Generalized Polygons with Doubly Transitive Ovoids

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Abstract

This thesis studies finite generalized quadrangles and hexagons which contain an ovoid. An ovoid is a set of mutually opposite points of maximum size. The first objective of the present work is to show that a generalized quadrangle or hexagon is a classical polygon if it contains an ovoid in the case of quadrangles or an ovoid-spread pairing in the case of hexagons, such that a group of given isomorphism type acts on the polygon in such a way that it leaves the ovoid invariant. The groups in use here are Suzuki and Ree groups.

The second part of the thesis is devoted to the problem of determining all groups which can act on a generalized quadrangle or generalized hexagon in such a way that they operate doubly transitively on an ovoid of this polygon. It will turn out that these groups are essentially the known examples of groups acting on a classical or semi-classical ovoid of a classical polygon.

In the first two chapters, meant as an introduction to the problem, constructions of the relevant generalized polygons are given and known results concerning the existence question of ovoids in known generalized quadrangles and hexagons are collected. In the following chapter, the known polarity of the symplectic quadrangle is presented and it is shown that the symplectic quadrangle can be reconstructed from the action of the Suzuki group on the absolute elements of this polarity. Then we show that any generalized quadrangle which contains an ovoid, such that a Suzuki group acts on the ovoid, is isomorphic to the symplectic quadrangle. In Chapter 4, analogous to the approach used for symplectic quadrangles, the known polarity of the split Cayley hexagon is described and it is shown that the split Cayley hexagon can be reconstructed from the action of the Ree group on the absolute points of the polarity. Then we show that any generalized hexagon that contains an ovoid-spread-pairing, on which a Ree group acts, is isomorphic to the classical split-Cayley hexagon. Both of these results are achieved without the use of classification results.

The last three chapters are devoted to the problem of determining all groups which can act doubly transitively on an ovoid of a generalized quadrangle or generalized hexagon. This chapter uses the classification of finite simple groups (via the classification of finite doubly transitive groups). For hexagons the result is that only unitary groups and Ree groups are possible. This result, together with the one obtained in the previous chapter and a theorem by Joris De Kaey provides that the generalized hexagon is classical and the ovoid is classical if the ovoid belongs to an ovoid-spread-pairing. The result for quadrangles is less smooth. A further restriction on the order of the

quadrangle is needed, namely that the number of points per line and the number of lines per point coincide and that this number is a prime power. This was not necessary in the case of the hexagons. With this additional assumption, we show that only orthogonal groups or Suzuki groups can act on these ovoids.

Zusammenfassung

In dieser Arbeit werden endliche verallgemeinerte Vierecke und Sechsecke untersucht, die ein Ovoid enthalten. Ein Ovoid ist eine Menge von sich paarweise gegenüberliegenden Punkten maximaler Größe. Das erste Ziel dieser Arbeit besteht einerseits darin, aus einer Untergruppe vorgegebenen Isomorphietyps der vollen Automorphismengruppe des Polygons, die das Ovoid invariant lässt, das Polygon zu rekonstruieren und damit zu zeigen, dass alle verallgemeinerten Vierecke und alle verallgemeinerten Sechsecke, die ein Ovoid beziehungsweise ein Ovoid-Spread-Paar besitzen, auf dem eine Gruppe des vorgegebenen Isomorphietyps operiert, isomorph zu einem bekannten verallgemeinerten Viereck beziehungsweise Sechseck sind. Die vorgegebenen Gruppen sind hierbei Suzuki- und Ree-Gruppen.

Der zweite Teil der Arbeit widmet sich der Frage, welche Gruppen auf einem verallgemeinerten Viereck oder verallgemeinerten Sechseck so operieren können, dass sie zweifach transitiv auf einem Ovoid des Vier- beziehungsweise Sechsecks operieren. Es wird sich herausstellen, dass diese Gruppen im wesentlichen die bekannten Ovoidstandgruppen der klassischen verallgemeinerten Vier- und Sechsecke sind.

In den ersten zwei Kapiteln werden kurz die bekannten Konstruktionen der benötigten verallgemeinerten Polygone beschrieben und ein kurzer Überblick über bekannte Resultate zur Frage der Existenz von Ovoiden in verallgemeinerten Polygonen gegeben. Im folgenden Kapitel wird die bekannte Polarität des symplektischen Vierecks in Charakteristik 2 vorgestellt. Die absoluten Punkte dieser Polarität bilden ein Ovoid. Es wird gezeigt, dass sich das gesamte verallgemeinerte Viereck aus der Wirkung der Suzuki-Gruppe auf den absoluten Punkten der Polarität rekonstruieren lässt. Dann wird gezeigt, dass ein beliebiges verallgemeinertes Viereck, das ein Ovoid enthält, auf dem eine Suzuki-Gruppe operiert, isomorph zum bekannten symplektischen Viereck ist. In Kapitel 5 wird analog zu diesem Vorgehen dazu die Polarität des Split-Cayley Sechsecks beschrieben und gezeigt, dass sich das Split-Cayley-Sechseck aus der Wirkung der Ree-Gruppe auf den absoluten Punkten rekonstruieren lässt. Es wird gezeigt, dass ein beliebiges verallgemeinertes Sechseck, das ein Ovoid-Spread-Paar enthält, auf dem eine Ree-Gruppe operiert, isomorph zum bekannten Split-Cayley Sechseck ist. Beide Resultate vermeiden die Verwendung von Klassifikationsresultaten.

In den letzten beiden Kapiteln wird die Frage untersucht, welche zweifach transitiven Gruppen auf einem Ovoid eines verallgemeinerten Vierecks oder verallgemeinerten Sechsecks operieren können. Diese Kapitel verwenden die

Klassifikation der endlichen einfachen Gruppen (über die Klassifikation der endlichen zweifach transitiven Gruppen). Für Sechsecke lautet das Resultat, dass nur unitäre Gruppen und Ree-Gruppen in Frage kommen. Zusammen mit dem Ergebnis aus dem vorhergehenden Kapitel und einem Ergebnis von Joris de Kaey liefert dieses Ergebnis dann auch den Isomorphietyp des verallgemeinerten Sechsecks, falls das ovoid zu einem Ovoid-Spread-Paar gehört. Das Ergebnis für Vierecke ist weniger glatt. Hier wird eine zusätzliche Einschränkung an die Ordnung des Vierecks gebraucht, die im Fall der Sechsecke nicht nötig war. Mit dieser Zusatzvoraussetzung wird gezeigt, dass für die auf dem Ovoid operierenden Gruppen nur orthogonale oder Suzuki-Gruppen in Frage kommen.

Introduction

There are two main questions which will be treated in this work:

- Suppose that we are given an ovoid of a generalized quadrangle or a generalized hexagon and suppose that there is a group of automorphisms of the polygon leaving the ovoid (in the case of quadrangles) or the ovoid-spread-pairing (in the case of hexagons) invariant such that this group is isomorphic to the subgroup of the automorphism group of a classical poygon occuring in this situation. Is this information sufficient to reconstruct the polygon?

- The classical polygons posess groups of automorphisms which act doubly transitivly on the ovoid. Are there other possibilities of groups acting doubly transitively on an ovoid of a generalized polygon?

The first question is answered completely. The second question is answered completely in the case of hexagons. In the case of quadrangles, there is an additional condition, which says that the the order of the polygon is (a, a) where a is a prime power.
The first question is treated in a way avoiding classification results, the treatment of the second question involves the classification of finite simple groups (via the classification of doubly transitive groups).

Chapter 1

Generalized Polygons

A **generalized n-gon** is a bipartite graph $\mathcal{G} = (V, E)$ satisfying the following conditions:

- The diameter of $\mathcal{G}$ is n.
- The girth of $\mathcal{G}$ is $2n$.
- For any vertex $v \in \mathcal{G}$, the set $\mathrm{D}_1(v)$ of vertices adjacent to v has size at least 3.

A **generalized polygon** is a generalized n-gon for some natural number n. The third property is called **thickness**. Graphs which satisfy the first and second property, but contain vertices with only two other vertices adjacent to them are called **weak generalized n-gons**. Since the graph is required to be bipartite, there exist a partition $V = V_1 \cup V_2$ into disjoint sets V_1 and V_2. If we call the vertices of V_1 points and the vertices contained in V_2 lines, this gives rise to an incidence structure with point set V_1 and line set V_2 whose incidence relation is the adjacency relation from $\mathcal{G}$. This incidence geometry is unique up to isomorphism and duality. We call $\{V_1, V_2\}$ the set of colors. Throughout this work, language from both graph theory (like "distance", "girth",...) and incidence geometry (like "intersection point", "joining line",...) will be used.

By definition, every graph automorphism either fixes both colors setwise or interchanges V_1 and V_2. With incidence geometry in mind, we call an a graph automorphism which fixes V_1 and V_2 an **automorphism** of the generalized polygon. An automorphism of a generalized polygon is also called a **collineation**. If a graph automorphism does not necessarily fix the set of colors, we speak of a **correlation**.

For any two points p and q of a generalized polygon, we have $|\mathrm{D}_1(p)| = |\mathrm{D}_1(q)|$ and for any two lines l and m, we have $|\mathrm{D}_1(l)| = |\mathrm{D}_1(m)|$. A generalized polygon Γ is called **of order** (s,t) if, for any point p and any line l of Γ, the number of points incident with l is $s+1$ and the number of lines incident with p is $t+1$.

1.1 Constructions of some generalized polygons

The intention of this section is to give a short overview about the generalized polygons which will appear in the subsequent parts of this work. The focus will be on the construction of these polygons and some properties which will be important for the study of ovoids in these polygons. The exposition follows [VM98a], to which the reader is referred for more detail and proofs of the claimed statements.

1.1.1 Classical quadrangles

Orthogonal and hermitian quadrangles

Let $\mathbb{K}$ be a field[1], σ an anti-automorphism of $\mathbb{K}$ of order at most 2 and let V be a vector space over $\mathbb{K}$. Furthermore, let $g\colon V \times V \to \mathbb{K}$ be a $(\sigma,1)$-linear form, i.e.

$$\forall v_1, v_2, w_1, w_2 \in V,\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}\colon$$

$$\begin{aligned} g(v_1\alpha_1 + v_2\alpha_2\,,\, w_1\beta_1 + w_2\beta_2) &= {\alpha_1}^{\sigma} g(v_1\,,\, w_1)\,\beta_1 + {\alpha_1}^{\sigma} g(v_1\,,\, w_2)\,\beta_2 \\ &\quad + {\alpha_2}^{\sigma} g(v_2\,,\, w_1)\,\beta_1 + {\alpha_2}^{\sigma} g(v_2\,,\, w_2)\,\beta_2\,. \end{aligned}$$

We define

$$f\colon V \times V \to \mathbb{K}\colon f(x,y) := g(x,y) + g(y,x)^{\sigma}$$

and

$$R := \left\{ v \in V \,\middle|\, f(v,w) = 0 \text{ for all } w \in V \right\}$$

Denote $\mathbb{K}_\sigma := \{t^\sigma - t \mid t \in \mathbb{K}\}$ and let

$$q\colon V \to \mathbb{K}/_{\mathbb{K}_\sigma}\colon x \mapsto q(x) = g(x\,,\,x) + \mathbb{K}_\sigma\,.$$

We call q a σ–**quadratic form**. The form q is called **anisotropic** over a subspace W of V if, for all $w \in W$, $q(w) = 0 \Leftrightarrow w = 0$. We call q non–degenerate if it is anisotropic over R. The **Witt index** of q is the dimension of a maximal subspace contained in $q^{-1}(0)$.

[1] $\mathbb{K}$ is not required to be commutative

Theorem 1 ([BT72]). *Let q be a non–degenerate σ–quadratic form of Witt index 2 on a vector space V over a field $\mathbb{K}$. Then the following incidence geometry (P, L, I) is a weak generalized quadrangle: P is the set of 1–dimensional subspaces of $q^{-1}(0)$, L is the set of 2–dimensional subspaces of $q^{-1}(0)$ and I is symmetrized containment. The quadrangle is thick if and only if V has dimension at least 5 or σ is not the identity.*

The quadrangle is called **orthogonal quadrangle** if $\sigma = 1$ and **hermitian quadrangle** if $\sigma \neq 1$. If the underlying field is finite, it is necessarily commutative and the quadrangle arises from a quadric or a hermitian variety in $\mathbf{PG}(d, \mathbb{F}_q)$ containing lines but no planes. Furthermore, there are exactly two classes of non–degenerate quadrics if d is odd (hyperbolic and elliptic quadrics) and one class of non–degenerate quadrics if d is even (parabolic quadrics) ([Art57], Chapter III, 6). The table below shows the Witt indices of these quadrics.

Quadric	Symbol	Witt index	dim V
hyperbolic	$\mathrm{Q}^+(d, q)$	$\frac{1}{2}(d+1)$	$d+1$
elliptic	$\mathrm{Q}^-(d, q)$	$\frac{1}{2}(d-1)$	$d+1$
parabolic	$\mathrm{Q}(d, q)$	$\frac{1}{2}d$	$d+1$

Consequently, the orthogonal quadrangles over a finite field of order q arise from $\mathrm{Q}^+(3, q)$ (this quadrangle is non–thick), $\mathrm{Q}^-(5, q)$ (this quadrangle is denoted by $\mathrm{Q}(5, q)$) and $\mathrm{Q}(4, q)$ (this quadrangle is denoted by $\mathrm{Q}(4, q)$).

If $\sigma \neq 1$, then q is necessarily a square and the involutory automorphism is given by $x \mapsto x^{\sqrt{q}}$. For every dimension d, there exists only one isomorphism class of hermitian varieties over $\mathbb{F}_q$. The projective dimension of maximal subspaces is $\frac{d-1}{2}$ for d odd and $\frac{d-2}{2}$ for d even. So the only values of d producing generalized quadrangles are $d = 3$ and $d = 4$. The resulting quadrangles are denoted by $\mathrm{H}(3, q)$ and $\mathrm{H}(4, q)$.

Symplectic quadrangles

Let $\mathbb{F}$ be any commutative field and consider the projective space $\mathbf{PG}(3, \mathbb{F})$ equipped with homogeneous coordinates. Furthermore, consider the polarity τ of $\mathbf{PG}(3, \mathbb{F})$ which maps the 1-dimensional subspace generated by (y_0, y_1, y_2, y_3) to the plane with equation $y_1X_0 - y_0X_1 + y_3X_2 - y_2X_3$. A line L of $\mathbf{PG}(3, \mathbb{F})$ is called totally isotropic if $L^\tau = L$.

Theorem 2 ([VM98a]). *The incidence geometry consisting of the points of $\mathbf{PG}(3, \mathbb{F})$ and the totally isotropic lines is a generalized quadrangle.*

These quadrangles are called **symplectic quadrangles** and are denoted by $\mathrm{W}(\mathbb{F}_q)$.

1.1.2 Classical generalized hexagons

Consider an incidence geometry of type $\mathcal{G} = \left(V_1, V_2, V_4^+, V_4^-\right)$ of type D_4 defined on the quadric $\mathrm{Q}^+(7,\mathbb{K})$. A triality is a mapping $\tau\colon \mathcal{G} \to \mathcal{G}$ of order 3 with $V_2 \to V_2$, $V_1 \to V_4^+$, $V_4^+ \to V_4^-$, $V_4^- \to V_1$. An absolute point is a point $p \in V_1 \cup V_4^+ \cup V_4^-$ satisfying $p^\tau I p$ [2]. An absolute line is a line $l \in V_2$ which is fixed by τ.

