Decomposing André-Neto supercharacters of Sylow p-subgroups of Lie Type D

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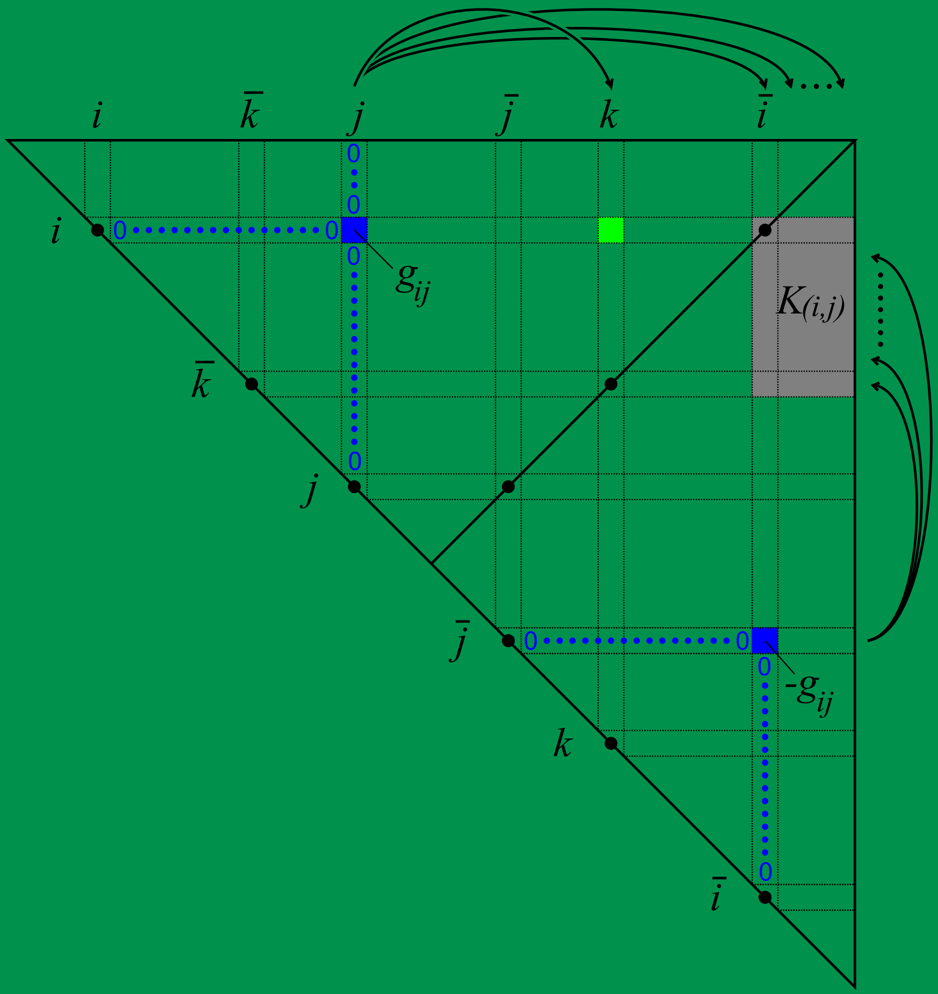

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Abstract

It is well known that the classification of irreducible characters for the group of upper unitriangular $n \times n$-matrices over a finite field with q elements, denoted $U_n(q)$, is a "wild" problem. N. Yan gave an approximation to a solution of this problem in terms of so-called supercharacters. These supercharacters are not necessarily irreducible but they satisfy many strong properties. For example they are pairwise orthogonal and contain every irreducible character as constituent. They can be classified in a pleasant combinatorial way. Methodically N. Yan's results are essentially obtained by investigating biorbits of a group operation, which is a coarser version of Kirillov's orbit method. More precisely, Yan's biorbits form disjoint unions of Kirillov orbits.

Hence N. Yan developed an easily accessible and elementary, but strong, theory for the investigation of $U_n(q)$. This thesis provides a generalization to the Sylow p-subgroups $D_n(q)$ of the orthogonal groups of Lie type D, defined over finite fields of characteristic p. The main result is the construction of a class of combinatorially described modules, the so-called hook-separated staircase modules. These modules are either orthogonal or isomorphic and contain all irreducible modules as constituents. This thesis provides also a generalization in the sense, that many of N. Yan's original results can be obtained as special cases.

The most important step is to generalize N. Yan's construction to abstract groups admitting a 1-cocycle. This generalization does not allow to consider biorbits, but only right orbits (or, if one prefers, left orbits). It is an extension of the original method to a more general class than algebra groups, for which P. Diaconis and I.M. Isaacs established a generalization carrying the full strength of biorbits. This is important, since Sylow p-subgroups of finite orthogonal groups of type D are not algebra groups.

The 1-cocycle approach is used to construct the mentioned hook-separated staircase modules. As a by-product the results provide a new and elementary proof of C.A.M. André's and A.M. Neto's supercharacter theory for $D_n(q)$ and a purely combinatorial and strong decomposition of their supercharacters into characters of hook-separated staircase modules.

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Likewise I thank Andreas Bächle and Wolfgang Rump, whose suggestions lead to the connection with 1-cocycles. Without them Section 2.1 in its current form would have been impossible.

Many thanks go to the members of the 'Abteilung für Darstellungstheorie' and more general of the IAZ and LExMath for the pleasant working atmosphere and the necessary distraction always at proper time.

Last but not least, I would like to thank my parents and my wife for their encouragement and their constant and invaluable support, which significantly contributed to the successful completion of this thesis.

Zusammenfassung

Bekanntermaßen ist die Klassifikation der irreduziblen Charaktere für die Gruppe der unitriangulären $n \times n$-Matrizen über einem Körper mit q Elementen, bezeichnet mit $U_n(q)$, ein "wildes" Problem. N. Yan konnte eine Approximation an die Lösung des Problems durch sogenannte Supercharaktere geben. Diese Supercharaktere sind zwar nicht notwendigerweise irreduzibel, haben aber dennoch einige starke Eigenschaften. Zum Beispiel sind sie paarweise orthogonal und enthalten jeden irreduziblen Charakter als Konstituenten. Außerdem lassen sie sich gut kombinatorisch klassifizieren. Methodisch gesehen erzielt N. Yan seine Resultate im Wesentlichen durch die Untersuchung von Biorbits einer Gruppenoperation, die eine gröbere Version von Kirillovs Orbit Methode darstellt. Genauer ausgedrückt bilden Yans Biorbits eine disjunkte Vereinigung von Kirillov Orbits.

Somit hat N. Yan für die Untersuchung von $U_n(q)$ eine einfach zugängliche und elementare, aber dennoch starke Theorie entwickelt. Diese Doktorarbeit bietet eine Verallgemeinerung des Ganzen für p-Sylowuntergruppe $D_n(q)$ von orthogonalen Gruppen des Lie Typs D, die über einem endlichen Körper der Charakteristik p definiert sind. Das Hauptergebnis ist die Konstruktion einer Klasse kombinatorisch beschriebener Moduln, der sogenannten *hook-separated staircase* Moduln. Diese Moduln sind entweder orthogonal oder isomorph und enthalten jeden irreduziblen Modul als Konstituent. Diese Doktorarbeit stellt auch in dem Sinn eine Verallgemeinerung dar, dass viele von N. Yans ursprünglichen Ergebnissen als Spezialfälle enthalten sind.

Der wichtigste Schritt ist N. Yans Methode zu verallgemeinern für abstrakte Gruppen, die einen 1-Kozykel zulassen. Diese Verallgemeinerung erlaubt jedoch nicht die Betrachtung von Biorbits, sondern nur von Rechtsorbits (oder, so bevorzugt, Linksorbits). Sie stellt eine Erweiterung der ursprünglichen Methode auf eine Klasse von Gruppen dar, allgemeiner als die der Algebra Gruppen, für welche P. Diaconis und I.M. Isaacs eine Verallgemeinerung entwickelten, die die volle Stärke der Biorbits hat. Dies ist wichtig, da p-Sylowuntergruppen endlicher orthogonaler Gruppen vom Lie Typ D keine Algebra Gruppen sind.

Der 1-Kozykel Ansatz wird verwendet, um die erwähnten *hook-separated staircase* Moduln zu konstruieren. Als Nebenprodukt enthalten die Ergebnisse einen neuen und elementaren Beweis für die Supercharaktertheorie von C.A.M André und A.M. Neto und eine ausschließlich kombinatorische, deutliche Zerlegung ihrer Supercharaktere in Charaktere von *hook-separated staircase* Moduln.

Danksagungen

Viele Menschen haben mich während der Erstellung meiner Dissertation unterstützt. Ihnen allen gebührt mein herzlicher Dank.

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Introduction

Let $U_n(q)$ denote the group of all upper unitriangular[1] $n \times n$-matrices over some finite field $\mathbf{F}_q$. Then q is a power of some prime p and $U_n(q)$ is a Sylow p-subgroup of the full linear group of invertible $n \times n$-matrices over the same field. Hence our group $U_n(q)$ is described by the two parameters n and q. We refer to $U_n(q)$ as *'unipotent linear groups'*. If one works with these groups one gets the impression that mainly the matrix size and only to a very small extend the size of the field plays a role.

In this spirit there exists a longstanding conjecture attributed to G. Higman[2] stating that for fixed n the number of conjugacy classes of $U_n(q)$ is given by a polynomial in q. This is one of these beautiful mathematical conjectures that are simple to state but very persistent towards a proof or a counter example. Up to now this conjecture is still open. All we know is, that the conjecture holds for $n \leq 13$. This was proven by A. Vera-López and J. Arregi using computer methods.[3] But why is it so difficult to count the conjugacy classes for unipotent linear groups? For other series of groups (for example the symmetric groups or the full linear groups) there are complete classifications of the conjugacy classes.

For the case of $U_n(q)$ it is a completely different story. The problem of determining all conjugacy classes for a fixed but arbitrary n is known to be 'wild', which means in a nutshell, that it is too difficult to be solved. Hence classical problems of ordinary representation theory (as determining the conjugacy classes or the irreducible characters) are out of reach for unipotent linear groups.

Therefore one has to find other methods to understand these series of groups. C.A.M. André[4] and later but independently N. Yan[5] could construct characters with wonderful properties, but which are in general not irreducible. First, their so called supercharacters[6] (which we call *André-Yan supercharacters*) form a partition of the irreducible characters in that sense, that every irreducible characters is contained in precisely one supercharacter. In particular this implies the supercharacters to be pairwise orthogonal. Second, there is a corresponding notion of superclasses (called *André-Yan superclasses*), which are unions of conjugacy classes. These two notions are connected by the fact, that supercharacters have constant values on

[1] i.e. triangular matrices, whose diagonal entries are 1
[2] c.f. [Hig]
[3] c.f. [VLA]
[4] c.f. [And1], [And2], [And3], [And4], [And5], [And6], and [And7]
[5] c.f. [Yan1] and [Yan2]
[6] the notion of supercharacters was later introduced by P. Diaconis and I. M. Isaacs in [DI].

superclasses. Hence by substituting *irreducible characters* with *supercharacters* and *conjugacy classes* with *superclasses* one can calculate a *supercharacter table* which is an approximation to the ordinary *character table*. Furthermore the supercharacters can be classified in a beautiful combinatorial way and the labels of this combinatorial description even know the dimensions of the supercharacters. Hence the supercharacter theory of André-Yan for unipotent linear groups is a powerful tool.

As usual with powerful mathematical tools an important task is to generalize them. One generalization was given by P. Diaconis and I. M. Isaacs for so called algebra groups.[7] They also formalised the main features of the theory for unipotent linear groups to the notion of a *supercharacter theory*. Besides they pointed out, that their theory for algebra groups (and hence also N. Yan's for unipotent linear groups) is a cruder version of Kirillov's orbit method and explain this relationship detailed.

At this stage we want to go into a bit more detail about Kirillov's orbit method. The main tool of N.Yan is to consider and interpret special classes of $U_n(q)$-$U_n(q)$-biorbits. Every such biorbit is a disjoint union of orbits under the conjugation operation. These orbits under the conjugation operation are the so called *Kirillov orbits* for the group $U_n(q)$. Using orbits of that kind A.A. Kirillov developed his renowned Kirillov orbit method for a huge class of not necessarily finite groups. It is a powerful tool most famous for providing a description of the equivalence classes of unitary irreducible representations of nilpotent Lie groups.[8] For unipotent linear groups A.A. Kirillov aimed at a construction of all irreducible complex characters using the orbit method.[9] Unfortunately this conjecture turned out to be wrong, as I.M. Issacs and D. Karagueuzian have shown.[10] The orbit method has its advantages, but it would go beyond the scope of this introduction to go into further detail. So far we have no classification of Kirillov orbits for unipotent linear groups. This is not surprising, as the problem of counting Kirillov orbits is equivalent to Higman's conjecture.

Another generalization of the supercharacter theory for unipotent linear groups was given by C.A.M. André and A.M. Neto for the Sylow p-subgroups of untwisted finite classical groups. Under 'Sylow p-Subgroups of *untwisted* finite classical groups' we understand $U_n(q)$ and subgroups of $U_n(q)$ leaving some bilinear form invariant, whereas in contrast 'Sylow p-Subgroups of *twisted* finite classical groups' are subgroups of $U_n(q^2)$ leaving some sesquilinear form invariant. C.A.M. André and A.M. Neto consider three series of classical groups, the orthogonal groups of type B and D and the symplectic groups (which have Lie type C). For the purpose of this thesis we only want to consider the series of orthogonal groups of type D. In order to have a bit more catchy denomination, throughout this thesis we call the series of 'Sylow p-subgroup of the finite orthogonal groups of type D' the '*unipotent orthogonal groups*' and abbreviate it until said otherwise with $D_n(q)$.

[7] c.f. [DI]
[8] c.f. [Kir1]; for more details on the orbit method see [Kir3].
[9] c.f. Conjecture 2.2.1 of [Kir2]
[10] c.f. [IK1] and [IK2]

C.A.M. André and A.M. Neto defined their supercharacters (which we call *André-Neto supercharacters*) for the unipotent orthogonal groups as restrictions from $U_n(q)$ to $D_n(q)$ of a special subclass of André-*Yan* supercharacters.[11] Appropriately they define their superclasses (which we call *André-Neto superclasses*) as the intersections with $D_n(q)$ of the André-*Yan* superclasses for $U_n(q)$. Hence André-Neto supercharacters are clearly constant on André-Neto superclasses. The whole work about supercharacters of C.A.M. André (including the parts written with his student A.M. Neto) is 'a tour de force in algebraic geometry'. These are the words N. Yan uses to describe C.A.M. André's work for $U_n(q)$.[12]. He can rightfully use these word, since his approach is way more simple and elementary. He is able to give his proofs by constructing a monomial representation for the complex group algebra of $U_n(q)$ and then determines and interprets orbits. A monomial representation is just a slight generalization of a permutation representations. Note further, that P. Diaconis' and I.M. Isaacs' work on algebra groups is also in this spirit.

The main goal of this thesis was to generalize the methods of N. Yan to the unipotent orthogonal groups. *This goal could be achieved!* The main step was to generalize N. Yan's construction to groups admitting a so called 1-*cocycle*. Since the unipotent orthogonal groups are no algebra groups, we could not use the generalization provided by P. Diaconis and I.M. Isaacs. A 1-cocycle is a map from a group G into an abelian group V on which G acts as automorphisms. Its main property is to connect the group operation of G with the operation of G on V. But unfortunately using this approach we loose some pleasant properties: In N. Yan's (and also in P. Diaconis' and I.M. Isaacs') work there is a left operation and a right operation and the main task is to determine biorbits. In our work we only have a right operation and right orbits. Or only a left operation and left orbits if one prefers that. In the language of representation theory this means, that we still can define a nice class of modules, so called orbit modules, but the task of understanding the homomorphisms between such orbit modules becomes more difficult. This also leads to some further differences. In the case of unipotent linear groups two orbit modules are always either isomorphic or there exists no nonzero homomorphism between them. This is in general wrong for the unipotent orthogonal groups and our notion of orbit modules. But we can find a special class of orbit modules having this property and the additional property, that all irreducible representations occur in some of these special orbit modules.

As a by-product we regain the results of C.A.M. André and A.M. Neto. In particular this implies we get a purely algebraic and combinatorial proof for their work, avoiding the use of algebraic geometry. We want to emphasize, that we have more than just a re-proof, since we gain a combinatorial decomposition of the André-Neto supercharacters into characters of orbit modules.[13] And these composition factors

[11] This is not the way they define the supercharacters in their articles, but we will show in this thesis, that the two descriptions are equivalent.

[12] c.f. the Introduction of [Yan2]

[13] in some cases the decomposition of the André-Neto supercharacters happens to be even into pair-

are either equal or orthogonal. Hence the isomorphism classes of these characters can be interpreted as the character theoretical side of a supercharacter theory. But we don't want to go as far as conjecturing a supercharacter theory, where the supercharacters are these isomorphism classes of characters, since we have no idea how superclasses could look like.

We want further to point out that a big difference between the André-Neto approach and our own, besides ours being way more elementary, is the fact, that their approach is heavily determined by the supercharacter theory for the unipotent linear groups, whereas ours is more intrinsic (the correspondence to the theory for $U_n(q)$ is not needed untill we try to recognize André-Neto supercharacters as objects in our theory).

We pointed out earlier that for $U_n(q)$ (and more general algebra groups) there is a close affiliation to the Kirillov orbit method. For our theory of working with the unipotent orthogonal groups the affiliation is by far not as close. The most apparent reason is the absence of biorbits having the property of 'obviously being unions of Kirillov orbits'.

To round off the thesis we at last consider the problem of 'counting the conjugacy classes for fixed matrix size n in dependency of the field size q'. This can be seen as a test for the quality of the approximation our theory provides. Hence, we determine these numbers for matrix size 8.[14] It turns out that this number is a polynomial in q, which is given explicitly. This is quite good, since we do not refine our results much to obtain this statement. State of art up to date is matrix size 14.[15] S.M. Goodwin, P. Mosch and G. Röhrle obtained their polynomials using computer methods, whereas our polynomial is obtained by hand in a combinatorial way, c.f. Appendix A.3.

A glimpse into the future:

While writing this thesis it turned out that many of the results for Sylow p-subgroups of orthogonal groups of type D can be generalized to the other series of Sylow p-subgroups of classical groups (twisted and untwisted). I will publish these results as soon as possible. Fore more details on the current stage of development see Chapter 'What is next?... on page 133.

An informal overview (focusing on pointing out important results and relating them):

First of all the author wants to remark, that he focused on clearness in his writing. Hence there are many repetitions and redundancies aiming on preserving the reading flow as best as possible and most of all to prevent the (unfortunately not completely avoidable) typographical errors from obstructing the understanding of important passages (as for example basic definitions).

wise orthogonal irreducible characters; a detailed discussion of the merits and limits of the decomposition can be found in the end of this thesis.

[14] note, that for unipotent orthogonal groups only even matrix sizes can occur.

[15] c.f. [GR] and [GMR]; the results for matrix size 8 where also obtained theoretically in [HLM]

This thesis is organized into three chapters and an appendix containing many examples. *Chapter 1* establishes the structural properties we need for the unipotent orthogonal groups, whereas *Chapter 2* provides the notions from general representation theory (additionally to a course in ordinary representation theory). The two chapters can be read independently of each other and are needed to understand *Chapter 3,* which is the centerpiece of this thesis.

Chapter 1 (Group theory):
Structure of unipotent orthogonal
groups and their pattern subgroups
(2 sections)

Chapter 2 (Representation theory):
theory of monomial representations
arising from 1-cocycles
(2 sections)

$\searrow \qquad \swarrow$

Chapter 3 (Representation theory, except Section 3.5):
Study of the unipotent orthogonal groups using
the method of 1-cocycle monomial representations
(6 sections)

Chapter 2 also contains the application of the theory to the groups $U_n(q)$. In particular we give an outline of the results of C.A.M. André and N. Yan for that case (using the methods of N. Yan) and a comparison to the results for the unipotent orthogonal group, which we will obtain in *Chapter 3*.

Some words on notation and our point of view:

Throughout the text we denote $U_n(q)$ with $A_n(\mathbf{F}_q)$ and the the unipotent orthogonal groups (of type D) with $D_n(\mathbf{F}_q)$.[16] The matrix size of the groups $D_n(\mathbf{F}_q)$ is $2n \times 2n$. It will turn out, that we have a subgroup structure of the form

$$(*) \qquad \cdots \leq A_n(\mathbf{F}_q) \leq D_n(\mathbf{F}_q) \leq A_{2n}(\mathbf{F}_q) \leq D_{2n}(\mathbf{F}_q) \leq \cdots$$

Having this structure in mind there are two possible approaches towards our theory. First, one could assume the reader to be familiar with N. Yan's theory for $A_n(\mathbf{F}_q)$ and try to *generalize* it to the case of $D_n(\mathbf{F}_q)$. Second, one could work out the theory for $D_n(\mathbf{F}_q)$ and later *specialize* it to the case of $A_n(\mathbf{F}_q)$. Usually the first approach is preferable, but in our situation this would only sparsely shorten our proofs but unnecessarily increase the obstacle for readers not familiar with N. Yan's work.

Hence we adopt the point of view of 'thinking $D_n(\mathbf{F}_q)$' and specialize to $A_n(\mathbf{F}_q)$ from time to time.

Requirements from the reader:

We require the reader to have some familiarity with the theory of the general linear group, see e.g. Chapter 2 of [AB] and with the knowledge of ordinary representation theory provided by a typical course with that name.

[16] we chose the notation according to the Lie types of the groups.

Overview of 'Chapter 1: The unipotent orthogonal groups $D_n(\mathbf{F}_q)$'

In *Section 1.1* we define the unipotent orthogonal groups of type D, denoted with the symbol $D_n(\mathbf{F}_q)$, derive basic properties and prove Theorem 1.1.19, which we will explain next. In contrast to $D_n(\mathbf{F}_q)$, for the groups $A_n(\mathbf{F}_q)$[17] every element is uniquely determined by its matrix entries above the diagonal and conversely each choosing of matrix entries above the diagonal leads to a unique group element. For $D_n(\mathbf{F}_q)$, which is a subgroup of $A_{2n}(\mathbf{F}_q)$, not all entries above the diagonal can be chosen arbitrary to get an element of $D_n(\mathbf{F}_q)$. Theorem 1.1.19 will allow us to consider a subset of matrix entries, such that elements of $D_n(\mathbf{F}_q)$ are completely determined by the values at these coordinates and conversely each choosing of values at these coordinates determines a unique group element. For the application of the methods from *Chapter 2* to the groups $D_n(\mathbf{F}_q)$ this is the key ingredient.

In *Section 1.2* we define and study pattern subgroups, culminating in Theorem 1.2.22, which provides all needed structure theory for pattern subgroups. This is also were the concept '*specialize from $D_n(\mathbf{F}_q)$ to $A_n(\mathbf{F}_q)$*' is applied first, as we derive the corresponding structure theory for the groups $A_n(\mathbf{F}_q)$ and their pattern subgroups from Theorem 1.2.22.

Overview of 'Chapter 2: Monomial representations arising from 1-cocycles'

In *Section 2.1* we define the notions *monomial representation* and 1-*cocycle*. Then we present our construction of a monomial representation from a 1-cocylce. With Theorem 2.1.11 we provide the construction for the most general case of abstract groups and arbitrary 1-cocylces, whereas Corollary 2.1.35 is a variant of the construction more suitable to the groups $D_n(\mathbf{F}_q)$ and $A_n(\mathbf{F}_q)$ utilizing the additional structure in these cases. The corollary can also be applied to other Sylow p-subgroups of classical groups, twisted and untwisted, as we will show in a forthcoming article.

In *Section 2.2* we apply Corollary 2.1.35 to the groups $A_n(\mathbf{F}_q)$, which implies constructing a 1-cocycle, and determine in Corollary 2.2.16 a nice explicit formula for the action of the arising monomial representation, the so-called *truncated column operation*. Afterwards we will give an overview on N. Yan's results and compare them to the corresponding results for $D_n(\mathbf{F}_q)$ of *Chapter 3*. Hence this also illustrates our strategy for the study of $D_n(\mathbf{F}_q)$ in *Chapter 3*.

Overview of 'Chapter 3: Study of a 1-cocycle monomial representation for $D_n(\mathbf{F}_q)$'

In *Section 3.1* we construct a 1-cocycle for arbitrary pattern subgroups of $D_n(\mathbf{F}_q)$ and describe the resulting monomial representation explicitly in Corollary 3.1.16. The operation has many similarities to the one of Corollary 2.2.16 and will (also) be called truncated column operation. This is the only section in which we consider representations of pattern subgroups, afterwards pattern subgroups will sparsely occur as stabilizers. But unfortunately not all occuring stabilizers are pattern subgroups.

[17] which is the group of upper unitriangular matrices (i.e. of upper triangular matrices with 1's on the diagonal and arbitrary entries above the diagonal).

Another important result of this section, is the extension of the truncated column operation of '$D_n(\mathbf{F}_q)$ on the complex group algebra of $D_n(\mathbf{F}_q)$' to '$A_{2n}(\mathbf{F}_q)$ on the complex group algebra of $D_n(\mathbf{F}_q)$'. This is done in Theorem 3.1.14.[18]

In *Section 3.2* we start our study of so called orbit modules, which are modules naturally occurring in monomial representations. In this section we investigate the case of so called staircase modules, which are the most pleasant orbit modules. We obtain a complete classification of these staircase modules in Theorem 3.2.29, which also includes a combinatorial algorithm for computing the dimensions of staircase modules.

Section 3.3 is in a way the technical heart of this thesis. There we study homomorphisms between orbit modules. We obtain that each orbit module is isomorphic to some staircase module in Corollary 3.3.15. Afterwards we consider the subclass of so called hook-separated staircase modules and show in Corollary 3.3.19, that every irreducible module is constituent of at least one hook-separated staircase module. Up to that point we tried to construct morphisms. In the rest of this section we investigate, under what circumstances there exist no nonzero homomorphism between orbit modules, i.e. we want to understand under which conditions characters of orbit modules are orthogonal. This investigation peaks in Theorem 3.3.32, which allows us to solve this problem combinatorially. In particular the theorem also allows us to determine a basis for the endomorphism ring of a hook-separated staircase module. With *Section 3.3* we also complete the technical part of the representation theoretic investigation of orbit modules. Afterwards we will reap the fruit of our labour, as we will mainly interpret our results from different points of view (with the exception of *Section 3.5* , where we derive the needed group theoretic results).

In *Section 3.4* we study the subgroup structure already mentioned in $(*)$. Here we relate the representation theory of $D_n(\mathbf{F}_q)$ to the representation theory of $A_n(\mathbf{F}_q)$ and $A_{2n}(\mathbf{F}_q)$ using their theories of orbit modules. We will show in Corollary 3.4.14 that an important class of modules (which will turn out to be the class of André-Neto supercharacters) has an interpretation as restrictions of André-Yan supercharaters.[19] Further in this section we pursue our strategy of *specialising from* $D_n(\mathbf{F}_q)$ *to* $A_n(\mathbf{F}_q)$ as we show in Corollary 3.4.10 that the representation theoretic results for $A_n(\mathbf{F}_q)$ of N. Yan presented in the end of *Section 2.2* can be derived easily from our theory for $D_n(\mathbf{F}_q)$. In particular this allows us also to derive (the few) results for $D_n(\mathbf{F}_q)$ obtained from $A_{2n}(\mathbf{F}_q)$ instead from $D_{2n}(\mathbf{F}_q)$.

In *Section 3.5* we obtain the needed group theoretic results for $D_n(\mathbf{F}_q)$. More precisely we define André-Neto superclasses and give a combinatorial labelling in Corollary 3.5.34. This was already done by C.A.M. Andé and A.M. Neto in their article [AN2]. Their approach is to study the superclasses algebro-geometric regarding

[18]the theorem also considers pattern subgroups.

[19]part of this is also contained in the work of C.A.M. André and A.M. Neto, but their work does not include the explicit combinatorial description we have.

them as varieties, whereas our approach is to study them combinatorial in the fashion of *Section 3.2*. The title of *Section 3.5* has an * to indicate, that the section is not necessary for understanding the rest of this thesis as we could use [AN2, Theorem 3.6] instead of Corollary 3.5.34. It is also noteworthy, that the methods developed in this section are not used elsewhere.

In *Section 3.6* we finally proof the result stated by the title of this thesis. We define André-Neto supercharacters following the work of C.A.M André and A.M. Neto, recognize them as objects of our theory in Proposition 3.6.16 and use our results to decompose them in Theorem 3.6.21. We finish the section (and the thesis) with a discussion of the strengths and weaknesses of our main theorem. The main strengths clearly are the combinatorial explicitness of the decomposition and the fact, that two orbit modules which are constituent of the same André-Neto supercharacter are either isomorphic or orthogonal.

There is also an appendix containing many examples. In *Appendix A.2* we illustrate the strengths and limitations of our work with concrete examples. In *Appendix A.3* we count the irreducible characters of $D_4(\mathbf{F}_q)$ of any fixed degree and show that the number of them are given by polynomials with non-negative integral coefficients in $q-1$. In particular this also shows that the number of irreducible all characters is given by a polynomial in q. All of these polynomials are determined explicitly.

A few words on notation

- If A is a matrix A_{ij} denotes the (i,j)-th matrix entry.
- e_{ij} denotes the matrix with entry 1 at the (i,j)-th position and 0 entries otherwise
- As usual δ_{ab} denotes the Kronecker delta.
- If f and g are maps, we use fg from time to time to abbreviate $f \circ g$.
- p is an odd prime and q a power of p.
- n is a positive integer. It usually denotes the Lie rank of the group $D_n(\mathbf{F}_q)$ or the matrix size of the group $A_n(\mathbf{F}_q)$.
- $N = 2n$ usually denotes the matrix size of the groups $D_n(\mathbf{F}_q)$ and $A_N(\mathbf{F}_q)$.
- The complex group algebra of a group G is denoted with $\mathbb{C}G$, but if the symbol denoting the group is longer we also use the notation $\mathbb{C}[G]$, as for example in $\mathbb{C}[D_n(\mathbf{F}_q)]$.

1. The unipotent orthogonal groups $D_n(\mathbf{F}_q)$

First, we want to introduce notions and techniques for Sylow p-subgroups of the Orthogonal groups $\mathrm{O}_N^+(\mathbf{F}_q)$, where N is an even positive integer, say $N = 2n$ and $\mathbf{F}_q$ denotes the Galois field with q elements, where $q = p^\ell$ with a prime p different from 2 and a positive integer ℓ. In Chapter 3 we will use the results of this chapter. The cornerstones of this chapter are the Theorems 1.1.19 and 1.2.22.

The reader is assumed to have some familiarity with the theory of the general linear group, see e.g. chapter 2 of [AB].

1.1. Definition and basic properties

In the theory of classical groups (see e.g. [Gro]) one view is to look at vector spaces with nondegenerate bilinear form (or more general quadratic form) and the groups of matrices, leaving such a bilinear form invariant. If the field is $\mathbf{F}_q$ and the dimension N of the vector space is even, say $N = 2n$, then there are only two inequivalent nondegenerate symmetric bilinear forms. Let us call them b^+ and b^-. They lead to the nonisomorphic groups $\mathrm{O}_N^+(\mathbf{F}_q)$ and $\mathrm{O}_N^-(\mathbf{F}_q)$. We only want to consider the first one.

Now let us make b^+ (or better: one possibility of b^+) a bit more precise. We define the $N \times N$-matrix

$$J := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

in order to give the following definition

1.1.1 Definition/Lemma. *Let $n \in \mathbb{N}$ and $N = 2n$. We define*

$$b^+ : \ \mathbf{F}_q^N \times \mathbf{F}_q^N \longrightarrow \mathbf{F}_q : \ (x, y) \longmapsto x^\top J y.$$

Then b^+ is a nondegenerate symmetric bilinear form with Witt index n; i.e. b^+ has a maximal isotropic[20] subspace W of dimension n.

[20] a subspace $W \leq V$ is called isotropic, if $b^+(x, y) = 0$ for all $x, y \in W$.

Proof. Let $W := \mathbf{F}_q - \operatorname{span}(e_1, \dots, e_n)$, where e_i denotes the i-th basis vector of the standard basis. We have $Je_i = e_{N+1-i}$ and since $N + 1 - i = 2n + 1 - i > n$ for $1 \leq i \leq n$ we get $b^+(e_j, e_i) = e_j^\top e_{2n+1-i} = 0$. Therefore W is an isotropic vector space of dimension n, and due to Witt's theorem this is also the largest possible dimension. qed. |

We will use the expression $N + 1 - i$ very often in future, so we systemize it:

1.1.2 Definition/Lemma. *Let $N \in \mathbb{N}$. We define the **mirror map***

$$\bar{\ }: \{1, \dots, N\} \longrightarrow \{1, \dots, N\} : i \longmapsto \bar{i} := N + 1 - i$$

The map mirrors every entry on $\frac{N+1}{2}$. It satisfies $\bar{\bar{i}} = i$ and

$$\forall i, j \in \{1, \dots, N\} : \qquad \begin{array}{ccc} i < j & \iff & \bar{i} > \bar{j}, \\ i = j & \iff & \bar{i} = \bar{j}. \end{array}$$

So we are able to operate with $\bar{\ }$ on whole inequalities, for example holds

$$\forall i, j, k \in \{1, \dots, N\} : \qquad i < j \leq k \qquad \iff \qquad \bar{k} \leq \bar{j} < \bar{i}$$

We will use this extensively.

Here is the definition of the orthogonal groups:

1.1.3 Definition. *Let $N = 2n$. We define the **(positive) orthogonal group** to be*

$$\mathrm{O}_N^+(\mathbf{F}_q) = \left\{ g \in \mathrm{GL}_N(\mathbf{F}_q) \;\middle|\; \forall v, w \in \mathbf{F}_q^N : \ b^+(gv, gw) = b^+(v, w) \right\}.$$

We are interested in the Sylow p-subgroups of the group $\mathrm{O}_N^+(\mathbf{F}_q)$. For the group $\mathrm{GL}_N(\mathbf{F}_q)$ one Sylow p-subgroup is given by the upper unitriangular matrices. Our (naive) approach is now, to look at the upper unitriangular orthogonal matrices. It will work, due to our choosing of b^+.

1.1.4 Definition. *We denote the set of upper unitriangular $N \times N$ matrices over the field $\mathbf{F}_q$ with $A_N(\mathbf{F}_q)$ and call it **unipotent linear group**. We define the **unipotent (positive) orthogonal group** $D_n(\mathbf{F}_q)$ by*

$$D_n(\mathbf{F}_q) := A_N(\mathbf{F}_q) \cap \mathrm{O}_N^+(\mathbf{F}_q)$$

Remark: The notation $A_N(\mathbf{F}_q)$ and $D_n(\mathbf{F}_q)$ may seem a bit odd, but it will make sense as soon as root systems will come into play.

We want to investigate if the group $D_n(\mathbf{F}_q)$ is indeed a Sylow p-subgroup (c.f. Corollary 1.1.26). First we rewrite the defining relations as a matrix equality. Therefore we introduce the notion of the right transpose of a matrix.

1.1.5 Lemma. *Let $N = 2n$ be in $\mathbb{N}$ and let A, B be $N \times N$ matrices over $\mathbf{F}_q$. We define $A^R := JA^\top J$, A^R is called the* ***right transpose of*** *A. We have*

$$(A+B)^R = A^R + B^R, \qquad (AB)^R = B^R A^R \qquad \text{and} \qquad (A^R)_{ij} = A_{\bar{j}\bar{i}},$$

where $1 \leq i, j \leq N$ and A_{ij} denotes the (i, j)-th matrix entry.

In particular it follows $(A^R)^R = A$.

Remark: A^R is obtained from A by taking the 'transpose' on the counter-diagonal.

Proof. We have

$$(AB)^R = J(AB)^\top J = JB^\top A^\top J \overset{J^2=1}{=} (JB^\top J)(JA^\top J) = B^R A^R.$$

Note, that $J = \sum_{k=1}^{N} e_{k\bar{k}}$. Using[21] $e_{ab}e_{cd} = \delta_{bc}e_{ad}$, we get

$$Je_{ij}J = \left(\sum_{k=1}^{N} \underbrace{e_{k\bar{k}}e_{ij}}_{=\delta_{\bar{k}i}e_{kj}} \right) J = e_{\bar{i}j}J = \sum_{k=1}^{N} \underbrace{e_{\bar{i}j}e_{k\bar{k}}}_{\delta_{jk}e_{\bar{i}\bar{k}}} = e_{\bar{i}\bar{j}}.$$

Thus $e_{ij}^R = Je_{ij}^\top J = Je_{ji}J = e_{\bar{j}\bar{i}}$. We get the result, using the equality $A = \sum_{i,j} A_{ij}e_{ij}$ and the additivity of the operation ^{R}. qed. |

1.1.6 Lemma. *Let $g \in \mathrm{Mat}_{N\times N}(\mathbf{F}_q)$. Then we have*

$$g \in \mathrm{O}_N^+(\mathbf{F}_q) \qquad \Longleftrightarrow \qquad gg^R = 1. \tag{1.1.7}$$

Proof. We have $g \in \mathrm{O}_N^+(\mathbf{F}_q)$ if and only if $\forall v, w \in \mathbf{F}_q^N :\ v^\top g^\top Jgw = v^\top Jw$. But this is equivalent to $g^\top Jg = J$, which is equivalent to $Jg^\top J = g^{-1}$, since $J^2 = 1$. qed. |

1.1.8 Remark. *If u is an upper unitriangular matrix the same is true for u^R.*[22]

Before we give the characterization of $D_N(\mathbf{F}_q)$ in terms of matrix entries we want to introduce a few sets of matrix entry coordinates.

1.1.9 Definition (Dictionary 'Sets of matrix entry coordinates'). *We define*

$$\begin{array}{lll@{\qquad}lll}
\square & := & \{(i,j) \mid 1 \leq i, j \leq N\} & \text{◹} & := & \{(i,j) \in \square \mid i < j\} \\
\triangledown & := & \{(i,j) \in \square \mid i < j < \bar{i}\} & \triangleleft & := & \{(i,j) \in \square \mid \bar{j} < i < j\} \\
\diagup & := & \{(i,j) \in \text{◹} \mid i = \bar{j}\} & \trianglelefteq & := & \{(i,j) \in \square \mid \bar{j} \leq i < j\} \\
\triangledown_{\mathrm{left}} & := & \{(i,j) \in \triangledown \mid j \leq n\} & \triangledown_{\mathrm{right}} & := & \{(i,j) \in \triangledown \mid j > n\} \\
\text{◸} & := & \{(i,j) \in \square \mid j < \bar{i}\} & \diagdown & := & \{(i,j) \in \square \mid i = j\}
\end{array}$$

[21] the notations e_{ij} and δ_{ij} were introduced in section 'A few words on notation' after the introduction.

[22] $u_{ij}^R = u_{\bar{j}\bar{i}}$. Since $i < j \Leftrightarrow \bar{j} < \bar{i}$ and $i = j \Leftrightarrow \bar{i} = \bar{j}$ hold, we see that the remark is true.

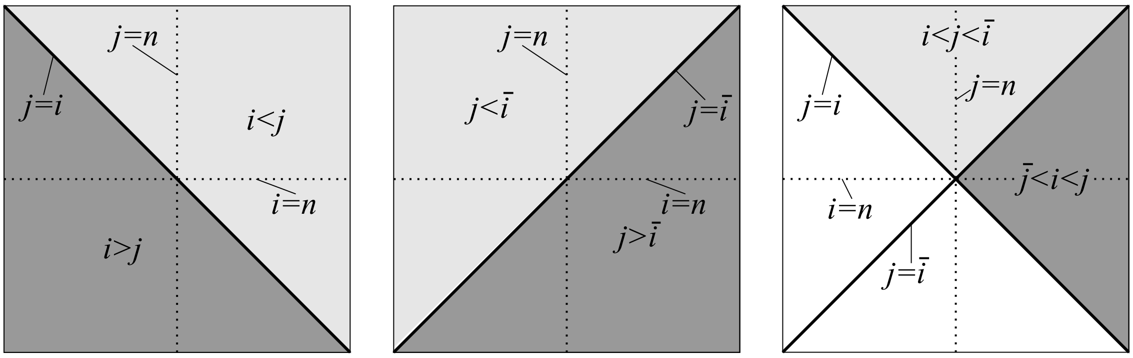


Figure 1.1.: Illustration to the Definition 1.1.9
The figures all show $\square$. Let (i,j) be an arbitrary coordinate in $\square$. Then the figures show, where (i,j) lies in $\square$, depending on the relations i and j satisfy.

The symbols for sets of matrix entry coordinates are chosen to remind of the sets of matrix entries they define. An illustration is given in Figure 1.1.

See also example 1.1.20: In the matrices in (1.1.21) and (1.1.23) the entries with coordinates in $\triangledown$ are coloured blue, the ones with coordinates in $\diagup$ are coloured green and so on.

1.1.10 Remark. *We will not need all of the position sets in this chapter. But we still present them all here for the purpose of clarity. Clearly, we have:*

$$\text{◹} = \triangledown \,\dot\cup\, \diagup \,\dot\cup\, \triangleleft, \qquad \dot\triangleleft = \triangleleft \,\dot\cup\, \diagup, \qquad \textit{and} \qquad \triangledown = \triangledown_{\ell} \,\dot\cup\, \triangledown_{r}.$$

1.1.11 Lemma. *We have*

$$D_n(\mathbf{F}_q) = A_N(\mathbf{F}_q) \cap \mathrm{O}_N^+(\mathbf{F}_q) = \Big\{ u \in A_N(\mathbf{F}_q) \;\Big|\; \forall (i,j) \in \triangledown \dot\cup \diagup : \; (uu^R)_{ij} = 0 \Big\}.$$

Proof. Since u and u^R are upper unitriangular matrices it follows, that uu^R is an upper unitriangular matrix, it suffices to prove, that $u \in D_n(\mathbf{F}_q)$ if and only if $(uu^R)_{ij} = 0$ for all $(i,j) \in \triangledown\dot\cup\diagup$.

Observing

$$(i,j) \in \triangledown \quad \Leftrightarrow \quad i < j < \bar{i} \quad \Leftrightarrow \quad \bar{\bar{i}} < \bar{j} < \bar{i} \quad \Leftrightarrow \quad (\bar{j}, \bar{i}) \in \triangleleft \tag{1.1.12}$$

and using

$$(uu^R)_{ij} \overset{\text{Lemma 1.1.5}}{=} \left((uu^R)^R\right)_{\bar{j}\bar{i}} \overset{\text{Lemma 1.1.5}}{=} \left((u^R)^R u^R\right)_{\bar{j}\bar{i}} \overset{\text{Lemma 1.1.5}}{=} (uu^R)_{\bar{j}\bar{i}}$$

we see, that $(uu^R)_{ij} = 0$ if and only if $(uu^R)_{\bar{j}\bar{i}} = 0$ and hence (since $\text{◹} = \triangledown\dot\cup\diagup\dot\cup\triangleleft$) we have $(uu^R)_{ij} = 0$ for all $(i,j) \in \text{◹}$ if and only if $(uu^R)_{ij} = 0$ for all $(i,j) \in \triangledown\dot\cup\diagup$.

Now, Lemma 1.1.6 implies the result. qed. |

1.1.13 Lemma. *Let u be in $A_N(\mathbf{F}_q)$. Then we have*

$$(uu^R)_{ij} = u_{ij} + u_{\bar{j}\bar{i}} + \sum_{i<r<j} u_{ir}u_{\bar{j}\bar{r}} \qquad \text{for all}(i,j) \in \square.$$

Proof. Note, that $u_{ir} = 0$ unless $i \le r$ and therefore $u_{\bar{j}\bar{r}} = 0$ unless $r \le j$. We have

$$(uu^R)_{ij} \overset{\text{Lemma 1.1.5}}{=} \sum_{r=1}^{N} u_{ir}u_{\bar{j}\bar{r}} \;=\; u_{ij} + u_{\bar{j}\bar{i}} + \sum_{i<r<j} u_{ir}u_{\bar{j}\bar{r}}.$$

qed. |

1.1.14 Lemma. *Let u be in $A_N(\mathbf{F}_q)$. Then for all $(r,\bar{r}) \in \diagup$ we have*

$$(uu^R)_{r\bar{r}} = 0 \qquad \Longleftrightarrow \qquad {\color{green}u_{r\bar{r}}} = -\sum_{r<l\le n} {\color{blue}u_{rl}u_{r\bar{l}}} \tag{1.1.15}$$

and for all $(r,s) \in \lhd$, which is equivalent to $(\bar{s},\bar{r}) \in \triangledown$, we have

$$(uu^R)_{\bar{s}\bar{r}} = 0 \qquad \Longleftrightarrow \qquad {\color{red}u_{rs}} = -{\color{blue}u_{\bar{s}\bar{r}}} - \sum_{\bar{s}<l<\bar{r}} {\color{blue}u_{\bar{s}l}}u_{r\bar{l}}, \tag{1.1.16}$$

where blue entries have coordinates in $\triangledown$, green ones in $\diagup$, red ones in $\lhd$ and from black ones we don't know it in generality.

Proof. We start with equation (1.1.15). We have, using Lemmas 1.1.11 and 1.1.13,

$$\begin{aligned} 0 = (uu^R)_{r\bar{r}} &= u_{r\bar{r}} + u_{\bar{\bar{r}}\bar{r}} + \sum_{r<l<\bar{r}} u_{rl}u_{\bar{\bar{r}}\bar{l}} \overset{\bar{n}=n+1}{=} 2u_{r\bar{r}} + \sum_{r<l\le n} u_{rl}u_{r\bar{l}} + \sum_{\bar{n}\le\bar{l}<\bar{r}} u_{r\bar{l}}u_{r\bar{\bar{l}}} \\ &= 2u_{r\bar{r}} + \sum_{r<l\le n} u_{rl}u_{r\bar{l}} + \sum_{r<\bar{l}\le n} u_{r\bar{l}}u_{r\bar{\bar{l}}} = 2u_{r\bar{r}} + 2\sum_{r<l\le n} u_{rl}u_{r\bar{l}} \end{aligned}$$

Note, that $(r,\bar{r}) \in \diagup$ implies $r \le n$ and $n+1 \le \bar{r}$. We get $r < l < \bar{r}$ and (acting with $^{-}$on the inequality) $r < \bar{l} < \bar{r}$. Hence we have $(r,l),(r,\bar{l}) \in \triangledown$.

Now, let us prove equation (1.1.16). We have, using Lemmas 1.1.11 and 1.1.13,

$$0 = (uu^R)_{\bar{s}\bar{r}} = u_{\bar{s}\bar{r}} + u_{rs} + \sum_{\bar{s}<l<\bar{r}} u_{\bar{s}l}u_{r\bar{l}}$$

We have $(\bar{s},\bar{r}) \in \triangledown$ due to equation (1.1.12). Now, $(\bar{s},\bar{r}) \in \triangledown$ implies $(\bar{s},l) \in \triangledown$ for $\bar{s} < l < \bar{r}$, since $\bar{s} < l < \bar{r} < s$. qed. |

1.1.17 Remark. *In Lemma 1.1.13 the map $(i,j) \mapsto (\bar{j},\bar{i})$ appears. Let us formalize this map and derive some properties. We define*

$$^{-}\colon\; \urtriangle \longrightarrow \urtriangle:\; (i,j) \longmapsto \overline{(i,j)} := (\bar{j},\bar{i}) \tag{1.1.18}$$

Obviously the map is an involution. We get $\overline{\diagup} = \diagup$ and (1.1.12) shows $\overline{\triangledown} = \lhd$ and $\overline{\lhd} = \triangledown$.

More precise the map^{-}: $\urtriangle \to \urtriangle$ mirrors elements of $\urtriangle$ on the counterdiagonal $\diagup$.

