underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$ by Prop. I.12. Which amounts to

$$(X\eta^-\mathsf{T}F)(XF\sigma^-F)(XF\mathsf{T}'\varepsilon^-) \;=\; ([XF\varepsilon]^{+1})(XFG\vartheta_n F)(XF\sigma^-F)([XF^{+1}]\varepsilon^-)\,;$$

i.e. to

$$X\eta^-\mathsf{T}F \;=\; ([X\eta^-F]^{+1})(XFG\vartheta_n F)\,.$$

Since $[X\eta^-F]^{+1} = [X\eta^-]^{+1}F$ and since ϑ_n is a transformation, the right hand side equals $(X\vartheta_n F)([(X\eta^-)^{+1}]F)$, and therefore we can continue the string of equivalent assertions with

$$X\eta^-\mathsf{T}F \;=\; (X\vartheta_n F)([(X\eta^-)^{+1}]F)\,;$$

i.e. with $X\vartheta_n F = 1$; i.e. with $X\vartheta_n = 1$; i.e. with X being an n-triangle. □

Definition III.13 A Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is said to be *closed* if every morphism $X \xrightarrow{f} Y$ therein can be completed to a 2-triangle; i.e. if for all morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$, there exists $U \in \operatorname{Ob} \mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_2^\#)$ with $(X \xrightarrow{f} Y) = (U_{1/0} \xrightarrow{u} U_{2/0})$. If this is the case, then also the Heller triangulation ϑ is called *closed*.

For instance, a Heller triangulated category whose idempotents split is closed; cf. Prop. I.23. Cf. §II.2.4.

Recall that ϑ' is transported from $(\mathcal{C}, \mathsf{T}, \vartheta)$ via F and G.

Lemma III.14 *If $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a closed Heller triangulated category, then $(\mathcal{C}', \mathsf{T}', \vartheta')$ is a closed Heller triangulated category.*

Proof. By Lemma III.9, it remains to prove closedness of $(\mathcal{C}', \mathsf{T}', \vartheta')$. Suppose given $X' \xrightarrow{u'} Y'$ in $\mathcal{C}'$. We have to prove that it can be prolonged to a 2-triangle. Using closedness of $(\mathcal{C}, \mathsf{T}, \vartheta)$, we find a 2-triangle

$$X'G \xrightarrow{u'G} Y'G \xrightarrow{v} Z \xrightarrow{w} X'G\,\mathsf{T}\ .$$

We *claim* that

$$M' \;:=\; (X' \xrightarrow{u'} Y' \xrightarrow{(Y'\varepsilon)(vF)} ZF \xrightarrow{(wF)(X'\sigma^- F)(X'\,\mathsf{T}'\,\varepsilon^-)} X'\,\mathsf{T}')$$

is a 2-triangle in $\mathcal{C}'$. By Lemma III.12.(1), it suffices to show that $M'G^\sigma$ is a 2-triangle in $\mathcal{C}$. Consider the periodic isomorphism with upper row $M'G^\sigma$ and lower row a 2-triangle

$$\begin{array}{ccccccc}
X'G & \xrightarrow{u'G} & Y'G & \xrightarrow{(Y'\varepsilon G)(vFG)} & ZFG & \xrightarrow{(wFG)(X'\sigma^- FG)(X'\,\mathsf{T}'\,\varepsilon^- G)(X'\sigma)} & X'G\,\mathsf{T} \\
\| & & \| & & \uparrow \wr\, Z\eta^- & & \| \\
X'G & \xrightarrow{u'G} & Y'G & \xrightarrow{v} & Z & \xrightarrow{w} & X'G\,\mathsf{T}\ .
\end{array}$$

In fact, we have $(Y'\varepsilon G)(vFG) = (Y'G\eta^-)(vFG) = v(Z\eta^-)$ and

$$\begin{array}{rcl}
(Z\eta^-)(wFG)(X'\sigma^- FG)(X'\,\mathsf{T}'\,\varepsilon^- G)(X'\sigma) & = & w(X'G\,\mathsf{T}\,\eta^-)(X'\sigma^- FG)(X'\,\mathsf{T}'\,\varepsilon^- G)(X'\sigma) \\
& = & w(X'\sigma^-)(X'\,\mathsf{T}'\,G\eta^-)(X'\,\mathsf{T}'\,\varepsilon^- G)(X'\sigma) \\
& = & w(X'\sigma^-)(X'\sigma) \\
& = & w\ .
\end{array}$$

This shows that $M'G^\sigma$ is a 2-triangle; cf. Lem. I.21.(4). This proves the *claim.* □

Remark III.15 *Suppose given $n \geqslant 0$.*

(1) *Given $X \in \operatorname{Ob} \mathcal{C}^+(\bar{\Delta}_n^\#)$, we have $(X\vartheta_n F)(X\rho) = XF\vartheta'_n$ in $\underline{\mathcal{C}'^+(\bar{\Delta}_n^\#)}$.*

(2) *Given $X' \in \operatorname{Ob} \mathcal{C}'^+(\bar{\Delta}_n^\#)$, we have $(X'\vartheta'_n G)(X'\sigma) = X'G\vartheta_n$ in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.*

Proof. Ad (1). We have

$$\begin{array}{rcl}
(X\vartheta_n F)(X\rho) & = & (X\vartheta_n F)(X\eta^-\,\mathsf{T}\,F)(XF\sigma^- F)(XF\,\mathsf{T}'\,\varepsilon^-) \\
& = & (X\vartheta_n F)([(X\eta^-)^{+1}]F)(XF\sigma^- F)([XF^{+1}]\varepsilon^-) \\
& = & ([X\eta^-]^{+1}F)(XFG\vartheta_n F)(XF\sigma^- F)([XF^{+1}]\varepsilon^-) \\
& = & ([X\eta^- F]^{+1})(XFG\vartheta_n F)(XF\sigma^- F)([XF^{+1}]\varepsilon^-) \\
& = & ([XF\varepsilon]^{+1})(XFG\vartheta_n F)(XF\sigma^- F)([XF^{+1}]\varepsilon^-) \\
& = & XF\vartheta'_n\ .
\end{array}$$

Ad (2). We have

$$\begin{array}{rcl}
(X'\vartheta'_n G)(X'\sigma) & = & ([X'\varepsilon]^{+1}G)(X'G\vartheta_n FG)(X'\sigma^- FG)([X'^{+1}]\varepsilon^- G)(X'\sigma) \\
& = & ([X'\varepsilon]^{+1}G)(X'G\vartheta_n FG)(X'\sigma^- FG)([X'^{+1}]G\eta)(X'\sigma) \\
& = & ([X'\varepsilon]^{+1}G)(X'G\vartheta_n FG)([X'G^{+1}]\eta)(X'\sigma^-)(X'\sigma) \\
& = & ([X'\varepsilon G]^{+1})(X'G\vartheta_n FG)([X'G^{+1}]\eta) \\
& = & ([X'G\eta^-]^{+1})(X'G\vartheta_n FG)([X'G^{+1}]\eta) \\
& = & ([X'G]^{+1}\eta^-)(X'G\vartheta_n FG)([X'G^{+1}]\eta) \\
& = & (X'G\vartheta_n)([X'G^{+1}]\eta^-)([X'G^{+1}]\eta) \\
& = & X'G\vartheta_n\ .
\end{array}$$

□

III.4.3 Some lemmata

Let $\mathcal{E}$ be a Frobenius category, let $\mathcal{B} \subseteq \mathcal{E}$ be its full subcategory of bijective objects; cf. e.g. Def. I.45. We use the notations and conventions of §I.6.2.3, in particular those of Ex. I.46.(2).

Let $n \geqslant 0$. Let $E \subseteq \bar{\Delta}_n^{\#}$ be a convex full subposet, i.e. whenever $\xi, \zeta \in E$ and $\lambda \in \bar{\Delta}_n^{\#}$ such that $\xi \leqslant \lambda \leqslant \zeta$, then $\lambda \in E$; cf. §I.2.2.2.1. For instance, $\Delta_n^{\Delta\nabla} \subseteq \bar{\Delta}_n^{\#}$ is such a convex full subposet; cf. Notation III.11.

A *pure square* in $\mathcal{E}$ is a commutative quadrangle (A, B, C, D) with pure short exact diagonal sequence $(A, B \oplus C, D)$; cf. §I.6.4.

Denote by $\mathcal{E}^{\square}(E) \subseteq \mathcal{E}(E)$ the full subcategory determined by

$$\operatorname{Ob}\mathcal{E}^{\square}(E) := \left\{ X \in \operatorname{Ob}\mathcal{E}(E) \quad : \quad \begin{array}{ll} 1) & X_{\alpha/\alpha} \text{ is in } \operatorname{Ob}\mathcal{B} \text{ for all } \alpha \in \bar{\Delta}_n \text{ such that } \alpha/\alpha \in E, \text{ and} \\ & X_{\alpha^{+1}/\alpha} \text{ is in } \operatorname{Ob}\mathcal{B} \text{ for all } \alpha \in \bar{\Delta}_n \text{ such that } \alpha^{+1}/\alpha \in E. \\ 2) & \text{For all } \delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1} \text{ in } \bar{\Delta}_n \\ & \text{such that } \gamma/\alpha,\ \gamma/\beta,\ \delta/\alpha \text{ and } \delta/\beta \text{ are in } E, \\ & \text{the quadrangle} \\ & \begin{array}{ccc} X_{\gamma/\beta} & \xrightarrow{x} & X_{\delta/\beta} \\ {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle x} \\ X_{\gamma/\alpha} & \xrightarrow[x]{} & X_{\delta/\alpha} \end{array} \\ & \text{is a pure square.} \end{array} \right\} .$$

A particular case of this definition has been considered in §I.4.1.

III.4.3.1 Cleaning the diagonal

Lemma III.16 *Suppose given $X \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$. Suppose given $\beta \in \bar{\Delta}_n$ such that $0 \leqslant \beta \leqslant 0^{+1}$.*

There exists $\tilde{X} \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ such that the following conditions (1a, 1b, 2) hold.

(1a) *We have $\tilde{X}_{\alpha/\alpha} = X_{\alpha/\alpha}$ for $0 \leqslant \alpha \leqslant 0^{+1}$ such that $\alpha \neq \beta$.*

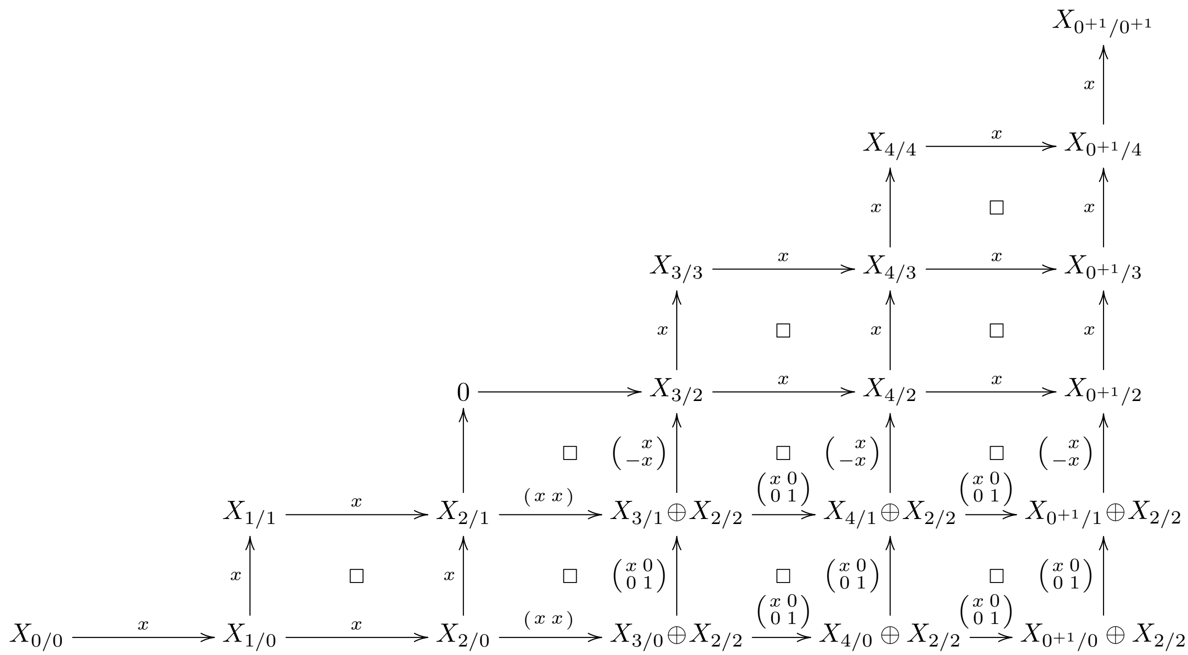
(1b) *We have $\tilde{X}_{\beta/\beta} = 0$.*

(2) *There exists an isomorphism $\tilde{X} \xrightarrow{\sim} X$ in $\underline{\mathcal{E}}(\bar{\Delta}_n^{\Delta\nabla})$.*

Proof. Pars pro toto, we consider the case $n = 4$ and $\beta = 2$. We display X as follows.

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & X_{0^{+1}/0^{+1}} \\
 & & & & & & & & & & {\scriptstyle x}\uparrow \\
 & & & & & & & & X_{4/4} & \xrightarrow{x} & X_{0^{+1}/4} \\
 & & & & & & & & {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle x} \\
 & & & & & & X_{3/3} & \xrightarrow{x} & X_{4/3} & \xrightarrow{x} & X_{0^{+1}/3} \\
 & & & & & & {\scriptstyle x}\uparrow & \square & {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle x} \\
 & & & & X_{2/2} & \xrightarrow{x} & X_{3/2} & \xrightarrow{x} & X_{4/2} & \xrightarrow{x} & X_{0^{+1}/2} \\
 & & & & {\scriptstyle x}\uparrow & \square & {\scriptstyle x}\uparrow & \square & {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle x} \\
 & & X_{1/1} & \xrightarrow{x} & X_{2/1} & \xrightarrow{x} & X_{3/1} & \xrightarrow{x} & X_{4/1} & \xrightarrow{x} & X_{0^{+1}/1} \\
 & & {\scriptstyle x}\uparrow & \square & {\scriptstyle x}\uparrow & \square & {\scriptstyle x}\uparrow & \square & {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle x} \\
X_{0/0} & \xrightarrow{x} & X_{1/0} & \xrightarrow{x} & X_{2/0} & \xrightarrow{x} & X_{3/0} & \xrightarrow{x} & X_{4/0} & \xrightarrow{x} & X_{0^{+1}/0}
\end{array}$$

Set $\tilde{X}$ to be the following diagram.



Using the Gabriel-Quillen-Laumon embedding theorem, we see that $\tilde{X}$ is actually an object of $\mathcal{E}^{\square}(\bar{\Delta}_n^{\vartriangle\triangledown})$; cf. §I.6.2.2, Lem. I.51.

Since $X_{2/2}$ is bijective, inserting the zero morphism on all copies of $X_{2/2}$ and the identity on all other summands yields an isomorphism $\tilde{X} \xrightarrow{\sim} X$ in $\underline{\mathcal{E}}(\bar{\Delta}_n^{\vartriangle\triangledown})$. ▫

Lemma III.17 *Suppose given* $X \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\vartriangle\triangledown})$.

There exists $X' \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\vartriangle\triangledown})$ *such that the following conditions* $(1, 2)$ *hold.*

(1) *We have* $X'_{\alpha/\alpha} = 0$ *for all* $\alpha \in \bar{\Delta}_n$ *such that* $0 \leqslant \alpha \leqslant 0^{+1}$.

(2) *There exists an isomorphism* $X' \xrightarrow{\sim} X$ *in* $\underline{\mathcal{E}}(\bar{\Delta}_n^{\vartriangle\triangledown})$.

Proof. This follows by application of Lemma III.16 consecutively for $\beta = 0$, $\beta = 1$, ..., $\beta = 0^{+1}$. ▫

III.4.3.2 Horseshoe lemma

Recall that $\mathcal{B}^{\mathrm{ac}}$ denotes the category of purely acyclic complexes with entries in $\mathcal{B}$, i.e. of complexes with entries in $\mathcal{B}$ that decompose into pure short exact sequences in $\mathcal{E}$; cf. §I.6.2.3.

Suppose given $Y \in \operatorname{Ob}\mathcal{E}$. An object B of $\mathcal{B}^{\mathrm{ac}}$ is called a *(both-sided) bijective resolution* of Y if Y is isomorphic to $\operatorname{Im}(B^0 \longrightarrow B^1)$. Note that a bijective resolution of a bijective object is split acyclic.

We have a full and dense functor ([2])

$$\begin{array}{ccc} \mathcal{B}^{\mathrm{ac}} & \xrightarrow{\hat{F}} & \mathcal{E} \\ B & \longmapsto & \operatorname{Im}(B^0 \longrightarrow B^1) \end{array}$$

We make the additional convention that if the image factorisation of a pure morphism d in $\mathcal{E}$ is chosen to be $d = \bar{d}\dot{d}$, then we choose the image factorisation $-d = \bar{d}(-\dot{d})$ over the same image object.

Pointwise application yields a functor $\mathcal{B}^{\mathrm{ac}}(\bar{\Delta}_n^{\vartriangle\triangledown}) \xrightarrow{\hat{F}} \mathcal{E}(\bar{\Delta}_n^{\vartriangle\triangledown})$, which is an abuse of notation.

[2] A functor induced by $\hat{F}$ will play the role of F of Setup III.10; cf. §III.4.4.2.2 below.

Suppose given $X \in \operatorname{Ob} \mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ such that $X_{\alpha/\alpha} = 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$.

In particular, $X_{\beta/\alpha} \longrightarrow X_{\gamma/\alpha} \longrightarrow X_{\gamma/\beta}$ is a pure short exact sequence for $0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant 0^{+1}$.

Recall that for $n \in \bar{\Delta}_n$, we have $n + 1 = 0^{+1}$; cf. §I.1.1.

Lemma III.18 *Suppose given a bijective resolution $C_{\alpha+1/\alpha}$ of $X_{\alpha+1/\alpha}$ for all $\alpha \in \bar{\Delta}_n$ such that $0 \leqslant \alpha \leqslant n$. Then there exists $B \in \operatorname{Ob}(\mathcal{B}^{\mathrm{ac}})^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ such that $(1, 2, 3)$ hold.*

(1) *We have $B\hat{F} \simeq X$ in $\mathcal{E}^{\square}(\bar{\Delta}_n^{\#})$.*

(2) *We have $B_{\alpha/\alpha} = 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$.*

(3) *We have $B_{\alpha+1/\alpha} = C_{\alpha+1/\alpha}$ for all $\alpha \in \bar{\Delta}_n$ such that $0 \leqslant \alpha \leqslant n$.*

If $n = 2$, and if we restrict to $\{1/0, 2/0, 2/1\} \subseteq \bar{\Delta}_2^{\Delta\nabla}$, we recover the classical horseshoe lemma in its bothsided Frobenius category variant.

Proof. For $0 \leqslant \alpha \leqslant n$, we denote $\left(C^0_{\alpha+1/\alpha} \xrightarrow{\bar{d}} X_{\alpha+1/\alpha}\right) := \left(C^0_{\alpha+1/\alpha} \longrightarrow C_{\alpha+1/\alpha}\hat{F} \xrightarrow{\sim} X_{\alpha+1/\alpha}\right)$.

By duality and by induction, it suffices to find a morphism $Y \longrightarrow X$ in $\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ such that (i, ii, iii) hold.

(i) We have $Y_{\beta/\alpha} \in \operatorname{Ob} \mathcal{B}$ for all $0 \leqslant \alpha \leqslant \beta \leqslant 0^{+1}$.

(ii) We have $Y_{\alpha/\alpha} = 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$.

(iii) We have $(Y_{\alpha+1/\alpha} \longrightarrow X_{\alpha+1/\alpha}) = (C^0_{\alpha+1/\alpha} \xrightarrow{\bar{d}} X_{\alpha+1/\alpha})$ for all $\alpha \in \bar{\Delta}_n$ such that $0 \leqslant \alpha \leqslant n$.

Note that any morphism $Y \longrightarrow X$ fulfilling (i, ii, iii) consists pointwise of pure epimorphisms, and that the kernel of such a morphism $Y \longrightarrow X$ taken in $\mathcal{E}(\bar{\Delta}_n^{\Delta\nabla})$ is in $\operatorname{Ob} \mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$.

To construct $Y \longrightarrow X$, we let

$$Y_{\gamma/\alpha} := \bigoplus_{\beta \in \bar{\Delta}_n,\, \alpha \leqslant \beta < \gamma} C^0_{\beta+1/\beta}$$

for $0 \leqslant \alpha \leqslant \gamma \leqslant 0^{+1}$. For $\gamma/\alpha \leqslant \gamma'/\alpha'$, the diagram morphism $Y_{\gamma/\alpha} \longrightarrow Y_{\gamma'/\alpha'}$ is stipulated to be identical on the summands $C^0_{\beta+1/\beta}$ with $\alpha' \leqslant \beta < \gamma$ and zero elsewhere. This yields $Y \in \operatorname{Ob} \mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$.

Given $0 \leqslant \alpha \leqslant \gamma \leqslant 0^{+1}$, we let $Y_{\gamma/\alpha} \longrightarrow X_{\gamma/\alpha}$ be defined as follows. For $0 \leqslant \beta \leqslant n$, we choose $Y_{\beta+1/\beta} \xrightarrow{e} X_{\beta+1/0}$ such that

$$(Y_{\beta+1/\beta} \xrightarrow{e} X_{\beta+1/0} \xrightarrow{x} X_{\beta+1/\beta}) = (Y_{\beta+1/\beta} \xrightarrow{\bar{d}} X_{\beta+1/\beta}) .$$

The component of the morphism

$$(Y_{\gamma/\alpha} \longrightarrow X_{\gamma/\alpha}) = \left(\bigoplus_{\beta \in \bar{\Delta}_n,\, \alpha \leqslant \beta < \gamma} C^0_{\beta+1/\beta} \longrightarrow X_{\gamma/\alpha} \right)$$

at β is defined to be the composite

$$(C^0_{\beta+1/\beta} \longrightarrow X_{\gamma/\alpha}) := (C^0_{\beta+1/\beta} \xrightarrow{e} X_{\beta+1/0} \xrightarrow{x} X_{\gamma/\alpha}) .$$

□

III.4.3.3 Applying $\hat{F}$ to a standard pure short exact sequence

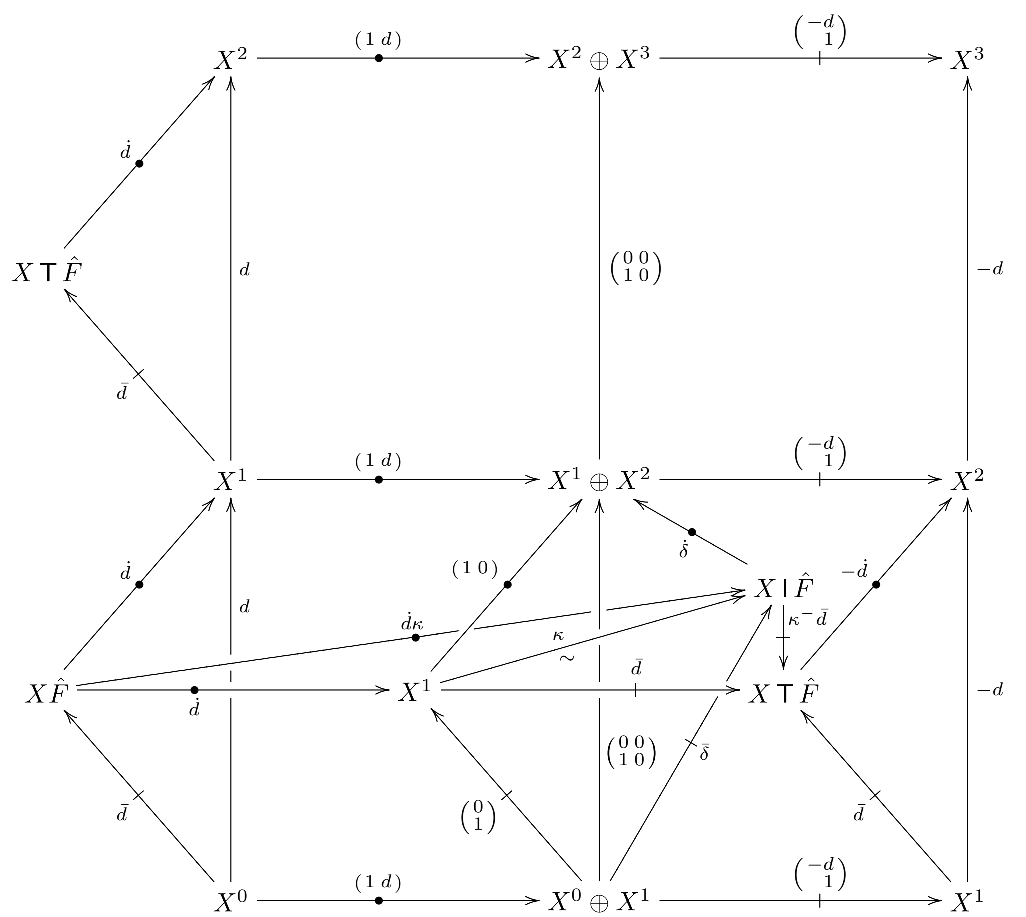
Recall that for $X \in \operatorname{Ob}\mathcal{B}^{\mathrm{ac}}$, we have chosen, in a functorial manner, a pure short exact sequence $X \rightarrowtail X\,\mathsf{I} \twoheadrightarrow X\,\mathsf{T}$ with a bijective middle term $X\,\mathsf{I}$, where the letter I stands for "injective"; cf. §I.6.2.3.

Lemma III.19 *Suppose given $X \in \operatorname{Ob}\mathcal{B}^{\mathrm{ac}}$. There exists an isomorphism of pure short exact sequences in $\mathcal{E}$ as follows.*

$$\begin{array}{ccccc} X\hat{F} & \rightarrowtail & X\,\mathsf{I}\,\hat{F} & \twoheadrightarrow & X\,\mathsf{T}\,\hat{F} \\ \| & & \downarrow \wr & & \| \\ X\hat{F} & \rightarrowtail & X^1 & \twoheadrightarrow & X\,\mathsf{T}\,\hat{F} \end{array}$$

Therein, the upper sequence results from an application of $\hat{F}$ to the pure short exact sequence $X \rightarrowtail X\,\mathsf{I} \twoheadrightarrow X\,\mathsf{T}$ in $\mathcal{B}^{\mathrm{ac}}$. The lower sequence is taken from the purely acyclic complex X.

Proof. Consider the following part of the pure short exact sequence $X \rightarrowtail X\,\mathsf{I} \twoheadrightarrow X\,\mathsf{T}$ in $\mathcal{B}^{\mathrm{ac}}$; cf. Ex. I.46.



We have added the image factorisations $X^0 \xrightarrow{\bar{d}} X\hat{F} \xrightarrow{\dot{d}} X^1$ and $X^0 \oplus X^1 \xrightarrow{\bar{\delta}} X\,\mathsf{I}\,\hat{F} \xrightarrow{\dot{\delta}} X^1 \oplus X^2$ of the respective differentials, resulting from an application of $\hat{F}$. Factoring the differential of $X\,\mathsf{T}$ as

$$\left(X^1 \xrightarrow{-d} X^2\right) = \left(X^1 \xrightarrow{\bar{d}} X\,\mathsf{T}\,\hat{F} \xrightarrow{-\dot{d}} X^2\right)$$

follows the additional convention made above.

Moreover, we have added the image factorisation $X^0 \oplus X^1 \xrightarrow{\binom{0}{1}} X^1 \xrightarrow{(1\,0)} X^1 \oplus X^2$ and, accordingly, the isomorphism $X^1 \xrightarrow[\sim]{\kappa} X\,\mathsf{I}\,\hat{F}$ that satisfies $\kappa\dot{\delta} = (1\,0)$ and $\binom{0}{1}\kappa = \bar{\delta}$.

The horizontal pure short exact sequence $X\hat{F} \xrightarrow{\dot{d}} X^1 \xrightarrow{\bar{d}} X\,\mathsf{T}\,\hat{F}$ lets all four arising parallelograms commute.

Now the sequence $X \xrightarrow{\bullet} X\mathsf{I} \xrightarrow{+} X\mathsf{T}$ maps to $X\hat{F} \xrightarrow{\dot{d}\kappa} X\mathsf{I}\hat{F} \xrightarrow{\kappa^-\bar{d}} X\mathsf{T}\hat{F}$, for the commutativities $(\dot{d}\kappa)\dot{\delta} = \dot{d}\begin{pmatrix}1 & d\end{pmatrix}$ and $\bar{\delta}(\kappa^-\bar{d}) = \begin{pmatrix}-d\\ 1\end{pmatrix}\bar{d}$ hold.

In particular, the sequence $(X \xrightarrow{\bullet} X\mathsf{I} \xrightarrow{+} X\mathsf{T})\hat{F}$ actually is purely short exact. □

III.4.4 Stable vs. classically stable

Let $\mathcal{E}$ be a Frobenius category, let $\mathcal{B} \subseteq \mathcal{E}$ be its full subcategory of bijective objects.

III.4.4.1 n-triangles in the stable category

Recall that $\mathcal{B}^{\mathrm{ac}}$ denotes the category of purely acyclic complexes with entries in $\mathcal{B}$; cf. §III.4.3.2. Let $\mathcal{B}^{\mathrm{sp\,ac}} \subseteq \mathcal{B}^{\mathrm{ac}}$ denote the subcategory of split acyclic complexes. Let $\underline{\underline{\mathcal{E}}} = \mathcal{B}^{\mathrm{ac}}/\mathcal{B}^{\mathrm{sp\,ac}}$ denote the stable category of $\mathcal{E}$; cf. Def. I.47. Let T be the automorphism on $\underline{\underline{\mathcal{E}}}$ that shifts a complex to the left by one position, inserting signs; cf. Ex. I.46.(1). Then $(\underline{\underline{\mathcal{E}}}, \mathsf{T})$ carries a Heller triangulation ϑ; cf. Cor. I.33. In fact, we may, and will, choose the tuple of isotransformations $\vartheta = (\vartheta_n)_{n \geqslant 0}$ constructed in the proof of Th. I.32.

Suppose given $n \geqslant 0$ and $X \in \mathrm{Ob}(\mathcal{B}^{\mathrm{ac}})^{\square}(\bar{\Delta}_n^{\#})$; cf. §I.4.1 or §III.4.3. Now X maps to an object $X \in \mathrm{Ob}\,\underline{\underline{\mathcal{E}}^+(\bar{\Delta}_n^{\#})}$; cf. Lem. I.69. Thus we have an isomorphism $[X]^{+1} \xrightarrow{X\vartheta_n} [X^{+1}]$ in $\underline{\underline{\mathcal{E}}^+(\bar{\Delta}_n^{\#})}$. By the construction in the proof of Th. I.32, there is a representative $[X]^{+1} \xrightarrow{X\theta} [X^{+1}]$ in $\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_n^{\#})$ of $X\vartheta_n$ such that in particular, there exists a morphism of pure short exact sequences

$$\begin{array}{ccccc} X_{i/0} & \xrightarrow[\bullet]{(x\;x)} & X_{i/i}\oplus X_{0^{+1}/0} & \xrightarrow[+]{\left(\begin{smallmatrix} x\\ -x\end{smallmatrix}\right)} & X_{0^{+1}/i} \\ \Vert & & \downarrow & & \downarrow {\scriptstyle X\hat{\theta}_{i/0}} \\ X_{i/0} & \xrightarrow{\bullet} & X_{i/0}\mathsf{I} & \xrightarrow{+} & X_{i/0}^{+1} \end{array}$$

for each $i \in [1, n]$; where $X\hat{\theta}_{i/0}$ is a representative in $\mathcal{B}^{\mathrm{ac}}$ for the morphism $X\theta_{i/0}$ in $\underline{\underline{\mathcal{E}}}$; where the upper pure short exact sequence stems from the diagram X; and where the lower pure short exact sequence is the standard one as in Ex. I.46.(1). In particular, $X\theta_{i/0}$ is an isomorphism in $\underline{\underline{\mathcal{E}}}$.