Theorem 3 ([Tit59]). *Let τ be a triality of $\mathcal{G}$. Suppose that one of the following hypotheses is satisfied:*

- *There exists at least one absolute point and every absolute point is incident with at least two absolute lines.*

- *There exists a cycle $(L_0, L_1, \ldots, L_d)$, $d > 2$ of absolute lines such that $L_i \cap L_{i+1} \neq \emptyset$ for all i and $L_d \cap L_0 \neq \emptyset$.*

Then the incidence geometry consisting of one sort of absolute points (i.e. the absolute elements in V_1 or the absolute elements in V_4^+ or the absolute elements in V_4^-) and absolute lines is a weak generalized hexagon. The isomorphism class is independent of the choice of the point set.

In order to give an explicit description of some trialities producing generalized hexagons, we introduce a trilinear form $\mathcal{T}\colon V \times V \times V \to \mathbb{K}$. This form describes the correspondence between points of different type. More precisely, if we start with a point $p = (x_0, x_1, \ldots, x_8) \in V_1$ and want to find the equations for the element of V_4^+ respectively V_4^- corresponding to that point, we plug the coordinates into the trilinear form and require that it is identically zero in the third respectively second argument. This trilinear

[2] Here, I denotes the incidence relation of the D_4-geometry, which is is defined as follows:

- Incidence between elements of V_1 and V_4^+ respectively V_4^- is symmetrized containment.

- An element of V_4^+ and an element of V_4^- are incident if and only if their intersection has dimension 3 (i. e. is a plane of $\mathrm{Q}(7,\mathbb{K})$)

form can be described by

$$\begin{aligned}\mathcal{T}(x,y,z) \quad = \quad & \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} + \begin{vmatrix} x_4 & x_5 & x_6 \\ y_4 & y_5 & y_6 \\ z_4 & z_5 & z_6 \end{vmatrix} \\ & + x_3\left(z_0y_4 + z_1y_5 + z_2y_6\right) + x_7\left(y_0z_4 + y_1z_5 + y_2z_6\right) \\ & + y_3\left(x_0z_4 + x_1z_5 + x_2z_6\right) + y_7\left(z_0x_4 + z_1x_5 + z_2x_6\right) \\ & + z_3\left(y_0x_4 + y_1x_5 + y_2x_6\right) + z_7\left(x_0y_4 + x_1y_5 + x_2y_6\right) \\ & - x_3y_3z_3 - x_7y_7z_7\,.\end{aligned}$$

Using this trilinear form, it is possible to label the points of V_4^+ and V_4^- in the same way as the elements of V_1.

Theorem 4 ([Tit59]). *Let σ be an automorphism of $\mathbb{K}$ of order 1 or 3. Then the map*

$$\tau_\sigma\colon V_1 \to V_4^+ \to V_4^- \to V_1\colon\ (x_j) \mapsto \left(x_j^\sigma\right) \mapsto \left(x_j^{\sigma^2}\right) \mapsto (x_j)$$

is a triality which satisfies the condition of Theorem 3 and therefore produces a generalized hexagon.

A triality as above is usually called triality of type I_σ. By taking $\sigma = 1$, we can construct at least one triality which produces a generalized hexagon over any commutative field $\mathbb{K}$. We call this hexagon the **split Cayley hexagon** over $\mathbb{K}$ and denote it by $\mathrm{H}(\mathbb{K})$.

We call a generalized hexagon arising from a triality with $\sigma \neq 1$ a **twisted triality hexagon** and denote it by $\mathrm{T}\left(\mathbb{K}, \mathbb{K}^\sigma, \sigma\right)$. If the field $\mathbb{K}$ is finite of order q, the field automorphism σ is (up to taking the inverse which does not change the isomorphism class of the hexagon) uniquely determined and we denote the hexagon by $\mathrm{T}\left(q^3, q\right)$.

Regarding the construction of ovoids, the following special property of split Cayley hexagons will be relevant. This property allows a realization of split Cayley hexagons in projective spaces of dimension 6.

Theorem 5 ([Tit59]). *The points and lines of $\mathrm{H}(\mathbb{K})$, considered as the absolute elements of the triality τ_{id} in V_1 respectively V_2, all lie in the hyperplane of $\mathbf{PG}(7,\mathbb{K})$ with equation $x_3 + x_7 = 0$. Conversely, every element of V_1 which is contained in that hyperplane belongs to $\mathrm{H}(\mathbb{K})$.*

Chapter 2

Ovoids, Spreads and Polarities in generalized $2m$-gons

2.1 Ovoids and Spreads

An **ovoid** in a generalized $2m$-gon Γ is a subset $\mathcal{O}$ of the point set which satisfies

- For all points p and q in $\mathcal{O}$, the distance of p and q is $2m$.
- For any vertex v of Γ, there exists an element of $\mathcal{O}$ at distance at most m from v.

A **spread** is a set $\mathcal{S}$ of lines of Γ such that

- For all lines l and m in $\mathcal{S}$, the distance of l and m is $2m$.
- For any vertex v of Γ, there exists an element of $\mathcal{S}$ at distance at most m from v.

An **ovoid-spread pairing** of a generalized $2m$-gon Γ is a pair $(\mathcal{O}, \mathcal{S})$ such that every element of $\mathcal{O}$ is incident with an element of $\mathcal{S}$ and vice versa.

Counting the number of vertices at distance at most m from the points of an ovoid and comparing these numbers to the total number of vertices of a $2m$-gon gives the following theorem (see [VM98a]):

Theorem 6. *A set of mutually opposite points of a finite* $2m$*-gon* Γ *of order* (s, t) *is an ovoid if and only if it contains*

1. $1 + st$ *points for* $m = 2$

2. $\dfrac{(1+s)(1+st+s^2t^2)}{1+s+st}$ *points for* $m = 3$

3. $1 + s^2t^2$ *points for* $m = 4$.

The number for generalized hexagons can be simplified. In [Off02], Offer proves the following statement:

Theorem 7. *If a generalized hexagon of order* (s, t) *contains an ovoid or a spread, then* $s = t$ *holds.*

This result is obtained by comparing two formulae for the number of points of an ovoid in a generalized hexagon resulting from two distinct ways of counting the points of the ovoid. Using this result, the second number in Theorem 6 can be simplified to $1 + s^3$.

2.2 Polarities of $2m$-gons and their absolute elements

A **duality** of a generalized $2m$-gon Γ is a graph-automorphism of Γ which acts non-trivially on the set of colors. A **polarity** is a duality which is an involution. An **absolute element** of a duality is an element which is incident with its image.

The following theorem (which can be found in [VM98a]) provides a source for ovoids (and spreads) in generalized $2m$-gons. The proof is well known. It is included because it illustrates the connection between polarities and ovoid-spread pairings.

Theorem 8. *Let ρ be a polarity of a generalized $2m$-gon Γ. Then the absolute points and lines of ρ form an ovoid-spread-pairing of Γ.*

Proof: To prove this, it is sufficient to show that the absolute points form an ovoid of Γ. Let v be any vertex of Γ. The image v^ρ can not be opposite v, so there exists a unique path

$$(v = x_0, x_1, \ldots, x_k = v^\rho)$$

of length $k < 2m$, k odd joining v and v^ρ. This path is inverted by ρ, so

$$\forall\, 0 \leq j \leq k\,;\; (x_j)^\rho = x_{k-j}.$$

Especially, $\left(x_{\frac{k-1}{2}}\right)^\rho = x_{\frac{k+1}{2}}$ and the flag $\left\{x_{\frac{k-1}{2}}, x_{\frac{k+1}{2}}\right\}$ is absolute. We have found an absolute element of each type at distance $\frac{k+1}{2} \leq m$. It remains to show that any two absolute points are opposite. Assume, to the contrary, that there are absolute points p and q at distance $k < 2m$ and let

$$(p = x_0, x_1, \ldots, x_{k-1}, x_k = q)$$

the unique path joining them. Since $d\,(p,q) = d\,(p^\rho, q^\rho)$, we can assume (upon interchanging p and q) that, $x_1 = p^\rho$. This implies that $(x_2)^\rho$ is incident with p. But now, the circuit

$$(p = x_0, x_1, \ldots, x_{k-1}, q, q^\rho, (x_{k-1})^\rho, (x_{k-2})^\rho, \ldots, (x_3)^\rho, (x_2)^\rho, p)$$

has length at most $k + 1 + (k-2) + 1 = 2k$ which is impossible.

$\square$

There are many ways to construct ovoids and spreads of generalized polygons. Some of them will be described on the following pages. However, there are generalized polygons which do not contain any ovoids or spreads and not every polygon containing an ovoid or a spread admits a polarity. The following results show that there are many polygons not admitting a polarity. The first restriction is an arithmetic one on the order of a generalized polygon admitting a polarity:

Theorem 9. *[CTP76] If Γ is a generalized $2m$-gon of order (s,t) and Γ admits a polarity, then $s = t$ and ms is a square.*

Moufang polygons and their polarities are completely classified in [TW02]. In the finite case, the only Moufang polygons admitting a polarity are given by the following two theorems which can also be found in [VM98a]:

Theorem 10. *A finite Moufang quadrangle is self-dual if and only if it is a symplectic quadrangle $\mathrm{W}(q)$ and q is a power of 2. It admits a polarity if and only if the field $\mathbb{F}_q$ admits a Tits endomorphism; this is the case precisely if $q = 2^{2k+1}$.*

Theorem 11. *A finite Moufang hexagon is self-dual if and only if it is a split Cayley hexagon $\mathrm{H}(q)$ and q is a power of 3. It admits a polarity if and only if the field $\mathbb{F}_q$ admits a Tits endomorphism; this is the case precisely if $q = 3^{2k+1}$.*

The ovoids and spreads arising from a polarity of a Moufang generalized polygon are called **semi classical ovoids** and **semi classical spreads**, respectively. For the purpose of this work, the connection between polarities, (special) ovoid-spread pairings and Tits endomorphisms is important. This is described in the following theorems due to Tits. The result is stated in its original form, also covering the infinite case. Therefore, it is a statement about mixed quadrangles $\mathrm{Q}(\mathbb{K}, \mathbb{K}', \mathbb{K}, \mathbb{K}')$ respectively mixed hexagons $\mathrm{H}(\mathbb{K}, \mathbb{K}')$ (see [VM98a], chapter 3.4.2 respectively chapter 3.5.3 for a definition). If the field $\mathbb{K}$ is perfect (for instance finite), then the only mixed quadrangle over that field is the symplectic quadrangle and the only mixed hexagon is the split Cayley hexagon.

Theorem 12. *[Tit62] The set of conjugacy classes of polarities (conjugacy with respect to $\mathrm{Aut}(\Gamma)$) of polarities of $\mathrm{Q}(\mathbb{K}, \mathbb{K}', \mathbb{K}, \mathbb{K}')$ is in natural bijective correspondence to the set of conjugacy classes (with respect to $\mathrm{Aut}(\mathbb{K})$) of Tits endomorphisms θ of $\mathbb{K}$ such that $\mathbb{K}^\theta = \mathbb{K}'$. Also, this set is in natural bijective correspondence with the set of isomorphism classes (with respect to $\mathrm{Aut}(\Gamma)$) of ovoid spread pairings arising from polarities.*

Theorem 13. *[Tit62] The set of conjugacy classes of polarities (conjugacy with respect to* $\mathrm{Aut}\,(\Gamma)$*) of polarities of* $\mathrm{H}\,(\mathbb{K}, \mathbb{K}')$ *is in natural bijective correspondence to the set of conjugacy classes (with respect to* $\mathrm{Aut}\,(\mathbb{K})$*) of Tits endomorphisms* θ *of* $\mathbb{K}$ *such that* $\mathbb{K}^\theta = \mathbb{K}'$*. Also, this set is in natural bijective correspondence with the set of isomorphism classes (with respect to* $\mathrm{Aut}\,(\Gamma)$*) of ovoid spread pairings arising from polarities.*

The focus of this text is on finite generalized polygons, where the situation becomes easier. A finite field $\mathbb{K}$ of characteristic 2 or 3 admits a Tits endomorphism (which automatically is an automorphism) if and only if the degree over the prime field is odd.

To see this, let $\mathbb{K} = \mathbb{F}_q$, $q = 2^d$ or $q = 3^d$. If d is odd, then the mapping

$$\theta\colon \mathbb{F}_q \to \mathbb{F}_q\colon x \mapsto x^k\,, k = 2^{\frac{d+1}{2}} \text{ respectively } k = 3^{\frac{d+1}{2}}$$

is a Tits endomorphism. If d is even and if we assume that there exists a Tits-endomorphism θ, then, since $\mathrm{Aut}\,(\mathbb{F}_q)$ is cyclic of order d, the square of θ, which is the Frobenius, would not generate $\mathrm{Aut}\,(\mathbb{F}_q)$ which is impossible.

If the field $\mathbb{K}$ is infinite, the property $\left(x^\theta\right)^\theta = x^2$ is not sufficient to determine θ uniquely, even if we fix the image of θ. As an example (taken from [VM98a]), consider the field $\mathbb{K}$ of Laurent series over $\mathbb{F}_2$ with variable t where the exponents of t are elements of $\mathbb{Z} + \sqrt{2}\mathbb{Z}$. The endomorphisms

$$\theta_\varepsilon\colon \mathbb{K} \to \mathbb{K}:\ f(x) \mapsto f\left(x^{\varepsilon\sqrt{2}}\right)\,,$$

for $\varepsilon \in \{-1, +1\}$, are Tits endomorphisms with $\mathbb{K}^{\theta_{+1}} = \mathbb{K}^{\theta_{-1}}$.

As a direct consequence, we have that, for $\Gamma = \mathrm{W}(q)$, $q = 2^{2k+1}$ or
$\Gamma = \mathrm{H}(q)$, $q = 3^{2k+1}$, any two polarities are conjugate, any two ovoid-spread pairings arising from a polarity are conjugate and two polarities having the same absolute elements coincide.

2.3 Ovoids in generalized quadrangles

Some other classes of ovoids and spreads will be important in the later part of this work. We will quickly review some known constructions. Examples of ovoids of classical generalized polygons can be constructed via the following two theorems which can be found in [VM98a]:

Theorem 14. *Let Γ be an orthogonal or hermitian quadrangle embedded in some projective space $\mathbf{PG}(d, \mathbb{K})$. Every hyperplane of $\mathbf{PG}(d, \mathbb{K})$ not containing lines of Γ intersects Γ in a point set which is an ovoid of Γ.*

In the finite case this construction produces ovoids of $\mathrm{Q}(4, q)$ and $\mathrm{H}(3, q^2)$.

Theorem 15. *Let Γ be a full subquadrangle of a generalized quadrangle Γ' and let p be a point of Γ' outside Γ. Then the set of points of Γ collinear with p in Γ' constitutes an ovoid $\mathcal{O}$ of Γ. If, moreover, Γ' is a Moufang quadrangle, then the stabilizer of $\mathcal{O}$ in the little projective group of Γ' acts doubly transitively on $\mathcal{O}$.*

Applying this theorem to $\mathrm{Q}(4, q) \subseteq \mathrm{Q}(5, q)$ respectively $\mathrm{H}(3, q^2) \subseteq \mathrm{H}(4, q^2)$ produces the same ovoids as Theorem 14. Ovoids which can be constructed as in Theorem 14 will be called **classical ovoids**. Ovoids constructed as in Theorem 15 are called subtended ovoids.

A construction of a spread of $\mathrm{Q}(5, q)$ will be given in the next section. The reason why that construction is placed there is that it also produces a spread of a generalized hexagon.