Let u be in $D_n(\mathbf{F}_q)$. Then equations (1.1.15) and (1.1.16) allow us to express the entries of u, which have coordinates in $\diagup \dot\cup \triangleleft$, as polynomial expressions of the entries with coordinates in $\triangledown$. More precisely, we have:

1.1.19 Theorem. *Let T_{ij} for $(i,j) \in \triangledown$ be variables. For every matrix coordinate $(r,s) \in \text{◹} \setminus \triangledown$ there exists a polynomial p_{rs} over $\mathbb{Z}/p\mathbb{Z}$ in variables T_{ij} with $(i,j) \in \triangledown$ such that*

$$\forall u \in D_n(\mathbf{F}_q) : \qquad u_{rs} = p_{rs}(u_{ij}),$$

where $p_{rs}(u_{ij})$ has the meaning of evaluating the polynomial p_{rs} at $(u_{ij})_{(i,j)\in\triangledown}$.

Conversely let λ_{ij} be arbitrary elements of $\mathbf{F}_q$ for all $(i,j) \in \triangledown$. Then there exists a unique element $u \in D_n(\mathbf{F}_q)$, such that $u_{ij} = \lambda_{ij}$ for all $(i,j) \in \triangledown$.

Before we give the proof, we first illustrate the theorem with the following example.

1.1.20 Example. *In the following we colour variables with coordinates in $\triangledown$ blue, variables with coordinates in $\diagup$ green and variables with coordinates in $\triangleleft$ red. The elements of $D_3(\mathbf{F}_q)$ are matrices of the form*

$$\begin{pmatrix} 1 & {\color{blue}u_{12}} & {\color{blue}u_{13}} & {\color{blue}u_{14}} & {\color{blue}u_{15}} & {\color{green}u_{16}} \\ & 1 & {\color{blue}u_{23}} & {\color{blue}u_{24}} & {\color{green}u_{25}} & {\color{red}u_{26}} \\ & & 1 & {\color{green}u_{34}} & {\color{red}u_{35}} & {\color{red}u_{36}} \\ & & & 1 & {\color{red}u_{45}} & {\color{red}u_{46}} \\ & & & & 1 & {\color{red}u_{56}} \\ & & & & & 1 \end{pmatrix} \tag{1.1.21}$$

We now use equations (1.1.13) and (1.1.14) to express the non-blue matrix entries using only blue matrix entries and non-blue matrix entries with coordinates on the left of the coordinates from which we started.

$$\begin{pmatrix} 1 & {\color{blue}u_{12}} & {\color{blue}u_{13}} & {\color{blue}u_{14}} & {\color{blue}u_{15}} & -{\color{blue}u_{12}}{\color{blue}u_{15}} - {\color{blue}u_{13}}{\color{blue}u_{14}} \\ & 1 & {\color{blue}u_{23}} & {\color{blue}u_{24}} & -{\color{blue}u_{23}}{\color{blue}u_{24}} & -{\color{blue}u_{15}} - {\color{blue}u_{12}}{\color{green}u_{25}} - {\color{blue}u_{13}}{\color{blue}u_{24}} - {\color{blue}u_{14}}{\color{blue}u_{23}} \\ & & 1 & 0 & -{\color{blue}u_{24}} - {\color{blue}u_{23}}{\color{green}u_{34}} & -{\color{blue}u_{14}} - {\color{blue}u_{12}}{\color{red}u_{35}} - {\color{blue}u_{13}}{\color{green}u_{34}} \\ & & & 1 & -{\color{blue}u_{23}} & -{\color{blue}u_{13}} - {\color{blue}u_{12}}{\color{red}u_{45}} \\ & & & & 1 & -{\color{blue}u_{12}} \\ & & & & & 1 \end{pmatrix}$$

Now we rewrite the remaining non-blue variables again using the two relations. Again all non-blue variables are expressed either only by blue ones or by non-blue variables with coordinates left of the coordinates we have expressed:

$$\begin{pmatrix} 1 & {\color{blue}u_{12}} & {\color{blue}u_{13}} & {\color{blue}u_{14}} & {\color{blue}u_{15}} & -{\color{blue}u_{12}}{\color{blue}u_{15}} - {\color{blue}u_{13}}{\color{blue}u_{14}} \\ & 1 & {\color{blue}u_{23}} & {\color{blue}u_{24}} & -{\color{blue}u_{23}}{\color{blue}u_{24}} & -{\color{blue}u_{15}} - {\color{blue}u_{12}} \cdot (-{\color{blue}u_{23}}{\color{blue}u_{24}}) - {\color{blue}u_{13}}{\color{blue}u_{24}} - {\color{blue}u_{14}}{\color{blue}u_{23}} \\ & & 1 & 0 & -{\color{blue}u_{24}} - {\color{blue}u_{23}} \cdot 0 & -{\color{blue}u_{14}} - {\color{blue}u_{12}} \cdot (-{\color{blue}u_{24}} - {\color{blue}u_{23}}{\color{green}u_{34}}) - {\color{blue}u_{13}} \cdot 0 \\ & & & 1 & -{\color{blue}u_{23}} & -{\color{blue}u_{13}} - {\color{blue}u_{12}} \cdot (-{\color{blue}u_{23}}) \\ & & & & 1 & -{\color{blue}u_{12}} \\ & & & & & 1 \end{pmatrix}$$

We see, that in the next step the process will terminate. We do the step now and simplify the equations after. It follows, that the element of $D_3(\mathbf{F}_q)$ are the matrices of the form

$$\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} & -u_{12}u_{15} - u_{13}u_{14} \\ & 1 & u_{23} & u_{24} & -u_{23}u_{24} & -u_{15} - u_{13}u_{24} - u_{14}u_{23} + u_{12}u_{23}u_{24} \\ & & 1 & 0 & -u_{24} & -u_{14} + u_{12}u_{24} \\ & & & 1 & -u_{23} & -u_{13} + u_{12}u_{23} \\ & & & & 1 & -u_{12} \\ & & & & & 1 \end{pmatrix}. \tag{1.1.22}$$

where $u_{12}, u_{13}, u_{14}, u_{15}, u_{23}, u_{24}$ are arbitrary elements of $\mathbf{F}_q$.

Here is a bigger example. The elements of $D_4(\mathbf{F}_q)$ are matrices of the form

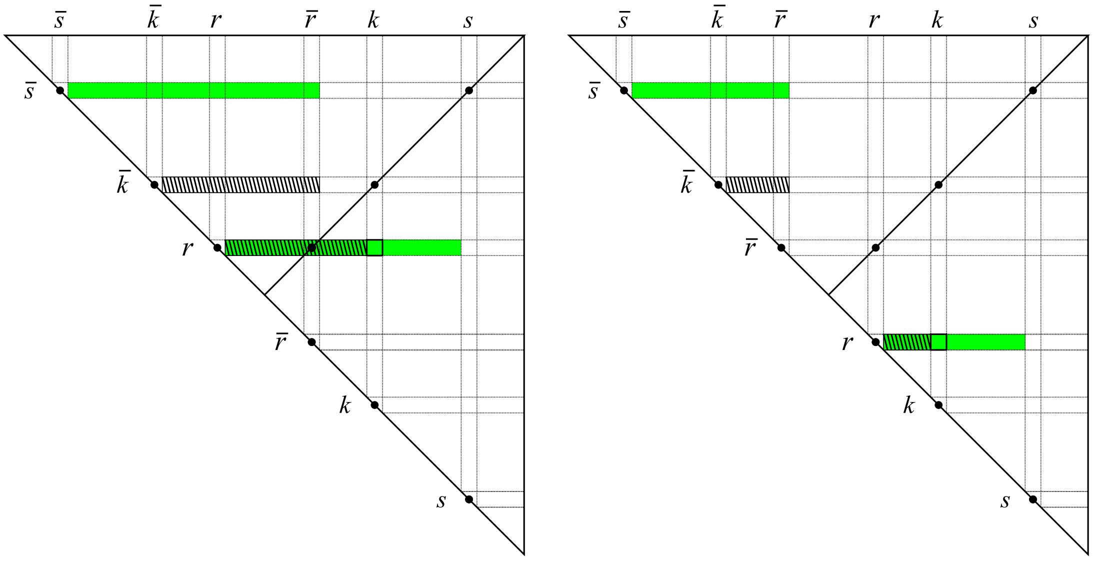
$$\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\ & 1 & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} & u_{28} \\ & & 1 & u_{34} & u_{35} & u_{36} & u_{37} & u_{38} \\ & & & 1 & u_{45} & u_{46} & u_{47} & u_{48} \\ & & & & 1 & u_{56} & u_{57} & u_{58} \\ & & & & & 1 & u_{67} & u_{68} \\ & & & & & & 1 & u_{78} \\ & & & & & & & 1 \end{pmatrix} \tag{1.1.23}$$

We again use equations (1.1.15) and (1.1.16) to rewrite the non-blue matrix entries:

$$\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & \begin{smallmatrix}-u_{12}u_{17} - u_{13}u_{16} \\ -u_{14}u_{15}\end{smallmatrix} \\ & 1 & u_{23} & u_{24} & u_{25} & u_{26} & -u_{23}u_{26} - u_{24}u_{25} & \begin{smallmatrix}-u_{17} - u_{12}u_{27} - u_{13}u_{26} \\ -u_{14}u_{25} - u_{15}u_{24} - u_{16}u_{23}\end{smallmatrix} \\ & & 1 & u_{34} & u_{35} & -u_{34}u_{35} & \begin{smallmatrix}-u_{26} - u_{23}u_{36} \\ -u_{24}u_{35} - u_{25}u_{34}\end{smallmatrix} & \begin{smallmatrix}-u_{16} - u_{12}u_{37} - u_{13}u_{36} \\ -u_{14}u_{35} - u_{15}u_{34}\end{smallmatrix} \\ & & & 1 & 0 & \begin{smallmatrix}-u_{35} \\ -u_{34}u_{45}\end{smallmatrix} & \begin{smallmatrix}-u_{25} - u_{23}u_{46} \\ -u_{24}u_{45}\end{smallmatrix} & \begin{smallmatrix}-u_{15} - u_{12}u_{47} \\ -u_{13}u_{46} - u_{14}u_{45}\end{smallmatrix} \\ & & & & 1 & -u_{34} & -u_{24} - u_{23}u_{56} & -u_{14} - u_{12}u_{57} - u_{13}u_{56} \\ & & & & & 1 & -u_{23} & -u_{13} - u_{12}u_{67} \\ & & & & & & 1 & -u_{12} \\ & & & & & & & 1 \end{pmatrix}$$

At this moment the reader will surely know how to proceed further. One can already see how the size of the polynomials will explode with increasing n. Our goal will be to get informations on the group algebra $\mathbb{C}[D_n(\mathbf{F}_q)]$ using, as much as possible, only the nice part $\triangledown$ of the matrices.

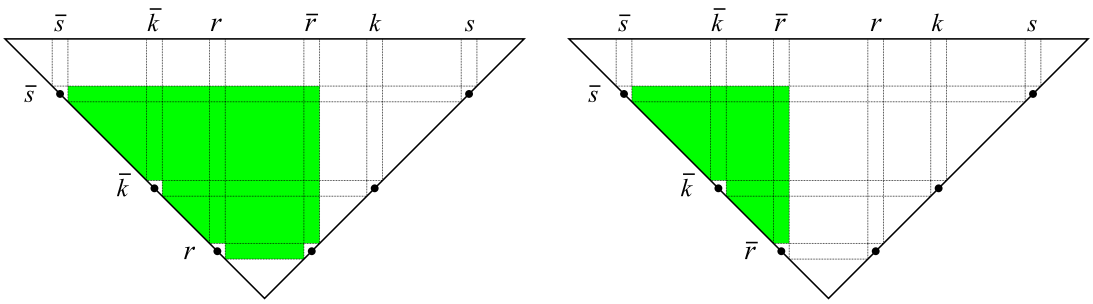
Proof of Theorem 1.1.19. We use the colouring of matrix entries of Lemma 1.1.14.



(a) Variant of the lemma, where $r \leq n$ (b) Variant of the lemma, where $r > n$

Figure 1.2.: To the proof of Theorem 1.1.19

The figure shows ◹. Equation (1.1.24) allows us to express a polynomial p_{rs} in variables and polynomials to the left of (r, s). The green boxes denote the coordinates of polynomials (resp. variables) in which we may express the polynomial p_{rs}. The black cross-hatched coordinates shows how a polynomial on the left of p_{rs}, say p_{rk}, is expressed in terms of polynomials on the left of it and variables. So, if we apply this observation recursively, we see, that p_{rs} depends only on the variables, which are green coloured in Figure 1.3.



(a) Variant, where $r \leq n$ (b) Variant, where $r > n$

Figure 1.3.: To corollary 1.1.25

The figure shows $\bigtriangledown$. The subset $R_{(r,s)}$ of $\bigtriangledown$ consists of the positions, which are coloured green.

If (r,s) is in $\diagup$, we have $(r,s) = (r,\bar{r})$ and $p_{r\bar{r}} = -\sum_{r<l\leq n} T_{rl}T_{r\bar{l}}$ is the desired polynomial due to formula (1.1.15).

Now let (r,s) be in $\triangleleft$. Suppose for any $(r,\bar{l})$ with $r < \bar{l} < s$ we have a polynomial $p_{r\bar{l}}$ over $\mathbb{Z}/p\mathbb{Z}$ in variables T_{ij}, such that $p_{r\bar{l}}(u_{ij}) = u_{r\bar{l}}$. Then, due to formula (1.1.16), the desired polynomial is given by

$$p_{rs} = -T_{\bar{s}\bar{r}} - \sum_{\bar{s}<l<\bar{r}} T_{\bar{s}l}p_{r\bar{l}}. \tag{1.1.24}$$

So we need to check, that we may make our assumption. We observe that all $(r,\bar{l})$, where $r < \bar{l} < s$, lie on the left of (r,s). If $(r,\bar{l})$ is in $\triangledown$ we take $p_{r\bar{l}} = T_{r\bar{l}}$, if $(r,\bar{l}) \in \diagup$ we use the polynomial $p_{r\bar{r}}$ calculated above.

If, now $(r,\bar{l})$ is in $\triangleleft$, we may again use formula (1.1.24) with $(r,\bar{l})$ instead of (r,s). This process terminates and we get the first part of our theorem.

In fact, we have proven a bit stronger version. As Figure 1.2 shows, we know which variables T_{ij} may appear in p_{rs} and in the case of $T_{\bar{s}\bar{r}}$ even a bit more. We state this observation in Corollary 1.1.25.

Conversely, suppose $\lambda_{ij} = u_{ij} \in \mathbf{F}_q$ is chosen arbitrary for every $(i,j) \in \triangledown$. Then the equations (1.1.15) and (1.1.16), evaluated at $(u_{ij})_{(i,j)\in\triangledown}$, turn out to be just linear equations. And we can see, there are exactly as many linear equations as there are blue and green variables. From the special form of the equations (variables appearing in them have coordinates left of the variable the equation expresses) it follows directly that the system of equations has exactly one solution. Now, Lemmas 1.1.11 and 1.1.14 imply the result. qed. |

1.1.25 Corollary* (Needed only for Lemma 3.3.16). *Let $(r,s) \in \triangleleft$. Then p_{rs} is a polynomial of the form*

$$p_{rs}(T_{ij}) = -T_{\bar{s}\bar{r}} + q_{rs}(T_{ij}),$$

where $q_{rs} \in (\mathbb{Z}/p\mathbb{Z})[T_{ij}]_{(i,j)\in R_{(r,s)}\smallsetminus(\bar{s},\bar{r})}$ with

$$R_{(r,s)} := \{(i,j) \in \triangledown \mid \bar{s} \leq i \leq r \text{ and } j \leq \bar{r}\}.$$

We make a second, more obvious, corollary of theorem

1.1.26 Corollary. *We have*

$$|D_n(\mathbf{F}_q)| = q^{n(n-1)} = q^{|\triangledown|},$$

and hence $D_n(\mathbf{F}_q)$ is a Sylow p-subgroup of $\mathrm{O}_n^+(\mathbf{F}_q)$.

Proof. From theorem 1.1.19 it is clear that $|D_n(\mathbf{F}_q)| = q^{|\triangledown|}$. So it remains to show $|\triangledown| = n(n-1)$, but it holds $|\triangledown| = \sum_{i=1}^{n-1} 2i = n(n-1)$. We have

$$|\,\mathrm{O}_N^+(\mathbf{F}_q)| = q^{n(n-1)}(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)(q^n-1)$$

due to [Gro, Theorem 9.11]. This yields $|\,\mathrm{O}_N^+(\mathbf{F}_q)|_p = q^{n(n-1)}$. qed. |

1.2. Pattern subgroups and structure theorem

First, we want to investigate how nontrivial subgroups with as most 0's as possible look. Such subgroups will be called root subgroups. More generally we then will consider what subgroups can be constructed by setting some of the blue variables equal to 0.

1.2.1 Lemma. *Let $(i, j) \in \nabla$. We set $x_{ij}(\lambda) := 1 + \lambda e_{ij} - \lambda e_{\bar{j}\bar{i}}$ for $\lambda \in \mathbf{F}_q$. Then*

$$\forall \lambda, \mu \in \mathbf{F}_q : \quad x_{ij}(\lambda)x_{ij}(\mu) = x_{ij}(\lambda + \mu)$$

and

$$X_{ij} := \Big\{ x_{ij}(\lambda) \;\Big|\; \lambda \in \mathbf{F}_q \Big\}$$

defines an abelian subgroup of $D_n(\mathbf{F}_q)$ of order q isomorphic to $(\mathbf{F}_q, +)$.

In view of theorem 1.1.19 the group X_{ij} consists of the elements of $D_n(\mathbf{F}_q)$ corresponding to the choice of $u_{ij} = \lambda$ for some $\lambda \in \mathbf{F}_q$ and $u_{rs} = 0$ for all $(r, s) \in \nabla \setminus \{(i, j)\}$.

Proof. Let $\lambda, \mu \in \mathbf{F}_q$. Using $i < j < \bar{i}$ and $i < \bar{j} < \bar{i}$[23] and also using $j \neq \bar{j}$[24] we get[25]

$$\begin{aligned} x_{ij}(\lambda)x_{ij}(\mu) &= (1 + \lambda e_{ij} - \lambda e_{\bar{j}\bar{i}})(1 + \mu e_{ij} - \mu e_{\bar{j}\bar{i}}) \\ &= 1 + \mu e_{ij} - \mu e_{\bar{j}\bar{i}} + \lambda e_{ij} + \lambda\mu\delta_{ji}e_{ij} - \lambda\mu\delta_{j\bar{j}}e_{i\bar{i}} - \lambda e_{\bar{j}\bar{i}} - \lambda\mu\delta_{\bar{i}i}e_{\bar{j}j} + \lambda\mu\delta_{\bar{i}\bar{j}}e_{\bar{j}\bar{i}} \\ &= 1 + (\lambda + \mu)e_{ij} - (\lambda + \mu)e_{\bar{j}\bar{i}} = x_{ij}(\lambda + \mu) \end{aligned}$$

So X_{ij} is an abelian group isomorphic to $(\mathbf{F}_q, +)$. It remains to show $x_{ij}(\lambda) \in D_n(\mathbf{F}_q)$. But this follows from the previous equation, since obviously $x_{ij}(\lambda)^R = x_{ij}(-\lambda)$ holds. qed. |

Now, we will give the name root a concrete meaning, starting with the following

1.2.2 Definition (Root System of type D_n)**.** *Let $\mathbb{R}^n$ be the usual n-dimensional euclidean vector space and let ε_i be the i-the standard basis vector. We call the set*

$$\Phi(D_n) = \Big\{ \pm(\varepsilon_i \pm \varepsilon_j) \;\Big|\; 1 \leq i \neq j \leq n \Big\} \subseteq \mathbb{R}^n \tag{1.2.3}$$

*the **root system of type** D_n. It is a root system in the usual sense (cf. [Hum, 9.2]). Its elements are called **roots**. We define the **set of positive roots** Φ^+ by*

$$\Phi^+(D_n) = \Big\{ \varepsilon_i \pm \varepsilon_j \;\Big|\; 1 \leq i < j \leq n \Big\}. \tag{1.2.4}$$

*Its elements are called **positive roots**.*

[23] $(i, j) \in \nabla$ implies $i < j < \bar{i}$. The application of $^-$ on that inequality yields $i = \bar{\bar{i}} < \bar{j} < \bar{i}$.
[24] $j = \bar{j}$ implies $j = 2n + 1 - j$. Hence $2j = 2n + 1$ would hold, a contradiction.
[25] applying multiple times the identity $e_{ab}e_{cd} = \delta_{bc}e_{ad}$

1.2.5 Definition/Lemma. *We define a bijection*

$$\pi : \ \nabla \to \Phi^+(D_n) : \ (i,j) \mapsto \begin{cases} \varepsilon_i - \varepsilon_j & \text{if } j \leq n, \\ \varepsilon_i + \varepsilon_{\bar{j}} & \text{if } j > n. \end{cases}$$

The bijection π relates positive roots and coordinates of matrix entries in ∇.

1.2.6 Definition (Root Subgroups)**.** *Let $\alpha \in \Phi^+(D_n)$ be a positive root we define the **root subgroup** X_α corresponding to the root α by*

$$X_\alpha := X_{\pi^{-1}(\alpha)},$$

where $X_{\pi^{-1}(\alpha)}$ is defined in Lemma 1.2.1.

1.2.7 Remark. *The name 'root subgroup' comes from C. Chevalley's construction of so called Chevalley groups. In Chevalley's construction one starts with a semisimple complex Lie algebra (for $\mathrm{SO}_N(\mathbf{F}_q)$ one would start with $\mathfrak{so}_{2n}(\mathbb{C})$). For every root space of the Lie algebra one gets a root subgroup. These root subgroups generate the Chevalley group and the root subgroups of a set of positive roots generate a Sylow p-subgroup of the Chevalley group. If one does the necessary identifications, one can obtain that these root subgroups are the X_α's defined above. One nice property of this way of looking at Chevalley groups is Chevalley's commutator formula, which expresses a whole lot of structural informations for any Chevalley group. For further reading on this topic the standard reference is [Ste].*

Note, that no further knowledge of Chevalley group is required to understand this text. It totally suffices to understand root subgroups as special subgroups of $D_n(\mathbf{F}_q)$.

We remark, that the $A_N(\mathbf{F}_q)$ corresponds to the root system A_{N-1}. So rigidly, we should write $A_{N-1}(\mathbf{F}_q)$ instead of $A_N(\mathbf{F}_q)$. But this notation would lead to more irritation than clearance, so we abuse our notation here.

Our next step is to define subsets $J \subset \nabla$ of matrix entry coordinates such that setting all entries with coordinates of $\nabla \setminus J$ to 0 leads to a subgroup of $D_n(\mathbf{F}_q)$, a so called pattern subgroup.

1.2.8 Definition (Closed subsets of Roots)**.** *Let Φ^+ be a system of positive roots, e.g. $\Phi^+(D_n)$ and let $J \subseteq \Phi^+$. We call J a **closed subset** of Φ^+ if holds*

$$\forall \alpha, \beta \in J : \qquad \alpha + \beta \in \Phi^+ \quad \Longrightarrow \quad \alpha + \beta \in J.$$

1.2.9 Definition (Pattern subgroups)**.** *Let $J \subseteq \nabla$ satisfy, that $\pi(J)$ is a closed subset of $\Phi^+(D_n)$. We then call J a **closed subset of** ∇. We define the **pattern subgroup** $D_n(\mathbf{F}_q)_J$ of $D_n(\mathbf{F}_q)$ associated to J by*

$$D_n(\mathbf{F}_q)_J := \Big\langle X_{ij} \ \Big| \ (i,j) \in J \Big\rangle.$$

In words: $D_n(\mathbf{F}_q)_J$ is the subgroup of $D_n(\mathbf{F}_q)$ generated by the root subgroups X_α, where α runs through $\pi(J)$.

Our next step is to prove some structural properties of the groups $D_n(\mathbf{F}_q)_J$. Most of them are well known (see [Ste]). For our presentation of $D_n(\mathbf{F}_q)_J$ we need some explicit information of the form '$D_n(\mathbf{F}_q)_J$ is defined by its positions of common zero entries and they are explicitly $\bigtriangledown \setminus J$'. In the previous definition we called a subset J of $\bigtriangledown$ closed if $\pi(J)$ is closed in $\Phi^+(D_n)$. We don't want to work all the time with the map π so we give a description of $\alpha + \beta \in \Phi^+(D_n)$ in terms of matrix entry coordinates:

1.2.10 Lemma. *We have*

$$(i,j),(j,k) \in \bigtriangledown \quad \Longrightarrow \quad (i,k) \in \bigtriangledown \text{ and } \pi(i,k) = \pi(i,j) + \pi(j,k), \tag{1.2.11}$$

$$\begin{matrix}(i,k),(j,\bar{k}) \in \bigtriangledown \\ \text{and } i < j\end{matrix} \quad \Longrightarrow \quad (i,\bar{j}) \in \bigtriangledown \text{ and } \pi(i,\bar{j}) = \pi(i,k) + \pi(j,\bar{k}). \tag{1.2.12}$$

Explanation: If $(i,j) \in \bigtriangledown$, then $\pi(i,j) \in \Phi^+(D_n)$. So, for example, (1.2.11) says: If we have roots α, β corresponding to $(i,j),(j,k)$ via $\pi(i,j) = \alpha, \pi(j,k) = \beta$. Then $\alpha + \beta$ is also a root and $\pi(i,k) = \alpha + \beta$.

Proof. Let us check (1.2.11) first. It holds $i < j < \bar{i}$ and $j < k < \bar{j}$. Acting with $\bar{\ }$ on the first inequality, we get $i < \bar{j} < \bar{i}$ and this yields $i < j < k < \bar{j} < \bar{i}$. It follows $(i,k) \in \bigtriangledown$.

Now, note that $(j,k) \in \bigtriangledown$ implies $j \leq n$. We get $\pi(i,j) = \varepsilon_i - \varepsilon_j$ and therefore

$$\begin{aligned}\pi(i,j) + \pi(j,k) &= \begin{cases}(\varepsilon_i - \varepsilon_j) + (\varepsilon_j - \varepsilon_k) & \text{if } k \leq n,\\ (\varepsilon_i - \varepsilon_j) + (\varepsilon_j + \varepsilon_{\bar{k}}) & \text{if } k > n.\end{cases}\\ &= \pi(i,k)\end{aligned}$$

Now, let us check (1.2.12). We have $i < k < \bar{j} < \bar{i}$ and therefore $(i,\bar{j}) \in \bigtriangledown$. Since either $k \leq n$ or $\bar{k} \leq n$ ($\Leftrightarrow k \geq n+1$) holds, we get

$$\begin{aligned}\pi(i,k) + \pi(j,\bar{k}) &= \begin{cases}(\varepsilon_i - \varepsilon_k) + (\varepsilon_j + \varepsilon_k) & \text{if } k \leq n,\\ (\varepsilon_i + \varepsilon_{\bar{k}}) + (\varepsilon_j - \varepsilon_{\bar{k}}) & \text{if } \bar{k} \leq n.\end{cases}\\ &= \varepsilon_i + \varepsilon_j = \pi(i,\bar{j}).\end{aligned}$$

qed. |

Lemma 1.2.10 has something like a 'reverse direction':

1.2.13 Lemma. *Let $\alpha, \beta, \alpha+\beta \in \Phi^+(D_n)$, then one of the following holds:*

1. $\{\alpha, \beta\} = \{\pi(i,j), \pi(j,k)\}$ *and* $\alpha + \beta = \pi(i,k)$ *for some* $(i,j),(j,k) \in \bigtriangledown$.

2. *$\{\alpha, \beta\} = \{\pi(i,k), \pi(j,\bar{k})\}$ and $\alpha + \beta = \pi(i,\bar{j})$ for some $(i,k),(j,\bar{k}) \in \triangledown$, where $i < j$.*

We want to remark, that this lemma states, that (1.2.11) *and* (1.2.12) *already know the combinatorics of the root system $\Phi^+(D_n)$ in terms of the addition operation.*

Proof. Suppose $\alpha, \beta \in \Phi^+(D_n)$, such that $\alpha + \beta \in \Phi^+(D_n)$. Then we may assume either $\alpha = \varepsilon_a - \varepsilon_b \wedge \beta = \varepsilon_c - \varepsilon_d$ or $\alpha = \varepsilon_a - \varepsilon_b \wedge \beta = \varepsilon_c + \varepsilon_d$ for some $i,j,k,l \in \mathbb{N}$.

In the first case we have either $b = c$ or $a = d$. If $b = c$ we set $a = i, b = c = j, d = \bar{k}$, if $a = d$ we set $c = i, d = a = j, b = \bar{k}$. In both cases the result follows.

In the second case we have either $b = c$ or $b = d$. If $b = c$ we set $a = i, b = c = j, d = \bar{k}$. If $b = d$ we must have either $a < c$ or $a > c$ (in case $a = c$ the expression $\alpha + \beta$ would be no root). If now $a < c$ we set $a = i, c = j, b = k, d = \bar{k}$ and if $c < a$ we set $c = i, a = j, b = \bar{k}, d = k$. In all of these cases the result follows. qed. |

1.2.14 Corollary. *A set J is a closed subset of $\triangledown$ if and only if*

$$\begin{array}{rcl} (i,j),(j,k) \in J & \Longrightarrow & (i,k) \in J \\ (i,k),(j,\bar{k}) \in J \text{ and } i < j & \Longrightarrow & (i,\bar{j}) \in J \end{array}$$

If $J \subseteq \text{◹}^{\because}$, then J is a closed subset of $\triangledown$ if and only if $(i,j),(j,k) \in J \Longrightarrow (i,k) \in J$.

Proof. Suppose J is closed in $\triangledown$. Then $\pi(J)$ is a closed subset of $\Phi^+(D_n)$. Now, let $(i,j),(j,k) \in J$ be arbitrary. Using Lemma 1.2.10 we get $(i,k) \in \triangledown$ and $\pi(i,k) = \pi(i,j) + \pi(j,k)$. But since both $\pi(i,j)$ and $\pi(j,k)$ are in $\pi(J)$ and $\pi(J)$ is a closed subset of $\Phi^+(D_n)$ we get $\pi(i,k)$ in $\pi(J)$. Hence we have $(i,k) \in J$. For the second implication we can argue completely analogous.

Conversely let us assume the two implications hold. Let $\alpha, \beta \in \pi(J)$, such that $\alpha + \beta \in \Phi^+(D_n)$. Due to Lemma 1.2.13, we may assume (without loss of generality), that

$$\alpha = \pi(i,j), \ \beta = \pi(j,k), \ \text{ and } \alpha + \beta = \pi(i,k)$$

or

$$\alpha = \pi(i,k), \ \beta = \pi(j,\bar{k}), \ i < j, \ \text{ and } \alpha + \beta = \pi(i,\bar{j}).$$

where $i,j,k \in \mathbb{N}$ are suitable chosen. Now, we have $(i,k) \in J$ (resp. $(i,\bar{j}) \in J$ in the second case). Hence $\alpha + \beta \in \pi(J)$ and thus J is a closed subset of $\triangledown$.

If $\text{◹}^{\because} \subseteq J$, then $(i,k),(j,\bar{k}) \in J$ is not possible, since then either k or $\bar{k}$ has to be bigger than n. qed. |

1.2.15 Remark. *The system of positive roots associated to the group $A_N(\mathbf{F}_q)$ is*

$$\Phi^+(A_{N-1}) := \left\{ \varepsilon_i - \varepsilon_j \;\middle|\; 1 \le i < j \le N-1 \right\} \subseteq \mathbb{R}^{N-1}.$$

There is also a corresponding set of matrix positions, namely ◹, and a bijection

$$\pi : \ \Phi^*(A_{N-1}) \longrightarrow \text{◹} : \ \varepsilon_i - \varepsilon_j \longmapsto (i,j).$$

Again, we say a J is a closed subset of ◹ if $\pi(J)$ is closed in $\Phi^+(A_{N-1})$, c.f. 1.2.8. The statement similarly to Corollary 1.2.14 for $A_N(\mathbf{F}_q)$ is

$$J \text{ is a closed subset of } \text{◹} \qquad \Longleftrightarrow \qquad (i,j),(j,k) \in J \Rightarrow (i,k) \in J.$$

As in the group $D_n(\mathbf{F}_q)$ we define the pattern subgroup associated to a closed subset $J \subseteq \text{◹}$ as

$$A_N(\mathbf{F}_q)_J := \left\langle \tilde{X}_{ij} \;\middle|\; (i,j) \in J \right\rangle,$$

where $\tilde{X}_{ij} := \{\tilde{x}_{ij}(\lambda) \mid \lambda \in \mathbf{F}_q\} \cong \mathbf{F}_q^+$ with $\tilde{x}_{ij}(\lambda) := 1 + \lambda e_{ij}$.

1.2.16 Definition. *For an $N \times N$ matrix A we call $\operatorname{supp}(A) := \{(i,j) \in \square \mid A_{ij} \neq 0\}$ the **support** of A. For an element $g \in A_N(\mathbf{F}_q)$ we call $\overline{\operatorname{supp}}(g) := \operatorname{supp}(g-1)$ the **essential support** of g. Clearly we have for $g \in A_N(\mathbf{F}_q)$*

$$\overline{\operatorname{supp}}(g) = \{(i,j) \in \text{◹} \mid g_{ij} \neq 0\}.$$

Let H be a subgroup of $A_N(\mathbf{F}_q)$. Then $\overline{\operatorname{supp}}(H) := \bigcup_{h \in H} \overline{\operatorname{supp}}(h)$.

1.2.17 Example. *Let $(i,j) \in \triangledown$, $\lambda \in \mathbf{F}_q^\times$ be arbitrary and let $u = x_{ij}(\lambda)$. Then we have $\overline{\operatorname{supp}}(u) = \{(i,j), \overline{(i,j)}\}$, where $\overline{(i,j)} = (\bar{j}, \bar{i})$ as in Remark 1.1.17.*

We give the following lemma, which seems extremely technical but is very useful, since Proposition 1.2.21 and Theorem 1.2.22 (where we need additionally Theorem 1.1.19) use it heavily.

1.2.18 Lemma. *Let $J \subset \triangledown$ be closed. We define $\bar{J} := \{(\bar{j}, \bar{i}) \mid (i,j) \in J\} \subseteq \lhd$. ($\bar{J}$ is the set J mirrored on $/$ as described in remark 1.1.17). Additionally we define the set $\dot{J} := \{(i, \bar{i}) \mid \exists k : (i,k), (i, \bar{k}) \in J\} \subseteq /$. We get*

$$(i,j),(j,k) \in J \cup \dot{J} \cup \bar{J} \qquad \Longrightarrow \qquad (i,k) \in J \cup \dot{J} \cup \bar{J}. \tag{1.2.19}$$

and

$$(i,j) \in J,\ (j,k) \in \bar{J} \qquad \Longrightarrow \qquad (i,k) \in \begin{cases} J & \text{if } i < \bar{k}, \\ \dot{J} & \text{if } i = \bar{k}, \\ \bar{J} & \text{if } i > \bar{k}. \end{cases} \tag{1.2.20}$$

In particular $s(J) := J \cup \dot{J} \cup \bar{J}$ is a closed subset of ◹ and hence we can define the pattern subgroup $A_N(\mathbf{F}_q)_{s(J)}$, c.f. Remark 1.2.15.

Proof. We have to do a case-by-case checking. In order to provide the reader with enough examples of the usage of the matrix coordinates, we do all of them explicitly:

(i,j)	(j,k)	Proof of the case:
$\in J$	$\in J$	This follows directly from Corollary 1.2.14.
	$\in \dot{J}$	We get $(j,k)=(j,\bar{j})$ and there exists an l, such that $(j,l),(j,\bar{l})\in J$. Since we have $(i,j),(j,l)\in J$, Corollary 1.2.14 implies $(i,l)\in J$. Therefore it holds $(i,l),(j,\bar{l})\in J$ and $i<j$, since $(i,j)\in J$. Hence Corollary 1.2.14 implies $(i,\bar{j})\in J$, but $\bar{j}=k$. Thus $(i,k)\in J$.
	$\in \bar{J}$	Here we have to show equation (1.2.20). Since $(j,k)\in\bar{J}$ we have $(i,j),(\bar{k},\bar{j})\in J$. If $k=\bar{i}$ we have $(i,k)=(i,\bar{i})$. Now $(i,j)\in J$ and $(i,\bar{j})=(\bar{k},\bar{j})\in J$ show that $(i,k)=(i,\bar{i})\in\dot{J}$. For the other two subcases we can simply apply Corollary 1.2.14.
$\in \dot{J}$	$\in J$	We have $(i,j)=(i,\bar{i})\in\diagup$. But $\diagup$ is a subset of $\lrtriangle$. Hence we get $i<\bar{i}$. On the other hand we have $(\bar{i},k)=(j,k)\in\triangledown$. This implies $\bar{i}<k<i$. But that is a contradiction, so the case can not appear.
	$\in \dot{J}$	In this case would hold $(i,j)=(i,\bar{i})$ and $(j,k)=(\bar{i},i)$. And both $(i,\bar{i})$ and $(i,\bar{i})$ would be in $\diagup$. So, as above, we get $i<\bar{i}<i$, which is again a contradiction. So this case does not appear.
	$\in \bar{J}$	We have $(i,j)=(i,\bar{i})$ and thus $(j,k)=(\bar{i},k)$. Since $(j,k)\in\bar{J}$ we get $(\bar{k},\bar{j})=(\bar{k},i)\in J$. But $(\bar{k},i)\in J$ and $(i,\bar{i})\in\dot{J}$ implies $(\bar{k},\bar{i})\in J$, due to the second case. Thus we have $(i,k)\in\bar{J}$.
$\in \bar{J}$	$\in J$	We have $(\bar{j},\bar{i})\in J$ and $(j,k)\in J$ So we get $\bar{j}<n$ and $j<n$. But action with $^-$ on the first inequality implies $j>\bar{n}=n+1$. So we have a contradiction and this case can not appear.
	$\in \dot{J}$	This time $(j,k)=(j,\bar{j})$ and $(j,\bar{j})\in\diagup$. So we get $j<n$. And again $(i,j)\in\bar{J}$ implies $j>n+1$. So this case also does not appear.
	$\in \bar{J}$	This one is very simple: $(i,j),(j,k)\in\bar{J}$ imply $(\bar{k},\bar{j}),(\bar{j},\bar{i})\in J$. But Corollary 1.2.14 then implies $(\bar{k},\bar{i})\in J$ and therefore $(i,k)\in\bar{J}$.

qed. |

1.2.21 Proposition. *We have*

$$\overline{\mathrm{supp}}\big(D_n(\mathbf{F}_q)_J\big)=J\cup\dot{J}\cup\bar{J}.$$

Proof. Let u,v be unitriangular matrices, such that $\overline{\mathrm{supp}}(u)$ and $\overline{\mathrm{supp}}(v)$ are subsets of $J\cup\dot{J}\cup\bar{J}$. Then we can write

$$u=1+\sum_{(i,j)\,\in\,\overline{\mathrm{supp}}(u)}u_{ij}e_{ij}\qquad\text{respectivly}\qquad v=1+\sum_{(r,s)\,\in\,\overline{\mathrm{supp}}(v)}v_{rs}e_{rs}$$

We calculate the product of u and v and get

$$uv=1+\sum_{(i,j)\,\in\,\overline{\mathrm{supp}}(u)}u_{ij}e_{ij}+\sum_{(r,s)\,\in\,\overline{\mathrm{supp}}(v)}v_{rs}e_{rs}+\sum_{\substack{(i,j)\,\in\,\overline{\mathrm{supp}}(u)\\(r,s)\,\in\,\overline{\mathrm{supp}}(v)}}u_{ij}v_{rs}\underbrace{e_{ij}e_{rs}}_{=\delta_{jr}e_{is}}.$$

Due to Lemma 1.2.18 $(i,j),(j,s) \in J \cup \dot{J} \cup \bar{J}$ implies $(i,s) \in J \cup \dot{J} \cup \bar{J}$. Thus we have

$$\overline{\operatorname{supp}}(u), \overline{\operatorname{supp}}(v) \subseteq J \cup \dot{J} \cup \bar{J} \qquad \Longrightarrow \qquad \overline{\operatorname{supp}}(uv) \subseteq J \cup \dot{J} \cup \bar{J}.$$

In Example 1.2.17 we saw, that each element $x_{ij}(\lambda)$ with $(i,j) \in J$, $\lambda \in \mathbf{F}_q$ has essential support in $J \cup \dot{J} \cup \bar{J}$. But $D_n(\mathbf{F}_q)_J$ is generated by this elements by definition. Hence we have $\overline{\operatorname{supp}}\big(D_n(\mathbf{F}_q)_J\big) \subseteq J \cup \dot{J} \cup \bar{J}$.

Conversely: If $(i,j) \in J$ and $\lambda \in \mathbf{F}_q^\times$ we have $\overline{\operatorname{supp}}\big(x_{ij}(\lambda)\big) = \{(i,j),(\bar{j},\bar{i})\}$, as seen in Example 1.2.17. Thus we get $J \cup \bar{J} \subseteq \overline{\operatorname{supp}}\big(D_n(\mathbf{F}_q)_J\big)$.

Let $(i,\bar{i}) \in \dot{J}$. Then there exists k such that $(i,k),(i,\bar{k}) \in J$. We calculate[26]

$$\begin{aligned} x_{ik}(\lambda)x_{i\bar{k}}(\mu) &= (1+\lambda e_{ik} - \lambda e_{\bar{k}\bar{i}})(1+\mu e_{i\bar{k}} - \mu e_{k\bar{i}}) \\ &= 1+\lambda e_{ik} - \lambda e_{\bar{k}\bar{i}} + \mu e_{i\bar{k}} + \lambda\mu e_{ik}e_{i\bar{k}} - \lambda\mu e_{\bar{k}\bar{i}}e_{i\bar{k}} - \mu e_{k\bar{i}} - \lambda\mu e_{ik}e_{k\bar{i}} + \lambda\mu e_{\bar{k}\bar{i}}e_{k\bar{i}} \\ &= 1+\lambda e_{ik} - \lambda e_{\bar{k}\bar{i}} + \mu e_{i\bar{k}} - \mu e_{k\bar{i}} - \lambda\mu e_{i\bar{i}} \end{aligned}$$

This yields $(i,\bar{i}) \in \overline{\operatorname{supp}}\big(D_n(\mathbf{F}_q)_J\big)$. qed. |

We get the following structure theorem. The second interpretation of $D_n(\mathbf{F}_q)_J$ stated in the theorem (the one about root subgroups) is a well known fact from the general theory of Chevalley groups (c.f. [Ste, Lemma 17]).

1.2.22 Theorem (Structure of $D_n(\mathbf{F}_q)_J$). *Let $J \subseteq \nabla$ be closed. Then we have*

$$D_n(\mathbf{F}_q)_J = \Big\{ u \in D_n(\mathbf{F}_q) \;\Big|\; \overline{\operatorname{supp}}(u) \cap \nabla \subseteq J \Big\} = \Big\{ \prod_{(i,j)\in J} x_{ij}(\lambda_{ij}) \;\Big|\; \lambda_{ij} \in \mathbf{F}_q \Big\}$$

where the product can be taken in arbitrary, but fixed, order.

Proof. We have $\{\prod_{(i,j)\in J} x_{ij}(\lambda_{ij}) \mid \lambda_{ij} \in \mathbf{F}_q\} \subseteq D_n(\mathbf{F}_q)_J$ after definition. Proposition 1.2.21 implies $D_n(\mathbf{F}_q)_J \subseteq \{u \in D_n(\mathbf{F}_q) \mid \overline{\operatorname{supp}}(u) \cap \nabla \subseteq J\}$. The result follows if we can show that all of these sets have equal size. Theorem 1.1.19 implies $|\{u \in D_n(\mathbf{F}_q) \mid \overline{\operatorname{supp}}(u) \cap \nabla \subseteq J\}| = q^{|J|}$. (This implies also corollary 1.2.23.) On the other hand $|\{\prod_{(i,j)\in J} x_{ij}(\lambda_{ij}) \mid \lambda_{ij} \in \mathbf{F}_q\}| \leq q^{|J|}$. Thus it remains to check, that the elements of this set are pairwise different. Suppose $\prod_{(i,j)\in J} x_{ij}(\lambda_{ij}) = \prod_{(i,j)\in J} x_{ij}(\mu_{ij})$, where the products are taken in the same order. We can write any $x_{ij}(\lambda_{ij})$ as $(1+\lambda_{ij}e_{ij})(1-\lambda_{ij}e_{\bar{j}\bar{i}})$. But now Proposition A.1.1 implies $\lambda_{ij} = \mu_{ij}$ for all $(i,j) \in \nabla$. qed. |

1.2.23 Corollary. *We have $|D_n(\mathbf{F}_q)_J| = q^{|J|}$.*

If we take $J = \nabla$ in theorem 1.2.22 we get

[26] for arbitrary $\lambda, \mu \in \mathbf{F}_q^\times$

1.2.24 Corollary. *We have*

$$D_n(\mathbf{F}_q) = \Big\{ \prod_{(i,j)\in\nabla} x_{ij}(\lambda_{ij}) \;\Big|\; \lambda_{ij} \in \mathbf{F}_q \Big\},$$

where the product can be taken in arbitrary, but fixed, order.

Theorem 1.2.22 has even more applications. We can consider $A_n(\mathbf{F}_q)$ as a subgroup of $D_n(\mathbf{F}_q)$ using the following embedding:

1.2.25 Lemma. *The map*

$$\varphi : \underbrace{A_n(\mathbf{F}_q)}_{n\times n\text{-matrices}} \longrightarrow \underbrace{D_n(\mathbf{F}_q)}_{2n\times 2n\text{-matrices}} \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^R \end{pmatrix}$$

is a group monomorphism[27] *with whom we can view $A_n(\mathbf{F}_q)$ as a subgroup of $D_n(\mathbf{F}_q)$.*

1.2.26 Remark. *Remember, that $\Phi^+(A_{n-1}) = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\} \subset \Phi^+(D_n)$ is the system of positive roots of type A_{n-1}. Clearly $\Phi^+(A_{n-1})$ is a closed subset of $\Phi^+(D_n)$. Thus we have $\operatorname{Im}\varphi = D_n(\mathbf{F}_q)_{\nabla^{\vdots}}$, since $\nabla^{\vdots} = \pi^{-1}\big(\Phi^+(A_{n-1})\big)$. Hence we also can view $A_n(\mathbf{F}_q)$ as a pattern subgroup of $D_n(\mathbf{F}_q)$. If we now consider a closed subset J of $\Phi^+(A_{n-1})$, then $\pi^{-1}(J) \subseteq \nabla^{\vdots}$ is a closed subset (of type D_n) of ∇ (apply Corollary 1.2.14).*

1.2.27 Corollary (Structure of $A_n(\mathbf{F}_q)_J$)**.** *We use the notation from Remark 1.2.15. Let J be a closed subset of ◹. Then we have*

$$A_n(\mathbf{F}_q)_J = \Big\{ g \in A_n(\mathbf{F}_q) \;\Big|\; \overline{\operatorname{supp}}(g) \subseteq J \Big\} = \Big\{ \prod_{(i,j)\in J} \tilde{x}_{ij}(\lambda_{ij}) \;\Big|\; \lambda_{ij} \in \mathbf{F}_q \Big\},$$

where the product can be taken in arbitrary, but fixed, order.