Let $X^{\vartheta} \in \mathrm{Ob}\,\underline{\underline{\mathcal{E}}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ be defined as periodic prolongation of the image of the diagram $X|_{\bar{\Delta}_n^{\Delta\nabla}}$ in $\mathrm{Ob}\,\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_n^{\Delta\nabla})$ with $X_{0^{+1}/i}$ isomorphically replaced via $X\theta_{i/0}$ by $X_{i/0}^{+1}$ for all $i \in [1, n]$. For short, the rightmost column of the image of $X|_{\bar{\Delta}_n^{\Delta\nabla}}$ becomes standardised; cf. §I.2.1.3. Using

$$\Big([X]^{+k} \xrightarrow{\sim} [X^{+1}]^{+(k-1)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} [X^{+k}]\Big)\Big|_{\bar{\Delta}_n^{\Delta\nabla}}$$

for $k \geqslant 0$, and similarly for $k \leqslant 0$, we obtain an isomorphism $X \xrightarrow{\omega} X^{\vartheta}$ in $\underline{\underline{\mathcal{E}}}^+(\bar{\Delta}_n^{\#})$ such that $\omega_{i/0} = 1_{X_{i/0}}$ and $\omega_{0^{+1}/i} = X\theta_{i/0}$ for $i \in [1, n]$; cf. Notation III.11.(1).

Lemma III.20 *Given $n \geqslant 0$ and $X \in \mathrm{Ob}(\mathcal{B}^{\mathrm{ac}})^{\square}(\bar{\Delta}_n^{\#})$, the periodic n-pretriangle X^{ϑ} is an n-triangle.*

The following proof is similar to the proof of Lemma III.12.

Proof. We have to show that $X^{\vartheta}\vartheta_n = 1$; cf. Def. I.5.(ii.2). Since ϑ_n is a transformation, we have a commutative quadrangle

$$\begin{array}{ccc} [X^{\vartheta}]^{+1} & \xrightarrow[\sim]{X^{\vartheta}\vartheta_n} & [(X^{\vartheta})^{+1}] \\ {\scriptstyle \wr}\uparrow{\scriptstyle [\omega]^{+1}} & & {\scriptstyle \wr}\uparrow{\scriptstyle [\omega^{+1}]} \\ [X]^{+1} & \xrightarrow[\sim]{X\vartheta_n} & [X^{+1}] \end{array}$$

in $\underline{\underline{\mathcal{E}}^+(\bar{\Delta}_n^{\#})}$. So we have to show that $(X\vartheta_n)[\omega^{+1}] = [\omega]^{+1}$. By Prop. I.12, it suffices to show that $(X\vartheta_n)|_{\dot{\Delta}_n}[\omega^{+1}]|_{\dot{\Delta}_n} = [\omega]^{+1}|_{\dot{\Delta}_n}$. Now $[\omega^{+1}]|_{\dot{\Delta}_n} = 1_{[X^{+1}]|_{\dot{\Delta}_n}}$ and, by construction, $(X\vartheta_n)|_{\dot{\Delta}_n} = [\omega]^{+1}|_{\dot{\Delta}_n}$. □

Corollary III.21 *The Heller triangulated category* $(\underline{\underline{\mathcal{E}}}, \mathsf{T}, \vartheta)$ *is closed.*

Cf. Definition III.13.

Proof. We can extend any morphism $X_{1/0} \longrightarrow X_{2/0}$ of $\mathcal{B}^{\mathrm{ac}}$ to an object of $(\mathcal{B}^{\mathrm{ac}})^{\Box}(\bar{\Delta}_2^{\#})$ by choosing $X_{1/0} \rightarrowtail X_{1/1}$ with $X_{1/1}$ bijective and by choosing $X_{0^{+1}/0} = 0$, then forming pushouts, then choosing $X_{2/1} \rightarrowtail X_{2/2}$ with $X_{2/2}$ bijective, etc. Dually in the other direction. Then we apply Lemma III.20. □

III.4.4.2 The classical stable category under an additional hypothesis

III.4.4.2.1 The hypothesis

Let $\underline{\mathcal{E}} := \mathcal{E}/\mathcal{B}$ denote the classical stable category of $\mathcal{E}$.

Suppose given a set $\mathcal{D}$ of *distinguished* pure short exact sequences in $\mathcal{E}$ such that the following conditions hold.

(i) The middle term of each distinguished pure short exact sequence is bijective.

(ii) For all $X \in \operatorname{Ob}\mathcal{E}$, there exists a unique distinguished pure short exact sequence with kernel term X.

(iii) For all $X \in \operatorname{Ob}\mathcal{E}$, there exists a unique distinguished pure short exact sequence with cokernel term X.

III.4.4.2.2 Consequences

We shall define an endofunctor T' of $\underline{\mathcal{E}}$.

On objects. Given $X \in \operatorname{Ob}\underline{\mathcal{E}} = \operatorname{Ob}\mathcal{E}$, there exists a unique distinguished pure short exact sequence with kernel term X. Let $X\,\mathsf{T}'$ be the cokernel term of this sequence.

On morphisms. The image under T' of the residue class in $\underline{\mathcal{E}}$ of a morphism $X \xrightarrow{f} Y$ in $\mathcal{E}$ is represented by the morphism $X\,\mathsf{T}' \xrightarrow{g} Y\,\mathsf{T}'$ in $\mathcal{E}$ if there exists a morphism of distinguished pure short exact sequences as follows.

$$\begin{array}{ccccc} X & \rightarrowtail & B & \twoheadrightarrow & X\,\mathsf{T}' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ Y & \rightarrowtail & C & \twoheadrightarrow & Y\,\mathsf{T}' \end{array}$$

Then T' is an automorphism of $\underline{\mathcal{E}}$; i.e. there exists an inverse T'^{-}, constructed dually, such that $\mathsf{T}'\,\mathsf{T}'^{-} = 1_{\underline{\mathcal{E}}}$ and $\mathsf{T}'^{-}\,\mathsf{T}' = 1_{\underline{\mathcal{E}}}$.

As usual, we shall write $X^{+1} := X\,\mathsf{T}'$ for $X \in \operatorname{Ob}\underline{\mathcal{E}}$; etc.

The functor

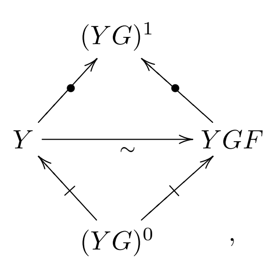
$$\begin{array}{ccc} \underline{\underline{\mathcal{E}}} & \xrightarrow{F} & \underline{\mathcal{E}} \\ B & \longmapsto & \operatorname{Im}(B^0 \longrightarrow B^1) \end{array}$$

induced by $\hat{F}$ is an equivalence; cf. §III.4.3.2, Lem. I.41. Splicing purely acyclic complexes from distinguished pure short exact sequences, we obtain an inverse equivalence $\underline{\underline{\mathcal{E}}} \xleftarrow{G} \underline{\mathcal{E}}$. Define $\mathsf{T}'\,G \xrightarrow{\sigma} G\,\mathsf{T}$ at $Y \in \operatorname{Ob}\underline{\mathcal{E}} = \operatorname{Ob}\mathcal{E}$ by letting

$$(Y\sigma^i) \;:=\; (-1_{(YG)^{i+1}})^i \,.$$

Note that $Y\sigma F = 1_{Y\,\mathsf{T}'\,GF} = 1_{YG\,\mathsf{T}\,F}$; cf. §III.4.3.2.

Suppose given $Y \in \operatorname{Ob}\mathcal{E}$. We have a commutative diagram



,

consisting of two image factorisations of the differential $(YG)^0 \longrightarrow (YG)^1$ and the induced isomorphism between the images $Y \xrightarrow{\sim} YGF$ that makes the upper and the lower triangle in this diagram commute.

The residue class in $\underline{\underline{\mathcal{E}}}$ of this induced morphism $Y \xrightarrow{\sim} YGF$ shall be denoted by $Y \xrightarrow[\sim]{Y\varepsilon} YGF$. Letting Y vary, this gives rise to an isotransformation $1_{\underline{\underline{\mathcal{E}}}} \xrightarrow[\sim]{\varepsilon} GF$. Since ε is a transformation, we have $(Y\varepsilon)(YGF\varepsilon) = (Y\varepsilon)(Y\varepsilon GF)$, whence $GF\varepsilon = \varepsilon GF$. Thus there is an isotransformation $FG \xrightarrow[\sim]{\eta} 1_{\underline{\underline{\mathcal{E}}}}$ such that $(\varepsilon G)(G\eta) = 1_G$ and $(F\varepsilon)(\eta F) = 1_F$. Namely, for η we may take the inverse image under F of $F\varepsilon^-$.

So we are in the situation of Setup III.10 of §III.4.2. Define $(\mathsf{T}\,F \xrightarrow{\rho} F\,\mathsf{T}')$ as in §III.4.2.

Proposition III.22

(1) *By transport from $(\underline{\underline{\mathcal{E}}}, \mathsf{T}, \vartheta)$ via F and G, we obtain a closed Heller triangulation ϑ' on $(\underline{\underline{\mathcal{E}}}, \mathsf{T}')$.*

(2) *Suppose given $X' \in \operatorname{Ob} \underline{\underline{\mathcal{E}}}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$. Then X' is an n-triangle if and only if $X'G^\sigma$ is an n-triangle.*

(3) *Suppose given $X \in \operatorname{Ob} \underline{\underline{\mathcal{E}}}^{+,\,\text{periodic}}(\bar{\Delta}_n^{\#})$. Then X is an n-triangle if and only if XF^ρ is an n-triangle.*

Proof. Assertion (1) follows by Lemmata III.9 and III.14; cf. Corollary III.21. Assertions $(2,3)$ follow by Lemma III.12. ▫

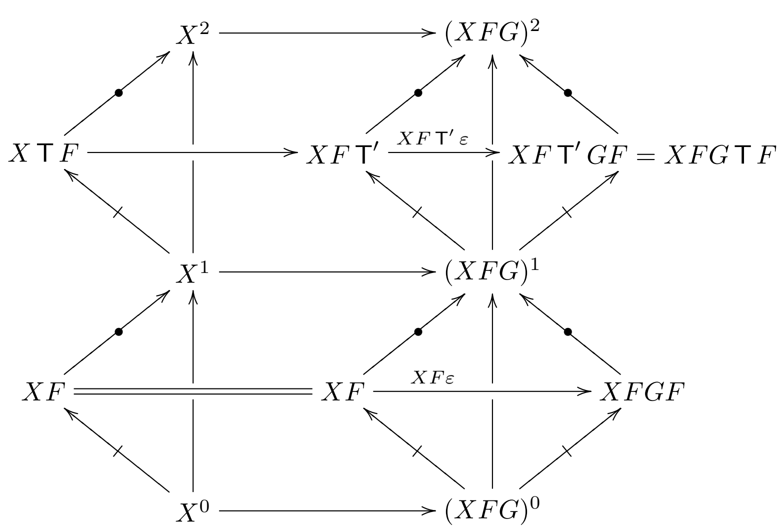
Recall that $XF = X\hat{F}$ in $\operatorname{Ob}\underline{\underline{\mathcal{E}}} = \operatorname{Ob}\mathcal{E}$ for $X \in \operatorname{Ob}\underline{\underline{\mathcal{E}}} = \operatorname{Ob}\mathcal{B}^{\text{ac}}$.

Lemma III.23 *Suppose given $X \in \operatorname{Ob}\mathcal{B}^{\text{ac}}$. We have a morphism of pure short exact sequences*

$$\begin{array}{ccccc} XF & \xrightarrow{\bullet} & X^1 & \xrightarrow{|} & X\,\mathsf{T}\,F \\ \| & & \downarrow & & \downarrow \\ XF & \xrightarrow{\bullet} & (XFG)^1 & \xrightarrow{|} & XF\,\mathsf{T}' \end{array}$$

in $\mathcal{E}$ such that its morphism $X\,\mathsf{T}\,F \longrightarrow XF\,\mathsf{T}'$ represents $X\rho$ in $\underline{\underline{\mathcal{E}}}$. Here, the upper pure short exact sequence is taken from the purely acyclic complex X; the lower pure short exact sequence is distinguished.

Proof. Given $X \in \operatorname{Ob}\mathcal{B}^{\text{ac}}$, we can form a commutative diagram in $\mathcal{E}$ as follows.



The morphisms $(XFG)^0 \xrightarrow{|} XF$, $XF \xrightarrow{\bullet} (XFG)^1$, $(XFG)^1 \xrightarrow{|} XF\,\mathsf{T}'$ and $XF\,\mathsf{T}' \xrightarrow{\bullet} (XFG)^2$ appear in distinguished pure short exact sequences. Moreover, by abuse of notation, we have written $XF\varepsilon$ resp. $XF\,\mathsf{T}'\varepsilon$ for representatives in $\mathcal{E}$ of the respective morphisms in $\underline{\underline{\mathcal{E}}}$.

The partially displayed morphism of complexes $X \longrightarrow XFG$ represents $X \xrightarrow{X\eta^-} XFG$ in $\underline{\underline{\mathcal{E}}}$, for F maps the morphism represented by $X \longrightarrow XFG$ to $XF\varepsilon = X\eta^- F$.

Therefore, the composite morphism

$$(X \top F \longrightarrow XF\top' \xrightarrow{XF\top'\varepsilon} XF\top' GF = XFG\top F)$$

from this diagram represents

$$(X \xrightarrow{X\eta^-} XFG)\top F\;;$$

note that there are no signs to be inserted at the respective pure epimorphisms of the image factorisation chosen by F; cf. §III.4.3.2. Thus the morphism $X\top F \longrightarrow XF\top'$ from this diagram represents

$$\big(X\top F \xrightarrow{X\eta^-\top F} XFG\top F = XF\top' GF \xrightarrow{XF\top'\varepsilon^-} XF\top'\big) = \big(X\top F \xrightarrow{X\rho} XF\top'\big)\,.$$

□

III.4.4.2.3 Standardisation by substitution of the rightmost column

We mimic the construction $X \longmapsto X^\vartheta$ made in §III.4.4.1, now for $\mathcal{E}$ instead of $\mathcal{B}^{\mathrm{ac}}$.

Denote by $\mathcal{E}^{\square}(\bar\Delta_n^{\Delta\nabla}) \xrightarrow{M'} \underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla})$ the residue class functor; cf. §III.4.3, §I.2.1.3, §I.4.1. Denote by $\underline{\mathcal{E}}^{+,(\square)}(\bar\Delta_n^{\Delta\nabla})$ the full subcategory of $\underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla})$ whose set of objects is given by

$$\operatorname{Ob}\underline{\mathcal{E}}^{+,(\square)}(\bar\Delta_n^{\Delta\nabla}) := \big(\operatorname{Ob}\mathcal{E}^{\square}(\bar\Delta_n^{\Delta\nabla})\big)M'\,.$$

So $\underline{\mathcal{E}}^{+,(\square)}(\bar\Delta_n^{\Delta\nabla})$ is defined to be the "full image" in $\underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla})$ of the residue class functor M'.

Suppose given $n \geqslant 0$ and $X \in \operatorname{Ob}\mathcal{E}^{\square}(\bar\Delta_n^{\Delta\nabla})$. Recall that $\dot\Delta_n = [1,n]$ is identified with $\{i/0 \,:\, i \in [1,n]\} \subseteq \bar\Delta_n^{\Delta\nabla}$. Write $X_{*/0} := X|_{\dot\Delta_n} = (i \longmapsto X_{i/0}) \in \operatorname{Ob}\mathcal{E}(\dot\Delta_n)$ and $X_{0^{+1}/*} := (i \longmapsto X_{0^{+1}/i}) \in \operatorname{Ob}\mathcal{E}(\dot\Delta_n)$, analogously for morphisms; analogously for objects in $\underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla})$ and their morphisms.

Let the isomorphism $X_{0^{+1}/*} \xrightarrow[\sim]{X\tau} X^{+1}_{*/0}$ in $\underline{\mathcal{E}}(\dot\Delta_n)$ be defined by morphisms of pure short exact sequences

$$\begin{array}{ccccc}
X_{i/0} & \xrightarrow{(x\;x)} & X_{i/i}\oplus X_{0^{+1}/0} & \xrightarrow{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} & X_{0^{+1}/i} \\
\| & & \downarrow & & \downarrow{\scriptstyle X\hat\tau_i} \\
X_{i/0} & \longrightarrow & B_{i/0} & \longrightarrow & X^{+1}_{i/0}
\end{array}$$

for $i \in [1,n]$; where $X\hat\tau_i$ is a representative in $\mathcal{E}$ for the morphism $X\tau_i$ in $\underline{\mathcal{E}}$; where the upper pure short exact sequence stems from the diagram X; and where the lower pure short exact sequence is distinguished.

In this way, we get an isotransformation τ between the functors $(-)_{0^{+1}/*}$ and $(-)^{+1}_{*/0}$ from $\underline{\mathcal{E}}^{+,(\square)}(\bar\Delta_n^{\Delta\nabla})$ to $\underline{\mathcal{E}}(\dot\Delta_n)$.

Let $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar\Delta_n^{\Delta\nabla})$ be the (in general not full) subcategory of $\underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla})$ given by the set of objects

$$\operatorname{Ob}\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar\Delta_n^{\Delta\nabla}) := \Big\{ Y \in \operatorname{Ob}\underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla}) \,:\, Y_{0^{+1}/*} = Y^{+1}_{*/0} \text{ in } \operatorname{Ob}\underline{\mathcal{E}}(\dot\Delta_n) \Big\}\,,$$

and by the set of morphisms

$${}_{\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar\Delta_n^{\Delta\nabla})}(Y,Y') = \{ f \in {}_{\underline{\mathcal{E}}^{+}(\bar\Delta_n^{\Delta\nabla})}(Y,Y') \,:\, f_{0^{+1}/*} = f^{+1}_{*/0} \text{ in } \underline{\mathcal{E}}(\dot\Delta_n)\}\,,$$

for $Y,\, Y' \in \operatorname{Ob}\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar\Delta_n^{\Delta\nabla})$.

Given $X \in \operatorname{Ob}\mathcal{E}^{\square}(\bar\Delta_n^{\Delta\nabla})$, we let $X^\tau \in \operatorname{Ob}\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar\Delta_n^{\Delta\nabla})$ be defined as the diagram X with $X_{0^{+1}/*}$ isomorphically replaced via $X\tau$ by $X^{+1}_{*/0}$. For short, the rightmost column of X becomes standardised to obtain X^τ.

Given $X, X' \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$, a morphism $X \xrightarrow{f} X'$ in $\underline{\mathcal{E}}^{+}(\bar{\Delta}_n^{\Delta\nabla})$ induces a morphism $X^{\tau} \xrightarrow{f^{\tau}} X'^{\tau}$ in $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\Delta\nabla})$. Namely, we let $f^{\tau}_{\beta/\alpha} := f_{\beta/\alpha}$ for $0 \leqslant \alpha \leqslant \beta \leqslant n$, and we let $f^{\tau}_{0^{+1}/*}$ be characterised by the commutative quadrangle

$$\begin{array}{ccc} X_{0^{+1}/*} & \xrightarrow[\sim]{X\tau} & X^{+1}_{*/0} \\ {\scriptstyle f_{0^{+1}/*}}\downarrow & & \downarrow{\scriptstyle f^{\tau}_{0^{+1}/*}} \\ X'_{0^{+1}/*} & \xrightarrow[\sim]{X'\tau} & X'^{+1}_{*/0} \end{array}$$

in $\underline{\mathcal{E}}(\dot{\Delta}_n)$. In particular, since τ is an isotransformation, we have $f^{\tau}_{0^{+1}/*} = (f^{\tau}_{*/0})^{+1}$.

Remark III.24 *The constructions made above define a functor*

$$\begin{array}{ccc} \underline{\mathcal{E}}^{+,\,(\square)}(\bar{\Delta}_n^{\Delta\nabla}) & \xrightarrow{(-)^{\tau}} & \underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\Delta\nabla}) \\ X & \longmapsto & X^{\tau}\,. \end{array}$$

III.4.4.2.4 n-triangles in the classical stable category

Proposition III.25 *Suppose given $n \geqslant 0$ and $X \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$.*

The periodic prolongation of $X^{\tau} \in \operatorname{Ob}\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\Delta\nabla})$ to an object of $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ is an n-triangle with respect to the triangulation ϑ' on $(\underline{\mathcal{E}}, \mathsf{T}')$ obtained as in Proposition III.22.

Proof. By Lemma III.17, there exists $X' \in \operatorname{Ob}\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ such that $X'_{\alpha/\alpha} = 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$ and such that X is isomorphic to X' in $\underline{\mathcal{E}}^{+}(\bar{\Delta}_n^{\Delta\nabla})$. By Remark III.24, the object X^{τ} is isomorphic to X'^{τ} in $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\Delta\nabla})$. Thus the periodic prolongation of X^{τ} is an n-triangle if and only if that of X'^{τ} is; cf. Lem. I.21.(4).

Therefore, we may assume that $X_{\alpha/\alpha} \simeq 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$.

Let $\tilde{X} \in \operatorname{Ob}(\mathcal{B}^{\mathrm{ac}})^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ be such that there exists an isomorphism $X \xrightarrow[\sim]{\hat{a}} \tilde{X}\hat{F}$ in $\mathcal{E}^{\square}(\bar{\Delta}_n^{\Delta\nabla})$ and such that $\tilde{X}_{\alpha/\alpha} = 0$ for all $0 \leqslant \alpha \leqslant 0^{+1}$; cf. Lemma III.18. Denote by $X \xrightarrow[\sim]{a} \tilde{X}F$ the isomorphism in $\underline{\mathcal{E}}^{+}(\bar{\Delta}_n^{\Delta\nabla})$ represented by $X \xrightarrow[\sim]{\hat{a}} \tilde{X}\hat{F}$.

Let $\tilde{\tilde{X}} \in \operatorname{Ob}(\mathcal{B}^{\mathrm{ac}})^{\square}(\bar{\Delta}_n^{\#})$ be such that $\tilde{\tilde{X}}|_{\bar{\Delta}_n^{\Delta\nabla}} = \tilde{X}$. By Lemma III.20, the periodic n-pretriangle $\tilde{\tilde{X}}^{\vartheta} \in \operatorname{Ob}\underline{\underline{\mathcal{E}}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ is an n-triangle. Note that $\tilde{\tilde{X}}^{\vartheta}$ depends only on $\tilde{X}$, not on the choice of $\tilde{\tilde{X}}$.

Thus, by Proposition III.22.(3), $\tilde{\tilde{X}}^{\vartheta}F^{\rho} \in \operatorname{Ob}\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$ is an n-triangle. Therefore, it suffices to show that X^{τ} and $\tilde{\tilde{X}}^{\vartheta}F^{\rho}|_{\bar{\Delta}_n^{\Delta\nabla}}$ are isomorphic in $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\Delta\nabla})$, for then their periodic prolongations are isomorphic in $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$, which in turn shows the periodic prolongation of X^{τ} to be an n-triangle; cf. Lem. I.21.(4).

We have a composite isomorphism

$$X^{\tau} \xleftarrow{\sim} X \xrightarrow[\sim]{a} \tilde{X}F = \tilde{\tilde{X}}F|_{\bar{\Delta}_n^{\Delta\nabla}} \xrightarrow{\sim} \tilde{\tilde{X}}^{\vartheta}F|_{\bar{\Delta}_n^{\Delta\nabla}} \xrightarrow{\sim} \tilde{\tilde{X}}^{\vartheta}F^{\rho}|_{\bar{\Delta}_n^{\Delta\nabla}}$$

in $\underline{\mathcal{E}}^{+}(\bar{\Delta}_n^{\Delta\nabla})$. We *claim* that it lies in $\underline{\mathcal{E}}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\Delta\nabla})$.

Suppose given $i \in [1, n]$. On $i/0$, this composite equals $a_{i/0}$. Thus we have to show that on $0^{+1}/i$, this composite equals $a_{i/0}^{+1} = a_{i/0}\,\mathsf{T}'$. Consider, to this end, the following morphisms of pure short exact sequences in $\mathcal{E}$.

$$
\begin{array}{ccccc}
X_{i/0} & \xrightarrow{\quad\bullet\quad} & (X_{i/0}G)^1 & \xrightarrow{\quad + \quad} & X_{i/0}\,\mathsf{T}' \\
\| & & \uparrow & & \uparrow {\scriptstyle X\hat{\tau}_i} \\
X_{i/0} & \xrightarrow{\quad x \quad} & X_{0^{+1}/0} & \xrightarrow{\quad -x \quad} & X_{0^{+1}/i} \\
\wr \downarrow {\scriptstyle \hat{a}_{i/0}} & & \wr \downarrow {\scriptstyle \hat{a}_{0^{+1}/0}} & & \wr \downarrow {\scriptstyle \hat{a}_{0^{+1}/i}} \\
\tilde{X}_{i/0}\hat{F} & \xrightarrow{\quad \tilde{x}\hat{F} \quad} & \tilde{X}_{0^{+1}/0}\hat{F} & \xrightarrow{\quad -\tilde{x}\hat{F} \quad} & \tilde{X}_{0^{+1}/i}\hat{F} \\
\| & & \downarrow & & \downarrow {\scriptstyle X\hat{\theta}_{i/0}\hat{F}} \\
\tilde{X}_{i/0}\hat{F} & \xrightarrow{\quad\bullet\quad} & \tilde{X}_{i/0}\,\mathsf{I}\,\hat{F} & \xrightarrow{\quad + \quad} & \tilde{X}_{i/0}\,\mathsf{T}\,\hat{F} \\
\| & & \downarrow & & \| \\
\tilde{X}_{i/0}\hat{F} & \xrightarrow{\quad\bullet\quad} & (\tilde{X}_{i/0})^1 & \xrightarrow{\quad + \quad} & \tilde{X}_{i/0}\,\mathsf{T}\,\hat{F} \\
\| & & \downarrow & & \downarrow \\
\tilde{X}_{i/0}\hat{F} & \xrightarrow{\quad\bullet\quad} & (\tilde{X}_{i/0}\hat{F}G)^1 & \xrightarrow{\quad + \quad} & \tilde{X}_{i/0}\hat{F}\,\mathsf{T}'
\end{array}
$$

The fourth sequence is purely short exact by Lemma III.19.

The first morphism from above arises by definition of $\hat{\tau}_i$; cf. §III.4.4.2.3. The second morphism is taken from $\hat{a}$. The third morphism arises by definition of $\hat{\theta}_{i/0}$ and an application of $\hat{F}$; cf. §III.4.4.1. The fourth morphism is given by Lemma III.19. The fifth morphism is given by Lemma III.23.

The first and the sixth pure short exact sequence are distinguished, and so the *claim* and hence the proposition follow.

Chapter IV

Comparison of spectral sequences involving bifunctors

IV.0 Introduction

To calculate $\mathrm{Ext}^*(X, Y)$, one can either resolve X projectively or Y injectively; the result is, up to isomorphism, the same. To show this, one uses the double complex arising when one resolves both X and Y; cf. [**10,** Chap. V, Th. 8.1].

Two problems in this spirit occur in the context of Grothendieck spectral sequences; cf. §§ IV.0.2, IV.0.3.

IV.0.1 Language

In §IV.3, we give a brief introduction to the Deligne-Verdier spectral sequence language; cf. [**57,** II.§4], [**12,** App.]; or, on a more basic level, cf. [**38,** Kap. 4]. This language amounts to considering a diagram $\mathrm{E}(X)$ containing all the images between the homology groups of the subquotients of a given filtered complex X, instead of, as is classical, only selected ones. This helps to gain some elbow room in practice: to govern the objects of the diagram $\mathrm{E}(X)$ we can make use of a certain short exact sequence; cf. §IV.3.4.

Dropping the E_1-terms and similar ones, we obtain the *proper* spectral sequence $\dot{\mathrm{E}}(X)$ of our filtered complex X. Amongst others, it contains all E_k-terms for $k \geqslant 2$ in the classical language; cf. §§ IV.3.6, IV.3.5.

IV.0.2 First comparison

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with enough injectives and an abelian category $\mathcal{C}$. Suppose given objects $X \in \mathrm{Ob}\,\mathcal{A}$ and $X' \in \mathrm{Ob}\,\mathcal{A}'$. Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor such that $F(X, -)$ and $F(-, X')$ are left exact. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be a left exact functor. Suppose further conditions to hold; see §IV.5.1.

We have a Grothendieck spectral sequence for the composition $G \circ F(X, -)$ and a Grothendieck spectral sequence for the composition $G \circ F(-, X')$. We evaluate the former at X' and the latter at X.

In both cases, the E_2-terms are $(\mathrm{R}^i G)(\mathrm{R}^j F)(X, X')$. Moreover, they both converge to $\big(\mathrm{R}^{i+j}(G \circ F)\big)(X, X')$. So the following assertion is well-motivated.

Theorem IV.31. *The proper Grothendieck spectral sequences just described are isomorphic; i.e.*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \simeq \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X) .$$

So instead of "resolving X' twice", we may just as well "resolve X twice".

In fact, the underlying double complexes are connected by a chain of double homotopisms, i.e. isomorphisms in the homotopy category as defined in [**10,** IV.§4], and rowwise homotopisms (the proof uses a chain $\bullet \xleftarrow{\text{double}} \bullet \xleftarrow{\text{roww.}} \bullet \xrightarrow{\text{roww.}} \bullet \xrightarrow{\text{double}} \bullet$). These morphisms then induce isomorphisms on the associated proper first spectral sequences.

IV.0.3 Second comparison

Suppose given abelian categories $\mathcal{A}$ and $\mathcal{B}'$ with enough injectives and abelian categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}'$ be a left exact functor. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor such that $G(Y, -)$ is left exact.

Let $B \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y, i.e. a complex B admitting a quasiisomorphism $\operatorname{Conc} Y \longrightarrow B$. Suppose that $G(B^k, -)$ is exact for all $k \geqslant 0$. Let $A \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$ be, say, an injective resolution of X. Suppose further conditions to hold; see §IV.6.1.

We have a Grothendieck spectral sequence for the composition $G(Y, -) \circ F$, which we evaluate at X. On the other hand, we can consider the double complex $G(B, FA)$, where the indices of B count rows and the indices of A count columns. To the first filtration of its total complex, we can associate the proper spectral sequence $\dot{\mathrm{E}}_{\mathrm{I}}\big(G(B, FA)\big)$.

If $\mathcal{B}$ has enough injectives and B is an injective resolution of Y, then in both cases the E_2-terms are a priori seen to be $(\mathrm{R}^i G)\big(Y, (\mathrm{R}^j F)(X)\big)$. So also the following assertion is well-motivated.

Theorem IV.34. *We have* $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) \simeq \dot{\mathrm{E}}_{\mathrm{I}}\big(G(B, FA)\big)$.

So instead of "resolving X twice", we may just as well "resolve X once and Y once".

The left hand side spectral sequence converges to $\big(\mathrm{R}^{i+j}(G(Y, -) \circ F)\big)(X)$. By this theorem, so does the right hand side one.

The underlying double complexes are connected by two morphisms of double complexes (in the directions $\bullet \longrightarrow \bullet \longleftarrow \bullet$) that induce isomorphisms on the associated proper spectral sequences.

Of course, Theorems IV.31 and IV.34 have dual counterparts.

IV.0.4 Results of Beyl and Barnes

Let R be a commutative ring. Let G be a group. Let $N \trianglelefteq G$ be a normal subgroup. Let M be an RG-module.

Beyl generalises Grothendieck's setup, allowing for a variant of a Cartan-Eilenberg resolution that consists of acyclic, but no longer necessarily injective objects [**7,** Th. 3.4]. We have documented Beyl's Theorem as Theorem IV.40 in our framework, without claiming originality.

Beyl uses his Theorem to prove that, from the E_2-term on, the Grothendieck spectral sequence for $RG\text{-Mod} \xrightarrow{(-)^N} RN\text{-Mod} \xrightarrow{(-)^{G/N}} R\text{-Mod}$ at M is isomorphic to the Lyndon-Hochschild-Serre spectral sequence, i.e. the spectral sequence associated to the double complex ${}_{RG}(\mathrm{Bar}_{G/N;R} \otimes_R \mathrm{Bar}_{G;R}\,, M)$; cf. [**7,** Th. 3.5], [**6,** §3.5]. This is now also a consequence of Theorems IV.31 and IV.34, as explained in §§ IV.8.2, IV.8.3.

Barnes works in a slightly different setup. He supposes given a commutative ring R, abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of R-modules, and left exact functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$, where F is supposed to have an exact left adjoint $J : \mathcal{B} \longrightarrow \mathcal{A}$ that satisfies $F \circ J = 1_{\mathcal{B}}$. Moreover, he assumes $\mathcal{A}$ to have ample injectives and $\mathcal{C}$ to have enough injectives. In this setup, he obtains a general comparison theorem. See [**3,** Sec. X.5, Def. X.2.5, Th. X.5.4].

Beyl [**7**] and Barnes [**3**] also consider cup products; in this article, we do not.

IV.0.5 Acknowledgements

Results of Beyl and Haas are included for sake of documentation that they work within our framework; cf. Theorem IV.40 and §IV.4. No originality from my part is claimed.

I thank B. Keller for directing me to [**46,** XII.§11]. I thank the referee for helping to considerably improve the presentation, and for suggesting Lemma IV.47 and §IV.8.2. I thank G. Carnovale and G. Hiss for help with Hopf algebras.