Ovoids and spreads also exist in finite generalized quadrangles which do not satisfy the Moufang condition. As an example, in [Pay71], Payne constructs a class of generalized quadrangles having order $(q-1, q+1)$ as follows: He starts with a quadrangle Γ of order (q, q), q a prime power having a regular point x_∞. Since any point of a symplectic quadrangle is a regular point, examples can be constructed starting with $\Gamma = \mathrm{W}(q)$. The new quadrangle ${}^{(x_\infty)}\Gamma$ has as its point set the points of Γ opposite x_∞, its lines are the lines of Γ not incident with x_∞ as well as the sets $\{x_\infty, a\}^{\perp\perp}$ for a point a opposite x_∞, incidence is defined the natural way. It is now easy to check that ${}^{(x_\infty)}\Gamma$ is a generalized quadrangle, and the sets $\{x_\infty, a\}^{\perp\perp}$, $a \in \mathrm{D}_4(x_\infty)$ form a spread of ${}^{(x_\infty)}\Gamma$.

The following list is provides an overview of results about the existence question of ovoids and spreads in known generalized quadrangles. The information is taken from [PT09]. Note that some existence questions remain open.

- The generalized quadrangle $\mathrm{Q}(4, q)$ always has ovoids. It has spreads if and only if q is even.
- The generalized quadrangle $\mathrm{Q}(5, q)$ has spreads but no ovoids.
- The generalized quadrangle $\mathrm{W}(q)$ always has spreads. It has ovoids if and only if q is even. [1]
- The generalized quadrangle $\mathrm{H}(3, q)$ has ovoids but no spreads. [2]
- The generalized quadrangle $\mathrm{H}(4, q)$ has no ovoids. The existence of spreads appears to be open.
- The generalized quadrangle $\mathrm{T}_2(\mathcal{O})$ has ovoids.The existence of spreads appears to be open.
- The generalized quadrangle $\mathrm{T}_3(\mathcal{O})$ has spreads but no ovoids.
- The generalized quadrangle $\mathrm{K}(q)$ has spreads but no ovoids.
- The generalized quadrangle ${}^{(x_\infty)}\mathcal{Q}$ has spreads. It has an ovoid if and only if $\mathcal{Q}$ has an ovoid containing x_∞.

[1] This follows from the statement about $\mathrm{Q}(4, q)$ by duality.
[2] This follows from the statement about $\mathrm{Q}(5, q)$ by duality.

2.4 Ovoids and spreads in generalized hexagons

In the finite case, there are two classes of classical generalized hexagons. One of them, the twisted triality hexagons, admit neither ovoids nor spreads (this follows from Theorem 7). For split Cayley hexagons, one class of ovoids inside the dual of the split Cayley hexagons was constructed by Thas in [Tha80] in the following way:

Theorem 16. *Let $\Gamma = \mathrm{H}(q)$ be embedded in the quadric $\mathbf{Q}\,(6,q)$ of $\mathbf{PG}\,(6,q)$. Let H be a hyperplane of $\mathbf{PG}\,(6,q)$ intersecting $\mathbf{Q}\,(6,q)$ in an elliptic quadric $\mathbf{Q}^-\,(5,q)$. Then the lines of Γ on H form a spread of Γ.*

Since the quadric $\mathbf{Q}^-\,(5,q)$ is a classical (orthogonal) generalized quadrangle, the spread of the generalized hexagon constructed in theorem 16 is also a spread of the generalized quadrangle by [Tha81].

An alternative construction of this spread (which can be found in [CTP76]) is accomplished by taking an orbit of length q^3+1 under a subgroup of the automorphism group of the hexagon isomorphic to $\mathbf{PSU}\,(3,q)$. A third construction of the same spread was given by Van Maldeghem in [VM98b]. These spreads of the classical hexagons are called **classical** or **hermitian** spreads.

If q is not a power of 3, then there is only one additional class of spreads of $\mathrm{H}(q)$ known to the author (see [BTVM98]). These spreads require $q \equiv 1 \bmod 3$ and arise from a distortion of the coordinates of lines of a hermitian spread.

Thas shows in [Tha81] that every ovoid of the hexagon is also an ovoid of the quadric $\mathbf{Q}\,(6,q)$ and conversely, every ovoid of $\mathbf{Q}\,(6,q)$ is an ovoid of the hexagon (embedded in $\mathbf{Q}\,(6,q)$ the standard way). Here, an "ovoid of $\mathbf{Q}\,(6,q)$" is a set of points of the quadric such that every plane of $\mathbf{Q}\,(6,q)$ is incident with exactly one of these points.

At a first glance, an ovoid of a generalized hexagon seems to carry no further structure - it is simply a set of mutually opposite points. However, the hermitian ovoids of the split Cayley hexagons can be furnished with a system of blocks, giving them the structure of a unital in the following sense:

For any $q \in \mathbb{N}$, a unital of order q is an incidence geometry (P, B, I) such that P consists of q^3+1 points, any block $b \in B$ is incident with $q+1$ points and any two distinct points are incident with a unique block. Another way of expressing this definition is: A unital of order $q \in \mathbb{N}$ is a $2-(q^3, q, 1)$-design.

We will from now on focus on the following problem: Suppose that we are given any generalized $2m$-gon Γ admitting a spread or ovoid and suppose that we know a certain group of automorphisms of Γ leaving the ovoid respectively the spread invariant. Is this enough to determine the isomorphism class of Γ?

Some work has been done in this direction. In [dK05], De Kaey proved that the hermitian ovoids, together with the action of their stabilizer inside the automorphism group of the hexagon, determine the hexagon uniquely up to isomorphism. More precisely:

Theorem 17. *If Γ is a generalized hexagon of order q admitting an ovoid $\mathcal{O}$ and if there is a group $G \leq \mathrm{Aut}_{\mathcal{O}}(\Gamma)$ inducing $\mathbf{PSU}(3,q)$ on $\mathcal{O}$, then Γ is dual to $\mathrm{H}(q)$ and the duality maps the ovoid to a hermitian spread.*

One objective of the present work is to obtain a similar result for the ovoids arising from polarities.

Chapter 3

Polarities of $\mathrm{W}(2^{2k+1})$ and Suzuki groups

3.1 Suzuki groups and their action on $\mathrm{W}(2^{2k+1})$

In order to go into some more detail, we introduce coordinates on $\mathrm{W}(q)$. For more information on this as well as general coordinatization theory, the reader is referred to [VM98a], Chapter 3. A quadratic quaternary ring corresponding to $\mathrm{W}(q)$ can be obtained by setting $R_1 = R_2 = \mathbb{F}_q$ and

$$\begin{aligned}
\psi_1\colon \quad & R_2 \times R_1 \times R_2 \times R_1 \to R_2\colon\ (k, a, l, a') \mapsto \quad & a^2k + l - 2aa' \\
\psi_2\colon \quad & R_1 \times R_2 \times R_1 \times R_2 \to R_1\colon\ (k, a, l, a') \mapsto \quad & -ak + l\,.
\end{aligned}$$

This means that a point of $\mathrm{W}(q)$ has coordinates in

$$\{(\infty)\} \cup \left\{(a) \,\middle|\, a \in \mathbb{F}_q\right\} \cup \left\{(k, b) \,\middle|\, k, b \in \mathbb{F}_q\right\} \cup \left\{(a, l, a') \,\middle|\, a, l, a' \in \mathbb{F}_q\right\}\,,$$

a line has coordinates in

$$\{[\infty]\} \cup \left\{[k] \,\middle|\, k \in \mathbb{F}_q\right\} \cup \left\{[a, l] \,\middle|\, a, l \in \mathbb{F}_q\right\} \cup \left\{[k, b, k'] \,\middle|\, k, b, k' \in \mathbb{F}_q\right\}\,,$$

and incidence is described as follows:

- The point (∞) is incident with all lines of $\left\{[k] \,\middle|\, a \in \mathbb{F}_q \cup \infty\right\}$.
- The line $[\infty]$ is incident with all points of $\left\{(a) \,\middle|\, a \in \mathbb{F}_q \cup \infty\right\}$.
- For $k \neq j$, a point $(x_1, \dots, x_k)$ and a line $[y_1, \dots, y_j]$ are incident if and only if $|k - j| = 1$ and all but the last coordinate coincide.
- A point (a, l, a') and a line $[k, b, k']$ are incident if and only if

$$\psi_1(k, a, l, a') = k' \qquad \text{and} \qquad \psi_2(k, a, l, a') = b\,.$$

The following tables describe the connection to homogeneous coordinates of the ambient projective space $\mathbf{PG}(3, q)$.

Coordinates in $W(q)$	Coordinates in $\mathbf{PG}(3, q)$
(∞)	$(1, 0, 0, 0)$
(a)	$(a, 0, 1, 0)$
(k, b)	$(-b, 0, k, 1)$
(a, l, a')	$(l - aa', 1, -a', -a)$

Lines of $W(q)$	Coordinates in $\mathbf{PG}(3, q)$
$[\infty]$	$\langle(1, 0, 0, 0), (0, 0, 1, 0)\rangle$
$[k]$	$\langle(1, 0, 0, 0), (0, 0, k, 1)\rangle$
$[a, l]$	$\langle(a, 0, 1, 0), (l, 1, 0, -a)\rangle$
$[k, b, k']$	$\langle(-b, 0, k, 1), (k', 1, -b, 0)\rangle$

We will now investigate the structure of the centralizer of a polarity in $\mathrm{Sp}_4(q)$ – for $q \neq 2$ the latter is the little projective group of $W(q)$ – centralizing a polarity. Note that, because we are working over a finite field, there is only one conjugacy class of polarities. Everything in this section follows [VM98a], chapter 7.6. It can be checked by direct computations that the following map is a polarity of $W(q)$:

$$\begin{array}{rcl} (\infty) & \leftrightarrow & [\infty] \\ (a) & \leftrightarrow & [a^\theta] \\ [a, b] & \leftrightarrow & (a^\theta, b^\theta) \\ (a, b, c) & \leftrightarrow & \left[a^\theta, b^{\theta^{-1}}, c^\theta\right] . \end{array}$$

The set of absolute points of this polarity (which form an ovoid) is

$$\mathcal{O} = \{\infty\} \cup \left\{ \left(a, a^{\theta+2} + b^{\theta}, b\right) \;\middle|\; a, b \in \mathbb{F}_q \right\} .$$

We call the subgroup of $\mathrm{Sp}_4(q)$ that centralizes ρ a Suzuki group and denote it by $\mathbf{Sz}(q)$. This definition coincides with other definitions of Suzuki groups that can be found in the literature. More precisely, the ovoid $\mathcal{O}$ defined above consists of points whose homogeneous coordinates in $\mathbf{PG}(3, q)$ are

$$\{(1, 0, 0, 0)\} \cup \left\{ \left(a^{\theta+2} + b^{\theta} - ab, 1, b, -a\right) \;\middle|\; a, b \in \mathbb{F}_q \right\} .$$

Swapping second and fourth coordinate produces the ovoid which is defined in [HB82], Theorem 3.3.
Note that this definition cannot be transferred to arbitrary fields without modifications. For infinite fields, the construction should start with a mixed quadrangle over a pair of fields of characteristic 2. Because an infinite field might have several non-conjugate Tits endomorphisms and the construction depends on the choice of such an endomorphism θ, we would have to speak about a Suzuki group with respect to θ.

The group $\mathbf{Sz}(q)$ acts doubly transitively on $\mathcal{O}$ ([Tit62] or [VM98a], Theorem 7.6.6). In our description, the stabilizer of the point ∞ in $\mathbf{Sz}(q)$ is a product TS where S is a Sylow 2-subgroup acting regularly on $\mathcal{O} \setminus \{\infty\}$ and T is a maximal torus normalizing S ([HB82], Theorem 3.3 or [VM98a], 7.6.7). Denoting the element of S mapping $(0, 0, 0)$ to $\left(a, a^{\theta+2} + b^{\theta}, b\right)$ by $\begin{pmatrix} a \\ b \end{pmatrix}$ turns S into a group with underlying set $\mathbb{F}_q^2$ and the following multiplication:

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d + ac^{\theta} \end{pmatrix} .$$

The group S acts on points and lines with 3 coordinates via

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} : (a, l, a') &\mapsto \left(a + x, l + x^{\theta+2} + y^{\theta} + a^2 x^{\theta}, a' + y + a x^{\theta}\right) \\ \begin{pmatrix} x \\ y \end{pmatrix} : [k, b, k'] &\mapsto \left[k + x^{\theta}, b + x^{\theta+1} + y + kx, k' + y^{\theta} + kx^2\right] \end{aligned}$$

The group T (the stabilizer of ∞ and $(0, 0)$) can be identified with the multiplicative group of $\mathbb{F}_q$, the action of $t \in \mathbb{F}_q$ on S is given by

$$h_t \colon \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} ta \\ t^{\theta+1} b \end{pmatrix} .$$

The group T acts on points and lines with 3 coordinates via

$$\begin{aligned} h_t\colon (a, l, a') &\mapsto \left(ta, t^{\theta+2}l, t^{\theta+1}a'\right) \\ h_t\colon [k, b, k'] &\mapsto \left[t^{\theta}k, t^{\theta+1}b, t^{\theta+2}k'\right] . \end{aligned}$$

In order to generate the group $\mathbf{Sz}(q)$, we need one additional involution ω_0 which does not fix ∞. It is more convenient to use coordinates of $\mathbf{PG}(3, q)$ here. Using these coordinates, ω_0 can be described by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(cf [HB82], 3.3).

We collect some facts about the structure of this subgroup for later usage.

- The inverse of an element of S is
$$\begin{pmatrix} a \\ b \end{pmatrix}^{-1} = \begin{pmatrix} a \\ b + a^{\theta+1} \end{pmatrix} .$$
- The square of an element of S is
$$\begin{pmatrix} a \\ b \end{pmatrix}^{2} = \begin{pmatrix} 0 \\ a^{\theta+1} \end{pmatrix} .$$
- The commutator of two elements of S is
$$\left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right] = \begin{pmatrix} 0 \\ a^{\theta}c + c^{\theta}a \end{pmatrix} .$$
- We define
$$Z = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \,\middle|\, b \in \mathbb{F}_q \right\} .$$
  This group is contained in the center of S. For $q \neq 2$, it equals the center of S and the commutator subgroup of S.

The stabilizer of the line $[0]$ is TZ. Because $\mathbf{Sz}(q)$ leaves the ovoid $\mathcal{O}$ and the spread $\mathcal{O}^{\rho}$ invariant there are at least two orbits of points and two orbits of lines. In order to see that there are exactly these four orbits note that each point which is not a point of the ovoid is incident with a unique line of the spread and that the group acts doubly transitively on the spread.

We define the geometry $\mathcal{Q}' = (P', L', \mathrm{I}')$ via

$$\begin{aligned} P' &:= \mathbf{Sz}(q)/_{TS} \cup \mathbf{Sz}(q)/_{TZ} \\ L' &:= \mathbf{Sz}(q)/_{TS} \cup \mathbf{Sz}(q)/_{TZ} \end{aligned}$$

and I' is defined by

$$\begin{aligned} TSg \, \mathrm{I}' \, TZh &\Leftrightarrow gh^{-1} \in TS \\ TSg \, \mathrm{I}' \, TSh &\Leftrightarrow gh^{-1} \in TS \\ TZg \, \mathrm{I}' \, TSh &\Leftrightarrow gh^{-1} \in TS \\ TZg \, \mathrm{I}' \, TZh &\Leftrightarrow gh^{-1} \in TZ\omega_0 T \end{aligned}$$

Instead of ω_0 (as defined above on page 28) one may use an arbitrary involution interchanging (∞) and $(0,0,0)$. The definition of incidence does not depend on the choice of ω because any such involution is contained in the coset $\omega_0 T$.