Proof. Remark 1.2.26 allows us to apply Theorem 1.2.22. Then we use the embedding of Lemma 1.2.25 to get the desired result. qed. |

1.2.28 Remark. *We will need the structure theory of $A_N(\mathbf{F}_q)$ and its pattern subgroups in Sections 3.1 and 3.6.*

1.2.29 Remark. *The previous corollary shows our point of view. Usually one would prove the results about the structure of $A_N(\mathbf{F}_q)$ and after that investigate the more complicated situation of $D_n(\mathbf{F}_q)$ using the results about $A_N(\mathbf{F}_q)$. We go the other way around, proving the results first for the more complicated situation and getting afterwards the corresponding results for $A_N(\mathbf{F}_q)$ for free. This will occur again in Chapter 3, where we will prove results about orbit modules of $D_n(\mathbf{F}_q)$ which will in the end imply the already known results for the orbit modules of $A_N(\mathbf{F}_q)$. A precise description of the known results, which we will re-obtain as corollaries, is given in the end of section 2.2.*

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[27] The map φ is well defined, since we have $\begin{pmatrix} g & 0 \\ 0 & g^R \end{pmatrix}^R = \begin{pmatrix} g & 0 \\ 0 & g^R \end{pmatrix}$.

2. Monomial representations arising from 1-cocycles

In [Yan1], [Yan2] N. Yan defined a monomial basis of $\mathbb{C}A_N(\mathbf{F}_q)$ using the dual space of the Lie algebra $\mathfrak{u}$ of $U = A_N(\mathbf{F}_q)$. This Lie algebra $\mathfrak{u}$ is given by the set of strictly upper triangular $N \times N$-matrices over the field $\mathbf{F}_q$, it is in fact an associative nilpotent $\mathbf{F}_q$-algebra of finite dimension. The bijective map $f : U \to \mathfrak{u}$ given by $f(g) = g - 1$ (with inverse $f^{-1}(X) = 1 + X$) relates the group U with the associative nilpotent algebra $\mathfrak{u}$.

In [DI] P. Diaconis and I.M. Isaacs turned the thing on its head. They saw, that the important part for the construction of N. Yan is in fact that U has the form of $1 + \mathfrak{u}$. Thus they started with an arbitrary associative nilpotent F-algebra J of finite dimension. They examined groups of the form $1 + J$. These groups are called F-algebra groups and consist of elements of the form $1 + x$, where 1 is just a formal one. The multiplication law is given by $(1 + x)(1 + y) = 1 + x + y + xy$. Again there exists a bijective map $f : 1 + J \to J$ with $f(g) = g - 1$ and (using Yan's construction) they defined a monomial basis for the complex group algebra of $1 + J$.

In this thesis we are mainly interested the group $D_n(\mathbf{F}_q)$ which is not an F-algebra group. In fact Sylow p-subgroups of classical groups are not F-algebra groups (with the obvious exception of $A_N(\mathbf{F}_q)$). Thus we need a generalization of the Diaconis-Isaacs-Yan construction. This generalisation will be provided by the construction of monomial representations using 1-cocycles. This method is in fact a true generalisation of the Diaconis-Isaacs-Yan construction (their maps f from U to $\mathfrak{u}$, respectively $1 + J$ to J are such 1-cocycles and the construction is the same).

It is further noteworthy, that there is a natural connection to Kirillov's orbit method (originally developed in [Kir1], see also [Kir3]). The Kirillov orbit method uses the adjoint operation of a Lie group (i.e. conjugation in the case of a matrix group) on the dual space of the associated Lie algebra to provide a linearisation. For F-algebra groups P. Diaconis and I.M. Isaacs pointed out that their construction is a cruder version of Kirillovs orbit method (c.f. [DI]). The connection is not as close in our case (since we can't consider bi-orbits, c.f. Remark 2.1.12 3. and 4.).

We also want to remark that B. Ackermann, R. Dipper and Q. Guo developed in [ADG] a more representation theoretic language for N. Yan's construction, which we shall use here as well.

2.1. Construction of the monomial representations

2.1.1 Notation. *Throughout this section let G always be a (not necessarily finite) group.*

We start by defining the most important actors of this section. We want to emphasize that at moment groups may have infinite order and representations infinite dimension.

2.1.2 Definition (Monomial Representation). *Let K be some field*[28] *and M a G-module with K-basis $\mathcal{B}$. then we call $\mathcal{B}$ a **monomial basis** (and thus M a **monomial representation**) if there exist*

(a) a group action $. : \ \mathcal{B} \times G \longrightarrow \mathcal{B} : \ (b, g) \longmapsto b.g$

(b) a function $\alpha : \mathcal{B} \times G \to K^{\times}$,

such that the module operation '$*$' of G on the basis $\mathcal{B}$ of M is given by

$$b * g = \alpha(b, g) b.g \qquad \text{for all } g \in G, b \in \mathcal{B}. \tag{2.1.3}$$

2.1.4 Remark. *Associativity of the module operation implies*

$$\alpha(b, gh) = \alpha(b, g)\alpha(b.g, h) \qquad \text{for all } b \in \mathcal{B} \text{ and } g, h \in G. \tag{2.1.5}$$

Conversely, let $\mathcal{B}$ be a set on which G acts as permutation group and let $\alpha : \mathcal{B} \times G \longrightarrow K^{\times}$ be a map satisfying (2.1.5)*. Then* (2.1.3) *defines a module operation on the K-vector space with basis $\mathcal{B}$. Clearly $\mathcal{B}$ is a monomial basis for this module.*

Our goal in this section is to provide a construction of interesting monomial representations using so-called 1-cocycles, which we define first.

2.1.6 Definition (1-cocycle). *Let V be an abelian group. Suppose G acts on V as automorphisms*[29]*. Then a map $f : G \longrightarrow V$, satisfying*

$$f(xg) = f(x)g + f(g) \qquad \text{for all } x, g \in G,$$

*is called a (right) 1-**cocycle**.*

2.1.7 Remark. *(a) Note that for a 1-cocylce it always holds $f(1) = 0$.*[30]

[28] in German 'fields' are called 'Körper'. Later we set $K = \mathbb{C}$. We use K since the more natural F would indicate towards $\mathbf{F}_q$.

[29] i.e. we have a group action on of G on V and for all $A, B \in V$ and $g \in G$ we have $(A + B)g = Ag + Bg$.

[30] $f(1) = f(1 \cdot 1) = f(1)1 + f(1) = 2f(1)$.

(b) *Equation 2.1.5 defines a multiplicative (i.e. we use '$\cdot$' instead of '$+$') 1-cocycle from G to $\mathrm{Maps}(\mathcal{B}, K)$. This is a well understood correspondence between monomial representations and 1-cocycles, c.f. pages 250-252 of [Kir4]. We want to go in the other direction and define monomial representations using 1-cocycles.*

2.1.8 Definition/Lemma. *Let V be an abelian group on which G acts from the right as automorphisms. Let $\hat{V}$ denote the character group $\mathrm{Hom}(V, K^\times)$. Then the group action of G on V induces a group action of G on $\hat{V}$ given by*

$$. \ : \hat{V} \times G \longrightarrow \hat{V} : \ (\varphi, g) \longmapsto \varphi.g,$$

where $\big(\varphi.g\big)(A) = \varphi(Ag^{-1})$ for all $A \in V$.

Proof. $\hat{\chi}.g(A+B) = \hat{\chi}\big((A+B)g^{-1}\big) = \hat{\chi}(Ag^{-1} + Bg^{-1}) = \hat{\chi}.g(A)\ \hat{\chi}.g(B)$. qed. |

2.1.9 Lemma (Artin's theorem, c.f. [Lan, VIII Theorem 4.1])**.** *Let V be an abelian group and K an arbitrary field. Then the character group*

$$\hat{V} = \mathrm{Hom}(V, K^\times) \subseteq K^V$$

is a linearly independent subset of the K-vector space K^V of K-valued functions on V.

2.1.10 Definition. *Let $K\hat{V}$ denote K-$\mathrm{span}\,\hat{V}$ seen as vector subspace of K^V. Due to Artin's theorem $\hat{V}$ is a basis of $K\hat{V}$.*

2.1.11 Theorem (Main theorem, general case)**.** *Let V be an abelian group on which G acts as automorphisms and $f : G \to V$ a 1-cocycle. We define the map $\alpha : \hat{V} \times G \longrightarrow K^\times$ by setting*

$$\alpha(\hat{\chi}, g) := \hat{\chi} f(g^{-1}) \qquad \text{for all } \hat{\chi} \in \hat{V} \text{ and } g \in G.$$

Then α satisfies equation (2.1.5) and hence we can make $K\hat{V}$ into a monomial representation with monomial basis $\hat{V}$ by extending the operation

$$\hat{\chi} * g := \alpha(\hat{\chi}, g)\hat{\chi}.g \qquad \text{for all } \hat{\chi} \in \hat{V} \text{ and } g \in G$$

*linearly. We say the G-module $(K\hat{V}, *)$ **arises from the 1-cocycle** f.*

Proof. Let $\hat{\chi} \in \hat{V}$ and $g, h \in G$ be arbitrary. Then we have

$$\begin{aligned}\alpha(\hat{\chi}, gh) &= \hat{\chi} f\big((gh)^{-1}\big) = \hat{\chi}\big(f(h^{-1}g^{-1})\big) = \hat{\chi}\big(f(h^{-1})g^{-1} + f(g^{-1})\big) \\ &= \hat{\chi}\big(f(h^{-1})g^{-1}\big)\hat{\chi}\big(f(g^{-1})\big) = \big(\hat{\chi}.g\big) f\big((h^{-1})\big)\alpha(\hat{\chi}, g) = \alpha(\hat{\chi}, g)\alpha(\hat{\chi}.g, h).\end{aligned}$$

Thus equation (2.1.5) is proven. We have

$$(\hat{\chi} * g) * h = \alpha(\hat{\chi}, g)(\hat{\chi}.g) * h = \underbrace{\alpha(\hat{\chi}, g)\alpha(\hat{\chi}.g, h)}_{= \alpha(\hat{\chi}, gh)} \underbrace{(\hat{\chi}.g).h}_{= \hat{\chi}.(gh)} = \hat{\chi} * (gh).$$

Thus $K\hat{V}$ is a monomial representation with monomial basis $\hat{V}$. qed. |

2.1.12 Remark. *Let us make some remarks to Theorem 2.1.11:*

1. *Proposition 2.1 of [Yan2] and Lemma 5.3 of [DI] are special cases of our theorem (See the introduction to this chapter for further details). Our proof is just Yan's proof (used also by Diaconis-Isaacs) applied to the more general situation.*

2. *All applies also for the 'left module situation', i.e. if G acts on V as automorphims from the left, f satisfies the formula $f(gx) = gf(x) + f(g)$ and the G-operation on the space K^G is given by $\big(g\varphi\big)(h) = \varphi(g^{-1}h)$.*

3. *In the case of $A_N(\mathbf{F}_q)$ (or more generally F-algebra groups) the considered 1-cocycles are in fact simultaneously 1-cocylces for the right module situation and the left module situation. Thus in these situations $K\mathcal{B}$ is a G-G-bimodule where both operations are acting monomially.*

4. *The monomial bimodule structure described in 3. does not carry over to the case of $D_n(\mathbf{F}_q)$ (and other series of classical groups, as will be discussed in a forthcoming article). The reason is, that our construction of the 1-cocycles in these cases only can handle either right multiplication or left multiplication (if one prefers) but not both simultaneously.*

5. *Let G be a finite group and suppose $F\mathcal{B}$ is a permutation module, which arises from f. Then all values $\alpha(\hat\chi, g)$ have to be equal to 1. Thus $\operatorname{im} f \subseteq \bigcap_{\hat\chi\in\hat V} \ker\hat\chi = (0)$. Hence in this situation we have $f = 0$.*

2.1.13 Definition/Lemma. *Let K be an arbitrary field. Then the K-vector space K^G of K-valued functions on G is a right G-module with operation given by*

$$K^G \times G \longrightarrow K^G : (\varphi, g) \longmapsto \varphi * g,$$

where $\varphi * g$ is the function on G with values in K defined by $\big(\varphi * g\big)(x) = \varphi(xg^{-1})$.

2.1.14 Remark. *Note, that for finite groups G the module K^G is isomorphic to KG with isomorphism given by*

$$\Phi : K^G \longrightarrow KG :\ \varphi \longmapsto \sum_{x\in G} \varphi(x)x.$$ [31]

2.1.15 Lemma. *Let K be a field, $f : G \longrightarrow V$ be a map and V an abelian group on which G acts from the right as automorphisms.*[32] *Then*

$$f^* :\ K^V \longrightarrow K^G :\ \varphi \longmapsto f^*(\varphi) = \varphi f$$

defines a K-linear map. If f is additionally surjective, then f^* is injective.

[31] $\Phi(\varphi)g = \sum_{x\in G}\varphi(x)xg = \sum_{y\in G}\varphi(yg^{-1})y = \sum_{y\in G}\varphi * g(y)y = \Phi(\varphi * g)$

[32] the last condition is not needed for the statement of the lemma. We still assume it to put the lemma in our context.

2.1.16 Corollary. *Let V be an abelian group on which G acts as automorphisms and $f : G \to V$ a 1-cocycle. Then $f^*|_{K\hat{V}}$ is a G-homomorphism, i.e. the K-linear map f^* satisfies*

$$f^*(\hat{\chi} * g) = f^*(\hat{\chi}) * g \qquad \text{for all } \hat{\chi} \in \hat{V} \text{ and } g \in G.$$

If f is surjective, then f^* is injective and hence $K\hat{V} \cong f^*(K\hat{V})$ is a submodule of K^G.

Proof. Let $\hat{\chi} \in \hat{V}$ and $x, g \in G$ be arbitrary. Then we have

$$\begin{aligned} \big(f^*(\hat{\chi}) * g\big)(x) &= f^*(\hat{\chi})(xg^{-1}) = \alpha\big(\hat{\chi}, (xg^{-1})^{-1}\big) = \alpha(\hat{\chi}, gx^{-1}) \\ &\overset{(2.1.5)}{=} \alpha(\hat{\chi}, g)\alpha(\hat{\chi}.g, x^{-1}) = \alpha(\hat{\chi}, g) f^*\big(\hat{\chi}.g\big)(x) = f^*\big(\hat{\chi} * g\big)(x) \end{aligned}$$

yielding the claim. qed. |

2.1.17 Remark.

1. *Suppose G acts on V trivially. Then f is a 1-cocycle if and only if it is a group homomorphism. In that case $f^* : \mathrm{Hom}(V, K^\times) \longrightarrow \mathrm{Hom}(G, K^\times)$. More explicitly we have $f^*(\hat{\chi})g = \hat{\chi} f(g^{-1}) f^*(\hat{\chi})$ for all $\hat{\chi} \in \hat{V}$ and $g \in G$.*
2. *Suppose f is bijective, G, V finite and $K = \mathbb{C}$. Then $\hat{V}$ is just the set of irreducible ordinary characters of V and thus we have $|\hat{V}| = |V| = |G| = \dim(\mathbb{C}^G)$. Thus in that situation we have $f^*(\mathbb{C}\hat{V}) = \mathbb{C}^G \cong \mathbb{C}G$ and thus we would have basis for the complex group algebra of G on which the usual right operation of G acts monomially. (We will construct such a 1-cocylce for $D_n(\mathbf{F}_q)$ in Section 3.1.)*

Up to now Theorem 2.1.11 is just a nice fact of 'abstract nonsense'. But if one wants to apply the theorem there are two main problems occurring. The first is 'How to describe the character group $\hat{V}$ of V?', the second is 'How to describe the action of G on $\hat{V}$ in such a way that it is easy to calculate it?'. We will solve both of this problems in the case where $K = \mathbb{C}$ and V is a finite dimensional vector space over some finite field $\mathbf{F}_q$.

2.1.18 Assumption. *For the remainder of this section let V be a finite dimensional vector space over some finite field $\mathbf{F}_q$ on which G acts as $\mathbf{F}_q$-vector space automorphisms.*

2.1.19 Definition. *Let V be an abelian group on which G acts as automorphisms, $f : G \longrightarrow V$ a surjective 1-cocycle and $\kappa : V \times V \longrightarrow \mathbf{F}_q$ a non-degenerate bilinear form. Then we call (f, κ) a* ***monomial linearisation*** *for G.*

Such bilinear forms κ always exist, as the following standard fact shows.

2.1.20 Lemma. *Let $\mathcal{B} = (v_1, \ldots, v_k)$ be an $\mathbf{F}_q$-basis of V. Then*

$$\kappa : V \times V \longrightarrow \mathbf{F}_q \qquad \text{defined by } \kappa(v_i, v_j) = \delta_{ij}$$

is a non-degenerate symmetric bilinear form.

Hence, having a monomial linearisation is not a property but a structure which can be chosen for every surjective 1-cocycle f as long as V is a finite dimensional $\mathbf{F}_q$-vector space. All such bilinear forms have beautiful properties.

2.1.21 Lemma (c.f. [Lan, XIII §5])**.** *Suppose G acts on V as $\mathbf{F}_q$-vector space automorphisms from the right and κ is a non-degenerate bilinear form. Then for every $g \in G$ there exists a uniquely determined $\mathbf{F}_q$-vector space automorphism g^*, such that*

$$\kappa(A, Bg) = \kappa(Ag^*, B) \qquad \textit{for all } A, B \in V.$$

The automorphism g^* is called the ***adjoint*** *or* ***transpose*** *of g. We further have*

$$(gh)^* = h^*g^*, \qquad \textit{and} \qquad (g^{-1})^* = (g^*)^{-1} \qquad \textit{for all } g, h \in G.$$

Proof. Because κ is non-degenerate the mapping $X \mapsto \kappa(A, X)$ from V to V^* is injective. Since V is finite dimensional hence also bijective. Thus we can write each element α of the dual space V^* as $X \mapsto \kappa(A, X)$ for a uniquely determined $A \in V$. Now $X \mapsto \kappa(A, Xg)$ is an element of the dual space and hence we have a uniquely defined element $C_A \in V$, such that $\kappa(C_A, X) = \kappa(A, Xg)$ for all $X \in V$. It is routine to check that the map $A \mapsto C_A$ is $\mathbf{F}_q$-linear.

Hence by setting $Ag^* := C_A$ we get the uniquely determined $\mathbf{F}_q$-linear linear map g^* satisfying $\kappa(Ag^*, B) = \kappa(A, Bg)$ for all $A, B \in V$. It is an easy exercise to check the remaining properties. qed. |

2.1.22 Lemma. *Suppose G acts on V as $\mathbf{F}_q$-vector space automorphisms from the right and κ is a non-degenerate bilinear form. Then we have*

(a) The set $G^* := \{g^* \mid g \in G\}$ is a subgroup of $\mathrm{Aut}_{\mathbf{F}_q}(V)$.

(b) The map $^* : \ G \longrightarrow G^* : \ g \longmapsto g^*$ defines a group anti-homomorphism with kernel $\tilde{G} := \{g \in G \mid Ag = A \text{ for all } A \in V\}$, which is the kernel of the group operation of G on V. Hence $G/\tilde{G}$ is anti isomorphic to G^*.

In particular

$$\tau : \ G \longrightarrow \mathrm{Aut}_{\mathbf{F}_q}(V) : \ g \longmapsto g^{-*} := (g^*)^{-1} = (g^{-1})^*$$

defines a group homomorphism from G with image G^* and kernel $\tilde{G}$.

Proof. The only statement not directly implied by Lemma 2.1.21 is that $g^* = 1$ if and only if $g \in \tilde{G}$. Now we have $g^* = 1$ if and only if $\kappa(A, B) = \kappa(Ag^*, B)$ for all $A, B \in V$. Since $\kappa(Ag^*, B) = \kappa(A, Bg)$, non-degeneracy of κ shows that $g^* = 1$ is equivalent to $Bg = B$ for all $B \in V$. qed. |

Using the group homomorphism τ of Lemma 2.1.22 we can define a new group operation of G on V:

2.1.23 Definition/Lemma. *Suppose G acts on V as $\mathbf{F}_q$-vector space automorphisms from the right and κ is a non-degenerate bilinear form. Then the map*

$$.: \; V \times G \longrightarrow V : \; (A, g) \longmapsto A.g := A\tau(g) = Ag^{-*}$$

defines a group action of G on V as $\mathbf{F}_q$-vector space automorphisms.

As promised we will now describe $\hat{V}$ by virtue of κ.

2.1.24 Notation. *Let $\vartheta : \mathbf{F}_q^+ \to \mathbb{C}^\times$ denote a once and for all fixed nontrivial linear character of the additive group of $\mathbf{F}_q$.*

2.1.25 Lemma. *Let V be a finite dimension vector space over some finite field $\mathbf{F}_q$[33] and let κ be a nondegenerate bilinear form. Then we have*

$$\hat{V} = \operatorname{Hom}(V, \mathbb{C}^\times) = \{\hat{\chi}_A \mid A \in V\},$$

where $\hat{\chi}_A : \; V \longrightarrow \mathbb{C}^\times : \; X \longmapsto \vartheta\kappa(A, X)$. In particular $\hat{\chi}_A = \hat{\chi}_B \implies A = B$.

Proof. It is easy to check that $\hat{\chi}_A$ is in $\hat{V}$. Now suppose we have $\hat{\chi}_A = \hat{\chi}_B$. Then

$$1 = \hat{\chi}_B(X)^{-1}\hat{\chi}_A(X) = \underbrace{\vartheta\kappa(B, X)^{-1}}_{=\vartheta(-\kappa(B,X))=\vartheta\kappa(-B,X)}\vartheta\kappa(A, X) = \vartheta\kappa(A - B, X) = \hat{\chi}_{A-B}(X) \quad (2.1.26)$$

for all $X \in V$. Suppose $A \neq B$. Nondegeneracy of κ implies that there exists $X \in V$ satisfying $\kappa(A - B, X) \neq 0$. But then $\kappa(A - B, \mathbf{F}_q X) = \mathbf{F}_q$ and thus we would have $\vartheta(\mathbf{F}_q) = \{1\}$ due to equation (2.1.26). But this is a contradiction to the assumption of ϑ being nontrivial. Thus the map $A \mapsto \hat{\chi}_A$ is injective. On the other hand we know that $|V| = |\hat{V}|$, since $\hat{V} = \operatorname{Irr}(V)$ for an abelian group[34]. The result follows. qed. |

Having established these structural properties we can now use the transfer of the action of G on V to the the adjoint group G^* by κ to describe the monomial action of G on $f^*(\mathbb{C}\hat{V})$ more explicitly.

2.1.27 Lemma. *Suppose G has a monomial linearisation. Then we have*

$$\hat{\chi}_A.g = \hat{\chi}_{A.g} \qquad \text{for all } A \in V \text{ and } g \in G,$$

with $\chi_A.g$ being defined in 2.1.8 and $A.g$ in 2.1.23.

Proof. For $B \in V$ we have

$$\big(\hat{\chi}_A.g\big)(B) = \hat{\chi}_A(Bg^{-1}) = \vartheta\kappa\big(A, Bg^{-1}\big) = \vartheta\kappa\big(Ag^{-*}, B\big) = \hat{\chi}_{A.g}(B)$$

yielding the claim. qed. |

[33]Thus in particular V is a finite abelian group.

[34]This argument only holds for a finite group V, which V is as a finite dimensional vector space over a finite field.

2.1.28 Reminder. *We use the notation* $\chi_A = \hat{\chi}_A f$.

2.1.29 Proposition (Main theorem for monomial linearisations)**.** *Suppose G has a monomial linearisation. Then $f^*(\mathbb{C}\hat{V}) = \mathbb{C}\text{-span}\{\chi_A \mid A \in V\}$ is a submodule of $\mathbb{C}^G$. Moreover $f^*(\hat{V}) = \{\chi_A \mid A \in V\}$ is a monomial basis of $f^*(\mathbb{C}\hat{V})$ and we have*

$$\chi_A * g = \chi_A(g^{-1})\chi_{A.g} \qquad \text{for all } A \in V, g \in G.$$

Thereby $\chi_A * g$ denotes the module operation on $\mathbb{C}^G$ from Definition/Lemma 2.1.13 and $A.g$ the group action of G on V from Definition/Lemma 2.1.23.

Proof. Note that $f^*(\hat{\chi}_A) = \chi_A$. By definition of a monomial linearisation we have a surjective 1-cocylce f. Hence the map f^* is an injective G-module homomorphism due to Corollary 2.1.16. Thus $f^*(\mathbb{C}\hat{V})$ is a submodule of $\mathbb{C}^G$ with monomial basis $f^*(\hat{V}) = \{\chi_A \mid A \in V\}$. Furthermore we have

$$\hat{\chi}_A * g \overset{2.1.11}{=} \underbrace{\alpha(\hat{\chi}_A, g)}_{=\hat{\chi}_A f(g^{-1}) = \chi_A(g^{-1})} \hat{\chi}_A.g = \chi_A(g^{-1})\hat{\chi}_A.g \overset{2.1.27}{=} \chi_A(g^{-1})\hat{\chi}_{A.g}.$$

Application of the injective G-module homomorphism f^* yields the result. qed.

2.1.30 Corollary. *Suppose G has a monomial linearisation. Let $A \in V$ and $g, h \in G$ be arbitrary and let $^- : \mathbb{C} \longrightarrow \mathbb{C}$ denote complex conjugation. Then we have*

(i) $\chi_A(1) = 1$

(ii) $\chi_{A.g}(g) = \overline{\chi_A(g^{-1})}$

(iii) $\chi_A\big((gh)^{-1}\big) = \chi_A(g^{-1})\chi_{A.g}(h^{-1})$

(iv) $\chi_{A.(gh)}(gh) = \chi_{A.g}(g)\chi_{A.(gh)}(h)$.

Proof. Let $A \in V$ and $g, h \in G$ be arbitrary. Note, that by Lemma 2.1.22 we have $1^* = 1$, which implies $A.1 = A1^{-*} = A$. Hence

$$\chi_A = \chi_A * 1 = \chi_A(1^{-1})\chi_{A.1} = \chi_A(1)\chi_A.$$

Thus $\chi_A(1) = 1$, which proves (i). Using associativity of the module operation we get

$$\chi_A\big((gh)^{-1}\big)\chi_{A.(gh)} = \chi_A*(gh) = (\chi_A*g)*h = \chi_A(g^{-1})\chi_{A.g}*h = \chi_A(g^{-1})\chi_{A.g}(h^{-1})\chi_{(A.g).h}.$$

Hence (iii) holds. Part (iii) with $h^{-1} = g$ leads to $1 = \chi_A(g^{-1})\chi_{A.g}(g)$. Since $\hat{\chi}_A$ is a linear character, the image of χ_A is always a root of unity. Hence we have $\chi_A(g^{-1})^{-1} = \overline{\chi_A(g^{-1})}$, which proves (ii). Finally, (iv) can be obtained by applying complex conjugation to (iii) and afterwards (ii). qed.

2.1.31 Warning. *It may be tempting to write $\overline{\chi_A(g^{-1})} = \chi_A(g)$. But this is **not** true in general, since χ_A is not a character of G.*

2.1.32 Remark. *For working with concrete examples it will be important to define κ in such a way, that the operation $A.g$ is easy to perform. We will see the first concrete example in Section 2.2, it is the example of $A_N(\mathbf{F}_q)$ considered by N.Yan in [Yan1],[Yan2].*

2.1.33 Proposition. *Suppose G has a monomial linearisation. If there exists a subgroup $U \leq G$, such that the map $f|_U$[35] is bijective. Then we have $\operatorname{Res}_U^G f^*(\mathbb{C}\hat{V}) \cong \mathbb{C}U$ where the isomorphism is given by $\chi_A \longmapsto [-A]$ with*

$$[A] := \frac{1}{|U|} \sum_{u \in U} \overline{\chi_A(u)} u.$$

In particular we have $[A]u = \chi_{A.u}(u)[A.u]$ and $\mathcal{B} := \{[A] \mid A \in V\}$ is a $\mathbb{C}$-basis of $\mathbb{C}U$.

Proof. The main ingredient of this proof is the 'diamond operation trick' developed by B. Ackermann, R. Dipper and Q. Guo in [ADG] for the case $G = U = A_N(\mathbf{F}_q)$.

Step 1: *Let u, v in U be arbitrary. We define*

$$u \diamond v := f|_U^{-1}\big(f|_U(u) + f|_U(v)\big).$$

Then $(U, \diamond)$ is an abelian group. Note, that $1_{(U,\diamond)} = 1_{(U,\cdot)}$.

We have $f|_U(1) = f(1) = 0 =$ the neutral element of $(V, +)$, due to Remark 2.1.7.

Step 2: *We have*

$$\operatorname{Irr}\big((U, \diamond)\big) = \{\chi_A \mid A \in V\}.$$

First of all we show that for $A \in V$ we have indeed $\chi_A \in \operatorname{Irr}(U, \diamond) = \operatorname{Hom}\big((U, \diamond), \mathbb{C}^\times\big)$. Let $u, v \in U$ be arbitrary. It holds

$$\chi_A(u \diamond v) = \hat{\chi}_A\big(f(u) + f(v)\big) \overset{\hat{\chi}_A \in \operatorname{Irr}(V)}{=} \hat{\chi}_A\big(f(u)\big)\hat{\chi}_A\big(f(v)\big) = \chi_A(u)\chi_A(v)$$

We consider the map $\varphi : \quad V \to \operatorname{Irr}\big((U, \diamond)\big) : \quad A \mapsto \chi_A$. Suppose $\chi_A = \chi_B$, for some $A, B \in V$. Hence we have $f^*(\hat{\chi}_A) = f^*(\hat{\chi}_B)$, which implies $\hat{\chi}_A = \hat{\chi}_B$, since f is surjective. Now, Lemma 2.1.25 implies $A = B$. Thus φ is injective. Since $|\operatorname{Irr}\big((U, \diamond)\big)| = |U| = |V|$, the step is proven.

Step 3: *The elements $[A]$ (where A runs through V) form a $\mathbb{C}$-basis of $\mathbb{C}U$.*

The elements $[A]$ are the central primitive idempotents of $\mathbb{C}(U, \diamond)$. They form a $\mathbb{C}$-basis for the center of $\mathbb{C}(U, \diamond)$, but since $(U, \diamond)$ is an abelian group the center of the group algebra equals the group algebra.

Step 4: *We have $[A]u = \chi_{A.u}(u)[A.u]$ for all $u \in U, A \in V$.*

Let $A \in V$ and $u, y \in U$ be arbitrary. Then

$$(*) \qquad \chi_A(yu^{-1}) \overset{2.1.30\,\text{(iii)}}{=} \chi_A(u^{-1})\chi_{A.u}(y) \overset{2.1.30\,\text{(ii)}}{=} \overline{\chi_{A.u}(u)}\chi_{A.u}(y)$$

[35] $f|_U$ denotes the restriciton of the map f to the subset U

and therefore also

$$\begin{aligned}[A]u &= \frac{1}{|U|}\sum_{x\in U}\overline{\chi_A(x)}xu \\ &\overset{y=xu}{=} \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(yu^{-1})}y \overset{(*)}{=} \chi_{A.u}(u)\frac{1}{|U|}\sum_{y\in U}\overline{\chi_{A.u}(y)}y = \chi_{A.u}(u)[A.u].\end{aligned}$$

Finishing the proof: Up to now we have proved the 'In particular' statements. Let Φ denote the mapping defined by $\Phi(\chi_A) = [-A]$. Using

$$(**) \qquad \overline{\chi_{A.u}(u)} = \overline{\vartheta\kappa\big(A.u, f(u)\big)} = \vartheta\kappa\big(-(A.u), f(u)\big) = \chi_{-(A.u)}(u)$$

and $-(A.u) = (-A).u$, which holds since the action '.' is via automorphisms, we get

$$\Phi(\chi_A * u) \overset{\substack{2.1.29\\2.1.30\text{ (ii)}}}{=} \overline{\chi_{A.u}(u)}\Phi(\chi_{A.u}) \overset{(**)}{=} \chi_{-A.u}(u)[-A.u] \overset{\text{Step 4}}{=} [-A]u = \Phi(\chi_A)u.$$

And since Φ maps a basis to a basis everything is proven. qed. |

We summarize this point of view in the following corollary. This corollary is the starting point for our investigation of $D_n(\mathbf{F}_q)$ in Section 3.1.

2.1.34 Reminder. *Suppose G has a monomial linearisation, i.e. we have a surjective 1-cocycle $f : G \longrightarrow V$ and a non-degenerate bilinear form $\kappa : V \times V \longrightarrow \mathbf{F}_q$. Additionally we fixed a nontrivial linear character $\vartheta : \mathbf{F}_q^+ \longrightarrow \mathbb{C}^\times$. Using this we form the expressions*

$$\chi_A(g) = \vartheta\kappa\big(A, f(g)\big) \qquad \textit{and} \qquad A.g = A(g^*)^{-1},$$

where g^* denotes the adjoint of g with respect to κ.

Recall that $(A, g) \mapsto A.g$ defines a group operation on V and $g \mapsto g^*$ a group anti-homomorphism. In particular we have $g^{-*} := (g^{-1})^* = (g^*)^{-1}$.

2.1.35 Corollary (Main Theorem, bijective subgroup version)**.** *We use the notation of Reminder 2.1.34. Suppose G has a monomial linearisation and suppose there exists a subgroup $U \le G$, such that the map $f|_U$ is bijective. Then the elements*

$$[A] = \frac{1}{|U|}\sum_{u\in U}\overline{\chi_A(u)}u \qquad \textit{where } A \textit{ runs through } V$$

form a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. By defining

$$[A] * g := \chi_{A.g}(g)[A.g] \qquad \textit{for all } g \in G, A \in V$$

we can make $\mathbb{C}U$ into a monomial G-module. The operation of U is given by the usual right multiplication of U on $\mathbb{C}U$, i.e.

$$[A] * u = [A]u = \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(y)}yu \qquad \textit{for all } u \in U, A \in V.$$

Proof. From Proposition 2.1.33 we have $f^*(\mathbb{C}\hat{V}) \cong \mathbb{C}U$ as $\mathbb{C}U$-modules and that $[A]u = \chi_{A.u}(u)[A.u]$ for all $u \in U$ and $A \in V$. Now $f^*(\mathbb{C}\hat{V})$ is in fact a G-module. Hence we can extend the U-module operation on $\mathbb{C}U$ to a G-module operation using the G-module operation on $f^*(\mathbb{C}\hat{V})$, which is obtained by defining $[A] * g := \chi_{A.g}(g)[A.g]$. qed. |

2.1.36 Warning. *Of course the operation of G on $\mathbb{C}U$ **cannot** be equal to the 'usual' one in which we see $\mathbb{C}U$ as a subset $\mathbb{C}G$ and use the operation in $\mathbb{C}G$. This is true because $\mathbb{C}H$ is never a $\mathbb{C}G$-submodule of $\mathbb{C}G$ if H is a proper subgroup of G.*

For monomial representations it is possible to define a natural class of submodules, called orbit modules, having very nice properties. The monomial representations considered in this thesis are all constructed using Corollary 2.1.35. Hence we give the definition of orbit modules adapted to the notation of the corollary.

2.1.37 Definition/Lemma (Orbit module)**.** *Suppose G has a monomial linearisation and U is a subgroup of G, such that $f|_U$ is bijective.*

*Let $A \in V$ be a arbitrary. Then we define the U**-orbit module** associated to A*

$$\mathbb{C}\mathcal{O}_U(A) := \mathbb{C}U\text{-module generated by } [A].$$

Then $\mathbb{C}\mathcal{O}_U(A)$ has a $\mathbb{C}$-basis given by

$$\big\{[C] \in \mathcal{B} \bigm| C \in \mathcal{O}_U(A)\big\}, \quad \textit{where} \quad \mathcal{O}_U(A) := \big\{A.g \bigm| g \in U\big\}$$

*is the **orbit** of A under the '.'-operation.*

By just substituting every 'U' with 'G' we define the G-orbit module $\mathbb{C}\mathcal{O}_G(A)$, which has $\mathbb{C}$-basis $\{[A] \mid A \in \mathcal{O}_G(A)\}$.

One main goal is to classify the orbit modules up to isomorphism. In Chapter 3 we will (almost) solve this problem for the case of $D_n(\mathbf{F}_q)$. A big advantage of this approach is, that we can apply some standard facts of group actions:

2.1.38 Lemma. *Let $A, B \in V$ be arbitrary, then*

(i) Two orbit modules $\mathbb{C}\mathcal{O}_U(A)$ and $\mathbb{C}\mathcal{O}_U(B)$ are either identical (if $A.g = B$ for some $g \in U$) or their intersection is (0).

(ii) Let $\mathrm{Stab}_U(A) := \{g \in U \mid A.g = A\}$. Then we have

$$\dim \mathbb{C}\mathcal{O}_U(A) = \frac{|U|}{|\operatorname{Stab}_U(A)|}.$$

The same holds for G-orbit modules by substituting every 'U' with 'G'.

Thus the Lemma reduces the problem of computing the dimensions of orbit modules to computing the sizes of stabilizers of patterns which usually is easier if we are working with concrete groups.

Of course the orbit modules are not irreducible in general. But if the monomial linearisation is nicely chosen the orbit modules can be seen as an approximation, which still provides strong and interesting results.

2.2. Application to the unipotent linear groups

In this section, we will use the methods developed in Section 2.1 to define - as a first example - the monomial operation given by N. Yan for $A_N(\mathbf{F}_q)$[36] (see [Yan1], [Yan2]). Of course this section could also be formulated for pattern subgroups of $A_N(\mathbf{F}_q)$. We avoid this technicality in favour for a clearer presentation of the ideas used[37].

We want to apply Corollary 2.1.35 to the situation where $G = U = A_N(\mathbf{F}_q)$. Hence we have to construct a bijective 1-cocycle. As a technical tool we will also need a non-degenerate bilinear form.

2.2.1 Notation. *Let* $\square := \{(i,j) \mid 1 \leq i,j \leq N\}$ *and* $\triangledown := \{(i,j) \mid 1 \leq i < j \leq N\}$[38].

2.2.2 Notation. *Let* $J \subseteq \square$ *be any subset. Let* $V_0 := \mathrm{Mat}_{N\times N}(\mathbf{F}_q)$. *Then we define*

$$V_J := \bigoplus_{(i,j)\in J} \mathbf{F}_q e_{ij}.$$

Clearly V_J *is a vector subspace of* V_0. *In this section we are mostly interested in* $V_{\triangledown}$.

2.2.3 Proposition. *Let* $G = A_N(\mathbf{F}_q)$. *Then matrix multiplication defines a group action of* U *on* $V := V_{\triangledown}$ *, where* G *acts as* $\mathbf{F}_q$*-vector space automorphisms and the map*

$$f: \ G \to V: \ g \mapsto g - 1.$$

defines a bijective (left and right) 1*-cocycle.*

Note, that V *is the set of all nilpotent upper triangular* $N \times N$*-matrices.*

Proof. The map f obviously is well defined and for $x, g \in G$ we have

$$f(xg) = xg - 1 = (x-1)g + (g-1) = f(x)g + f(g)$$

and

$$f(gx) = gx - 1 = g(x-1) + (g-1) = gf(x) + f(g)$$

Further f is obviously bijective with $X \mapsto 1 + X$ being the inverse map. The only part of the statement not following directly from the properties of matrix addition and multiplication is, weather for $X \in V, g \in U$ we have $Xg \in V$. But we can write $X = h - 1$ for some $h \in G$. Then it holds $Xg = (h-1)g = hg - g$ which, as the difference of two elements of G, clearly is in V. qed. |

[36] this is the group of upper unitrianglular $N \times N$-matrices over the finite field $\mathbf{F}_q$

[37] and in order to keep Chapter 2 readable independently of Chapter 1.

[38] $\square$ and $\triangledown$ were defined in 1.1.9. We consider them as sets of matrix coordinate positions. An illustration of these sets is given in Figure 1.1.

We already have enough to apply Theorem 2.1.11, but we can do better, since there also exists a transfer structure, which we describe next.

2.2.4 Notation. *We set* $V_0 = \mathrm{Mat}_{N\times N}(\mathbf{F}_q)$ *and define*

$$\kappa : \ V_0 \times V_0 \to \mathbf{F}_q : \ (X, Y) \mapsto \mathrm{tr}(X^\top Y).$$

The map κ *is called* ***trace form****.*

2.2.5 Warning. *The trace form is only almost the non-degenerate we need as it as it is defined on all matrices instead of only the strictly upper triangular ones. Hence we will use the restriction of* κ *to the space* $V \times V$ *instead, denoted with* $\kappa|_{V\times V}$*. Of course it remains to be checked if* $\kappa|_{V\times V}$ *is non-degenerate.*

2.2.6 Remark. *Any* $A \in V_0$ *can be written in the form* $A = \sum_{(i,j)\in\square} A_{ij} e_{ij}$*.*

We recall the properties of κ.

2.2.7 Lemma. *We have*

(a) κ *is a non-degenerate symmetric bilinear form.*

(b) For every $A \in V_0$ *and* $(i, j) \in \square$ *we have*

$$\kappa(A, e_{ij}) = A_{ij}.$$

(c) For every $A, B, C \in V_0$ *we have*

$$\kappa(B^\top A, C) = \kappa(A, BC) = \kappa(AC^\top, B).$$

Proof. Let us start with part (b). We have

$$\kappa(A, e_{ij}) = \mathrm{tr}(A^\top e_{ij}) = \sum_{r=1}^{N} (A^\top e_{ij})_{rr} = \sum_{r,s=1}^{N} \underbrace{A^\top_{rs}}_{=A_{sr}} \underbrace{(e_{ij})_{sr}}_{=\delta_{is}\delta_{jr}} = A_{ij}$$

It is straightforward to check (c) and, that κ is an symmetric bilinear form. Let $A \in V_0$ satisfy $\kappa(A, C) = 0$ for all $C \in V_0$. Then part (b) implies $0 = \kappa(A, e_{ij}) = A_{ij}$ for all $(i, j) \in \square$. Thus $A = 0$ and κ is non-degenerate. qed. |

From part (b) we directly get the following corollary:

2.2.8 Corollary. *Let* $A, B \in V_0$*, then* $\kappa(A, B) = \sum_{(i,j)\in\square} A_{ij} B_{ij}$*.*

2.2.9 Lemma. *Let $J \subseteq \square$ be any subset. Then we have*

$$V_J^{\perp} = V_{\square \smallsetminus J} \qquad \textit{and} \qquad V_0 = V_J \oplus V_J^{\perp},$$

where $V_J^{\perp}$ denotes the orthogonal space of V_J with respect to the traceform κ.[39]

In particular $V_{◹}^{\perp}$ is the set of lower triangular matrices over $\mathbf{F}_q$.

Proof. Clearly we have $V_0 = V_J \oplus V_{\square \smallsetminus J}$. It remains to check $V_J^{\perp} = V_{\square \smallsetminus J}$. Corollary 2.2.8 implies that $V_{\square \smallsetminus J} \subseteq V_J^{\perp}$. On the other hand suppose $\kappa(A, X) = 0$ for all $X \in V_J$. Then also for all $(i, j) \in J$ we have $0 = \kappa(A, e_{ij}) = A_{ij}$ due to Lemma 2.2.7(b). qed. |

2.2.10 Corollary. *Let $J \subseteq \square$ be any subset, then $\kappa|_{V_J \times V_J}$ is nondegenerate.*

Proof. If $A \in V_J$, such that $\kappa(A, B) = 0$ for all $B \in V_J$, then $A \in V_J \cap V_J^{\perp} = (0)$. qed. |

2.2.11 Remark. *We have shown, that $(f, \kappa|_{V \times V})$ is a monomial linearisation for G.*

Note further that $\kappa|_{V \times V}$ is the bilinear form described in Lemma 2.1.20, with respect to the basis e_{ij} of V, where (i, j) runs through $◹$.

We have yet to determine how $A.g$ is given for $(f, \kappa|_{V \times V})$.

2.2.12 Reminder. *Let (f_0, κ_0) be a monomial linearisation. In Definition/Lemma 2.1.23 we defined a group operation by setting $A.g := A(g^*)^{-1}$ for all $g \in G$, $A \in V$, where g^* denotes the adjoint of g with respect to the non-degenerate bilinear form κ_0.*

2.2.13 Lemma. *In case of the monomial linearisation $(f, \kappa|_{V \times V})$ for $G = A_N(\mathbf{F}_q)$ we have*

$$A.g = \pi(Ag^{-\top}) \qquad \textit{for all } A \in V, g \in G,$$

where $g^{-\top} := (g^{-1})^{\top} = (g^{\top})^{-1}$ and $\pi : V_0 = V \oplus V^{\perp} \longrightarrow V$ denotes the projection to the first component.

Note, that applying π to a matrix A means putting all entries outside of $◹$ to zero.

Proof. Using Lemma 2.2.7(c) and Corollary 2.2.8 we get

$$\kappa(A, Bg) = \kappa(\underbrace{Ag^{\top}}_{\notin V}, B) = \kappa\big(\underbrace{\pi(Ag^{\top})}_{\in V}, B\big) \qquad \text{for all } A, B \in V, g \in G.$$

These are just the defining equations for the adjoint of g, i.e. Lemma 2.1.21 implies $Ag^* = \pi(Ag^{\top})$. Also Lemma 2.1.21 implies $A.g = A(g^{-1})^* = \pi(Ag^{-\top})$. qed. |

Now all pieces are in place to apply Corollary 2.1.35, which for our purpose is the most important result of Section 2.1. This yields essentially Proposition 2.1 of [Yan2] but in the language of Ackermann-Dipper-Guo of [ADG].

[39] i.e. $V_J^{\perp} = \{X \in V_0 \mid \kappa(X, A) = 0 \text{ for all } A \in V_J\}$.

2.2.14 Theorem (Yan). *Let $G = A_N(\mathbf{F}_q)$ and $V = V_{\triangledown}$. Then the elements*

$$[A] = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_A(g)}g \qquad \textit{where } A \textit{ runs through } V$$

form a monomial $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}G$ where the action is given by

$$[A]g = \chi_{A.g}(g)[A.g] \qquad \textit{for all } g \in G, A \in V$$

and $A.g := \pi(Ag^{-\top})$ defines a group action of G on V.

In order to understand how $A.g$ can be calculated we want to remind of the following very basic fact about matrix multiplication which is the key step of understanding the operation.

2.2.15 Lemma. *Let $A \in V_0$ and $(i,j) \in \square$ be arbitrary. Then we have*

(i) $Ae_{ji} =$ *j-th column of A written in the i-th column,*

(ii) $e_{ji}A =$ *i-th row of A written in the j-th row.*

Proof. We have

$$Ae_{ji} = \sum_{r,s=1}^{N} A_{rs}\underbrace{e_{rs}e_{ji}}_{=\delta_{sj}e_{ri}} = \sum_{r=1}^{N} A_{rj}e_{ri} = j\text{-th column of } A \text{ written in the } i\text{-th column.}$$

Part (ii) is proved completely analogous. qed.

2.2.16 Corollary. *Let $\lambda \in \mathbf{F}_q$, $(i,j) \in \triangledown$ and define $\tilde{x}_{ij}(\lambda) := 1 + \lambda e_{ij}$. Then, we have for $A \in V$*

$$[A]\tilde{x}_{ij}(\lambda) = \vartheta(A_{ij}\lambda)[A.\tilde{x}_{ij}(\lambda)],$$

where $A.\tilde{x}_{ij}(\lambda)$ arises from A by taking A, adding $-\lambda$ times the j-th column of A to the i-th column of A and then putting 0's at any entry, whose coordinates lie not in $\triangledown$.

Proof. We set $x = \tilde{x}_{ij}(\lambda)$. We have $A.g = \pi(Ax^{-\top}) = A - \lambda\pi(Ae_{ji})$. Lemma 2.2.15 implies the statement about the column operation and the application of π produces the 0's. We calculate, using $f(x) = \lambda e_{ij}$,

$$\chi_{A.x}(x) = \vartheta\kappa\big(\underbrace{A.x}_{\equiv Ax^{-\top} (\mathrm{mod}\, V^{\perp})}, f(x)\big) = \vartheta\kappa\big(Ax^{-\top}, \lambda e_{ij}\big) = \vartheta\big(\lambda(Ax^{-\top})_{ij}\big) = \vartheta(\lambda A_{ij}),$$

since the j-th columns of A and $Ax^{-\top}$ are the same. qed.