Conventions

Throughout these conventions, let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{A}$ be an additive category, let $\mathcal{B}$ and $\mathcal{B}'$ be abelian categories, and let $\mathcal{E}$ be an exact category in which all idempotents split.

- For $a,\, b \in \mathbf{Z}$, we write $[a, b] := \{c \in \mathbf{Z} \ :\ a \leqslant c \leqslant b\}$, $[a, b[\, := \{c \in \mathbf{Z} \ :\ a \leqslant c < b\}$, etc.
- Given $I \subseteq \mathbf{Z}$ and $i \in \mathbf{Z}$, we write $I_{\geqslant i} := \{j \in I \ :\ j \geqslant i\}$ and $I_{<i} := \{j \in I \ :\ j < i\}$.
- The disjoint union of sets A and B is denoted by $A \sqcup B$.
- Composition of morphisms is written on the right, i.e. $\xrightarrow{a}\xrightarrow{b} \;=\; \xrightarrow{ab}$.
- Functors act on the left. Composition of functors is written on the left, i.e. $\xrightarrow{F}\xrightarrow{G} \;=\; \xrightarrow{G\circ F}$
- Given objects X, Y in $\mathcal{C}$, we denote the set of morphisms from X to Y by ${}_{\mathcal{C}}(X, Y)$.
- The category of functors from $\mathcal{C}$ to $\mathcal{D}$ and transformations between them is denoted by $[\![\mathcal{C}, \mathcal{D}]\!]$.
- Denote by $\mathrm{C}(\mathcal{A})$ the category of complexes

$$X \;=\; (\cdots \xrightarrow{d} X^{i-1} \xrightarrow{d} X^{i} \xrightarrow{d} X^{i+1} \xrightarrow{d} \cdots)$$

with values in $\mathcal{A}$. Denote by $\mathrm{C}^{[0}(\mathcal{A})$ the full subcategory of $\mathrm{C}(\mathcal{A})$ consisting of complexes X with $X^i = 0$ for $i < 0$. We have a full embedding $\mathcal{A} \xrightarrow{\mathrm{Conc}} \mathrm{C}^{[0}(\mathcal{A})$, where, given $X \in \mathrm{Ob}\,\mathcal{A}$, the complex $\mathrm{Conc}\,X$ has entry X at position 0 and zero elsewhere.

- Given a complex $X \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})$ and $k \in \mathbf{Z}$, we denote by $X^{\bullet+k}$ the complex that has differential $X^{i+k} \xrightarrow{(-1)^k d} X^{i+1+k}$ between positions i and $i+1$. We also write $X^{\bullet-1} := X^{\bullet+(-1)}$ etc.

- Suppose given a full additive subcategory $\mathcal{M} \subseteq \mathcal{A}$. Then $\mathcal{A}/\mathcal{M}$ denotes the quotient of $\mathcal{A}$ by $\mathcal{M}$, which has the same objects as $\mathcal{A}$, and which has as morphisms residue classes of morphisms of $\mathcal{A}$, where two morphisms are in the same residue class if their difference factors over an object of $\mathcal{M}$.

- A morphism in $\mathcal{A}$ is *split* if it isomorphic, as a diagram on $\bullet \longrightarrow \bullet$, to a morphism of the form $X \oplus Y \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} X \oplus Z$. A complex $X \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})$ is *split* if all of its differentials are split.

- An *elementary split acyclic* complex in $\mathrm{C}(\mathcal{A})$ is a complex of the form

$$\cdots \longrightarrow 0 \longrightarrow T \xrightarrow{1} T \longrightarrow 0 \longrightarrow \cdots,$$

where the entry T is at positions k and $k+1$ for some $k \in \mathbf{Z}$. A *split acyclic* complex is a complex isomorphic to a direct sum of elementary split acyclic complexes, i.e. a complex isomorphic to a complex of the form

$$\cdots \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} T^i \oplus T^{i+1} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} T^{i+1} \oplus T^{i+2} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} T^{i+2} \oplus T^{i+3} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} \cdots$$

Let $\mathrm{C}_{\mathrm{sp\,ac}}(\mathcal{A}) \subseteq \mathrm{C}(\mathcal{A})$ denote the full additive subcategory of split acyclic complexes. Let $\mathrm{K}(\mathcal{A}) := \mathrm{C}(\mathcal{A})/\mathrm{C}_{\mathrm{sp\,ac}}(\mathcal{A})$ denote the homotopy category of complexes with values in $\mathcal{A}$. Let $\mathrm{K}^{[0}(\mathcal{A})$ denote the image of $\mathrm{C}^{[0}(\mathcal{A})$ in $\mathrm{K}(\mathcal{A})$. A morphism in $\mathrm{C}(\mathcal{A})$ is a *homotopism* if its image in $\mathrm{K}(\mathcal{A})$ is an isomorphism.

- We denote by $\mathrm{Inj}\,\mathcal{B} \subseteq \mathcal{B}$ the full subcategory of injective objects.

- Concerning exact categories, introduced by QUILLEN **[52,** p. 15], we use the conventions of §I.6.2. In particular, a commutative quadrangle in $\mathcal{E}$ being a pullback is indicated by

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ C & \longrightarrow & D, \end{array}$$

a commutative quadrangle being a pushout by

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ C & \longrightarrow & D. \end{array}$$

- Given $X \in \mathrm{Ob}\,\mathrm{C}(\mathcal{E})$ with pure differentials, and given $k \in \mathbf{Z}$, we denote by $\mathrm{Z}^k X$ the kernel of the differential $X^k \longrightarrow X^{k+1}$, by $\mathrm{Z}'^k X$ the cokernel of the differential $X^{k-1} \longrightarrow X^k$, and by $\mathrm{B}^k X$ the image of the differential $X^{k-1} \longrightarrow X^k$. Furthermore, we have pure short exact sequences $\mathrm{B}^k X \rightarrowtail \mathrm{Z}^k X \twoheadrightarrow \mathrm{H}^k X$ and $\mathrm{H}^k X \rightarrowtail \mathrm{Z}'^k X \twoheadrightarrow \mathrm{B}^{k+1} X$.

- A morphism $X \longrightarrow Y$ in $\mathrm{C}(\mathcal{E})$ between complexes X and Y with pure differentials is a *quasiisomorphism* if H^k applied to it yields an isomorphism for all $k \in \mathbf{Z}$. A complex X with pure differentials is *acyclic* if $\mathrm{H}^k X \simeq 0$ for all $k \geqslant 0$. Such a complex is also called a *purely acyclic* complex.

- Suppose that $\mathcal{B}$ has enough injectives. Given a left exact functor $\mathcal{B} \xrightarrow{F} \mathcal{B}'$, an object $X \in \mathrm{Ob}\,\mathcal{B}$ is F-*acyclic* if $\mathrm{R}^i F X \simeq 0$ for all $i \geqslant 1$. In other words, X is F-acyclic if for an injective resolution $I \in \mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{B})$ of X (and then for all such injective resolutions), we have $\mathrm{H}^i F I \simeq 0$ for all $i \geqslant 1$.

- By a module, we understand a left module, unless stated otherwise. If A is a ring, we abbreviate ${}_A(-,=) := {}_{A\text{-Mod}}(-,=) = \mathrm{Hom}_A(-,=)$.

IV.1 Double and triple complexes

We fix some notations and sign conventions.

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let $\mathrm{C}(\mathcal{A}) \xrightarrow{H} \mathcal{B}$ be an additive functor.

IV.1.1 Double complexes

IV.1.1.1 Definition

A *double complex* with entries in $\mathcal{A}$ is a diagram

$$X \;=\; \begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
\cdots & \xrightarrow{d} & X^{i+2,j} & \xrightarrow{d} & X^{i+2,j+1} & \xrightarrow{d} & X^{i+2,j+2} & \xrightarrow{d} & \cdots \\
 & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
\cdots & \xrightarrow{d} & X^{i+1,j} & \xrightarrow{d} & X^{i+1,j+1} & \xrightarrow{d} & X^{i+1,j+2} & \xrightarrow{d} & \cdots \\
 & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
\cdots & \xrightarrow{d} & X^{i,j} & \xrightarrow{d} & X^{i,j+1} & \xrightarrow{d} & X^{i,j+2} & \xrightarrow{d} & \cdots \\
 & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in $\mathcal{A}$ such that $dd = 0$, $\partial\partial = 0$ and $d\partial = \partial d$ everywhere. As morphisms between double complexes, we take all diagram morphisms. Let $\mathrm{CC}(\mathcal{A})$ denote the category of double complexes. We may identify $\mathrm{CC}(\mathcal{A}) = \mathrm{C}(\mathrm{C}(\mathcal{A}))$.

The double complexes considered in this §IV.1.1 are stipulated to have entries in $\mathcal{A}$.

Let $\mathrm{CC}^{\llcorner}(\mathcal{A}) := \mathrm{C}^{[0}(\mathrm{C}^{[0}(\mathcal{A}))$ be the category of *first quadrant double complexes*, consisting of double complexes X such that $X^{i,j} = 0$ whenever $i < 0$ or $j < 0$.

Given a double complex X and $i \in \mathbf{Z}$, we let $X^{i,*} \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})$ denote the complex that has entry $X^{i,j}$ at position $j \in \mathbf{Z}$, the differentials taken accordingly; $X^{i,*}$ is called the ith *row* of X.

Similarly, given $j \in \mathbf{Z}$, $X^{*,j} \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})$ denotes the jth *column* of X.

IV.1.1.2 Applying H in different directions

Given $X \in \mathrm{Ob}\,\mathrm{CC}(\mathcal{A})$, we let $H(X^{*,-}) \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})$ denote the complex that has $H(X^{*,j})$ at position $j \in \mathbf{Z}$, and as differential $H(X^{*,j}) \longrightarrow H(X^{*,j+1})$ the image of the morphism $X^{*,j} \longrightarrow X^{*,j+1}$ of complexes under H. Similarly, $H(X^{-,*}) \in \mathrm{Ob}\,\mathrm{C}(\mathcal{A})$ has $H(X^{j,*})$ at position $j \in \mathbf{Z}$.

In other words, a "$*$" denotes the index direction to which H is applied, a "$-$" denotes the surviving index direction. For short, "$*$" before "$-$".

IV.1.1.3 Concentrated double complexes

Given a complex $U \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$, we denote by $\operatorname{Conc}_2 U \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{A})$ the double complex whose 0th row is given by U, and whose other rows are zero; i.e. given $j \in \mathbf{Z}$, then $(\operatorname{Conc}_2 U)^{i,j}$ equals U^j if $i = 0$, and 0 otherwise, the differentials taken accordingly. Similarly, $\operatorname{Conc}_1 U \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{B})$ denotes the double complex whose 0th column is given by U, and whose other columns are zero.

IV.1.1.4 Row- and columnwise notions

A morphism $X \xrightarrow{f} Y$ of double complexes is called a *rowwise homotopism* if $X^{i,*} \xrightarrow{f^{i,*}} Y^{i,*}$ is a homotopism for all $i \in \mathbf{Z}$. Provided $\mathcal{A}$ is abelian, it is called a *rowwise quasiisomorphism* if $X^{i,*} \xrightarrow{f^{i,*}} Y^{i,*}$ is a quasiisomorphism for all $i \in \mathbf{Z}$.

A morphism $X \xrightarrow{f} Y$ of double complexes is called a *columnwise homotopism* if $X^{*,j} \xrightarrow{f^{*,j}} Y^{*,j}$ is a homotopism for all $j \in \mathbf{Z}$. Provided $\mathcal{A}$ is abelian, it is called a *columnwise quasiisomorphism* if $X^{*,j} \xrightarrow{f^{*,j}} Y^{*,j}$ is a quasiisomorphism for all $j \in \mathbf{Z}$.

Provided $\mathcal{A}$ is abelian, a double complex X is called *rowwise split* if $X^{i,*}$ is split for all $i \in \mathbf{Z}$; a short exact sequence $X' \longrightarrow X \longrightarrow X''$ of double complexes is called *rowwise split short exact* if $X'^{i,*} \longrightarrow X^{i,*} \longrightarrow X''^{i,*}$ is split short exact for all $i \in \mathbf{Z}$.

A double complex X is called *rowwise split acyclic* if $X^{i,*}$ is a split acyclic complex for all $i \in \mathbf{Z}$. It is called *columnwise split acyclic* if $X^{*,j}$ is a split acyclic complex for all $j \in \mathbf{Z}$.

IV.1.1.5 Horizontally and vertically split acyclic double complexes

An *elementary horizontally split acyclic* double complex is a double complex of the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & T^{i+1} & = & T^{i+1} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & T^{i} & = & T^{i} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & .
\end{array}$$

A *horizontally split acyclic* double complex is a double complex isomorphic to a direct sum of elementary horizontally split acyclic double complexes, i.e. to one of the form

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & T^{i+1,j} \oplus T^{i+1,j+1} & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} & T^{i+1,j+1} \oplus T^{i+1,j+2} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \left(\begin{smallmatrix} \partial & 0 \\ 0 & \partial \end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} \partial & 0 \\ 0 & \partial \end{smallmatrix}\right)} & & \\
\cdots & \longrightarrow & T^{i,j} \oplus T^{i,j+1} & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} & T^{i,j+1} \oplus T^{i,j+2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & .
\end{array}
$$

An *elementary vertically split acyclic* double complex is a double complex of the form

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & T^{i} & \xrightarrow{d} & T^{i+1} & \longrightarrow & \cdots \\
 & & \| & & \| & & \\
\cdots & \longrightarrow & T^{i} & \xrightarrow{d} & T^{i+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & .
\end{array}
$$

A *vertically split acyclic* double complex is a double complex isomorphic to a direct sum of elementary vertically split acyclic double complexes, i.e. to one of the form

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & T^{i+1,j} \oplus T^{i+2,j} & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)} & T^{i+1,j+1} \oplus T^{i+2,j+1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} & & \\
\cdots & \longrightarrow & T^{i,j} \oplus T^{i+1,j} & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)} & T^{i,j+1} \oplus T^{i+1,j+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & .
\end{array}
$$

A horizontally split acyclic double complex is in particular rowwise split acyclic. A vertically split acyclic double complex is in particular columnwise split acyclic.

A double complex is called *split acyclic* if it is isomorphic to the direct sum of a horizontally and a vertically split acyclic double complex. Let $\mathrm{CC}_{\mathrm{sp\,ac}}(\mathcal{A})$ denote the full additive subcategory of split acyclic double complexes. Let

$$\mathrm{KK}(\mathcal{A}) \;:=\; \mathrm{CC}(\mathcal{A})/\mathrm{CC}_{\mathrm{sp\,ac}}(\mathcal{A})\;;$$

cf. [**10,** IV.§4]. A morphism in $\mathrm{CC}(\mathcal{A})$ that is mapped to an isomorphism in $\mathrm{KK}(\mathcal{A})$ is called a *double homotopism.*

> A speculative aside. The category $\mathrm{K}(\mathcal{A})$ is Heller triangulated; cf. Def. I.5.(i), Th. I.32. Such a Heller triangulation hinges on two induced shift functors, one of them induced by the shift functor on $\mathrm{K}(\mathcal{A})$. Now $\mathrm{KK}(\mathcal{A})$ carries two shift functors, and so there might be more isomorphisms between induced shift functors one can fix. How can the formal structure of $\mathrm{KK}(\mathcal{A})$ be described?

IV.1.1.6 Total complex

Let $\mathrm{KK}^{\llcorner}(\mathcal{A})$ be the full image of $\mathrm{CC}^{\llcorner}(\mathcal{A})$ in $\mathrm{KK}(\mathcal{A})$.

The *total complex* $\mathrm{t}X$ of a double complex $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ is given by the complex

$$\mathrm{t}X \;=\; \Big(X^{0,0} \xrightarrow{(d\;\partial)} X^{0,1} \oplus X^{1,0} \xrightarrow{\left(\begin{smallmatrix} d & \partial & 0 \\ 0 & -d & -\partial \end{smallmatrix}\right)} X^{0,2} \oplus X^{1,1} \oplus X^{2,0} \xrightarrow{\left(\begin{smallmatrix} d & \partial & 0 & 0 \\ 0 & -d & -\partial & 0 \\ 0 & 0 & d & \partial \end{smallmatrix}\right)} X^{0,3} \oplus X^{1,2} \oplus X^{2,1} \oplus X^{3,0} \longrightarrow \cdots \Big)$$

in $\mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$. Using the induced morphisms, we obtain a total complex functor $\mathrm{CC}^{\llcorner}(\mathcal{A}) \xrightarrow{\mathrm{t}} \mathrm{C}^{[0}(\mathcal{A})$. Since t maps elementary horizontally or vertically split acyclic double complexes to split acyclic complexes, it induces a functor $\mathrm{KK}^{\llcorner}(\mathcal{A}) \xrightarrow{\mathrm{t}} \mathrm{K}^{[0}(\mathcal{A})$. If, in addition, $\mathcal{A}$ is abelian, the total complex functor maps rowwise quasiisomorphisms and columnwise quasiisomorphisms to quasiisomorphisms, as one sees using the long exact homology sequence and induction on a suitable filtration.

Note that we have an isomorphism $U \xrightarrow{\sim} \mathrm{t}\,\mathrm{Conc}_1 U$, natural in $U \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$, having entries 1_{U_0}, 1_{U_1}, -1_{U_2}, -1_{U_3}, 1_{U_4}, etc. Moreover, $U = \mathrm{t}\,\mathrm{Conc}_2 U$, natural in $U \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$.

IV.1.1.7 The homotopy category of first quadrant double complexes as a quotient

Lemma IV.1 *The residue class functor* $\mathrm{CC}(\mathcal{A}) \longrightarrow \mathrm{KK}(\mathcal{A})$*, restricted to* $\mathrm{CC}^{\llcorner}(\mathcal{A}) \longrightarrow \mathrm{KK}^{\llcorner}(\mathcal{A})$*, induces an equivalence*

$$\mathrm{CC}^{\llcorner}(\mathcal{A})/\big(\mathrm{CC}_{\mathrm{sp\,ac}}(\mathcal{A}) \cap \mathrm{CC}^{\llcorner}(\mathcal{A})\big) \ \xrightarrow{\sim}\ \mathrm{KK}^{\llcorner}(\mathcal{A}) .$$

Proof. We have to show faithfulness; i.e. that if a morphism $X \longrightarrow Y$ in $\mathrm{CC}^{\llcorner}(\mathcal{A})$ factors over a split acyclic double complex, then it factors over a split acyclic double complex that lies in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. By symmetry and additivity, it suffices to show that if a morphism $X \longrightarrow Y$ in $\mathrm{CC}^{\llcorner}(\mathcal{A})$ factors over a horizontally split acyclic double complex, then it factors over a horizontally split acyclic double complex that lies in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. Furthermore, we may assume $X \longrightarrow Y$ to factor over an elementary horizontally split acyclic double complex S concentrated in the columns k and $k+1$ for some $k \in \mathbf{Z}$. We may assume that $S^{i,j} = 0$ for $i < 0$ and $j \in \mathbf{Z}$. If $k < 0$, and in particular, if $k = -1$, then $X \longrightarrow Y$ is zero because $S \longrightarrow Y$ is zero, so that in this case we may assume $S = 0$. On the other hand, if $k \geqslant 0$, then $S \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. ▫

Cf. also the similar Remark IV.2.

IV.1.2 Triple complexes

IV.1.2.1 Definition

Let $\mathrm{CCC}(\mathcal{A}) := \mathrm{C}(\mathrm{C}(\mathrm{C}(\mathcal{A})))$ be the category of *triple complexes.* A triple complex Y has entries $Y^{k,\ell,m}$ for $k,\ \ell,\ m\ \in\ \mathbf{Z}$.

We denote the differentials in the three directions by $Y^{k,\ell,m} \xrightarrow{d_1} Y^{k+1,\ell,m}$, $Y^{k,\ell,m} \xrightarrow{d_2} Y^{k,\ell+1,m}$ and $Y^{k,\ell,m} \xrightarrow{d_3} Y^{k,\ell,m+1}$, respectively.

Let $k,\ \ell,\ m\ \in\ \mathbf{Z}$. We shall use the notation $Y^{-,\ell,=}$ for the double complex having at position (k,m) the entry $Y^{k,\ell,m}$, differentials taken accordingly. Similarly the complex $Y^{k,\ell,*}$ etc.

Given a triple complex $Y \in \mathrm{Ob}\,\mathrm{CCC}(\mathcal{A})$, we write $HY^{-,=,*} \in \mathrm{Ob}\,\mathrm{CC}(\mathcal{A})$ for the double complex having at position (k,ℓ) the entry $H(Y^{k,\ell,*})$, differentials taken accordingly.

Denote by $\mathrm{CCC}^{\llcorner}(\mathcal{A}) \subseteq \mathrm{CCC}(\mathcal{A})$ the full subcategory of *first octant triple complexes;* i.e. triple complexes Y having $Y^{k,\ell,m} = 0$ whenever $k < 0$ or $\ell < 0$ or $m < 0$.

IV.1.2.2 Planewise total complex

For $Y \in \mathrm{Ob}\,\mathrm{CCC}^{\llcorner}(\mathcal{A})$ we denote by $\mathrm{t}_{1,2}Y \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ the *planewise total complex* of Y, defined for $m \in \mathbf{Z}$ as

$$(\mathrm{t}_{1,2}Y)^{*,m} \ :=\ \mathrm{t}(Y^{-,=,m})\ ,$$

with the differentials of $\mathrm{t}_{1,2}Y$ in the horizontal direction being induced by the differentials in the third index direction of Y, and with the differentials of $\mathrm{t}_{1,2}Y$ in the vertical direction being given by the total complex differentials. Explicitly, given $k,\ \ell\ \geqslant\ 0$, we have

$$(\mathrm{t}_{1,2}Y)^{k,\ell} = \bigoplus_{i,j\,\geqslant 0,\; i+j\,=\,k} Y^{i,j,\ell}\ .$$

By means of induced morphisms, this furnishes a functor

$$\begin{array}{rcl} \mathrm{CCC}^{\swarrow}(\mathcal{A}) & \xrightarrow{\mathrm{t}_{1,2}} & \mathrm{CC}^{\llcorner}(\mathcal{A}) \\ Y & \longmapsto & \mathrm{t}_{1,2}Y\ . \end{array}$$

IV.2 Cartan-Eilenberg resolutions

We shall use QUILLEN's language of exact categories [**52,** p. 15] to deal with Cartan-Eilenberg resolutions [**10,** XVII.§1], as it has been done by MAC LANE already before this language was available; cf. [**46,** XII.§11]. The assertions in this section are for the most part wellknown.

IV.2.1 A remark

Remark IV.2 *Let $\mathcal{A}$ be an additive category. Then* $\mathrm{C}^{[0}(\mathcal{A})/\big(\mathrm{C}^{[0}(\mathcal{A}) \cap \mathrm{C}_{\mathrm{sp\,ac}}(\mathcal{A})\big) \longrightarrow \mathrm{K}^{[0}(\mathcal{A})$ *is an equivalence.*

Proof. Faithfulness is to be shown. A morphism $X \longrightarrow Y$ in $\mathrm{C}^{[0}(\mathcal{A})$ that factors over an elementary split acyclic complex of the form $(\cdots \longrightarrow 0 \longrightarrow T = T \longrightarrow 0 \longrightarrow \cdots)$ with T in positions k and $k+1$ is zero, provided $k < 0$. □

IV.2.2 Exact categories

Concerning the terminology of exact categories, introduced by QUILLEN [**52,** p. 15], we refer to §I.6.2.

Let $\mathcal{E}$ be an exact category in which all idempotents split. An object $I \in \mathrm{Ob}\,\mathcal{E}$ is called *relatively injective,* or a *relative injective* (relative to the set of pure short exact sequences, that is), if ${}_{\mathcal{E}}(-, I)$ maps pure short exact sequences of $\mathcal{E}$ to short exact sequences. We say that $\mathcal{E}$ has *enough relative injectives,* if for all $X \in \mathrm{Ob}\,\mathcal{E}$, there exists a relative injective I and a pure monomorphism $X \rightarrowtail I$.

In case $\mathcal{E}$ is an abelian category, with all short exact sequences stipulated to be pure, then we omit "relative" and speak of "injectives" etc.

Definition IV.3 Suppose given a complex $X \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{E})$ with pure differentials. A *relatively injective complex resolution* of X is a complex $I \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{E})$, together with a quasiisomorphism $X \longrightarrow I$, such that the following properties are satisfied.

(1) The object entries of I are relatively injective.

(2) The differentials of I are pure.

(3) The quasiisomorphism $X \longrightarrow I$ consists of pure monomorphisms.

We often refer to such a relatively injective complex resolution just by I.

A *relatively injective object resolution*, or just a *relatively injective resolution,* of an object $Y \in \operatorname{Ob}\mathcal{E}$ is a relatively injective complex resolution of $\operatorname{Conc} Y$.

A *relatively injective resolution* is the complex of a relatively injective object resolution of some object in $\mathcal{E}$.

Remark IV.4 *Suppose that $\mathcal{E}$ has enough relative injectives. Every complex $X \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{E})$ with pure differentials has a relatively injective complex resolution $I \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{E})$.*

In particular, every object $Y \in \operatorname{Ob}\mathcal{E}$ has a relatively injective resolution $J \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{E})$.

Proof. Let $X^0 \rightarrowtail I^0$ be a pure monomorphism into a relatively injective object I^0. Forming a pushout along $X^0 \rightarrowtail I^0$, we obtain a pointwise purely monomorphic morphism of complexes $X \longrightarrow X'$ with $X'^0 = I^0$ and $X'^k = X^k$ for $k \geqslant 2$. By considering its cokernel, we see that it is a quasiisomorphism. So we may assume X^0 to be relatively injective.

Let $X^1 \rightarrowtail I^1$ be a pure monomorphism into a relatively injective object I^1. Form a pushout along $X^1 \rightarrowtail I^1$ etc. $\square$

Remark IV.5 *Suppose given $X \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{E})$ with pure differentials such that $\mathrm{H}^k X \simeq 0$ for $k \geqslant 1$. Suppose given $I \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{E})$ such that I^k is purely injective for $k \geqslant 0$, and such that the differential $I^0 \xrightarrow{d} I^1$ has a kernel in $\mathcal{E}$. Then the map*

$$ {}_{\mathrm{K}^{[0}(\mathcal{E})}(X, I) \longrightarrow {}_{\mathcal{E}}\big(\operatorname{Kern}(X^0 \xrightarrow{d} X^1), \operatorname{Kern}(I^0 \xrightarrow{d} I^1)\big) $$

that sends a representing morphism of complexes to the morphism induced on the mentioned kernels, is bijective.

Suppose $\mathcal{E}$ to have enough relative injectives. Let $\mathcal{I} \subseteq \mathcal{E}$ denote the full subcategory of relative injectives. Let $\mathrm{C}^{[0,\mathrm{res}}(\mathcal{I})$ denote the full subcategory of $\mathrm{C}^{[0}(\mathcal{I})$ consisting of complexes X with pure differentials such that $\mathrm{H}^k X \simeq 0$ for $k \geqslant 1$. Let $\mathrm{K}^{[0,\mathrm{res}}(\mathcal{I})$ denote the image of $\mathrm{C}^{[0,\mathrm{res}}(\mathcal{I})$ in $\mathrm{K}(\mathcal{E})$.

Remark IV.6 *The functor $\mathrm{C}^{[0,\mathrm{res}}(\mathcal{I}) \longrightarrow \mathcal{E}$, $X \longmapsto \mathrm{H}^0(X)$, induces an equivalence*

$$ \mathrm{K}^{[0,\mathrm{res}}(\mathcal{I}) \xrightarrow{\sim} \mathcal{E} \,. $$

Proof. This functor is dense by Remark IV.4, and full and faithful by Remark IV.5. $\square$

Remark IV.7 (exact Horseshoe Lemma)
Given a pure short exact sequence $X' \longrightarrow X \longrightarrow X''$ and relatively injective resolutions I' of X' and I'' of X'', there exists a relatively injective resolution I of X and a pointwise split short exact sequence $I' \longrightarrow I \longrightarrow I''$ that maps under H^0 to $X' \longrightarrow X \longrightarrow X''$.

Proof. Choose pure monomorphisms $X' \rightarrowtail I'^0$ and $X'' \rightarrowtail I''^0$ into relative injectives I'^0 and I''^0. Embed them into a morphism from the pure short exact sequence $X' \rightarrowtail X \twoheadrightarrow X''$ to the split short exact sequence $I' \xrightarrow{(1\,0)} I' \oplus I'' \xrightarrow{\binom{0}{1}} I''$. Insert the pushout T of $X' \xrightarrow{\bullet} X$ along $X' \xrightarrow{\bullet} I'^0$ and the pullback of $I'^0 \oplus I''^0 \xrightarrow{+} I''^0$ along $X'' \xrightarrow{\bullet} I''^0$ to see that $X \longrightarrow I'^0 \oplus I''^0$ is purely monomorphic. So we can take the cokernel $\mathrm{B}^1 I' \longrightarrow \mathrm{B}^1 I \longrightarrow \mathrm{B}^1 I''$ of this morphism of pure short exact sequences. Considering the cokernels on the commutative triangle $(X, T, I'^0 \oplus I''^0)$ of pure monomorphisms, we obtain a bicartesian square $(T,\ I'^0 \oplus I''^0,\ \mathrm{B}^1 I',\ \mathrm{B}^1 I)$ and conclude that the sequence of cokernels is itself purely short exact. So we can iterate. ▫

IV.2.3 An exact category structure on $\mathrm{C}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category with enough injectives.

Remark IV.8 *The following conditions on a short exact sequence* $X' \longrightarrow X \longrightarrow X''$ *in* $\mathrm{C}(\mathcal{A})$ *are equivalent.*

(1) *All connectors in its long exact homology sequence are equal to zero.*

(2) *The sequence* $\mathrm{B}^k X' \longrightarrow \mathrm{B}^k X \longrightarrow \mathrm{B}^k X''$ *is short exact for all* $k \in \mathbf{Z}$.

(3) *The morphism* $\mathrm{Z}^k X \longrightarrow \mathrm{Z}^k X''$ *is epimorphic for all* $k \in \mathbf{Z}$.

(3′) *The morphism* $\mathrm{Z}'^k X' \longrightarrow \mathrm{Z}'^k X$ *is monomorphic for all* $k \in \mathbf{Z}$.

(4) *The diagram*

$$\begin{array}{ccccc}
\mathrm{B}^k X' & \longrightarrow & \mathrm{Z}^k X' & \longrightarrow & \mathrm{H}^k X' \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{B}^k X & \longrightarrow & \mathrm{Z}^k X & \longrightarrow & \mathrm{H}^k X \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{B}^k X'' & \longrightarrow & \mathrm{Z}^k X'' & \longrightarrow & \mathrm{H}^k X''
\end{array}$$

has short exact rows and short exact columns for all $k \in \mathbf{Z}$.

Proof. We consider the diagram in (4) as a (horizontal) short exact sequence of (vertical) complexes and regard its long exact homology sequence. Taking into account that all assertions are supposed to hold for all $k \in \mathbf{Z}$, we can employ the long exact homology sequence on $X' \longrightarrow X \longrightarrow X''$ to prove the equivalence of (1), (2), (3) and (4).

Now the assertion (1) $\Longleftrightarrow$ (3) is dual to the assertion (1) $\Longleftrightarrow$ (3′). ▫

Remark IV.9 *The category* $\mathrm{C}(\mathcal{A})$*, equipped with the set of short exact sequences that have zero connectors on homology as pure short exact sequences, is an exact category with enough relatively injective objects in which all idempotents split. With respect to this exact category structure on* $\mathrm{C}(\mathcal{A})$*, a complex is relatively injective if and only if it is split and has injective object entries.*

Cf. [**46,** XII.§11], where pure short exact sequences are called *proper*. A relatively injective object in $\mathrm{C}(\mathcal{A})$ is also referred to as an *injectively split complex*. To a relatively injective resolution of a complex $X \in \operatorname{Ob} \mathrm{C}(\mathcal{A})$, we also refer as a *Cartan-Eilenberg-resolution,* or, for short, as a *CE-resolution* of X; cf. [**10,** XVII.§1]. A *CE-resolution* is a CE-resolution of some complex. Considered as a double complex, it is in particular rowwise split and has injective object entries.

Given a morphism $X \xrightarrow{f} X'$ in $\mathrm{C}(\mathcal{A})$, CE-resolutions J of X and J' of X', a morphism $J \xrightarrow{\hat{f}} J'$ in $\mathrm{CC}(\mathcal{A})$ such that $(J^{i,j} \xrightarrow{\hat{f}^{i,j}} J'^{i,j}) = (0 \longrightarrow 0)$ for $i < 0$ and such that

$$\mathrm{H}^0(J^{*,-} \xrightarrow{\hat{f}^{*,-}} J'^{*,-}) \;=\; (X \xrightarrow{f} X')$$

is called a *CE-resolution* of $X \xrightarrow{f} X'$. By Remarks IV.9 and IV.6, each morphism in $\mathrm{C}(\mathcal{A})$ has a CE-resolution.