Lemma 1. *The geometries* $\mathrm{W}(q)$ *and* $\mathcal{Q}'$ *are isomorphic.*

Proof: Let $\mathrm{W}(q) = (P, L, I)$. The desired isomorphism is provided by the mappings

$$\begin{aligned} \Phi &: P' \to P : TSg \mapsto (\infty)^g , \; TZg \mapsto (0)^g \\ \Psi &: L' \to L : TSg \mapsto [\infty]^g , \; TZg \mapsto [0]^g . \end{aligned}$$

Consider the restriction Ψ' of the map Ψ to $\mathbf{Sz}(q)/_{TS}$ and the restriction Φ' of Φ to $\mathbf{Sz}(q)/_{TS}$. Then, the pairs (Ψ', Φ) and (Ψ, Φ') provide isomorphisms of the corresponding subgeometries by [Str92]. It remains to be shown that $TZg \, \mathrm{I}' \, TZh$ if and only if $\Phi(TZg) \, \mathrm{I} \, \Psi(TZh)$. To this end, let $(TZg, TZh) \in P' \times L'$. The images $\Phi(TZg) = (0)^g$ and $\Psi(TZh) = [0]^h$ are incident if and only if

$$\begin{aligned} (0)^g \, \mathrm{I} \, [0]^h &\Leftrightarrow (0)^{gh^{-1}} \, \mathrm{I} \, [0] \\ &\Leftrightarrow \exists z \in Z \colon (0)^{gh^{-1}z} = (0,0) \\ &\Leftrightarrow \exists z \in Z \colon (0)^{gh^{-1}z\omega} = (0) \\ &\Leftrightarrow \exists z \in Z \colon gh^{-1}z\omega \in TZ \\ &\Leftrightarrow gh^{-1} \in TZ\omega Z \\ &\Leftrightarrow TZg \, \mathrm{I}' \, TZh \end{aligned}$$

$\square$

3.2 Characterization of W(q)

Consider a generalized quadrangle $\mathcal{Q}$ having parameters suitable order and containing an ovoid. Assume further that there is a group G of automorphisms of $\mathcal{Q}$ leaving $\mathcal{O}$ invariant isomorphic to $\mathbf{Sz}(q)$, the Suzuki-group over the field $\mathbb{F}_q$. It will be shown that $\mathcal{Q}$ is isomorphic to the classical quadrangle W(q). More precisely:

Theorem 18. *Let $\mathcal{Q}$ be a finite generalized quadrangle having parameters (q, q) for some number $q = 2^{2k+1}$ and containing an ovoid $\mathcal{O}$. Assume further that there is a group G of automorphisms of $\mathcal{Q}$ leaving $\mathcal{O}$ invariant isomorphic to $\mathbf{Sz}(q)$. Then $\mathcal{Q}$ is isomorphic to* W(q).

This will be done by showing that the action of $\mathbf{Sz}(q)$ on $\mathcal{Q}$ corresponds to the action of $\mathbf{Sz}(q)$ on W(q).

Lemma 2. *Let S be a Sylow 2-subgroup of G. Then S fixes an element ∞.*

Proof: The existence of a fixed point follows from the fact that the number $q + 1$ of points of $\mathcal{O}$ is odd while S is a 2-group.

$\square$

Lemma 3. *If S is not transitive on $\mathcal{O} \setminus \{\infty\}$ then there exists $g \in Z \setminus \{1\}$ fixing more than $q + 1$ points of the ovoid.*

Proof: Assume that S has more than one orbit on $\mathcal{O} \setminus \{\infty\}$. As any orbit has less than q^2 elements, the stabilizer of any element of $\mathcal{O}$ in S is nontrivial. Using the fact that each $g \in S \setminus Z$ has nontrivial square, and that each square lies in Z, we find

$$\forall x \in \mathcal{O} \setminus \{\infty\} : \operatorname{Stab}_S(x) \cap Z \neq \{1\} .$$

We define the set

$$M := \{(x, g) \in (\mathcal{O} \setminus \{\infty\}) \times (Z \setminus \{1\}) \mid x^g = x\} .$$

Double counting leads to

$$\sum_{g \in Z \setminus \{1\}} |\operatorname{Fix}(g)| = |M| = \sum_{x \in \mathcal{O} \setminus \{\infty\}} |\operatorname{Stab}_Z(x)| - 1$$

which gives us

$$(q-1)\max_{g\in Z\setminus\{1\}}|\mathrm{Fix}\,(g)|\geq q^2\min_{x\in\mathcal{O}\setminus\{\infty\}}(|\mathrm{Stab}_Z\,(x)|-1)\geq q^2(2-1)=q^2$$

and finally

$$\max_{g\in Z\setminus\{1\}}|\mathrm{Fix}\,(g)|\geq\frac{q^2}{q-1}>q+1\,.$$

□

Lemma 4. *In the situation of Lemma 3, the subgeometry of points and lines fixed by g is a (thick) subquadrangle.*

Proof: Since $|D_1(\infty)|=q+1$ and S is a 2-group, there exists a line l incident with ∞ which is fixed by g. For any $x\in\mathcal{O}\setminus\{\infty\}$ which is a fixed point of g, let (l,v_x,w_x,x) denote the shortest path connecting x and l. There are points x_0 and x_1 with $v_{x_0}\neq v_{x_1}$ because all the lines w_x are distinct (so there are more than $q+1$ of these) and each point is incident with $q+1$ lines. The path $(x_0,w_{x_0},v_{x_0},l,v_{x_1},w_{x_1},x_1)$ is irreducible and by adding a path of length 3 from x_0 to w_{x_1} we obtain an apartment which is fixed by g. By [VM98a], Theorem 4.4.2, the subgeometry of points and lines fixed by g is a weak subquadrangle. Thickness follows because the number of fixed vertices adjacent to a vertex fixed by g is necessarily odd.

□

Lemma 5. *S acts transitively on $\mathcal{O}\setminus\{\infty\}$.*

Proof: Assume, to the contrary, that S is not transitive and let g be as in Lemma 3 and Lemma 4. Let (s,t) denote the order of the subquadrangle $\mathcal{Q}'$ of vertices fixed by g. By [Pay73], the restriction $st<q$ holds. The points of $\mathcal{O}$ contained in $\mathcal{Q}'$ form an ovoid of $\mathcal{Q}'$ consisting of $1+st$ points (see [VM98a], 7.2.3). But the number of fixed points of g is greater than $q+1$ by Lemma 3.

□

We need some algebraic information in order to proceed with the determination of the isomorphism class of the quadrangle. The following lemmas provide information about the possible stabilizers of elements outside $\mathcal{O}$.

Lemma 6.

$$\Phi \colon \mathbb{F}_q* \to \mathbb{F}_q* \colon x \mapsto x^{\theta+1}$$

is bijective.

Proof: The inverse of $\theta+1$ is $\theta-1$ since

$$\left(x^{\theta+1}\right)^{(\theta-1)} = x^{\theta^2-1} = x\,.$$

□

Lemma 7. *The only T-invariant subgroup of S are* $\{1\}$, Z *and* S.

Proof: Assume that $H \neq \{1\}$ is a T-invariant subgroup of S.

1. If H contains an element $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a \neq 0$, then H also contains $\begin{pmatrix} a \\ b \end{pmatrix}^2 = \begin{pmatrix} 0 \\ a^{\theta+1} \end{pmatrix}$. Applying the action of T, we find $Z \leq H$ because

$$\begin{pmatrix} 0 \\ a^{\theta+1} \end{pmatrix}^T = \left\{ \begin{pmatrix} 0 \\ a^{\theta+1} t^{\theta+1} \end{pmatrix} \,\middle|\, t \in \mathbb{F}_q* \right\}.$$

   Since Z is contained in the center of S, the subgroup H contains the element $\begin{pmatrix} a \\ 0 \end{pmatrix}$. Because $\begin{pmatrix} a \\ 0 \end{pmatrix}^T = \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix} \,\middle|\, f \in \mathbb{F}_q \right\}$, we obtain $H = S$.

2. If H does not contain an element $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a \neq 0$ but H is nontrivial, then H contains an element $\begin{pmatrix} 0 \\ b \end{pmatrix}$ with $b \neq 0$ and T-invariance yields $H = Z$.

□

The normalizer of S in G is TS where T is a maximal torus having order $q-1$. Because ∞ is the only point of $\mathcal{O} \setminus \{\infty\}$ which is fixed by S, the point ∞ is also fixed by TS and TS acts on $\mathcal{O} \setminus \{\infty\}$.

Lemma 8. *There is a point $n \in \mathcal{O} \setminus \{\infty\}$ and a path*

$$(\infty, l, p, m, n)$$

such that the following statements hold:

1. *S fixes l.*

2. *$TS_n = T$.*

3. *$S_p = Z$ and $(TS)_p = TZ$.*

Proof: 1. The number of lines incident with ∞ is $q+1$, therefore the 2-group S fixes at least one of them.

2. The stabilizer of a point $z \in \mathcal{O} \setminus \{\infty\}$ is a group of order $q-1$, therefore $(TS)_z$ is a Hall $2'$-subgroup of TS. By the Schur-Zassenhaus Theorem ([Rob96] ,9.2.1) the groups T and $(TS)_z$ are conjugate. Hence, there exists $n \in \mathcal{O} \setminus \{\infty\}$ satisfying $(TS)_n = T$.

3. Each line of the set $\mathcal{M} := D_2(l) \backslash D_1(\infty)$ is incident with exactly one point of $\mathcal{O} \setminus \{\infty\}$. The action of TS on $\mathcal{O} \setminus \{\infty\}$ therefore induces an action of TS on the set $\mathcal{M}$, the group T and the stabilizer of any line $h \in \mathcal{M}$ are conjugate. Now let h be the (unique) line of $\mathcal{M}$ incident with n and let p be the unique point incident with both l and h. We have $(TS)_h = T$ and $(TS)_p \supset T$, furthermore $(TS)_p$ contains q elements of S (because there are q lines incident with p distinct from l and the action of S on $\mathcal{M}$ is regular). Hence, using Lemma 7 we find $S_p = Z$. The stabilizer $(TS)_p$ is of the form TU with $U = (TS)_p \cap S$. Hence, we get $(TS)_p = TZ$.

□

Lemma 9. *If $q \neq 2$ then the action of G on $\mathcal{O}$ is doubly transitive.*

Proof: Assume that the whole group G fixes ∞. Since $|\mathrm{D}_1(\infty)| = q+1$, any Sylow 2-subgroup fixes at least one line incident with ∞. There are q^2+1 Sylow 2-subgroups, so there exists at least one line $a \in \mathrm{D}_1(\infty)$ which is fixed by two Sylow 2-subgroups. Any two distinct Sylow 2-subgroups, however, generate G, so G fixes the line a. We obtain a homomorphism

$$\Phi\colon G \to \mathrm{Sym}\left(\left(\mathrm{D}_1\left(a\right) \setminus \infty\right)\right) .$$

Replacing l with a in the proof of Lemma 8, we obtain a path

$$(\infty, a, x, b, y) \text{ with } y \in \mathcal{O} \setminus \{\infty\}$$

such that:

1. $\mathrm{Stab}_S\left(x\right) = Z$
2. S acts transitively on $\mathrm{D}_1\left(a\right) \setminus \{\infty\}$

The subgroup Z is normal in S and S acts transitively on $\mathrm{D}_1\left(a\right) \setminus \{\infty\}$, therefore Z fixes $\mathrm{D}_1\left(a\right) \setminus \{\infty\}$ pointwise. The homomorphism Φ has nonzero kernel and, because the group G is simple, $\ker\left(\Phi\right) = G$. Therefore, the group G can not act transitively on $\mathrm{D}_2\left(a\right) \setminus \mathrm{D}_1\left(\infty\right)$ and, consequently, G can not act transitively on $\mathcal{O}$. But this contradicts Lemma 5.

□

Note that the restriction $q > 2$ is not relevant for the purpose presented here because it is well-known that there is only one quadrangle with parameters $(2, 2)$ (see e.g. [VM98a] or [Str07]).

As all the pieces occurring in the reconstruction of W(q) from the action of $\mathbf{Sz}(q)$ coincide, we can deduce the following lemma:

Lemma 10. *$\mathcal{Q}$ and $\mathcal{Q}'$ are isomorphic.*

Proof: The desired isomorphism is provided by the mappings

$$\begin{aligned} \Phi \ &: \ P' \to P \ : \ TSg \mapsto \infty^g,\ TZg \mapsto p^g \\ \Psi \ &: \ L' \to L \ : \ TSg \mapsto l^g,\ TZg \mapsto m^{\omega g}. \end{aligned}$$

□

Corollary 1. $\mathrm{W}(q)$ *and* $\mathcal{Q}$ *are isomorphic.*

A different characterization of symplectic quadrangles over perfect fields of characteristic 2 using the Suzuki Groups was presented by van Maldeghem in [VM97]. He studies the Suzuki-Tits ovoid together with its intersections with planes of the ambient projective space. He calls these intersections "circles" and associates to each circle a "corner". The ovoid, equipped with the system of "circles" and "corners" satisfies a certain list of properties. Conversely, he shows that if, for any set $\mathcal{P}$, there is a set of subsets $\mathcal{S}$ (called circles) and a map $\partial\colon \mathcal{S} \to \mathcal{P}$ such that $(\mathcal{P}, \mathcal{S}, \partial)$ satisfies these properties, then $\mathcal{P}$ can be embedded in a projective space $\mathbf{PG}\,(3, \mathbb{K})$ over a perfect field of characteristic 2 admitting a Tits automorphism, such that $\mathcal{P}$ is the set of absolute points of a polarity of a symplectic quadrangle $\mathrm{W}(\mathbb{K})$ in $\mathbf{PG}\,(3, \mathbb{K})$ and the circles are the plane sections of $\mathcal{P}$ in $\mathbf{PG}\,(3, \mathbb{K})$.