2.2.17 Example. *We consider $U = A_3(\mathbf{F}_q)$. Then we have*

$$V = \left\{ \begin{pmatrix} \cdot & A_{12} & A_{13} \\ & \cdot & A_{23} \\ & & \cdot \end{pmatrix} \middle| A_{12}, A_{13}, A_{23} \in \mathbf{F}_q \right\},$$

where dots are only there for orientation and indicate zeros.

For $A \in V$ and $\lambda \in \mathbf{F}_q$ we get, using Corollary 2.2.16,

$$\begin{aligned} \left[\begin{pmatrix} \cdot & A_{12} & A_{13} \\ & \cdot & A_{23} \\ & & \cdot \end{pmatrix} \right] \tilde{x}_{23}(\lambda) &= \vartheta(A_{23}\lambda) \left[\pi \underbrace{\begin{pmatrix} \cdot & A_{12} - \lambda A_{13} & A_{13} \\ & -\lambda A_{23} & A_{23} \\ & & \cdot \end{pmatrix}}_{= A\tilde{x}_{23}(\lambda)^{-\top}} \right] \\ &= \vartheta(A_{23}\lambda) \left[\begin{pmatrix} \cdot & A_{12} - \lambda A_{13} & A_{13} \\ & \cdot & A_{23} \\ & & \cdot \end{pmatrix} \right]. \end{aligned}$$

2.2.18 Notation. *We call the elements of V* ***patterns****. Keeping Corollary 2.2.16 in mind, we call the monomial action of G on $\mathbb{C}G$ and also the corresponding permutation operation on V* ***truncated column operation****.*

2.2.19 Remark. *We make a few remarks:*

(i) Of course, Proposition 2.2.3 remains true, proofs included, if we take a pattern subgroup instead of the full group $A_N(\mathbf{F}_q)$.

(ii) In case of $G = A_N(\mathbf{F}_q)$ we also have

$$f(gx) = gf(x) + f(g).$$

This would lead to a monomial basis $\mathcal{B}$, defined exactly as in our variant[40], working for left operation instead of one for right operation, more precisely Theorem 2.2.14 then would state

$$g[A] = \chi_{g.A}(g)[g.A] \qquad \text{for all } g \in G, A \in V,$$

where $g.A := \pi(g^{-\top}A)$. But then $\mathcal{B}$ is even a monomial basis for both the left and the right actions of the bimodule $\mathbb{C}G$. This is a curiosity of the case $A_N(\mathbf{F}_q)$.

Having established the case $A_N(\mathbf{F}_q)$ up to that point we will provide an overview over some important results regarding the orbit modules. This will illustrate the strategy we use in Chapter 3 to investigate the orbit modules for $D_n(\mathbf{F}_q)$. We also want to compare the results for the cases $A_N(\mathbf{F}_q)$ and $D_n(\mathbf{F}_q)$. To all results we state, we will also explain the situation in case $D_n(\mathbf{F}_q)$.

[40] of course the general theory has an obvious analogue if multiplications act from the left.

2.2.20 Notation. *For the remainder of this section let* $V := V_{◹}$*. We also want to remind, that the elements of* V *are called patterns.*

2.2.21 Definition. *Let* A *be a pattern. A matrix position* $(i, j) \in ◹$ *is called* ***main condition*** *of* A *if* $A_{ij} \neq 0$ *and all entries in the* i*-th row on the right of* A_{ij} *are zero. The set of all main conditions of* A *is denoted by* $\mathrm{main}(A)$.

2.2.22 Definition/Lemma. *Let* A *be a pattern. We define the* ***verge*** *of* A *to be the pattern*

$$\mathrm{verge}(A) := \sum_{(i,j)\in\mathrm{main}(A)} A_{ij} e_{ij}.$$

If A *and* B *lie in the same orbit, then Corollary 2.2.3 implies* $\mathrm{verge}(A) = \mathrm{verge}(B)$*. Patterns satisfying* $\mathrm{verge}(A) = A$ *are called* ***verge patterns****.*

2.2.23 Definition/Lemma. *Patterns with the property, that in every column lies at most one main condition are called* ***staircase patterns****. Orbit modules generated by staircase patterns*[41] *are called* ***staircase modules****.*

The class of staircase modules has the advantage that the group operation is in that situation the easiest to understand.

2.2.24 Theorem (Classification of staircase modules, part of [Yan2, Theorem 3.2])**.** *Every staircase module has a basis consisting of patterns. In this basis there exists precisely one verge pattern.*

Comparison to $D_n(\mathbf{F}_q)$***:***[42]*Theorem 3.2.29 gives a classification of the staircase modules in Type* D*. But the verge patterns are not enough to do the classification. We need the so called core patterns. Another huge difference is that in Type* A *every staircase module has a basis element whose stabilizer is a pattern subgroups, whereas the same does not hold in Type D.*

The next step is to prove, that these staircase modules are up to isomorphism all orbit modules.

2.2.25 Proposition (mainly [Yan2, Proposition 2.2], part of [Yan2, Theorem 3.2])**.** *Every orbit module is isomorphic to some staircase module.*

Comparison to $D_n(\mathbf{F}_q)$***:*** *In Type* D *the same holds (c.f. Corollary 3.3.15).*

How do the irreducible complex characters relate to orbit modules?

2.2.26 Theorem ([Yan2, Theorem 2.4] and [Yan2, Corollary 2.7])**.** *Every irreducible complex character is constituent of precisely one staircase module.*

[41] and thus having a basis labelled only by of staircase patterns.

[42] We will refer to the case of $D_n(\mathbf{F}_q)$ as 'Type D' and to the case of $A_N(\mathbf{F}_q)$ as 'Type A'.

***Comparison to** $D_n(\mathbf{F}_q)$**:** Unfortunately this statement does not hold in Type D. But if one considers the subclass of 'hook-free staircase modules' Corollaries 3.3.19 and 3.3.43 state that representatives of isomorphism classes of hookfree staircase modules satisfy the statement. As a second variant we could consider the sums of all hookfree staircase modules with the same verge, the so called hookfree verge modules (c.f. Definition 3.3.24).*

The next step is the notion of a supercharacter theory. This structural concept can be seen as an approximation to the character table, where we have the irreducible, which are constant on conjugacy classes and admit some orthogonality relations. For a supercharacter theory we look at (not necessarily irreducible) characters (*supercharacters*) being constant on equal many unions of conjugacy classes (*superclasses*) admitting some orthogonality relations. A precise definition and further explanations are given in the beginning of Section 3.6.

2.2.27 Notation (Giving supercharacters a name)**.** *Characters of staircase modules are called **André-Yan supercharacters**.*[43]

***Comparison to** $D_n(\mathbf{F}_q)$**:** The corresponding concept of supercharacters is given by the characters of hookfree verge modules. We call them **André-Neto supercharacters**. This is not the exact way these characters were defined originally by C.A.M André and A.M. Neto, but it is an equivalent description, c.f. Definition 3.6.13 for the original definition and Proposition 3.6.16 for the proof of the equivalence.*

2.2.28 Lemma. *Let $g, h \in A_N(\mathbf{F}_q)$. Then the following statements are equivalent:*

(i) *There exists $x, y \in A_N(\mathbf{F}_q)$, such that $g - 1 = x(h-1)y$,*

(ii) $A_N(\mathbf{F}_q)(g-1)A_N(\mathbf{F}_q) = A_N(\mathbf{F}_q)(h-1)A_N(\mathbf{F}_q)$,

(iii) $g \in C_h^{\mathrm{AY}} := 1 + A_N(\mathbf{F}_q)(h-1)A_N(\mathbf{F}_q)$.

Note, that the biorbits $A_N(\mathbf{F}_q)(g-1)A_N(\mathbf{F}_q)$ are subsets of $V_{\triangleleft}$ and, that the sets C_h^{AY} form a partition of $A_N(\mathbf{F}_q)$, since (i) describes an equivalence relation.

2.2.29 Remark. *The three equivalent statements in the previous lemma, are the three different point of views we can use. The first compares g and h using an equivalence relation, the second by looking at the $A_N(\mathbf{F}_q)$-$A_N(\mathbf{F}_q)$ biorbits of $V_{\triangledown}$, and the third by defining explicit subsets of $A_N(\mathbf{F}_q)$, which are unions of conjugacy classes.*

[43] As remarked in the introduction, these characters were first introduced by C.A.M André, c.f. [And1] up to [And7], but N. Yan's work, which we follow here, is way more elementary and simple. The most important reason for this notation (apart from honouring the contributors) is to distinguish the supercharacters for the group $A_N(\mathbf{F}_q)$ (i.e. André-Yan supercharacters) from the ones for the group $D_n(\mathbf{F}_q)$ (called André-Neto supercharacters) by name.

2.2.30 Definition (Superclasses). *The sets $C_g^{\mathrm{AY}} = \{1 + x(g-1)y \mid x, y \in A_N(\mathbf{F}_q)\}$, where g runs through $A_N(\mathbf{F}_q)$, are called **André-Yan superclasses**.*

Note, that André-Yan superclasses are unions of conjugacy classes.[44]

***Comparison to** $D_n(\mathbf{F}_q)$**:** The corresponding concept of superclasses is given by the sets $C_u^{\mathrm{AN}} := C_u^{\mathrm{AY}} \cap D_n(\mathbf{F}_q)$, where g runs trough $D_n(\mathbf{F}_q)$. Clearly the sets C_u^{AN} form a partition of $D_n(\mathbf{F}_q)$ consisting of unions of conjugacy classes.*

*We call the sets C_g^{AN} **André-Neto superclasses**.*

2.2.31 Proposition ([Yan2, Corollary 2.6])**.** *André-Yan supercharacters are André-Yan superclass functions.*

***Comparison to** $D_n(\mathbf{F}_q)$**:** André-Neto supercharacters are André-Neto superclass functions, c.f. Corollary 3.6.19.*

Another way to express Theorem 2.2.26 and Proposition 2.2.31 is the following:

2.2.32 Theorem. *The André-Yan supercharacters and the André-Yan superclasses form a supercharacter theory in the sense of Definition/Lemma 3.6.2.*

***Comparison to** $D_n(\mathbf{F}_q)$**:** The André-Neto supercharacters and the André-Neto superclasses form a supercharacter theory in the sense of Definition/Lemma 3.6.2.*

To finish this section we want to point out, that the results for $D_n(\mathbf{F}_q)$ we indicated here are not all of those, which we shall establish. In particular the main result of this thesis is not stated here.

[44] $xgx^{-1} = 1 + x(g-1)x^{-1}$.

3. Study of a 1-cocycle monomial representation for $D_n(\mathbf{F}_q)$

In Section 2.1 we have established a tool to construct a monomial basis for the complex group algebra of a finite group. In this chapter we want to apply the method to the case of $D_n(\mathbf{F}_q)$. Having constructed a monomial basis we will study the $D_n(\mathbf{F}_q)$-orbit modules arising. At the end of the chapter we will use this theory to decompose the so called André-Neto supercharacters into characters of orbit modules in a pure combinatorial way. Also we will make a statement concerning orthogonality of the occurring constituents. A nice feature of the theory is, that we are able to reprove the statements of C.A.M. André and A.M. Neto concerning their supercharacters in an elementary and combinatorial way.

We assume the reader to be familiar with Chapters 1 and 2.

3.1. Construction of a monomial linearisation

In this section we construct a monomial linearisation for pattern subgroups $D_n(\mathbf{F}_q)_J$ of $D_n(\mathbf{F}_q)$. After Section 3.1 we will always work with the full group $D_n(\mathbf{F}_q)$. The construction of a monomial basis can be obtained in the more general setting of pattern subgroups. This more generalized approach does not add much technicality. Thus we won't restrict ourselves in this section.

3.1.1 Notation. *Throughout this chapter we fix the following notation:*

$$U = D_n(\mathbf{F}_q), \qquad J \text{ a closed}^{45} \text{ subset of } \triangledown \qquad \text{and} \qquad U_J = D_n(\mathbf{F}_q)_J.^{46}$$

Note, that $U = U_\triangledown$, thus everything stated for U_J in particular holds for U.

3.1.2 Notation. *Throughout this chapter we fix the following notation:*

$$G = A_N(\mathbf{F}_q), \qquad s(J) := J\dot{\cup}\dot{J}\dot{\cup}\bar{J}^{47} \qquad \text{and} \qquad G_{s(J)} = A_N(\mathbf{F}_q)_{s(J)}.^{48}$$

[45] closed in the sense of Definition 1.2.9

[46] c.f. Theorem 1.2.22.

[47] $\bar{J} := \{(\bar{j},\bar{i}) \mid (i,j) \in J\}$ and $\dot{J} := \{(i,\bar{i}) \mid \exists k : (i,k),(i,\bar{k}) \in J\}$, c.f. Lemma 1.2.18

[48] c.f. Corollary 1.2.27.

Note, that $G_{s(J)}$ is a well defined pattern subgroup of $A_N(\mathbf{F}_q)$, since $s(J)$ is a closed subset of ◹ in the sense of Remark 1.2.15 as Lemma 1.2.18 shows.

Moreover $G = G_{s(\triangledown)}$, thus everything stated for $G_{s(J)}$ in particular holds for G.

3.1.3 Remark. *Don't get confused by notation: U_J is just the set of elements in the unipotent orthogonal group U having support $s(J)$, whereas $G_{s(J)}$ is the set of elements of unipotent linear group G having support $s(J)$.[49] So $U_J = G_{s(J)} \cap U$.*

3.1.4 Remark (Goal of section). *We want to apply Corollary 2.1.35 to the situation $G_{s(J)}$ with U_J as bijective subgroup . This amounts to constructing two mathematical objects. First, a surjective 1-cocycle $f : G_{s(J)} \longrightarrow V_J$ for some $\mathbf{F}_q$-vector space V_J on which $G_{s(J)}$ acts as automorphisms, such that $f|_{U_J}$ is a bijective map. Second, a non-degenerate bilinear form $\kappa_J : V_J \times V_J \to \mathbf{F}_q$ which allows to describe the monomial operation explicitly. Hence the bilinear form is an auxiliary tool, which turns out to be very helpful.*

The first step in the construction is to obtain an 1-cocycle for $G_{s(J)}$. For that purpose we have to construct an abelian group V_J on which $G_{s(J)}$ acts as $\mathbf{F}_q$-vector space automorphisms. We start with two technical lemmas (which in fact carry the main ingredient why our approach actually works).

3.1.5 Reminder. *Let $A \in V_0$. Then $\operatorname{supp}(A) = \{(i,j) \in \square \mid A_{ij} \neq 0\}$.[50] If $J \subseteq \triangledown$ then $V_J = \bigoplus_{(i,j)\in J} \mathbf{F}_q e_{ij}$. Furthermore $\kappa : V_0 \times V_0 \longrightarrow \mathbf{F}_q : (A,B) \longmapsto \operatorname{tr}(A^\top B)$ is a non-degenerate bilinear form, satisfying $\kappa|_{V_J\times V_J}$ is non-degenerate.*

3.1.6 Lemma. *Let $A \in V_J$ and $g \in G_{s(J)}$. Then we have*

$$\operatorname{supp}(Ag) \subseteq s(J) \qquad \textit{and} \qquad \operatorname{supp}(Ag^\top) \subseteq \text{◸}.$$

Proof. Since $A \in V_J$, we have $\operatorname{supp}(A) \subseteq s(J)$ and hence $\overline{\operatorname{supp}}(1+A) \subseteq s(J)$, where $1 + A \in G$. But then $1 + A \in G_{s(J)}$ due to Corollary 1.2.27, yielding $g + Ag = (1+A)g \in G_{s(J)}$, which implies $\operatorname{supp}(Ag) \subseteq s(J)$, since $\overline{\operatorname{supp}}(g) \subseteq s(J)$.

For the second part let us write $g = 1 + \sum_{(i,j)\in s(J)} g_{ij}e_{ij}$. Now Lemma 2.2.15 implies that $Ag^\top$ is obtained from A by taking A and adding several columns (times some scalar factors) to columns on the left of the original columns. qed.

3.1.7 Definition. *Let $\pi : V_0 = V_J \oplus V_J^\perp \longrightarrow V_J$ denote the projection map to the first component.[51]*

3.1.8 Lemma. *Suppose $A, B \in V_0$, such that $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$. Then we have*

$$\kappa\big(A,B\big) = \kappa\big(\pi(A),B\big) = \kappa\big(A,\pi(B)\big) = \kappa\big(\pi(A),\pi(B)\big) = \kappa|_{V_J\times V_J}\big(\pi(A),\pi(B)\big).$$

[49] c.f. Proposition 1.2.21 and Theorem 1.2.27

[50] c.f. Defintion 1.2.16.

[51] remember: $V_J^\perp$ is the orthogonal space of V_J with respect to κ, c.f Lemma 2.2.9.

Proof. We have

$$\kappa(A,B) \overset{\text{Cor 2.2.8}}{=} \sum_{(i,j)\in\square} A_{ij}B_{ij} = \sum_{(i,j)\in J} A_{ij}B_{ij}.$$

The result follows, since $\pi(A) = \sum_{(i,j)\in J} A_{ij}e_{ij}$ and $\pi(B) = \sum_{(i,j)\in J} B_{ij}e_{ij}$. qed. |

3.1.9 Definition/Lemma (Group actions on V_J)**.** *We define the group action*

$$V_J \times G_{s(J)} \longrightarrow V_J : \ (A,g) \longmapsto A \circ g := \pi(Ag),$$

where the elements of the group $G_{s(J)}$ act as ($\mathbf{F}_q$-vector space) automorphisms.

Proof. We have to show $A \circ (gh) = (A \circ g) \circ h$. Let $B \in V_J$ be arbitrary. Using Lemma 3.1.6 we get $\operatorname{supp}(Bh^\top) \cap \operatorname{supp}(Ag) \subseteq J$. Thus Lemma 3.1.8 yields

$$\kappa\big(Bh^\top, Ag\big) = \kappa\big(Bh^\top, \underbrace{\pi(Ag)}_{=A\circ g}\big)$$

Using Lemma 2.2.7 we get

$$\kappa\big(B, A(gh)\big) = \kappa\big(B, (A\circ g)h\big) \qquad \text{for all } B \in V_J.$$

which implies $A(gh) \equiv (A \circ g)h \pmod{V_J^\perp}$. We apply π on both equations and get $A \circ (gh) = (A \circ g) \circ h$, which is the desired result.

Furthermore g acts as $\mathbf{F}_q$-vector space automorphisms, since both π and right multiplication with a matrix are $\mathbf{F}_q$-linear mappings. qed. |

3.1.10 Definition. *Let $f : \ G_{s(J)} \longrightarrow V_J$ denote the map given by $\pi|_{G_{s(J)}}$.*

3.1.11 Lemma. *Let $x, g \in G_{s(J)}$ be arbitrary, then we have*

$$f(x)g \equiv (x-1)g \pmod{V_J^\perp}.$$

In particular we have $f(x) \equiv x - 1 \pmod{V_J^\perp}$.[52]

Proof. We have

$$f(x) = \pi(x) = \pi(x) - \underbrace{\pi(1)}_{=0} = \pi(x-1)$$

and hence $f(x) \equiv x - 1 \pmod{V_J^\perp}$. For every $A \in V_J$ we have

$$\begin{aligned}\kappa\big(A, f(x)g\big) &\overset{\substack{\text{Lemma}\\ 2.2.7(c)}}{=} \kappa\big(Ag^\top, f(x)\big) \overset{\substack{\text{Lemmas}\\ 3.1.6\ \&\ 3.1.8}}{=} \kappa\big(\underbrace{\pi(Ag^\top)}_{\in V_J}, \underbrace{f(x)}_{\equiv x-1 \pmod{V_J^\perp}}\big)\\ &= \kappa\big(\pi(Ag^\top), x-1\big) \overset{\substack{\text{Lemmas}\\ 3.1.6\ \&\ 3.1.8}}{=} \kappa\big(Ag^\top, x-1\big) = \kappa\big(A, (x-1)g\big).\end{aligned}$$

But this implies $f(x)g \equiv (x-1)g \pmod{V_J^\perp}$. qed. |

[52] note, that $\operatorname{supp}(x-1) \subseteq s(J)$. This allows in many situations to apply Lemma 3.1.8.

3.1.12 Proposition. *We define* $f: G_{s(J)} \longrightarrow V_J$ *via* $f = \pi|_{G_{s(J)}}$.[53] *Then we have*

$$f(xg) = f(x) \circ g + f(g) \qquad \textit{for all } x, g \in G_{s(J)}.$$

Additionally f *is surjective and* $f|_{U_J}$ *bijective. In particular* f *is a surjective* 1*-cocycle.*

Proof of Proposition 3.1.12. Let $x, g \in G_{s(J)}$ be arbitrary. We have

$$f(xg) \overset{\substack{\text{Lemma}\\ \text{3.1.11(ii)}\\ \text{`in particular'}}}{\equiv} xg - 1 = (x-1)g + g - 1 \overset{\substack{\text{Lemma}\\ \text{3.1.11(ii)}}}{\equiv} f(x)g + f(g) \qquad \left(\operatorname{mod} V_J^{\perp}\right).$$[54]

Applying π we get, that f is a 1-cocycle. From Theorems 1.1.19 and 1.2.22 it follows that $f|_{U_J}$ is bijective. In particular this implies that f is surjective. qed.

3.1.13 Remark. *Due to Corollary 2.2.10 the bilinear form* $\kappa|_{V_J \times V_J}$ *is non-degenerate. Hence we have shown that* $(f, \kappa|_{V_J \times V_J})$ *is a monomial linearisation for* $G_{s(J)}$.

Note further that $\kappa|_{V_J \times V_J}$ *is the bilinear form described in Lemma 2.1.20, with respect to the basis* e_{ij} *of* V*, where* (i, j) *runs through* J.

Now we can state the theorem which is the cornerstone of this thesis.

3.1.14 Theorem (Fundamental theorem)**.** *The elements*

$$[A] = \frac{1}{|U_J|} \sum_{u \in U_J} \overline{\chi_A(u)} u \qquad \textit{where } A \textit{ runs through } V_J$$

form a $\mathbb{C}$*-basis for the complex group algebra* $\mathbb{C}U_J$*. By defining*

$$[A] * g := \chi_{A.g}(g)[A.g] \qquad \textit{for all } g \in G_{s(J)}, A \in V_J$$

we can make $\mathbb{C}U_J$ *into a monomial* $G_{s(J)}$*-module, where* $A.g := \pi(Ag^{-\top})$ *defines a group action of* $G_{s(J)}$ *on* V_J*. The restriction of the* $*$*-operation to* U_J *is given by usual right multiplication of* U_J *on* $\mathbb{C}U_J$*, i.e.*

$$[A] * u = [A]u = \frac{1}{|U_J|} \sum_{y \in U_J} \overline{\chi_A(y)} yu \qquad \textit{for all } u \in U_J, A \in V_J.$$

We again want to emphasize: the operation of $G_{s(J)}$ *on* $\mathbb{C}U_J$ *is not the 'usual right multiplication', since* $\mathbb{C}U_J$ *is not a* $\mathbb{C}G_{s(J)}$*-right module (c.f. Warning 2.1.36).*

[53] in matrix terms the map f is given by $f(g)_{ij} = g_{ij}$ if $(i, j) \in \triangledown$ and $f(g)_{ij} = 0$ otherwise.

[54] comparing this calculation to the one in the proof Proposition 2.2.3 we see that that only difference is, that equalities are substituted by congruences modulo $V_J^{\perp}$.

Proof. We have proven that $(f, \kappa|_{V_J \times V_J})$ is a monomial linearisation, satisfying that $f|_{U_J}$ is a bijective map. Hence we can apply Corollary 2.1.35. Every statement of the theorem follows directly from the corollary with the only exception being the formula $A.g = \pi(Ag^{-\top})$. As stated in Reminder 2.1.34, by definition we have $A.g = Ag^{-*}$. Thereby g^* denotes the adjoint of g with respect to $\kappa_J := \kappa|_{V_J \times V_J}$, where g is seen as the element of $\mathrm{End}_{\mathbf{F}_q}(V_J)$ given by $A \longmapsto Ag$ for all $A \in V_J$.

Hence it remains to show that the element $g^{-*} \in \mathrm{End}_{\mathbf{F}_q}(V_J)$ is given by $Ag^{-*} = \pi(Ag^{-\top})$, i.e. that we have $\kappa_J(Ag^{-*}, B) = \kappa_J\big(\pi(Ag^{-\top}), B\big)$ for all $B \in V_J$. Now, let $B \in V_J$ be arbitrary and set $y := g^{-1}$. Then we have

$$(*) \qquad \kappa_J(Ay^*, B) \overset{\text{Definition}}{=} \kappa_J(A, By) = \kappa(A, By) \overset{\text{Lemma 2.2.7(c)}}{=} \kappa(Ay^\top, B)$$

Due to Lemma 3.1.6 we have $\mathrm{supp}(B) \subseteq J$ and $\mathrm{supp}(Ag^{-\top}) \subseteq \not\nabla$. Hence we may apply Lemma 3.1.6 get

$$(**) \qquad \kappa_J(Ay^*, B) \overset{(*)}{=} \kappa(Ay^\top, B) \overset{\text{Lemma 3.1.8}}{=} \kappa_J\big(\pi(Ay^\top), B\big).$$

And since $y = g^{-1}$ everything is proven. qed. |

3.1.15 Remark. *In particular we have shown in* $(**)$ *of the previous proof that we have*

$$\kappa(A.g^{-1}, B) = \kappa(A, B \circ g) \qquad \textit{for all } A, B \in V_J, g \in G_{s(J)}.$$

Next, we state the variant of Corollary 2.2.16 in case $D_n(\mathbf{F}_q)$. This is the point of view with whom we will work. In Corollary 3.1.16 we forget the operation of $G_{s(J)}$ and make a statement only involving $\mathbb{C}U_J$.

3.1.16 Corollary (Truncated column operation for $D_n(\mathbf{F}_q)_J$)**.** *Let* $\lambda \in \mathbf{F}_q$ *and* $(i,j) \in J$ *be arbitrary. As in Chapter 1 we denote with* $x_{ij}(\lambda)$ *the root subgroup element of* U_J *given by* $1 + \lambda e_{ij} - \lambda e_{\bar{j}\bar{i}}$*. Then we have for all* $A \in V_J$

$$[A]x_{ij}(\lambda) = \vartheta(A_{ij}\lambda)[A.x_{ij}(\lambda)],$$

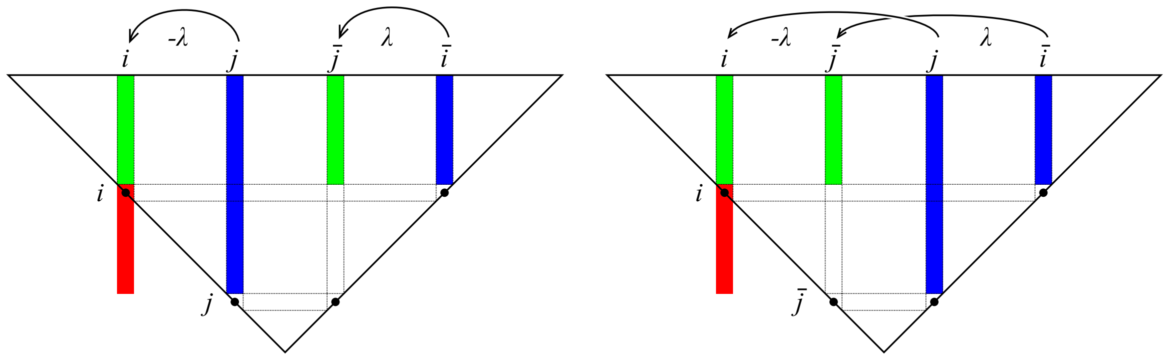
where $A.x_{ij}(\lambda)$ *arises from* A *by taking* A*, adding* $-\lambda$ *times the* j*-th column of* A *to the* i*-th column of* A*, adding* λ *times the* $\bar{i}$*-th column of* A *to the* $\bar{j}$*-th column of* A *and then putting* 0*'s at any entry, whose coordinates lie not in* J*. An Illustration is given in Figure 3.1.*[55]

Proof. Set $x = x_{ij}(\lambda)$. We have $A.x = \pi(A - \lambda A e_{ji} + \lambda A e_{\bar{j}\bar{i}})$. Thus the statement how $A.x$ arises from A follows from Lemma 2.2.15. Using $f\big(x_{ij}(\lambda)\big) = \lambda e_{ij}$, we obtain

$$\chi_{A.x}(x) = \vartheta\kappa\big(A.x, \lambda e_{ij}\big) = \vartheta\big(\lambda (A.x)_{ij}\big) = \vartheta(\lambda A_{ij}),$$

since the j-th columns of A and $A.x$ are the same $(j \neq i, \bar{j})$. qed. |

[55] $x_{ij}(\lambda) := 1 + \lambda e_{ij} - \lambda e_{\bar{j}\bar{i}}$ is an element of the root subgroup $X_{ij} \subseteq U_J$. It suffices to look at root subgroups, since U_J is generated by root subgroups, c.f. Theorem 1.2.22.



(a) Variant, where $j \leq n$ (b) Variant, where $j > n$

Figure 3.1.: The truncated column operation, Corollary 3.1.16
The figures show ∇. The green entries are the positions, at which the values may change. The red entries are truncated (i.e. we put zeros there). The Position which is red and lies on the diagonal of the matrix plays a role for the scalar $\vartheta(A_{ij}\lambda)$.

3.1.17 Example. *We consider $U = D_3(\mathbf{F}_q)$. Then we have*

$$V = \left\{ \left(\begin{array}{cccccc} \cdot & a_{12} & a_{13} & a_{14} & a_{15} & \cdot \\ & \cdot & a_{23} & a_{24} & \cdot & \\ & & \cdot & \cdot & & \\ & & & & & \end{array} \right) \;\middle|\; \begin{array}{c} a_{12}, a_{13}, a_{14} \\ a_{15}, a_{23}, a_{24} \end{array} \in \mathbf{F}_q \right\}$$

and for $A \in V$ and $\lambda \in \mathbf{F}_q$ we get, using Corollary 3.1.16,

$$\begin{aligned} &\left[\left(\begin{array}{cccccc} \cdot & A_{12} & A_{13} & A_{14} & A_{15} & \cdot \\ & \cdot & A_{23} & A_{24} & \cdot & \\ & & \cdot & \cdot & & \\ & & & & & \end{array} \right) \right] x_{23}(\lambda) \\ &= \vartheta(A_{23}\lambda) \left[\pi \underbrace{\left(\begin{array}{cccccc} \cdot & A_{12} - \lambda A_{13} & A_{13} & A_{14} + \lambda A_{15} & A_{15} & \cdot \\ & -\lambda A_{23} & A_{23} & A_{24} & \cdot & \\ & & \cdot & \cdot & & \\ & & & & & \end{array} \right)}_{Ax_{23}(\lambda)^{-\top}} \right] \\ &= \vartheta(A_{23}\lambda) \left[\left(\begin{array}{cccccc} \cdot & A_{12} - \lambda A_{13} & A_{13} & A_{14} + \lambda A_{15} & A_{15} & \cdot \\ & \cdot & A_{23} & A_{24} & \cdot & \\ & & \cdot & \cdot & & \\ & & & & & \end{array} \right) \right]. \end{aligned}$$

3.1.18 Remark. *This also shows why we called in Notation 2.2 elements of V_J ‘patterns’ and the operations $(A, g) \mapsto A.g$ and $([A], g) \mapsto [A]g$ ‘truncated column operation’.*

3.1.19 Remark. *Corollary 3.1.16 describes how the Group U_J acts on our monomial basis of $\mathbb{C}U_J$ by looking at elements of root subgroups, which are the buildings blocks of the group. For the corresponding building blocks $\tilde{x}_{ij}(\lambda)$[56] of $G_{s(J)}$ we can do the same. We get*

$$[A]\tilde{x}_{ij}(\lambda) = \vartheta(A_{ij}\lambda)[A.\tilde{x}_{ij}(\lambda)] \qquad \text{for all } A \in V_J,$$

where $A.\tilde{x}_{ij}(\lambda)$ arises from A by taking A, adding $-\lambda$ times the j-th column to the i-th column of A and then putting zeros to all positions not in J.

3.2. Classification of staircase orbit modules

3.2.1 Notation. *During this section we write $U = D_n(\mathbf{F}_q)$ and $G = A_N(\mathbf{F}_q)$.*

We attempt to find a classification of U-orbit modules, wherein we want to have a combinatorial formula to calculate the dimension of each module and to make statements of common irreducible constituents of such modules. For now, we restrict ourselves to the case of what will be called staircase U-modules. We will provide a labelling for these staircase U-modules and a combinatorial formula to calculate the dimension of such a module. In Section 3.3 we will show that every U-orbit module is isomorphic to some (not necessarily unique) staircase U-module. So, in particular we know the dimension of any U-orbit module (at least in principle, since we first need to find an isomorphic staircase U-module). The main result of this section is Theorem 3.2.29.

3.2.2 Notation. *If $J = \triangledown$, we write V instead of $V_\triangledown$ and $V^\perp$ instead of $V_\triangledown^\perp$.*

3.2.3 Reminder. *Due to Definition/Lemma 2.1.37 the U-orbit module $\mathbb{C}\mathcal{O}_U(A)$ has $\mathbb{C}$-basis $\mathcal{O}_U(A) = \{A.u \mid u \in U\}$. Hence for classifying U-orbit modules we only need to determine the U-orbits of V under the permutation operation '.', which saves us to keep notice of the linear characters $\chi_{A.u}$.*

3.2.4 Definition/Lemma. *Suppose A is a pattern. Then we call $(i,j) \in \triangledown$ a* ***main condition*** *of A if and only if A_{ij} is the rightmost non-zero entry in the i-th row. We call a main condition (i,j)* ***left main condition*** *if $(i,j) \in \triangledown^{\cdot\cdot}$, i.e. $j \leq n$, and* ***right main condition*** *if $(i,j) \in {}^{\cdot\cdot}\triangledown$, i.e. $j > n$. We define*

$$\begin{aligned} \mathrm{main}(A) &:= \big\{(i,j) \in \triangledown \bigm| (i,j) \textit{ is a main condition of } A\big\}, \\ \mathrm{l.main}(A) &:= \big\{(i,j) \in \triangledown \bigm| (i,j) \textit{ is a left main condition of } A\big\} \subseteq \triangledown^{\cdot\cdot}, \\ \mathrm{r.main}(A) &:= \big\{(i,j) \in \triangledown \bigm| (i,j) \textit{ is a right main condition of } A\big\} \subseteq {}^{\cdot\cdot}\triangledown. \end{aligned}$$

Of course we have $\mathrm{main}(A) = \mathrm{l.main}(A) \mathbin{\dot\cup} \mathrm{r.main}(A)$.

[56] $(i,j) \in s(J)$ and $\lambda \in \mathbf{F}_q$, compare Corollary 1.2.27

3.2.5 Lemma. *We have* $\mathrm{main}(A.g) = \mathrm{main}(A)$ *for all* $g \in G$. *Of course the statement remains true if we use* $\mathrm{l.main}(A)$ *or* $\mathrm{r.main}(A)$ *instead of* $\mathrm{main}(A)$.

Proof. This a direct consequence of Remark 3.1.19. qed. |

3.2.6 Definition. *Let* A *be a pattern. Lemma 3.2.5 allows us to define*

$$\mathrm{main}\left(\mathbb{C}\mathcal{O}_U(A)\right) = \mathrm{main}\left(\mathcal{O}_U(A)\right) = \mathrm{main}\left(\mathbb{C}\mathcal{O}_G(A)\right) = \mathrm{main}\left(\mathcal{O}_G(A)\right) := \mathrm{main}(A).$$

We can do the same for $\mathrm{l.main}(A)$ *and* $\mathrm{r.main}(A)$.

3.2.7 Definition. *Let* A *be a pattern. We call* A *a* ***staircase pattern****, if the elements of* $\mathrm{main}(A)$ *lie in different columns. Analogously, we call an* U*-orbit (or* G*-orbit) module* M *a* ***staircase*** U***-module*** *(or* ***staircase*** G***-modules****) if the elements of* $\mathrm{main}(M)$ *lie in different columns.*

We have already proved a stronger invariant than main, since not only the coordinates of the main conditions but also the concrete entries of A at main conditions remain the same. Thus we have:

3.2.8 Definition/Lemma. *Let* A *be a staircase pattern, then we define the* ***verge*** *of* A *as the pattern*

$$\mathrm{verge}(A) := \sum_{(i,j)\in\mathrm{main}(A)} A_{ij} e_{ij}.$$

We have $\mathrm{verge}(A.g) = \mathrm{verge}(A)$ *for all* $g \in G$.

Staircase patterns A*, where* $A = \mathrm{verge}(A)$*, are called* ***verge patterns****.*

3.2.9 Remark. *Let* A, B *be staircase patterns and suppose* $\mathrm{verge}(A) = \mathrm{verge}(B)$. *In Section 3.4 we will show that we have* $\mathcal{O}_G(A) = \mathcal{O}_G(B)$. *But the same is not true for* U*-orbits: In case* $U = D_2(\mathbf{F}_q)$ *all* U*-orbits are singletons. Therefore we have* q^2 U*-orbits. But we only have* $2q-1$ *different verges.*

Hence for the U*-orbits structure we need a different invariant.*

3.2.10 Definition. *Suppose* A *is a staircase pattern. Then we call* $(i,j) \in \nabla^{\cdot\cdot}$ *a* ***minor condition*** *of* A *if and only if* $(i,\bar{j})$ *is a right main condition. We define*

$$\mathrm{minor}(A) := \left\{(i,j) \in \nabla \;\middle|\; (i,j) \textit{ is a minor condition of } A\right\} \subseteq \nabla^{\cdot\cdot}.$$

Note: Since $\mathrm{minor}(A)$ *depends only on* $\mathrm{r.main}(A)$ *it is also invariant under truncated column operation.*

We want to label the staircase U-modules. This is done by finding a certain 'normal form' in the U-orbit of an arbitrary staircase pattern. The main tool in finding the normal form is provided by three 'combinatorial moves' on patterns. These will allow us to (in principle explicitly) calculate the 'normal form' of a given staircase pattern.

3.2.11 Lemma (First combinatorial move for U-orbits). *Let A be a staircase pattern and (i,k) a right main condition of A, such that*

$$A_{ak} = 0 \quad \textit{for all } (a,k) \in \nabla \setminus \{(i,k)\}.$$

Let $\bar{k} < j < k$ and $\lambda \in \mathbf{F}_q$ be arbitrary. Then $(i,j), (\bar{k},\bar{j}) \in \nabla$, $x := x_{\bar{k}\bar{j}}(A_{ik}^{-1}\lambda) = 1 + A_{ik}^{-1}\lambda e_{\bar{k}\bar{j}} - A_{ik}^{-1}\lambda e_{jk}$ is an element of U, and $A.x \in \mathcal{O}_U(A)$ satisfies:

(i) $[A.x]_{ij} = A_{ij} + \lambda$,

(ii) $[A.x]_{ab} = A_{ab}$ *for all $(a,b) \in \nabla$, except $(a,b) = (i,j)$ or lies in the $\bar{k}$-th column.*

An illustration is given in Figure 3.2(a).

Proof. Since $(i,k) \in \nabla$, we have $i < k < \bar{i}$ and thus $i < \bar{k} < j < k < \bar{i}$. Therefore it holds $(i,j) \in \nabla$. We have $\bar{j} < \bar{\bar{k}} = k$, hence $(\bar{k},\bar{j}) \in \nabla$ and $x \in U$. Using Corollary 3.1.16, we see from Figure 3.2(a), that the result holds. qed. |

3.2.12 Lemma (Second combinatorial move for U-orbits). *Let A be a staircase pattern, (i,k) a right main condition of A and $i < j < \bar{k}$, such that*

$$A_{ak} = 0 \textit{ for all } (a,k) \in \nabla \setminus \{(i,k)\} \quad \textit{and} \quad A_{a\bar{j}} = 0 \textit{ for all } (a,\bar{j}) \in \nabla.$$

Let $\lambda \in \mathbf{F}_q$ be arbitrary. Then $(i,j), (j,k) \in \nabla$, $x := x_{jk}(-A_{ik}^{-1}\lambda) = 1 - A_{ik}^{-1}\lambda e_{jk} + A_{ik}^{-1}\lambda e_{\bar{k}\bar{j}}$ is an element of U, and $A.x \in \mathcal{O}_U(A)$ is given by

$$[A.x]_{ab} = \begin{cases} A_{ij} + \lambda & (a,b) = (i,j), \\ A_{ab} & (a,b) \neq (i,j). \end{cases}$$

An illustration is given in Figure 3.2(b).

Proof. Since $(i,k) \in \nabla$, we have $i < k < \bar{i}$ and thus $i < j < \bar{k} < \bar{i}$. It holds $(i,j) \in \nabla$. We have $j < \bar{k} < k < \bar{j}$, since (i,k) is a right main condition. Hence $(j,k) \in \nabla$ and $x \in U$. Using Corollary 3.1.16, the result follows from Figure 3.2(b). qed. |

3.2.13 Lemma (Third combinatorial move for U-orbits). *Let A be a staircase pattern and (i,k) a left main condition of A and $i < j < k$, such that*

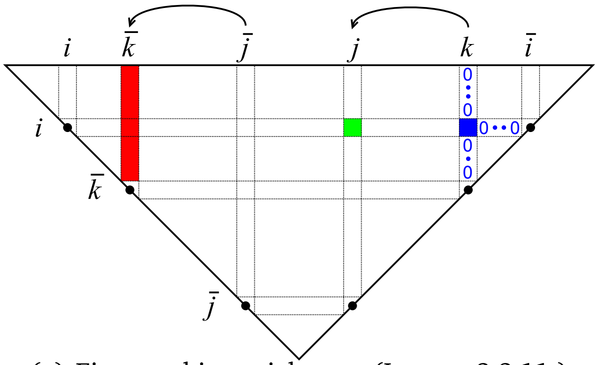
$$A_{a\bar{j}} = 0 \quad \textit{for all } (a,\bar{j}) \in \nabla.$$

Let $\lambda \in \mathbf{F}_q$ be arbitrary. Then $(i,j), (j,k) \in \nabla$, $x := x_{\bar{k}\bar{j}}(A_{ik}^{-1}\lambda) = 1 + A_{ik}^{-1}\lambda e_{\bar{k}\bar{j}} - A_{ik}^{-1}\lambda e_{jk}$ is an element of U, and $A.x \in \mathcal{O}_U(A)$ has the following properties:

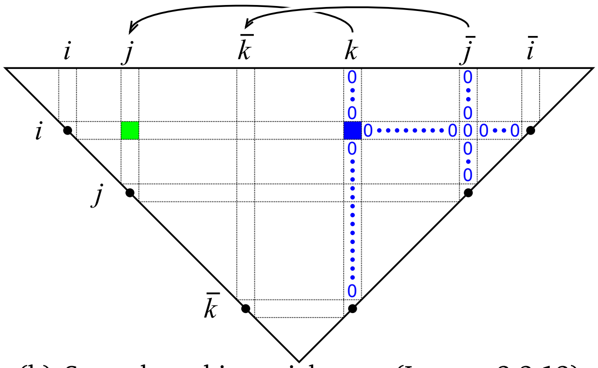
(i) $[A.x]_{ab} = A_{ab}$ *for all $(a,b) \in \nabla$, except $(a,b) = (a,j)$ and $A_{ak} \neq 0$,*

(ii) $[A.x]_{ij} = A_{ij} + \lambda$.

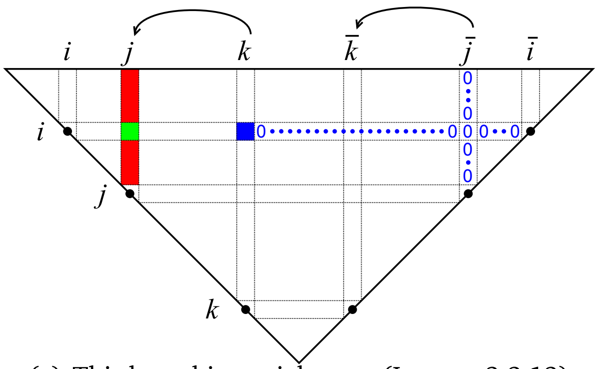
An illustration is given in Figure 3.2(c).



(a) First combinatorial move (Lemma 3.2.11)



(b) Second combinatorial move (Lemma 3.2.12)



(c) Third combinatorial move (Lemma 3.2.13)

Explanation: The figures show $\bigtriangledown$, the blue boxes denote entries (in these cases here exactly one) that are nonzero. The Blue 0's denote entries, which have to be 0. All the blue entries are the conditions needed to apply the move. The move produces arbitrary chosen entries in the green boxes at the cost of the red boxes. We do not really know how the red entries look after the move. The rest of the boxes remain the same. The arrows indicate how x operates on A by truncated column operation.

Figure 3.2.: Illustration of the three combinatorial moves

Proof. Since $(i,k) \in \triangledown$, we have $i < k < \bar{i}$ and thus $i < j < k < \bar{i}$. Therefore it holds $(i,j) \in \triangledown$. We have $j < k < \bar{k} < \bar{j}$, hence $(j,k) \in \triangledown$ and $x \in U$. Using Corollary 3.1.16, we see from Figure 3.2(c), that the result holds. qed. |

As our next step we will describe the aforementioned 'normal form'. We define so called core patterns, which will play the role.

3.2.14 Definition. *Let A be a staircase pattern.*

*(i) We call $(i,j) \in \triangledown$ a **supplementary condition** of A if (i,j) is on the left of some minor condition or some left main condition of A, in the same column as some minor condition of A and is not itself a minor or main condition. We define*

$$\operatorname{suppl}(A) := \big\{(i,j) \in \triangledown \;\big|\; (i,j) \textit{ is a supplementary condition of } A\big\} \subseteq \triangledown^{\ddots}.$$

*(ii) We define the **core** of a A to be*

$$\operatorname{core}(A) := \operatorname{main}(A) \;\cup\; \operatorname{minor}(A) \;\cup\; \operatorname{suppl}(A).$$

Note that their union is disjoint.

*(iii) We call A a **core pattern** if* $\operatorname{supp}(A) \subseteq \operatorname{core}(A)$.

Note: Since main *and* minor *are invariant under truncated column operation,* suppl *and* core *are also invariant under truncated column operation.*

3.2.15 Definition. *Let $(i,j) \in \triangledown$. We define the **lower hook** associated to (i,j) as the subset of $\triangledown$ given by*

$$H^{\ell}_{(i,j)} := \big\{(a,b) \in \triangledown \;\big|\; a = \bar{j}\big\} \cup \big\{(a,b) \in \triangledown \;\big|\; a > i \textit{ and } b = j\big\}.$$

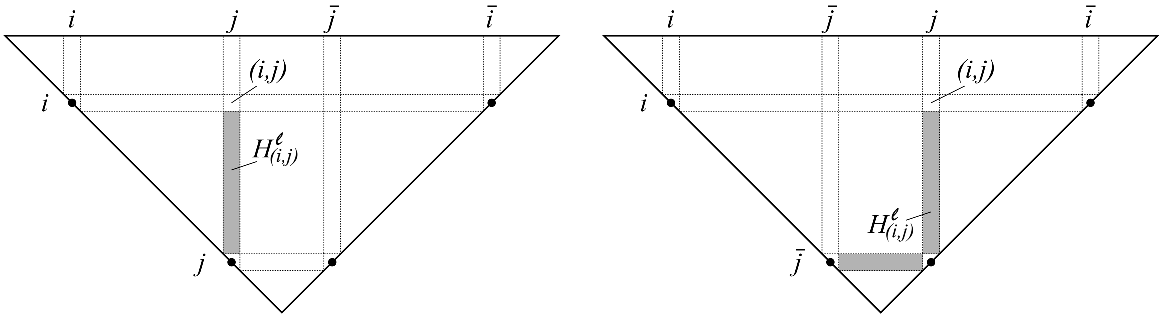
An Illustration is given in Figure 3.3.[57]

3.2.16 Definition. *Let A be a staircase pattern. Then we define the **flock of hooks** associated to A by*

$$\overline{\triangledown}(A) := \bigcup_{(i,j)\in\operatorname{main}(A)} H^{\ell}_{(i,j)}.$$

Note that $\overline{\triangledown}(A)$ depends only on $\operatorname{main}(A)$ *and not on A itself.*

[57] In Definition 3.5.26 we will also define upper hooks, whereas in Definition 3.3.17 we define hooks. Hence for avoiding ambiguity we called this one lower hook.