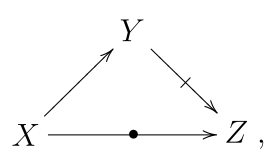
Proof of Remark IV.9. We *claim* that $\mathrm{C}(\mathcal{A})$, equipped with the said set of short exact sequences, is an exact category. We verify the conditions (Ex $1, 2, 3$) listed in §I.6.2. The conditions (Ex $1^\circ, 2^\circ, 3^\circ$) then follow by duality.

Note that by Remark IV.8.(3′), a monomorphism $X \longrightarrow Y$ in $\mathrm{C}(\mathcal{A})$ is pure if and only if $\mathrm{Z}'^k(X \longrightarrow Y)$ is monomorphic in $\mathcal{A}$ for all $k \in \mathbf{Z}$.

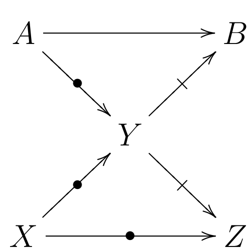
Ad (Ex 1). To see that a split monomorphism is pure, we may use additivity of the functor Z'^k for $k \in \mathbf{Z}$.

Ad (Ex 2). To see that the composition of two pure monomorphisms is pure, we may use Z'^k being a functor for $k \in \mathbf{Z}$.

Ad (Ex 3). Suppose given a commutative triangle



in $\mathrm{C}(\mathcal{A})$. Applying the functor Z'^k to it, for $k \in \mathbf{Z}$, we conclude that $\mathrm{Z}'^k(X \longrightarrow Y)$ is monomorphic, whence $X \longrightarrow Y$ is purely monomorphic. So we may complete to



in $\mathrm{C}(\mathcal{A})$ with (X, Y, B) and (A, Y, Z) pure short exact sequences. Applying Z'^k to this diagram, we conclude that $\mathrm{Z}'^k(A \longrightarrow B)$ is a monomorphism for $k \in \mathbf{Z}$, whence $A \longrightarrow B$ is a pure monomorphism.

This proves the *claim.*

Note that idempotents in $\mathrm{C}(\mathcal{A})$ are split since $\mathrm{C}(\mathcal{A})$ is also an abelian category.

We *claim* relative injectivity of complexes with split differentials and injective object entries. By a direct sum decomposition, and using the fact that any monomorphism from an elementary split acyclic complex with injective entries to an arbitrary complex is split, we are reduced to showing that a pure monomorphism from a complex with a single nonzero injective entry, at position 0, say, to an arbitrary complex is split. So suppose given $I \in \operatorname{Ob}\operatorname{Inj}\mathcal{A}$, $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$ and a pure monomorphism $\operatorname{Conc} I \rightarrowtail X$. Using Remark IV.8.(3′), we may choose a retraction to the composite $(I \longrightarrow X^0 \longrightarrow \mathrm{Z}'^0 X)$. This yields a retraction to $I \longrightarrow X^0$ that composes to 0 with $X^{-1} \longrightarrow X^0$, which can be employed for the sought retraction $X \longrightarrow \operatorname{Conc} I$. This proves the *claim.*

Let $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$. We *claim* that there exists a pure monomorphism from X to a relatively injective complex. Since $\mathcal{A}$ has enough injectives, by a direct sum decomposition we are reduced to finding a pure monomorphism from X to a split complex. Consider the following morphism φ_k of complexes for $k \in \mathbf{Z}$,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & X^k & \xrightarrow{(1\,0)} & X^k \oplus \mathrm{Z}'^k X & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle (1\,p)} & & \uparrow & & \\
\cdots & \longrightarrow & X^{k-2} & \xrightarrow{d} & X^{k-1} & \xrightarrow{d} & X^k & \xrightarrow{d} & X^{k+1} & \longrightarrow & \cdots ,
\end{array}$$

where $X^k \xrightarrow{p} \mathrm{Z}'^k X$ is taken from X. The functor Z'^k maps it to the identity. We take the direct sum of the upper complexes over $k \in \mathbf{Z}$ and let the morphisms φ_k be the components of a morphism φ from X to this direct sum. At position k, this morphism φ is monomorphic because φ_k is. Moreover, $\mathrm{Z}'^k(\varphi)$ is a monomorphism because $\mathrm{Z}'^k(\varphi_k)$ is. Hence φ is purely monomorphic by condition (3′) of Remark IV.8. This proves the *claim.* ▫

Remark IV.10 *Write* $\mathcal{E} := \mathrm{C}(\mathcal{A})$. *Given* $\ell \geqslant 0$, *we have a homology functor* $\mathcal{E} \xrightarrow{\mathrm{H}^\ell} \mathcal{A}$, *which induces a functor* $\mathrm{C}(\mathcal{E}) \xrightarrow{\mathrm{C}(\mathrm{H}^\ell)} \mathrm{C}(\mathcal{A})$. *Suppose given a purely acyclic complex* $X \in \operatorname{Ob}\mathrm{C}(\mathcal{E})$. *Then* $\mathrm{C}(\mathrm{H}^\ell)X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$ *is acyclic.*

Proof. This follows using the definition of pure short exact sequences, i.e. Remark IV.8.(1). ▫

IV.2.4 An exact category structure on $\mathrm{C}^{[0}(\mathcal{A})$

Write $\mathrm{CC}^{\llcorner,\mathrm{CE}}(\operatorname{Inj}\mathcal{A})$ for the full subcategory of $\mathrm{CC}^{\llcorner}(\mathcal{A})$ whose objects are CE-resolutions. Write $\mathrm{KK}^{\llcorner,\mathrm{CE}}(\operatorname{Inj}\mathcal{A})$ for the full subcategory of $\mathrm{KK}^{\llcorner}(\mathcal{A})$ whose objects are CE-resolutions.

Remark IV.11 *The category* $\mathrm{C}^{[0}(\mathcal{A})$, *equipped with the short exact sequences that lie in* $\mathrm{C}^{[0}(\mathcal{A})$ *and that are pure in* $\mathrm{C}(\mathcal{A})$ *in the sense of Remark* IV.9 *as pure short exact sequences, is an exact category wherein idempotents are split. It has enough relative injectives, viz. injectively split complexes that lie in* $\mathrm{C}^{[0}(\mathcal{A})$.

Proof. To show that it has enough relative injectives, we replace φ_0 in the proof of Remark IV.9 by $X \xrightarrow{\varphi_0'} \operatorname{Conc} X^0$, defined by $X_0 \xrightarrow{1_{X_0}} X_0$ at position 0. ▫

IV.2.5 The Cartan-Eilenberg resolution of a quasiisomorphism

Abbreviate $\mathcal{E} := \mathrm{C}(\mathcal{A})$, which is an exact category as in Remark IV.9. Consider $\mathrm{CC}^{\llcorner}(\mathcal{A}) \subseteq \mathrm{C}^{[0}(\mathcal{E})$, where the second index of $X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{A})$ counts the positions in $\mathcal{E} = \mathrm{C}(\mathcal{A})$; i.e. when X is viewed as a complex with values in $\mathcal{E}$, its entry at position k is given by $X^{k,*} \in \mathcal{E} = \mathrm{C}(\mathcal{A})$.

Remark IV.12 *Suppose given a split acyclic complex* $X \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$. *There exists a horizontally split acyclic CE-resolution* $J \in \operatorname{Ob} \mathrm{CC}^{\llcorner,\mathrm{CE}}(\operatorname{Inj} \mathcal{A})$ *of* X.

Proof. This holds for an elementary split acyclic complex, and thus also in the general case by taking a direct sum. □

Lemma IV.13 *Suppose given* $X \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{A})$ *with pure differentials when considered as an object of* $\mathrm{C}^{[0}(\mathcal{E})$, *and with* $\mathrm{H}^k(X^{*,-}) \simeq 0$ *in* $\mathrm{C}^{[0}(\mathcal{A})$ *for* $k \geqslant 1$.

Suppose given $J \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\operatorname{Inj} \mathcal{A})$ *with split rows* $J^{k,*}$ *for* $k \geqslant 1$. *In other words,* J *is supposed to consist of relative injective object entries when considered as an object of* $\mathrm{C}^{[0}(\mathcal{E})$.

Then the map

$$(*) \qquad {}_{\mathrm{KK}^{\llcorner}(\mathcal{A})}(X, J) \xrightarrow{\mathrm{H}^0((-)^{*,-})} {}_{\mathrm{K}^{[0}(\mathcal{A})}\big(\mathrm{H}^0(X^{*,-}),\, \mathrm{H}^0(J^{*,-})\big)$$

is bijective.

Proof. First, we observe that by Remark IV.5, we have

$$(**) \qquad {}_{\mathrm{K}^{[0}(\mathcal{E})}(X, J) \xrightarrow[\sim]{\mathrm{H}^0((-)^{*,-})} {}_{\mathcal{E}}\big(\mathrm{H}^0(X^{*,-}),\, \mathrm{H}^0(J^{*,-})\big)\ .$$

So it remains to show that $(*)$ is injective. Let $X \xrightarrow{f} J$ be a morphism that vanishes under $(*)$. Then $\mathrm{H}^0(X^{*,-}) \longrightarrow \mathrm{H}^0(J^{*,-})$ factors over a split acyclic complex $S \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$; cf. Remark IV.2. Let K be a horizontally split acyclic CE-resolution of S; cf. Remark IV.12. By Remark IV.5, we obtain a morphism $X \longrightarrow K$ that lifts $\mathrm{H}^0(X^{*,-}) \longrightarrow S$ and a morphism $K \longrightarrow J$ that lifts $S \longrightarrow \mathrm{H}^0(J^{*,-})$. The composite $X \longrightarrow K \longrightarrow J$ vanishes in $\mathrm{KK}^{\llcorner}(\mathcal{A})$. The difference

$$(X \xrightarrow{f} J) - (X \longrightarrow K \longrightarrow J)$$

lifts $\mathrm{H}^0(X^{*,-}) \xrightarrow{0} \mathrm{H}^0(J^{*,-})$. Hence by $(**)$, it vanishes in $\mathrm{K}^{[0}(\mathcal{E})$ and so a fortiori in $\mathrm{KK}^{\llcorner}(\mathcal{A})$. Altogether, $X \xrightarrow{f} J$ vanishes in $\mathrm{KK}^{\llcorner}(\mathcal{A})$. □

Proposition IV.14 *The functor* $\mathrm{CC}^{\llcorner,\mathrm{CE}}(\operatorname{Inj} \mathcal{A}) \xrightarrow{\mathrm{H}^0((-)^{*,-})} \mathrm{C}^{[0}(\mathcal{A})$ *induces an equivalence*

$$\mathrm{KK}^{\llcorner,\mathrm{CE}}(\operatorname{Inj} \mathcal{A}) \xrightarrow[\sim]{\mathrm{H}^0((-)^{*,-})} \mathrm{K}^{[0}(\mathcal{A})\ .$$

Proof. By Lemma IV.13, this functor is full and faithful. By Remark IV.4, it is dense. □

Corollary IV.15 *Suppose given* $X, X' \in \operatorname{Ob} \mathcal{C}^{[0}(\mathcal{A})$. *Let* J *be a CE-resolution of* X. *Let* J' *be a CE-resolution of* X'. *If* X *and* X' *are isomorphic in* $\mathrm{K}^{[0}(\mathcal{A})$, *then* J *and* J' *are isomorphic in* $\mathrm{KK}^{\llcorner}(\mathcal{A})$.

The following lemma is to be compared to Remark IV.12.

Lemma IV.16 *Suppose given an acyclic complex* $X \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$. *There exists a rowwise split acyclic CE-resolution* J *of* X. *Each CE-resolution of* X *is isomorphic to* J *in* $\mathrm{KK}^{\llcorner}(\mathcal{A})$.

Proof. By Corollary IV.15, it suffices to show that there exists a rowwise split acyclic CE-resolution of X. Recall that a CE-resolution of an arbitrary complex $Y \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$ can be constructed by a choice of injective resolutions of $\mathrm{H}^k Y$ and $\mathrm{B}^k Y$ for $k \in \mathbf{Z}$, followed by an application of the abelian Horseshoe Lemma to the short exact sequences $\mathrm{B}^k Y \longrightarrow \mathrm{Z}^k Y \longrightarrow \mathrm{H}^k Y$ for $k \in \mathbf{Z}$ and then to $\mathrm{Z}^k Y \longrightarrow Y^k \longrightarrow \mathrm{B}^{k+1} Y$ for $k \in \mathbf{Z}$; cf. [**10,** Chap. XVII, Prop. 1.2]. Since $\mathrm{H}^k X = 0$ for $k \in \mathbf{Z}$, we may choose the zero resolution for it. Applying this construction, we obtain a rowwise split acyclic CE-resolution. $\square$

Given $X \xrightarrow{f} X'$ in $\mathrm{C}^{[0}(\mathcal{A})$, a morphism $J \xrightarrow{\hat{f}} J'$ in $\mathrm{CC}^{\llcorner}(\mathcal{A})$ is called a *CE-resolution* of $X \xrightarrow{f} X'$ if $\mathrm{H}^0(\hat{f}^{*,-}) \simeq f$, as diagrams of the form $\bullet \longrightarrow \bullet$. By Remark IV.5, given CE-resolutions J of X and J' of X', there exists a CE-resolution $J \xrightarrow{\hat{f}} J'$ of $X \xrightarrow{f} X'$.

Proposition IV.17 *Let $X \xrightarrow{f} X'$ be a quasiisomorphism in $\mathrm{C}^{[0}(\mathcal{A})$. Let $J \xrightarrow{\hat{f}} J'$ be a CE-resolution of $X \xrightarrow{f} X'$. Then $\hat{f}$ can be written as a composite in $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\operatorname{Inj} \mathcal{A})$ of a rowwise homotopism, followed by a double homotopism.*

Proof. Choose a pointwise split monomorphism $X \xrightarrow{a} A$ into a split acyclic complex X. We can factor

$$(X \xrightarrow{f} X') \quad = \quad \left(X \xrightarrow{(f\ a)} X' \oplus A \xrightarrow{\binom{1}{0}} X'\right) ,$$

so that $(f\ a)$ is a pointwise split monomorphism. Let B be a CE-resolution of A. Choosing a CE-resolution b of a, we obtain the factorisation

$$(J \xrightarrow{\hat{f}} J') \quad = \quad \left(J \xrightarrow{(\hat{f}\ b)} J' \oplus B \xrightarrow{\binom{1}{0}} J'\right) .$$

Since $X' \oplus A \xrightarrow{\binom{1}{0}} X'$ is a homotopism, $J' \oplus B \xrightarrow{\binom{1}{0}} J$ is a double homotopism; cf. Corollary IV.15. Hence $\hat{f}$ is a composite of a rowwise homotopism and a double homotopism if and only if this holds for $(\hat{f}\ b)$. So we may assume that f is pointwise split monomorphic, so in particular, monomorphic.

By Proposition IV.14, we may replace the given CE-resolution $\hat{f}$ by an arbitrary CE-resolution of f between J and an arbitrarily chosen CE-resolution of X' without changing the property of being a composite of a rowwise homotopism and a double homotopism for this newly chosen CE-resolution of f.

Let $X \xrightarrow{f} X' \longrightarrow \bar{X}$ be a short exact sequence in $\mathrm{C}^{[0}(\mathcal{A})$. Since f is a quasiisomorphism, $\bar{X} \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$ is acyclic. Let $\bar{J}$ be a rowwise split acyclic CE-resolution of $\bar{X}$; cf. Lemma IV.16. The short exact sequence $X \xrightarrow{f} X' \longrightarrow \bar{X}$ is pure by acyclicity of $\bar{X}$; cf. Remark IV.8.(1). Hence by the exact Horseshoe Lemma, there exists a rowwise split short exact sequence $J \longrightarrow \tilde{J}' \longrightarrow \bar{J}$ of CE-resolutions that maps to $X \xrightarrow{f} X' \longrightarrow \bar{X}$ under $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Remark IV.7. Since $\bar{J}$ is rowwise split acyclic and since the sequence $J \longrightarrow \tilde{J}' \longrightarrow \bar{J}$ is rowwise split short exact, $J \longrightarrow \tilde{J}'$ is a rowwise homotopism. Since $J \longrightarrow \tilde{J}'$ is a CE-resolution of $X \xrightarrow{f} X'$, this proves the proposition. $\square$

IV.3 Formalism of spectral sequences

We follow essentially Verdier [**57,** II.4]; cf. [**12,** App.]; on a more basic level, cf. [**38,** Kap. 4].

Let $\mathcal{A}$ be an abelian category.

IV.3.1 Pointwise split and pointwise finitely filtered complexes

Let $\mathbf{Z}_\infty := \{-\infty\} \sqcup \mathbf{Z} \sqcup \{\infty\}$, considered as a linearly ordered set, and thus as a category. Write $]\alpha, \beta] := \{\sigma \in \mathbf{Z}_\infty \,:\, \alpha < \sigma \leqslant \beta\}$ for α, $\beta \in \mathbf{Z}_\infty$ such that $\alpha \leqslant \beta$; etc.

Given $X \in \operatorname{Ob} [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!]$, the morphism of X on $\alpha \leqslant \beta$ in $\mathbf{Z}_\infty$ shall be denoted by $X(\alpha) \xrightarrow{x} X(\beta)$.

An object $X \in \operatorname{Ob} [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!]$ is called a *pointwise split and pointwise finitely filtered complex (with values in $\mathcal{A}$)*, provided (SFF 1, 2, 3) hold.

(SFF 1) We have $X(-\infty) = 0$.

(SFF 2) The morphism $X(\alpha)^i \xrightarrow{x^i} X(\beta)^i$ is split monomorphic for all $i \in \mathbf{Z}$ and all $\alpha \leqslant \beta$ in $\mathbf{Z}_\infty$.

(SFF 3) For all $i \in \mathbf{Z}$, there exist β_0, $\alpha_0 \in \mathbf{Z}$ such that $X(\alpha)^i \xrightarrow{x^i} X(\beta)^i$ is an identity whenever $\alpha \leqslant \beta \leqslant \beta_0$ or $\alpha_0 \leqslant \alpha \leqslant \beta$ in $\mathbf{Z}_\infty$.

The pointwise split and pointwise finitely filtered complexes with values in $\mathcal{A}$ form a full subcategory $\mathrm{SFFC}(\mathcal{A}) \subseteq [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!]$.

Suppose given a pointwise split and pointwise finitely filtered complex X with values in $\mathcal{A}$ for the rest of the present §IV.3.

Let $\alpha \in \mathbf{Z}_\infty$. Write $\bar{X}(\alpha) := \operatorname{Cokern}\big(X(\alpha - 1) \longrightarrow X(\alpha)\big)$ for $\alpha \in \mathbf{Z}$. Given $i \in \mathbf{Z}$, we obtain $X(\alpha)^i \simeq \bigoplus_{\sigma \in]-\infty, \alpha]} \bar{X}(\sigma)^i$, which is a finite direct sum. We identify along this isomorphism. In particular, we get as a matrix representation for the differential

$$\big(X(\alpha)^i \xrightarrow{d} X(\alpha)^{i+1}\big) \;=\; \left(\bigoplus_{\sigma \in]-\infty, \alpha]} \bar{X}(\sigma)^i \xrightarrow{(d^i_{\sigma,\tau})_{\sigma,\tau}} \bigoplus_{\tau \in]-\infty, \alpha]} \bar{X}(\tau)^{i+1} \right),$$

where $d^i_{\sigma,\tau} = 0$ whenever $\sigma < \tau$; a kind of lower triangular matrix.

IV.3.2 Spectral objects

Let $\bar{\mathbf{Z}}_\infty := \mathbf{Z}_\infty \times \mathbf{Z}$. Write $\alpha^{+k} := (\alpha, k)$, where $\alpha \in \mathbf{Z}_\infty$ and $k \in \mathbf{Z}$. Let $\alpha^{+k} \leqslant \beta^{+\ell}$ in $\bar{\mathbf{Z}}_\infty$ if $k < \ell$ or ($k = \ell$ and $\alpha \leqslant \beta$), i.e. let $\bar{\mathbf{Z}}_\infty$ be linearly ordered via a lexicographical ordering. We have an automorphism $\alpha^{+k} \longmapsto \alpha^{+k+1}$ of the poset $\bar{\mathbf{Z}}_\infty$, to which we refer as *shift*. Note that $-\infty^{+k} = (-\infty)^{+k}$.

We have an order preserving injection $\mathbf{Z}_\infty \longrightarrow \bar{\mathbf{Z}}_\infty$, $\alpha \longmapsto \alpha^{+0}$. We use this injection as an identification of $\mathbf{Z}_\infty$ with its image in $\bar{\mathbf{Z}}_\infty$, i.e. we sometimes write $\alpha := \alpha^{+0}$ by abuse of notation.

Let $\bar{\mathbf{Z}}^{\#}_\infty := \{(\alpha, \beta) \in \bar{\mathbf{Z}}_\infty \times \bar{\mathbf{Z}}_\infty \,:\, \beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}\}$. We usually write $\beta/\alpha := (\alpha, \beta) \in \bar{\mathbf{Z}}^{\#}_\infty$; reminiscent of a quotient. The set $\bar{\mathbf{Z}}^{\#}_\infty$ is partially ordered by $\beta/\alpha \leqslant \beta'/\alpha' :\Longleftrightarrow (\beta \leqslant \beta'$ and

$\alpha \leqslant \alpha'$). We have an automorphism $\beta/\alpha \longmapsto (\beta/\alpha)^{+1} := \alpha^{+1}/\beta$ of the poset $\bar{\mathbf{Z}}^{\#}_{\infty}$, to which, again, we refer as *shift*.

We write $\mathbf{Z}^{\#}_{\infty} := \{\beta/\alpha \in \bar{\mathbf{Z}}^{\#}_{\infty} \,:\, -\infty \leqslant \alpha \leqslant \beta \leqslant \infty\}$. Note that any element of $\bar{\mathbf{Z}}^{\#}_{\infty}$ can uniquely be written as $(\beta/\alpha)^{+k}$ for some $\beta/\alpha \in \mathbf{Z}^{\#}_{\infty}$ and some $k \in \mathbf{Z}$.

We shall construct the *spectral object* $\mathrm{Sp}(X) \in \mathrm{Ob}\,[\![\bar{\mathbf{Z}}^{\#}_{\infty}, \mathrm{K}(\mathcal{A})]\!]$. The morphism of $\mathrm{Sp}(X)$ on $\beta/\alpha \leqslant \beta'/\alpha'$ in $\bar{\mathbf{Z}}^{\#}_{\infty}$ shall be denoted by $X(\beta/\alpha) \xrightarrow{x} X(\beta'/\alpha')$.

We require that

$$\Big(X\big((\beta/\alpha)^{+k}\big) \xrightarrow{x} X\big((\beta'/\alpha')^{+k}\big)\Big) \;=\; \Big(X(\beta/\alpha) \xrightarrow{x} X(\beta'/\alpha')\Big)^{\bullet+k}$$

for $\beta/\alpha \leqslant \beta'/\alpha'$ in $\bar{\mathbf{Z}}^{\#}_{\infty}$; i.e., roughly put, that $\mathrm{Sp}(X)$ be periodic up to shift of complexes.

Define

$$X\big(\beta/\alpha\big) \;:=\; \mathrm{Cokern}\,\big(X(\alpha) \xrightarrow{x} X(\beta)\big)$$

for $\beta/\alpha \in \mathbf{Z}^{\#}_{\infty}$. By periodicity, we conclude that $X\big(\alpha/\alpha\big) = 0$ and $X\big(\alpha^{+1}/\alpha\big) = 0$ for all $\alpha \in \bar{\mathbf{Z}}_{\infty}$.

Write

$$D^i_{\beta/\alpha,\,\beta'/\alpha'} \;:=\; (d^i_{\sigma,\tau})_{\sigma\in]\alpha,\beta],\;\tau\in]\alpha',\beta']} \;:\; X(\beta/\alpha)^i \longrightarrow X(\beta'/\alpha')^{i+1}$$

for $i \in \mathbf{Z}$ and $\beta/\alpha,\ \beta'/\alpha' \in \mathbf{Z}^{\#}_{\infty}$.

Given $-\infty \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \infty$ and $i \in \mathbf{Z}$, we let

$$\begin{aligned}
\Big(X(\beta/\alpha)^i \xrightarrow{x^i} X(\gamma/\alpha)^i\Big) &:= \Big(X(\beta/\alpha)^i \xrightarrow{(1\,0)} X(\beta/\alpha)^i \oplus X(\gamma/\beta)^i\Big)\\
\Big(X(\gamma/\alpha)^i \xrightarrow{x^i} X(\gamma/\beta)^i\Big) &:= \Big(X(\beta/\alpha)^i \oplus X(\gamma/\beta)^i \xrightarrow{\binom{0}{1}} X(\gamma/\beta)^i\Big)\\
\Big(X(\gamma/\beta)^i \xrightarrow{x^i} X(\alpha^{+1}/\beta)^i\Big) &:= \Big(X(\gamma/\beta)^i \xrightarrow{D^i_{\gamma/\beta,\,\beta/\alpha}} X(\beta/\alpha)^{i+1}\Big)\,.
\end{aligned}$$

By periodicity up to shift of complexes, this defines $\mathrm{Sp}(X)$. The construction is functorial in $X \in \mathrm{Ob}\,\mathrm{SFFC}(\mathcal{A})$.

IV.3.3 Spectral sequences

Let $\bar{\mathbf{Z}}^{\#\#}_{\infty} := \{(\gamma/\alpha, \delta/\beta) \in \bar{\mathbf{Z}}^{\#}_{\infty} \times \bar{\mathbf{Z}}^{\#}_{\infty} \,:\, \delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1}\}$. Given $(\gamma/\alpha, \delta/\beta) \in \bar{\mathbf{Z}}^{\#\#}_{\infty}$, we usually write $\delta/\beta /\!\!/ \gamma/\alpha := (\gamma/\alpha, \delta/\beta)$. The set $\bar{\mathbf{Z}}^{\#\#}_{\infty}$ is partially ordered by

$$\delta/\beta /\!\!/ \gamma/\alpha \leqslant \delta'/\beta' /\!\!/ \gamma'/\alpha' \;\;:\Longleftrightarrow\;\; (\gamma/\alpha \leqslant \gamma'/\alpha' \text{ and } \delta/\beta \leqslant \delta'/\beta')\,.$$

Define the *spectral sequence* $\mathrm{E}(X) \in \mathrm{Ob}\,[\![\bar{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{A}]\!]$ of X by letting its value on

$$\delta/\beta /\!\!/ \gamma/\alpha \;\leqslant\; \delta'/\beta' /\!\!/ \gamma'/\alpha'$$

in $\bar{\mathbf{Z}}^{\#\#}_{\infty}$ be the morphism that appears in the middle column of the diagram

$$\begin{array}{ccccc}
\mathrm{H}^0\big(X(\gamma/\alpha)\big) & \longrightarrow & \mathrm{E}(\delta/\beta /\!\!/ \gamma/\alpha)(X) & \longrightarrow & \mathrm{H}^0\big(X(\delta/\beta)\big)\\
\downarrow{\scriptstyle \mathrm{H}^0(x)} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \mathrm{H}^0(x)}\\
\mathrm{H}^0\big(X(\gamma'/\alpha')\big) & \longrightarrow & \mathrm{E}(\delta'/\beta' /\!\!/ \gamma'/\alpha')(X) & \longrightarrow & \mathrm{H}^0\big(X(\delta'/\beta')\big)\,.
\end{array}$$

Given $\delta/\beta/\!\!/\gamma/\alpha \in \bar{\mathbf{Z}}_\infty^{\#\#}$ and $k \in \mathbf{Z}$, we also write

$$\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(X) \;:=\; \mathrm{E}\big((\delta/\beta)^{+k}/\!\!/(\gamma/\alpha)^{+k}\big)(X)\;.$$

Altogether,

$$\begin{array}{ccccccc} [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!] & \supseteq & \mathrm{SFFC}(\mathcal{A}) & \longrightarrow & [\![\bar{\mathbf{Z}}_\infty^{\#}, \mathrm{K}(\mathcal{A})]\!] & \longrightarrow & [\![\bar{\mathbf{Z}}_\infty^{\#\#}, \mathcal{A}]\!] \\ & & X & \longmapsto & \mathrm{Sp}(X) & \longmapsto & \mathrm{E}(X)\;. \end{array}$$

IV.3.4 A short exact sequence

Lemma IV.18 *Given $\varepsilon^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \varepsilon \leqslant \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$, we have a short exact sequence*

$$\mathrm{E}(\varepsilon/\beta/\!\!/\gamma/\alpha)(X) \xrightarrow{e} \mathrm{E}(\varepsilon/\beta/\!\!/\delta/\alpha)(X) \xrightarrow{e} \mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\alpha)(X)\;.$$

Proof. See Lemma I.26. ▫

Lemma IV.19 *Given $\varepsilon^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \varepsilon \leqslant \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$, we have a short exact sequence*

$$\mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\alpha)(X) \xrightarrow{e} \mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\beta)(X) \xrightarrow{e} \mathrm{E}(\alpha^{+1}/\gamma/\!\!/\delta/\beta)(X)\;.$$

Proof. Apply the functor induced by $\beta/\alpha \longmapsto \alpha^{+1}/\beta$ to $\mathrm{Sp}(X)$. Then apply Lemma I.26. ▫

The short exact sequence in Lemma IV.18 is called a *fundamental short exact sequence (in first notation)*, the short exact sequence in Lemma IV.19 is called a *fundamental short exact sequence (in second notation)*. They will be used without further comment.

IV.3.5 Classical indexing

Let $1 \leqslant r \leqslant \infty$ and let $p,\, q \in \mathbf{Z}$. Denote

$$\mathrm{E}_r^{p,q} \;=\; \mathrm{E}_r^{p,q}(X) \;:=\; \mathrm{E}(-p-1+r/-p-1/\!\!/-p/-p-r)^{+p+q}(X)\;,$$

where $i + \infty := \infty$ and $i - \infty := -\infty$ for all $i \in \mathbf{Z}$.

Example IV.20 The short exact sequences in Lemmata IV.18, IV.19 allow to derive the exact couples of Massey. Write $\mathrm{D}_r^{i,j} = \mathrm{D}_r^{i,j}(X) := \mathrm{E}(-i/-\infty/\!\!/-i-r+1/-\infty)^{+i+j}(X)$ for $i,\, j \in \mathbf{Z}$ and $r \geqslant 1$. We obtain an exact sequence

$$\mathrm{D}_r^{i,\,j} \xrightarrow{e} \mathrm{D}_r^{i-1,\,j+1} \xrightarrow{e} \mathrm{E}_r^{i+r-2,\,j-r+2} \xrightarrow{e} \mathrm{D}_r^{i+r-1,\,j-r+2} \xrightarrow{e} \mathrm{D}_r^{i+r-2,\,j-r+3}$$

by Lemmata IV.18, IV.19.

IV.3.6 Comparing proper spectral sequences

Let $X \xrightarrow{f} Y$ be a morphism in SFFC($\mathcal{A}$), i.e. a morphism of pointwise split and pointwise finitely filtered complexes with values in $\mathcal{A}$. Write $\mathrm{E}(X) \xrightarrow{\mathrm{E}(f)} \mathrm{E}(Y)$ for the induced morphism on the spectral sequences.

For $\alpha,\ \beta\ \in\ \bar{\mathbf{Z}}_\infty$, we write $\alpha \mathrel{\dot{<}} \beta$ if

$$\big(\alpha < \beta\big) \quad \text{or} \quad \big(\alpha = \beta \quad \text{and} \quad \alpha \in \{\infty^{+k}\ :\ k \in \mathbf{Z}\} \cup \{-\infty^{+k}\ :\ k \in \mathbf{Z}\}\big)\ .$$

We write

$$\dot{\bar{\mathbf{Z}}}{}^{\#\#}_\infty \ :=\ \{\delta/\beta/\!\!/\gamma/\alpha \in \bar{\mathbf{Z}}^{\#\#}_\infty\ :\ \delta^{-1} \leqslant \alpha \mathrel{\dot{<}} \beta \leqslant \gamma \mathrel{\dot{<}} \delta \leqslant \alpha^{+1}\}\ .$$

We write

$$\dot{\mathrm{E}} \ =\ \dot{\mathrm{E}}(X) \ :=\ \mathrm{E}(X)|_{\dot{\bar{\mathbf{Z}}}{}^{\#\#}_\infty} \ \in\ \mathrm{Ob}\,[\![\dot{\bar{\mathbf{Z}}}{}^{\#\#}_\infty, \mathcal{A}]\!]$$

for the *proper spectral sequence* of X; analogously for the morphisms.