Chapter 4

Polarities of $\mathrm{H}(3^{2k+1})$ and Ree groups

4.1 Ree groups and their action on $\mathrm{H}(3^{2k+1})$

The classical action of $\mathbf{Ree}(q)$ on $\mathrm{H}(q)$ has been described in [VM98a]. This section will use the standard embedding of $\mathrm{H}(q)$ in $\mathbf{PG}\,(6,q)$. The points of the hexagon are the points of the quadric $\mathbf{Q}\,(6,q)$ with equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2\,.$$

The lines of the hexagon are the lines on $\mathbf{Q}\,(6,q)$ whose grassmannian coordinates (see [VM98a], **2.3.18** for a definition) satisfy the equations

$$\begin{array}{rclcrclcrcl} p_{12} & = & p_{34}, & \quad & p_{20} & = & p_{35}, & \quad & p_{01} & = & p_{36} \\ p_{03} & = & p_{56}, & \quad & p_{13} & = & p_{64}, & \quad & p_{23} & = & p_{45}\,. \end{array}$$

Coordinates are assigned to points and lines in the following way:

Coordinates in $\mathrm{H}(q)$	Coordinates in $\mathbf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a^2+a''l+aa'a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$

Lines of $\mathrm{H}(q)$	Coordinates in $\mathbf{PG}(6,q)$
$[\infty]$	$\Big\langle(1,0,0,0,0,0,0)\,,(0,0,0,0,0,0,1)\Big\rangle$
$[k]$	$\Big\langle(1,0,0,0,0,0,0)\,,(0,0,0,0,0,1,-k)\Big\rangle$
$[a,l]$	$\Big\langle(a,0,0,0,0,0,1)\,,(-l,1,0,-a,0,a^2,1)\Big\rangle$
$[k,b,k']$	$\Big\langle(b,0,0,0,0,1,-k)\,,(k',k,1,b,0,0,b^2)\Big\rangle$
$[a,l,a',l']$	$\Big\langle(-l-aa',1,0,-a,0,a^2,-a')\,,$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\Big\rangle$
$[k,b,k',b',k'']$	$\Big\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\Big\rangle$

The corresponding operations are

$$
\begin{aligned}
\Psi_1(k,a,l,a',l',a'') &= a^3k + l - 3a''a^2 + 3aa'\,,\\
\Psi_2(k,a,l,a',l',a'') &= a^2k + a' - 2aa''\,,\\
\Psi_3(k,a,l,a',l',a'') &= a^3k^2 + l' - kl - 3a^2a''k - 3a'a''\,,\\
\Psi_4(k,a,l,a',l',a'') &= -ak + a''
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_1(a,k,b,k',b',k'') &= ak + b\,,\\
\Phi_2(a,k,b,k',b',k'') &= a^3k^2 + k' + kk'' + 3a^2kb + 3bb' + 3ab^2\,,\\
\Phi_3(a,k,b,k',b',k'') &= a^2k + b' + 2ab\,,\\
\Phi_4(a,k,b,k',b',k'') &= -a^3k + k'' - 3ba^2 - 3ab'\,.
\end{aligned}
$$

A point (a,l,a',l',a'') and a line $[k,b,k',b',k'']$ are incident if and only if

$$
\begin{aligned}
\Psi_1(k,a,l,a',l',a'') &= k''\,,\\
\Psi_2(k,a,l,a',l',a'') &= b'\,,\\
\Psi_3(k,a,l,a',l',a'') &= k'\,,\\
\Psi_4(k,a,l,a',l',a'') &= b\,.
\end{aligned}
$$

The hexagon admits the following polarity ρ : The hat-rack of the coordinatization is mapped to its dual and on points and lines with five coordinates

ρ is given by

$$\begin{aligned}
(\infty)^\rho &= [\infty] \\
[\infty]^\rho &= (\infty) \\
(a_1)^\rho &= \left[a_1^\theta\right] \\
[l_1]^\rho &= \left(l_1^{\theta^{-1}}\right) \\
(a_1,l_1)^\rho &= \left[a_1^{\theta^{-1}}, l_1^\theta\right] \\
[l_1,a_1]^\rho &= \left(l_1^\theta, a_1^{\theta^{-1}}\right) \\
(a_1,l_1,a_2)^\rho &= \left[a_1^\theta, l_1^{\theta^{-1}}, a_2^\theta\right] \\
[l_1,a_1,l_2]^\rho &= \left(l_1^\theta, a_1^{\theta^{-1}}, l_2^\theta\right) \\
(a_1,l_1,a_2,l_2)^\rho &= \left[a_1^{\theta^{-1}}, l_1^\theta, a_2^{\theta^{-1}}, l_2^\theta,\right] \\
[l_1,a_1,l_2,a_2]^\rho &= \left(l_1^\theta, a_1^{\theta^{-1}}, l_2^\theta, a_2^{\theta^{-1}}, l_3^\theta\right) \\
(a_1,l_1,a_2,l_2,a_3)^\rho &= \left[a_1^\theta, l_1^{\theta^{-1}}, a_2^\theta, l_2^{\theta^{-1}}, a_3^\theta\right] \\
[l_1,a_1,l_2,a_2,l_3]^\rho &= \left(l_1^{\theta^{-1}}, a_1^\theta, l_2^{\theta^{-1}}, a_2^\theta, l_3^{\theta^{-1}}\right).
\end{aligned}$$

The set of absolute points of ρ is the ovoid

$$\{\infty\} \cup \left\{\left(a_1, a_3^\theta - a_1^{\theta+3}, a_2, a_1^{2\theta+3} + a_2^\theta + a_1^\theta a_3^\theta, a_3\right) :\ a_1,a_2,a_3 \in \mathbb{F}_q\right\}.$$

This is the ovoid used in [Tit95].

$$\sigma(a,b,c)\colon \quad \begin{pmatrix} a_1 \\ l_1 \\ a_2 \\ l_2 \\ a_3 \end{pmatrix} \mapsto \begin{pmatrix} a_1 + a \\ l_1 + c^\theta - a^\theta a_1^3 - a^{\theta+3} \\ a_2 + b - c a_1 + a^\theta a_1 \\ l_2 + b^\theta + a^{2\theta} a_1^3 + a^\theta l + a^{2\theta+3} + a^\theta c^\theta \\ a_3 + c + a^\theta a_1 \end{pmatrix} \tag{4.1}$$

$$\begin{bmatrix} l_1 \\ a_1 \\ l_2 \\ a_2 \\ l_3 \end{bmatrix} \mapsto \begin{bmatrix} l_1 + a^\theta \\ a_1 + c - a l_1^3 - a^{\theta+1} \\ l_2 + b^\theta - c^\theta l_1 + a^3 a_1^2 \\ a_2 + b + a^2 l_1^3 + a a_2 + a^{2\theta+1} + ac \\ l_3 + c^\theta + a^3 l_1 \end{bmatrix} \tag{4.2}$$

The Ree-group $\mathbf{Ree}(q)$ now is the intersection of of $\mathbf{G_2}\,(\boldsymbol{q})$ (for $q \neq 3$, this is the little projective group of $\mathrm{H}(q)$) with the centralizer of ρ (in the full correlation group of $\mathrm{H}(q)$). The action of $\mathbf{Ree}(q)$ on $\mathcal{O}$ is doubly transitive. The stabilizer of ∞ is TS, with T being a maximal torus and the two-point-stabilizer of ∞ and $(0,0,0,0,0)$, and S being the Sylow 3-subgroup. S acts regularly on $\mathcal{O}\setminus\{\infty\}$; denoting the element mapping $(0,0,0,0,0)$ to $\left(a_1, a_3^\theta - a_1^{\theta+3}, a_2, a_1^{2\theta+3}+a_2^\theta+a_1^\theta a_3^\theta, a_3\right)$ by $\begin{pmatrix} a_1 \\ a_3 \\ a_2 - a_1a_3 \end{pmatrix}$ the group S has $\mathbb{F}_q^3$ as its underlying set and the following multiplication:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e+ad^\theta \\ c+f+ae-bd-ad^{\theta+1} \end{pmatrix}.$$

The action of $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in S$ on points and lines having five coordinates is given by

$$\begin{array}{rcl} (a_1,l_1,a_2,l_2,a_3) & \mapsto & \begin{pmatrix} a_1+a \\ l_1+b^\theta-a^\theta a_1^3-a^{\theta+3} \\ a_2+c+ab-ba_1+a^\theta a_1^2 \\ l_2+c^\theta-a^\theta b^\theta+a^{2\theta}a_1^3+a^\theta l_1+a^{2\theta+3} \\ a_3+b+a^\theta a_1 \end{pmatrix}^{\mathrm{T}} \\ {[l_1,a_1,l_2,a_2,l_3]} & \mapsto & \begin{bmatrix} l_1+a^\theta \\ a_1+b-al_1^3-a^{\theta+1} \\ l_2+c^\theta+a^\theta b^\theta-b^\theta l_1+a^3l_1^2 \\ a_2+c-ab+a^2l_1+aa_1+a^{\theta+2} \\ l_3+b^\theta+a^3l_1 \end{bmatrix}^{\mathrm{T}} \end{array} \qquad (\star)$$

The torus T consists of the mappings

$$h_t, t\in\mathbb{F}_q\setminus\{1\} \;:\; \begin{array}{rcl} (a_1,l_1,a_2,l_2,a_3) & \mapsto & \left(a_1t, l_1t^{\theta+3}, a_2^{\theta+2}, l_2^{2\theta+3}, a_3^{\theta+1}\right) \\ {[l_1,a_1,l_2,a_2,l_3]} & \mapsto & \left[l_1t^\theta, a_1t^{\theta+1}, l_2^{2\theta+3}, a_2^{\theta+2}, l_3^{\theta+3}\right] \end{array}$$

and acts on S via

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^{h_t} = \begin{pmatrix} at \\ bt^{\theta+1} \\ ct^{\theta+2} \end{pmatrix}.$$

Since the vertices (∞) and $[\infty]$ are fixed by TS, the set

$$M := \mathrm{D}_3\left((\infty)\right)\setminus \mathrm{D}_2\left([\infty]\right)$$

is TS-invariant. For any element $x \in M$ there is a unique element of $\mathcal{O}^\rho$ at distance 2 from x, therefore the action of S on M is regular. From this we can deduce that there are exactly six G-orbits on $\mathrm{H}(q)$ (three point orbits and three line orbits). A system of representatives for these orbits is $R_p \cup R_l$

$$\text{where } R_p := \{(\infty), (0), (0,0)\} \text{ and } R_l := \{[\infty], [0], [0,0]\}.$$

The group T fixes R_p and R_l point-wise. From ($\star$) it can be observed that the stabilizer of (0) in S is the group

$$S' := \left\{ \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}, \ b, c \in \mathbb{F}_q \right\},$$

this group is simultaneously the stabilizer of $[0]$. For $q \neq 3$, it is the commutator subgroup. The stabilizer of $(0,0)$ has to satisfy the conditions $a = 0$ and $b^\theta - a^{\theta+3} = 0$, so it is the group

$$Z := \left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}, \ c \in \mathbb{F}_q \right\},$$

which is simultaneously the stabilizer of $[0,0]$ and will be denoted by Z (it is the center of S).

Lemma 11. *1. The inverse of an element of S is*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^{-1} = \begin{pmatrix} -a \\ -b + a^{\theta+1} \\ -c \end{pmatrix}$$

2. The third power of an element of S

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^{3} = \begin{pmatrix} 0 \\ 0 \\ -a^{\theta+2} \end{pmatrix}$$

3. The commutator of two elements of S is

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] = \begin{pmatrix} 0 \\ ax^\theta - xa^\theta \\ xb - ay + ax^{\theta+1} - xa^{\theta+1} + x^2a^\theta - a^2x^\theta \end{pmatrix}$$

4. *The center of S is the subgroup*

$$Z = \left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \,\middle|\, c \in \mathbb{F}_q \right\}$$

5. *If $q \neq 3$, the commutator subgroup of S is the subgroup*

$$\left\{ \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \,\middle|\, b, c \in \mathbb{F}_q \right\}$$

For $q = 3$, the commutator subgroup is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \,\middle|\, c \in \mathbb{F}_q \right\}$$

In both cases, this group is abelian.

Proof: The first 4 statements follow from direct calculations. In order to see that the commutator subgroup for $q \neq 3$ is indeed as claimed, it remains to prove that $A := \left\{ ax^\theta - xa^\theta \,\middle|\, x, a \in \mathbb{F}_q \right\} = \mathbb{F}_q$. In order to see that, choose $y \in \mathbb{F}_q \setminus \mathbb{F}_3$ and let $a = \left(y + y^{-1}\right)^{-1} \left(y - y^{-1}\right)^\theta$ and $x = \left(y + y^{-1}\right)^\theta \left(y - y^{-1}\right)^{-1}$. Both terms are well defined because $\mathbb{F}_{3^{2k+1}}$ does not contain any square roots of -1, so $y + y^{-1} \neq 0$ and $y \notin \mathbb{F}_3$ implies $y - y^{-1} \neq 0$. With these choices of a and x, we get $ax^\theta - xa^\theta = 1$. It follows that $\mathbb{F}_3 \subset A$. For any $y \in \mathbb{F}_q \setminus \mathbb{F}_3$, let now $a = (y+1)^{-1} (y-1)^\theta$ and $x = (y+1)^\theta (y+1)^{-1}$. Thus $ax^\theta - xa^\theta = y$.

□

Lemma 12. *For $q \neq 3$, there is only one T-invariant subgroup of S of order q^2, this group is S'. Furthermore, if a T-invariant subgroup U contains an element $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a \neq 0$, then $U = S$. The only T-invariant subgroups of S of order q are*

$$B := \left\{ \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix},\ b \in \mathbb{F}_q \right\} \text{ and } Z := \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix},\ z \in \mathbb{F}_q \right\}.$$

Proof: Suppose that H is a T-invariant subgroup containing an element

$$g = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \; a \neq 0\,.$$

Calculating g^3, we get $g^3 = \begin{pmatrix} 0 \\ 0 \\ -a^{\theta+2} \end{pmatrix}$. Applying the action of T yields

$$H \supseteq \left\{ \left(g^3\right)^{h_t} \,\middle|\, t \in \mathbb{F}_q* \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ -t^{\theta+2} a^{\theta+2} \end{pmatrix} \,\middle|\, t \in \mathbb{F}_q* \right\},$$

The map $\theta + 2$ is surjective, so H contains the center of S. We choose $t \in \mathbb{F}_q$ with $t^\theta \neq t$ (here we need $q \neq 3$) and calculate the commutator

$$\begin{aligned} \left[g, g^{h_t}\right] &= \begin{pmatrix} 0 \\ a\,(ta)^\theta - (ta)\,a^\theta \\ (ta)\,b - a\left(t^{\theta+1}b\right) + a\,(ta)^{\theta+1} - (ta)\,a^{\theta+1} + (ta)^2\,a^\theta - a^2\,(ta)^\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ a^{\theta+1}\left(t^\theta - t\right) \\ (ta)\,b - a\left(t^{\theta+1}b\right) + a\,(ta)^{\theta+1} - (ta)\,a^{\theta+1} + (ta)^2\,a^\theta - a^2\,(ta)^\theta \end{pmatrix}. \end{aligned}$$

Since H also contains the center of S, the group H contains an element $\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$ with $x \neq 0$ and, consequently, the set

$$X := \left\{ \begin{pmatrix} 0 \\ t^{\theta+1} x \\ 0 \end{pmatrix} \,\middle|\, t \in \mathbb{F}_q* \right\}.$$

The subset of $\mathbb{F}_q$ additively generated by the images of $\theta + 1$ contains all squares because

$$\forall f \in \mathbb{F}_q\colon\; \left(f^{(\theta-1)}\right)^{\theta+1} = f^{\theta^2-1} = f^2\,.$$

The subset additively generated by all squares is $\mathbb{F}_q$ because it contains the set

$$\left\{(f+1)^2 - (f-1)^2 \,\middle|\, f \in \mathbb{F}_q\right\} = \left\{4f \,\middle|\, f \in \mathbb{F}_q\right\}.$$

Consequently, H contains the subgroup $\begin{pmatrix} 0 \\ \mathbb{F}_q \\ 0 \end{pmatrix}$, therefore H contains S'.

As we have seen above, no T-invariant subgroup of order q contains an element $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a \neq 0$, so all subgroups of interest are contained in S'.

Assume that H is a T-invariant subgroup containing $g = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$, $b \neq 0$.

Because

$$g\, g^{h_{-1}} = \begin{pmatrix} 0 \\ 2b \\ 0 \end{pmatrix}$$

we observe that $B \subset H$.