(a) Variant, where $j \leq n$ (b) Variant, where $j > n$

Figure 3.3.: Illustrations to Definition 3.2.15

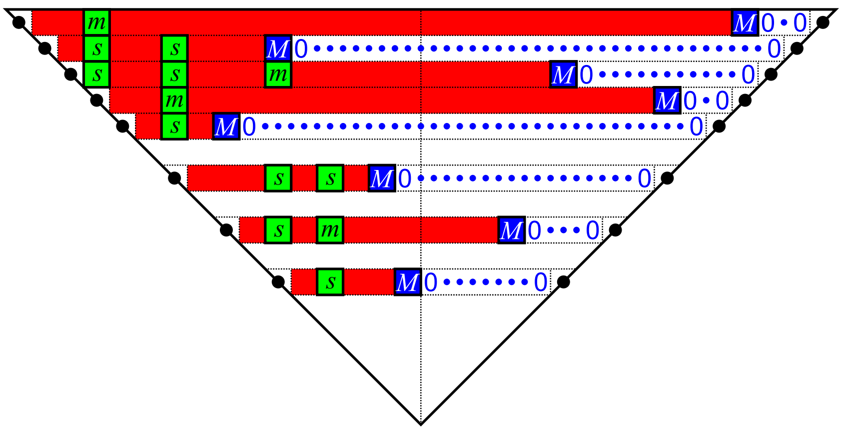
3.2.17 Lemma. *In the U-orbit of a staircase pattern exits a core pattern.*

More precisely: There exists a fixed ordering $\prec$ on $\forall(A)$, such that for all staircase patterns A there exist $\lambda_{ij} \in \mathbf{F}_q$, where (i,j) runs through $\forall(A)$, such that the element

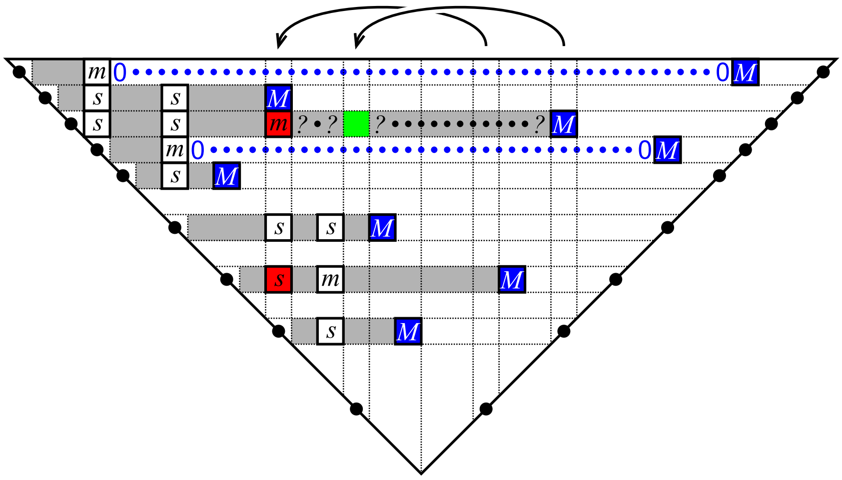
$$y := \prod_{(i,j)\in\forall(A)} x_{ij}(\lambda_{ij}), \qquad \text{where the product is taken according to } \prec,$$

satisfies that $A.y$ is a core pattern.

Proof. We consider an arbitrary staircase pattern A.

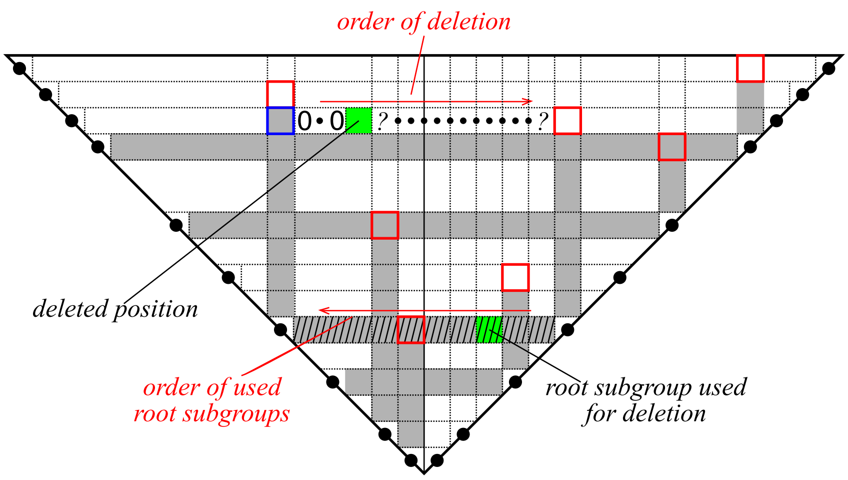


In the picture M denotes main conditions, m minor conditions and s supplementary conditions. We have to find an element in the U-orbit of A, where all entries at matrix positions which are coloured red are 0. We start by applying the first combinatorial move to delete entries lying between a main condition and the minor condition in the same row.

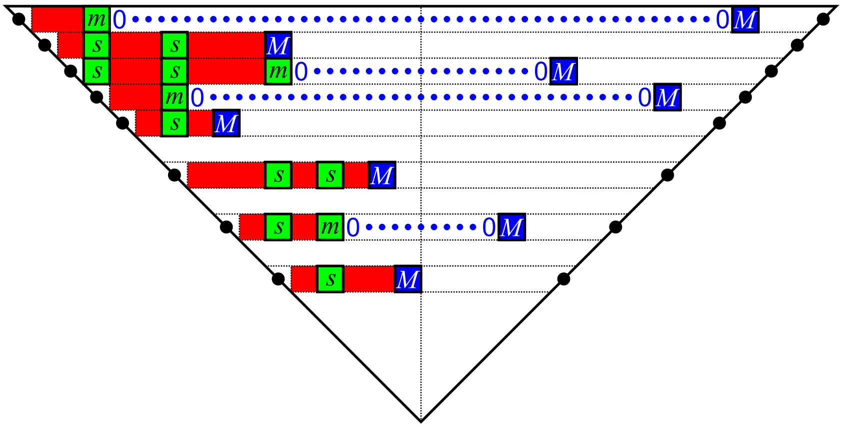


As one can see from the picture we can proceed inductively, starting with the row having the rightmost main condition. By doing this we ensure that we can use in each step the first combinatorial move.

We want to keep track on the root subgroups used in this process.

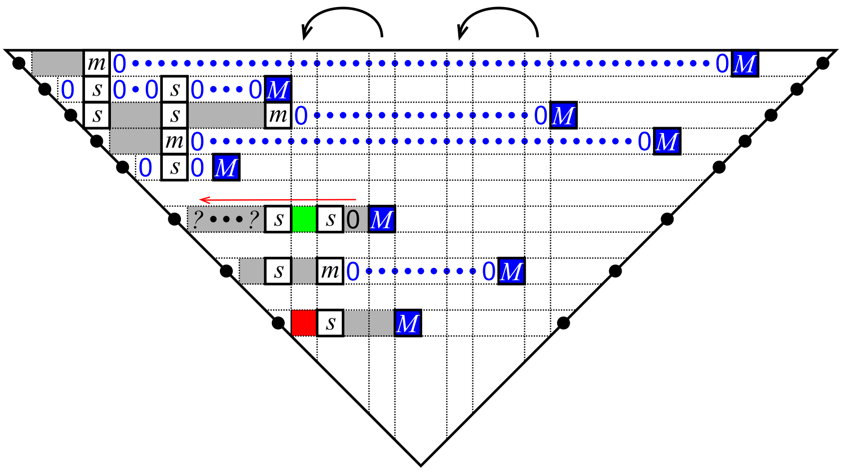


In the picture the main conditions of A are the red outlined boxes, whereas the grey coloured boxes form $\forall\!\!\!\!\mid(A)$. As one can see to delete the entries between a main condition and the corresponding minor condition, we need the root subgroups associated to matrix positions lying on the horizontal part of the lower hook associated to the main condition. Hence we can find an element in the U-orbit of A of the following form:

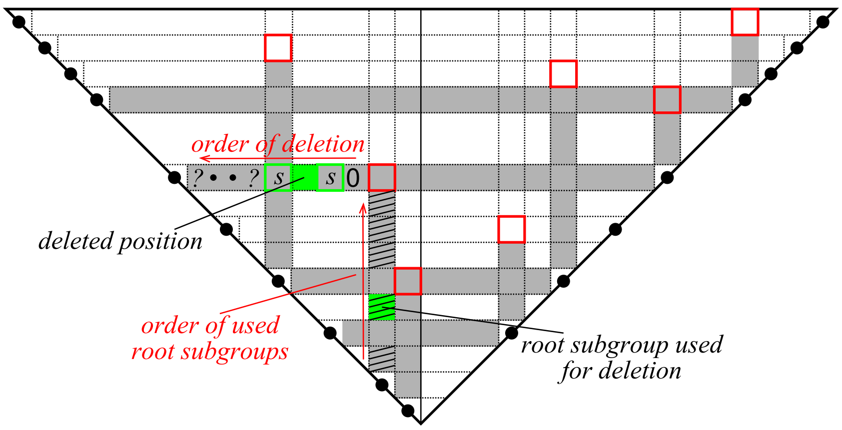


(3.2.18)

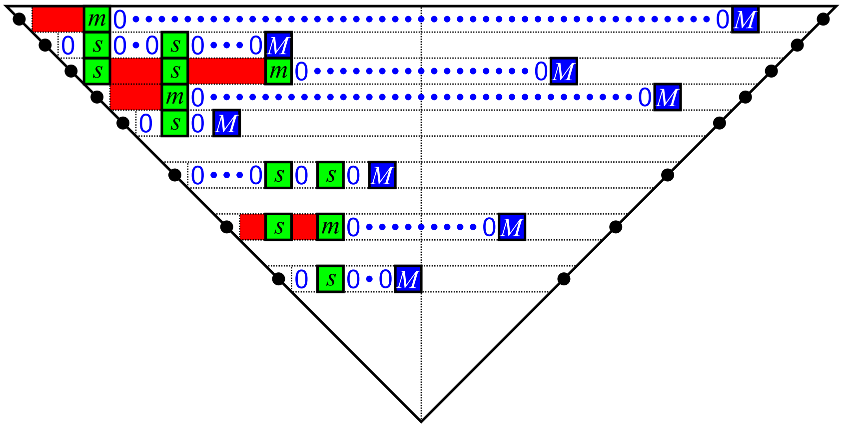
The next step is to use the third combinatorial move to delete entries which are on the left of left main conditions and are itself not supplementary conditions. Again we proceed inductively, but this time starting with the row having the leftmost main condition.



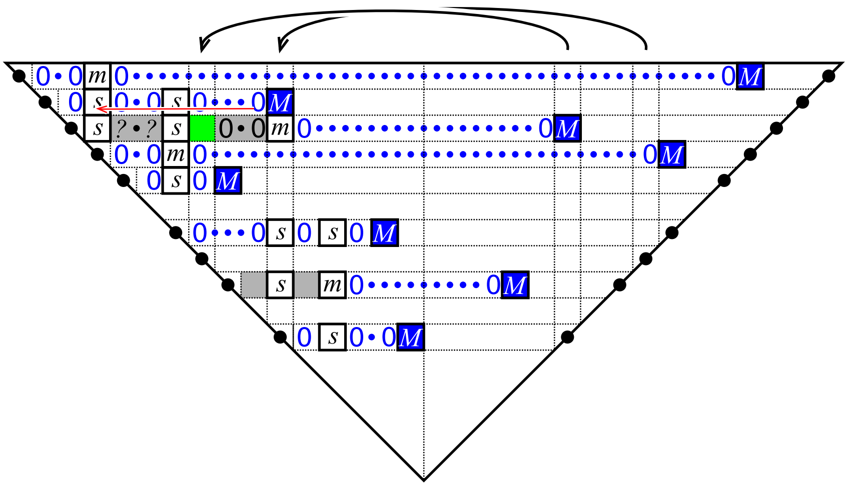
Again we want to keep track on the root subgroups used for that process.



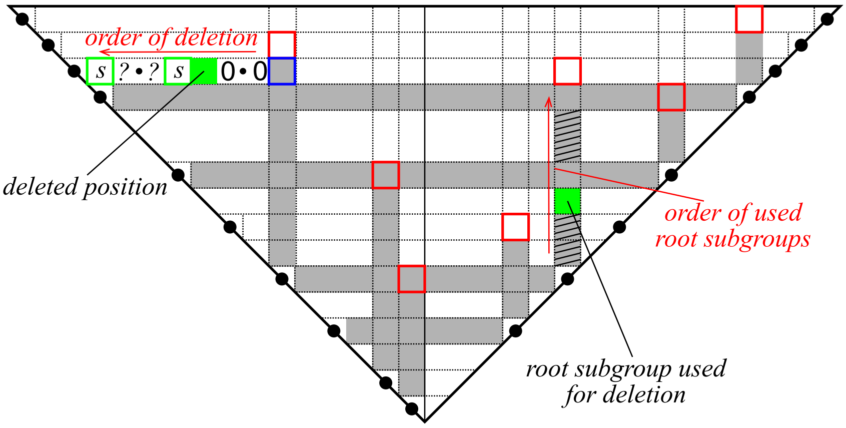
This time we use the root subgroups associated to matrix positions lying below some left main condition but not in the horizontal part of some lower hook. Hence we can find an element in the U-orbit of A of the following form:



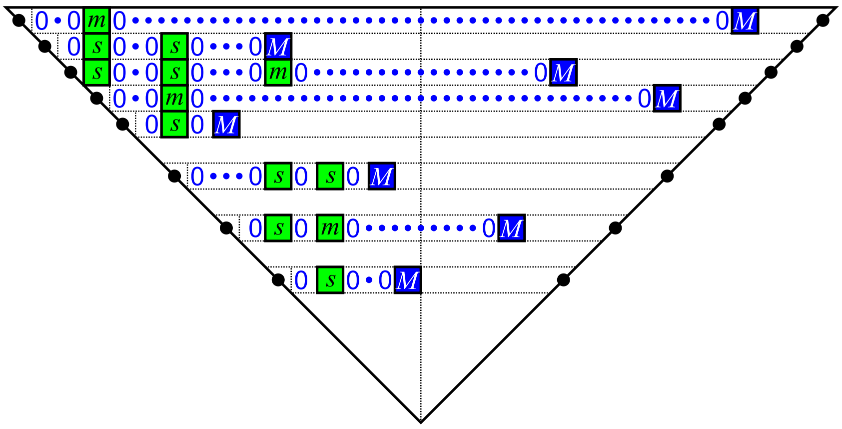
It remains to delete the entries lying on the left of minor conditions. In order to do this we apply the second combinatorial move, as one can see in the following picture:



Again we keep track on the used root subgroups.



This time we use the root subgroups associated to positions lying on the vertical part of the lower hook but being on no horizontal part of a lower hook. Hence we can find an element in the U-orbit of A of the desired type



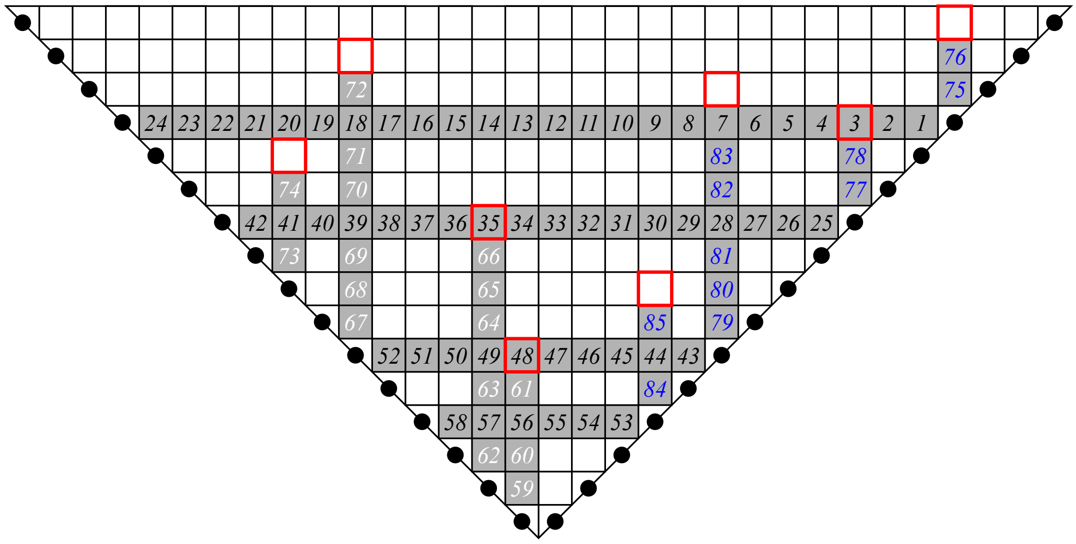
By construction we used root subgroups associated to every matrix position in $\overline{\forall}(A)$, but each one only once according to an order $\prec$, completely independent of the pattern A, c.f. Example 3.2.20. qed. |

3.2.19 Corollary. *Let A be a staircase pattern. Suppose all entries of A between minor and main conditions are equal to zero, i.e. A is of the form* (3.2.18). *Then the core pattern $C \in \mathcal{O}_U(A)$ obtained by Lemma 3.2.17 is given by*

$$C \quad = \sum_{(i,j)\,\in\,\mathrm{core}(A)} A_{ij} e_{ij}.$$

Proof. Let $y \in U$ be the element obtained using the procedure of Lemma 3.2.17. Hence $A.y$ is a core pattern. The first step of the procedure of Lemma 3.2.17 is to use the first conjugation move to obtain an element in $\mathcal{O}_U(A)$ of the form (3.2.17). As we start with such an element all λ_{ij}'s used so far are equal to 1, hence y consists only of root subgroups corresponding to the second and third conjugation move. But as the proof of Lemma 3.2.17 shows the second and third conjugation moves do not change the values at minor or supplementary conditions, proving the claim. qed. |

3.2.20 Example. *For the example used to illustrate the proof of Lemma 3.2.17 the order $\prec$ is given as in the following picture.*



The red outlined boxes are the main conditions of the pattern A, the numbers show the order $\prec$ on $\forall\!\!\!\!\!\top(A)$. Black coloured numbers correspond to root subgroups operating as 'first combinatorial move', white ones to the ones acting as 'third combinatorial move' and blue ones to the ones acting as 'second combinatorial move'.

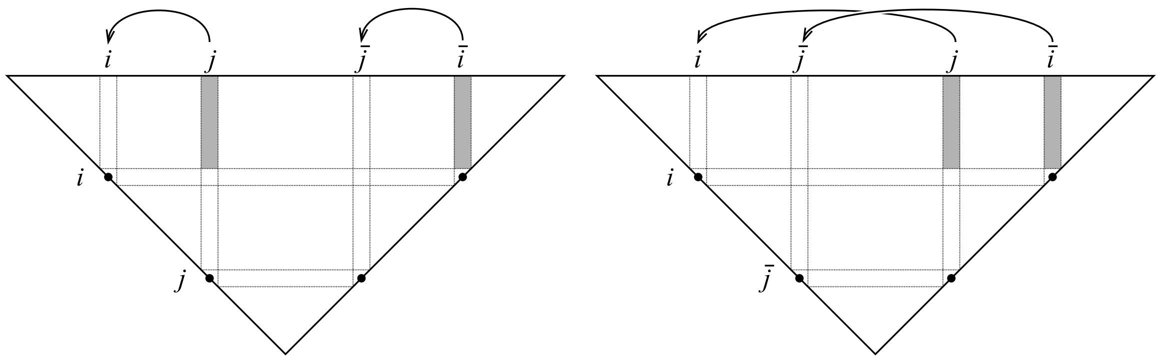
3.2.21 Remark. *We will show later, that U-orbits of staircase pattern are* uniquely *determined by their core pattern. Then in fact the proof of Lemma 3.2.17 provides an algorithm of how to calculate* the *'normal form' of an element A, i.e. to calculate in which staircase U-orbit a given staircase pattern lies.*

As already indicated in the previous remark, we would like to know, that there is a *unique* core pattern in each U-orbit. Our next step is to prove this uniqueness and to provide a combinatorial formula for the dimension of staircase U-orbit modules. In order to do this we need some lower bound for the stabilizer.

3.2.22 Lemma. *Let A be be a staircase pattern. Then $J(A) := \triangledown \setminus \forall\!\!\!\!\!\top(A)$ is a closed subset of $\triangledown$. If A is a verge pattern, then*

$$U_{J(A)} \leq \mathrm{Stab}_U(A).$$

If A is an arbitrary staircase patterns we have $|U_{J(A)}| \leq |\,\mathrm{Stab}_U(A)|$.



(a) Variant, where $j \leq n$

(b) Variant, where $j > n$

Explanation: The figures show how $A.x_{ij}(\lambda)$ arises from A by truncated column operation. In particular we see that $A.x_{ij}(\lambda)$ can differ from A only if A has some non-zero entry at a grey coloured position.

Figure 3.4.: Illustration to the proof of Lemma 3.2.22

Proof. **Step 1:** *$J(A)$ is a closed subset of $\triangledown$.*

We apply Corollary 1.2.14: Let first $(i, j), (j, k) \in J(A)$. Suppose $(i, k) \notin J(A)$, then there either exists some main condition $(l, k) \in \triangledown$, such that $i > l$, or some main condition $(l, \bar{i}) \in \triangledown$. In the first case (j, k) would be in $H^{\ell}_{(l,k)}$, since $j > i > l$, in the second case (i, j) would be in $H^{\ell}_{(l,\bar{i})}$. Both is a contradiction.

Second, let $(i, k), (j, \bar{k}) \in J(A)$, such that $i < j$. Suppose $(i, \bar{j}) \notin J(A)$, then there either exists a main condition $(l, \bar{j}) \in \triangledown$, such that $i > l$, or some main condition $(l, \bar{i}) \in \triangledown$. In the first case $(j, \bar{k})$ would be in $H^{\ell}_{(l,\bar{j})}$, in the second case (i, k) would be in $H^{\ell}_{(l,\bar{i})}$. Again both is a contradiction. Thus $J(A)$ is a closed subset of $\triangledown$.

Step 2: *Let A be a core pattern, $(i, j) \in J(A)$ and $x = x_{ij}(\lambda)$ for some $\lambda \in \mathbf{F}_q$. Then*

$$(A.x)_{ij} = A_{ij} \qquad \textit{for } (i, j) \in \operatorname{core}(A).$$

If A is additionally a verge pattern then $x \in \operatorname{Stab}_U(A)$. Hence Theorem 1.2.22 implies $U_{J(A)} \leq \mathrm{Stab}_{\mathrm{U}}(\mathrm{A})$ for verge patterns A.

Due to Definition/Lemma 3.2.8 we have $(A.x)_{ij} = A_{ij}$ as long as $(i, j) \in \operatorname{main}(A)$. As Figure 3.4 shows the pattern $A.x$ can differ from A only if A has some non-zero entries in the $\bar{i}$-th column or in the $\bar{j}$-th column above the i-th row. Since $(i, j) \in J(A)$ we have $(i, j) \notin \overline{\triangledown}(A)$. Hence there is no main condition of A in the $\bar{i}$-th column and no main condition of A in the j-th column above the i-th row. If A is a verge pattern, then $\operatorname{supp}(A) \subseteq \operatorname{main}(A)$, which implies $A.x = A$. If A is only a core pattern, then non-zero entries can also be at minor or supplementary conditions. We can have a minor or supplementary condition in the grey coloured entries of Figure 3.4 only if we have $j \leq n$, since $\operatorname{minor}(A) \cup \operatorname{suppl}(A) \subseteq \nabla^{\because}$. Hence A can differ from $A.x$ only in column i. Since there is no main condition in the $\bar{i}$-th column there is no minor condition of A in the i-th column and hence also no supplementary condition.

Step 3: *Let A be a core pattern. For every $u \in U_{J(A)}$ let y_u denote the element of Lemma 3.2.17, satisfying that $(A.u).y_u$ is a core pattern. Then $uy_u \in \operatorname{Stab}_U(A)$ and hence we may define the map*

$$\varphi : \; U_{J(A)} \longrightarrow \operatorname{Stab}_U(A) : u \longmapsto uy_u.$$

Let $u \in U_{J(A)}$ be arbitrary. We set $A_0 := \operatorname{verge}(A)$ and $A_1 := A - \operatorname{verge}(A)$. Note that $J(A) = J(\operatorname{verge}(A))$. Hence $U_{J(A)} \leq \operatorname{Stab}_U(A_0)$ by Step 2 and thus we have

$$A.u = (A_0 + A_1).u = A_0 + A_1.u.$$

Now $A_1.u$ can differ from A_1 only at entries, which lie in the same row but on the left of a matrix coordinate of $\operatorname{main}(A_1) \subseteq \operatorname{minor}(A) \cup \operatorname{suppl}(A)$. Hence $A.u = A_0 + A_1.u$ is of the form (3.2.18). Due to Corollary 3.2.19 we get

$$A.\varphi(u) \quad = \sum_{(i,j) \in \operatorname{core}(A)} (A.u)_{ij} e_{ij} \quad \overset{\text{Step 2}}{=} \sum_{(i,j) \in \operatorname{core}(A)} A_{ij} e_{ij} \quad \overset{A \text{ core pattern}}{=} \quad A.$$

Step 4: *The map φ is injective and hence we have $|U_{J(A)}| \leq |\operatorname{Stab}_U(A)|$.*

Due to Theorem 1.2.22 every element $u \in U_{J(A)}$ can be written uniquely as

$$u \; = \prod_{(i,j) \in \nabla \setminus \blacktriangledown(A)} x_{ij}(\lambda_{ij}^{(u)}) \tag{3.2.23}$$

if we take the product in some fixed but arbitrary order $\prec_{J(A)}$ on $J(A)$. On the other hand, due to Lemma 3.2.17, every element y_u has an expression of the form

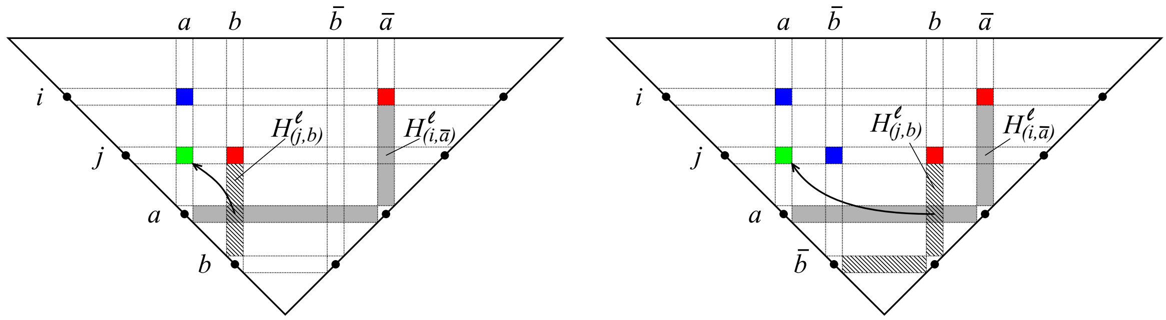
$$y_u \; = \prod_{(i,j) \in \blacktriangledown(A)} x_{ij}(\lambda_{ij}^{(u)})$$

with product taken in the order $\prec$. Hence the products

$$uy_u \; = \Big(\prod_{(i,j) \in \nabla \setminus \blacktriangledown(A)} x_{ij}(\lambda_{ij}^{(u)}) \Big) \Big(\prod_{(r,s) \in \blacktriangledown(A)} x_{rs}(\lambda_{rs}^{(u)}) \Big)$$

are all taken in the same order. Suppose $u, v \in U_{J(A)}$ satisfy $uy_u = vy_v$. Then Theorem 1.2.22 implies $\lambda_{ij}^{(u)} = \lambda_{ij}^{(v)}$ for all $(i,j) \in \nabla$ and hence in particular for all $(i,j) \in J(A) \subseteq \nabla$. The uniqueness of (3.2.23) implies then $u = v$. qed. |

3.2.24 Corollary. *Let A be a pattern. Then we have $|\mathcal{O}_U(A)| \leq q^{|\blacktriangledown(A)|}$.*



(a) Variant of the lemma, where $k \leq n$ (b) Variant of the lemma, where $k > n$

Figure 3.5.: Illustration to the proof of Lemma 3.2.27

The figures show $\triangledown$. They illustrate how a supplementary condition arises from a coordinate, which lies on two hooks.

Proof. Since $J(A) = \triangledown \smallsetminus {\triangledown\!\!\!\!\!=}(A)$ is a closed subset of $\triangledown$, we have $|U_{J(A)}| = q^{|\triangledown|-|{\triangledown\!\!\!\!\!=}(A)|}$ due to Corollary 1.2.23. Using the Orbit-Stabilizer Theorem we get

$$|\mathcal{O}_U(A)| = \frac{|U|}{|\operatorname{Stab}_U(A)|} \overset{\text{Lemma 3.2.22}}{\leq} \frac{|U|}{U_J(A)} = \frac{q^{|\triangledown|}}{q^{|\triangledown|-|{\triangledown\!\!\!\!\!=}(A)|}} = q^{|{\triangledown\!\!\!\!\!=}(A)|},$$

proving the claim. qed.

3.2.25 Remark. *We will later show, that equality holds, i.e.* $U_{J(A)} = \operatorname{Stab}_U(A)$ *for a verge pattern. In particular this shows, that* $\operatorname{Stab}_U(A)$ *is a pattern subgroup in that case. For general core patterns this is not true.*

Next we will prove two auxiliary lemmas.

3.2.26 Lemma. *Let A be a staircase pattern and (i,j) a main condition of A. Then*

$$\left|H^{\ell}_{(i,j)}\right| = \left|\left\{(a.b) \in \triangledown \;\middle|\; (a,b) \text{ is left of } (i,j) \text{ and not a minor condition}\right\}\right|.$$

Proof. This can directly be seen from Figure 3.3. qed.

3.2.27 Lemma. *Let A be a staircase pattern and $(a,b) \in \triangledown$. Then we define*

$$\operatorname{noh}_A(a,b) := \left|\left\{(i,j) \in \operatorname{main}(A) \;\middle|\; (a,b) \in H^{\ell}_{(i,j)}\right\}\right|.$$

Thereby $\operatorname{noh}(a,b)$ *is suggesting for 'number of lower hooks associated to main conditions of A on which (a,b) lies'. We have* $\operatorname{noh}(ab) \leq 2$ *and*

$$|\operatorname{suppl}(A)| = \left|\left\{(a,b) \in \triangledown \;\middle|\; \operatorname{noh}_A(a,b) = 2\right\}\right|.$$

Proof. Let $(a,b) \in \nabla(A)$. Then (a,b) lies on $H^{\ell}_{(i,j)}$ if and only if $a = \bar{j}$ or $b = j \wedge a > i$. Since in each column lies at most one main condition, a position (a,b) can lie on no hook at all, on only one hook or on two hooks. Hence $\mathrm{noh}_A(a,b) \leq 2$.

Now let (a,b) satisfy $\mathrm{noh}_A(a,b) = 2$. Then there exist main conditions $(i,\bar{a}) \in \nabla$ and $(j,b) \in \nabla$, where $j < a$. Since $a \leq n$ the position $(i,\bar{a})$ is a right main condition and (i,a) a minor condition. Now $a < b, \bar{b} < \bar{a}$. Thus (j,a) is a supplementary condition (See also Figure 3.5). Now let (a_1,b_1) and (a_2,b_2) be two positions, which lie each on two hooks. Then there are main conditions $(i_1,\bar{a}_1),(i_2,\bar{a}_2),(j_1,b_1),(j_2,b_2)$ of A, where $j_1 < a_1$ and $j_2 < a_2$ and (j_1,a_1) and (j_2,a_2) are supplementary conditions. Suppose $(j_1,a_1) = (j_2,a_2)$. Then the main conditions (j_1,b_1) and (j_2,b_2) lie in the same row, hence they are equal. Thus we have $(a_1,b_1) = (a_2,b_2)$. On the other hand, if (c,d) is a supplementary condition of A, then there are main conditions (c,j), where $j > d$ and $(i,\bar{d})$. We directly see, that $(c,d) \in H^{\ell}_{(c,j)} \cap H^{\ell}_{(i,\bar{d})}$. Thus there are exactly as many supplementary positions, as there are positions, which lie on two hooks. qed. |

The next lemma provides a possibility to easily count $|\rotatebox[origin=c]{180}{$\mathbb{A}$}(A)|$ from the pattern A. An example is given in 3.2.31 where $|\rotatebox[origin=c]{180}{$\mathbb{A}$}(A)| = \#$ of black 0's.

3.2.28 Lemma. *Let A be a staircase pattern. Then*

$$|\rotatebox[origin=c]{180}{$\mathbb{A}$}(A)| = \left| \left\{ (i,j) \in \nabla \;\middle|\; \begin{array}{c} (i,j) \textit{ is in the same row and strictly on the} \\ \textit{left of some main condition of } A \textit{ but itself neither} \\ \textit{a minor nor a supplementary condition of } A \end{array} \right\} \right| .$$

Proof. Using Lemma 3.2.26, we have

$$\sum_{(i,j)\in \mathrm{main}(A)} \left| H^{\ell}_{(i,j)} \right| = \left| \left\{ (a,b) \in \nabla \;\middle|\; \begin{array}{c} (i,j) \text{ is left of some main condition of } A \\ \text{but itself not a minor condition of } A \end{array} \right\} \right| .$$

Since $\rotatebox[origin=c]{180}{$\mathbb{A}$}(A) = \{(a,b) \mid \mathrm{noh}_A(a,b) = 1 \text{ or } 2\}$,Lemma 3.2.27 implies

$$|\rotatebox[origin=c]{180}{$\mathbb{A}$}(A)| \quad = \sum_{(i,j)\in \mathrm{main}(A)} \left| H^{\ell}_{(i,j)} \right| - |\mathrm{suppl}(A)|$$

and hence the result follows. qed. |

Now we can reap the fruit of our labour.

3.2.29 Theorem (Classification Staircase U-modules). *Let $M = \mathbb{C}\mathcal{O}_U(A)$ be a staircase U-module. Then there exists precisely one core pattern C in $\mathcal{O}_U(A)$ and the dimension of M is given by $q^{|\rotatebox[origin=c]{180}{$\mathbb{A}$}(M)|}$.*

In particular this implies, that core patterns form a labelling set for staircase U-modules.

Remark: *Lemma 3.2.28 provides a nice way to calculate $|\rotatebox[origin=c]{180}{$\mathbb{A}$}(A)|$.*

Proof. We consider $\mathcal{V}(A) := \{B \in V \mid \operatorname{verge}(B) = \operatorname{verge}(A)\}$.

Let $h := |\{(i,j) \in \nabla \mid (i,j)$ is on the left of some main condition of $A\}|$, then we have $|\mathcal{V}(A)| = q^h$. Further let $S := \{C \mid C$ core pattern with $\operatorname{verge}(C) = \operatorname{verge}(A)\}$. Remember that the verge is invariant under truncated column operation. Hence Lemma 3.2.17 allows us to find a subset S_0 of S, such that

$$\mathcal{V}(A) = \bigsqcup_{C \in S_0} \mathcal{O}_U(C),$$

where $\bigsqcup$ denotes disjoint union. Note that $|S| = q^{|\operatorname{minor}(A)|+|\operatorname{suppl}(A)|}$, which implies $|\mathcal{V}(A)| = q^h = |S| q^{|\triangledown(A)|}$ due to Lemma 3.2.28. Hence we get

$$|S| q^{|\triangledown(A)|} = |\mathcal{V}(A)| = \sum_{C \in S_0} |\mathcal{O}_U(C)| \leq \sum_{C \in S} |\mathcal{O}_U(C)| = |S||\mathcal{O}_U(C)| \overset{\text{Corollary 3.2.24}}{\leq} |S| q^{|\triangledown A|}.$$

Thus $S_0 = S$ and $|\mathcal{O}_U(A)| = q^{|\triangledown(A)|}$. But $S = S_0$ implies that there is exactly one core pattern in each U-orbit. qed. |

3.2.30 Corollary. *Let A be a staircase pattern. Then*

$$\operatorname{Stab}_U\big(\operatorname{verge}(A)\big) = U_{J(A)} \qquad \textit{and} \qquad |\operatorname{Stab}_U(A)| = |U_{J(A)}|.$$

Remark: $Stab_U(A)$ is in general not a pattern subgroup of U for core patterns.

Proof. Using the Orbit-Stabilizer Theorem and Theorem 3.2.29 we get $|\operatorname{Stab}_U(A)| = q^{|\nabla|-|\triangledown(A)|} = |U_{J(A)}|$. Lemma 3.2.22 implies the result. qed. |

We close the section, with an example to illustrate the combinatorics.

3.2.31 Example. *Let C be a core pattern of $\in D_6(\mathbf{F}_q)$. We make the assumption, that* $\operatorname{main}(C) = \{(1,6),(2,8),(3,9)\}$. *Then* $\operatorname{l.main}(C) = \{(1,6)\}$, $\operatorname{r.main}(C) = \{(2,8),(3,9)\}$, $\operatorname{minor}(C) = \{(2,5),(3,4)\}$ *and* $\operatorname{suppl} = \{(1,4),(1,5),(2,5)\}$. *Hence we have*

$$C = \left(\begin{array}{cccccc|cccccc}
\cdot & 0 & 0 & \color{green}{C_{14}} & \color{green}{C_{15}} & \color{red}{C_{16}} & & & & & & \cdot \\
 & \cdot & 0 & \color{green}{C_{24}} & \color{blue}{C_{25}} & 0 & 0 & \color{red}{C_{28}} & & & \cdot & \\
 & & \cdot & \color{blue}{C_{34}} & 0 & 0 & 0 & 0 & \color{red}{C_{39}} & \cdot & & \\
 & & & \cdot & & & & & \cdot & & & \\
 & & & & \cdot & & & \cdot & & & & \\
 & & & & & \cdot & \cdot & & & & &
\end{array}\right),$$

where the red entries are entries at main conditions (they need to be nonzero!), whereas blue entries are entries at minor conditions and green entries are entries at supplementary conditions (green and blue entries may very well be zero). Note, that the black 0's are only included explicitly to simplify the counting for the dimension of $\mathbb{C}\mathcal{O}_U(C)$, every not explicitly written entry of course is also a zero. The $|$ simply indicates the symmetry axis of the pattern. We get $\dim \mathbb{C}\mathcal{O}_U(C) = q^{\#\text{black } 0\text{'s}} = q^9$ for any choice of $\color{green}{C_{14}}, \color{green}{C_{15}}, \color{green}{C_{24}}, \color{blue}{C_{25}}, \color{blue}{C_{34}} \in \mathbf{F}_q$ *and* $\color{red}{C_{16}}, \color{red}{C_{28}}, \color{red}{C_{39}} \in \mathbf{F}_q^{\times}$.

3.3. Homomorphisms between orbit modules

As previously let $U = D_n(\mathbf{F}_q)$. In this section we want first to give reasons why staircase patterns are not an exotic special case, but rather the general case. More precisely we will show, that every U-orbit module is isomorphic to some staircase module. Then we will introduce a special class of staircase modules (the so called hook-separated staircase modules) and prove, that any irreducible module is constituent of some hook-separated staircase module. By then we have shown, that hook-separated staircase modules carry all interesting information of the representation theory of U. Hence we may restrict our study to hook-separated staircase modules.

The remainder of this section is devoted to the question of determining homomorphisms between hook-separated staircase modules. In particular we want to investigate, when there are no nontrivial homomorphisms.[58] We develop with Corollary 3.3.23 a criterion, that allows us to attack the problem of orthogonality. Complementing this criterion, Theorem 3.3.32 shows, that hook-separated staircase modules, which are not orthogonal, are in fact isomorphic. Together this amounts to the statement 'hook-separated U-orbit modules are either isomorphic or orthogonal'. We close the section by determining a basis for the endomorphism ring of a hook-separated staircase module.

From general theory we know, that every $\varphi \in \mathrm{End}_{\mathbb{C}U}(\mathbb{C}U)$ is of the form

$$\lambda_a : \mathbb{C}U \longrightarrow \mathbb{C}U : x \longmapsto ax, \qquad \text{for a unique } a \in \mathbb{C}U.$$

Using Remark 2.2.19, we get in the case of $G = U = A_N(\mathbf{F}_q)$, that left multiplication of U on $\mathcal{B}$ is monomial, where the operation is given by 'truncated row operation'. Hence U-orbit modules are mapped to U-orbit modules. This fails to be true in general in the case of $U = D_n(\mathbf{F}_q)$. But we still have:

3.3.1 Lemma. *Let $g \in U$ and $A \in V$, then the map $\lambda_g|_{\mathbb{C}\mathcal{O}_U(A)} : \mathbb{C}\mathcal{O}_U(A) \to \mathrm{Im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U(A)})$ is a $\mathbb{C}U$-isomorphism.*

Let us try to understand the image of λ_g.

3.3.2 Lemma. *Let $A \in V$ and $g \in U$. Then we have*

$$\lambda_g([A]) \overset{\text{Def.}}{=} g[A] = \frac{1}{|U|}\sum_{y\in U}\overline{\vartheta\kappa\big(g^{-\top}A, y\big)}y.$$

Proof. Let $A \in V$ and $g \in U$ be arbitrary. We have

$$\lambda_g([A]) = g[A] = \frac{1}{|U|}\sum_{z\in U}\overline{\chi_A(z)}gz \overset{y=gz}{=} \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(g^{-1}y)}y$$

[58] Two $\mathbb{C}U$-modules having no nontrivial $\mathbb{C}U$-homomorphism between them are called orthogonal.

and

$$\chi_A(g^{-1}y) \;=\; \vartheta\kappa\big(A,\; \underbrace{f(g^{-1}y)}_{\equiv g^{-1}y \,(\mathrm{mod}\, V^\perp)}\;\big) \;=\; \vartheta\kappa\big(A, g^{-1}y\big) \stackrel{\text{Lemma 2.2.7}(c)}{=} \vartheta\kappa\big(g^{-\top}A, y\big).$$

Thus, the result follows. qed.

3.3.3 Remark. *Suppose $\vartheta\kappa(g^{-\top}A, x) = \alpha\chi_C(x)$ for some $C \in V$ and $\alpha \in \mathbb{C}$. Then we would have $\lambda_g([A]) = \overline{\alpha}[C]$. Hence the left operation would be monomial. Unfortunately this is* ***not true*** *in general as the following example shows:*

3.3.4 Example. *Let $U = D_3(\mathbf{F}_q)$ and suppose $a, \lambda \in \mathbf{F}_q^\times$. Then we have*

$$x_{12}(\lambda)\left[\left(\begin{array}{ccc|ccc} \cdot & & & & a & \cdot \\ & \cdot & & & \cdot & \\ & & \cdot & \cdot & & \end{array}\right)\right]$$

$$= \sum_{b,c\in\mathbf{F}_q} \frac{1}{q}\vartheta(bca^{-1}\lambda^{-1}) \left[\left(\begin{array}{ccc|ccc} \cdot & & & & a & \cdot \\ & \cdot & c & d & \cdot & \\ & & \cdot & \cdot & & \end{array}\right)\right].$$

Note, that we used the notational conventions of Example 3.2.31. To calculate this equation we used the methods, which will be developed in the proof of Lemma 3.3.16.

Before we intensify our investigation of the left multiplication, we want to introduce the 'truncated row operation'. This will simplify the notation used later on.

3.3.5 Definition/Lemma (Truncated row operation)**.** *The map*

$$U \times V \longrightarrow V : \;(u, A) \longmapsto g.A := \pi(g^{-\top}A)$$

defines a group operation, called ***truncated row operation****.*

Note that the elements of U act as $\mathbf{F}_q$-vector space automorphisms on V.

Proof. We have to show $(gh).A = g.(h.A)$.

Clearly we have $h^{-\top}A \equiv h.A \;\big(\mathrm{mod}\, V^\perp\big)$ and hence also

$$\kappa(h^{-\top}A - h.A, Y) = 0 \qquad \text{for all } Y \in V.$$

If $X \in V$ and $g \in U$ we always have $g^{-1}X \in V$ by Lemma 2.2.15. Hence we get

$$\kappa(h^{-\top}A - h.A, g^{-1}X) = 0 \qquad \text{for all } X \in V.$$

But $\kappa(h^{-\top}A - h.A, g^{-1}X) = \kappa(g^{-\top}h^{-\top}A - g^{-\top}(h.A), X)$ due to Lemma 2.2.7(c). Thus

$$\underbrace{g^{-\top}h^{-\top}}_{=\,(gh)^{-\top}}A \equiv g^{-\top}(h.A) \quad \big(\mathrm{mod}\, V^\perp\big).$$

We apply π to get $(gh).A = g.(h.A)$. qed.

As for truncated column operation we study how it operates explicitly for root subgroups, i.e. for the building blocks of our group U:

3.3.6 Corollary (Truncated row operation). *Let $A \in V, \lambda \in \mathbf{F}_q$ and $(i,j) \in \triangledown$.*

(i) If $(i,j) \in {}^{\ddots}\!\triangledown$, then $x_{ij}(\lambda).A = A$,

(ii) If $(i,j) \in \triangledown^{\ddots}$, then $x_{ij}(\lambda).A$ is obtained from A by taking A and adding $-\lambda$ times the i-th row of A to the j-th row of A and afterwards putting 0's at any entry whose coordinates lie not in $\triangledown$.

Proof. We have $x_{ij}(\lambda).A = \pi\big((1+\lambda e_{ji} - \lambda e_{\overline{ij}})A\big)$. Combining Definition/Lemma 3.3.5 with Lemma 2.2.15 we get the result. qed. |

From the proof we also get

3.3.7 Remark. *Let $A \in V$, $\lambda \in \mathbf{F}_q$ and $(i,j) \in \triangledown^{\ddots}$. Then $x_{ij}(\lambda)^{-\top}A$ is obtained from A by taking A and adding $-\lambda$ times the i-th row of A to the j-th row of A.*

3.3.8 Notation. *We write $\mathcal{O}_U^{\text{row}}(A)$ and $\operatorname{Stab}_U^{\text{row}}(A)$ in situations where we consider orbits or stabilizers defined for 'truncated row operation'. Conversely we want to emphasize that $\mathcal{O}_U(A)$ and $\operatorname{Stab}_U(A)$ are only used for the 'truncated column operation'.*

3.3.9 Remark. *From Corollary 3.3.6 we get*

$$\mathcal{O}^{\text{row}}_{D_n(\mathbf{F}_q)}(A) = \mathcal{O}^{\text{row}}_{A_n(\mathbf{F}_q)}(A),$$

where $A_n(\mathbf{F}_q)$ is interpreted as a subgroup of $D_n(\mathbf{F}_q)$ using embedding φ of Lemma 1.2.25.