Lemma IV.21 *If* $\mathrm{E}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-2)^{+k}(f)$ *is an isomorphism for all* $\alpha \in \mathbf{Z}$ *and all* $k \in \mathbf{Z}$*, then* $\dot{\mathrm{E}}(f)$ *is an isomorphism.*

Proof. Claim 1. We have an isomorphism $\mathrm{E}(\gamma/\beta-1/\!\!/\beta/\beta-2)^{+k}(f)$ for all $k \in \mathbf{Z}$, all $\beta \in \mathbf{Z}$ and all $\gamma \in \mathbf{Z}$ such that $\gamma > \beta$. We have an isomorphism $\mathrm{E}(\beta+1/\beta-1/\!\!/\beta/\alpha-1)^{+k}(f)$ for all $k \in \mathbf{Z}$, all $\beta \in \mathbf{Z}$ and all $\alpha \in \mathbf{Z}$ such that $\alpha < \beta$.

The assertions follow by induction using the exact sequences

$$\mathrm{E}(\gamma+2/\gamma/\!\!/\gamma+1/\beta)^{+k-1} \xrightarrow{e} \mathrm{E}(\gamma/\beta-1/\!\!/\beta/\beta-2)^{+k} \xrightarrow{e} \mathrm{E}(\gamma+1/\beta-1/\!\!/\beta/\beta-2)^{+k} \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{E}(\beta{+}1/\beta{-}1/\!\!/\beta/\alpha{-}2)^{+k} \xrightarrow{e} \mathrm{E}(\beta{+}1/\beta{-}1/\!\!/\beta/\alpha{-}1)^{+k} \xrightarrow{e} \mathrm{E}(\beta{-}1/\alpha{-}2/\!\!/\alpha{-}1/\alpha{-}3)^{+k+1}\ .$$

Claim 2. We have an isomorphism $\mathrm{E}(\gamma/\beta-1/\!\!/\beta/\alpha-1)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\ \beta,\ \gamma\ \in\ \mathbf{Z}$ such that $\alpha < \beta < \gamma$.

We proceed by induction on $\gamma-\alpha$. By Claim 1, we may assume that $\alpha < \beta-1 < \beta+1 < \gamma$. Consider the image diagram

$$\mathrm{E}(\gamma-1/\beta-1/\!\!/\beta/\alpha-1)^{+k} \overset{e}{\twoheadrightarrow} \mathrm{E}(\gamma/\beta-1/\!\!/\beta/\alpha-1)^{+k} \overset{e}{\rightarrowtail} \mathrm{E}(\gamma/\beta-1/\!\!/\beta/\alpha)^{+k}\ .$$

Claim 3. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\ \beta,\ \gamma,\ \delta\ \in\ \mathbf{Z}$ such that $\alpha < \beta \leqslant \gamma < \delta$.

We may assume that $\gamma-\beta \geqslant 1$, for $\mathrm{E}(\delta/\beta/\!\!/\beta/\alpha)^{+k} = 0$. We proceed by induction on $\gamma-\beta$. By Claim 2, we may assume that $\gamma-\beta \geqslant 2$. Consider the short exact sequence

$$\mathrm{E}(\delta/\beta/\!\!/\gamma-1/\alpha)^{+k} \overset{e}{\rightarrowtail} \mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k} \overset{e}{\twoheadrightarrow} \mathrm{E}(\delta/\gamma-1/\!\!/\gamma/\alpha)^{+k}\ .$$

Claim 4. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\ \beta,\ \gamma,\ \delta\ \in\ \mathbf{Z}_\infty$ such that $\alpha < \beta \leqslant \gamma < \delta$.

In view of Claim 3, it suffices to choose $\tilde{\alpha} \in \mathbf{Z}$ small enough such that $\mathrm{E}(\delta/\beta/\!\!/\gamma/\tilde{\alpha})^{+k}(f) = \mathrm{E}(\delta/\beta/\!\!/\gamma/-\infty)^{+k}(f)$; etc.

Claim 5. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbf{Z}_\infty$ such that $\alpha \mathrel{\dot{<}} \beta \leqslant \gamma \mathrel{\dot{<}} \delta$.

In view of Claim 4, it suffices to choose $\tilde{\beta} \in \mathbf{Z}$ small enough such that $\mathrm{E}(\delta/\tilde{\beta}/\!\!/\gamma/-\infty)^{+k}(f) = \mathrm{E}(\delta/-\infty/\!\!/\gamma/-\infty)^{+k}(f)$; etc.

Claim 6. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\ \beta,\ \gamma,\ \delta \in \bar{\mathbf{Z}}_\infty$ such that $-\infty \leqslant \delta^{-1} \leqslant \alpha \mathrel{\dot{<}} \beta \leqslant \gamma \leqslant \infty < -\infty^{+1} \leqslant \delta \leqslant \alpha^{+1}$.

In view of Claim 5, it suffices to consider the short exact sequence

$$\mathrm{E}(\infty/\beta/\!\!/\gamma/\delta^{-1})^{+k} \xrightarrow{e} \mathrm{E}(\infty/\beta/\!\!/\gamma/\alpha)^{+k} \xrightarrow{e} \mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}\ .$$

Claim 7. The morphism $\dot{\mathrm{E}}(f)$ is an isomorphism.

Suppose given $\alpha,\ \beta,\ \gamma,\ \delta \in \bar{\mathbf{Z}}_\infty$ such that $\delta^{-1} \leqslant \alpha \mathrel{\dot{<}} \beta \leqslant \gamma \mathrel{\dot{<}} \delta \leqslant \alpha^{+1}$. Via a shift, we may assume that we are in the situation of Claim 5 or of Claim 6. ▫

IV.3.7 The first spectral sequence of a double complex

Let $\mathcal{A}$ be an abelian category. Let $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. Given $n \in \mathbf{Z}_\infty$, we write $X^{[n,*}$ for the double complex arising from X by replacing $X^{i,j}$ by 0 for all $i \in [0,n[$. We define a pointwise split and pointwise finitely filtered complex $\mathrm{t_I}X$, called the *first filtration of* $\mathrm{t}X$, by letting $\mathrm{t_I}X(\alpha) := \mathrm{t}X^{[-\alpha,*}$ for $\alpha \in \mathbf{Z}_\infty$; and by letting $\mathrm{t_I}X(\alpha) \longrightarrow \mathrm{t_I}X(\beta)$ be the pointwise split inclusion $\mathrm{t}X^{[-\alpha,*} \longrightarrow \mathrm{t}X^{[-\beta,*}$ for $\alpha,\ \beta \in \mathbf{Z}_\infty$ such that $\alpha \leqslant \beta$. Let $\mathrm{E_I} = \mathrm{E_I}(X) := \mathrm{E}(\mathrm{t_I}X)$. This construction is functorial in $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. Note that $\overline{\mathrm{t_I}X}(\alpha) = X^{-\alpha,k+\alpha}$.

We record the following wellknown lemma in the language we use here.

Lemma IV.22 *Let $\alpha \in]-\infty,0]$. Let $k \in \mathbf{Z}$ such that $k \geqslant -\alpha$. We have*

$$\begin{array}{rcl}\mathrm{E_I}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k}(X) & = & \mathrm{H}^{k+\alpha}(X^{-\alpha,*}) \\ \mathrm{E_I}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-2)^{+k}(X) & = & \mathrm{H}^{-\alpha}\big(\mathrm{H}^{k+\alpha}(X^{-,*})\big)\ ,\end{array}$$

naturally in $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$.

Proof. The first equality follows by $\mathrm{E_I}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k} = \mathrm{H}^k\mathrm{t_I}X(\alpha/\alpha-1) = \mathrm{H}^{k+\alpha}(X^{-\alpha,*})$.

The morphism $\mathrm{t_I}X(\alpha/\alpha-1) \longrightarrow \mathrm{t_I}X\big((\alpha-2)^{+1}/\alpha-1\big) = \mathrm{t_I}X\big(\alpha-1/\alpha-2\big)^{\bullet+1}$ from $\mathrm{Sp}(\mathrm{t_I}X)$ is at position $k \geqslant 0$ given by

$$\overline{\mathrm{t_I}X}(\alpha)^k = X^{-\alpha,k+\alpha} \xrightarrow{(-1)^\alpha\,\partial} X^{-\alpha+1,k+\alpha} = \overline{\mathrm{t_I}X}(\alpha-1)^{k+1}\ ;$$

cf. §IV.1.1.6. In particular, the morphisms

$$\mathrm{E_I}(\alpha+1/\alpha/\!\!/\alpha+1/\alpha)^{+k-1} \xrightarrow{e} \mathrm{E_I}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k} \xrightarrow{e} \mathrm{E_I}(\alpha-1/\alpha-2/\!\!/\alpha-1/\alpha-2)^{+k+1}$$

are given by

$$\mathrm{H}^{k+\alpha}(X^{-\alpha-1,*}) \xrightarrow{(-1)^{\alpha+1}\mathrm{H}^{k+\alpha}(\partial)} \mathrm{H}^{k+\alpha}(X^{-\alpha,*}) \xrightarrow{(-1)^{\alpha}\mathrm{H}^{k+\alpha}(\partial)} \mathrm{H}^{k+\alpha}(X^{-\alpha+1,*}) .$$

Now the second equality follows by the diagram

$$\begin{array}{ccccc}
 & & \mathrm{E_I}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-2)^{+k} & & \\
 & \nearrow^{e} & & \searrow^{e} & \\
\mathrm{E_I}(\alpha/\alpha-1/\!\!/\alpha/\alpha-2)^{+k} & & & & \mathrm{E_I}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-1)^{+k} \\
 & \searrow^{e} & & \nearrow^{e} & \\
\mathrm{E_I}(\alpha+1/\alpha/\!\!/\alpha+1/\alpha)^{+k-1} \xrightarrow{e} & & \mathrm{E_I}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k} & \xrightarrow{e} & \mathrm{E_I}(\alpha-1/\alpha-2/\!\!/\alpha-1/\alpha-2)^{+k+1} .
\end{array}$$

□

Remark IV.23 *Let* $X \xrightarrow{f} Y$ *be a rowwise quasiisomorphism in* $\mathrm{CC}^{\llcorner}(\mathcal{A})$. *Then* $\mathrm{E_I}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ *is an isomorphism for* $\delta^{-1} \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant \alpha^{+1}$ *in* $\bar{\mathbf{Z}}_\infty$ *and* $k \in \mathbf{Z}$.

Proof. It suffices to show that the morphism $\mathrm{Sp}(\mathrm{t_I}f)$ in $[\![\bar{\mathbf{Z}}_\infty^{\#}, \mathrm{K}(\mathcal{A})]\!]$ is pointwise a quasiisomorphism. To have this, it suffices to show that $\mathrm{t}f^{[k,*}$ is a quasiisomorphism for $k \geqslant 0$. But $f^{[k,*}$ is a rowwise quasiisomorphism for $k \geqslant 0$; cf. §IV.1.1.6. □

Lemma IV.24 *The functor* $\mathrm{CC}^{\llcorner}(\mathcal{A}) \xrightarrow{\dot{\mathrm{E}}_\mathrm{I}} [\![\dot{\bar{\mathbf{Z}}}_\infty^{\#\#}, \mathcal{A}]\!]$ *factors over*

$$\mathrm{KK}^{\llcorner}(\mathcal{A}) \xrightarrow{\dot{\mathrm{E}}_\mathrm{I}} [\![\dot{\bar{\mathbf{Z}}}_\infty^{\#\#}, \mathcal{A}]\!] .$$

Proof. By Lemma IV.1, we have to show that $\dot{\mathrm{E}}_\mathrm{I}$ annihilates all elementary horizontally split acyclic double complexes in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ and all elementary vertically split acyclic double complexes in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$.

Let $U \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ be an elementary vertically split acyclic double complex concentrated in rows i and $i+1$, where $i \geqslant 0$. Let $V \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ be an elementary horizontally split acyclic double complex concentrated in columns j and $j+1$, where $j \geqslant 0$.

Since V is rowwise acyclic, $\mathrm{E_I}$ annihilates V by Remark IV.23, whence so does $\dot{\mathrm{E}}_\mathrm{I}$.

Suppose given

$$(*) \qquad -\infty \leqslant \alpha \mathrel{\dot{<}} \beta \leqslant \gamma \mathrel{\dot{<}} \delta \leqslant \infty$$

in $\bar{\mathbf{Z}}_\infty$ and $k \in \mathbf{Z}$. We *claim* that the functor $\mathrm{E_I}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}$ annihilates U. We may assume that $\beta < \gamma$. Note that $\mathrm{E_I}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(U)$ is the image of

$$\mathrm{H}^k\big(\mathrm{t_I}U(\gamma/\alpha)\big) \longrightarrow \mathrm{H}^k\big(\mathrm{t_I}U(\delta/\beta)\big) .$$

The double complex $U^{[-\delta,*}/U^{[-\beta,*}$ is columnwise acyclic except possibly if $-\beta = i+1$ or if $-\delta = i+1$. The double complex $U^{[-\gamma,*}/U^{[-\alpha,*}$ is columnwise acyclic except possibly if $-\alpha = i+1$ or if $-\gamma = i+1$. All three remaining combinations of these exceptional cases are excluded by $(*)$, however. Hence $\mathrm{E_I}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(U) = 0$. This proves the *claim.*

Suppose given

$$(**)\qquad \delta^{-1} \leqslant \alpha \mathrel{\dot{<}} \beta \leqslant \gamma \leqslant \infty \leqslant -\infty^{+1} \leqslant \delta \leqslant \alpha^{+1}\ .$$

in $\bar{\mathbf{Z}}_\infty$ and $k \in \mathbf{Z}$. We *claim* that the functor $\mathrm{E_I}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}$ annihilates U. We may assume that $\beta < \gamma$ and that $\delta^{-1} < \alpha$. Note that $\mathrm{E_I}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}(U)$ is the image of

$$\mathrm{H}^k\big(\mathrm{t_I} U(\gamma/\alpha)\big) \longrightarrow \mathrm{H}^{k+1}\big(\mathrm{t_I} U(\beta/\delta^{-1})\big)\ .$$

The double complex $U^{[-\beta,*}/U^{[-(\delta^{-1}),*}$ is columnwise acyclic except possibly if $-(\delta^{-1}) = i+1$ or if $-\beta = i+1$. The double complex $U^{[-\gamma,*}/U^{[-\alpha,*}$ is columnwise acyclic except possibly if $-\gamma = i+1$ or if $-\alpha = i+1$. Both remaining combinations of these exceptional cases are excluded by $(**)$, however. Hence $\mathrm{E_I}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}(U) = 0$. This proves the *claim.*

Both claims taken together show that $\dot{\mathrm{E}}_\mathrm{I}$ annihilates U. ▫

IV.4 Grothendieck spectral sequences

IV.4.1 Certain quasiisomorphisms are preserved by a left exact functor

Suppose given abelian categories $\mathcal{A}$, $\mathcal{B}$, and suppose that $\mathcal{A}$ has enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a left exact functor.

Remark IV.25 *Suppose given an F-acyclic object $X \in \mathrm{Ob}\,\mathcal{A}$ and an injective resolution $I \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{A})$ of X. Let $\mathrm{Conc}\,X \xrightarrow{f} I$ be its quasiisomorphism. Then $\mathrm{Conc}\,FX \xrightarrow{Ff} FI$ is a quasiisomorphism.*

Proof. This follows since F is left exact and since $\mathrm{H}^i(FI) \simeq (\mathrm{R}^i F)X \simeq 0$ for $i \geqslant 1$. ▫

Remark IV.26 *Suppose given a complex $U \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$ consisting of F-acyclic objects. There exists an injective complex resolution $I \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{A})$ of U such that its quasiisomorphism $U \xrightarrow{f} I$ maps to a quasiisomorphism $FU \xrightarrow{Ff} FI$.*

Proof. Let $J \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A})$ be a CE-resolution of U; cf. Remark IV.9. Since the morphism of double complexes $\mathrm{Conc}_2\,U \longrightarrow J$ is a columnwise quasiisomorphism consisting of monomorphisms, taking the total complex, we obtain a quasiisomorphism $U \longrightarrow \mathrm{t}J$ consisting of monomorphisms. By F-acyclicity of the entries of U, the image $\mathrm{Conc}_2\,FU \longrightarrow FJ$ under F is a columnwise quasiisomorphism, too; cf. Remark IV.25. Hence F maps the quasiisomorphism $U \longrightarrow \mathrm{t}J$ to the quasiisomorphism $FU \longrightarrow F\mathrm{t}J$. So we may take $I := \mathrm{t}J$. ▫

Lemma IV.27 *Suppose given a complex $U \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$ consisting of F-acyclic objects and an injective complex resolution $I \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{A})$ of U. Let $U \xrightarrow{f} I$ be its quasiisomorphism. Then $FU \xrightarrow{Ff} FI$ is a quasiisomorphism.*

Proof. Let $U \longrightarrow I'$ be a quasiisomorphism to an injective complex resolution I' that is mapped to a quasiisomorphism by F; cf. Remark IV.26. Since $U \longrightarrow I'$ is a quasiisomorphism, the induced map ${}_{\mathrm{K}(\mathcal{A})}(U, I) \longleftarrow {}_{\mathrm{K}(\mathcal{A})}(I', I)$ is surjective, so that there exists a morphism $I' \longrightarrow I$ such that $(U \longrightarrow I' \longrightarrow I) = (U \xrightarrow{f} I)$ in $\mathrm{K}(\mathcal{A})$. Since, moreover, $U \xrightarrow{f} I$ is a quasiisomorphism, $I' \longrightarrow I$ is a homotopism. Since $FU \longrightarrow FI'$ is a quasiisomorphism and $FI' \longrightarrow FI$ is a homotopism, we conclude that $FU \longrightarrow FI$ is a quasiisomorphism. ▫

IV.4.2 Definition of the Grothendieck spectral sequence functor

Suppose given abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be left exact functors.

A (F, G)-*acyclic resolution* of $X \in \operatorname{Ob} \mathcal{A}$ is a complex $A \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$, together with a quasiisomorphism $\operatorname{Conc} X \longrightarrow A$, such that the following hold.

(A 1) The object A^i is F-acyclic for $i \geqslant 0$.

(A 2) The object A^i is $(G \circ F)$-acyclic for $i \geqslant 0$.

(A 3) The object FA^i is G-acyclic for $i \geqslant 0$.

An object $X \in \operatorname{Ob} \mathcal{A}$ that possesses an (F, G)-acyclic resolution is called (F, G)-*acyclicly resolvable.* The full subcategory of (F, G)-acyclicly resolvable objects in $\mathcal{A}$ is denoted by $\mathcal{A}_{(F,G)}$.

A complex $A \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$, together with a quasiisomorphism $\operatorname{Conc} X \longrightarrow A$, is called an *$F$-acyclic resolution* of $X \in \operatorname{Ob} \mathcal{A}$ if (A 2) holds.

Remark IV.28 *If F carries injective objects to G-acyclic objects, then* (A 1) *and* (A 3) *imply* (A 2).

Proof. Given $i \geqslant 0$, we let I be an injective resolution of A^i, and $\tilde{I}$ the acyclic complex obtained by appending A^i to I in position -1. Since A^i is F-acyclic, the complex $F\tilde{I}$ is acyclic; cf. Remark IV.25. Note that $F\mathrm{B}^0\tilde{I} \simeq FA^i$ is G-acyclic by assumption. Since

$$(\mathrm{R}^k G)F\tilde{I}^j \longrightarrow (\mathrm{R}^k G)F\mathrm{B}^{j+1}\tilde{I} \longrightarrow (\mathrm{R}^{k+1} G)F\mathrm{B}^j\tilde{I}$$

is exact in the middle for $j \geqslant 0$ and $k \geqslant 1$, we may conclude by induction on j and by G-acyclicity assumption on $F\tilde{I}^j$ that $F\mathrm{B}^j\tilde{I}$ is G-acyclic for $j \geqslant 0$. In particular, we have $(\mathrm{R}^1 G)(F\mathrm{B}^j\tilde{I}) \simeq 0$ for $j \geqslant 0$, whence

$$GF\mathrm{B}^j\tilde{I} \longrightarrow GF\tilde{I}^j \longrightarrow GF\mathrm{B}^{j+1}\tilde{I}$$

is short exact for $j \geqslant 0$. We conclude that $(G \circ F)\tilde{I}$ is acyclic. Hence A^i is $(G \circ F)$-acyclic. ▫

To see Remark IV.28, one could also use a Grothendieck spectral sequence, once established.

Remark IV.29 *Suppose given $X \in \operatorname{Ob} \mathcal{A}$, an injective resolution I of X and an F-acyclic resolution A of X. Then there exists a quasiisomorphism $A \longrightarrow I$ that is mapped to 1_X by H^0. Moreover, any morphism $A \xrightarrow{u} I$ that is mapped to 1_X by H^0 is a quasiisomorphism and is mapped to a quasiisomorphism $FA \xrightarrow{Fu} FI$ by F.*

Proof. Let I' be an injective complex resolution of A such that its quasiisomorphism $A \longrightarrow I'$ is mapped to a quasiisomorphism by F; cf. Remark IV.26. We use the composite quasiisomorphism $\operatorname{Conc} X \longrightarrow A \longrightarrow I'$ to resolve X by I'.

To prove the first assertion, note that there is a homotopism $I' \longrightarrow I$ resolving 1_X; whence the composite $(A \longrightarrow I' \longrightarrow I)$ is a quasiisomorphism resolving 1_X.

To prove the second assertion, note that the induced map ${}_{\mathrm{K}(\mathcal{A})}(A, I) \longleftarrow {}_{\mathrm{K}(\mathcal{A})}(I', I)$ is surjective, whence there is a factorisation $(A \longrightarrow I' \longrightarrow I) = (A \xrightarrow{u} I)$ in $\mathrm{K}(\mathcal{A})$ for some morphism $I' \longrightarrow I$, which, since resolving 1_X as well, is a homotopism. In particular, $A \xrightarrow{u} I$ is a quasiisomorphism. Finally, since $FI' \longrightarrow FI$ is a homotopism, also $FA \xrightarrow{Fu} FI$ is a quasiisomorphism. ▫

Alternatively, in the last step of the preceding proof we could have invoked Lemma IV.27.

The following construction originates in [**10,** XVII.§7] and [**17,** Th. 2.4.1]. In its present form, it has been carried out by Haas in the classical framework [**21**]. We do not claim any originality.

I do not know whether the use of injectives in $\mathcal{A}$ in the following construction can be avoided; in any case, it would be desirable to do so.

We set out to define the *proper Grothendieck spectral sequence functor*

$$\mathcal{A}_{(F,G)} \xrightarrow{\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}} [\![\dot{\bar{\mathbf{Z}}}^{\#\#}_{\infty}, \mathcal{C}]\!] .$$

We define $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}$ *on objects.* Suppose given $X \in \operatorname{Ob}\mathcal{A}_{(F,G)}$. Choose an (F,G)-acyclic resolution $A_X \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ of X. Choose a CE-resolution $J_X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ of FA_X. Let $\mathrm{E}^{\mathrm{Gr}}_{F,G}(X) := \mathrm{E}_{\mathrm{I}}(GJ_X) = \mathrm{E}(\mathrm{t}_{\mathrm{I}}GJ_X) \in \operatorname{Ob}[\![\bar{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$ be the *Grothendieck spectral sequence* of X with respect to F and G. Accordingly, let

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(X) \;:=\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \;=\; \dot{\mathrm{E}}(\mathrm{t}_{\mathrm{I}}GJ_X) \;\in\; \operatorname{Ob}[\![\dot{\bar{\mathbf{Z}}}^{\#\#}_{\infty}, \mathcal{C}]\!]$$

be the *proper Grothendieck spectral sequence* of X with respect to F and G.

We define $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}$ *on morphisms.* Suppose given $X \in \operatorname{Ob}\mathcal{A}_{(F,G)}$, and let A_X and J_X be as above. Choose an injective resolution $I_X \in \operatorname{Ob}\mathrm{C}^{[0}(\operatorname{Inj}\mathcal{A})$ of X. Choose a quasiisomorphism $A_X \xrightarrow{p_X} I_X$ that is mapped to 1_X by H^0 and to a quasiisomorphism by F; cf. Remark IV.29. Choose a CE-resolution $K_X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ of FI_X. Choose a morphism $J_X \xrightarrow{q_X} K_X$ in $\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ that is mapped to Fp_X by $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Remark IV.6.

Note that $J_X \xrightarrow{q_X} K_X$ can be written as a composite in $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\operatorname{Inj}\mathcal{B})$ of a rowwise homotopism, followed by a double homotopism; cf. Proposition IV.17. Hence, so can $GJ_X \xrightarrow{Gq_X} GK_X$. Thus $\dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_X)} \dot{\mathrm{E}}_{\mathrm{I}}(GK_X)$ is an isomorphism; cf. Remark IV.23, Lemma IV.24.

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}_{(F,G)}$. Choose a morphism $I_X \xrightarrow{f'} I_Y$ in $\mathrm{C}^{[0}(\mathcal{A})$ that is mapped to f by H^0. Choose a morphism $K_X \xrightarrow{f''} K_Y$ in $\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ that is mapped to Ff' by $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Remark IV.6. Let

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(X \xrightarrow{f} Y) \;:=\; \left(\dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \xrightarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_X)} \dot{\mathrm{E}}_{\mathrm{I}}(GK_X) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(Gf'')} \dot{\mathrm{E}}_{\mathrm{I}}(GK_Y) \xleftarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_Y)} \dot{\mathrm{E}}_{\mathrm{I}}(GJ_Y)\right) .$$

The procedure can be adumbrated as follows.

$$\begin{array}{ccccc} & K_X & \xrightarrow{f''} & & K_Y \\ {}^{q_X}\nearrow & & & {}^{q_Y}\nearrow & \\ J_X & & & J_Y & \end{array}$$

$$\begin{array}{ccccc} & I_X & \xrightarrow{f'} & & I_Y \\ {}^{p_X}\nearrow & & & {}^{p_Y}\nearrow & \\ A_X & & & A_Y & \end{array}$$

$$X \xrightarrow{f} Y$$

We show that this defines a functor $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G} : \mathcal{A}_{(F,G)} \longrightarrow [\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$. We need to show independence of the construction from the choices of f' and f'', for then functoriality follows by appropriate choices.

Let $I_X \xrightarrow{\tilde{f}'} I_Y$ and $K_X \xrightarrow{\tilde{f}''} K_Y$ be alternative choices. The residue classes of f' and $\tilde{f}'$ in $\mathrm{K}^{[0}(\mathcal{A})$ coincide, whence so do the residue classes of Ff' and $F\tilde{f}'$ in $\mathrm{K}^{[0}(\mathcal{B})$. Therefore, the residue classes of f'' and $\tilde{f}''$ in $\mathrm{KK}^{\llcorner}(\mathcal{B})$ coincide; cf. Proposition IV.14. Hence, so do the residue classes of Gf'' and $G\tilde{f}''$ in $\mathrm{KK}^{\llcorner}(\mathcal{C})$. Thus $\dot{\mathrm{E}}_{\mathrm{I}}(Gf'') = \dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{f}'')$; cf. Lemma IV.24.

We show that alternative choices of A_X, I_X and p_X, and of J_X, K_X and q_X, yield isomorphic proper Grothendieck spectral sequence functors.

Let $\tilde{A}_X \xrightarrow{\tilde{p}_X} \tilde{I}_X$ and $\tilde{J}_X \xrightarrow{\tilde{q}_X} \tilde{K}_X$ be alternative choices, where X runs through $\mathrm{Ob}\,\mathcal{A}_{(F,G)}$.

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}_{(F,G)}$. We resolve the commutative quadrangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \| \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{A}$ to a commutative quadrangle

$$\begin{array}{ccc} I_X & \xrightarrow{f'} & I_Y \\ {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle u_Y} \\ \tilde{I}_X & \xrightarrow{\tilde{f}'} & \tilde{I}_Y \end{array}$$

in $\mathrm{K}^{[0}(\mathcal{A})$, in which u_X and u_Y are homotopisms; cf. Remark IV.6. Then we resolve the commutative quadrangle

$$\begin{array}{ccc} FI_X & \xrightarrow{Ff'} & FI_Y \\ {\scriptstyle Fu_X}\downarrow & & \downarrow{\scriptstyle Fu_Y} \\ F\tilde{I}_X & \xrightarrow{F\tilde{f}'} & F\tilde{I}_Y \end{array}$$

in $\mathrm{K}^{[0}(\mathcal{B})$ to a commutative quadrangle

$$\begin{array}{ccc} K_X & \xrightarrow{f''} & K_Y \\ {\scriptstyle v_X}\downarrow & & \downarrow{\scriptstyle v_Y} \\ \tilde{K}_X & \xrightarrow{\tilde{f}''} & \tilde{K}_Y \end{array}$$

in $\mathrm{KK}^{\llcorner}(\mathcal{B})$; cf. Proposition IV.14. Therein, v_X and v_Y are each composed of a rowwise homotopism, followed by a double homotopism; cf. Proposition IV.17. So are Gv_X and Gv_Y. An application of $\dot{\mathrm{E}}_{\mathrm{I}}\big(G(-)\big)$ yields the sought isotransformation, viz.

$$\Big(\dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \xrightarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_X)} \dot{\mathrm{E}}_{\mathrm{I}}(GK_X) \xrightarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(Gv_X)} \dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{K}_X) \xleftarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{q}_X)} \dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{J}_X)\Big)$$

at $X \in \mathrm{Ob}\,\mathcal{A}_{(F,G)}$; cf. Remark IV.23, Lemma IV.24.

Finally, we recall the starting point of the whole enterprise.

Remark IV.30 ([10, XVII.§7], [17, Th. 2.4.1]) *Suppose given $X \in \mathrm{Ob}\,\mathcal{A}_{(F,G)}$ and $k, \ell \in \mathbf{Z}_{\geqslant 0}$. We have*

$$\begin{array}{lcl} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(-k+1/-k-1/\!\!/-k/-k-2)^{+k+\ell}(X) & \simeq & (\mathrm{R}^k G)(\mathrm{R}^\ell F)(X) \\ \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(\infty/-\infty/\!\!/\infty/-\infty)^{+k+\ell}(X) & \simeq & \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(X)\;, \end{array}$$

naturally in X.

Proof. Keep the notation of the definition of $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}$.