□

We define the following geometry $\mathcal{H}' = (P, L, I)$:

$$\begin{array}{rcl} P & := & G/_{TS} \cup G/_{TS'} \cup G/_{TZ} \\ L & := & G/_{TS} \cup G/_{TS'} \cup G/_{TZ} \end{array}$$

where incidence is defined as follows

$$\begin{array}{rcl} TSg \,\mathrm{I}\, TSh & \Leftrightarrow & gh^{-1} \in TS \\ TSg \,\mathrm{I}\, TS'h & \Leftrightarrow & gh^{-1} \in TS' \\ TS'g \,\mathrm{I}\, TSh & \Leftrightarrow & gh^{-1} \in TS' \\ TS'g \,\mathrm{I}\, TZh & \Leftrightarrow & gh^{-1} \in TZ \\ TZg \,\mathrm{I}\, TS'h & \Leftrightarrow & gh^{-1} \in TZ \\ TZg \,\mathrm{I}\, TZh & \Leftrightarrow & gh^{-1} \in TZ\omega Z \end{array}$$

and ω is an (arbitrary) involution interchanging (∞) and $(0,0,0,0,0)$. As in the case of quadrangles, the definition of incidence does not depend on the choice of ω because any such involution is contained in the coset ωT.

Lemma 13. *The geometries $\mathcal{H}'$ and $\mathrm{H}(q)$ are isomorphic.*

Proof: The isomorphism is provided by

$$\begin{aligned} TZg &\mapsto (0,0)^g \\ TS'g &\mapsto (0)^g \\ TSg &\mapsto (\infty)^g \\ TZg &\mapsto [0,0]^g \\ TS'g &\mapsto [0]^g \\ TSg &\mapsto [\infty]^g \end{aligned}$$

Let

$$\begin{array}{llll} P_1' := & G/_{TZ} & P_1 := & (0,0)^G \\ P_2' := & G/_{TS} & P_2 := & (\infty)^G \\ P_3' := & G/_{TS'} & P_3 := & (0)^G \\ L_1' = & G/_{TS'} & L_1 := & [0]^G \\ L_2' = & G/_{TS} & L_2 := & [\infty]^G \\ L_3' = & G/_{TZ} & L_3 := & [0,0]^G \end{array}$$

From [Str92] we can deduce that the subgeometries

$$\begin{aligned} \mathcal{S}_1 &= \left(P_1 \cup P_2, L_1, \mathrm{I}'|_{(P_1\cup P_2)\times L_1}\right), \\ \mathcal{S}_2 &= \left(P_2, L_1 \cup L_2, \mathrm{I}'|_{P_2\times (L_1\cup L_2)}\right), \\ \mathcal{S}_3 &= \left(P_2 \cup P_3, L_1, \mathrm{I}'|_{(P_2\cup P_3)\times L_1}\right) \text{ and} \\ \mathcal{S}_4 &= \left(P_3, L_2 \cup L_3, \mathrm{I}'|_{P_2\times (L_2\cup L_3)}\right) \end{aligned}$$

of $\mathcal{H}$ are isomorphic to the corresponding subgeometries of $\mathrm{H}(q)$ (i.e. the subgeometries with the same indices). It remains to be shown that $TZg\,\mathrm{I}'\,TZh$ if and only if $TZg^\Phi\,\mathrm{I}\,TZh^\Psi$. To this end, let $(TZg, TZh) \in P' \times L'$. The images $TZg^\Phi = (0)^g$ and $TZh^\Psi = (0)^h$ are incident if and only if

$$\begin{aligned} (0,0)^g\,\mathrm{I}\,[0,0]^h &\Leftrightarrow (0,0)^{gh^{-1}}\,\mathrm{I}\,[0,0] \\ &\Leftrightarrow \exists z \in Z\colon\ (0,0)^{gh^{-1}z} = [0,0,0] \\ &\Leftrightarrow \exists z \in Z\colon\ (0,0)^{gh^{-1}z\omega} = [0,0] \\ &\Leftrightarrow \exists z \in Z\colon gh^{-1}z\omega \in TZ \\ &\Leftrightarrow gh^{-1} \in TZ\omega Z \\ &\Leftrightarrow TZg\,\mathrm{I}'\,TZh \end{aligned}$$

$\square$

4.2 Characterization of $\mathrm{H}(3^{2k+1})$

Consider a generalized hexagon $\mathcal{H}$ having an ovoid-spread-pairing $(\mathcal{O}, \mathcal{S})$. By Offer's Theorem ([Off02]), the number of points on each line and the number of lines through each point are equal. Assume further that $\mathrm{Aut}_{(\mathcal{O},\mathcal{S})}(\mathcal{H})$ contains a group G isomorphic to $\mathbf{Ree}(q)$, the Ree-group over the field $\mathbb{F}_q$. We will show that $\mathcal{H}$ is isomorphic to the split Cayley hexagon of order q.

Theorem 19. *Let $\mathcal{H}$ be a generalized hexagon containing an ovoid-spread-pairing $(\mathcal{O}, \mathcal{S})$ and let G be a subgroup of $\mathrm{Aut}_{(\mathcal{O},\mathcal{S})}(\mathcal{H})$ isomorphic to $\mathbf{Ree}(q)$. Then $\mathcal{H}$ is isomorphic to $\mathrm{H}(q)$.*

To begin with, we show that the action of G on $\mathcal{O}$ is transitive.

Lemma 14. *The Sylow 3-subgroup S of G has a fixed point ∞.*

Proof: The existence of a point fixed by S follows from the fact that $\mathcal{O}$ has $q^3 + 1$ points and S is a 3-group.

□

Recall the definition of S' from page 42 (for $q = 3$, it is not the commutator subgroup).

Lemma 15. *If S is not transitive on $\mathcal{O} \setminus \{\infty\}$, then $S_x \cap S' \neq \{1\}$ holds for each $x \in \mathcal{O} \setminus \{\infty\}$.*

Proof: As any orbit has less than q^3 elements, $\forall x \in \mathcal{O} \setminus \{\infty\} : \ S_x \neq \{1\}$ holds. Using $\forall g \in S \setminus \{1\} \ : \ g \notin S' \Rightarrow g^3 \in S' \setminus \{1\}$ we find

$$\forall x \in \mathcal{O} \setminus \{\infty\} : \ S_x \cap S' \neq \{1\}.$$

□

Lemma 16. *If there exist $g \in S$ such that g fixes more than $q + 1$ elements of $\mathcal{O}$, then the subgeometry $\mathcal{H}'$ of $\mathcal{H}$ consisting of points and lines fixed by g is a subhexagon.*

Proof: Let $g \in S$ such that g fixes a set A containing more than $q + 1$ elements of $\mathcal{O}$. The subgeometry $\mathcal{H}'$ of $\mathcal{H}$ consisting of points and lines fixed by g contains the point ∞, the set A and at least one line incident with each of the points of A. For each $y \in A$, let

$$(\infty, l_y, p_y, m_y, q_y, n_y, y)$$

be a path joining y and ∞ such that l_y is fixed by g. Then all elements of the path are fixed. Since there are more than $q+1$ elements in the set A, there exist elements $y, z \in A$ such that $m_y \neq m_y$. Therefore, $\mathcal{H}'$ contains a path of length at least 8 and an apartment. By [VM98a], Theorem 4.4.2, $\mathcal{H}'$ is a weak subhexagon. It is thick because the number of fixed vertices adjacent to some fixed vertex must be congruent to 1 modulo 3.

□

Lemma 17. *Any non–trivial element g of S fixes less than $q\sqrt{q}$ elements of $\mathcal{O} \setminus \{\infty\}$.*

Proof: Assume, to the contrary, that there exists $g \in S$ which fixes each point in a subset $A \subset \mathcal{O} \setminus \{\infty\}$ with $|A| \geq q\sqrt{q}$. By Lemma 16, the element g fixes a subhexagon $\mathcal{H}'$ containing A. Let (r, s) denote the order of $\mathcal{H}'$. The intersection of A and $\mathcal{H}'$ consists of points of $\mathcal{H}'$ which are mutually opposite and for each point x of $\mathcal{H}'$ not contained in $\mathcal{O}$, the unique point of $\mathcal{O}$ at distance 2 from x is also contained in A, because $\mathcal{O}$ is invariant under g. So $\mathcal{O} \cap \mathcal{H}'$ is an ovoid of $\mathcal{H}'$ and we necessarily have $r = s$ by Theorem 7. This implies that $|\mathcal{O} \cap \mathcal{H}'| = r^3 + 1$ and we get $|A| \leq r^3$. If $r > \sqrt{q}$, a result due to Thas ([Tha76]) implies that $\mathcal{H}' = \mathcal{H}$, and g is in the kernel of the action, which is impossible. The fact that $r = \sqrt{q}$ is impossible because q is not a square completes the proof of this lemma.

□

Lemma 18. *If S is not transitive on $\mathcal{O} \setminus \{\infty\}$, then $|S_x \cap S'| \geq \sqrt{3q}$ holds for each $x \in \mathcal{O} \setminus \{\infty\}$.*

Proof: Assume, by means of contradiction, that there is an element $x \in \mathcal{O} \setminus \{\infty\}$ such that $|S_x \cap S'| < \sqrt{q}$. It follows that

$$\left| x^{S'} \right| = \frac{|S'|}{|S' \cap S_x|} > \frac{q^2}{\sqrt{q}} = q\sqrt{q}.$$

By Lemma 15, there is a nontrivial element of S' fixing x. Since the group S' is abelian, it fixes the orbit $x^{S'}$ pointwise. This is impossible by Lemma 17. Since q is an odd power of 3, the statement follows.

□

Lemma 19. *S acts transitively on $\mathcal{O} \setminus \{\infty\}$.*

Proof: Assuming that the action of S is not transitive, we define the map

$$\Phi \,:\, S' \setminus \{1\} \to \mathcal{P}\left(\mathcal{O} \setminus \{\infty\}\right) \,:\, g \mapsto \{x \in \mathcal{O} \setminus \{\infty\} \mid x^g = x\}$$

and the set

$$M := \{(g,x) \in (S' \setminus \{1\}) \times (\mathcal{O} \setminus \{\infty\}) \,|x^g = x\}\,.$$

Counting the elements of M in two different ways, we obtain

$$\sum_{g \in S' \setminus \{1\}} |\Phi(g)| = |M| = \sum_{x \in \mathcal{O} \setminus \{\infty\}} |S_x \cap S' \setminus \{1\}|\,.$$

Since Lemma 18 implies $\forall x \in \mathcal{O} \setminus \{\infty\} : |S_x \cap S'| \geq \sqrt{3q}$, we get

$$(q^2 - 1) \max_{g \in S' \setminus \{1\}} |\Phi(g)| \geq q^3(\sqrt{3q} - 1),$$

which leads to $\max\limits_{g \in S' \setminus \{1\}} |\Phi(g)| \geq q\sqrt{q}$ contradicting Lemma 17. The lemma is proved.

□

Corollary 2. *G acts transitively on $\mathcal{S}$.*

It should be remarked that the result of the preceding lemma can also be obtained by using the classification of maximal subgroups of $\mathbf{Ree}(q)$ ([Asc87], [LN85] or [Kle88]). For $q > 3$, the maximal subgroups are isomorphic to one of the following:

Subgroup	Structure	Order
U_1	$[q^3] : Z_{q-1}$	$(q-1)\,q^3$
U_2	$2 \times L_2(q)$	$q(q-1)$
U_3	$\left(2^2 \times D_{\frac{q+1}{2}}\right) : 3$	$6(q+1)$
U_4	$Z_{q+\sqrt{3q}+1} : Z_6$	$6(q+\sqrt{3q}+1)$
U_5	$Z_{q-\sqrt{3q}+1} : Z_6$	$6(q-\sqrt{3q}+1)$
U_6	$\mathbf{Ree}\left(q^{\frac{1}{\alpha}}\right)$	$q^{\frac{3}{\alpha}}\left(q^{\frac{3}{\alpha}}+1\right)\left(q^{\frac{1}{\alpha}}-1\right)$

The groups U_1-U_6 can be described geometrically: U_1 is the stabilizer of a point, U_2 is the stabilizer of a block of the Ree-Unital, U_3, U_4 and U_6 are stabilizers of intersections of the polar unital with a classical unital (the unitals have either $q+\sqrt{3q}+1$, $q-\sqrt{3q}+1$ or $q+1$ points in common), U_6 is the stabilizer of a subunital (defined over the field with $q^{\frac{1}{\alpha}}$ elements).

The stabilizer of a point is contained in one of these maximal subgroups and the length of the orbit is larger than or equal to the index of this maximal subgroup in $\mathbf{Ree}(q)$. The order of any other maximal subgroup is strictly smaller than the order of the Sylow 3-normalizer which has index q^3+1. This implies that the action is transitive and the stabilizer of a point is isomorphic to the Sylow 3-normalizer.

The normalizer of S in G is TS where T is a maximal torus having order $q-1$. Because ∞ is the only fixed point of S, the point ∞ is also fixed by TS and TS acts on $\mathcal{O} \setminus \{\infty\}$. The stabilizer of a point $z \in \mathcal{O} \setminus \{\infty\}$ is a group of order $q-1$, therefore $(TS)_z$ is a Hall $3'$-subgroup of TS and by the Schur-Zassenhaus Theorem T and $(TS)_x$ are conjugate. Hence, there exists $n \in \mathcal{O} \setminus \{\infty\}$ satisfying $(TS)_n = T$.

Lemma 20. *S acts transitively on* $\mathrm{D}_1(\infty) \setminus \mathcal{S}$.

Proof: Let l_∞ be the line contained in $\mathcal{S}$ adjacent to ∞. Each line of the set $\mathrm{D}_3(\infty) \setminus \mathrm{D}_2(l_\infty)$ is at distance 2 from a unique element of $\mathcal{S} \setminus \{l_\infty\}$ and, consequently, S acts transitively on this set. Therefore, S acts transitively on $D_1(\infty) \setminus \mathcal{S}$.

$\square$

Corollary 3. *T fixes l_∞.*

Each point of the set $\mathcal{M} := D_3(l_\infty) \setminus D_2(\infty)$ is at distance 2 from exactly one point of $\mathcal{O} \setminus \{\infty\}$. The action of TS on $\mathcal{O} \setminus \{\infty\}$ therefore induces an action of TS on the set $\mathcal{M}$ and the stabilizer of any point $m \in \mathcal{M}$ and T are conjugate. Let

$$P := (\infty, l_\infty, p_1, l_1, p_2, l_2, n)$$

be the (unique) path connecting l and n and let

$$A := (\infty, l, p_1, l_1, p_2, l_2, n, l_n, p_3, l_3, p_4, l_4)$$

be the apartment containing P and the line of $\mathcal{S}$ incident with n. Since S acts transitively on $D_3(\infty) \setminus D_2(l_\infty)$ and $D_3(l_\infty) \setminus D_2(\infty)$, we have $(TS)_{p_2} = (TS)_{l_3} = T$ and, consequently, $(TS)_{p_1} \supset T$ as well as $(TS)_{l_1} \supset T$. Furthermore, the stabilizers of p_1 and l_1 are T-invariant.

Lemma 21. *The stabilizer of p_1 in S is S' and $(TS)_{p_1} = TS'$.*

Proof: The stabilizer of p_1 in S has order q^2, because there are q^2 points of $\mathcal{M}$ at distance 2 from p_1 and the action of S on $\mathcal{M}$ is regular. By Lemma 12, The only subgroup of S of order q^2 which is invariant under conjugation by T is S', hence $S_{p_1} = S'$ and $(TS)_{p_1} = TS'$.