3.3.10 Remark. *In general we have $g.(A.u) \neq (g.A).u$. For example*

$$\left(x_{12}(\lambda). \left(\begin{array}{ccc|ccc} \cdot & & & & a & \cdot \\ & \cdot & & & \cdot & \\ & & \cdot & \cdot & & \end{array}\right)\right).x_{23}(\mu) = \left(\begin{array}{ccc|ccc} \cdot & & & \mu a & a & \cdot \\ & \cdot & & & \cdot & \\ & & \cdot & \cdot & & \end{array}\right)$$

whereas

$$x_{12}(\lambda). \left(\left(\begin{array}{ccc|ccc} \cdot & & & & a & \cdot \\ & \cdot & & & \cdot & \\ & & \cdot & \cdot & & \end{array}\right).x_{23}(\mu) \right) = \left(\begin{array}{ccc|ccc} \cdot & & & \mu a & a & \cdot \\ & \cdot & & -\lambda\mu a & \cdot & \\ & & \cdot & \cdot & & \end{array}\right)$$

We have used the same conventions on notation as in Example 3.2.31.

We return to the investigation of the left operation.

3.3.11 Proposition. *Let $g \in U$ and $A \in V$, such that* $\operatorname{supp}(g^{-\top}A) \subseteq \text{◸}$. *Then we have*

$$\lambda_g([B]) = \chi_{g.B}(g)[g.B] \qquad \textit{for all } B \in \mathcal{O}_U(A).$$

In particular, this implies $\operatorname{Im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U(A)}) = \mathbb{C}\mathcal{O}_U(g.A)$.

Proof. Let $B \in \mathcal{O}_U(A)$. From Corollary 2.2.15 we see, that $g^{-\top}B$ arises from B by taking B and adding several rows to other rows (times some scalars). Hence it only depends on the main conditions of B if $\operatorname{supp}(g^{-\top}B) \subseteq \text{◸}$, but $\operatorname{main}(B) = \operatorname{main}(A)$, hence we get $\operatorname{supp}(g^{-\top}B) \subseteq \text{◸}$ for all $B \in \mathcal{O}_U(A)$.

$$\kappa(g^{-\top}B, y) = \kappa(g^{-\top}B, y-1) + \kappa(g^{-\top}B, 1) = \kappa(g^{-\top}B, y-1) + \kappa(B, g^{-1})$$

Note that $\operatorname{supp}(g^{-\top}B) \cap \operatorname{supp}(y-1) \subseteq \nabla$ and $\operatorname{supp}(B) \cap \operatorname{supp}(g^{-1}) \subseteq \nabla$ and remember that in this section $J = \nabla$. Hence we may apply Lemma 3.1.8, which yields

$$(*) \qquad \vartheta\kappa\big(g^{-\top}B, y\big) = \vartheta\kappa\big(\pi(g^{-\top}B), \pi(y)\big)\vartheta\kappa\big(B, \pi(g^{-1})\big) \overset{f=\pi|_G}{=} \chi_{g.B}(y)\chi_B(g^{-1}).$$

Hence, using Lemma 3.3.2, we have

$$g[B] = \overline{\chi_B(g^{-1})}\frac{1}{|U|}\sum_{y \in U} \overline{\chi_{g.B}(y)}y = \overline{\chi_B(g^{-1})}[g.B].$$

It remains to show $\overline{\chi_B(g^{-1})} = \chi_{g.B}(g)$ or equivalently $\chi_{g.B}(g)\chi_B(g^{-1}) = 1$. We have

$$1 = \chi_B(1) = \vartheta\kappa\big(B, f(1)\big) \overset{\text{Lemma 3.1.8}}{=} \vartheta\kappa\big(B, 1\big) = \vartheta\kappa\big(g^{-\top}B, g\big) \overset{(*)}{=} \chi_{g.B}(g)\chi_B(g^{-1}),$$

which finishes the proof. qed. |

3.3.12 Remark. *Hence at least under some conditions we get an analogue of the formula in Remark 2.2.19(ii) for the group $D_n(\mathbf{F}_q)$.*

The proposition has very nice consequences:

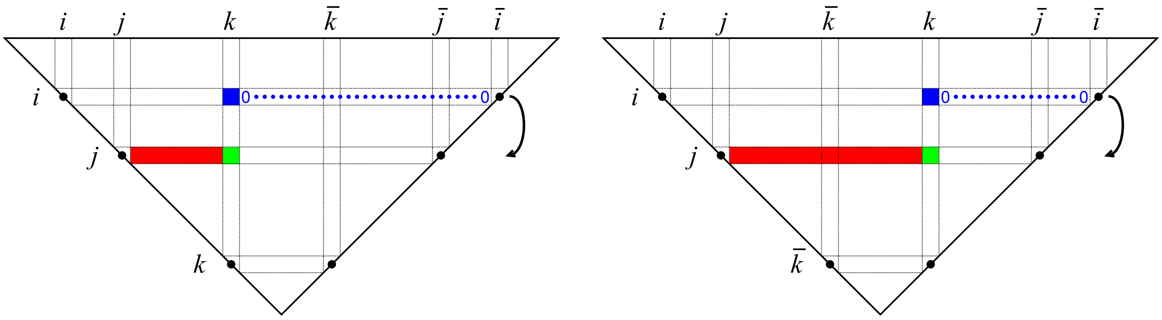
3.3.13 Corollary. *Let $A \in V$ and $g \in U$ be, such that* $\operatorname{supp}(g^{-\top}A) \subseteq \text{◸}$. *Then we have*

$$g.(B.u) = (g.B).u \qquad \textit{for all } B \in \mathcal{O}_U(A), u \in U.$$

Proof. We have

$$\mathbb{C}[(g.B).u] = \mathbb{C}[g.B]u = \mathbb{C}(g[B])u = \mathbb{C}g([B]u) = \mathbb{C}g([B.u]) = \mathbb{C}[g.(B.u)]$$

and hence $(g.B).u = g.(B.u)$. qed. |



(a) Variant, where $k \leq n$ (b) Variant, where $k > n$

Explanation: The figures show $\bigtriangledown$, the blue boxes denote entries (in these cases here exactly one) that are nonzero. The Blue 0's denote entries, which have to be 0. All the blue entries are the conditions needed to apply the move. The move produces arbitrary chosen entries in the green boxes at the cost of the red boxes. We do not really know how the red entries look after the move. The rest of the boxes remain the same. The arrows indicate how x operates on A by truncated row operation.

Figure 3.6.: Illustrations to Corollary 3.3.14

3.3.14 Corollary (Simple row move)**.** *Let A be a pattern and $(i,k) \in \mathrm{main}(A)$. If $i < j < \min(k, \bar{k})$ and $\lambda \in \mathbf{F}_q$, then $(i,j),(j,k) \in \bigtriangledown$, thus $x := x_{ij}(-\lambda A_{ik}^{-1}) \in D_n(\mathbf{F}_q)$ and we have*

$$\mathbb{C}\mathcal{O}_U(A) \cong \mathbb{C}\mathcal{O}_U(x.A),$$

where

(i) $(x.A)_{jk} = A_{jk} + \lambda$

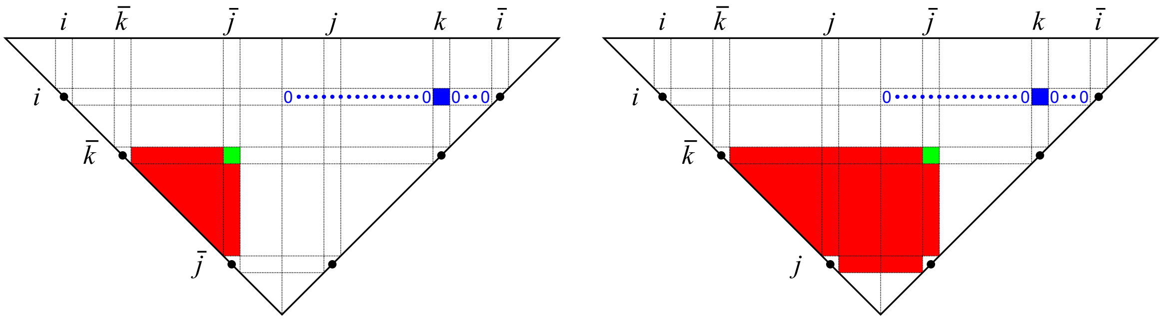
(ii) $(x.A)_{ab} = A_{ab}$*, except* (a,b) *is on the left of* (j,k)*.*

An illustration is given in Figure 3.6.

Proof. The proof follows directly from Lemma 2.2.15(ii) and Figure 3.6. qed. |

3.3.15 Corollary. *Every U-orbit module is isomorphic to a (not necessarily unique) staircase module and the isomorphism is given by left multplication with a group element.*

Proof. Corollary 3.3.14 allows us to delete main conditions in the same column, which are below of other main conditions, while new ones occur only on the left of that column. Thus, by proceeding from right to left we can obtain a staircase pattern. qed. |



(a) Variant, where $\bar{j} \leq n$ (b) Variant, where $\bar{j} > n$

Explanation: The figures show $\bigtriangledown$, the blue boxes denote entries (in these cases here exactly one) that are nonzero. The Blue 0's denote entries, which have to be 0. Left multiplication produces arbitrary chosen entries in the green boxes at the cost of the red boxes. We do not really know how the red entries look after the move. The rest of the boxes remain the same. Note that $\lambda_x(A)$ is a linear combination of patterns of the shown type.

Figure 3.7.: Illustrations to Lemma* 3.3.16

Unfortunately the situation is not always as pleasant as in Proposition 3.3.11 as we have already seen in Example 3.3.4. But Corollary 3.3.15 allows us to restrict ourselves to the investigation of staircase U-orbits, and due to Theorem 3.2.29 we may suppose, the module is generated by a core pattern. Making these assumptions, we can formulate the next lemma, which in view of Corollary 3.3.14, has a very weak statement at first sight. But nevertheless the Lemma is the key result for the proof of Corollary 3.3.19.

We should also mention, that Corollary 3.3.19, which is our main result for the investigation of the left operation in the 'ugly case', can also be derived from the main theorem of this thesis using, Theorem 3.6.20 and Definition/Lemma 3.6.2 (i) $\implies$ (iii). Hence we could in principle omit our proof of Corollary* 3.3.19 and the road to that Corollary, i.e. Lemma* 3.3.16 and Proposition* 3.3.18. This is, why we added an * to all of these results. In the end we decided to still include the discussion of the 'ugly case' of the left operation, since it gives us on the one hand some structural insight into the statement of Corollary 3.3.19 and on the other hand it illustrates the structural differences between the theories of $D_n(\mathbf{F}_q)$ and $A_n(\mathbf{F}_q)$.

3.3.16 Lemma*. *Let A be a core pattern and (i,k) a right main condition of A. In contrast to Corollary 3.3.14, let $\bar{k} < j < k$ (i.e. $(j,k) \in \lhd$). Let $\lambda \in \mathbf{F}_q$ and set $x := x_{ij}(-\lambda A_{ik}^{-1})$. Then we have*

$$\lambda_x(A) = \sum_{C\in\mathcal{I}} \mu_C[C], \qquad \textit{for unique } \mu_C \in \mathbb{C},$$

where $C \in \mathcal{I}$ if and only if

(i) $A_{ab} = C_{ab}$ *for all* $(a,b) \notin R_{(j,k)}$ *(c.f. Corollary 1.1.25 for the definition of* $R_{(j,k)}$*)*

(ii) $C_{\bar{k}\bar{j}} = A_{\bar{k}\bar{j}} + \lambda$.

An illustration is given in Figure 3.7.

Proof. Since $x[A] \in \mathbb{C}U$ there are unique $\mu_C \in \mathbb{C}$, where C runs through V, such that

$$x[A] = \sum_{C \in V} \mu_C [C].$$

We want to show, that $\mu_C = 0$ if $C \notin \mathcal{I}$.

Step 1: *For every* $C \in V$*, we have*

$$\mu_C = \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(x^{-\top}A, y)} \chi_C(y).$$

Using Lemma 3.3.2, we get

$$\frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(x^{-\top}A, y)} y = x[A] = \sum_{B \in V} \mu_B [B] = \frac{1}{|U|} \sum_{y \in U} \left(\sum_{B \in V} \mu_B \overline{\chi_B(y)} \right) y.$$

Thus we have $\overline{\vartheta\kappa(x^{-\top}A, y)} = \sum_{B \in V} \mu_B \overline{\chi_B(y)}$, from which we get for every $C \in V$, that

$$\mu_C = \sum_{B \in V} \mu_B \langle \chi_C, \chi_B \rangle = \frac{1}{|U|} \sum_{\substack{B \in V \\ y \in U}} \mu_B \chi_C(y) \overline{\chi_B(y)} = \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(x^{-\top}A, y)} \chi_C(y).$$

Step 2: *Due to theorem 1.1.19, we may consider* $\psi(y) := \overline{\vartheta\kappa(x^{-\top}A, y)} \chi_C(y)$ *as a map from* V *to* $\mathbb{C}$*. Let* $J := \nabla \setminus R_{(j,k)}$ *and* $K := R_{(j,k)} \setminus \{(\bar{k}, \bar{j})\}$*.*

Then, for all $y \in U, C \in V$*, we have*

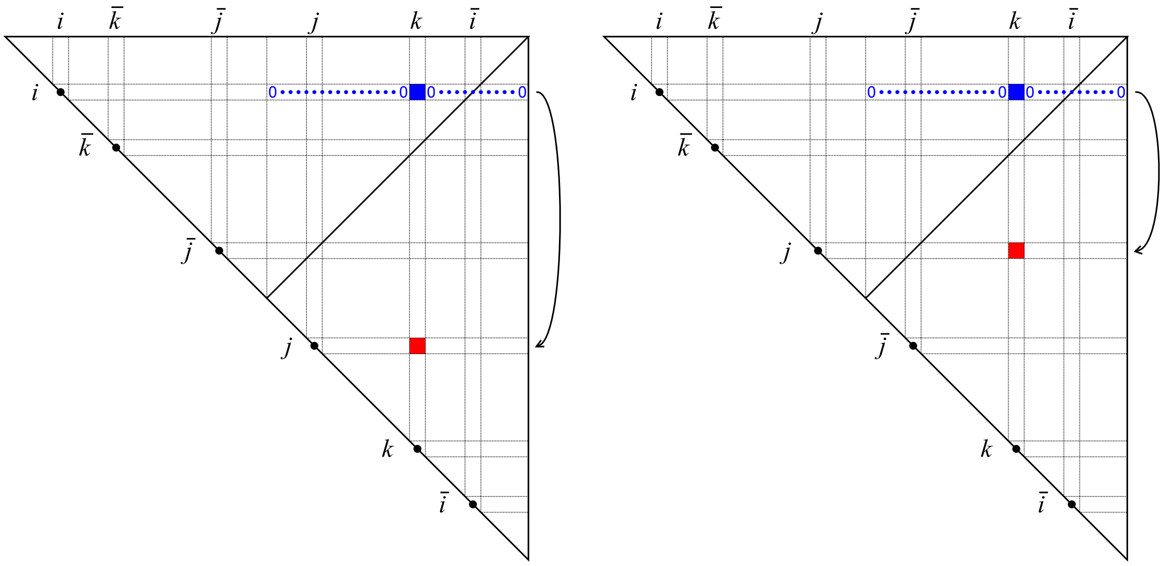
$$\psi(y) = \prod_{(a,b) \in J} \vartheta\big((C - A)_{ab} y_{ab}\big) \cdot \vartheta\big((C_{\bar{k}\bar{j}} - \lambda - A_{\bar{k}\bar{j}}) y_{\bar{k}\bar{j}}\big) \cdot \psi_K(y),$$

where $\psi_K : V \longrightarrow \mathbb{C}$ *is a map, which depends only on* V_K*.*

On the one hand we have

$$\chi_C(y) = \vartheta\kappa\big(C, \underbrace{f(y)}_{\equiv y \pmod{V^\perp}}\big) = \vartheta\kappa\big(C, y\big) \overset{\text{Cor. 2.2.8}}{=} \prod_{(a,b) \in \nabla} \vartheta(C_{ab} y_{ab}).$$

On the other hand we use Lemma 2.2.15, Corollary 2.2.8, and and our assumption, that A is a core pattern (see also Figure 3.8) to get



(a) Variant, where $\bar{j} \leq n$ (b) Variant, where $\bar{j} > n$

Explanation: The figures shows how $x^{-\top}A$ arises from A.

Figure 3.8.: Illustrations to the proof of Lemma* 3.3.16

$$\vartheta_K\big(x^{-\top}A, y\big) = \vartheta_K\big(A, y\big)\vartheta(-\lambda y_{jk}) = \left(\prod_{(a,b)\in\triangledown} \vartheta(A_{ab}y_{ab})\right)\vartheta(-\lambda y_{jk}).$$

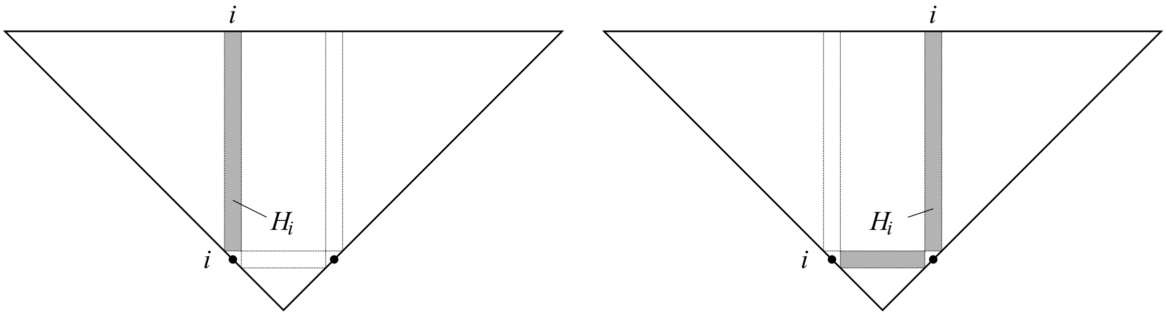
Since $(j,k) \in \triangleleft$ we can use Corollary 1.1.25 to write

$$\vartheta(-\lambda y_{jk}) = \vartheta(\lambda y_{\bar{k}\bar{j}})\vartheta\big(\lambda q_{jk}(y)\big).$$

Because $y \mapsto \vartheta\big(\lambda q_{jk}(y)\big)$ depends only on V_K we can put all of the parts together to receive the statement of Step 2.

Finishing of the proof: Suppose $I, J, K \subseteq \triangledown$, such that $I = J\dot{\cup}K$. Then we have $V_I = V_J \oplus V_K$. If ψ_J and ψ_K are maps from V to $\mathbb{C}$, where ψ_J depends only on V_J and ψ_K depends only on V_K. Then $\psi_I := \psi_J \cdot \psi_K$ is also a map from V to $\mathbb{C}$ and we have

$$\frac{1}{|V_I|}\sum_{X\in V_I}\psi_I(X) = \frac{1}{|V_J||V_K|}\sum_{\substack{X\in V_J\\ Y\in V_K}}\psi_J(X)\psi_K(Y) = \left(\frac{1}{|V_J|}\sum_{X\in V_J}\psi_J(X)\right)\left(\frac{1}{|V_K|}\sum_{Y\in V_K}\psi_I(Y)\right).$$



(a) Variant where $i \leq n$ (b) Variant where $i > n$

Figure 3.9.: Illustration to Definition 3.3.17

Applying this argument (multiple times) and using Steps 1 and 2 we conclude

$$\mu_C = \left(\prod_{(a,b)\in J} \frac{1}{q}\sum_{x\in\mathbf{F}_q} \vartheta\big((C-A)_{ab}x\big)\right)\cdot\left(\frac{1}{q}\sum_{x\in\mathbf{F}_q}\vartheta\big((C-\lambda-A)_{\bar{k}\bar{j}}x\big)\right)\cdot\left(\frac{1}{|V_K|}\sum_{X\in V_K}\psi(X)\right)$$
$$= \left(\prod_{(a,b)\in J}\delta_{C_{ab},A_{ab}}\right)\cdot\delta_{C_{\bar{k},\bar{j}},A_{\bar{k},\bar{j}}+\lambda}\cdot\left(\frac{1}{|V_K|}\sum_{X\in V_K}\psi(X)\right).$$

Thus $\mu_C = 0$ for all $C \notin \mathcal{I}$. qed. |

3.3.17 Definition. *Let $1 \leq i \leq N$. We define the i-**th hook** of ∇ to be*

$$H_i := \big\{(a,b)\in\nabla \;\big|\; b = i \text{ or } a = \bar{i}\big\},$$

*c.f. Figure 3.9 for an illustration. Let A be an arbitrary pattern. Then A is called **hook-separated**, if on every hook H_i of ∇ lies at most one main condition of A. Clearly hook-separated patterns are always staircase patterns. A U-orbit module generated by a hook-separated pattern is called **hook-separated staircase module**.*

All patterns of Appendix A.2 and A.3 are examples of hook-separated staircase patterns.

3.3.18 Proposition*. *Let A be a staircase pattern and S an irreducible constituent of $\mathbb{C}\mathcal{O}_U(A)$. Then there exists a hook-separated core pattern C, such that S is a constituent of $\mathbb{C}\mathcal{O}_U(C)$.*

Proof. Suppose A is a staircase pattern, which is not hook-separated. Let k be maximal, such that H_k contains two main conditions. Since A is a staircase pattern, then $k > n$ and the two main conditions must be of the form (i,k) and $(\bar{k},\bar{j})$. We call the main condition $(\bar{k},\bar{j})$ the name $\mathrm{mhmc}(A)$ (short for 'maximal hook main condition').

Auxiliary result : If S is an irreducible constituent of $\mathbb{C}\mathcal{O}_U(A)$, then there exists a staircase pattern B, such that

(i) S is constituent of $\mathbb{C}\mathcal{O}_U(B)$,

(ii) B is a hook-separated staircase pattern or $\mathrm{mhmc}(B) \prec \mathrm{mhmc}(A)$,

where $\prec$ is the total order on $\bigtriangledown$ given by

$$(a,b) \prec (c,d) \qquad \Longleftrightarrow \qquad a > c \ \vee \ (a = c \wedge b < d),$$

i.e. we have $(a,b) \prec (c,d)$ if and only if (a,b) lies in a row below of (c,d) or they are in the same row, but (c,d) lies on the right of (a,b).

Then the statement of the proposition follows by induction.

Thus we prove the auxiliary result. Since $(j,k) \in \lhd$ we can apply Proposition 3.3.16. We observe that $\mathbb{C}\mathcal{O}_U(A) \subseteq \bigoplus_C \mathbb{C}\mathcal{O}_U(C)$ for some pattern C satisfying

(a) A and C are equal above the $\bar{k}$-th row

(b) C has in the $\bar{k}$-th row no main condition or one, which it is left of $(\bar{k},\bar{j})$

Thus S is an constituent of $\mathbb{C}\mathcal{O}_U(C)$ for one of these C. Now, C is not necessarily a staircase pattern. Due to Corollary 3.3.15 (and the proof of it), there exists a staircase pattern B, such that the properties (a), (b) and (i) hold. It remains to proof (ii). If B is hook-separated nothing is to do, otherwise consider $\mathrm{mhmc} = (\bar{\ell},r)$. Since B is not hook-separated there exists a main condition of B of the form (a,ℓ).

Suppose first $\bar{\ell} < \bar{k}$. Because $(a,\ell) \in \bigtriangledown$, we have $a < \bar{\ell} < \bar{a}$. Thus, we have $a < \bar{\ell} < \bar{k}$. Hence (a,ℓ) and $(\bar{\ell},r)$ lie above the $\bar{k}$-th row. This implies, that they are also main conditions of A, but since $k < \ell$, this would imply that $(\bar{k},\bar{j}) \neq \mathrm{mhmc}(A)$, a contradiction. Thus we may assume $\bar{k} \leq \bar{\ell}$. If $\bar{\ell} > \bar{k}$, we have $\mathrm{mhmc}(B) = (\bar{\ell},r) \prec (\bar{k},\bar{j}) = \mathrm{mhmc}(B)$. Thus it remains to check (ii) if $\bar{\ell} = \bar{k}$. In that case we have, due to (ii), $\mathrm{mhmc}(B) = (\bar{k},r)$ with $r < \bar{j}$, which implies $\mathrm{mhmc}(B) \prec \mathrm{mhmc}(B)$. qed. |

Hence, if we try to find Irreducibles it suffices to consider hook-separated staircase modules:

3.3.19 Corollary*. *Every irreducible $\mathbb{C}U$-module is constituent of some hook-separated staircase module.*

This far we used homomorphisms given by left operation to reduce the amount of U-orbit modules to a much smaller class (hook-separated staircase modules). In particular the results of this and the previous section give us a classification of all of these modules in terms of hook-separated core patterns. We even have a combinatorial formula for calculating the dimensions of these U-orbit modules.

At moment we have no idea in how many hook-separated U-orbit modules a fixed irreducible module S is constituent of. The most preferable situation would be, if every such S is constituent of exactly one hook-separated staircase module, but this unfortunately is wrong. But we will prove, that we can divide the hook-separated staircase modules into series (given by the verge of the core patterns generating the modules), such that each simple module is constituent of exactly one of those series.

3.3.20 Notation. *Let A, B be patterns. Then we set* $\operatorname{Stab}_U(A,B) := \operatorname{Stab}_{\mathrm{U}}(A) \cap \operatorname{Stab}_U(B)$.

We will extensively use the following criterion for the orthogonality of U-orbit modules.

3.3.21 Lemma. *Let A and B be patterns. Then we have*

$$\operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{O}_U(B)\big) = (0)$$

if and only if for all $C \in \mathcal{O}_U(A)$ and $D \in \mathcal{O}_U(B)$ holds

$$\operatorname{Hom}_{\operatorname{Stab}_U(C,D)}\big(\mathbb{C}[C], \mathbb{C}[D]\big) = (0).$$

Proof. We prove this using character theory. Let ψ_A denote the character of $\mathbb{C}\mathcal{O}_U(A)$ and ψ_B the character of $\mathbb{C}\mathcal{O}_U(B)$. Then we have

$$\begin{aligned}
\langle \psi_A, \psi_B \rangle_U &= \frac{1}{|U|} \sum_{u \in U} \overline{\psi_A(u)} \psi_B(u) = \frac{1}{|U|} \sum_{u \in U} \Bigg(\sum_{\substack{C \in \mathcal{O}_U(A) \\ C.u = C}} \overline{\chi_C(u)} \Bigg) \Bigg(\sum_{\substack{D \in \mathcal{O}_U(A) \\ D.u = D}} \chi_D(u) \Bigg) \\
&= \frac{1}{|U|} \sum_{\substack{C \in \mathcal{O}_U(A) \\ D \in \mathcal{O}_U(B)}} \sum_{\substack{u \in U \\ C.u = C \\ D.u = D}} \overline{\chi_C(u)} \chi_D(u) = \frac{1}{|U|} \sum_{\substack{C \in \mathcal{O}_U(A) \\ D \in \mathcal{O}_U(B)}} \sum_{u \in \operatorname{Stab}_U(C,D)} \overline{\chi_C(u)} \chi_D(u) \\
&= \sum_{\substack{C \in \mathcal{O}_U(A) \\ D \in \mathcal{O}_U(B)}} \frac{|\operatorname{Stab}_U(C,D)|}{|U|} \langle \chi_C, \chi_D \rangle_{\operatorname{Stab}_U(C,D)}.
\end{aligned}$$

Now χ_C is the caracter of the $\mathbb{C}\operatorname{Stab}_U(C,D)$-module $\mathbb{C}[C]$, whereas χ_D is the character of the $\mathbb{C}\operatorname{Stab}_U(C,D)$-module $\mathbb{C}[D]$. Thus, since we are writing a non-negative integer as a sum of non-negative integers, we get

$$\langle \psi_A, \psi_B \rangle_U = 0 \qquad \Longleftrightarrow \qquad \forall C \in \mathcal{O}_U(A), D \in \mathcal{O}_U(B) : \ \langle \chi_C, \chi_D \rangle_{\operatorname{Stab}_U(C,D)} = 0.$$

This is obviously equivalent to the statement of the lemma. qed. |

3.3.22 Lemma. *Let A and B be patterns and $y \in U$. Then we have*

$$\operatorname{Hom}_{\mathbb{C}\operatorname{Stab}_U(A,B)}\big(\mathbb{C}[A], \mathbb{C}[B]\big) \cong \operatorname{Hom}_{\mathbb{C}\operatorname{Stab}_U(A.y,B.y)}\big(\mathbb{C}[A.y], \mathbb{C}[B.y]\big)$$

as $\mathbb{C}$-vector spaces.

Proof. We set $S := \mathrm{Stab}_U(A,B)$. Then we have $\mathrm{Stab}_U(A.y,B.y) = y^{-1}Sy$. Note, that χ_A is the character of $\mathbb{C}[A]$, χ_B the one of $\mathbb{C}[B]$, where both are interpreted as $\mathbb{C}S$-modules and $\chi_{A.y}$ the one of $\mathbb{C}[A.y]$, $\chi_{B.y}$ the one of $\mathbb{C}[B.y]$, where both are interpreted as $\mathbb{C}(y^{-1}Sy)$- modules. We have

$$\begin{aligned}\langle\chi_{A.y},\chi_{B.y}\rangle_{y^{-1}Sy} &= \frac{1}{|S|}\sum_{g\in S}\overline{\chi_{A.y}(y^{-1}gy)}\chi_{B.y}(y^{-1}gy)\\ &= \frac{1}{|S|}\sum_{g\in S}\vartheta\kappa\big((B-A).y, f(y^{-1}gy)\big)\\ &\overset{\text{Remark 3.1.15}}{=} \frac{1}{|S|}\sum_{g\in S}\vartheta\kappa\big(B-A, f(y^{-1}gy)\circ y^{-1}\big)\end{aligned}$$

Since $f(y^{-1}gy)\circ y^{-1} = f(y^{-1}g) - f(y^{-1}) = f(y^{-1})\circ g + f(g) - f(y^{-1})$, we get

$$\langle\chi_{A.y},\chi_{B.y}\rangle_{y^{-1}Sy} = \frac{1}{|S|}\sum_{g\in S}\vartheta\kappa\big(B-A, f(y^{-1})\circ g\big)\vartheta\kappa\big(B-A, f(g)-f(y^{-1})\big)$$

Now $\vartheta\kappa\big(B-A, f(y^{-1})\circ g\big) = \vartheta\kappa\big((B-A).g^{-1}, f(y^{-1})\big)$ due to Remark 3.1.15. But

$$(B-A).g^{-1} = B.g^{-1} - A.g^{-1} \overset{g\in S}{=} B-A.$$

Putting this together, we get

$$\langle\chi_{A.y},\chi_{B.y}\rangle_{y^{-1}Sy} = \frac{1}{|S|}\sum_{g\in S}\vartheta\kappa\big(B-A, f(g)\big) = \frac{1}{|S|}\sum_{g\in S}\overline{\chi_A(u)}\chi_B(u) = \langle\chi_A,\chi_B\rangle_S.$$

Since all involved modules are one dimensional, the homomorphism spaces are either equal to (0) or isomorphic to $\mathbb{C}$. Hence the result follows. qed. |

Putting together Lemmas 3.3.21 and 3.3.22 we obtain

3.3.23 Corollary (Homomorphism criterion)**.** *Let A and B be patterns. Then we have*

$$\mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A),\mathbb{C}\mathcal{O}_U(B)\big) = (0)$$

if and only if

$$\mathrm{Hom}_{\mathrm{Stab}_U(A,C)}\big(\mathbb{C}[A],\mathbb{C}[C]\big) = (0) \qquad \textit{for all } C\in\mathcal{O}_U(B).$$

3.3.24 Definition/Lemma. *Let A be a staircase pattern.*

(i) Let $\mathcal{V}(A) := \{B \mid B$ is a pattern with $\mathrm{verge}(B) = \mathrm{verge}(A)\}$.

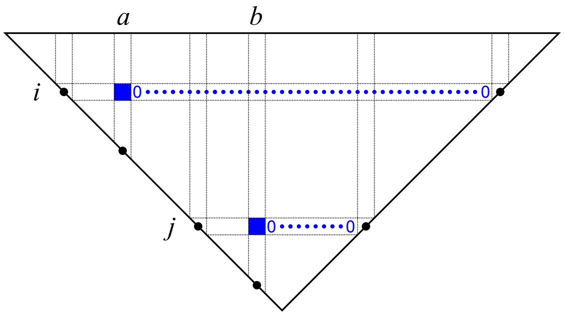


Figure 3.10.: Illustrations of $I(A)$ 3.3.32

Explanation: A position (i,j) is an element of $I(A)$ if and only if there exist main conditions (i,a) and (j,b), such that (i,a) lies in the north-west of (j,b).

(ii) *Let* $\mathbb{C}\mathcal{V}(A) := \mathbb{C}\text{-span}\{[B] \mid B \in \mathcal{V}(A)\}$. *Then* $\mathbb{C}\mathcal{V}(A)$ *is a* $\mathbb{C}U$*-right module*[59] *(due to Definition/Lemma 3.2.8), which we call the* ***verge module*** *of* A.

(iii) *Let A be hook-separated staircase. We define the closed subset*

$$I(A) := \left\{ (i,j) \in \nabla \;\middle|\; \begin{array}{c} \textit{there exist main conditions } (i,a) \\ \textit{and } (j,b) \textit{ of } A\textit{, such that } a < b \end{array} \right\}$$

of ∇. *Clearly we have* $I(A) = I\big(\operatorname{verge}(A)\big)$. *Hence we may define* $U_{\mathcal{V}(A)} := U_{I(A)}$, *which is a pattern subgroup of* U, *called the* ***verge subgroup of*** A. *An Illustration, how* $I(A)$ *can be read off from the pattern is given in Figure 3.10.*

Proof. We have to show that $I(A)$ is closed. Using Corollary 1.2.14 it remains to prove '$(i,j),(j,k) \in I(A) \implies (i,k) \in I(A)$'. If $(i,j),(j,k) \in I(A)$, then there exist $(i,a),(j,b),(k,c) \in \operatorname{main}(A)$, such that $a < b < c$. Hence $(i,k) \in I(A)$. qed.

3.3.25 Lemma. *Let* $C \in V$ *and* $x \in U$, *such that* $\operatorname{supp}(x^{-\top}C) \subseteq \text{◸}$. *Then*

$$\chi_{x.C}(xy) = \chi_C(y)\chi_{x.C}(x) \qquad \textit{for all } y \in U.$$

Proof. We have

$$\chi_{x.C}(xy)\overline{\chi_{x.C}(x)} = \vartheta\kappa\big(x.C, \underbrace{f(xy) - f(x)}_{\equiv xy - x \pmod{V^\perp}}\big) = \vartheta\kappa\big(\pi(x^{-\top}C), xy - x)\big)$$

Since $\operatorname{supp}(xy - x) \subseteq \text{◹}$ we have $\operatorname{supp}(x^{-\top}C) \cap \operatorname{supp}(xy - x) \subseteq \nabla$. Hence we may apply Lemma 3.1.8, yielding

$$\chi_{x.C}(xy)\overline{\chi_{x.C}(x)} = \vartheta\kappa\big(x^{-\top}C, x(y-1)\big) = \vartheta\kappa\big(C, \underbrace{y - 1}_{\equiv f(y) \pmod{V^\perp}}\big) = \chi_C(y).$$

[59] and in fact also an $A_N(\mathbf{F}_q)$-right module

Since $\chi_{x.C}(y)$ is a complex root of unity we have $\overline{\chi_{x.C}(x)} = \chi_{x.C}(x)^{-1}$, which finishes the proof. qed. |

3.3.26 Proposition. *Let A be a hook-separated staircase pattern. Then the verge module $\mathbb{C}\mathcal{V}(A)$ is a $\mathbb{C}U_{\mathcal{V}(A)}$-$\mathbb{C}U$-bimodule (where the left operation is given by restricting the left multiplication in $\mathbb{C}U$ to $\mathbb{C}U_{\mathcal{V}(A)}$) and $\mathcal{V}(A)$ is a monomial basis of $\mathbb{C}\mathcal{V}(A)$ for the operations of $U_{\mathcal{V}(A)}$ and U. Explicitly we have*

$$g[B] = \chi_{g.B}(g)[g.B] \qquad \text{for all } B \in \mathcal{V}(A) \text{ and } g \in U_{\mathcal{V}(A)}.$$

Proof. Let $(i,j) \in I(A)$, $\lambda \in \mathbf{F}_q$ and $B \in \mathcal{V}(A)$. Remember that $x_{ij}(\lambda) = 1+\lambda e_{ij} - \lambda e_{\overline{ji}}$. Combining Remark 3.3.7 and Figure 3.10, using also $\operatorname{verge}(B) = \operatorname{verge}(A)$, we get

$$\operatorname{supp}\left(x_{ij}(\lambda)^{-\top}B\right) \subseteq \not\nabla \qquad \text{and} \qquad \operatorname{verge}\left(x_{ij}(\lambda).B\right) = \operatorname{verge}(A).$$

Thus we may apply Proposition 3.3.11 to get

$$x_{ij}(\lambda)[B] = \chi_{x_{ij}(\lambda).B}(x_{ij}(\lambda))[x_{ij}(\lambda).B] \quad \text{with} \quad x_{ij}(\lambda).B \in \mathcal{V}(A). \tag{3.3.27}$$

Hence we have proven the statement of the proposition for a set of generators for the group $U_{\mathcal{V}(A)}$. It remains to show that the same holds for arbitrary elements.

Suppose $y[B] = \chi_{y.B}(y)[y.B]$ with $y.B \in \mathcal{V}(A)$ for all $B \in \mathcal{V}(A)$. Abbreviating $x := x_{ij}(\lambda)$, we will show for all $B \in \mathcal{V}(A)$ that

$$xy[B] = \chi_{(xy).B}(xy)[(xy).B] \quad \text{with} \quad (xy).B \in \mathcal{V}(A). \tag{3.3.28}$$

Clearly we have

$$xy[B] = \chi_{y.B}(y)x[y.B] \overset{(3.3.27),\ \text{since } y.B\in\mathcal{V}(A)}{=} \chi_{y.B}(y)\chi_{x.(y.B)}(x)[(xy).B)],$$

since $x.(y.B) = (xy).B$. We have shown earlier in this proof that $\operatorname{supp}(x^{-\top}C) \subseteq \not\nabla$ for all $C \in \mathcal{V}(A)$ and we have $y.B \in \mathcal{V}(A)$ by assumption. Hence, using Lemma 3.3.25 with $C = y.B$, we get (3.3.28).

Finally, let $g \in U_{\mathcal{V}(A)}$ be arbitrary. We may express $g = \prod_{(i,j)\in\nabla} x_{ij}(\lambda_{ij})$ due to Theorem 1.2.22. Using (3.3.27) and (3.3.28) we finish the proof via Induction. qed. |

3.3.29 Remark. *To Proposition 3.3.26.*

(i) We want to emphasize that the proposition in particular implies

$$\operatorname{verge}(x.B) = \operatorname{verge}(A) \qquad \text{for all } x \in U_{\mathcal{V}(A)} \text{ and } B \in \mathcal{V}(A).$$

(ii) We will show later, that every homomorphism between hook-separated staircase U-orbits can be realised by left multiplication with some element of $\mathbb{C}[U_{\mathcal{V}(A)}]$.

From Proposition 3.3.26 we get a stronger version of Corollary 3.3.13, proven identically.

3.3.30 Corollary. *Let A be a hook-separated staircase pattern. Then we have*

$$x.(B.u) = (x.B).u \qquad \textit{for all } x \in U_{\mathcal{V}(A)},\ B \in \mathcal{V}(A),\ u \in U.$$

Combining Proposition 3.3.26 with Theorem 3.2.29 we get

3.3.31 Corollary (Decomposition of verge modules)**.** *Let A be a hook-separated staircase pattern. Then we have*

$$\mathbb{C}\mathcal{V}(A) \quad = \bigoplus_{\substack{C \text{ core pattern, such that} \\ \mathrm{verge}(C)=\mathrm{verge}(A)}} \mathbb{C}\mathcal{O}_U(C).$$

Now we can prove a strong theorem from which we can easily derive many beautiful consequences.

3.3.32 Theorem. *Suppose A and B are hook-separated staircase patterns, such that*

$$\mathrm{Hom}_{\mathbb{C}\,\mathrm{Stab}_U(A,B)}\big(\mathbb{C}[A], \mathbb{C}[B]\big) \neq (0).\text{[60]}$$

Then we have $\mathrm{verge}(A) = \mathrm{verge}(B)$ *and there exists some $x \in U_{\mathcal{V}(A)}$, such that*

$$x.A = B \qquad \textit{and thus we also have } x\mathbb{C}\mathcal{O}_U(A) = \mathbb{C}\mathcal{O}_U(B).$$

We give the proof of the theorem in a series of lemmas.

3.3.33 Lemma. *If X and Y are patterns, which are equal above the j-th row and $g \in U$. Then $g.X$ and $g.Y$ are also equal above the j-th row.*

Proof. Due to Corollary 3.3.6 the pattern $g.X$ arises from X by taking X and adding several rows (times some scalar) to lower rows (and then truncating entries outside $\triangledown$). But $X_{ab} = Y_{ab}$ for $(a, b) \in \triangledown$ with $a < j$. Hence $(g.X)_{ab} = (g.Y)_{ab}$ for $(a, b) \in \triangledown$ satisfying $a < j$. qed. |

We introduce the 'weak inductive statement':

(WIS): Let A be a core pattern and D a staircase pattern. Suppose further that A and D are hook-separated. If $1 \leq i \leq n$ is a positive integer, satisfying

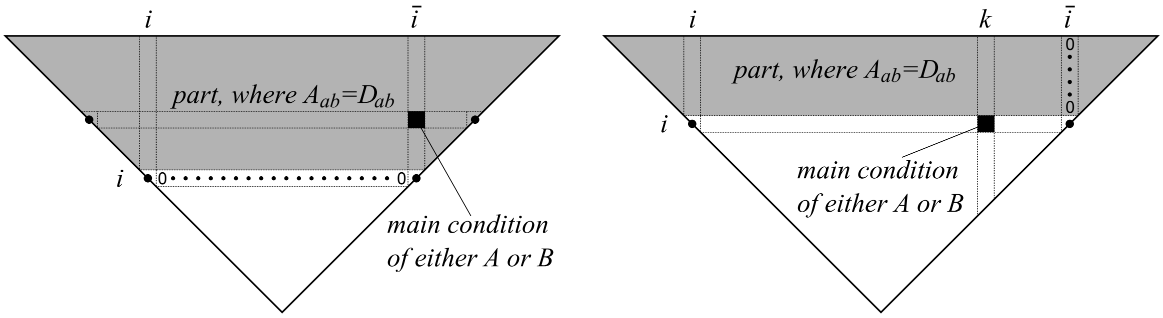
(i) $A_{ab} = D_{ab}$ for all $(a, b) \in \triangledown$ with $a < i$, i.e. A equals D in all rows strictly to the north of row i.

(ii) A has no nonzero entry in the $\bar{\imath}$-th column

(iii) $\mathrm{Hom}_{\mathbb{C}\,\mathrm{Stab}_U(A,D)}\big(\mathbb{C}[A], \mathbb{C}[D]\big) \neq (0)$

Then exists $x \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(D)}$, such that $(x.A)_{ab} = D_{ab}$ for all $(a, b) \in \triangledown$ with $a \leq i$.

[60] Note that $\mathrm{Hom}_{\mathbb{C}\,\mathrm{Stab}_U(A,B)}\big(\mathbb{C}[A], \mathbb{C}[B]\big) \neq (0)$ if and only if $\mathbb{C}[A] \cong \mathbb{C}[B]$ as $\mathbb{C}\,\mathrm{Stab}_U(A, B)$-modules.



(a) to Lemma 3.3.34

(b) to Step 2 of Lemma 3.3.38

Figure 3.11.: Illustrations to the proof of Theorem 3.3.32

3.3.34 Lemma. *(WIS) implies Theorem 3.3.32.*

Proof. Due to Lemma 3.3.21, we may suppose, that A is a core pattern. Our task is to construct an element $x \in U_{\mathcal{V}(A)}$, such that $x.A = B$. Everything else then follows from Proposition 3.3.26. We will prove the following inductive statement:

(IS): *Let $1 \leq i \leq n$ be a positive integer. Suppose there exists $y \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$, such that $(y.A)_{ab} = B_{ab}$ for all $(a,b) \in \nabla$, satisfying $a < i$. Then there exists $x \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$, such that $(x.A)_{ab} = B_{ab}$ for all $(a,b) \in \nabla$, satisfying $a \leq i$.*

Clearly Theorem 3.3.32 follows from (IS) by induction. We set $D := y^{-1}.B$.

We want to apply (WIS) to $D = y^{-1}.B$, hence we have to check the requirements. Lemma 3.3.33 implies that $D = A$ strictly north of the i-th row, hence (WIS)(i) holds. Suppose A has a nonzero entry in the $\bar{i}$-th column, say $(a, \bar{i})$, then according to Figure 3.11(a) we have $a < i$. Hence $A_{a\bar{i}} = D_{a\bar{i}} \neq 0$. Thus $(a, \bar{i})$ is a main condition of A and D, since A is a core pattern and $(a, \bar{i}) \in \nabla$. Because A and D are hook-separated, the i-th row of A and D are then equal to zero. In this case nothing would be to prove. Hence we may assume that (WIS)(ii) holds. By Proposition 3.3.26 we have $D \in \mathcal{V}(B)$, since $y \in U_{\mathcal{V}(B)}$. Note, that $\mathrm{Stab}_U(B) = \mathrm{Stab}_U(D)$ due to Corollary 3.3.30, since $y \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$. Hence $\mathrm{Stab}_U(A, D) = \mathrm{Stab}_U(A, B)$. Since $\lambda_{y^{-1}}$ is a $\mathbb{C}\,\mathrm{Stab}_U(A, D)$-isomorphism from $\mathbb{C}[B]$ to $\mathbb{C}[D]$ also (WIS)(iii) holds.

Thus it exists $h \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$ such that $(h.A)_{ab} = D_{ab}$ for all $(a,b) \in \nabla$, satisfying $a \leq i$. We again apply Lemma 3.3.33, but this time taking $g = y$, $j = i+1$, $X = h.A$, and $Y = D$. Hence we get for all $(a,b) \in \nabla$, satisfying $a \leq i$:

$$((yh).A)_{ab} = (y.(h.A))_{ab} \overset{\text{Lemma 3.3.33}}{=} (y.D)_{ab} = (y.(y^{-1}.B))_{ab} = B_{ab}.$$

And thus $yh \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$ satisfies (IS). qed.

3.3.35 Notation. *Until finishing the proof of Lemma 3.3.41 we use the following notation: Let $(i_1, k_1), (i_2, k_2), \dots, (i_\ell, k_\ell)$ denote the main conditions of A strictly north of the i-th row, i.e. satisfying $i_r < i$, arranged, such that $k_1 < k_2 < \cdots < k_\ell$.*

3.3.36 Definition. *Let $S := \{(i,j) \mid i < j < \bar{i} \text{ and } j \neq k_1, \dots, k_\ell\}$, i.e. S is the set of matrix positions of ∇ lying in the i-th row, but not lying strictly below a main condition of A.*[61] *We define $V_S := \bigoplus_{(i,j)\in S} \mathbf{F}_q e_{ij}$, which has $\mathbf{F}_q$-basis $\mathcal{B}_S := \{e_{is} \mid (i,s) \in S\}$.*

3.3.37 Definition/Lemma. *For every $i < j < \bar{i}$, we* recursively *define elements α_j in $V_S^* = \operatorname{Hom}_{\mathbf{F}_q}(V_S, \mathbf{F}_q)$ via*

$$\alpha_j = \begin{cases} e_{ij}^* & \text{if } j \neq k_1, \dots k_\ell, \\ -\sum_{i<b<k_r} A_{i_r b} A_{i_r k_r}^{-1} \alpha_b & \text{if } j = k_r \text{ for some } r, \end{cases}$$

where as usual e_{ij}^* is given by its values $e_{ij}^*(e_{ib}) = \delta_{jb}$ on the basis e_{ib} of V_S, where (i,b) runs through S.