We shall prove the first isomorphism. By Lemma IV.22, we have

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(-k+1/-k-1/\!\!/-k/-k-2)^{+k+\ell}(X) \;\simeq\; \mathrm{H}^k(\mathrm{H}^\ell(GJ_X^{-,*}))\;.$$

Since J_X is rowwise split, we have $\mathrm{H}^\ell(GJ_X^{-,*}) \simeq G(\mathrm{H}^\ell J_X^{-,*})$. Note that $\mathrm{H}^\ell J_X^{-,*}$ is an injective resolution of $\mathrm{H}^\ell FA_X$; cf. Remark IV.8.(1). By Remark IV.29, $\mathrm{H}^\ell FA_X \xrightarrow[\sim]{\mathrm{H}^\ell Fp_X} \mathrm{H}^\ell FI_X \simeq (\mathrm{R}^\ell F)(X)$. So

$$\mathrm{H}^k(\mathrm{H}^\ell(GJ_X^{-,*})) \;\simeq\; \mathrm{H}^k(G(\mathrm{H}^\ell J_X^{-,*})) \;\simeq\; (\mathrm{R}^k G)(\mathrm{H}^\ell FA_X) \;\simeq\; (\mathrm{R}^k G)(\mathrm{R}^\ell F)(X)\;.$$

We shall prove naturality of the first isomorphism. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}_{(F,G)}$. Consider the following commutative diagram. Abbreviate $E := \dot{\mathrm{E}}(-k+1/-k-1/\!\!/-k/-k-2)^{+k+\ell}$.

$$\begin{array}{ccccccc}
E(\mathrm{t_I}GJ_X) & \xrightarrow[\sim]{E(\mathrm{t_I}Gq_X)} & E(\mathrm{t_I}GK_X) & \xrightarrow{E(\mathrm{t_I}Gf'')} & E(\mathrm{t_I}GK_Y) & \xleftarrow[\sim]{E(\mathrm{t_I}Gq_Y)} & E(\mathrm{t_I}GJ_Y) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{H}^k\mathrm{H}^\ell GJ_X^{-,*} & \xrightarrow[\sim]{\mathrm{H}^k\mathrm{H}^\ell Gq_X^{-,*}} & \mathrm{H}^k\mathrm{H}^\ell GK_X^{-,*} & \xrightarrow{\mathrm{H}^k\mathrm{H}^\ell Gf''^{-,*}} & \mathrm{H}^k\mathrm{H}^\ell GK_Y^{-,*} & \xleftarrow[\sim]{\mathrm{H}^k\mathrm{H}^\ell Gq_Y^{-,*}} & \mathrm{H}^k\mathrm{H}^\ell GJ_Y^{-,*} \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{H}^k G\mathrm{H}^\ell J_X^{-,*} & \xrightarrow[\sim]{\mathrm{H}^k G\mathrm{H}^\ell q_X^{-,*}} & \mathrm{H}^k G\mathrm{H}^\ell K_X^{-,*} & \xrightarrow{\mathrm{H}^k G\mathrm{H}^\ell f''^{-,*}} & \mathrm{H}^k G\mathrm{H}^\ell K_Y^{-,*} & \xleftarrow[\sim]{\mathrm{H}^k G\mathrm{H}^\ell q_Y^{-,*}} & \mathrm{H}^k G\mathrm{H}^\ell J_Y^{-,*} \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
(\mathrm{R}^k G)\mathrm{H}^\ell FA_X & \xrightarrow[\sim]{(\mathrm{R}^k G)\mathrm{H}^\ell Fp_X} & (\mathrm{R}^k G)\mathrm{H}^\ell FI_X & \xrightarrow{(\mathrm{R}^k G)\mathrm{H}^\ell Ff'} & (\mathrm{R}^k G)\mathrm{H}^\ell FI_Y & \xleftarrow[\sim]{(\mathrm{R}^k G)\mathrm{H}^\ell Fp_Y} & (\mathrm{R}^k G)\mathrm{H}^\ell FA_Y \\
 & & \downarrow\wr & & \downarrow\wr & & \\
 & & (\mathrm{R}^k G)(\mathrm{R}^\ell F)(X) & \xrightarrow{(\mathrm{R}^k G)(\mathrm{R}^\ell F)(f)} & (\mathrm{R}^k G)(\mathrm{R}^\ell F)(Y) & &
\end{array}$$

We shall prove the second isomorphism. By Lemma IV.27, the quasiisomorphism $FA_X \longrightarrow \mathrm{t}J_X$ maps to a quasiisomorphism $GFA_X \longrightarrow \mathrm{t}GJ_X \simeq G\mathrm{t}J_X$. By Lemma IV.27, the quasiisomorphism $A_X \xrightarrow{p_X} I_X$ maps to a quasiisomorphism $GFA_X \xrightarrow{GFp_X} GFI_X$. So

$$\begin{aligned} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(\infty/-\infty/\!\!/\infty/-\infty)^{+k+\ell}(X) \;&\simeq\; \mathrm{H}^{k+\ell}(\mathrm{t}GJ_X) \;\simeq\; \mathrm{H}^{k+\ell}(G\mathrm{t}J_X) \;\simeq\; \mathrm{H}^{k+\ell}(GFA_X) \\ &\simeq\; \mathrm{H}^{k+\ell}(GFI_X) \;\simeq\; \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(X)\;. \end{aligned}$$

We shall prove naturality of the second isomorphism. Consider the following diagram. Abbreviate $\tilde{E} := \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(\infty/-\infty/\!\!/\infty/-\infty)^{+k+\ell}$.

$$\begin{array}{ccccccc}
\tilde{E}(\mathrm{t_I}GJ_X) & \xrightarrow[\sim]{\tilde{E}(\mathrm{t_I}Gq_X)} & \tilde{E}(\mathrm{t_I}GK_X) & \xrightarrow{\tilde{E}(\mathrm{t_I}Gf'')} & \tilde{E}(\mathrm{t_I}GK_Y) & \xleftarrow[\sim]{\tilde{E}(\mathrm{t_I}Gq_Y)} & \tilde{E}(\mathrm{t_I}GJ_Y) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{H}^{k+\ell}\mathrm{t}GJ_X & \xrightarrow[\sim]{\mathrm{H}^{k+\ell}\mathrm{t}Gq_X} & \mathrm{H}^{k+\ell}\mathrm{t}GK_X & \xrightarrow{\mathrm{H}^{k+\ell}\mathrm{t}Gf''} & \mathrm{H}^{k+\ell}\mathrm{t}GK_Y & \xleftarrow[\sim]{\mathrm{H}^{k+\ell}\mathrm{t}Gq_Y} & \mathrm{H}^{k+\ell}\mathrm{t}GJ_Y \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{H}^{k+\ell}G\mathrm{t}J_X & \xrightarrow[\sim]{\mathrm{H}^{k+\ell}G\mathrm{t}q_X} & \mathrm{H}^{k+\ell}G\mathrm{t}K_X & \xrightarrow{\mathrm{H}^{k+\ell}G\mathrm{t}f''} & \mathrm{H}^{k+\ell}G\mathrm{t}K_Y & \xleftarrow[\sim]{\mathrm{H}^{k+\ell}G\mathrm{t}q_Y} & \mathrm{H}^{k+\ell}G\mathrm{t}J_Y \\
\uparrow\wr & & \uparrow & & \uparrow & & \uparrow\wr \\
\mathrm{H}^{k+\ell}GFA_X & \xrightarrow[\sim]{\mathrm{H}^{k+\ell}GFp_X} & \mathrm{H}^{k+\ell}GFI_X & \xrightarrow{\mathrm{H}^{k+\ell}GFf'} & \mathrm{H}^{k+\ell}GFI_Y & \xleftarrow[\sim]{\mathrm{H}^{k+\ell}GFp_Y} & \mathrm{H}^{k+\ell}GFA_Y \\
& & \downarrow\wr & & \downarrow\wr & & \\
& & \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(X) & \xrightarrow{(\mathrm{R}^{k+\ell}(G\circ F))(f)} & \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(Y) & &
\end{array}$$

□

IV.4.3 Haas transformations

The following transformations have been constructed in the classical framework by Haas **[21]**. We do not claim any originality.

IV.4.3.1 Situation

Consider the following diagram of abelian categories, left exact functors and transformations,

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \\
{\scriptstyle U}\downarrow & \overset{\mu}{\nearrow\!\!\!\nearrow} & {\scriptstyle V}\downarrow & \overset{\nu}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\
\mathcal{A}' & \xrightarrow{F'} & \mathcal{B}' & \xrightarrow{G'} & \mathcal{C}'
\end{array}\;,$$

i.e. $F' \circ U \xrightarrow{\mu} V \circ F$ and $G' \circ V \xrightarrow{\nu} W \circ G$. Suppose that the conditions $(1, 2, 3)$ hold.

(1) The categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and $\mathcal{B}'$ have enough injectives.

(2) The functors U and V carry injectives to injectives.

(3) The functor F carries injective to G-acyclic objects. The functor F' carries injective to G'-acyclic objects.

We have $\mathcal{A}_{(F,G)} = \mathcal{A}$ since an injective resolution is an (F,G)-acyclic resolution. Likewise, we have $\mathcal{A}'_{(F',G')} = \mathcal{A}'$.

Note in particular the case $U = 1_{\mathcal{A}}$, $V = 1_{\mathcal{B}}$ and $W = 1_{\mathcal{C}}$.

We set out to define the *Haas transformations*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}\big(U(-)\big) \xrightarrow{\mathrm{h}^{\mathrm{I}}_{\mu}} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}\big(\,-\,\big) \xrightarrow{\mathrm{h}^{\mathrm{II}}_{\nu}} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,W\circ G}\big(\,-\,\big)\,,$$

where $\mathrm{h}^{\mathrm{I}}_{\mu}$ depends on F, F', G', U, V and μ, and where $\mathrm{h}^{\mathrm{II}}_{\nu}$ depends on F, G, G', V, W and ν.

IV.4.3.2 Construction of the first Haas transformation

Given $T \in \operatorname{Ob}\mathcal{A}$, we let $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(T)$ be defined via an injective resolution I_T of T and via a CE-resolution J_T of FI_T; cf. §IV.4.2.

Given $T' \in \operatorname{Ob}\mathcal{A}'$, we let $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(T')$ be defined via an injective resolution $I'_{T'}$ of T' and via a CE-resolution $J'_{T'}$ of $F'I'_{T'}$; cf. §IV.4.2.

We define $\mathrm{h}^{\mathrm{I}}_{\mu}$. Let $X \in \operatorname{Ob}\mathcal{A}$. By Remark IV.5, there is a unique morphism $I'_{UX} \xrightarrow{h'X} UI_X$ in $\mathrm{K}^{[0}(\mathcal{A}')$ that maps to 1_{UX} under H^0. Let $J'_{UX} \xrightarrow{h''X} VJ_X$ be the unique morphism in $\mathrm{KK}^{\llcorner}(\mathcal{B}')$ that maps to the composite morphism $\Big(F'I'_{UX} \xrightarrow{F'h'X} F'UI_X \xrightarrow{\mu} VFI_X\Big)$ in $\mathrm{K}^{[0}(\mathcal{B}')$ under $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Lemma IV.13. Let the *first Haas transformation* be defined by

$$\Big(\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(UX) \xrightarrow{\mathrm{h}^{\mathrm{I}}_{\mu}X} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(X)\Big) \quad := \quad \Big(\mathrm{E}_{\mathrm{I}}(G'J'_{UX}) \xrightarrow{\mathrm{E}_{\mathrm{I}}(G'h''X)} \mathrm{E}_{\mathrm{I}}(G'VJ_X)\Big)\,.$$

We show that $\mathrm{h}^{\mathrm{I}}_{\mu}$ *is a transformation.* Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{A}$. Let $I_X \xrightarrow{f'} I_Y$ resolve $X \xrightarrow{f} Y$. Let $J_X \xrightarrow{f''} J_Y$ resolve $FI_X \xrightarrow{f'} FI_Y$. Let $I'_{UX} \xrightarrow{\tilde{f}'} I'_{UY}$ resolve $UX \xrightarrow{Uf} UY$. Let $J'_{UX} \xrightarrow{\tilde{f}''} J_{UY}$ resolve $F'I_{UX} \xrightarrow{F'\tilde{f}'} F'I_{UY}$. The quadrangle

$$\begin{array}{ccc} UX & = & UX \\ {\scriptstyle Uf}\downarrow & & \downarrow{\scriptstyle Uf} \\ UY & = & UY \end{array}$$

commutes in $\mathcal{A}'$. Hence, by Remark IV.5, applied to I'_{UX} and UI_Y, the resolved quadrangle

$$\begin{array}{ccc} I'_{UX} & \xrightarrow{h'X} & UI_X \\ {\scriptstyle \tilde{f}'}\downarrow & & \downarrow{\scriptstyle Uf'} \\ I'_{UY} & \xrightarrow[h'Y]{} & UI_Y \end{array}$$

commutes in $\mathrm{K}^{[0}(\mathcal{A}')$. Hence both quadrangles in

$$\begin{array}{ccccc} F'I'_{UX} & \xrightarrow{F'h'X} & F'UI_X & \xrightarrow{\mu} & VFI_X \\ {\scriptstyle F'\tilde{f}'}\downarrow & & \downarrow{\scriptstyle F'Uf'} & & \downarrow{\scriptstyle VFf'} \\ F'I'_{UY} & \xrightarrow[F'h'Y]{} & F'UI_Y & \xrightarrow[\mu]{} & VFI_Y \end{array}$$

commute in $\mathrm{K}^{[0}(\mathcal{B}')$. By Lemma IV.13, applied to J'_{UX} and VJ_Y, the outer quadrangle in the latter diagram can be resolved to the commutative quadrangle

$$\begin{array}{ccc} J'_{UX} & \xrightarrow{h''X} & VJ_X \\ {\scriptstyle \tilde{f}''}\downarrow & & \downarrow{\scriptstyle Vf''} \\ J'_{UY} & \xrightarrow[h''Y]{} & VJ_Y \end{array}$$

in $\mathrm{KK}^{\llcorner}(\mathcal{B}')$. Applying $\mathrm{E}_{\mathrm{I}}\big(G'(-)\big)$ and employing the definitions of $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}$, $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}$ and $\mathrm{h}^{\mathrm{I}}_{\mu}$, we obtain the sought commutative diagram

$$\begin{array}{ccc} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(UX) & \xrightarrow{\mathrm{h}^{\mathrm{I}}_{\mu}X} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(X) \\ {\scriptstyle \dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(Uf)}\downarrow & & \downarrow{\scriptstyle \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(f)} \\ \dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(UY) & \xrightarrow[\mathrm{h}^{\mathrm{I}}_{\mu}Y]{} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(Y) \end{array}$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}']\!]$.

IV.4.3.3 Construction of the second Haas transformation

We maintain the notation of §IV.4.3.2.

Given $X \in \mathrm{Ob}\,\mathcal{A}$, we let the *second Haas transformation* be defined by

$$\Big(\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G\circ V}(X) \xrightarrow{\mathrm{h}^{\mathrm{II}}_{\nu}X} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,W\circ G}(X)\Big) \quad := \quad \Big(\dot{\mathrm{E}}_{\mathrm{I}}(G'VJ_X) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(\nu)} \dot{\mathrm{E}}_{\mathrm{I}}(WGJ_X)\Big) \; .$$

It is a transformation since ν is.

IV.5 The first comparison

IV.5.1 The first comparison isomorphism

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with enough injectives and an abelian category $\mathcal{C}$. Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be an additive functor.

Suppose given objects $X \in \mathrm{Ob}\,\mathcal{A}$ and $X' \in \mathrm{Ob}\,\mathcal{A}'$. Suppose the following properties to hold.

(a) The functor $F(-, X') : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact.

(a′) The functor $F(X, -) : \mathcal{A}' \longrightarrow \mathcal{B}$ is left exact.

(b) The functor G is left exact.

(c) The object X possesses a $\big(F(-, X'), G\big)$-acyclic resolution $A \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$.

(c′) The object X' possesses a $\big(F(X, -), G\big)$-acyclic resolution $A' \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A}')$.

Moreover, the resolutions appearing in (c) and (c′) are stipulated to have the following properties.

(d) For all $k \geqslant 0$, the quasiisomorphism $\operatorname{Conc} X \longrightarrow A$ is mapped to a quasiisomorphism $\operatorname{Conc} F(X, A'^k) \longrightarrow F(A, A'^k)$ under $F(-, A'^k)$.

(d′) For all $k \geqslant 0$, the quasiisomorphism $\operatorname{Conc} X' \longrightarrow A'$ is mapped to a quasiisomorphism $\operatorname{Conc} F(A^k, X') \longrightarrow F(A^k, A')$ under $F(A^k, -)$.

The conditions (d, d′) are e.g. satisfied if $F(-, A'^k)$ and $F(A^k, -)$ are exact for all $k \geqslant 0$.

Theorem IV.31 (first comparison) *The proper Grothendieck spectral sequence for the functors $F(X, -)$ and G, evaluated at X', is isomorphic to the proper Grothendieck spectral sequence for the functors $F(-, X')$ and G, evaluated at X; i.e.*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \;\simeq\; \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X)$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$.

Proof. Let J_A, $J_{A'}$, $J_{A,A'} \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\operatorname{Inj} \mathcal{B})$ be CE-resolutions of the complexes $F(A, X')$, $F(X, A')$, $\mathrm{t}F(A, A') \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$, respectively.

The quasiisomorphism $\operatorname{Conc} X \longrightarrow A$ induces a morphism $F(\operatorname{Conc} X, A') \longrightarrow F(A, A')$, yielding $F(X, A') \longrightarrow \mathrm{t}F(A, A')$, which is a quasiisomorphism since $\operatorname{Conc} F(X, A'^k) \longrightarrow F(A, A'^k)$ is a quasiisomorphism for all $k \geqslant 0$ by (d).

Choose a CE-resolution $J_{A'} \longrightarrow J_{A,A'}$ of $F(X, A') \longrightarrow \mathrm{t}F(A, A')$; cf. Remark IV.6. Since the morphism $F(X, A') \longrightarrow \mathrm{t}F(A, A')$ is a quasiisomorphism, $J_{A'} \longrightarrow J_{A,A'}$ is a composite in $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\operatorname{Inj} \mathcal{B})$ of a rowwise homotopism and a double homotopism; cf. Proposition IV.17. So is $GJ_{A'} \longrightarrow GJ_{A,A'}$. Hence, by Remark IV.23 and by Lemma IV.24, we obtain an isomorphism of the proper spectral sequences of the first filtrations of the total complexes,

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \;=\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A'}) \xrightarrow{\sim} \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A,A'}) \,.$$

Likewise, we have an isomorphism

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X) \;=\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A}) \xrightarrow{\sim} \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A,A'}) \,.$$

We compose to an isomorphism $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \xrightarrow{\sim} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X)$ as sought. ▫

IV.5.2 Naturality of the first comparison isomorphism

We narrow down the assumptions just as we have done for the introduction of the Haas transformations in §IV.4.3.1 in order to be able to express, in this narrower case, a naturality of the first comparison isomorphism from Theorem IV.31.

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with enough injectives and an abelian category $\mathcal{C}$.

Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be an additive functor.

Suppose that the following properties hold.

(a) The functor $F(-,X') : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact for all $X' \in \operatorname{Ob}\mathcal{A}'$.

(a′) The functor $F(X,-) : \mathcal{A}' \longrightarrow \mathcal{B}$ is left exact for all $X \in \operatorname{Ob}\mathcal{A}$.

(b) The functor G is left exact.

(c) For all $X' \in \operatorname{Ob}\mathcal{A}'$, the functor $F(-,X')$ carries injective objects to G-acyclic objects.

(c′) For all $X \in \operatorname{Ob}\mathcal{A}$, the functor $F(X,-)$ carries injective objects to G-acyclic objects.

(d) The functor $F(I,-)$ is exact for all $I \in \operatorname{Ob}\operatorname{Inj}\mathcal{A}$.

(d′) The functor $F(-,I')$ is exact for all $I' \in \operatorname{Ob}\operatorname{Inj}\mathcal{A}'$.

Proposition IV.32 *Suppose given $X \xrightarrow{x} \tilde{X}$ in $\mathcal{A}$ and $X' \in \operatorname{Ob}\mathcal{A}'$. Note that we have a transformation $F(x,-) : F(X,-) \longrightarrow F(\tilde{X},-)$. The following quadrangle, whose vertical isomorphisms are given by the construction in the proof of Theorem* IV.31*, commutes.*

$$\begin{array}{ccc}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') & \xrightarrow{\mathrm{h}^{\mathrm{I}}_{F(x,-)}X'} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(\tilde{X},-),G}(X') \\
\downarrow\wr & & \downarrow\wr \\
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X) & \xrightarrow{\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(x)} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(\tilde{X})
\end{array}$$

For the definition of the first Haas transformation $\mathrm{h}^{\mathrm{I}}_{F(x,-)}$, see §IV.4.3.2.

An analogous assertion holds with interchanged roles of $\mathcal{A}$ and $\mathcal{A}'$.

Proof of Proposition IV.32. Let I resp. $\tilde{I}$ be an injective resolution of X resp. $\tilde{X}$ in $\mathcal{A}$. Let $I \xrightarrow{\hat{x}} \tilde{I}$ be a resolution of $X \xrightarrow{x} \tilde{X}$. Let I' be an injective resolution of X' in $\mathcal{A}'$.

Let $J^{(X)}_{I'}$ resp. $J^{(\tilde{X})}_{I'}$ be a CE-resolution of $F(X,I')$ resp. $F(\tilde{X},I')$.

Let $J_{I,I'}$ resp. $J_{\tilde{I},I'}$ be a CE-resolution of $\mathrm{t}F(I,I')$ resp. $\mathrm{t}F(\tilde{I},I')$.

Let J_I resp. $J_{\tilde{I}}$ be a CE-resolution of $F(I,X')$ resp. $F(\tilde{I},X')$.

We have a commutative diagram

$$\begin{array}{ccc}
F(X,I') & \xrightarrow{F(x,I')} & F(\tilde{X},I') \\
\downarrow & & \downarrow \\
\mathrm{t}F(I,I') & \xrightarrow{\mathrm{t}F(\hat{x},I')} & \mathrm{t}F(\tilde{I},I') \\
\uparrow & & \uparrow \\
F(I,X') & \xrightarrow{F(\hat{x},X')} & F(\tilde{I},X')
\end{array}$$

in $\mathrm{C}^{[0}(\mathcal{B})$, hence in $\mathrm{K}^{[0}(\mathcal{B})$. By Proposition IV.14, it can be resolved to a commutative diagram

$$\begin{array}{ccc}
J_{I'}^{(X)} & \longrightarrow & J_{I'}^{(\tilde{X})} \\
\downarrow & & \downarrow \\
J_{I,I'} & \longrightarrow & J_{\tilde{I},I'} \\
\uparrow & & \uparrow \\
J_I & \longrightarrow & J_{\tilde{I}}
\end{array}$$

in $\mathrm{KK}^{\llcorner}(\mathcal{B})$. Application of $\dot{\mathrm{E}}_{\mathrm{I}}\big(G(-)\big)$ yields the result; cf. Lemma IV.24.

We refrain from investigating naturality of the first comparison isomorphism in G.

IV.6 The second comparison

IV.6.1 The second comparison isomorphism

Suppose given abelian categories $\mathcal{A}$ and $\mathcal{B}'$ with enough injectives, and abelian categories $\mathcal{B}$ and $\mathcal{C}$.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B}'$ be an additive functor. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor.

Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$. Let $B \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y, i.e. suppose a quasiisomorphism $\operatorname{Conc} Y \longrightarrow B$ to exist. Suppose the following properties to hold.

(a) The functor F is left exact.

(b) The functor $G(Y,-)$ is left exact.

(c) The object X possesses an $(F,\, G(Y,-))$-acyclic resolution $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$.

(d) The functor $G(B^k,-)$ is exact for all $k \geqslant 0$.

(e) The functor $G(-,I')$ is exact for all $I' \in \operatorname{Ob}\operatorname{Inj}\mathcal{B}'$.

Remark IV.33 *Suppose given a morphism $D \xrightarrow{f} D'$ in $\mathrm{CC}^{\llcorner}(\mathcal{C})$. If $\mathrm{H}^{\ell}(f^{-,*})$ is a quasiisomorphism for all $\ell \geqslant 0$, then f induces an isomorphism*

$$\dot{\mathrm{E}}_{\mathrm{I}}(D) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(f)} \dot{\mathrm{E}}_{\mathrm{I}}(D')$$

of proper spectral sequences.

Proof. By Lemma IV.21, it suffices to show that $\mathrm{E}_{\mathrm{I}}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-2)^{+k}(f)$ is an isomorphism for all $\alpha \in \mathbf{Z}$ and all $k \in \mathbf{Z}$. By Lemma IV.22, this amounts to isomorphisms $\mathrm{H}^k\mathrm{H}^{\ell}(f^{-,*})$ for all $k,\, \ell \geqslant 0$, i.e. to quasiisomorphisms $\mathrm{H}^{\ell}(f^{-,*})$ for all $\ell \geqslant 0$. ▫

Consider the double complex $G(B,FA) \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{C})$, where the indices of B count rows and the indices of A count columns. To the first filtration of its total complex, we can associate the proper spectral sequence $\dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA)) \in \operatorname{Ob}[\![\dot{\bar{\mathbf{Z}}}_{\infty}^{\#\#},\mathcal{C}]\!]$.

Theorem IV.34 (second comparison) *The proper Grothendieck spectral sequence for the functors F and $G(Y,-)$, evaluated at X, is isomorphic to $\dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA))$; i.e.*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA))$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\mathcal{C}]\!]$.

Proof. Let $J' \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B}')$ be a CE-resolution of FA. By definition, $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) = \dot{\mathrm{E}}_{\mathrm{I}}(G(Y,J'))$. By Remark IV.33, it suffices to find $D \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{C})$ and two morphisms of double complexes

$$G(B,FA) \xrightarrow{\;u\;} D \xleftarrow{\;v\;} G(Y,J')$$

such that $\mathrm{H}^{\ell}(u^{-,*})$ and $\mathrm{H}^{\ell}(v^{-,*})$ are quasiisomorphisms for all $\ell \geqslant 0$.

Given a complex $U \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$, recall that we denote by $\mathrm{Conc}_2 U \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{B})$ the double complex whose row number 0 is given by U, and whose other rows are zero.

We have a diagram

$$G(B,\mathrm{Conc}_2\, FA) \longrightarrow G(B,J') \longleftarrow G(\mathrm{Conc}\, Y,J')$$

in $\mathrm{CCC}^{\llcorner}(\mathcal{C})$. Let $\ell \geqslant 0$. Application of $\mathrm{H}^{\ell}\big((-)^{-,=,*}\big)$ yields a diagram

$$(*) \qquad \mathrm{H}^{\ell}\big(G(B,\mathrm{Conc}_2\, FA)^{-,=,*}\big) \longrightarrow \mathrm{H}^{\ell}\big(G(B,J')^{-,=,*}\big) \longleftarrow \mathrm{H}^{\ell}\big(G(\mathrm{Conc}\, Y,J')^{-,=,*}\big)$$

in $\mathrm{CC}^{\llcorner}(\mathcal{C})$. We have

$$\mathrm{H}^{\ell}\big(G(B,\mathrm{Conc}_2\, FA)^{-,=,*}\big) \;\simeq\; G\Big(B\,,\,\mathrm{H}^{\ell}\big((\mathrm{Conc}_2\, FA)^{-,*}\big)\Big) \;=\; G\big(B,\mathrm{Conc}\,\mathrm{H}^{\ell}(FA)\big)$$

and

$$\mathrm{H}^{\ell}\big(G(B,J')^{-,=,*}\big) \;\simeq\; G\big(B,\mathrm{H}^{\ell}(J'^{-,*})\big)\;,$$

since the functor $G(B^k,-)$ is exact for all $k \geqslant 0$ by (d), or, since the CE-resolution J is rowwise split. Since the CE-resolution J' is rowwise split, we moreover have

$$\mathrm{H}^{\ell}\big(G(\mathrm{Conc}\, Y,J')^{-,=,*}\big) \;\simeq\; G\big(\,\mathrm{Conc}\, Y,\mathrm{H}^{\ell}(J'^{-,*})\big)\;.$$

So the diagram $(*)$ is isomorphic to the diagram

$$(**) \qquad G\big(B,\mathrm{Conc}\,\mathrm{H}^{\ell}(FA)\big) \longrightarrow G\big(B,\mathrm{H}^{\ell}(J'^{-,*})\big) \longleftarrow G\big(\,\mathrm{Conc}\, Y,\,\mathrm{H}^{\ell}(J'^{-,*})\big)\;,$$

whose left hand side morphism is induced by the quasiisomorphism $\mathrm{Conc}\,\mathrm{H}^{\ell}(FA) \longrightarrow \mathrm{H}^{\ell}(J'^{-,*})$, and whose right hand side morphism is induced by the quasiisomorphism $\mathrm{Conc}\, Y \longrightarrow B$.

By exactness of $G(B^k,-)$ for $k \geqslant 0$, the left hand side morphism of $(**)$ is a rowwise quasiisomorphism. Since $\mathrm{H}^{\ell}(J'^{k,*})$ is injective, the functor $G(-,\mathrm{H}^{\ell}(J'^{k,*}))$ is exact by (e), and therefore the right hand side morphism of $(**)$ is a columnwise quasiisomorphism. Thus an application of t to $(**)$ yields two quasiisomorphisms; cf. §IV.1.1.6. Hence, also an application of t to $(*)$ yields two quasiisomorphisms in the diagram

$$\mathrm{tH}^{\ell}\big(G(B,\mathrm{Conc}_2\, FA)^{-,=,*}\big) \longrightarrow \mathrm{tH}^{\ell}\big(G(B,J')^{-,=,*}\big) \longleftarrow \mathrm{tH}^{\ell}\big(G(\mathrm{Conc}\, Y,J')^{-,=,*}\big)\;.$$

Note that $\mathrm{t} \circ \mathrm{H}^\ell\big((-)^{-,=,*}\big) = \mathrm{H}^\ell\big((-)^{-,*}\big) \circ \mathrm{t}_{1,2}$, where $\mathrm{t}_{1,2}$ denotes taking the total complex in the first and the second index of a triple complex; cf. §IV.1.2.2. Hence we have a diagram

$$\mathrm{H}^\ell\Big(\big(\mathrm{t}_{1,2}G(B, \mathrm{Conc}_2\, FA)\big)^{-,*}\Big) \longrightarrow \mathrm{H}^\ell\Big(\big(\mathrm{t}_{1,2}G(B, J')\big)^{-,*}\Big) \longleftarrow \mathrm{H}^\ell\Big(\big(\mathrm{t}_{1,2}G(\mathrm{Conc}\, Y, J')\big)^{-,*}\Big)$$

consisting of two quasiisomorphisms. This diagram in turn, is isomorphic to

$$\mathrm{H}^\ell\Big(G(B, FA)^{-,*}\Big) \longrightarrow \mathrm{H}^\ell\Big((\mathrm{t}_{1,2}G(B, J'))^{-,*}\Big) \longleftarrow \mathrm{H}^\ell\Big(\big(G(Y, J')\big)^{-,*}\Big) ,$$

where the left hand side morphism is obtained by precomposition with the isomorphism $G(B, FA^k) \xrightarrow{\sim} \mathrm{t}\,\mathrm{Conc}_1\, G(B, FA^k) = (\mathrm{t}_{1,2}G(B, \mathrm{Conc}_2\, FA))^{-,k}$, where $k \geqslant 0$; cf. §IV.1.1.6.

Hence we may take

$$\big(G(B, FA) \xrightarrow{u} D \xleftarrow{v} G(B, J')\big) \quad := \quad \Big(G(B, FA) \longrightarrow \mathrm{t}_{1,2}G(B, J') \longleftarrow G(Y, J')\Big) .$$

□

IV.6.2 Naturality of the second comparison isomorphism

Again, we narrow down the assumptions just as we have done for the introduction of the Haas transformations in §IV.4.3.1 to express a naturality of the second comparison isomorphism from Theorem IV.34.

Suppose given abelian categories $\mathcal{A}$ and $\mathcal{B}'$ with enough injectives, and abelian categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given additive functors $\mathcal{A} \underset{\tilde{F}}{\overset{F}{\rightrightarrows}} \mathcal{B}'$ and a transformation $F \xrightarrow{\varphi} \tilde{F}$. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor.

Suppose given a morphism $X \xrightarrow{x} \tilde{X}$ in $\mathcal{A}$ and an object $Y \in \mathrm{Ob}\,\mathcal{B}$. Let $B \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y, i.e. suppose a quasiisomorphism $\mathrm{Conc}\, Y \longrightarrow B$ to exist. Suppose the following properties to hold.

(a) The functors F and $\tilde{F}$ are left exact and carry injective to $G(Y, -)$-acyclic objects.

(b) The functor $G(Y, -)$ is left exact.

(c) The functor $G(B^k, -)$ is exact for all $k \geqslant 0$.

(d) The functor $G(-, I')$ is exact for all $I' \in \mathrm{Ob}\,\mathrm{Inj}\,\mathcal{B}'$.

Let $A \xrightarrow{a} \tilde{A}$ in $\mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{A})$ be an injective resolution of $X \xrightarrow{x} \tilde{X}$ in $\mathcal{A}$. Note that we have a commutative quadrangle

$$\begin{array}{ccc} G(B, FA) & \xrightarrow{G(B, \varphi A)} & G(B, \tilde{F}A) \\ {\scriptstyle G(B, Fa)}\downarrow & & \downarrow{\scriptstyle G(B, \tilde{F}a)} \\ G(B, F\tilde{A}) & \xrightarrow{G(B, \varphi\tilde{A})} & G(B, \tilde{F}\tilde{A}) \end{array}$$

in $\mathrm{CC}^{\llcorner}(\mathcal{C})$.

Note that once chosen injective resolutions A of X and $\tilde{A}$ of $\tilde{X}$, the image of $G(B, Fa)$ in $\mathrm{KK}^{\llcorner}(\mathcal{C})$ does not depend on the choice of the resolution $A \xrightarrow{a} \tilde{A}$ of $X \xrightarrow{x} \tilde{X}$, for $\mathrm{C}^{[0}(\mathcal{A}) \xrightarrow{G(B,F(-))} \mathrm{CC}^{\llcorner}(\mathcal{C})$ maps an elementary split acyclic complex to an elementary horizontally split acyclic complex.

Lemma IV.35 *The quadrangle*

$$\begin{array}{ccc} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) & \xrightarrow{\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(x)} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(\tilde{X}) \\ \wr\downarrow & & \downarrow\wr \\ \dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA)) & \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(G(B,Fa))} & \dot{\mathrm{E}}_{\mathrm{I}}(G(B,F\tilde{A})) \end{array}$$

commutes, where the vertical isomorphisms are those constructed in the proof of Theorem IV.34.