$\square$

Lemma 22. *The stabilizer of l_1 in S is Z and $(TS)_{l_1} = TZ$.*

Proof: The stabilizer of l_1 is a T-invariant subgroup of TS containing q elements of S. By Lemma 12, $(S)_{l_1} \in \{B, Z\}$. Furthermore, we also have $(S)_{p_4} \in \{B, Z\}$. Assume $(TS)_{l_1} = (TS)_{p_4} = B$. Since h_{-1} is in the kernel of the action of T on B, every element of B commutes with h_{-1}. Let $\mathcal{H}'$ be the subgeometry fixed by h_{-1}. It contains the apartment A. Since $\mathcal{H}'$ contains an apartment, by [VM98a], Theorem 4.4.2, $\mathcal{H}'$ is a weak subhexagon. Since

$$\forall b \in B\colon\ \left(p_2^b\right)^{h_{-1}} = \left(p_2^{h_{-1}}\right)^b = p_2^b\,,$$

and

$$\forall b \in B\colon\ \left(l_3^b\right)^{h_{-1}} = \left(l_3^{h_{-1}}\right)^b = l_3^b\,,$$

all points incident with l_1 and all lines incident with p_4 are contained in $\mathcal{H}'$. Since $d(l_1, p_4) = 5$, [VM98a], Theorem 4.4.2 implies that h_{-1} is in the kernel of the action.

Assume now that $(S)_{p_4} = Z$ and $(S)_{l_1} = B$. As above, the subgeometry fixed by h_{-1} contains the apartment A and all lines incident with l_1. The group B acts transitively on on the points incident with l_4 distinct from ∞, so h_{-1} fixes $\mathrm{D}_1(l_4)$ pointwise. By [VM98a], Theorem 4.4.2, subgeometry fixed by h_{-1} is a full weak subpolygon. But it also contains $q+1$ opposite lines, because h_{-1} must fix all the elements of the ovoid at distance 2 from the B-orbit of p_2 and the lines of the spread incident with them. This is impossible by Lemma 23.

It remains to treat the case $(S)_{p_4} = B$ and $(S)_{l_1} = Z$. This is the dual situation to the case treated before. Here, h_{-1} fixes a weak subhexagon containing $\mathrm{D}_1(p_4)$ and $\mathrm{D}_1(p_1)$ as well as $q+1$ points of the ovoid. This is impossible by Lemma 23.

□

Lemma 23. *Let Γ be a weak generalized hexagon having parameters $(1, q)$ and let (x, y) be a pair of opposite points. Then any third point z is contained in $\mathrm{D}_4(x) \cup \mathrm{D}_4(y)$ and this union is disjoint.*

Dually, let Γ be a weak generalized hexagon having parameters $(q, 1)$ and let (l, m) be a pair of opposite lines. Then any third line n is contained in $\mathrm{D}_4(x) \cup \mathrm{D}_4(y)$ and this union is disjoint.

Proof: We show that the union is disjoint first. Assume, to the contrary, that there is a point $z \in \mathrm{D}_4(x) \cap \mathrm{D}_4(y)$. Let (z, l_1, p, l_2, y) be the path joining z and y. Since $|\mathrm{D}_1(l_2)| = 2$, the point p is at distance 4 from x and we have constructed a pentagon.

Now, because all the sets are disjoint,

$$|\{x\} \cup \{y\} \cup \mathrm{D}_4(x) \cup \mathrm{D}_4(y)| = 1+1+t(t+1)+t(t+1) = 2\left(1+t+t^2\right),$$

which is the total number of points of Γ (see e.g. [VM98a] Corollary 1.5.5).

□

As all the pieces occurring in the reconstruction of $H(q)$ from the action of $\mathbf{Ree}(q)$ coincide, we can deduce the following lemma:

Lemma 24. *$\mathcal{H}$ and $\mathcal{H}'$ are isomorphic.*

Proof: The isomorphism is provided by

$$\begin{aligned} TZg &\mapsto p_2^{\omega g} && (4.3) \\ TS'g &\mapsto p_1^{g} && (4.4) \\ TSg &\mapsto \infty^{g} && (4.5) \\ TZg &\mapsto l_1^{g} && (4.6) \\ TS'g &\mapsto l_2^{\omega g} && (4.7) \\ TSg &\mapsto l^{g} && (4.8) \\ & && (4.9) \end{aligned}$$

□

Corollary 4. $H(q)$ *and $\mathcal{H}$ are isomorphic.*

Chapter 5

Ovoids having doubly transitive groups

In this part, we will consider a finite generalized quadrangle or a finite generalized hexagon containing an ovoid which admits a group of automorphisms of the polygon leaving the ovoid invariant and acting doubly transitively on the ovoid. We will study the following questions: What are the possible isomorphism types for this group? What can be said about the isomorphism class of the quadrangle or hexagon?

5.1 Generalized hexagons

It will be shown that a group acting doubly transitively on an ovoid of a generalized hexagon is a unitary group or a Ree group. The result is summarized in the following theorem. The rest of this section will give a proof of the result.

Theorem 20. *Let Γ be a generalized hexagon containing an ovoid $\mathcal{O}$. Let A be a group of automorphisms of Γ acting doubly transitively on $\mathcal{O}$. Then, the group A contains a subgroup isomorphic to* $\mathbf{PSU}\,(3,q)$ *or A contains a subgroup isomorphic to* $\mathbf{Ree}(q)$, *$q = 3^{2k+1}$for some $q \in \mathbb{N}$.*

Contrasting the first part of this work, where the use of classification results (e.g. the classification of maximal subgroups of of Ree groups) was not necessary, we will use the classification of finite simple groups (via the classification of doubly transitive groups).

We use the following well known theorem:

Theorem 21 (Burnside, 1911, Theorem 4.1B in [DM96])**.** *The socle of a finite doubly transitive group is either a regular elementary abelian p-group, or a nonregular nonabelian simple group.*

In the first case, we say that the group is of affine type. In the second case, we say that the group is of almost simple type. Let G be the socle of A. In a first step, it will be shown that the group G can not be elementary abelian. This will be done in Lemma 25 using an elementary arithmetic argument. It follows that G must be a nonabelian simple group. The nonabelian finite simple groups having a faithful doubly transitive permutation representation have been classified in [CKS76] and [Fei80]. A table of these groups is included in the appendix. In a second step, we will go through the list of almost simple doubly transitive groups and investigate the isomorphism types of groups case by case until only the known possibilities, namely $\mathbf{PSU}\,(3,q)$ and $\mathbf{Ree}\,(q)$ remain.

By Theorem 7, the number of points incident with a line and the number of lines incident with a point coincide and Γ has order (a,a). For $a=2$, there are exactly two isomorphism classes of generalized hexagons, the split Cayley hexagon over $\mathbb{F}_2$ and its dual ([CT85]). For the determination of the isomorphism class of Γ, it is therefore sufficient to restrict our attention to $a>2$.

The following lemma shows that the group can not be of affine type.

Lemma 25. *If, for a natural number $a>0$, the number a^3+1 is a prime power, then $a=2$.*

Proof: Assume that $a^3+1=p^k$ for some prime p. The number a^3+1 can be decomposed into $(a+1)(a^2-a+1)$, so $a+1=p^r$ and $a^2-a+1=p^s$ with $r\leq s$. This implies that a^2-a+1 is divisible by $a+1$. Carrying out the division, we get

$$\frac{a^2-a+1}{a+1}=a-2+\frac{3}{a+1}.$$

This is a natural number if and only if $a+1=3$.

□

Knowing that G is of almost simple type and excluding all groups whose degree is not of the form q^3+1, we are left with the following possibilities:

- $\mathbf{PSL}\,(d,q)$
- $\mathbf{Sp}\,(2d,2)$
- $\mathbf{Sz}(q)$
- $\mathbf{PSU}\,(3,q)$
- $\mathbf{Ree}(q)$
- $\mathbf{A}_{\boldsymbol{n}}$

Now we need two technical lemmas which will be used in the analysis of the almost simple cases.

Lemma 26. *Let x, y be two distinct points of $\mathcal{O}$. There exists a non-empty set of at most a^2-1 points of $\mathcal{O}$ which is invariant under the the stabilizer of x and y inside the group* $\mathrm{Aut}_{\mathcal{O}}(\Gamma)$*.*

Proof: Let H be the stabilizer of two arbitrary distinct points $x,\, y \in \mathcal{O}$. The set $\mathcal{T} := \mathrm{D}_3\,(x) \cap \mathrm{D}_3\,(y)$ consists of $a+1$ elements and is H-invariant. Since for any vertex of the (H-invariant) set

$$\mathcal{M} := \mathrm{D}_1\,(\mathcal{T}) \setminus (\mathrm{D}_2\,(x) \cup \mathrm{D}_2\,(y))$$

there exists a unique element of $\mathcal{O}$ at distance 2 and since

$$|\mathcal{M}| = |\mathcal{T}|\,(a-1)\,,$$

the set $\mathrm{D}_2\,(\mathcal{M}) \cap \mathcal{O}$ is an H-invariant set of size at most a^2-1.

□

Lemma 27. *If a subgroup $H \leq \mathrm{Aut}_{\mathcal{O}}(\Gamma)$ fixes a flag (x, l) with $x \in \mathcal{O}$, then there exists an H-invariant equivalence relation R on $\mathcal{O}\backslash\{x\}$ with equivalence classes of size a.*

Proof: Consider the set

$$\mathcal{M} := \mathrm{D}_3(l) \setminus \mathrm{D}_2(x)$$

of points at distance 4 from x whose projection onto x is l. Let ψ_l be the mapping which maps any point y of $\mathcal{M}$ to the (unique) point of $\mathcal{O} \setminus \{x\}$ at distance 2 from y. Since any element of $\mathcal{O} \setminus \{x\}$ is at distance 5 from l, the mapping ψ is surjective, therefore bijective. Now we define an equivalence relation R on $\mathcal{O} \setminus \{x\}$ via

$$R \;:=\; \left\{(u, v) \in (\mathcal{O} \setminus \{x\}) \times (\mathcal{O} \setminus \{x\}) \,\middle|\, d\left(\psi^{-1}(u), \psi^{-1}(v)\right) \leq 2\right\}.$$

Any automorphism of Γ which fixes the flag (x, l) and leaves $\mathcal{O} \setminus \{x\}$ invariant preserves R-equivalence.

□

We will now start to look at the possibilities for the isomorphism classes of G case by case.

Lemma 28. *The socle of the doubly transitive group is not isomorphic to* $\mathbf{PSL}(d, q)$.

Proof: Assume, to the contrary, that G contains a subgroup isomorphic to $\mathbf{PSL}(d, q)$. There are two distinct doubly transitive actions of $\mathbf{PSL}(d, q)$ (see [CKS76] for a proof), namely the action of $\mathbf{PSL}(d, q)$ on the points and the action on the hyperplanes of the corresponding projective space. However, these actions are quasi-equivalent. The degree of a doubly transitive action of $\mathbf{PSL}(d, q)$ is $\dfrac{q^d - 1}{q - 1}$, this means we have $a^3 + 1 = \dfrac{q^d - 1}{q - 1}$ for some prime power $q = p^e$. We get

$$a^3 + 1 = \sum_{j=0}^{d-1} q^j\,,$$

this gives us

$$a^3 = q\left(\sum_{j=0}^{d-2} q^j\right) \equiv 1 \bmod q\,.$$

From this equation, we can observe the following facts:

- q is a divisor of a^3
- q is not a divisor of a
- p is a divisor of a.

We treat the cases $d = 2$, $d = 3$ and $d > 3$ separately.

1. Let $d = 2$.
   Since we need
   $$a^3 + 1 = \frac{q^2 - 1}{q - 1} = q + 1\,,$$
   we have $q = a^3$. The action of the stabilizer of two points $x, y \in \mathcal{O}$ on $\mathcal{O} \setminus \{x, y\}$ has one orbit if $\mathbb{F}_q$ is perfect and two orbits of length $\frac{q-1}{2}$ otherwise. Lemma 26 ensures the existence of an invariant set of size
   $$a^2 - 1 < \frac{a^3 - 1}{2} = \frac{q - 1}{2}\,.$$
   This is a contradiction.
2. Let $d = 3$.
   As above, $q = p^e$ for some prime p. We have $a^3 + 1 = \dfrac{q^3 - 1}{q - 1} = q^2 + q + 1$ and, consequently, $a^3 = q(q + 1)$. Since the numbers q and $q + 1$ are relatively prime, both q and $q + 1$ are cubes. This is impossible.
3. Let $d > 3$.
   Here, $q < a$. Let S_p be a Sylow p-subgroup of G. Since p divides a, the p-group S_p fixes (at least) one point $x \in \mathcal{O}$. The number of lines of Γ incident with x is $a + 1$, so there exists (at least) one line l which is fixed by S_p. We are now able to apply Lemma 27 and we obtain an H-invariant equivalence relation R on $\mathcal{O} \setminus \{x\}$ whose equivalence classes each consist of a elements of $\mathcal{O} \setminus \{x\}$. Let H be a two point stabilizer in S_p:
   $$H := (S_p)_{x,y} = \left\{ g \in S_p \,\middle|\, y^g = y \right\}\,.$$
   The orbits of H have the following sizes:
   $$1, 1, \ldots, 1, q, q^2, \ldots, q^{n-2}, q^{n-1}\,,$$
   where the number of orbits of size one is q. Since y is fixed, the equivalence class C_1 containing y is H-invariant, so it is a union of

H-orbits. If all the orbits of size one were contained in C_1, the size a of C_1 would be a sum

$$a = q + \sum_{n=0}^{m} q^{e_n}$$

where all exponents e_n are at least one. Thus a would be divisible by q. This is impossible. This yields the existence of a point z which is fixed by H and which is not contained in C_1. The equivalence classes C_1 containing y and C_2 containing z are both H-invariant and therefore are unions of H-orbits. Since $a > q$, both C_1 and C_2 contain nontrivial orbits. Let M denote the maximal length of a H-orbit contained in $C_1 \cup C_2$, so we have $M = q^k$ for some $k \geq 2$. We assume, without loss of generality, that the orbit of length M is contained in C_1. Since all the lengths of nontrivial orbits are distinct, this yields

$$a = |C_1| \geq M = q^k > q + \sum_{j=2}^{k-1} q^j \geq |C_2| = a$$

and we have reached a contradiction.

□

Lemma 29. *The socle of the doubly transitive group can not be isomorphic to* $\mathbf{Sz}(q)$.