Note that the elements $A_{i_r k_r}$ are invertible, since the (i_r, k_r)'s are main conditions.

3.3.38 Lemma. *Let A be a hook-separated core pattern, $i \leq n$ a positive integer and suppose A has no nonzero entry in the $\bar{i}$-th column.*

Now, let B be a hook-separated staircase pattern equal to A above the i-th row.[62]*, such that*

$$\operatorname{Hom}_{\mathbb{C}\operatorname{Stab}_U(A,B)}(\mathbb{C}[A], \mathbb{C}[B]) \neq (0)$$

Then we have

$$B_{ij} = A_{ij} + \sum_{\substack{1 \leq r \leq \ell \\ j < k_r}} (A_{ik_r} - B_{ik_r}) \alpha_{k_r}(e_{ij}) \qquad \text{for all } (i,j) \in S,$$

i.e. the entries B_{ij}, with $(i,j) \in S$, are determined by the entries B_{ik_r} below of main conditions (and the ones of A) using a formula completely independent of B.

Proof. We divide the proof in several steps:

Step 1: *Obviously $J_i := \{(a,b) \in \nabla \mid a = i\}$ is a closed subset of ∇ and $U_i := U_{J_i}$ a commutative pattern subgroup of U. If $x = \prod_{i<j<\bar{i}} x_{ij}(\lambda_j) \in U_i$, then for every $i < b < \bar{i}$ the (i,b)-th matrix entry of x is given by λ_b.*

Let $x, y \in U_i$ and $(i,j) \in \nabla$, then

$$(xy)_{ij} = \sum_r x_{ir} y_{rj} \overset{y_{rj} \neq 0 \text{ only if } r = i \text{ or } r = j}{=} x_{ij} + y_{ij}.$$

[61] (i,j) may be very well a main condition of A.

[62] i.e. $A_{ab} = B_{ab}$ for all $(a,b) \in \nabla$, with $a < i$

Thus on every $(a,b) \in \operatorname{supp}(U_i)$ the group product is taken by adding the entries, hence the result follows.

Step 2: *Since A equals B above the i-th row, $(i_1,k_1),\ldots,(i_\ell,k_\ell)$ are also the main conditions of B, satisfying $i_r < i$. Let $v \in V_S$, then*

$$x := \prod_{i<j<\bar{i}} x_{ij}\big(\alpha_j(v)\big) \qquad \textit{is an element of } \operatorname{Stab}_U(A,B).$$

Note, that the $\bar{i}$-th column of A is equal to zero by assumption. From Figure 3.11(b) we can see, that the same holds for the $\bar{i}$-th column of B. Using Corollary 3.1.16 we have, that $A.x$ is obtained from A by adding some multiples of columns of A to the i-th column of A and then truncating all the entries not in ∇. Thus we have to check, if $(Ax^{-\top})_{ai} = A_{ai}$ for all $a < i$. We make the same argument for B. Hence we also have to check $\sum_j B_{aj} x_{ij}^{-1} = (Bx^{-\top})_{ai} = B_{ai}$ for all $a < i$. But since B equals A above the i-th row this second system of equations is in fact equal to $(Ax^{-\top})_{ai} = A_{ai}$ for all $a < i$, which remains to be checked.

If $a \neq i_1,\ldots i_\ell$ the complete a-th row of A is zero and thus also the one of $A.x$. So suppose $a = i_r$ for some $1 \leq r \leq \ell$. We have

$$\begin{aligned}
(A.x)_{i_r i} &= (Ax^{-\top})_{i_r i} = \sum_j A_{i_r j} \underbrace{x_{ij}^{-1}}_{=0 \text{ if } i>j} \overset{(i_r,k_r)\,\in\,\mathrm{main}(A)}{=} \sum_{i\leq j\leq k_r} A_{i_r j} x_{ij}^{-1} \\
&= A_{i_r i} + \sum_{i<j<k_r} A_{i_r j} x_{ij}^{-1} + A_{i_r k_r} x_{i i_r}^{-1} \\
&\overset{\text{Step 1}}{=} A_{i_r i} + \sum_{i<j<k_r} A_{i_r j}\big(-\alpha_j(v)\big) + A_{i_r k_r} \sum_{i<b<k_r} A_{i_r b} A_{i_r k_r}^{-1} \alpha_b(v) = A_{i_r i}.
\end{aligned}$$

Thus we have $x \in \operatorname{Stab}_U(A,B)$.

Finishing the proof. Remember that $\operatorname{Hom}_{\mathbb{C}\operatorname{Stab}_U(A,B)}(\mathbb{C}[A],\mathbb{C}[B]) \neq (0)$ if and only if $\mathbb{C}[A] \cong \mathbb{C}[B]$ as $\mathbb{C}\operatorname{Stab}_U(A,B)$-module. Hence $\mathbb{C}[A]$ and $\mathbb{C}[B]$ have the same character, i.e. $\chi_A(g) = \chi_B(g)$ for all $g \in \operatorname{Stab}_U(A,B)$. In particular we have $\chi_A(x) = \chi_B(x)$, which is equivalent to $\chi_{B-A}(x) = 1$, with x as defined above. Since the $v \in V_S$ defining x is arbitrary we get

$$\vartheta\Big(\sum_{i<b<\bar{i}} (B_{ib} - A_{ib})\alpha_b(v)\Big) = 1 \qquad \text{for all } v \in V_S.$$

In particular the equation holds for $v = \lambda e_{ij}$, where $(i,j) \in S$ and $\lambda \in \mathbf{F}_q$. Since all the α_b's are $\mathbf{F}_q$-linear we get

$$\vartheta\Big(\lambda\Big(\sum_{i<b<\bar{i}} (B_{ib} - A_{ib})\alpha_b(e_{ij})\Big)\Big) = 1 \qquad \text{for all } \lambda \in \mathbf{F}_q, (i,j) \in S.$$

But since ϑ is nontrivial this can only be true if

$$\sum_{i<b<\bar{i}} (B_{ib} - A_{ib})\alpha_b(e_{ij}) = 0 \qquad \text{for all } (i,j) \in S.$$

Now $\alpha_b(e_{ij}) \neq 0$ only if $b = j$ or $b = k_r$, where $j < k_r$. We get

$$0 = (B_{ij} - A_{ij})\underbrace{\alpha_j(e_{ij})}_{=1} + \sum_{\substack{1\leq r\leq \ell \\ j<k_r}} (B_{ik_r} - A_{ik_r})\alpha_{k_r}(e_{ij}).$$

This finishes the proof. qed. |

3.3.39 Remark. *The lemma is in the same methodical spirit as the statement 'a linear mappings is determined by its values on a basis': We try to reduce the amount of information needed to understand the image of a (nontrivial) $\mathbb{C}\,\mathrm{Stab}_U(A,B)$-homomorphism between the $\mathbb{C}\,\mathrm{Stab}_U(A,B)$-rightmodules $\mathbb{C}[A]$ and $\mathbb{C}[B]$. In simplified terms we could say that such mappings are determined by their values at positions below of main conditions.*

3.3.40 Lemma. *Suppose A, D and i satisfy the assumptions of (WIS). Then either $A_{ab} = D_{ab}$ for $(a,b) \in \triangledown$ satisfying $a \leq i$, or A and D have identical main condition in the i-th row, say (i,k), and it holds $A_{ik} = D_{ik}$.*

Proof. If the i-th row of A and D is zero, then we have $A_{ab} = D_{ab}$ for $(a,b) \in \triangledown$, such that $a \leq i$. Hence we may assume that not both i-th rows are zero. Let (i,a) (resp. (i,b)) denote if existent the main condition of A (resp. B) lying in the i-th row. If only one of these exist we call it (i,k), if both exist we call the right one (i,k). Applying Lemma 3.3.38 with $B = D$, we get

$$D_{ik} - A_{ik} \;=\; \sum_{\substack{1\leq r\leq \ell \\ k<k_r}} \underbrace{(A_{ik_r} - D_{ik_r})}_{\substack{=0,\text{ since } (i,k_r) \text{ lies right of the main} \\ \text{conditions } (i,a) \text{ and } (i,b)}} \alpha_{k_r}(e_{ij}) \;=\; 0.$$

Hence the Lemma is proven. qed. |

Now we can finish the proof of Theorem 3.3.32 by proving (WIS).

3.3.41 Lemma. *The statement (WIS) holds (and hence also Theorem 3.3.32).*

Proof. We assume A, D and i satisfy the assumptions of (WIS).

Step 1: *Let $i \leq n$. Then $\tilde{J}_i := \{(a,b) \in \triangledown \mid b = i\}$ is a closed subset of $\triangledown$ and $\tilde{U}_i := U_{\tilde{J}_i}$ a commutative pattern subgroup of U. If $x = \prod_{a<i} x_{ai}(\lambda_a) \in \tilde{U}_i$, then for every $b < i$ the (b,i)-th matrix entry of x is given by λ_b.*

This is proven completely analogous to Step 1 of the proof of Lemma 3.3.38.

Step 2: *Let (i,k) be as in Lemma 3.3.40 and*

$$g := \prod_{\substack{1\le r\le \ell\\ k_r<k}} x_{i_r i}(-\mu_r) \qquad \textit{where } \mu_r := A_{i_r k_r}^{-1}\Bigg(D_{ik_r} - A_{ik_r} - \sum_{\substack{1\le s\le \ell\\ k_r<k_s<k}} \mu_s A_{i_s k_r}\Bigg).$$

Then we have $g \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$ and $(g.A)_{ik_r} = D_{ik_r}$ for all $1 \le r \le \ell$.

Using Lemma 3.3.40 we have $g \in U_{\mathcal{V}(A)} \cap U_{\mathcal{V}(B)}$. Due to Corollary 3.3.6 the elements A and $g.A$ differ only on the i-th row. Note, that $g \in \tilde{U}_i$. Due to Step 1, we have

$$g_{ai}^{-1} = \begin{cases} 1 & \text{if } a = i, \\ \mu_s & \text{if } a = i_s \text{ for some } s\text{, such that } k_s < k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $j = k_r$ for some r, such that $k_r < k$, then

$$\begin{aligned}(g.A)_{ik_r} &\overset{(i,k_r)\in\nabla}{=} (g^{-\top}A)_{ik_r} \;=\; \sum_a g_{ai}^{-1} A_{ak_r} \\ &= \;\; A_{ik_r} + \sum_{\substack{1\le s\le \ell\\ k_s<k}} \mu_s \underbrace{A_{i_s k_r}}_{\neq 0 \text{ only if } k_r \le k_s} \;=\; A_{ik_r} + \mu_r A_{i_r k_r} + \sum_{\substack{1\le s\le \ell\\ k_r<k_s<k}} \mu_s A_{i_s k_r} \;=\; D_{ik_r}.\end{aligned}$$

If $k_r > k$ we have $(g.A)_{ik_r} = A_{ik_r} = 0 = D_{ik_r}$, since (i,k) is a main condition of A and B and (i,k_r) lies on the right of (i,k). Hence we get

$$(g.A)_{ik_r} = D_{ik_r} \qquad \text{for all } 1 \le r \le \ell.$$

Finishing the proof: Clearly both $B = D$ and $B = g.A$ meet the requirements of Lemma 3.3.38. Hence we have

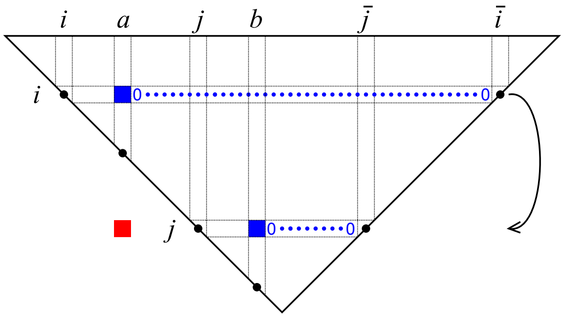
$$(g.A)_{ij} \overset{\text{Lemma 3.3.38}}{=} A_{ij} + \sum_{\substack{1\le r\le \ell\\ j<k_r}} \big(A_{ik_r} - \underbrace{(g.A)_{ik_r}}_{=D_{ik_r}}\big)\alpha_{k_r}(e_{ij}) \overset{\text{Lemma 3.3.38}}{=} D_{ij}.$$

The result follows. qed. |

We get a bunch of corollaries out of Theorem 3.3.32. For the first two, we just apply Corollary 3.3.23 and Theorem 3.3.32:

3.3.42 Corollary. *Suppose A and B are hook-separated staircase patterns, such that* $\operatorname{verge}(A) \neq \operatorname{verge}(B)$. *Then*

$$\operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{O}_U(B)\big) = (0).$$



Explanation: The figure shows $x_{ij}(\lambda)^{-\top}A$ arises from A. By applying π we get $x_{ij}(\lambda).A = A$.

Figure 3.12.: Illustrations to Definition/Lemma 3.3.44

3.3.43 Corollary. *Let A and B be hook-separated staircase patterns. Then we either have*

$$\mathbb{C}\mathcal{O}_U(A) \cong \mathbb{C}\mathcal{O}_U(B) \qquad \textit{or} \qquad \operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{O}_U(B)\big) = (0).$$

So either $\mathbb{C}\mathcal{O}_U(A)$ and $\mathbb{C}\mathcal{O}_U(B)$ have precisely the same irreducible constituents or they have no irreducible constituents in common, i.e. their characters are orthogonal.

Let A be a staircase pattern. We want to construct a basis of $\operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big)$ (see Proposition 3.3.49). In order to do this we investigate the group action of $U_{\mathcal{V}(A)}$ on $\mathcal{V}(A)$ (i.e. for example we will determine Stabilizers). At first view, the description looks very technical. But in fact we can use it to easily read off the dimension of $\operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big)$ from the picture of A. We will illustrate this in the many examples of Appendix A.2.

3.3.44 Definition/Lemma. *Let A be a hook-separated staircase pattern. We define*

$$T(A) := \big\{(i,j) \in \triangledown \;\big|\; \exists a,b:\; (i,a),(j,b) \in \operatorname{main}(A), a < b, a \leq j\big\} \subseteq I(A).$$

Then $T(A)$ is a closed subset of $\triangledown$, depending only on $\operatorname{verge}(A)$, and we have

$$U^{\mathrm{Stab}}_{\mathcal{V}(A)} := U_{T(A)} = \operatorname{Stab}^{\mathrm{row}}_{U_{\mathcal{V}(A)}}(A),$$

where $\operatorname{Stab}^{\mathrm{row}}_{U_{\mathcal{V}(A)}}(A) := \{g \in U_{\mathcal{V}(A)} \mid g.A = A\}$. We even have

$$U^{\mathrm{Stab}}_{\mathcal{V}(A)} = \{g \in U_{\mathcal{V}(A)} \mid g.B = B \textit{ for all } B \in \mathcal{V}(A)\}.$$

Since $T(A)$ depends only on $\operatorname{verge}(A)$.

Proof. It is easy to check, using Corollary 1.2.14, that $T(A)$ is closed. It remains to show that $U_{T(A)} = \mathrm{Stab}^{\mathrm{row}}_{U_{\mathcal{V}(A)}}(A)$. Let $(i,j) \in T(A)$ and $\lambda \in \mathbf{F}_q$ be arbitrary, then $x_{ij}(\lambda).A = A$ holds. (See Figure 3.12 for an illustration.) Thus a system of generators of $U_{T(A)}$ is included in $\mathrm{Stab}^{\mathrm{row}}_{U_{\mathcal{V}(A)}}(A)$ and hence also the whole group $U_{T(A)}$.

Conversely, let $1 \neq x \in \mathrm{Stab}^{\text{left op.}}_{U_{\mathcal{V}(A)}}(A)$ be arbitrary. We will show, that $x^{-1} \in U_{T(A)}$. Using Theorem 1.2.22, we have to show that $\mathrm{supp}(x^{-1}) \cap \nabla \subseteq T(A)$. We suppose the contrary. Let $(i,j) \in \mathrm{supp}(x^{-1}) \cap \nabla \smallsetminus T(A)$. Since $x \in U_{\mathcal{V}(A)}$, we have main conditions (i,k_i) and (j,k_j) of A, such that $j < k_i < k_j$. Otherwise (i,j) would be in $T(A)$. We may assume that (i,j) is chosen, such that k_i is maximal $\big($i.e. if $(r,s) \in \mathrm{supp}(x^{-1}) \cap \nabla \smallsetminus T(A)$ with corresponding main conditions (r,k_r) and (s,k_s), such that $s < k_r < k_s$ we have $k_r \leq k_i\big)$. We have $x.A = A$, hence we get

$$\begin{aligned} A_{jk_i} &= (x.A)_{jk_i} = \sum_r (x^{-\top})_{jr} A_{rk_j} = \sum_r \underbrace{(x^{-1})_{rj}}_{\neq 0 \text{ only if } r \leq j} \underbrace{A_{rk_i}}_{\neq 0 \text{ only if } k_i < \bar{r}} \\ &\overset{j<k_i<\bar{r}}{=} A_{jk_i} + \sum_{\substack{r, \text{ such that} \\ r<j<\bar{r}}} (x^{-1})_{rj} A_{rk_i} = A_{jk_i} + \sum_{\substack{r, \text{ such that} \\ (r,j)\in\nabla}} (x^{-1})_{rj} A_{rk_i}. \end{aligned}$$

We will show that $(i,j) \neq (r,j) \in \nabla$ implies $(x^{-1})_{rj} A_{rk_i} = 0$. Using this, we can simplify the above formula to $0 = (x^{-1})_{ij} A_{ik_i}$. But (i,k_i) is a main condition of A, hence $A_{ik_i} \neq 0$. But then we have $(i,j) \notin \mathrm{supp}(x^{-1})$, which is a contradiction.

We finish the proof by showing $(i,j) \neq (r,j) \in \nabla \implies (x^{-1})_{rj} A_{rk_i} = 0$. There are two cases, $(r,j) \in T(A)$ and $(r,j) \notin T(A)$. In the first case, $(r,j) \in T(A)$, we have main conditions (r,k_r) and (j,k_j) of A, such that $k_r < k_j$ and $k_r \leq j$. But $j < k_i$, thus (r,k_i) lies on the right of the main condition (r,k_r) of A. Hence $A_{rk_i} = 0$ and the first case is proven. In the second case, $(r,j) \in T(A)$, we have $j < k_r < k_j$. Due to maximality of k_i, we get $k_r \leq k_i$. The case $k_i = k_r$ is forbidden since $i \neq r$ and A is a staircase pattern, hence we get $k_r < k_i$. Now (r,k_i) is on the right of the main condition (r,k_r) of A, thus $A_{rk_i} = 0$. qed. |

3.3.45 Corollary. *Let A be a staircase pattern. Then we have*

$$U^{\mathrm{Stab}}_{\mathcal{V}(A)} \trianglelefteq U_{\mathcal{V}(A)}.$$

Proof. We consider an arbitrary group operation of some abstract group G on some set S, where the operation of $g \in G$ on $s \in S$ is written as '$s.g$'.

Let $K := \{g \in G \mid s.g = s \text{for all } s \in S\}$. Then it is a basic (and easily provable) fact from the theory of group actions, that $K \trianglelefteq G$: Let $k \in K$ and $g \in G$. Then we have

$$s.(gkg^{-1}) = (\underbrace{(s.g)}_{\in S}.k).g^{-1} = (s.g).g^{-1} = s \qquad \text{for all } s \in S.$$

Hence $K \trianglelefteq G$. In Definition/Lemma 3.3.44 we have shown, that $U^{\mathrm{Stab}}_{\mathcal{V}(A)} = \mathrm{Stab}_{U_{\mathcal{V}(A)}}(B)$ for all $B \in \mathcal{V}(A)$. Hence the result follows. qed. |

3.3.46 Definition/Lemma. *We define*

$$E(A) := I(A) \setminus T(A) = \big\{(i,j) \in \triangledown \;\big|\; \exists a, b: \; (i,a), (j,b) \in \operatorname{main}(A), j < a < b\big\}.$$

Then

$$R(A) := \Big\{ \prod_{(i,j)\in E(A)} x_{ij}(\lambda_{ij}) \;\Big|\; \lambda_{ij} \in \mathbf{F}_q \Big\}$$

is a system of left coset representatives for $U^{\mathrm{Stab}}_{\mathcal{V}(A)}$ in $U_{\mathcal{V}(A)}$, i.e. $U_{\mathcal{V}(A)} = \bigsqcup_{x\in R(A)} xU^{\mathrm{Stab}}_{\mathcal{V}(A)}$.

Proof. This is a direct consequence of the fact that $I(A)$ and $T(A)$ are closed, where $T(A)$ is included in $I(A)$, and Theorem 1.2.22. qed. |

3.3.47 Remark. *The sets $T(A)$, $E(A)$, and $R(A)$ depend only on the main conditions of A, thus they only depend on $\operatorname{verge}(A)$ and not on A itself.*

3.3.48 Example. *We consider the hook-separated staircase pattern*

$$A = \left(\begin{array}{cccccc|cccccc}
\cdot & ? & \color{red}{A_{1,3}} & & & & & & & & & \cdot \\
 & \cdot & ? & ? & \color{red}{A_{2,5}} & & & & & & \cdot & \\
 & & \cdot & ? & ? & ? & ? & ? & \color{red}{A_{3,9}} & \cdot & & \\
 & & & \cdot & & & & & \cdot & & & \\
 & & & & \cdot & \color{red}{A_{5,6}} & & \cdot & & & & \\
 & & & & & \cdot & \cdot & & & & & \\
 & & & & & & & & & & &
\end{array}\right).$$

where '$\cdot$' and empty positions denote 0, $\operatorname{main}(A) = \{(1,3),(2,5),(3,9),(5,6)\}$, and '?' denotes arbitrary entries. Note further that $|$ indicates the symmetry axis of $\triangledown$.

Then we have $I(A) = \{(1,2),(1,3),(1,5),(2,3),(2,5)\}$, $T(A) = \{(1,3),(1,5),(2,5)\}$, and $E(A) = \{(1,2),(2,3)\}$.

3.3.49 Proposition. *Let A be a hook-separated staircase pattern. Then we have*

$$\mathcal{B}^{\mathrm{Hom}}_{\mathcal{V}(A)} := \big\{\lambda_x|_{\mathbb{C}\mathcal{O}_U(A)} \;\big|\; x \in R(A)\big\} \text{ is a } \mathbb{C}\text{-basis for } \operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big),$$

where again $\lambda_x : \mathbb{C}U \longrightarrow \mathbb{C}U$ means left multiplication by x for $x \in U$.

Proof. Let $d := \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big)$ and for a pattern B, let ψ_B be the character of $\mathbb{C}\mathcal{O}_U(B)$. We have

$$\begin{aligned} d &= \sum_{\substack{B \text{ core pattern} \\ \mathrm{verge}(B)=\mathrm{verge}(A)}} \langle \psi_A, \psi_B \rangle_U \\ &\overset{\substack{\text{main formula in} \\ \text{proof of 3.3.21}}}{=} \sum_{\substack{B \text{ core pattern} \\ \mathrm{verge}(B)=\mathrm{verge}(A)}} \sum_{\substack{C \in \mathcal{O}_U(A) \\ D \in \mathcal{O}_U(B)}} \frac{|\operatorname{Stab}_U(C,D)|}{|U|} \langle \chi_C, \chi_D \rangle_{\operatorname{Stab}_U(C,D)} \\ &= \sum_{\substack{C \in \mathcal{O}_U(A) \\ D \in \mathcal{V}(A)}} \frac{|\operatorname{Stab}_U(C,D)|}{|U|} \langle \chi_C, \chi_D \rangle_{\operatorname{Stab}_U(C,D)} \\ &\overset{\substack{\text{last formula in} \\ \text{proof of 3.3.22}}}{=} \sum_{D \in \mathcal{V}(A)} \frac{|\mathcal{O}_U(A)||\operatorname{Stab}_U(A,D)|}{|U|} \underbrace{\langle \chi_A, \chi_D \rangle_{\operatorname{Stab}_U(A,D)}}_{=1 \text{ or } 0}. \end{aligned}$$

Due to Corollary 3.3.23 and Theorem 3.3.32 we have $\langle \chi_A, \chi_D \rangle_{\operatorname{Stab}_U(A,D)} = 1$ if and only if there exists an element $x \in U_{\mathcal{V}(A)}$, such that $D = x.A$. But then, $\operatorname{Stab}_U(A,D) = \operatorname{Stab}_U(A)$. And, since $|U| = |\mathcal{O}_U(A)||\operatorname{Stab}_U(A)|$, we get

$$d = |\mathcal{O}^{\text{row}}_{U_{\mathcal{V}(A)}}(A)| = \frac{|U_{\mathcal{V}(A)}|}{|\operatorname{Stab}^{\text{row}}_{U_{\mathcal{V}(A)}}(A)|} \overset{\text{Defi./Lemma 3.3.46}}{=} \frac{|U_{\mathcal{V}(A)}|}{|U_{T(A)}|} = q^{|I(A)|-|T(A)|} = q^{|E(A)|}.$$

Thus it remains to show that $\mathcal{B}$ is linearly independent. Thus let μ_x for $x \in R(A)$ be complex numbers, such that $\sum_{x \in R(A)} \mu_x \lambda_x|_{\mathbb{C}\mathcal{O}_U(A)} = 0$. Evaluated at $[A]$ we get $\sum_{x \in R(A)} \mu_x \chi_{x.A}(x)[x.A] = 0$, due to Proposition 3.3.26. We show that the $x.A$'s are pairwise different, then the result follows, since the patterns form a $\mathbb{C}$-basis of $\mathbb{C}U$. Thus suppose, we have $x, y \in R(A)$, such that $x.A = y.A$. Then we have $y^{-1}x \in \operatorname{Stab}^{\text{left op.}}_{U_{\mathcal{V}(A)}}(A) = U_{T(A)}$ and thus $x = y$. qed. |

Of course we have:

3.3.50 Corollary. *Let A be a hook-separated staircase pattern. Then we have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}U} \big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big) = q^{|E(A)|}.$$

3.3.51 Corollary. *Let A be a hook-separated staircase pattern, such that $E(A) = \emptyset$. Then all orbit modules $\mathbb{C}\mathcal{O}_U(B)$ with $\mathrm{verge}(B) = \mathrm{verge}(A)$ are irreducible and pairwise non-isomorphic.*

Proof. If A and B satisfy $\mathrm{verge}(A) = \mathrm{verge}(B)$, we have

$$\dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}U} \big(\mathbb{C}\mathcal{O}_U(B)\big) \leq \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}U} \big(\mathbb{C}\mathcal{O}_U(B), \mathbb{C}\mathcal{V}(B)\big) = q^{|E(B)|} \overset{\text{Remark 3.3.47}}{=} q^{|E(A)|} = 1.$$

This implies the statement about irreducibility and that every orbit module occurs exactly once as irreducible constituent of the verge module. qed. |

3.3.52 Remark. *The situation of Corollary 3.3.51 occurs in particular if* $\mathrm{main}(A) = \{(i_1, k_1), (i_2, k_2), \dots, (i_\ell k_\ell)\}$*, such that we have* $i_1 < i_2 < \cdots < i_\ell$ *and* $k_1 > k_2 > \cdots > k_\ell$*. In everyday language this means that the main conditions are heading from bottom left to upper right in the picture of* A*. For an illustration c.f. Example A.2.1.*

We want to close this section by determining a basis for the endomorphism ring of a hook-separated staircase module. The basis is obtained by simply taking the elements from the basis $\mathcal{B}^{\mathrm{Hom}}_{\mathcal{V}(A)}$ of $\mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big)$ which have image in $\mathbb{C}\mathcal{O}_U(A)$:

3.3.53 Proposition. *Let* A *be a hook-separated staircase pattern. Then*

$\mathcal{B}^{\mathrm{End}}_{\mathcal{O}_U(A)} := \big\{\lambda_x|_{\mathbb{C}\mathcal{O}_U(A)} \bigm| x \in R(A) \text{ and } x.A \in \mathcal{O}_U(A)\big\}$ *is a* $\mathbb{C}$*-basis for* $\mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big)$.

Proof. Clearly the elements of $\mathcal{B}^{\mathrm{End}}_{\mathcal{O}_U(A)} \subseteq \mathcal{B}^{\mathrm{Hom}}_{\mathcal{V}(A)}$ are linearly independent. It remains to check that they generate the endomorphism ring. Hence let $\varphi \in \mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big)$ be arbitrary. Since $\mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big) \subseteq \mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big)$ we may write

$$\varphi = \sum_{x \in R(A)} \alpha_x \lambda_x|_{\mathbb{C}\mathcal{O}_U(A)} \qquad \text{for some } \alpha_x \in \mathbb{C}.$$

It remains to show, that $\alpha_x = 0$ for all $x \in R(A)$, satifying $x.A \notin \mathcal{O}_U(A)$. We have

$$\begin{aligned}
\varphi([A]) &= \sum_{x \in R(A)} \alpha_x x[A] \overset{\text{Prop. 3.3.26}}{=} \sum_{x \in R(A)} \alpha_x \chi_{x.A}(x)[x.A] \\
&= \sum_{\substack{x \in R(A) \\ x.A \in \mathcal{O}_U(A)}} \alpha_x \chi_{x.A}(x)[x.A] \quad + \sum_{\substack{x \in R(A) \\ x.A \notin \mathcal{O}_U(A)}} \alpha_x \chi_{x.A}(x)[x.A].
\end{aligned}$$

Since $\varphi([A]) \in \mathbb{C}\mathcal{O}_U(A)$, we have

$$\sum_{\substack{x \in R(A) \\ x.A = B}} \alpha_x \underbrace{\chi_{x.A}(x)}_{=\chi_B(x)} = 0 \qquad \text{for every } B \in \mathcal{V}(A) \smallsetminus \mathcal{O}_U(A).$$

We show $|\{x \in R(A) \mid x.A = B\}| \leq 1$ for every $B \in \mathcal{V}(A)$. Then we get $\alpha_x = 0$ for all $x \in R(A)$, satisfying $x.A \notin \mathcal{O}_U(A)$, and the result follows.

Hence suppose $x, y \in R(A)$ satisfy $x.A = B = y.A$. Then we have $(x^{-1}y).A = A$, i.e. $x^{-1}y \in \mathrm{Stab}^{\mathrm{row}}_{U_{\mathcal{V}(A)}}(A) = U_{T(A)}$, due to 3.3.46, which also implies $x = y$. qed. |

In general, we cannot determine the dimension of these endomorphism rings combinatorially, but only by doing calculations. Examples of endomorphism rings of orbit modules can be found in the Appendix A.2.

3.3.54 Corollary. *Let A be a hook-separated staircase pattern.*
We set $S_{\mathcal{O}_U(A)} := \{x \in U_{\mathcal{V}(A)} \mid x.A \in \mathcal{O}_U(A)\}$. Then $U^{\mathrm{Stab}}_{\mathcal{V}(A)} \trianglelefteq S_{\mathcal{O}_U(A)}$ and we have

$$\mathbb{C}[S_{\mathcal{O}_U(A)}/U^{\mathrm{Stab}}_{\mathcal{V}(A)}] \cong \mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big),$$

where $\mathbb{C}[S_{\mathcal{O}_U(A)}/U^{\mathrm{Stab}}_{\mathcal{V}(A)}]$ denotes the group algebra of the quotient group $S_{\mathcal{O}_U(A)}/U^{\mathrm{Stab}}_{\mathcal{V}(A)}$. The isomorphism is given by

$$xU^{\mathrm{Stab}}_{\mathcal{V}(A)} \longmapsto \lambda_x|_{\mathbb{C}\mathcal{O}_U(A)}.$$

Proof. Clearly for $x \in U_{\mathcal{V}(A)}$ we have

$$\lambda_x|_{\mathbb{C}\mathcal{O}_U(A)} \in \mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big) \quad\iff\quad x.\mathcal{O}_U(A) = \mathcal{O}_U(A) \quad\iff\quad x \in S_{\mathcal{O}_U(A)}.$$

Now, Proposition 3.3.53 implies the result. qed.

3.3.55 Remark. *Unfortunately the group $S_{\mathcal{O}_U(A)}$ is in general **not** a pattern subgroup. This is a pity, since otherwise $\mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big)$ would have algebra generators of the form $\lambda_{x_{ij}(\lambda)}|_{\mathbb{C}\mathcal{O}_U(A)}$, i.e. it would suffice to check for which $x_{ij}(\lambda)$ we have $x_{ij}(\lambda).A \in \mathcal{O}_U(A)$. For an example illustrating this situation, c.f. A.2.11.*

It seems very likely that the following conjecture holds. The conjecture matches nicely with the general philosophy of 'q is just a parameter'.

3.3.56 Conjecture. *Let A be a hook-separated staircase pattern. Then we have*

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big) = q^a \qquad \text{for some integer } a.$$

3.4. Relating the linear and the orthogonal case

In this section we want to relate N. Yan's theory for the unipotent linear groups to our theory for the unipotent orthogonal groups. On the one hand we can embed our theory for $D_n(\mathbf{F}_q)$ in the theory of $A_N(\mathbf{F}_q)$.[63] This embedding is crucial in Section 3.6 for the understanding of André-Neto supercharacters, as a special class of characters orbit modules of $A_N(\mathbf{F}_q)$ restricted to $D_n(\mathbf{F}_q)$.

On the other hand we will show, that we also can embed the theory for $A_n(\mathbf{F}_q)$ into our theory for $D_n(\mathbf{F}_q)$. It will turn out, that the character theoretical statements for $A_n(\mathbf{F}_q)$ presented at the end of Section 2.2 can be derived as corollaries of our theory.[64] Hence this shows that our approach is a true generalization of the approach for $A_n(\mathbf{F}_q)$.

[63] we already have established a monomial action of $A_N(\mathbf{F}_q)$ on the complex group algebra of $D_n(\mathbf{F}_q)$ in Section 3.1, so this should not be a surprise.
[64] for the results speaking about superclasses we will postpone this to Section 3.6.

3.4.1 Notation. *Throughout this section we have to work with the monomial linearisation for different groups at once. Hence we have to introduce a bit of notation to distinguish between the different cases. Let*

$$H := A_n(\mathbf{F}_q), \qquad U := D_n(\mathbf{F}_q) \qquad \text{and} \qquad G := A_N(\mathbf{F}_q).$$

Using the isomorphism $\varphi : H \longrightarrow U_{\dot\triangledown}$ *of Lemma 1.2.25 we can understand* H *as a subgroup of* U*, thus we have* $H \leq U \leq G$*. To distinguish between the different monomial representations we index '*V*', '*π*', '*f*', '*$[A]$*', '*χ_A*', '*$.$*', '*$\circ$*' with indices* H, U, G *and the traceform '*κ*' according to the matrix size with* n *or* N*. Hence for example*

$$A._G g = \pi_G(Ag^{-\top}), \quad f_U(xg) = f_U(x) \circ_U g + f_U(g), \quad \text{or} \quad \chi_A^H(g) = \vartheta\kappa_n\big(A, f_H(g)\big)$$

can all be clearly put into the right context. Recall that '$\circ_G$*' and '*$\circ_H$*' denote just matrix multiplication.*

We want to relate H with U.

3.4.2 Definition. *We define*

$$\rho : \ \mathrm{Mat}_{N\times N}(\mathbf{F}_q) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbf{F}_q) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto A.$$

3.4.3 Proposition. *The map* $\rho|_U : U \longrightarrow H$ *is a group epimorphism with kernel* $U_{\dot\triangledown}$*, the map* $\rho|_{V_U} : \ V_\triangledown \longrightarrow V_H$ *is an* $\mathbf{F}_q$*-vector space epimorphism, and we have*

$$\rho(A._U u) = \rho(A)._H \rho(u) \qquad \text{for all } A \in V_{\dot\triangledown}, u \in U.$$

In particular $\rho|_{V_U} : \ V_{\dot\triangledown} \longrightarrow V_H$ *is bijective.*

Proof. Let A, B, C, D be $n \times n$-matrices over $\mathbf{F}_q$. Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^R = \begin{pmatrix} D^R & B^R \\ C^R & A^R \end{pmatrix}.$$

Using this identity we get

$$U = \left\{ \begin{pmatrix} g & v \\ & g^{-R} \end{pmatrix} \;\middle|\; \begin{array}{c} g \in A_n(\mathbf{F}_q) \text{ arbitrary}, \\ v \in \mathrm{Mat}_{n\times n}(\mathbf{F}_q), \text{ s.t. } gv^R + vg^R = 0 \end{array} \right\}$$

It is then clear that $\rho|_U$ defines a group epimorphism. An element $\begin{pmatrix} g & v \\ & g^{-R} \end{pmatrix}$ is in the kernel of the map $\rho|_U$ if and only if $g = 1$, i.e. if and only if $\mathrm{supp}\begin{pmatrix} g & v \\ & g^{-R} \end{pmatrix} \cap \triangledown \subseteq \dot\triangledown$. Hence Theorem 1.2.22 implies $\ker \rho|_U = U_{\dot\triangledown}$.

Clearly the map $\rho|_{V_U} : V_{\triangledown} \longrightarrow V_H$ is an $\mathbf{F}_q$-vector space epimorphism. To finish the proof let $u \in U$ and $A \in V_{\ddot{\nabla}}$ be arbitrary. Note that

$$A = \begin{pmatrix} \rho(A) & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad u = \begin{pmatrix} \rho(u) & v \\ & \rho(u)^{-R} \end{pmatrix} \qquad \text{for some } v \in \mathrm{Mat}_{n\times n}(\mathbf{F}_q).$$

Hence we have

$$A\,\underbrace{u^{-\top}}_{=(u^R)^\top} = A \begin{pmatrix} \rho(u)^{-1} & v^R \\ & \rho(u)^R \end{pmatrix}^{\!\top} = \begin{pmatrix} \rho(A) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho(u)^{-\top} & \\ (v^R)^\top & (\rho(u)^R)^\top \end{pmatrix} = \begin{pmatrix} \rho(A)\rho(u)^{-\top} & 0 \\ 0 & 0 \end{pmatrix}.$$

The application of $\rho\pi_U$ on both sides yields

$$\rho(A._U u) = \rho\pi_U(Au^{-\top}) = \rho \begin{pmatrix} \pi_H\big(\rho(A)\rho(u)^{-\top}\big) & 0 \\ 0 & 0 \end{pmatrix} = \rho(A)._H\rho(u).$$

qed.

3.4.4 Corollary. *We have*

$$\rho\big(\mathcal{O}_U(A)\big) = \mathcal{O}_H(\rho A) \qquad \textit{for all } A \in V_{\ddot{\nabla}}.$$

Before we draw some more corollaries from the proposition we prove the following

3.4.5 Lemma. *We have*

$$\chi^U_A(u) = \chi^H_{\rho A}(\rho u) \qquad \textit{for all } A \in V_{\ddot{\nabla}}, u \in U.$$

Proof. Let $u \in U$ and $A \in V_{\ddot{\nabla}}$ be arbitrary. Note, that

$$f_U(u)_{ij} = \begin{cases} u_{ij} & \text{if } (i,j) \in \triangledown, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad f_H(\rho u) = \begin{cases} u_{ij} & \text{if } (i,j) \in \ddot{\nabla}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{supp}(A) \subseteq \ddot{\nabla}$ we get

$$\kappa_N\big(A, f_U(u)\big) \overset{\text{Cor. 2.2.8}}{=} \sum_{(i,j)\in\ddot{\nabla}} A_{ij}u_{ij} \overset{\text{Cor. 2.2.8}}{=} \kappa_n\big(\rho A, f_H(\rho u)\big).$$

Applying ϑ on the equation yields the result. qed.

3.4.6 Corollary. *The map $\rho|_U$ induces a 1-preserving ring epimorphism*

$$\hat\rho : \mathbb{C}U \longrightarrow \mathbb{C}H : \sum_{u\in U} \lambda_u u \longmapsto \sum_{u\in U} \lambda_u \rho(u).$$

and we have $\hat\rho([A]_U) = [\rho(A)]_H$ for all $A \in V_{\ddot{\nabla}}$.

Proof. It is a standard method to extend a group epimorphism by linearity to gain a ring epimorphism between the corresponding group algebras. Hence let $A \in V_{\triangledown}$ be arbitrary. We have

$$\hat{\rho}([A]_U) = \frac{1}{|U|}\sum_{u\in U}\underbrace{\overline{\chi_A^U(u)}}_{=\chi_{\rho A}^H(\rho u)}\rho u = \underbrace{\frac{|\ker\rho|}{|U|}}_{=\frac{1}{|H|}}\sum_{h\in H}\overline{\chi_{\rho A}^H(h)}h = [\rho A]_H.$$

qed. |

3.4.7 Corollary. *We define* $\mathbb{C}V_{\triangledown} := \bigoplus_{A\in V_{\triangledown}} \mathbb{C}[A]_U$.[65] *Then*

$$\mathrm{Res}_H^U \mathbb{C}V_{\triangledown} \cong \mathbb{C}H \qquad \text{as } \mathbb{C}H\text{-rightmodule,}$$

where the isomorphism is given by $\hat{\rho}|_{\mathbb{C}V_{\triangledown}}$.[66]

In particular we have $\mathrm{Res}_H^U \mathbb{C}\mathcal{O}_U(A) \cong \mathbb{C}\mathcal{O}_H(\rho A)$ *for all* $A \in V_{\triangledown}$.

Proof. This is a direct consequence from the previous corollary. qed. |

3.4.8 Notation. *For* $A \in V_X$ *let* ψ_A^X *denote the character of the orbit module* $\mathbb{C}\mathcal{O}_X(A)$*, where* X *may be* H, U *or* G.

3.4.9 Proposition. *We have*

$$\psi_A^U(u) = \psi_{\rho A}^H(\rho u) \qquad \text{for all } A \in V_{\triangledown},\ u \in U.$$

And hence we have

$$\langle\psi_A^U, \psi_B^U\rangle_U = \langle\psi_A^H, \psi_B^H\rangle_H \qquad \text{for all } A, B \in V_{\triangledown}.$$

Proof. Let $A \in V_{\triangledown}$ and $u \in U$ be arbitrary. Note, that $B._U u = B$ if and only if $\rho(B)._H\rho(u) = \rho(B)$ since $\rho|_{V_{\triangledown}}$ is injective and $\rho(B._U u) = \rho(B)._H\rho(u)$. We have

$$\psi_A^U(u) \;=\; \sum_{\substack{B\in\mathcal{O}_U(A)\\ B._U u=B}} \underbrace{\chi_B^U(u)}_{=\chi_{\rho B}^H(\rho u)} \;=\; \sum_{\substack{B\in\mathcal{O}_U(A)\\ \rho(B)._H\rho(u)=\rho(B)}} \chi_{\rho B}^H(\rho u) \;=\; \sum_{\substack{X\in\mathcal{O}_H(\rho A)\\ X._H\rho(u)=X}} \chi_X^H(\rho u) \;=\; \psi_{\rho A}^H(\rho u).$$

Now let $A, B \in V_{\triangledown}$ be arbitrary. We have

$$\langle\psi_A^U, \psi_B^U\rangle_U = \frac{1}{|U|}\sum_{u\in U}\underbrace{\overline{\psi_A^U(u)}\psi_B^U(u)}_{=\overline{\psi_{\rho A}^H(\rho u)}\psi_{\rho B}^H(\rho u)} = \frac{|\ker\rho|}{|U|}\sum_{h\in H}\overline{\psi_{\rho A}^H(h)}\psi_{\rho B}^H(h) = \langle\psi_A^H, \psi_B^H\rangle_H.$$

qed. |

[65]Clearly $\mathbb{C}V_{\triangledown}$ is a $\mathbb{C}U$-rightmodule due to Theorem 3.1.14.

[66]note, that $\mathbb{C}V_{\triangledown}$ is even an ideal of $\mathbb{C}U$ due to Proposition 3.3.11 and hence, using Remark 2.2.19, we can interpret the isomorphism even as $\mathbb{C}H$-$\mathbb{C}H$-bimodule isomorphism.

Now we can derive Yan's results (precisely Proposition 2.2.25 and Theorems 2.2.24 and 2.2.26) from our theory:

3.4.10 Corollary (Obtaining the results of $A_n(\mathbf{F}_q)$ from the theory of $D_n(\mathbf{F}_q)$)**.** *The statement 'Each $A_n(\mathbf{F}_q)$-orbit module is isomorphic a to unique staircase $A_n(\mathbf{F}_q)$-module (labelled by its verge) and every irreducible character of $A_n(\mathbf{F}_q)$ is constituent of precisely one staircase $A_n(\mathbf{F}_q)$-module.' can be derived from the theory for $D_n(\mathbf{F}_q)$.*

Proof. Note that the $[A]_U$'s, where A runs through V_{∇} form a monomial basis for both the left and the right operation of $\mathbb{C}U$ due to Proposition 3.3.11. Let M be an orbit module. Then $M \cong \operatorname{Res}^U_H \mathbb{C}\mathcal{O}_U(A)$ for some $A \in V_{\nabla}$ due to Corollary 3.4.7. Corollary 3.3.15 then implies that we may assume A to be a core pattern. Since $\operatorname{supp}(A) \subseteq \nabla$, the pattern A is in fact a verge pattern (minor and supplementary conditions only can occur if there are some main conditions in the right part of $\triangledown$). Hence from Corollary 3.3.42 and and Proposition 3.4.9 it follows that different staircase modules are orthogonal, which means that every irreducible complex character is constituent of at most one staircase module. Of course every irreducible complex character is constituent of at least one staircase module, since the orbit modules decompose $\mathbb{C}H$. qed. |

Now we want to relate U and G.

3.4.11 Proposition ($\mathbb{C}U$ as a submodule of $\mathbb{C}G$)**.** *The map*

$$\iota : \ \mathbb{C}U \longrightarrow \mathbb{C}G : \quad [A]_U = \frac{1}{|U|}\sum_{u\in U}\overline{\chi^U_A(u)}u \quad \longmapsto \quad [A]_G = \frac{1}{|G|}\sum_{g\in G}\overline{\chi^G_A(g)}g$$

is an injective $\mathbb{C}G$-rightmodule homomorphism[67] *with* $\operatorname{Im}(\iota) = M_{\triangledown} := \bigoplus_{A\in V_{\triangledown}} \mathbb{C}[A]_G$.

Obviously this embedding is not the canonical embedding $\mathbb{C}U \hookrightarrow \mathbb{C}G$ which arises from the embedding $U \hookrightarrow G$. The canonical action of G on $\mathbb{C}U$ is not G-invariant (see Corollary 2.1.35 and Warning 2.1.36).

Proof. Since ι maps a basis to a linearly independent set, it remains to show

$$\iota([A]_U g) = \iota([A]_U)g \qquad \text{for every } g \in G \text{ and } A \in V_U = V_{\triangledown}.$$

We have for every $B \in V_{\triangledown}$ and $g \in G$

$$\chi^U_B(g) = \vartheta\kappa_N\big(B, \underbrace{f_U(g)}_{=\pi_U(g)}\big) \overset{\text{Lemma 3.1.8}}{=} \vartheta\kappa_N(B,g) \overset{\text{Cor. 2.2.8}}{=} \vartheta\kappa_N\big(B, f_G(g)\big) = \chi^G_B(g)$$

and

$$B._U g = \pi_U(Bg^{-\top}) \overset{\text{since } \operatorname{supp}(Bg^{-\top})\subseteq\triangledown}{=} \pi_G(Bg^{-\top}) = B._G g.$$

[67]the $\mathbb{C}G$ rightoperation on $\mathbb{C}U$ was defined in Theorem 3.1.14.