Proof. Let $J' \xrightarrow{\hat{a}} \tilde{J}'$ be a CE-resolution of $FA \xrightarrow{Fa} F\tilde{A}$. Consider the following commutative diagram in $\mathrm{CC}^{\llcorner}(\mathcal{C})$.

$$\begin{array}{ccc} G(Y,J') & \xrightarrow{G(Y,\hat{a})} & G(Y,\tilde{J}') \\ \downarrow & & \downarrow \\ \mathrm{t}_{1,2}G(B,J') & \xrightarrow{\mathrm{t}_{1,2}G(B,\hat{a})} & \mathrm{t}_{1,2}G(B,\tilde{J}') \\ \uparrow & & \uparrow \\ G(B,FA) & \xrightarrow{G(B,Fa)} & G(B,F\tilde{A}) \end{array}$$

An application of $\dot{\mathrm{E}}_{\mathrm{I}}$ yields the result. ▫

Lemma IV.36 *The quadrangle*

$$\begin{array}{ccc} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) & \xrightarrow{\mathrm{h}^{\mathrm{I}}_{\varphi}X} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{\tilde{F},G(Y,-)}(X) \\ \wr\downarrow & & \downarrow\wr \\ \dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA)) & \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(G(B,\varphi A))} & \dot{\mathrm{E}}_{\mathrm{I}}(G(B,\tilde{F}A)) \end{array}$$

commutes, where the vertical morphisms are those constructed in the proof of Theorem IV.34.

For the definition of the first Haas transformation $\mathrm{h}^{\mathrm{I}}_{F(x,-)}$, see §IV.4.3.2.

Proof. Let $J' \xrightarrow{\hat{\varphi}} \breve{J}'$ be a CE-resolution of $FA \xrightarrow{F\varphi} \tilde{F}A$. Consider the following commutative diagram in $\mathrm{CC}^{\llcorner}(\mathcal{C})$.

$$\begin{array}{ccc} G(Y,J') & \xrightarrow{G(Y,\hat{\varphi})} & G(Y,\breve{J}') \\ \downarrow & & \downarrow \\ \mathrm{t}_{1,2}G(B,J') & \xrightarrow{\mathrm{t}_{1,2}G(B,\hat{\varphi})} & \mathrm{t}_{1,2}G(B\breve{J}') \\ \uparrow & & \uparrow \\ G(B,FA) & \xrightarrow{G(B,\varphi A)} & G(B,\tilde{F}A) \end{array}$$

An application of $\dot{\mathrm{E}}_{\mathrm{I}}$ yields the result. ▫

We refrain from investigating naturality of the second comparison isomorphism in Y.

IV.7 Acyclic CE-resolutions

We record BEYL's Theorem [7, Th. 3.4] (here Theorem IV.40) in order to document that it fits in our context. The argumentation is entirely due to BEYL [7, Sec. 3], so we do not claim any originality.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Suppose $\mathcal{A}$ and $\mathcal{B}$ to have enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be left exact functors.

IV.7.1 Definition

Let $T \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$. In this §IV.7, a CE-resolution of T will synonymously (and not quite correctly) be called an *injective CE-resolution,* to emphasise the fact that its object entries are injective.

We regard $\mathrm{C}^{[0}(\mathcal{B})$ as an exact category as in Remarks IV.9 and IV.11.

Definition IV.37 A double complex $B \in \mathrm{CC}^{\llcorner}(\mathcal{B})$ is called a *G-acyclic CE-resolution* of T if the following conditions are satisfied.

(1) We have $\mathrm{H}^0(B^{*,-}) \simeq T$ and $\mathrm{H}^k(B^{*,-}) \simeq 0$ for all $k \geqslant 1$.

(2) The morphism of complexes $B^{k,*} \longrightarrow B^{k+1,*}$, consisting of vertical differentials of B, is a pure morphism for all $k \geqslant 0$.

(3) The object $\mathrm{B}^{\ell}(B^{k,*})$ is G-acyclic for all $k,\ \ell \geqslant 0$.

(4) The object $\mathrm{Z}^{\ell}(B^{k,*})$ is G-acyclic for all $k,\ \ell \geqslant 0$.

A *G-acyclic CE-resolution* is a G-acyclic CE-resolution of some $T \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$.

From $(3,4)$ and the short exact sequence $\mathrm{Z}^{\ell}(B^{k,*}) \longrightarrow B^{k,\ell} \longrightarrow \mathrm{B}^{\ell+1}(B^{k,*})$, we conclude that $B^{k,\ell}$ is G-acyclic for all $k,\ \ell \geqslant 0$.

From $(3,4)$ and the short exact sequence $\mathrm{B}^{\ell}(B^{k,*}) \longrightarrow \mathrm{Z}^{\ell}(B^{k,*}) \longrightarrow \mathrm{H}^{\ell}(B^{k,*})$, we conclude that $\mathrm{H}^{\ell}(B^{k,*})$ is G-acyclic for all $k,\ \ell \geqslant 0$.

Example IV.38 An injective CE-resolution of T is in particular a G-acyclic CE-resolution of T.

Note that given $Y \in \operatorname{Ob} \mathrm{C}(\mathcal{B})$ and $\ell \in \mathbf{Z}$, we have $\mathrm{Z}^{\ell} GY \simeq G\mathrm{Z}^{\ell} Y$, whence the universal property of the cokernel $\mathrm{H}^{\ell} GY$ of $GY^{\ell-1} \longrightarrow \mathrm{Z}^{\ell} GY$ induces a morphism $\mathrm{H}^{\ell} GY \longrightarrow G\mathrm{H}^{\ell} Y$. This furnishes a transformation $\mathrm{H}^{\ell}(GX^{k,*}) \xrightarrow{\theta X} G\mathrm{H}^{\ell}(X^{k,*})$, natural in $X \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{B})$.

Remark IV.39 *If B is a G-acyclic CE-resolution, then $\mathrm{H}^{\ell}(GB^{-,*}) \xrightarrow{\theta B} G\mathrm{H}^{\ell}(B^{-,*})$ is an isomorphism for all $\ell \geqslant 0$.*

Proof. The sequences

$$\begin{array}{ccccc} G\mathrm{B}^\ell(B^{k,*}) & \longrightarrow & G\mathrm{Z}^\ell(B^{k,*}) & \longrightarrow & G\mathrm{H}^\ell(B^{k,*}) \\ G\mathrm{Z}^{\ell-1}(B^{k,*}) & \longrightarrow & GB^{k,\ell-1} & \longrightarrow & G\mathrm{B}^\ell(B^{k,*}) \end{array}$$

are short exact for $k,\ \ell \geqslant 0$ by G-acyclicity of $\mathrm{B}^\ell(B^{k,*})$ resp. of $\mathrm{Z}^{\ell-1}(B^{k,*})$. In particular, the cokernel of $GB^{k,\ell-1} \longrightarrow G\mathrm{Z}^\ell(B^{k,*})$ is given by $G\mathrm{H}^\ell(B^{k,*})$. ▫

IV.7.2 A theorem of Beyl

Let $X \in \mathrm{Ob}\,\mathcal{A}_{(F,G)}$. Let $A \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$ be a (F,G)-acyclic resolution of X. Let $B \in \mathrm{CC}^{\llcorner}(\mathcal{B})$ be a G-acyclic CE-resolution of FA.

Theorem IV.40 (Beyl, [**7**, Th. 3.4]) *We have an isomorphism of proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(X) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}(GB)$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_\infty, \mathcal{C}]\!]$.

Proof. Since the proper Grothendieck spectral sequence is, up to isomorphism, independent of the choice of an injective CE-resolution, as pointed out in §IV.4.2, our assertion is equivalent to the existence of an injective CE-resolution J of FA such that $\dot{\mathrm{E}}_{\mathrm{I}}(GJ) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}(GB)$. So by Remark IV.33, it suffices to show that there exists an injective CE-resolution J of FA and a morphism $B \longrightarrow J$ that induces a quasiisomorphism $\mathrm{H}^\ell(GB^{-,*}) \longrightarrow \mathrm{H}^\ell(GJ^{-,*})$ for all $\ell \geqslant 0$. By Remark IV.39 and Example IV.38, it suffices to show that $G\mathrm{H}^\ell(B^{-,*}) \longrightarrow G\mathrm{H}^\ell(J^{-,*})$ is a quasiisomorphism for all $\ell \geqslant 0$.

By the conditions $(1, 2)$ on B and by G-acyclicity of $\mathrm{H}^\ell(B^{k,*})$ for $k,\ \ell \geqslant 0$, the complex $\mathrm{H}^\ell(B^{-,*})$ is a G-acyclic resolution of $\mathrm{H}^\ell(FA)$; cf. Remark IV.10.

By Remark IV.4, there exists $J \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathrm{Inj}\,\mathcal{B})$ with vertical pure morphisms and split rows, and a morphism $B \longrightarrow J$ consisting rowwise of pure monomorphisms such that $\mathrm{H}^k(B^{*,-}) \longrightarrow \mathrm{H}^k(J^{*,-})$ is an isomorphism of complexes for all $k \geqslant 0$. In particular, the composite $(\mathrm{Conc}_2\, FA \longrightarrow B \longrightarrow J)$ turns J into an injective CE-resolution of FA.

Let $\ell \geqslant 0$. Since B is a G-acyclic and J an injective CE-resolution of FA, both $\mathrm{Conc}\,\mathrm{H}^\ell(FA) \longrightarrow \mathrm{H}^\ell(B^{-,*})$ and $\mathrm{Conc}\,\mathrm{H}^\ell(FA) \longrightarrow \mathrm{H}^\ell(J^{-,*})$ are quasiisomorphisms. Hence $\mathrm{H}^\ell(B^{-,*}) \longrightarrow \mathrm{H}^\ell(J^{-,*})$ is a quasiisomorphism, too. Now Lemma IV.27 shows that $G\mathrm{H}^\ell(B^{-,*}) \longrightarrow G\mathrm{H}^\ell(J^{-,*})$ is a quasiisomorphism as well. ▫

IV.8 Applications

We will apply Theorems IV.31 and IV.34 in various algebraic situations. In particular, we will re-prove a theorem of Beyl; viz. Theorem IV.53 in §IV.8.3.

In several instances below, we will make tacit use of the fact that a left exact functor between abelian categories respects injectivity of objects provided it has an exact left adjoint.

IV.8.1 A Hopf algebra lemma

We will establish Lemma IV.47 in §IV.8.1.4, needed to prove an acyclicity that enters the proof of the comparison result Theorem IV.52 in §IV.8.2 for Hopf algebra cohomology, which in turn allows to derive comparison results for group cohomology and Lie algebra cohomology; cf. §§ IV.8.3, IV.8.4.

IV.8.1.1 Definition

Let R be a commutative ring. Write $\otimes := \otimes_R$. A *Hopf algebra over* R is an R-algebra H together with R-algebra morphisms $H \xrightarrow{\varepsilon} R$ *(counit)* and $H \xrightarrow{\Delta} H \otimes H$ *(comultiplication)*, and an R-linear map $H \xrightarrow{S} H$ *(antipode)* such that the following conditions (i–iv) hold.

Write $x\Delta = \sum_i xu_i \otimes xv_i$ for $x \in H$, where u_i and v_i are chosen maps from H to H, and where i runs over a suitable indexing set. Note that $\sum_i (r \cdot x + s \cdot y)u_i \otimes (r \cdot x + s \cdot y)v_i = r \cdot (\sum_i xu_i \otimes xv_i) + s \cdot (\sum_i yu_i \otimes yv_i)$ for $x, y \in H$ and $r, s \in R$, whereas u_i and v_i are not necessarily R-linear maps.

The elegant Sweedler notation [55, §1.2] for the images under $\Delta(\Delta \otimes 1)$ etc. led the author, being new to Hopf algebras, to confusion in a certain case. So we will express them in these more naive terms.

Write $H \otimes H \xrightarrow{\nabla} H$, $x \otimes y \longmapsto x \cdot y$ and $R \xrightarrow{\eta} H$, $r \longmapsto r \cdot 1_H$. Write $H \otimes H \xrightarrow{\tau} H \otimes H$, $x \otimes y \longmapsto y \otimes x$.

(i) We have $\Delta(\varepsilon \otimes \mathrm{id}_H) = (x \longmapsto 1_R \otimes x)$, i.e. $\sum_i xu_i\varepsilon \cdot xv_i = x$ for $x \in H$.

(i′) We have $\Delta(\mathrm{id}_H \otimes \varepsilon) = (x \longmapsto x \otimes 1_R)$, i.e. $\sum_i xu_i \cdot xv_i\varepsilon = x$ for $x \in H$.

(ii) We have $\Delta(\mathrm{id}_H \otimes \Delta) = \Delta(\Delta \otimes \mathrm{id}_H)$, i.e. $\sum_{i,j} xu_i \otimes xv_iu_j \otimes xv_iv_j = \sum_{i,j} xu_iu_j \otimes xu_iv_j \otimes xv_i$ for $x \in H$.

(iii) We have $\Delta(S \otimes \mathrm{id}_H)\nabla = \varepsilon\eta$, i.e. $\sum_i xu_iS \cdot xv_i = x\varepsilon \cdot 1_H$ for $x \in H$.

(iii′) We have $\Delta(\mathrm{id}_H \otimes S)\nabla = \varepsilon\eta$, i.e. $\sum_i xu_i \cdot xv_iS = x\varepsilon \cdot 1_H$ for $x \in H$.

(iv) We have $S^2 = \mathrm{id}_H$.

In particular, imposing (iv), we stipulate a Hopf algebra to have an *involutive* antipode.

IV.8.1.2 Some basic properties

In an attempt to be reasonably self-contained, we recall some basic facts on Hopf algebras needed for Lemma IV.47 below; cf. [**55,** Ch. IV], [**1,** §2], [**47,** §§1-3]. In doing so, we shall use direct arguments.

Suppose given a Hopf algebra H over R.

Remark IV.41 ([**55,** Prop. 4.0.1], [**1,** Th. 2.1.4], [**47,** 3.4.2])

The following hold.

(1) *We have* $\sum_i (x\cdot y)u_i \otimes (x\cdot y)v_i = \sum_{i,j}(xu_i\cdot yu_j)\otimes(xv_i\cdot yv_j)$ *for* $x,\, y\,\in\, H$.

(2) *We have* $1_H S = 1_H$.

(3) *We have* $(x\cdot y)S = yS\cdot xS$ *for* $x,\, y\,\in\, H$.

(4) *We have* $S\varepsilon = \varepsilon$.

(5) *We have* $\Delta(S\otimes S)\tau = S\Delta$, *i.e.* $\sum_i xu_iS\otimes xv_iS = \sum_i xSv_i\otimes xSu_i$ *for* $x\in H$.

(6) *We have* $x\cdot y = \sum_i \Big(\sum_j (xu_i)u_j\cdot y\cdot (xu_i)v_jS\Big)\cdot xv_i$ *for* $x,\, y\,\in\, H$.

(6′) *We have* $y\cdot x = \sum_i xu_i\cdot \Big(\sum_j (xv_i)u_jS\cdot y\cdot (xv_i)v_j\Big)$ *for* $x,\, y\,\in\, H$.

(7) *We have* $\sum_i xv_i\cdot xu_iS = x\varepsilon\cdot 1_H$ *for* $x\in H$.

(7′) *We have* $\sum_i xv_iS\cdot xu_i = x\varepsilon\cdot 1_H$ *for* $x\in H$.

Proof. Ad (1). Given $x,\, y\,\in\, H$, we obtain

$$\textstyle\sum_i (xy)u_i\otimes(xy)v_i \;=\; (xy)\Delta \;=\; x\Delta\cdot y\Delta \;=\; \sum_{i,j}(xu_i\cdot yu_j)\otimes(xv_i\cdot yv_j)\;.$$

Ad (2). Remarking that $1_H\Delta = 1_H\otimes 1_H$, we obtain

$$1_H S \;=\; 1_H\Delta(S\otimes \mathrm{id}_H)\nabla \;\overset{\text{(iii)}}{=}\; 1_H\varepsilon\cdot 1_H \;=\; 1_H\;.$$

Ad (3). Given $x,\, y\,\in\, H$, we obtain

$$\begin{aligned}
(x\cdot y)S &\overset{2\times\text{(i}'\text{)}}{=} \textstyle\sum_{i,k}(xu_i\cdot xv_i\varepsilon\cdot yu_k\cdot yv_k\varepsilon)S\\
&\overset{\text{(iii}'\text{)}}{=} \textstyle\sum_{i,j,k}(xu_i\cdot yu_k\cdot yv_k\varepsilon)S\cdot xv_iu_j\cdot xv_iv_jS\\
&\overset{\text{(iii}'\text{)}}{=} \textstyle\sum_{i,j,k,\ell}(xu_i\cdot yu_k)S\cdot xv_iu_j\cdot yv_ku_\ell\cdot yv_kv_\ell S\cdot xv_iv_jS\\
&\overset{2\times\text{(ii)}}{=} \textstyle\sum_{i,j,k,\ell}(xu_iu_j\cdot yu_ku_\ell)S\cdot xu_iv_j\cdot yu_kv_\ell\cdot yv_kS\cdot xv_iS\\
&\overset{(1)}{=} \textstyle\sum_{i,j,k}(xu_i\cdot yu_k)u_jS\cdot (xu_i\cdot yu_k)v_j\cdot yv_kS\cdot xv_iS\\
&\overset{\text{(iii)}}{=} \textstyle\sum_{i,k}(xu_i\cdot yu_k)\varepsilon\cdot yv_kS\cdot xv_iS\\
&= \textstyle\sum_{i,k}(yu_k\varepsilon\cdot yv_k)S\cdot (xu_i\varepsilon\cdot xv_i)S\\
&\overset{2\times\text{(i)}}{=} yS\cdot xS\;.
\end{aligned}$$

Ad (4). Note that $(y\varepsilon\cdot z)\varepsilon = y\varepsilon\cdot z\varepsilon = (y\cdot z)\varepsilon$ for $y,\, z\,\in\, H$. Given $x\in H$, we obtain

$$xS\varepsilon \overset{\text{(i)}}{=} (\textstyle\sum_i xu_i\varepsilon\cdot xv_i)S\varepsilon \;=\; (\sum_i xu_i\varepsilon\cdot xv_iS)\varepsilon \;=\; (\sum_i xu_i\cdot xv_iS)\varepsilon \overset{\text{(iii}'\text{)}}{=} (x\varepsilon\cdot 1_H)\varepsilon \;=\; x\varepsilon\;.$$

Ad (5). Given $x \in H$, we obtain

$$
\begin{array}{rcl}
x\Delta(S\otimes S)\tau & \overset{\text{(i)}}{=} & \sum_i (xu_i\varepsilon \cdot xv_i)\Delta(S\otimes S)\tau \\
& = & \sum_i (xu_i\varepsilon \cdot 1_H)\Delta \cdot xv_i\Delta(S\otimes S)\tau \\
& \overset{\text{(iii)}}{=} & \sum_{i,j} (xu_iu_jS \cdot xu_iv_j)\Delta \cdot xv_i\Delta(S\otimes S)\tau \\
& = & \sum_{i,j} xu_iu_jS\Delta \cdot xu_iv_j\Delta \cdot xv_i\Delta(S\otimes S)\tau \\
& \overset{\text{(ii)}}{=} & \sum_{i,j} xu_iS\Delta \cdot xv_iu_j\Delta \cdot xv_iv_j\Delta(S\otimes S)\tau \\
& = & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_ju_k \otimes xv_iu_jv_k)\cdot(xv_iv_jv_\ell S \otimes xv_iv_ju_\ell S) \\
& = & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_ju_k \cdot xv_iv_jv_\ell S \otimes xv_iu_jv_k \cdot xv_iv_ju_\ell S) \\
& \overset{\text{(ii)}}{=} & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jv_kv_\ell S \otimes xv_iv_ju_k \cdot xv_iv_jv_ku_\ell S) \\
& \overset{\text{(ii)}}{=} & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jv_kS \otimes xv_iv_ju_ku_\ell \cdot xv_iv_ju_kv_\ell S) \\
& \overset{\text{(iii')}}{=} & \sum_{i,j,k} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jv_kS \otimes xv_iv_ju_k\varepsilon \cdot 1_H) \\
& = & \sum_{i,j,k} xu_iS\Delta \cdot (xv_iu_j \cdot (xv_iv_jv_k \cdot xv_iv_ju_k\varepsilon)S \otimes 1_H) \\
& \overset{\text{(i)}}{=} & \sum_{i,j} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jS \otimes 1_H) \\
& \overset{\text{(iii')}}{=} & \sum_i xu_iS\Delta \cdot (xv_i\varepsilon \cdot 1_H \otimes 1_H) \\
& = & \sum_i (xu_i \cdot xv_i\varepsilon)S\Delta \\
& \overset{\text{(i')}}{=} & xS\Delta\ .
\end{array}
$$

Ad (6). Given $x,\, y \in H$, we obtain

$$
x\cdot y \overset{\text{(i')}}{=} \textstyle\sum_i xu_i \cdot y \cdot xv_i\varepsilon \overset{\text{(iii)}}{=} \sum_{i,j} xu_i \cdot y \cdot xv_iu_jS \cdot xv_iv_j \overset{\text{(ii)}}{=} \sum_{i,j} xu_iu_j \cdot y \cdot xu_iv_jS \cdot xv_i\ .
$$

Ad (6′). Given $x \in H$, we obtain

$$
y\cdot x \overset{\text{(i)}}{=} \textstyle\sum_i xu_i\varepsilon \cdot y \cdot xv_i \overset{\text{(iii')}}{=} \sum_{i,j} xu_iu_j \cdot xu_iv_jS \cdot y \cdot xv_i \overset{\text{(ii)}}{=} \sum_{i,j} xu_i \cdot xv_iu_jS \cdot y \cdot xv_iv_j\ .
$$

Ad (7). Given $x \in H$, we have

$$
\textstyle\sum_i xv_i\cdot xu_iS \overset{\text{(iv)}}{=} \sum_i xS^2v_i\cdot xS^2u_iS \overset{\text{(5)}}{=} \sum_i xSu_iS\cdot xSv_iS^2 \overset{\text{(iv)}}{=} \sum_i xSu_iS\cdot xSv_i \overset{\text{(iii)}}{=} xS\varepsilon\cdot 1_H \overset{\text{(4)}}{=} x\varepsilon\cdot 1_H\ .
$$

Ad (7′). Given $x \in H$, we have

$$
\textstyle\sum_i xv_iS\cdot xu_i \overset{\text{(iv)}}{=} \sum_i xS^2v_iS\cdot xS^2u_i \overset{\text{(5)}}{=} \sum_i xSu_iS^2\cdot xSv_iS \overset{\text{(iv)}}{=} \sum_i xSu_i\cdot xSv_iS \overset{\text{(iii')}}{=} xS\varepsilon\cdot 1_H \overset{\text{(4)}}{=} x\varepsilon\cdot 1_H\ .
$$

□

In the present §IV.8.1, we shall refer to the assertions Remark IV.41.(1–7′) just by (1–7′).

IV.8.1.3 Normality

Suppose given a Hopf algebra H over R, and an R-subalgebra $K \subseteq H$. Suppose H and K to be flat as modules over R.

Note that $K \otimes K \longrightarrow H \otimes H$ is injective. We will identify $K \otimes K$ with its image.

The R-subalgebra $K \subseteq H$ is called a *Hopf-subalgebra* if $K\Delta \subseteq K \otimes K$ and $KS \subseteq K$. In this case, we may and will suppose the maps u_i and v_i to restrict to maps from K to K.

Suppose $K \subseteq H$ to be a Hopf-subalgebra. It is called *normal,* if for all $a \in K$ and all $x \in H$, we have

$$\textstyle\sum_i xu_i \cdot a \cdot xv_i S \;\in\; K \qquad \text{and} \qquad \sum_i xu_i S \cdot a \cdot xv_i \;\in\; K\;.$$

An ideal $I \subseteq H$ is called a *Hopf ideal* if $I\Delta \subseteq I \otimes H + H \otimes I$ (where we have identified $I \otimes H$ and $H \otimes I$ with their images in $H \otimes H$), $I\varepsilon = 0$ and $IS \subseteq I$. In this case, the quotient H/I carries a Hopf algebra structure via

$$\begin{array}{lcllcl} H/I & \xrightarrow{\varepsilon} & R\,, & x+I & \longmapsto & x\varepsilon \\ H/I & \xrightarrow{\Delta} & H/I \otimes H/I\,, & x+I & \longmapsto & \sum_i (xu_i + I) \otimes (xv_i + I) \\ H/I & \xrightarrow{S} & H/I\,, & x+I & \longmapsto & xS + I\,. \end{array}$$

Suppose $K \subseteq H$ to be a normal Hopf subalgebra. Write $K^+ := \mathrm{Kern}(K \xrightarrow{\varepsilon} R)$. By $(6, 6', 3, 4)$ and by writing

$$k\Delta \;=\; \left(\textstyle\sum_i (ku_i - ku_i\varepsilon) \otimes kv_i\right) + 1 \otimes k$$

for $k \in K^+$, the ideal $HK^+ = K^+H$ is a Hopf ideal in H.

IV.8.1.4 Some remarks and a lemma

Suppose given a Hopf algebra H over R and a normal Hopf-subalgebra $K \subseteq H$. Suppose H and K to be flat as modules over R.

Write $\bar{H} := H/HK^+$. Given $x \in H$, write $\bar{x} := x + HK^+ \in \bar{H}$ for its residue class.

Let N', N, M, M' and Q be H-modules. Let P be an $\bar{H}$-module, which we also consider as an H-module via $H \longrightarrow \bar{H}$, $x \longmapsto \bar{x}$.

We write ${}_K(N, M) = {}_K(N|_K, M|_K)$ for the R-module of K-linear maps from N to M.

Remark IV.42 *Given $f \in {}_R(N, M)$ and $x \in H$, we define $x \cdot f \in {}_R(N, M)$ by*

$$[n](x \cdot f) \;:=\; \textstyle\sum_i xu_i \cdot [xv_i S \cdot n]f$$

for $n \in N$. This defines a left H-module structure on ${}_R(N, M)$.

Formally, squared brackets mean the same as parentheses. Informally, squared brackets are to accentuate the arguments of certain maps.

Proof. We *claim* that $x' \cdot (x \cdot f) = (x' \cdot x) \cdot f$ for x, $x' \in H$. Suppose given $n \in N$. We obtain

$$\begin{array}{rcl} [n](x' \cdot (x \cdot f)) & = & \sum_i x'u_i \cdot [x'v_i S \cdot n](x \cdot f) \\ & = & \sum_{i,j} x'u_i \cdot xu_j \cdot [xv_j S \cdot x'v_i S \cdot n]f \\ & \overset{(3)}{=} & \sum_{i,j} (x'u_i \cdot xu_j) \cdot [(x'v_i \cdot xv_j)S \cdot n]f \\ & \overset{(1)}{=} & \sum_i (x' \cdot x)u_i \cdot [(x' \cdot x)v_i S \cdot n]f \\ & = & [n]((x' \cdot x) \cdot f)\,. \end{array}$$

We *claim* that $1_H \cdot f = f$. Suppose given $n \in N$. We obtain

$$[n](1_H \cdot f) \;=\; \sum_i 1_H u_i \cdot [1_H v_i S \cdot n]f \;=\; 1_H \cdot [1_H S \cdot n]f \overset{(2)}{=} [n]f \;,$$

remarking that $1_H \Delta = 1_H \otimes 1_H$. □

I owe to G. Hiss the hint to improve a previous weaker version of Corollary IV.45 below by means of the following Remark IV.43.

Denote by

$$M^K \;:=\; \{m \in M \;:\; a \cdot m = a\varepsilon \cdot m \text{ for all } a \in K\}$$

the fixed point module of M under K.

Remark IV.43 *Letting $\bar{x} \cdot m := x \cdot m$ for $x \in H$ and $m \in M^K$, we define an $\bar{H}$-module structure on M^K.*

Proof. The value of the product $\bar{x} \cdot m$ does not depend on the chosen representative x of $\bar{x}$ since, given $y \in H$, $a \in K^+$ and $m \in M^K$, we have

$$y \cdot a \cdot m \;=\; y \cdot a\varepsilon \cdot m \;=\; 0 \;.$$

It remains to be shown that given $x \in H$ and $m \in M^K$, the element $x \cdot m$ lies in M^K. In fact, given $a \in K$, we obtain

$$\begin{aligned} a \cdot x \cdot m &\overset{(6')}{=} \sum_i xu_i \cdot \Big(\sum_j (xv_i)u_j S \cdot a \cdot (xv_i)v_j\Big) \cdot m \\ &= \sum_i xu_i \cdot \Big(\sum_j (xv_i)u_j S \cdot a \cdot (xv_i)v_j\Big)\varepsilon \cdot m \\ &= \sum_{i,j} xu_i \cdot xv_i u_j S\varepsilon \cdot a\varepsilon \cdot xv_i v_j \varepsilon \cdot m \\ &\overset{(4)}{=} \sum_{i,j} xu_i \cdot xv_i u_j \varepsilon \cdot a\varepsilon \cdot xv_i v_j \varepsilon \cdot m \\ &\overset{(\mathrm{ii})}{=} \sum_{i,j} xu_i u_j \cdot xu_i v_j \varepsilon \cdot a\varepsilon \cdot xv_i \varepsilon \cdot m \\ &\overset{(\mathrm{i}')}{=} \sum_i xu_i \cdot a\varepsilon \cdot xv_i \varepsilon \cdot m \\ &\overset{(\mathrm{i}')}{=} a\varepsilon \cdot x \cdot m \;. \end{aligned}$$

□

Remark IV.44 *We have $({}_R(N, M))^K = {}_K(N, M)$, as subsets of ${}_R(N, M)$.*

Proof. The module $({}_R(N, M))^K$ consists of the R-linear maps $N \xrightarrow{f} M$ that satisfy

$$\sum_i xu_i \cdot [xv_i S \cdot n]f \;=\; x\varepsilon \cdot [n]f \;.$$

for $x \in H$ and $n \in N$. The module ${}_K(N, M)$ consists of the R-linear maps $N \xrightarrow{f} M$ that satisfy

$$[x \cdot n]f \;=\; x \cdot [n]f$$

for $x \in H$ and $n \in N$. By (iii′), we have $({}_R(N, M))^K \supseteq {}_K(N, M)$.

It remains to show that $({}_R(N,M))^K \subseteq {}_K(N,M)$. Given $f \in ({}_R(N,M))^K$, $x \in H$ and $n \in N$, we obtain

$$\begin{aligned} x\cdot[n]f & \overset{\text{(i')}}{=} \textstyle\sum_i xu_i \cdot xv_i\varepsilon\cdot[n]f \\ & = \textstyle\sum_i xu_i\cdot[xv_i\varepsilon\cdot n]f \\ & \overset{\text{(iii)}}{=} \textstyle\sum_{i,j} xu_i\cdot[xv_iu_jS\cdot xv_iv_j\cdot n]f \\ & \overset{\text{(ii)}}{=} \textstyle\sum_{i,j} xu_iu_j\cdot[xu_iv_jS\cdot xv_i\cdot n]f \\ & = \textstyle\sum_i xu_i\varepsilon\cdot[xv_i\cdot n]f \\ & \overset{\text{(i)}}{=} [x\cdot n]f\ . \end{aligned}$$

□

Corollary IV.45 *Given $f \in {}_K(N,M)$ and $x \in H$, we define $\bar{x}\cdot f \in {}_K(N,M)$ by*

$$[n](\bar{x}\cdot f) \;:=\; \textstyle\sum_i xu_i\cdot[xv_iS\cdot n]f$$

for $n \in N$. This defines a left $\bar{H}$-module structure on ${}_K(N,M)$.

Proof. By Remark IV.42, we may apply Remark IV.43 to ${}_R(N,M)$. By Remark IV.44, the assertion follows. □

Remark IV.46 *Given $f \in {}_K(N,M)$, $x \in H$, and H-linear maps $N' \xrightarrow{\nu} N$, $M \xrightarrow{\mu} M'$, we obtain*

$$\nu(\bar{x}\cdot f)\mu \;=\; \bar{x}\cdot(\nu f\mu)\ .$$

Proof. Given $n' \in N'$, we obtain

$$[n']\big(\nu(\bar{x}\cdot f)\mu\big) \;=\; \big(\textstyle\sum_i xu_i\cdot[xv_iS\cdot n'\nu]f\big)\mu \;=\; \textstyle\sum_i xu_i\cdot[xv_iS\cdot n'](\nu f\mu) \;=\; [n'](\bar{x}\cdot(\nu f\mu))\ .$$ □

The following Lemma IV.47 has been suggested by the referee, and has been achieved with the help of G. Carnovale. It is reminiscent of [**54,** Cor. 4.3], but easier. It resembles a bit a Fourier inversion.

Note that the right $\bar{H}$-module structure on $\bar{H}$ induces a left $\bar{H}$-module structure on ${}_R(\bar{H},M)$.