Proof: Assume, to the contrary, that G contains a doubly transitive subgroup isomorphic to $\mathbf{Sz}(q)$. Then $a^3 + 1 = q^2 + 1$ for some $q = 2^{2k+1}$. Thus a is a power of 2, more precisely $a = q^{\frac{2}{3}}$. Let S_2 be the Sylow 2-subgroup of G. It fixes one point x of $\mathcal{O}$ and acts regularly on the complement $\mathcal{O} \setminus \{x\}$. Let T be the stabilizer of x and an additional point $y \in \mathcal{O}$ (so T is a maximal torus normalizing S_2). Let l be a line incident with x which is fixed by S_2. Such a line exists because the number of lines incident with x is odd and S_2 is a 2-group. Let T_l be the stabilizer of l in T. This group T_l also fixes all elements of the T-orbit of l because T is abelian. Let k be the length of the T-orbit of l. Neither $a+1$ nor a is a divisor of $q-1 = |T|$, therefore $k \leq a-1$. We obtain

$$|T_l| = \frac{|T|}{k} = \frac{q-1}{k} \geq \frac{q-1}{a-1} > q^{\frac{1}{3}}.$$

Let z be the unique point incident with l at distance 4 from y and let U be the stabilizer of z in S_2. Since S_2 acts regularly on $\mathcal{O} \setminus \{x\}$ and, consequently, on $\mathrm{D}_3(l) \setminus \mathrm{D}_2(x)$, the order of the group U is a^2. Let $\pi : S_2 \to S_2/S_2'$ be the canonical projection map and let $V := U^\pi$. Then U is isomorphic to a subgroup of the additive group of $\mathbb{F}_q$. The group T_l fixes the path connecting l and y, so its action by conjugation on S_2 leaves U invariant and induces an action of T_l on V. Let N be the normalizer of V in T. Clearly, we have $T_l \leq N$ and , for any $g, h \in N$,

$$V(g+h) = Vg + Vh = V + V = V\,,$$

so the subring of $\mathrm{End}(V)$ generated by N is equal to N. It follows that N is a subfield $\mathbb{F}$ of $\mathbb{F}_q$. Since

$$|N| \geq |T_l| > q^{\frac{1}{3}}\,,$$

the degree of the field extension $\mathbb{F} \colon \mathbb{F}_q$ is at most 2. Since q is an odd power of 2, the degree can not be two and we obtain $N = T$. The T-invariant subgroups of S_2 are $\{1\}, S_2'$ and S_2. None of these groups has order $a^2 = q^{\frac{4}{3}}$ and we have reached a contradiction.

□

Putting the pieces together, we obtain the following result.

Lemma 30. *The socle of the doubly transitive group is isomorphic to* $\mathbf{PSU}(3,q)$ *or* $\mathbf{Ree}(q)$.

Proof: By Burnside's theorem (cf [Cam99]) and Lemma 25, G is a non-abelian simple group. According to the classification of almost simple two-transitive groups (see e.g [Cam99], Table 7.4), the possible groups are:

1. $\mathbf{PSL}(d,q)$
2. $\mathbf{Sp}(2d,2)$, $d \geq 3$
3. $\mathbf{Sz}(q)$
4. $\mathbf{A}_n$
5. $\mathbf{PSU}(3,q)$
6. $\mathbf{Ree}(q)$

Any other group in the list does not meet the requirement that the degree is of the form $a^3 + 1$. We go through the possibilities case by case.

1. **PSL** (d, q): This is impossible by Lemma 28.
2. **Sp** $(2d, 2)$: The possible degrees of this group are $2^{2d-1} + 2^{d-1}$ and $2^{2d-1} - 2^{d-1}$. Let

$$a^3 + 1 = (a+1)(a^2 - a + 1) = 2^{2d-1} \pm 2^{d-1} = 2^{d-1}(2^d \pm 1)\,.$$

   Since $a^2 - a + 1$ is odd for any $a \in \mathbb{N}$, we can deduce that 2^{d-1} is a divisor of $a + 1$. Therefore, necessarily $a + 1 \geq 2^{d-1}$ and

$$a^3+1 \geq \left(2^{d-1} - 1\right)^3+1 = 2^{3d-3}-3\cdot 2^{2d-2}+3\cdot 2^{d-1} > 2^{2d-2}\left(2^{d-1} - 3\right)+2^d\,.$$

   For $d > 3$, the term $\left(2^{d-1} - 3\right)$ is greater than 2 and the degree does not fit.
3. $\mathbf{Sz}(q)$ This is impossible by Lemma 29.
4. $\mathbf{A}_{\boldsymbol{n}}$. This is impossible by Lemma 26.

That leaves **PSU** $(3, q)$ and $\mathbf{Ree}(q)$ as the only possibilities.

□

Corollary 5. *The generalized hexagon* Γ *is isomorphic to* $\mathrm{H}(q)$*.*

Proof: This follows from Theorem 17 and Theorem 19.

□

This completes the proof of Theorem 20.

5.2 Generalized quadrangles

Classical Ovoids with doubly transitive groups

We have seen that the ovoids arising from a polarity of $\mathrm{W}(q)$, $q = 2^{2k+1}$ admit a doubly transitive group isomorphic to $\mathbf{Sz}(q)$. There are 2 other known classes of ovoids admitting a doubly transitive group in generalized quadrangles:

- Consider the quadrangle $\mathrm{Q}(4,q)$ and an ovoid arising from a hyperplane of the ambient projective space as in Theorem 14. The full automorphism group of $\mathrm{Q}(4,q)$ is isomorphic to a semi-direct product $\mathbf{PO}\,(4,q) : \mathrm{Aut}\,(\mathbb{F}_q)$ ([VM98a], Proposition 4.6.3). The stabilizer of a hyperplane of $\mathbf{PG}\,(4,q)$ in this group contains a group which induces a group isomorphic to $\mathbf{PSL}\,(2,q^2)$ on the hyperplane. [1] This group acts doubly transitively on the ovoid.
- Consider the quadrangle $\mathrm{H}(3,q^2)$ and an ovoid arising from a hyperplane of the ambient projective space as in Theorem 14. The full automorphism group of $\mathrm{H}(3,q^2)$ is isomorphic to a semi-direct product $\mathbf{PGU}\,(4,q) : \mathrm{Aut}\,(\mathbb{F}_{q^2})$ ([VM98a], Proposition 4.6.3). The stabilizer of a hyperplane of $\mathbf{PG}\,(4,q)$ in this group contains a group which induces a group isomorphic to $\mathbf{PSU}\,(3,q)$ on the hyperplane. This group acts doubly transitively on the ovoid.
- Consider the quadrangle $\mathrm{W}(q)$ and its polar ovoid. The group $\mathbf{Sz}(q)$ acts doubly transitively on the ovoid.

It will be shown that, if the number of points on a line and the number of of lines through a point coincide, the only possible isomorphism classes of doubly transitive groups are in the list above. More precisely:

Possible Groups

Theorem 22. *Let p be a prime and let Γ be a generalized quadrangle of order (a,a) such that a is a power of p. Furthermore, let $\mathcal{O}$ be an ovoid of Γ and let G be a group of automorphisms acting doubly transitively on $\mathcal{O}$. Then, G contains a doubly transitive subgroup isomorphic to one of the following:*

- $\mathbf{PSL}\,(2,q)$ *for some* $q = p^n$
- $\mathbf{Sz}(q)$ *for* $q = 2^{2n+1}$.

[1] The orthogonal group $\mathbf{\Omega}(3,q)$ is isomorphic to $\mathbf{PSL}\left(2,q^2\right)$, see e.g.[KL90], Proposition 2.1.9

All cases actually occur.

We use Burnside's theorem and get that the socle is either a regular elementary abelian group or a nonregular almost simple group.

Let G be the socle of H. In a first step, it will be shown that the group G can not be of affine type unless the number $st+1$ is a Fermat prime. This will be done in Lemma 31 by applying Catalan's conjecture. It follows that G must be a nonabelian simple group. The nonabelian finite simple groups having a faithful doubly transitive permutation representation have been classified in [CKS76] and [Fei80]. A table of these groups is included in the appendix. In a second step, we will go through the list of almost simple doubly transitive groups and investigate the isomorphism types of groups case by case until only the known possibilities, namely the linear groups $\mathbf{PSL}(2, q^2)$ and the Suzuki groups $\mathbf{Sz}(q)$ remain.

The following lemma shows that the group can not be of affine type unless we have one of two special cases.

Lemma 31. *Let Γ be a generalized quadrangle having order (a, a) such that a is a power of a prime number. Let further $\mathcal{O}$ be an ovoid of Γ and let G be a group of automorphisms acting doubly transitively on $\mathcal{O}$. Then G is not of affine type.*

Proof: Suppose that a is a power of a prime p and that $a^2+1 = p^k+1 = q^l$ for some prime q and integers $k > 1$ and $l > 0$. If $l > 1$ then, according to Mihăilescu's proof of Catalan's conjecture [Mih04], the only solution of $q^l - p^k = 1$ is $q = 3$, $p = 2$, $l = 2$, $k = 3$. The only possible values for (s, t) in this case are $(s, t) = (4, 2)$ and $(s, t) = (2, 4)$ and the quadrangle can not have order (a, a).

□

Knowing that G is of almost simple type and excluding all groups whose degree is obviously not of the form $a^2 + 1$, we are left with the following possibilities:

- $\mathbf{PSL}(d, q)$
- $\mathbf{Sp}(2d, 2)$, $d \geq 3$
- $\mathbf{Sz}(q)$
- $\mathbf{A}_n$

Note that Ree groups are not possible because their degree $3^{2k+1}+1$ does not have the form a^2+1.

Lemma 32. *$\mathbf{Sp}(2d,2)$ and $\mathbf{PSL}(d,q)$, $d>2$ is impossible.*

Proof:
- The degree of $\mathbf{Sp}(2d,2)$ is $2^{2d-1} \pm 2^{d-1}$, $d \geq 3$. This number must be equal to p^n+1 for some prime p. Since this is not a power of 2, we get $p \neq 2$. Hence, we get

$$\begin{aligned} p^n - 1 &= 2^{2d-1} \pm 2^{d-1} - 2 \qquad \text{and} \\ (p-1)\sum_{k=0}^{n-1} p^k &= 2\left(2^{2d-2} \pm 2^{d-2} - 1\right) \end{aligned}$$

  Since $k>1$, both factors on the left hand side of the equation are divisible by 2. The term on the right hand side of the equation, however is not divisible by 4.
- The degree of $\mathbf{PSL}(d,q)$ is $\frac{q^d-1}{q-1}$. We obtain

$$\begin{aligned} p^k + 1 &= \sum_{l=0}^{d-1} q^l \text{ and} \\ p^k &= \sum_{l=1}^{d-1} q^l = q\sum_{l=0}^{d-2} q^l \end{aligned}$$

  It follows that q divides p^k, and, consequently, that q is a power of p. For $d>2$, the term $\sum_{l=0}^{d-2} q^l$ is not divisible by p.

□

Now, we will treat the alternating groups.

Lemma 33. *For $t>2$, the group G is not $t+2$-transitive.*

Proof: We choose a point x of the quadrangle which does not belong to the ovoid $\mathcal{O}$. Then, since any line of the quadrangle is incident with a unique element of $\mathcal{O}$, there are $t+1$ points $x_1, \dots x_{t+1}$ of the ovoid at distance 2 from x. If an automorphism $\phi \in G$ of the quadrangle fixes the set $\left\{x_i \,\middle|\, 1 \leq i \leq t\right\}$ pointwise, then x is mapped to a point $x^\phi \in M := \bigcap_{1\leq i\leq t} \mathrm{D}_2(x_i)$. Since $M \subset (\mathrm{D}_2(x_1) \cap \mathrm{D}_2(x_2))$, the set M contains at most $t+1$ elements. The image ${x_{t+1}}^\phi$ of x_{t+1} is contained in $\mathrm{D}_2\left(x^\phi\right) \cap \mathcal{O}$. This set consists of $t+1$ elements and contains the set $\left\{x_i \,\middle|\, 1 \leq i \leq t\right\}$. It follows that the image of x_{t+1} is determined by the image of x. Therefore, the orbit of x_{t+1} has size at most $t+1$ and G is not $t+1$-transitive.

□

Corollary 6. *For $a \neq 2$, the group G does not contain a doubly transitive subgroup isomorphic to an alternating group on a^2+1 letters.*

For $(s,t) = (2,2)$, the quadrangle is isomorphic to W(2) and the full automorphism group is isomorphic to the symmetric group on 5 letters. Therefore, the restriction $(s,t) \neq (2,2)$ in the previous lemma is necessary.

This completes the proof of Theorem 22.

I would like to add some comments on why my proof of Theorem 22 requires that the number of points on a line and the number of lines through a point are the same. If this assumption is dropped and we start with a quadrangle of order (s,t) such that both numbers are powers of a prime p, the following additional cases occur:

1. Of course there is an additional classical case, the quadrangle $\mathrm{H}(3,q^2)$.
2. $st = 8$. The order of the quadrangle is $(2,4)$ or $(4,2)$. In both cases, there is a unique isomorphism class of generalized quadrangles of this order (see e.g [Str07]). If $(s,t) = (2,4)$, then the quadrangle is isomorphic to $\mathrm{Q}(4,2)$. This quadrangle does not contain ovoids. If $(s,t) = (4,2)$, then the quadrangle is isomorphic to $\mathrm{H}(3,2)$. The full automorphism group of this quadrangle is $\mathbf{PGU}(4,4) : \mathrm{Aut}(\mathbb{F}_4)$ ([CCN+85]). Calculations in GAP show that this group does not contain a subgroup isomorphic to any doubly transitive affine group on 9 elements.
3. $st+1$ is a Fermat prime. In this case, I could not exclude the possibility that there is an affine group acting doubly transitively on an ovoid of the polygon. All presently known Fermat primes (i.e. 3, 5, 17, 257 and 65537) have been checked. The result of the calculations is that no affine group can act doubly transitively on ovoids of this order. The calculation for 65537 was done in MAPLE.
4. $st = 3^{2k+1}$. In this case, I could not exclude the possibility that a Ree group acts doubly transitively on an ovoid of the polygon.

5.3 Tables of doubly transitive groups

The following tables contain all doubly transitive finite groups. The original sources are [Hup57], [Her85], [Her74] and [Cam81]. The tables in the form presented here can be found in [Cam99]. The affine action of a group containing $\mathbf{SL}(2,5)$ as a normal subgroup of a point stabilizer on 3^4 elements seems to be missing there. It is contained in Hering's list ([Her74]).

5.3.1 Affine groups

Group	Degree	Condition
$\mathbf{SL}(d,q) \leq H \leq \mathbf{\Gamma L}(d,q)$	q^d	
$\mathbf{Sp}(d,q) \trianglelefteq H$	q^{2d}	$d \geq 2$
$\mathbf{G_2}(\boldsymbol{q}) \trianglelefteq H$	q^6	q even
$\mathbf{SL}(2,3) \trianglelefteq H$	q	$q \in \{5^2, 7^2, 11^2, 23^2\}$
$2^{1+4} \trianglelefteq H$	3^4	
$\mathbf{SL}(2,5) \trianglelefteq H$	q	$q \in \{3^4 11^2, 19^2, 29^2, 59^2\}$
$\mathbf{A_6}$	2^4	
$\mathbf{A_7}$	2^4	
$\mathbf{PSU}(3,3)$	2^6	
$\mathbf{SL}(2,13)$	3^6	

5.3.2 Socles of almost simple groups

Group	Degree	Condition
$\mathbf{A}_{\boldsymbol{n}}$	n	$n \geq 5$
$\mathbf{PSL}\,(d,q)$	$\dfrac{q^d-1}{q-1}$	$d \geq 2,\ (d,q) \notin \{(2,2),(2,3)\}$
$\mathbf{Sp}\,(2d,2)$	$2^{2d-1}+2^{d-1}$	$d \geq 3$
$\mathbf{Sp}\,(2d,2)$	$2^{2d-1}-2^{d-1}$	$d \geq 3$
$\mathbf{PSU}\,(3,q)$	q^3+1	$q \geq 3$
$\mathbf{Sz}(q)$	q^2+1	$q = 2^{2k+1} > 2$
$\mathbf{Ree}\,(q)$	q^3+1	$q = 3^{2k+1} > 3$
$\mathbf{PSL}\,(2,11)$	11	
M_{11}	11	
M_{11}	12	
M_{12}	12	
$\mathbf{A}_{\mathbf{7}}$	15	
M_{22}	22	
M_{23}	23	
M_{24}	24	
$\mathbf{PSL}\,(2,8)$	28	
HS	176	
Co_3	276	

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