Lemma IV.47 *We have the following mutually inverse isomorphisms of $\bar{H}$-modules.*

$$\begin{array}{rcl} {}_K(H,M) & \overset{\Phi}{\underset{\sim}{\longrightarrow}} & {}_R(\bar{H},M) \\ f & \longmapsto & (\bar{x}\mapsto \sum_i xu_i\cdot[xv_iS]f) \\ & & \\ {}_K(H,M) & \overset{\Psi}{\underset{\sim}{\longleftarrow}} & {}_R(\bar{H},M) \\ (x\mapsto \sum_j xv_j\cdot[\,\overline{xu_jS}\,]g) & \longleftarrow\!\shortmid & g \end{array}$$

Proof. We *claim* that Φ is a welldefined map. We have to show that $f\Phi$ is welldefined, i.e. that its value at $\bar{x}$ does not depend on the representing element x. Suppose given $y \in H$ and $a \in K^+$. We obtain

$$\begin{array}{rcl} \sum_i (ya)u_i \cdot [(ya)v_i S]f & \overset{(1)}{=} & \sum_{i,j} yu_i \cdot au_j \cdot [(yv_i \cdot av_j)S]f \\ & \overset{(3)}{=} & \sum_{i,j} yu_i \cdot au_j \cdot [av_j S \cdot yv_i S]f \\ & = & \sum_{i,j} yu_i \cdot au_j \cdot av_j S \cdot [yv_i S]f \\ & \overset{(\text{iii}')}{=} & \sum_i yu_i \cdot a\varepsilon \cdot [yv_i S]f \\ & = & 0\,. \end{array}$$

We *claim* that Φ is $\bar{H}$-linear. Suppose given $y \in H$ and $x \in H$. We obtain

$$\begin{array}{rcl} [\bar{x}]((\bar{y}f)\Phi) & = & \sum_i xu_i \cdot [xv_i S](\bar{y}f) \\ & = & \sum_{i,j} xu_i \cdot yu_j \cdot [yv_j S \cdot xv_i S]f \\ & \overset{(3)}{=} & \sum_{i,j} xu_i \cdot yu_j \cdot [(xv_i \cdot yv_j)S]f \\ & \overset{(1)}{=} & \sum_i (x \cdot y)u_i \cdot [(x \cdot y)v_i S]f \\ & = & [\bar{x}](\bar{y}(f\Phi))\,. \end{array}$$

We *claim* that Ψ is a welldefined map. We have to show that $g\Psi$ is K-linear. Suppose given $a \in K$ and $x \in H$. Note that $au_i \in K$ for all i, whence also $au_i S \in K$, and therefore $au_i S \equiv_{HK^+} au_i S\varepsilon \cdot 1_H$. We obtain

$$\begin{array}{rcl} [a \cdot x](g\Psi) & = & \sum_j (a \cdot x)v_j \cdot [\,\overline{(a \cdot x)u_j S}\,]g \\ & \overset{(1)}{=} & \sum_{i,j} av_i \cdot xv_j \cdot [\,\overline{(au_i \cdot xu_j)S}\,]g \\ & \overset{(3)}{=} & \sum_{i,j} av_i \cdot xv_j \cdot [\,\overline{xu_j S} \cdot \overline{au_i S}\,]g \\ & = & \sum_{i,j} av_i \cdot xv_j \cdot [\,\overline{xu_j S} \cdot \overline{au_i S\varepsilon}\,]g \\ & \overset{(4)}{=} & \sum_{i,j} au_i\varepsilon \cdot av_i \cdot xv_j \cdot [\,\overline{xu_j S}\,]g \\ & \overset{(\text{i})}{=} & \sum_j a \cdot xv_j \cdot [\,\overline{xu_j S}\,]g \\ & = & a \cdot [x](g\Psi)\,. \end{array}$$

We *claim* that $\Phi\Psi = \mathrm{id}_{K(H,M)}$. Suppose given $x \in H$. We obtain

$$\begin{array}{rcl} [x](f\Phi\Psi) & = & \sum_j xv_j \cdot [\,\overline{xu_j S}\,](f\Phi) \\ & = & \sum_{i,j} xv_j \cdot xu_j S u_i \cdot [xu_j S v_i S]f \\ & \overset{(5)}{=} & \sum_{i,j} xv_j \cdot xu_j v_i S \cdot [xu_j u_i S^2]f \\ & \overset{(\text{iv})}{=} & \sum_{i,j} xv_j \cdot xu_j v_i S \cdot [xu_j u_i]f \\ & \overset{(\text{ii})}{=} & \sum_{i,j} xv_j v_i \cdot xv_j u_i S \cdot [xu_j]f \\ & \overset{(7)}{=} & \sum_j xv_j\varepsilon \cdot [xu_j]f \\ & \overset{(\text{i})}{=} & [x]f\,. \end{array}$$

We *claim* that $\Psi\Phi = \mathrm{id}_{{}_R(\bar{H},M)}$. Suppose given $x \in H$. We obtain

$$\begin{aligned}
[\bar{x}](g\Psi\Phi) &= \textstyle\sum_i xu_i \cdot [xv_iS](g\Psi) \\
&= \textstyle\sum_{i,j} xu_i \cdot xv_iSv_j \cdot [\,\overline{xv_iSu_jS}\,]g \\
&\overset{(5)}{=} \textstyle\sum_{i,j} xu_i \cdot xv_iu_jS \cdot [\,\overline{xv_iv_jS^2}\,]g \\
&\overset{(\mathrm{iv})}{=} \textstyle\sum_{i,j} xu_i \cdot xv_iu_jS \cdot [\,\overline{xv_iv_j}\,]g \\
&\overset{(\mathrm{ii})}{=} \textstyle\sum_{i,j} xu_iu_j \cdot xu_iv_jS \cdot [\,\overline{xv_i}\,]g \\
&\overset{(\mathrm{iii}')}{=} \textstyle\sum_i xu_i\varepsilon \cdot [\,\overline{xv_i}\,]g \\
&\overset{(\mathrm{i})}{=} [\bar{x}]g\ .
\end{aligned}$$

Finally, it follows by $\bar{H}$-linearity of Φ and by $\Psi = \Phi^{-1}$ that Ψ is $\bar{H}$-linear. ▫

The tensor product $N \otimes M$ is an H-module via Δ. Note that R is an H-module via ε. Note that $R \otimes M \simeq M \simeq M \otimes R$ as H-modules by $(\mathrm{i}, \mathrm{i}')$.

Remark IV.48 (cf. [6, Lemma 3.5.1]**)** *We have mutually inverse isomorphisms of R-modules*

$$\begin{array}{rcl}
{}_{\bar{H}}(P,\ {}_K(Q,M)) & \xrightarrow[\sim]{\alpha} & {}_H(P\otimes Q, M) \\
f & \longmapsto & (p\otimes q \mapsto [q](pf)) \\
{}_{\bar{H}}(P,\ {}_K(Q,M)) & \xleftarrow[\sim]{\beta} & {}_H(P\otimes Q, M) \\
(p \mapsto (q \mapsto [p\otimes q]g)) & \longleftarrow & g\ ,
\end{array}$$

natural in $P \in \mathrm{Ob}\,\bar{H}$-Mod*,* $Q \in \mathrm{Ob}\,H$-Mod *and* $M \in \mathrm{Ob}\,H$-Mod.

Proof. We *claim* that α is welldefined. We have to show that $f\alpha$ is H-linear. Suppose given $x \in H$. We obtain

$$\begin{aligned}
x\cdot(p\otimes q) &= \textstyle\sum_i \overline{xu_i}\cdot p \otimes xv_i \cdot q \\
&\overset{f\alpha}{\longmapsto} \textstyle\sum_i [xv_i\cdot q]((\overline{xu_i}\cdot p)f) \\
&= \textstyle\sum_i [xv_i\cdot q](\overline{xu_i}\cdot (pf)) \\
&= \textstyle\sum_{i,j} xu_iu_j \cdot [xu_iv_jS \cdot xv_i \cdot q](pf) \\
&\overset{(\mathrm{ii})}{=} \textstyle\sum_{i,j} xu_i \cdot [xv_iu_jS \cdot xv_iv_j \cdot q](pf) \\
&\overset{(\mathrm{iii})}{=} \textstyle\sum_i xu_i \cdot [xv_i\varepsilon \cdot q](pf) \\
&\overset{(\mathrm{i}')}{=} x\cdot [q](pf) \\
&= x\cdot [p\otimes q](f\alpha)\ .
\end{aligned}$$

We *claim* that β is welldefined. First, we have to show that $[p](g\beta)$ is K-linear. Suppose given $a \in K$. We obtain

$$a\cdot q \overset{[p](g\beta)}{\longmapsto} [p\otimes a\cdot q]g \overset{(\mathrm{i})}{=} \textstyle\sum_i [\overline{au_i\varepsilon}\cdot p\otimes av_i\cdot q]g \;=\; \sum_i [\overline{au_i}\cdot p \otimes av_i \cdot q]g \;=\; a\cdot[p\otimes q]g\ .$$

Second, we have to show that $g\beta$ is $\bar{H}$-linear. Suppose given $x \in H$. We obtain

$$\begin{array}{rcl} \bar{x}\cdot p & \overset{g\beta}{\longmapsto} & (q \mapsto [\bar{x}\cdot p \otimes q]g) \\ & \overset{\text{(i)}}{=} & (q \mapsto \sum_i [\overline{xu_i \cdot xv_i\varepsilon} \cdot p \otimes q)]g) \\ & \overset{\text{(iii}'\text{)}}{=} & (q \mapsto \sum_{i,j} [\overline{xu_i} \cdot p \otimes xv_iu_j \cdot xv_iv_jS \cdot q)]g) \\ & \overset{\text{(ii)}}{=} & (q \mapsto \sum_{i,j} [\overline{xu_iu_j} \cdot p \otimes xu_iv_j \cdot xv_iS \cdot q)]g) \\ & = & (q \mapsto \sum_i xu_i \cdot [p \otimes xv_iS \cdot q]g) \\ & = & \bar{x}\cdot(q \mapsto [p\otimes q]g)\ . \end{array}$$

Finally, α and β are mutually inverse. ▫

Corollary IV.49 *We have* ${}_{\bar{H}}(P, M^K) \simeq {}_{\bar{H}}(P, {}_K(R, M)) \simeq {}_H(P, M)$ *as R-modules, natural in P and M.*

Proof. Note that $M \simeq {}_R(R, M)$ as H-modules, whence $M^K \simeq {}_K(R, M)$ as $\bar{H}$-modules by Remarks IV.43, IV.44. Now the assertion follows from Remark IV.48, letting $Q = R$. ▫

IV.8.2 Comparing Hochschild-Serre-Hopf with Grothendieck

Let R be a commutative ring. Suppose given a Hopf algebra H over R (with involutive antipode) and a normal Hopf-subalgebra $K \subseteq H$; cf. §IV.8.1.3. Write $\bar{H} := H/HK^+$. Suppose H, K and $\bar{H}$ to be projective as modules over R. Suppose H to be projective as a module over K.

Let $B \in \operatorname{Ob} \mathrm{C}(H\text{-Mod})$ be a projective resolution of R over H. Let $\bar{B} \in \operatorname{Ob} \mathrm{C}(\bar{H}\text{-Mod})$ be a projective resolution of R over $\bar{H}$. Note that since $\bar{H}$ is projective over R, $\bar{B}|_R \in \operatorname{Ob} \mathrm{C}(R\text{-Mod})$ is a projective resolution of R over R. Let M be an H-module.

By Corollary IV.45 and by Remark IV.46, we have a biadditive functor

$$\begin{array}{ccccc} (H\text{-Mod})^\circ & \times & H\text{-Mod} & \xrightarrow{U} & \bar{H}\text{-Mod} \\ (X & , & X') & \longmapsto & U(X, X') := {}_K(X, X')\ . \end{array}$$

Write

$$\begin{array}{ccccc} (\bar{H}\text{-Mod})^\circ & \times & \bar{H}\text{-Mod} & \xrightarrow{V} & R\text{-Mod} \\ (Y & , & Y') & \longmapsto & V(Y, Y') := {}_{\bar{H}}(Y, Y') \end{array}$$

for the usual Hom-functor.

In particular, we shall consider the functors

$$\begin{array}{ccccc} H\text{-Mod} & \xrightarrow{U(R,-)} & \bar{H}\text{-Mod} & \xrightarrow{V(R,-)} & R\text{-Mod} \\ X & \longmapsto & U(R, X) \simeq X^K & & \\ & & Y & \longmapsto & V(R, Y) \simeq Y^{\bar{H}}\ . \end{array}$$

On the other hand, we shall consider the double complex

$$D(M) \;=\; D^{-,=}(M) \;:=\; V\big(\bar{B}_-\,,\, U(B_=\,,\, M)\big) \;=\; {}_{\bar{H}}\big(\bar{B}_-\,,\, {}_K(B_=\,,\, M)\big)\ .$$

Note that $D(M)$ is isomorphic in $\mathrm{CC}^{\llcorner}(R\text{-Mod})$ to ${}_H\big(\bar{B}_- \otimes_R B_=\,,\ M\big)$, naturally in M; cf. Remark IV.48.

Lemma IV.50 *The $\bar{H}$-module $U(H,M)$ is $V(R,-)$-acyclic.*

Proof. By Lemma IV.47, this amounts to showing that ${}_R(\bar{H},M)$ is $V(R,-)$-acyclic, which in turn amounts to showing that $V\big(\bar{B}\,,\ {}_R(\bar{H},M)\big) = {}_{\bar{H}}\big(\bar{B}\,,\ {}_R(\bar{H},M)\big)$ has vanishing cohomology in degrees $\geqslant 1$. Now,

$$ {}_{\bar{H}}\big(\bar{B}\,,\ {}_R(\bar{H},M)\big) \;\simeq\; {}_R(\bar{H}\otimes_{\bar{H}}\bar{B}\,,\ M) \;\simeq\; {}_R(\bar{B}\,,\ M)\,, $$

whose cohomology in degree $i \geqslant 1$ is $\mathrm{Ext}^i_R(R,M) \simeq 0$. □

Lemma IV.51 *Given a projective H-module P, the $\bar{H}$-module $U(P,M)$ is $V(R,-)$-acyclic.*

Proof. It suffices to show that $U(\coprod_\Gamma H,\ M) \simeq \prod_\Gamma U(H,\ M)$ is $V(R,-)$-acyclic for any indexing set Γ. By Lemma IV.50, it remains to be shown that $\mathrm{R}^i V(R,\prod_\Gamma Y)$ is isomorphic to $\prod_\Gamma \mathrm{R}^i V(R,Y)$ for a given $\bar{H}$-module Y and for $i \geqslant 1$. Having chosen an injective resolution J of Y, we may choose the injective resolution $\prod_\Gamma J$ of $\prod_\Gamma Y$. Then

$$ \mathrm{R}^i V(R,\textstyle\prod_\Gamma Y) \;\simeq\; \mathrm{H}^i V(R,\prod_\Gamma J) \;\simeq\; \mathrm{H}^i \prod_\Gamma V(R,J) \;\simeq\; \prod_\Gamma \mathrm{H}^i V(R,J) \;\simeq\; \prod_\Gamma \mathrm{R}^i V(R,Y)\,. $$

□

Theorem IV.52 *The proper spectral sequences*

$$ \dot{\mathrm{E}}_{\mathrm{I}}(D(M)) \qquad \text{and} \qquad \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),\,V(R,-)}(M) $$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_\infty,\ R\text{-Mod}\,]\!]$), naturally in $M \in \mathrm{Ob}\,H\text{-Mod}$.

Proof. To apply Theorem IV.31 with, in the notation of §IV.5.1,

$$ \Big(\mathcal{A}\times\mathcal{A}' \xrightarrow{\;F\;} \mathcal{B} \xrightarrow{\;G\;} \mathcal{C}\Big) \;=\; \Big((H\text{-Mod})^\circ \times H\text{-Mod} \xrightarrow{\;U\;} \bar{H}\text{-Mod} \xrightarrow{\;V(R,-)\;} R\text{-Mod}\Big)\,, $$

and with $X = R$ and $X' = M$, we verify the conditions (a–d′) of loc. cit. in this case.

Ad (c). We *claim* that B is a $\big(U(-,M),\ V(R,-)\big)$-acyclic resolution of R. We have to show that $U(B_i,M)$ is $V(R,-)$-acyclic for $i \geqslant 0$; cf. §IV.4.2. Since B_i is projective over H, this follows by Lemma IV.51. This proves the *claim*.

Ad (c′). Let I be an injective resolution of M over H. We *claim* that I is a $\big(U(R,-),\ V(R,-)\big)$-acyclic resolution of M. We have to show that $U(R,I^i)$ is $V(R,-)$-acyclic for $i \geqslant 0$. In fact, by Corollary IV.49, $U(R,I^i)$ is an injective $\bar{H}$-module. This proves the *claim*.

Ad (d, d′). We *claim* that $U(B_i,-)$ and $U(-,I^i)$ are exact for $i \geqslant 0$; cf. §IV.5.1. The former follows from H being projective over K. The latter is a consequence of $I^i|_K$ being injective in K-Mod by exactness of $K\text{-Mod} \xrightarrow{H\otimes_K -} H\text{-Mod}$. This proves the *claim*.

So an application of Theorem IV.31 yields

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(M) \;\simeq\; \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R)\;.$$

To apply Theorem IV.34 with, in the notation of §IV.6.1,

$$\Big(\mathcal{A} \xrightarrow{F} \mathcal{B}'\;,\;\; \mathcal{B}\times\mathcal{B}' \xrightarrow{G} \mathcal{C}\Big) \;=\; \Big((H\text{-Mod})^\circ \xrightarrow{U(-,M)} \bar{H}\text{-Mod}\;,\;\; (\bar{H}\text{-Mod})^\circ \times \bar{H}\text{-Mod} \xrightarrow{V} \mathcal{C}\Big)\;,$$

and with $X = R$ and $Y = R$, we verify the conditions (a–e) of loc. cit. in this case.

Ad (c). We have already remarked that B is a $\big(U(-,M),\, V(R,-)\big)$-acyclic resolution of R.

Ad (d). As a resolution of R over $\bar{H}$, we choose $\bar{B}$.

So an application of Theorem IV.34 yields

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}\Big(V\big(\bar{B}_-\,,\; U(B_=\,,\, M)\big)\Big)\;.$$

Naturality in $M \in \mathrm{Ob}\, H\text{-Mod}$ remains to be shown. Suppose given $M \xrightarrow{m} \tilde{M}$ in H-Mod. Note that the requirements of §IV.5.2 are met. By Proposition IV.32, with roles of $\mathcal{A}$ and $\mathcal{A}'$ interchanged, we have the following commutative quadrangle.

$$\begin{array}{ccc}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(M) & \xrightarrow{\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(m)} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(\tilde{M}) \\
\uparrow\wr & & \uparrow\wr \\
\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R) & \xrightarrow{\mathrm{h}^{\mathrm{I}}_{U(-,m)}R} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,\tilde{M}),V(R,-)}(R)
\end{array}$$

Note that the requirements of §IV.6.2 are met. By Lemma IV.36, we have the following commutative quadrangle.

$$\begin{array}{ccc}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R) & \xrightarrow{\mathrm{h}^{\mathrm{I}}_{U(-,m)}R} & \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,\tilde{M}),V(R,-)}(R) \\
\downarrow\wr & & \downarrow\wr \\
\dot{\mathrm{E}}_{\mathrm{I}}\Big(V\big(\bar{B}_-\,,\; U(B_=\,,\, M)\big)\Big) & \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(V(\bar{B}_-\,,\,U(B_=,m)))} & \dot{\mathrm{E}}_{\mathrm{I}}\Big(V\big(\bar{B}_-\,,\; U(B_=\,,\, \tilde{M})\big)\Big)
\end{array}$$

□

IV.8.3 Comparing Lyndon-Hochschild-Serre with Grothendieck

Let R be a commutative ring. Let G be a group and let $N \trianglelefteqslant G$ be a normal subgroup. Write $\bar{G} := G/N$. Let M be an RG-module. Write $\mathrm{Bar}_{G;R} \in \mathrm{Ob}\,\mathrm{C}(RG\text{-Mod})$ for the bar resolution of R over RG, having $(\mathrm{Bar}_{G;R})_i = RG^{\otimes(i+1)}$ for $i \geqslant 0$, the tensor product being taken over R.

Note that RG is a Hopf algebra over R via

$$\begin{array}{cclccc}
RG & \xrightarrow{\Delta} & RG \otimes RG\;, & g & \longmapsto & g \otimes g \\
RG & \xrightarrow{S} & RG\;, & g & \longmapsto & g^{-1} \\
RG & \xrightarrow{\varepsilon} & R\;, & g & \longmapsto & 1\;,
\end{array}$$

where $g \in G$; cf. §IV.8.1.1. Moreover, RN is a normal Hopf subalgebra of RG such that $RG/(RG)(RN)^+ \simeq R\bar{G}$; cf. §IV.8.1.3.

Note that RG, RN and $R\bar{G}$ are projective over R, and that RG is projective over RN.

We have functors $RG\text{-Mod} \xrightarrow{(-)^N} R\bar{G}\text{-Mod} \xrightarrow{(-)^{\bar{G}}} R\text{-Mod}$, taking respective fixed points.

Theorem IV.53 (Beyl, [**7,** Th. 3.5]) *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{(-)^N,\,(-)^{\bar{G}}}(M) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\Big({}_{RG}\big((\mathrm{Bar}_{\bar{G};R})_- \otimes_R (\mathrm{Bar}_{G;R})_= \,,\; M\big)\Big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\, R\text{-Mod}\,]\!]$*), naturally in* $M \in \operatorname{Ob} RG\text{-Mod}$.

> Beyl uses his Theorem IV.40 to prove Theorem IV.53. We shall re-derive it from Theorem IV.52, which in turn relies on the Theorems IV.31 and IV.34.

Proof. This follows by Theorem IV.52. □

IV.8.4 Comparing Hochschild-Serre with Grothendieck

Let R be a commutative ring. Let $\mathfrak{g}$ be a Lie algebra over R that is free as an R-module. Let $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be an ideal such that $\mathfrak{n}$ and $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{n}$ are free as R-modules. Let M be a $\mathfrak{g}$-module, i.e. a $\mathcal{U}(\mathfrak{g})$-module. Write $\mathrm{Bar}_{\mathfrak{g};R} \in \operatorname{Ob} \mathrm{C}(\mathcal{U}(\mathfrak{g})\text{-Mod})$ for the Chevalley-Eilenberg resolution of R over $\mathcal{U}(\mathfrak{g})$, having $(\mathrm{Bar}_{\mathfrak{g};R})_i = \mathcal{U}(\mathfrak{g}) \otimes_R \wedge^i \mathfrak{g}$ for $i \geqslant 0$; cf. [**10,** XIII.§7] or [**59,** Th. 7.7.2].

Note that $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra over R via

$$\begin{array}{lllcl}
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\Delta} & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\,, & g \longmapsto & g \otimes 1 + 1 \otimes g \\
\mathcal{U}(\mathfrak{g}) & \xrightarrow{S} & \mathcal{U}(\mathfrak{g})\,, & g \longmapsto & -g \\
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\varepsilon} & R\,, & g \longmapsto & 0\,,
\end{array}$$

where $g \in \mathfrak{g}$; cf. §IV.8.1.1.

Note that $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{n})$ and $\mathcal{U}(\bar{\mathfrak{g}})$ are projective over R, and that $\mathcal{U}(\mathfrak{g})$ is projective over $\mathcal{U}(\mathfrak{n})$; cf. [**59,** Cor. 7.3.9].

We have functors $\mathcal{U}(\mathfrak{g})\text{-Mod} \xrightarrow{(-)^{\mathfrak{n}}} \mathcal{U}(\bar{\mathfrak{g}})\text{-Mod} \xrightarrow{(-)^{\bar{\mathfrak{g}}}} R\text{-Mod}$, taking respective annihilated submodules; cf. [**59,** p. 221].

Theorem IV.54 *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{(-)^{\mathfrak{n}},\,(-)^{\bar{\mathfrak{g}}}}(M) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\Big({}_{\mathcal{U}(\mathfrak{g})}\big((\mathrm{Bar}_{\bar{\mathfrak{g}};R})_- \otimes_R (\mathrm{Bar}_{\mathfrak{g};R})_= \,,\; M\big)\Big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\, R\text{-Mod}\,]\!]$*), naturally in* $M \in \operatorname{Ob} \mathcal{U}(\mathfrak{g})\text{-Mod}$.

> Cf. Barnes, [**3,** Sec. IV.4, Ch. VII].

Proof. This follows by Theorem IV.52. □

IV.8.5 Comparing two spectral sequences for a change of rings

The following application is taken from [**10,** XVI.§6].

Let R be a commutative ring. Let $A \xrightarrow{\varphi} B$ be a morphism of R-algebras. Consider the functors $A\text{-Mod} \xrightarrow{{}_A(B,-)} B\text{-Mod}$ and $(B\text{-Mod})^\circ \times B\text{-Mod} \xrightarrow{{}_B(-,=)} R\text{-Mod}$.

Let X be an A-module, let Y be a B-module.

We shall compare two spectral sequences with E_2-terms $\mathrm{Ext}^i_B(Y, \mathrm{Ext}^j_A(B,X))$, converging to $\mathrm{Ext}^{i+j}_A(Y,X)$. If one views $X{\Uparrow}^B_A := {}_A(B,X)$ as a way to induce from A-Mod to B-Mod, this measures the failure of the Eckmann-Shapiro-type formula $\mathrm{Ext}^i_B(Y, X{\Uparrow}^B_A) \overset{?}{\simeq} \mathrm{Ext}^i_A(Y,X)$, which holds if B is projective over A.

Let $I \in \mathrm{Ob}\,\mathrm{C}^{[0}(A\text{-Mod})$ be an injective resolution of X. Let $P \in \mathrm{Ob}\,\mathrm{C}^{[0}(B\text{-Mod})$ be a projective resolution of Y.

Proposition IV.55 *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{{}_A(B,-),\ {}_B(Y,-)}(X) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\Big({}_B\big(P_-,\ {}_A(B,I^=)\big)\Big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_\infty,\ R\text{-Mod}\,]\!]$*).*

Proof. To apply Theorem IV.34, if suffices to remark that for each injective A-module I', the B-module ${}_A(B,I')$ is injective, and thus ${}_B(Y,-)$-acyclic. ▫

Remark IV.56 The functor ${}_A(B,-)$ can be replaced by ${}_A(M,-)$, where M is an A-B-bimodule that is flat over B.

IV.8.6 Comparing two spectral sequences for Ext and ⊗

Let R be a commutative ring. Let S be a ring. Let A be an R-algebra. Let M be an R-S-bimodule. Let X and X' be A-modules. Assume that X is flat over R. Assume that $\mathrm{Ext}^i_R(M,X') \simeq 0$ for $i \geqslant 1$.

Example IV.57 Let T be a discrete valuation ring, with maximal ideal generated by t. Let $R = T/t^\ell$ for some $\ell \geqslant 1$. Let $S = T/t^k$, where $1 \leqslant k \leqslant \ell$. Let G be a finite group, and let $A = RG$. Let $M = S$. Let X and X' be RG-modules that are both finitely generated and free over R.

Consider the functors

$$(A\text{-Mod})^\circ \times A\text{-Mod} \xrightarrow{{}_A(-,=)} R\text{-Mod} \xrightarrow{{}_R(M,-)} S\text{-Mod}$$

Proposition IV.58 *The proper Grothendieck spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{{}_A(X,-),\ {}_R(M,-)}(X') \qquad \text{and} \qquad \dot{\mathrm{E}}^{\mathrm{Gr}}_{{}_A(-,X'),\ {}_R(M,-)}(X)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\ S\text{-Mod}]\!]$*).*

Both have E_2-terms $\mathrm{Ext}^i_R\big(M,\ \mathrm{Ext}^j_A(X,X')\big)$ and converge to $\mathrm{Ext}^{i+j}_A(X\otimes_R M, X')$. In particular, in the situation of Example IV.57, both have E_2-terms $\mathrm{Ext}^i_R\big(S,\ \mathrm{Ext}^j_{RG}(X,X')\big)$ and converge to $\mathrm{Ext}^{i+j}_{RG}(X/t^k, X')$.

Proof of Proposition IV.58*.* To apply Theorem IV.31, we comment on the conditions in §IV.5.1.

(c) Given a projective A-module P, we want to show that the R-module ${}_A(P,X')$ is ${}_R(M,-)$-acyclic. We may assume that $P=A$, which is to be viewed as an A-R-bimodule. Now, we have $\mathrm{Ext}^i_R\big(M,\ {}_A(A,X')\big)\simeq \mathrm{Ext}^i_R(M,X')\simeq 0$ for $i\geqslant 1$ by assumption.

(c′) Given an injective A-module I', the R-module ${}_A(X,I')$ is injective since X is flat over R by assumption. □

IV.8.7 Comparing two spectral sequences for $\mathcal{E}xt$ of sheaves

Let $T\xrightarrow{f} S$ be a flat morphism of ringed spaces, i.e. suppose that

$$\mathcal{O}_T\otimes_{f^{-1}\mathcal{O}_S} -\ :\ f^{-1}\mathcal{O}_S\text{-Mod}\longrightarrow \mathcal{O}_T\text{-Mod}$$

is exact. Consequently, $f^*:\mathcal{O}_S\text{-Mod}\longrightarrow\mathcal{O}_T\text{-Mod}$ is exact.

Given $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{F}'$, we abbreviate ${}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}') := \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}')\in\mathrm{Ob}\,R\text{-Mod}$ and ${}_{\mathcal{O}_S}(\!(\mathcal{F},\mathcal{F}')\!) := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}')\in\mathrm{Ob}\,\mathcal{O}_S\text{-Mod}$.

Let $\mathcal{F}$ be an $\mathcal{O}_S$-module that has a locally free resolution $\mathcal{B}\in\mathrm{Ob}\,\mathrm{C}(\mathcal{O}_S\text{-Mod})$; cf. [**23,** Prop. III.6.5]. Let $\mathcal{G}\in\mathrm{Ob}\,\mathcal{O}_T\text{-Mod}$. Let $\mathcal{A}\in\mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{O}_T\text{-Mod})$ be an injective resolution of $\mathcal{G}$.

Consider the functors $\mathcal{O}_T\text{-Mod}\xrightarrow{f_*}\mathcal{O}_S\text{-Mod}$ and $(\mathcal{O}_S\text{-Mod})^\circ\times\mathcal{O}_S\text{-Mod}\xrightarrow{{}_{\mathcal{O}_S}(\!(-,=)\!)}\mathcal{O}_S\text{-Mod}$.

Proposition IV.59 *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{f_*,\ {}_{\mathcal{O}_S}(\!(\mathcal{F},-)\!)}(\mathcal{G}) \qquad \text{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\big({}_{\mathcal{O}_S}(\!(\mathcal{B}_-, f_*\mathcal{A}^=)\!)\big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\ \mathcal{O}_S\text{-Mod}]\!]$*).*

In particular, both spectral sequences have E_2-terms $\mathcal{E}xt^i_{\mathcal{O}_S}\big(\mathcal{F},\ (\mathrm{R}^j f_*)(\mathcal{G})\big)$ and converge to $(\mathrm{R}^{i+j}\Gamma\!\!\Gamma_{\mathcal{F}})(\mathcal{G})$, where $\Gamma\!\!\Gamma_{\mathcal{F}}(-) := {}_{\mathcal{O}_S}(\!(\mathcal{F}, f_*(-))\!)\simeq f_*\,{}_{\mathcal{O}_T}(\!(f^*\mathcal{F},-)\!)$. For example, if $S=\{*\}$ is a one-point-space and if we write $R:=\mathcal{O}_S(S)$, then we can identify $\mathcal{O}_S\text{-Mod}=R\text{-Mod}$. If, in this case, $\mathcal{F}=R/rR$ for some $r\in R$, then $\Gamma\!\!\Gamma_{R/rR}(\mathcal{G})\simeq\Gamma(T,\mathcal{G})[r]:=\{g\in\mathcal{G}(T)\ :\ rg=0\}$.

Proof of Proposition IV.59*.* To apply Theorem IV.34, we comment on the conditions in §IV.6.1.

(c) Since f_* maps injective $\mathcal{O}_T$-modules to injective $\mathcal{O}_S$-modules by flatness of $T \xrightarrow{f} S$, the complex $\mathcal{A}$ is an $\big(f_*,\ {}_{\mathcal{O}_S}((\mathcal{F},-))\big)$-acyclic resolution of $\mathcal{G}$.

(e) If $\mathcal{I}$ is an injective $\mathcal{O}_S$-module and $U \subseteq S$ is an open subset, then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module; cf. [**23,** Lem. III.6.1]. Hence ${}_{\mathcal{O}_S}((-,\mathcal{I}))$ turns a short exact sequence of $\mathcal{O}_S$-modules into a sequence that is short exact as a sequence of abelian presheaves, and hence a fortiori short exact as a sequence of $\mathcal{O}_S$-modules. In other words, the functor ${}_{\mathcal{O}_S}((-,\mathcal{I}))$ is exact. $\square$