Hence we have

$$\iota([A]_U g) = \chi^U_{A.{}_U g}(g)\ \iota\big([A.{}_U g]_U\big) = \chi^U_{A.{}_U g}(g)[A.{}_U g]_G = \chi^G_{A.{}_G g}(g)[A.{}_G g]_G = [A]_G g.$$

And since $[A]_G = \iota([A]_U)$ the result follows. qed. |

As a corollary from the proof we get

3.4.12 Corollary. *We have*

$$B.{}_U g = B.{}_G g \quad \textit{and} \quad \chi^U_B(g) = \chi^G_B(g) \qquad \textit{for all } B \in V_\triangledown \textit{ and } g \in G.$$

3.4.13 Remark. *In Proposition 3.4.11 we have shown that we can view $\mathbb{C}U$ as a submodule of $\mathbb{C}G$ by virtue of the embedding ι. We have already seen in Warning 2.1.36 that this embedding cannot be the usual embedding induced by the inclusion map. But the embedding ι fits very well with the Yan-basis for $\mathbb{C}G$ described in Theorem 2.2.14.*

This allows us a nice interpretation of the verge modules:

3.4.14 Corollary. *Let $A \in V_\triangledown$ be a staircase pattern. Then we have*

$$\mathcal{V}(A) = \mathcal{O}_G(A) \qquad \textit{and} \qquad \mathbb{C}\mathcal{V}(A) \cong \operatorname{Res}^G_U \mathbb{C}\mathcal{O}_G(A) \quad \textit{as } \mathbb{C}U\textit{-right modules.}$$

Proof. We only need to check $\mathcal{V}(A) = \mathcal{O}_G(A)$. This is essentially N. Yan's Theorem 3.2 of [Yan2], but stated in the variant of our Theorem 2.2.24. But one can obtain this also on a few minutes thought (compare also the proof of Lemma 3.2.17 of which this proof is a simple variant.): The operation is explicitly of G on $V_\triangledown$ is explicitly described in Corollary 2.2.16. Let us consider some staircase pattern. Clearly the verge of the pattern is fixed under truncated column operation with elements of G. Let $(i_1, k_1), \dots, (i_\ell, k_\ell)$ denote the main conditions of the pattern ordered from right to left according to the column they lie in, i.e. such that $k_1 > k_2 > \cdots > k_\ell$. Row-by-row (first row i_1, then row i_2, ...) we then delete each non-zero entry on the left of some main condition using truncated column operation. For deleting these entries we use for the i_r-th row only elements of the form $1 + \lambda e_{ak_r}$. Since we can use elements of said form only the i_r-th row is affected by the truncated column operation.

We could also get the result by embedding $D_n(\mathbf{F}_q)$ into $D_{2n}(\mathbf{F}_q)$ and applying Corollary 3.4.10. From this point of view the verge modules of $D_n(\mathbf{F}_q)$ are just special cases of orbit modules of $D_{2n}(\mathbf{F}_q)$, whose support is part of ⧩. Being more precise we are talking of those staircase orbit modules of $D_{2n}(\mathbf{F}_q)$ whose support is included in the $\triangledown$-part of the ◹-part of ⧩. qed. |

3.5. André-Neto superclasses*

This section is dedicated to prove Corollary 3.5.34. It is not a new result (c.f. [AN2, Theorem 3.6]). Thus this section has the small asterisk to indicate that the reader can omit this section by just assuming the cited result. But even if the result itself is not new, the proof is. The original proof by C.A.M. André and A.M. Neto investigates some 'basic subvarieties of the Lie Algebra of $D_n(\mathbf{F}_q)$', whereas our proof uses some 'combinatorics on pictures' (very similar to the method in Section 3.2) to derive the same result. Thus with this section we provide the completion of our elementary approach and show the beauty and power within the pictures we use.

Following N. Yan we define the superclasses used for the theory of $A_N(\mathbf{F}_q)$. The superclasses used by C.A.M. André and A.M. Neto will then arise as intersections of the type $A_N(\mathbf{F}_q)$ superclasses with $D_n(\mathbf{F}_q)$.

3.5.1 Notation. *As in the previous section we set $G := A_N(\mathbf{F}_q)$ and $U := D_n(\mathbf{F}_q)$.*

3.5.2 Lemma. *Let $g, h \in G$. Then the following statements are equivalent:*

(i) *There exists $x, y \in G$, such that $g - 1 = x(h-1)y$,*

(ii) *$G(g-1)G = G(h-1)G$,*

(iii) *$g \in C_h^{\mathrm{AY}} := 1 + G(h-1)G$.*

Note, that the biorbits $G(g-1)G$ are subsets of $V_{\triangledown}$ and, that the sets C_h^{AY} form a partition of G, since (i) describes an equivalence relation.

3.5.3 Definition. *The sets $C_g^{\mathrm{AY}} = \{1 + x(g-1)y \mid x, y \in G\}$, where g runs through G, are called **André-Yan superclasses**.*

Note, that André-Yan superclasses are unions of conjugacy classes.[68]

3.5.4 Remark. *The three equivalent statements in the previous lemma are the three different points of view we can use. The first compares g and h using an equivalence relation, the second by looking at the G-G biorbits of $V_{\triangledown}$, and the third by defining explicit subsets of G, which are unions of conjugacy classes.*

3.5.5 Definition. *The sets $C_u^{\mathrm{AN}} := C_u^{\mathrm{AY}} \cap U$, where u runs trough U, are called **André-Neto superclasses**.*

3.5.6 Remark. *Let u and v be elements of U. Then Lemma 3.5.2 implies that u and v lie in the same AN-superclass if and only if there exists $g, h \in G$, such that $u - 1 = g(v-1)h$.*

Clearly we have

3.5.7 Lemma. *André-Neto superclasses are unions of conjugacy classes.*

[68] $xgx^{-1} = 1 + x(g-1)x^{-1}$.

The goal of this section is to provide a system of representatives for André-Neto superclasses. For André-Yan-superclasses we have the following:

3.5.8 Definition. *Let $A \in V_{\urtriangle}$. If A satisfies that in each row and each column lies at most one non-zero matrix entry then we call A an **André-Yan pattern**.*

3.5.9 Definition. *An element $g \in G$ is called G-**label** if and only if the pattern $g - 1 \in V_{\urtriangle}$ is an André-Yan pattern (AY-pattern for short).*

3.5.10 Theorem (Theorem 3.1 of [Yan2])**.** *The André-Yan patterns form a system of representatives of G-G biorbits on $V_{\urtriangle}$.*[69]

Using Lemma 3.5.2 we can change the point of view to get

3.5.11 Corollary. *The G-labels form a system of representatives of André-Yan superclasses. The André-Yan superclasses can be indexed using André-Yan patterns.*

3.5.12 Definition. *If $h \in G$ is an element of the André-Yan superclass C_g^{AY}, where g is a G-label, then we call g the G-**label of** h.*

Hence the G-label of an element determines the André-Yan superclass of the element.

André-Neto superclasses are defined as intersections of André-Yan superclasses with the group U. Hence in order to determine labels for the André-Neto superclasses we have to determine the G-labels of the elements of U.

For elements of a certain type we can easily read off the G-label. The next step is to precisely define what 'certain type' means. Unfortunately this implies that we have to fix some notation.

3.5.13 Definition. *Let $A \in V_{\urtriangle}$. We call $(i,j) \in \urtriangle$ a **superclass condition of** A if either*

1. *$(i,j) \in \triangledown$ and A_{ij} is the left-most non-zero entry of A in the i-th row* **-or-**
2. *$(i,j) \in \mathrel{\triangleleft\!\!/} = \triangleleft \cup /$ and A_{ij} is the down-most non-zero entry of A in j-th row.*

We also define the set $\mathrm{sc}(A) := \{(i,j) \in \urtriangle \mid (i,j) \text{ is a superclass condition of } A\}$. .

3.5.14 Example. *We consider*

$$A = \begin{pmatrix} \cdot & 0 & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ & \cdot & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ & & \cdot & 0 & 0 & 0 & 0 \\ & & & \cdot & A_{45} & 0 & A_{47} \\ & & & & \cdot & 0 & 0 \\ & & & & & \cdot & A_{67} \\ & & & & & & \cdot \end{pmatrix},$$

[69] G acts from left and right on $V_{\urtriangle}$ by matrix multiplication.

which is an element of $V_{\triangledown}$. As usual '$\cdot$', '0' and positions left empty denote 0. Red coloured entries A_{ij} are non-zero elements of $\mathbf{F}_q$, black coloured entries A_{ij} are arbitrary elements of $\mathbf{F}_q$. Entries whose background is shaded grey lie on $\diagup$. Hence in this example we have

$$\operatorname{sc}(A) = \{(1,3),(2,3),(4,5),(2,6),(6,7)\}.$$

The superclass conditions should be thought of as an superclass analogue for the main conditions. We also have a corresponding analogue for staircase pattern and the verge of a staircase pattern.

3.5.15 Definition. *Let $A \in V_{\triangledown}$. Then we call A* ***sc-staircase pattern*** *if in each row and column of $\operatorname{sc}(A)$ lies at most one coordinate.*

3.5.16 Example. *Keeping the notation of Example 3.5.14 we see that the element*

$$B = \begin{pmatrix} \cdot & 0 & 0 & A_{14} & A_{15} & 0 & A_{17} \\ & \cdot & A_{23} & A_{24} & A_{25} & 0 & A_{27} \\ & & \cdot & 0 & 0 & 0 & 0 \\ & & & \cdot & A_{45} & 0 & A_{47} \\ & & & & \cdot & 0 & 0 \\ & & & & & \cdot & A_{67} \\ & & & & & & \cdot \end{pmatrix}$$

has superclass conditions $\operatorname{sc}(B) = \{(1,4),(2,3),(4,5),(6,7)\}$. Hence B is an sc-staircase pattern. However the element A of Example 3.5.14 is ***not*** *an sc-staircase pattern, since the elements $(1,3)$ and $(2,3)$ lie in the same column. (Alternatively we could pick the elements $(2,3)$ and $(2,6)$, which lie in the same row.)*

3.5.17 Definition/Lemma. *Let $A \in V_{\triangledown}$. Then the element*

$$\operatorname{sc-verge}(A) := \sum_{(i,j)\in \operatorname{sc}(A)} A_{ij} e_{ij} \in V_{\triangledown}$$

is called the ***superclass verge*** *of A. Clearly we have*

A *is sc-staircase pattern* *if and only if* $\operatorname{sc-verge}(A)$ *is André-Yan pattern.*

3.5.18 Example. *We keep the notation of 3.5.14 we have*

$$\operatorname{sc-verge}(A) = \begin{pmatrix} \cdot & 0 & A_{13} & 0 & 0 & 0 & 0 \\ & \cdot & A_{23} & 0 & 0 & A_{26} & 0 \\ & & \cdot & 0 & 0 & 0 & 0 \\ & & & \cdot & A_{45} & 0 & 0 \\ & & & & \cdot & 0 & 0 \\ & & & & & \cdot & A_{67} \\ & & & & & & \cdot \end{pmatrix}.$$

Hence the sc-verge *of some pattern* $A \in V_{\triangledown}$ *consists of the 'west-border' of the* $\triangledown$*-part of* A *and the 'south-border' of the* $\triangleleft\!\!\!/ = \triangleleft \cup /$*-part of* A*. Clearly* $\text{sc-verge}(A)$ *is not an André-Neto pattern. On the other hand we have for the element* B *of Example 3.5.16*

$$\text{sc-verge}(B) = \begin{pmatrix} \cdot & 0 & 0 & A_{14} & 0 & 0 & 0 \\ & \cdot & A_{23} & 0 & 0 & 0 & 0 \\ & & \cdot & 0 & 0 & 0 & 0 \\ & & & \cdot & A_{45} & 0 & 0 \\ & & & & \cdot & 0 & 0 \\ & & & & & \cdot & A_{67} \\ & & & & & & \cdot \end{pmatrix},$$

which clearly is an André-Neto pattern.

3.5.19 Remark. *Clearly we have for* $A \in V_{\triangledown}$*:*

$$A \text{ is André-Yan pattern} \iff 1 + A \text{ is } G\text{-label} \iff \begin{array}{c} A \text{ is sc-staircase pattern} \\ \text{and } A = \text{sc-verge}(A) \end{array}$$

We mentioned earlier that for elements of a certain type it is easy to read off the G-label. The next proposition makes all of this precise.

3.5.20 Proposition. *Let* $g \in G$ *satisfy that* $g - 1$ *is an sc-staircase pattern. Then*

$$g_0 := 1 + \text{sc-verge}(g-1) = 1 + \sum_{(i,j)\in \text{sc}(g-1)} g_{ij} e_{ij} \quad \in G$$

is the G*-label of* g*, i.e.* g_0 *is a* G*-label and* $g \in C^{\text{AY}}_{g_0}$*.*

Proof. Using Remark 3.5.19 we see that g_0 is a G-label. It remains to show that g and g_0 are in the same André-Yan superclass. Hence we have to show that there exists elements $x, y \in G$, such that $g_0 - 1 = x(g-1)y$.

The proof of this is just a variant of the proof of 3.4.14. As always let $\tilde{x}_{ij}(\lambda) = 1 + \lambda e_{ij}$, where $\lambda \in \mathbf{F}_q$ and $(i,j) \in \triangledown$, i.e. the $\tilde{x}_{ij}(\lambda)$'s are the root subgroup elements of G.

Now, suppose A is an element of $V_{\triangledown}$. Lemma 2.2.15 implies

1. $\tilde{x}_{ij}(\lambda)A$ arises from A by taking A and adding λ times the j-th row to the i-th row (which lies strictly north of the j-th row).

2. $A\tilde{x}_{ij}(\lambda)$ arises from A by taking A and adding λ times the i-th column to the j-th column (which lies strictly east of the i-th column).

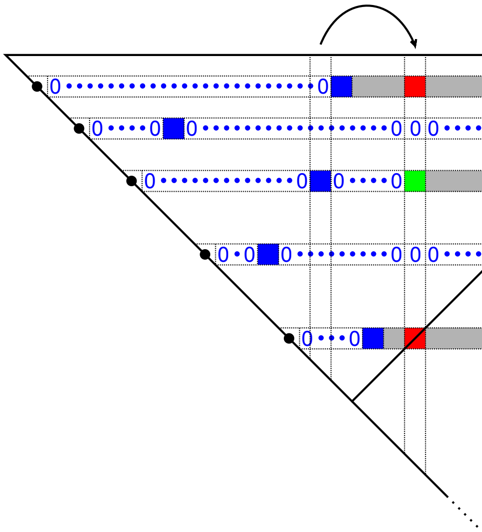


Figure 3.13.: Illustration to the proof of Lemma 3.5.20

We will use 1. to delete all entries of $g-1$ lying strictly to the north of some matrix coordinate $(i,j) \in \mathrm{sc}(A) \cap \triangleleft$ and 2. to delete all entries of $g-1$ lying strictly to the east of some matrix coordinate $(i,j) \in \mathrm{sc}(A) \cap \triangledown$. Since no two elements of $\mathrm{sc}(A)$ lie in the same row or column we will thereby not delete any entry of $g-1$ whose coordinates lie in $\mathrm{sc}(\mathrm{A})$.

Now we want to delete all entries 'to the east of $\mathrm{sc}(A) \cap \triangledown$'. We may write $\mathrm{sc}(g-1) \cap \triangledown = \{(i_1,k_1),\dots,(i_\ell,k_\ell)\}$ where the elements $(i_1,k_1),\dots,(i_\ell,k_\ell)$ are ordered in such a way that $k_1 < k_2 < \cdots < k_\ell$, i.e. according to their column from left to right. Starting with $r=1$ we apply the following move for each r (assuming we already applied the move for $1,\dots,r-1$): Using only multiplication with elements $\tilde{x}_{k_r b}(\lambda)$ from the right we can delete all entries in row i_r on the east of (i_r,k_r) without changing entries outside the i_r-th row. Figure 3.13 illustrates this process.

Analogously (using multiplication with elements $\tilde{x}_{ij}(\lambda)$ from the left) we can delete all entries of $g-1$ lying in the same column as (but to the north of) some matrix coordinate of $\mathrm{sc}(g-1) \cap \triangleleft$.

As already mentioned the whole process does not affect the entries with coordinates in $\mathrm{sc}(g-1)$. On the other hand let $(i,j) \in \triangledown$ be some matrix coordinate for which $(g-1)_{ij} = g_{ij} \neq 0$. Clearly (i,j) is either an element of $\triangledown$ or of $\triangleleft$. In case $(i,j) \in \triangledown$ the matrix coordinate (i,j) is either an element of $\mathrm{sc}(g-1) \cap \triangledown$ or lies in the same row, but east as some element of $\mathrm{sc}(g-1)$, i.e. is deleted in the process. In case $(i,j) \in \triangleleft$ the matrix coordinate (i,j) is either an element of $\mathrm{sc}(g-1) \cap \triangleleft$ or lies in the same column but north of some element of $\mathrm{sc}(g-1) \cap \triangleleft$, i.e. is deleted in the process.

We have shown that $g_0 - 1$ lies in the same G-G biorbit as does $g-1$. Due to

Lemma 3.5.2 this is equivalent to g and g_0 being in the same André-Yan superclass, which finishes the proof. qed. |

Keeping Proposition 3.5.20 in mind the G-label of an element u heavily depends on $\mathrm{sc}(u-1)$. If u is an element of U, then the set $\mathrm{sc}(u-1)$ has strong properties, which we will describe in Lemma 3.5.22.

3.5.21 Lemma. *Let $u \in U$ and $(i,j) \in \text{◹}$.*

(a) If $u_{ib} = 0$ for $i < b < j$, then it holds $u_{\bar{b}\bar{i}} = 0$ for $\bar{i} < \bar{b} < \bar{j}$ and $u_{\bar{j}\bar{i}} = -u_{ij}$

(b) If $u_{aj} = 0$ for $i < a < j$, then it holds $u_{\bar{j}\bar{a}} = 0$ for $\bar{j} < \bar{a} < \bar{i}$ and $u_{\bar{j}\bar{i}} = -u_{ij}$

Proof. (a) Let $i < b \leq j$. We have

$$0 = (uu^R)_{ib} \overset{\text{Lemma 1.1.13}}{=} u_{ib} + u_{\bar{b}\bar{i}} + \sum_{i<r<b} \underbrace{u_{ir}}_{\substack{=0 \\ \text{since } b \leq j}} u_{\bar{b}\bar{r}} = u_{ib} + u_{\bar{b}\bar{i}}.$$

So it follows $u_{\bar{b}\bar{i}} = 0$ if $b < j$ and $u_{\bar{j}\bar{i}} = -u_{ij}$ if $b = j$.

(b) The proof is completely analogous to (a), we just expand $0 = (u^R u)_{aj}$.

qed. |

3.5.22 Lemma. *Let $u \in U$. Then $f(u) \in V_{\nabla} \subset V_{\text{◹}}$, hence $\mathrm{sc}\left(f(u)\right)$ is a well-defined expression, and we have*

$$\mathrm{sc}(u-1) \cap \nabla = \mathrm{sc}\left(f(u)\right), \quad \mathrm{sc}(u-1) \cap \diagup = \emptyset \quad \textit{and} \quad \mathrm{sc}(u-1) \cap \triangleleft = \overline{\mathrm{sc}\left(f(u)\right)},$$

where $\overline{\mathrm{sc}\left(f(u)\right)}$ is the image of the set $\mathrm{sc}\left(f(u)\right)$ under the mirror map $^-$ from Remark 1.1.17, i.e. for a set $J \subseteq \text{◹}$ we have $\bar{J} = \{(\bar{j}, \bar{i}) \mid (i,j) \in J\}$.

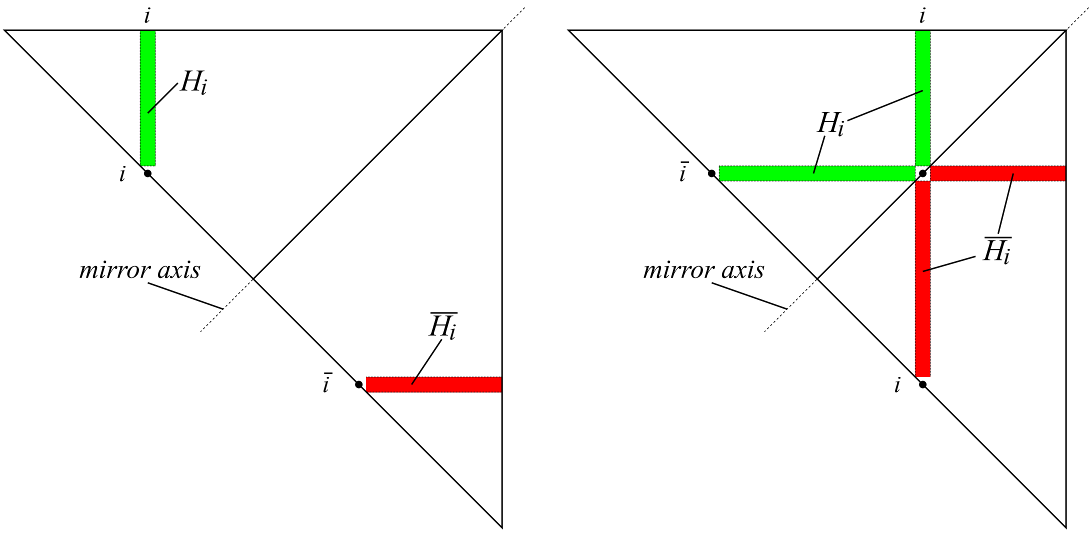
In particular we have $\mathrm{sc}(u-1) = \mathrm{sc}\left(f(u)\right) \cup \overline{\mathrm{sc}\left(f(u)\right)}$.

Proof. By definition of sc and f we have $\mathrm{sc}(u-1) \cap \nabla = \mathrm{sc}\left(f(u)\right)$. Lemma 3.5.21 implies $\mathrm{sc}(u-1) \cap \triangleleft = \overline{\mathrm{sc}\left(f(u)\right)}$. At last we check $\mathrm{sc}(u-1) \cap \diagup = \emptyset$. Suppose $(i, \bar{i}) \in \mathrm{sc}(u-1) \cap \diagup$. Then (1.1.15) implies there exists $(i,j) \in \mathrm{sc}(u-1) \cap \nabla$. But we already have shown that $\mathrm{sc}(u-1) \cap \triangleleft = \overline{\mathrm{sc}(u-1) \cap \nabla}$. Hence $(\bar{j}, \bar{i}) \in \mathrm{sc}(u-1) \cap \triangleleft$. In particular we get $u_{\bar{j}\bar{i}} \neq 0$, which is a contradiction to $(i, \bar{i}) \in \mathrm{sc}(u-1)$. qed. |

3.5.23 Definition. *Analogously to the definitions for patterns we make the following definitions:*

1. *$I \subseteq \text{◹}$ is called **sc-staircase set** if in each row and each column lies at most one element of I.*

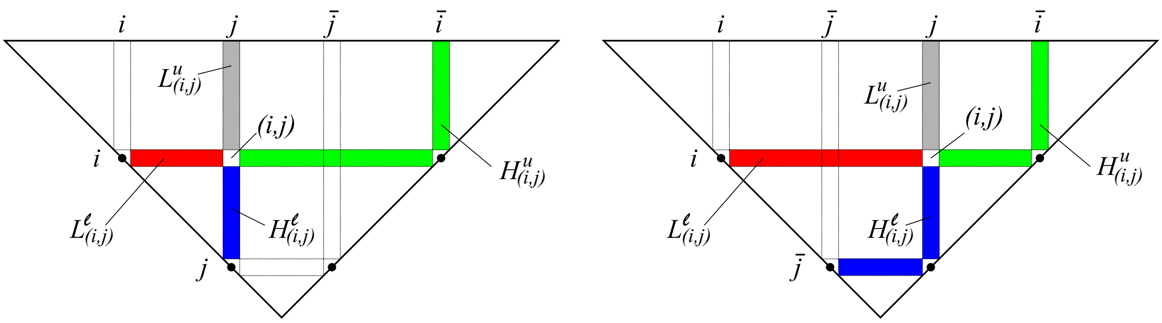
2. *$J \subseteq \nabla$ is called **hook-separated set** if on each hook H_i (defined in 3.3.17, c.f. also Figure 3.14) lies at most one element of J.*



(a) Variant, where $i \leq n$ (b) Variant, where $i > n$

Explanation: The pictures show H_i and $\overline{H_i}$. As one can see $\overline{H_i}$ is obtained from H_i by mirroring on the counter diagonal.

Figure 3.14.: Mirrored hooks



(a) Variant, where $j \leq n$ (b) Variant, where $j > n$

Figure 3.15.: Illustrations to Definition 3.5.26

Clearly $A \in V_{\triangleleft}$ is an sc-staircase pattern if and only if $\mathrm{sc}(A)$ is an sc-staircase set and $B \in V_{\triangledown}$ is a hook-separated pattern if and only if $\mathrm{main}(B)$ is a hook-separated set.

3.5.24 Lemma. *Let $J \subseteq \triangledown$ be arbitrary. Then we have*

$$J \text{ is hook-separated} \qquad \text{if and only if} \qquad J \cup \bar{J} \text{ is sc-staircase},$$

where $\bar{J} = \{(\bar{j}, \bar{i}) \mid (i,j) \in J\} \subseteq \triangleleft$.

Proof. Figure 3.14 shows how $H_i \cup \overline{H_i}$ arises from H_i, proving the claim. qed. |

3.5.25 Corollary. *Let $u \in U$ be arbitrary. Then we have*

$$\mathrm{sc}(u-1) \text{ is sc-staircase} \qquad \text{if and only if} \qquad \mathrm{sc}\big(f(u)\big) \text{ is hook-separated.}$$

Proof. Due to Lemma 3.5.22 we have $\mathrm{sc}(u-1) = \mathrm{sc}\big(f(u)\big) \cup \overline{\mathrm{sc}\big(f(u)\big)}$. Hence Lemma 3.5.24 implies $\mathrm{sc}(u-1)$ sc-staircase if and only if $\mathrm{sc}(f(u))$ hook-separated. qed. |

Our next goal is to show that for every $u \in U$ there exists $v \in U$ lying in the same U-conjugacy class as u, such that $\mathrm{sc}(v-1)$ is an sc-staircase pattern. This will allow us to apply Proposition 3.5.20. Corollary 3.5.25 allows us to follow the strategy of the earlier sections to just investigate the image under f, i.e. elements of $V_{\triangledown}$. In order to do this we need two combinatorial moves on pictures, called ‘first’ and ‘second conjugation move’. For the algebraic description of the conjugation moves we need the following notations.

3.5.26 Definition. *Let (i,j) be in $\triangledown$.*
We define the ***upper leg associated to*** *(i,j) as the subset of $\triangledown$ given by*

$$L^u_{(i,j)} := \big\{(a,b) \in \triangledown \;\big|\; a < i \text{ and } b = j\big\},$$

the ***lower leg associated to*** *(i,j) as the subset of $\triangledown$ given by*

$$L^\ell_{(i,j)} := \big\{(a,b) \in \triangledown \;\big|\; a = i \text{ and } b < j\big\},$$

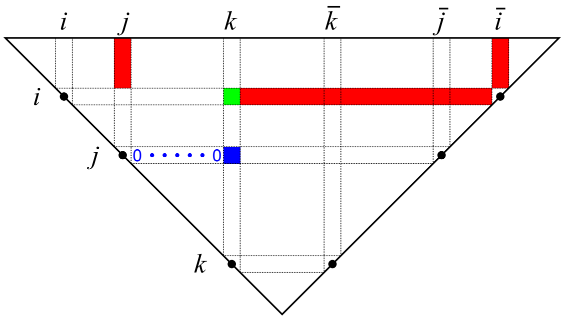
the ***upper hook associated to*** *(i,j) as the subset of $\triangledown$ given by*

$$H^u_{(i,j)} := \big\{(a,b) \in \triangledown \;\big|\; a = i \text{ and } b > j\big\} \cup \big\{(a,b) \in \triangledown \;\big|\; a < i \text{ and } b = \bar{i}\big\}$$

and we remember, that the lower hook associated to (i,j) was defined as

$$H^\ell_{(i,j)} := \big\{(a,b) \in \triangledown \;\big|\; a = \bar{j} \text{ and } b < j\big\} \cup \big\{(a,b) \in \triangledown \;\big|\; a > i \text{ and } b = j\big\}.$$

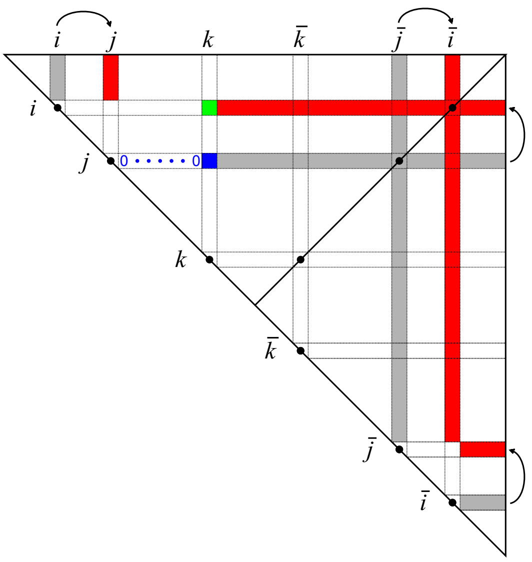
Note: Illustrations are given in Figure 3.15.



(a) Variant of the lemma, where $k \leq n$ (b) Variant of the lemma, where $k > n$

Figure 3.16.: The first conjugation move, Lemma 3.5.27

The figures show $\triangledown$, the blue boxes denote all entries (in these cases here exactly one) that are non zero. The Blue 0's denote 0 entries. All the blue entries are the conditions needed to apply the move. The move produces arbitrary chosen entries in the green boxes at the cost of the red boxes. There we don't really know how the red entries look after the move. The rest of the boxes remain the same.



(a) Variant, where $k \leq n$ (b) Variant, where $k > n$

Figure 3.17.: To the proof of Lemma 3.5.27

Explanation: The pictures shows how $x(g-1)x^{-1}$ rises from $g-1$. The arrows show which rows (resp. columns) are added (times some scalar) to which rows (resp. columns). Grey entries are unknown, the colors red, blue and green have the same meaning as in Figure 3.16.

3.5.27 Lemma (First conjugation move)**.** *Let u be an element of U and let (j,k) be in $\bigtriangledown$, such that*

1. $f(u)_{ab} = 0$ *for all* $(a,b) \in L^{\ell}_{(j,k)}$,
2. $f(u)_{jk} \neq 0$.

Let $(i,k) \in \bigtriangledown$ be above (j,k), i.e. $i < j$, and let λ be an arbitrary element of $\mathbf{F}_q$. Then $x := x_{ij}(u_{jk}^{-1}\lambda) = 1 + u_{jk}^{-1}\lambda e_{ij} - u_{jk}^{-1}\lambda e_{\bar{j}\bar{i}}$ is an element of U and $xux^{-1} \in U$ has the following properties:

(i) $[f(xux^{-1})]_{ab} = f(u)_{ab}$ *if* (a,b) *is in* $\bigtriangledown \smallsetminus \big(\{(i,k)\} \cup H^u_{(i,k)} \cup L^u_{(i,j)}\big)$,

(ii) $[f(xux^{-1})]_{ik} = f(u)_{ik} + \lambda$.

Note: An illustration of the lemma is given in Figure 3.16.

Proof. We have $i < j < k < \bar{i}$ and therefore $(i,j) \in \bigtriangledown$. Then especially x is an element of the root subgroup $X_{ij} \subseteq U$. Note that $f(u)_{ab} = u_{ab}$ for all $(a,b) \in \bigtriangledown$. It holds

$$xgx^{-1} - 1 = x(g-1)x^{-1}.$$

Thus xux^{-1} is obtained from u in the following way (remember: $x^{-1} = x_{ij}(-u_{jk}^{-1}\lambda)$):

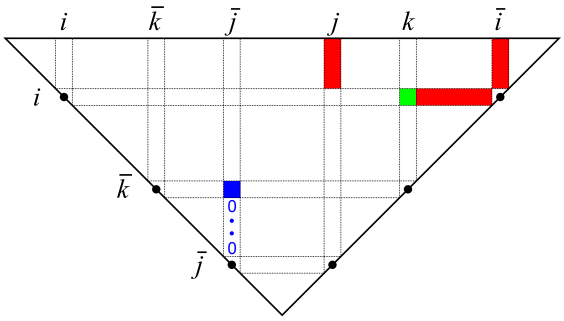
1. Delete the diagonal entries from u, that is $u \mapsto u - 1$.
2. Take $u - 1$, add $u_{jk}^{-1}\lambda$ times the j-th row to the i-th row and add $-u_{jk}^{-1}\lambda$ times the $\bar{i}$-th row to $\bar{j}$-th row of $g-1$, that is $g - 1 \mapsto x(g-1)$.
3. Take $x(u-1)$, add $-u_{jk}^{-1}\lambda$ times the i-th column to the j-th column and add $u_{jk}^{-1}\lambda$ times the $\bar{j}$-th column to $\bar{i}$-th column of $x(u-1)$, that is $x(u-1) \mapsto x(u-1)x^{-1}$.
4. Put ones on the diagonal, that is $x(u-1)x^{-1} \mapsto x(u-1)x^{-1} + 1 = xux^{-1}$.

The proof follows from Figure 3.17. qed. |

3.5.28 Remark. *The use of f in the formulation of Lemma 3.5.27 may seem a bit artificial. But in this way we can see that everything fits to our philosophy to manipulate the $\bigtriangledown$ part of the group elements. The mapping f is just the technical expression of this philosophy.*

3.5.29 Lemma (Second conjugation move)**.** *Let u be an element of U and let $(\bar{k},\bar{j})$ be in $\bigtriangledown$, such that it holds*

1. $f(u)_{ab} = 0$ *for all* $(a,b) \in H^{\ell}_{(\bar{k},\bar{j})}$,
2. $f(u)_{\bar{k}\bar{j}} \neq 0$.

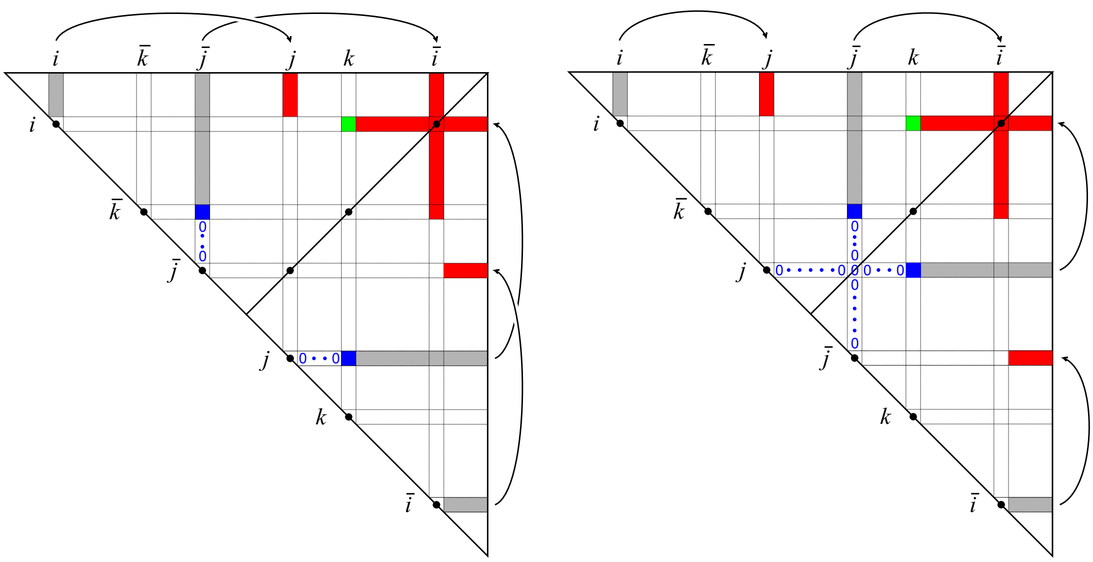


(a) Variant of the lemma, where $k \leq n$

(b) Variant of the lemma, where $k > n$

Figure 3.18.: The second conjugation move, Lemma 3.5.29

The figures show ∇, the blue boxes denote entries (in these cases here exactly one) that are non zero. The Blue 0's denote 0 entries. All the blue entries are the conditions needed to apply the move. The move produces arbitrary chosen entries in the green boxes at the cost of the red boxes. There we don't really know how the red entries look after the move. The rest of the boxes remain the same.



(a) Variant, where $k \leq n$

(b) Variant, where $k > n$

Figure 3.19.: To the proof of Lemma 3.5.29

Explanation: The pictures shows how $x(u-1)x^{-1}$ rises from $u - 1$. The arrows show which rows (resp. columns) are added (times some scalar) to which rows (resp. columns). Grey entries are unknown, the colors red, blue and green have the same meaning as in Figure 3.18.

Let $(i,k) \in \nabla$ and $\lambda \in \mathbf{F}_q^\times$ be arbitrary. Then $x := x_{ij}(-u_{k\bar{j}}^{-1}\lambda) = 1 - u_{k\bar{j}}^{-1}\lambda e_{ij} + u_{k\bar{j}}^{-1}\lambda e_{\bar{j}\bar{i}}$ is an element of U and $xux^{-1} \in U$ has the following properties:

(i) $[f(xux^{-1})]_{ab} = f(u)_{ab}$ *if (a,b) is in $\nabla \setminus \big(\{(i,k)\} \cup H^u_{(i,k)} \cup L^u_{(i,j)}\big)$,*

(ii) $[f(xux^{-1})]_{ik} = f(u)_{ik} + \lambda$.

Note: An illustration of the lemma is given in Figure 3.18.

Proof. We proceed as in the proof of Lemma 3.5.27 to get to the point, were we can use the pictures in Figure 3.19. In variant (a) we apply Lemma 3.5.21 (b) to get the blue entries below the counterdiagonal. In variant (b) we first check $u_{j\bar{j}} = 0$, c.f. (1.1.15). Then we can apply Lemma 3.5.21 parts (a) and (b) to obtain the other blue entries below the counterdiagonal. Now, the proof follows from the pictures. qed. |

We can use the two conjugation moves to prove the following Lemma, which will later allow us to apply Proposition 3.5.20

3.5.30 Lemma. *Let $u \in U$ be arbitrary, then there exists an element v in the same U-conjugacy class*[70] *as u, such that $\operatorname{sc}\big(f(v)\big)$ is hook-separated.*

Proof. In the pictures in Figure 3.16 and Figure 3.18 the red coloured positions denote matrix coordinates of entries whose value we do not know after applying one of the conjugation moves. These red positions are either in the same row but on the right of the entry, which may change completely as we want (the green one), or in rows lying above of the one including the green position. Hence it is obvious that we should apply the conjugation moves row by row going upwards, starting with the lowest row and in each row going from left to right.

Hence let $u \in U$ be arbitrary. Starting with $i = n-1$ (that is the lowest non-empty row of ∇) we manipulate the pattern $f(u)$ as follows:

1. We consider the i-th row of ∇ consisting of $(i,i+1),(i,i+2),\dots,(i,\bar{i}-1)$.
2. Heading from left to right for each (i,a) we
   a) delete the matrix entry at position (i,a) if (i,a) lies strictly above some non-zero entry using the first conjugation move.
   b) delete the matrix entry at position (i,a) if on the horizontal part of the hook H_a is some non-zero entry (i.e. one of the form $(\bar{a},b)$ for some b) using the second conjugation move.
   c) otherwise we do not change the matrix entry at position (i,a).
3. We repeat the procedure for row $i-1$.

[70] i.e. $v = xux^{-1}$ for some $x \in U$

Note that after we arrived at some position (i, a) during the procedure we will not change the entry at the position again as the two conjugation moves do only affect the current entry and entries which appear later in the procedure.

After finishing the procedure we have produced some pattern $f(v)$, which is the image under f for some v in the same U-conjugacy class as u. Thereby u is conjugate to v since we did only apply the two conjugation moves.

Now we want to prove that $\mathrm{sc}(f(v))$ is indeed hook-separated. Hence suppose $(i, j) \in \mathrm{sc}(f(v))$ and suppose there exists some $(a, b) \in \mathrm{sc}(f(v))$, such that $f(v)_{ab} \neq 0$, such that (i, j) and (i, b) lie on the same hook. By definition of the set $\mathrm{sc}(f(v))$ the two coordinates (i, j) and (a, b) cannot lie in the same row. Hence they lie either in the same column (in this situation the entry at the higher position would have been deleted during the procedure using the first conjugation move) or one lies on the horizontal part and one on the vertical part of some hook (in this situation the one on the vertical part would have been deleted during the procedure using the second conjugation move). Hence $\mathrm{sc}(f(v))$ is hook-separated. qed. |

3.5.31 Definition. *Hook-separated verge patterns are called* ***André-Neto patterns*** *(AN-patterns for short).*

3.5.32 Remark. *With the definition of André-Neto patterns Cor. 3.5.25 takes the form*

$$\text{sc-verge}(u-1) \textit{ is André-Yan pattern} \quad \Longleftrightarrow \quad \text{sc-verge}\big(f(u)\big) \textit{ is André-Neto pattern}.$$

3.5.33 Theorem. *Let $u \in U$ be arbitrary. Then the G-label of u is of the form*

$$\rho(A) \;:=\; 1 \quad + \sum_{(i,j)\in \mathrm{main}(A)} A_{ij}e_{ij} - A_{ij}e_{\overline{ji}} \qquad \textit{for some André-Neto pattern } A.$$

Conversely for every André-Neto pattern A the element $f^{-1}(A) \in U$ has G-label $\rho(A)$.

Proof. Using Lemma 3.5.30 we may assume that $\mathrm{sc}\big(f(u)\big)$ is hook-separated. Then Corollary 3.5.25 implies that $\mathrm{sc}(u-1)$ is sc-staircase, which allows us to apply Proposition 3.5.20. We get

$$G\text{-label of } u \;=\; 1 \;+ \sum_{(i,j)\in \mathrm{sc}(u-1)} u_{ij}e_{ij} \overset{\text{Lemma 3.5.22}}{=} 1 \;+ \sum_{(i,j)\in \mathrm{sc}(f(u))} u_{ij}e_{ij} + u_{\overline{ji}}e_{\overline{ji}}.$$

And Lemma 3.5.21 implies $u_{\overline{ji}} = -u_{ij}$ for $(i, j) \in \mathrm{sc}(u-1) \cap \triangledown$. Hence by setting $A := \sum_{(i,j)\in \mathrm{sc}(f(u))} u_{ij}e_{ij}$ the first part of the theorem is proven.

The 'Conversely...' part follows from the fact that $f(f^{-1}(A)) = A$ is hook-separated and the first part of the proof. qed. |

3.5.34 Corollary (Theorem 3.6 from [AN2]). *The André-Neto patterns are in bijection with the André-Neto superclasses.*

Proof. Corollary 3.5.11 implies that each André-Neto superclass (which is a subsets of a unique André-Yan superclass) has a well defined G-label. Hence the G-label defines an injective map

$$\Phi : \{\text{André-Neto superclasses}\} \longrightarrow \{\text{André-Yan patterns}\}.$$

But due to Theorem 3.5.33 we have $\mathrm{im}(\Phi) = \{\text{André-Neto patterns}\}$. qed. |

3.5.35 Remark. *C.A.M. André and A.M. Neto describe their labels in an equivalent but a bit different way. They consider 'basic pairs' (D, ϕ), where D is a special type of $\Phi^+(D_n)$[71] and $\phi : D \to \mathbf{F}_q^\times$ is some map. In Definition/Lemma 1.2.5 we have shown that we can identify $\Phi^+(D_n)$ with the set $\bigtriangledown$. Considering the positive roots as elements $\bigtriangledown$ the their definition of a basic pair reads as follows. We call (D, ϕ) a basic pair if D is a subset of $\bigtriangledown$ satisfying that $D \cup \bar{D}$ is sc-staircase and $\phi : D \to \mathbf{F}_q^\times$ is a map. For a basic pair let $A_{(D,\phi)} := \sum_{(i,j)\in D} \phi(i,j)e_{ij}$. Lemma 3.5.24 implies that $A_{(D,\phi)}$ is an André-Neto pattern. Clearly $(D, \phi) \mapsto A_{(D,\phi)}$ maps basic pairs bijectively into André-Neto patterns.*

3.6. Decomposition of André-Neto supercharacters

In this section we first establish the notion of a supercharacter theory for a finite group, introduced by P. Diaconis and I.M. Isaacs in [DI]. Paper [DI] also contains a wonderful overview over the beauties and possibilities of a supercharacter theory. We will show, that the characters of hook-separated verge modules form a supercharacter theory. In fact they form the one of C.A.M. André and A.M. Neto (see [AN1] for large primes $p = \mathrm{char}(\mathbf{F}_q)$ and [AN2], [AN3] for the general case). Adopting that point of view Corollaries 3.3.31 and 3.3.43 turn out to provide a decomposition of the André-Neto supercharacters into characters of U-orbit modules (which have the property of being either orthogonal or isomorphic) and where the decomposition is given by an explicit 'combinatorial algorithm'.

3.6.1 Definition. *Let G be a finite group. Suppose we have a partition $\mathcal{K}$ of G (into nonempty parts) and a set of (nonzero) complex characters $\mathcal{X}$ of G, such that*

(i) $|\mathcal{X}| = |\mathcal{K}|$

(ii) The characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$

(iii) The elements of $\mathcal{X}$ are pairwise orthogonal.

*Then we call $(\mathcal{X}, \mathcal{K})$ a **pre-supercharacter theory** for G. To every function $\varphi : G \to \mathbb{C}$ satisfying (ii) we refer as a **superclass function**. Clearly the superclass functions form a $\mathbb{C}$-vector space.*

[71] $\Phi^+(D_n)$ is the set of positive roots of the root system of type D_n

3.6.2 Definition/Lemma (Supercharacter theory, see [DI, Lemma 2.1]). *Let G be a finite group and $(\mathcal{X}, \mathcal{K})$ a pre-supercharater theory. For every $\chi \in \mathcal{X}$ we define $\operatorname{Irr}(\chi)$ to be the set of irreducible constituents of χ. We set $\sigma_\chi := \sum_{\psi \in \operatorname{Irr}(\chi)} \psi(1)\psi$ for each $\chi \in \mathcal{X}$. Then the following statements are equivalent:*

(i) The set $\{1\}$ is a member of $\mathcal{K}$.

(ii) $\bigcup_{\chi \in \mathcal{X}} \operatorname{Irr}(\chi) = \operatorname{Irr}(G)$ and every supercharacter χ is a constant multiple of σ_χ.

(iii) Every irreducible character ψ of G is constituent of at least one character $\chi \in \mathcal{X}$.

*Suppose one of the three statements (and thus all of them) holds. Then we call $(\mathcal{X}, \mathcal{K})$ a **supercharacter theory** of G. To the characters $\chi \in \mathcal{X}$ we refer as **supercharacters** and to the elements of $\mathcal{K}$ as **superclasses**.*

Proof. (i) $\implies$ (ii): We set $M := \operatorname{Irr}(G) \smallsetminus \bigcup_{\chi \in \mathcal{X}} \operatorname{Irr}(\chi)$ and $\sigma_M := \sum_{\psi \in M} \psi(1)\psi$. Suppose $\{1\} \in \mathcal{K}$. Then the regular character reg_G is a superclass function of G. (Recall that $\operatorname{reg}_G(g) = 0$ for $1 \neq g \in G$ and $\operatorname{reg}_G(1) = |G|$.) The characters $\chi \in \mathcal{X}$ are linearly independent (they are orthogonal, cf. Definition 3.6.1(iii)). Since $|\mathcal{X}| = |\mathcal{K}|$, the set $\mathcal{X}$ is basis of the vector space of superclass functions. Hence, we have

$$\sigma_M + \sum_{\chi \in \mathcal{X}} \sigma_\chi = \operatorname{reg}_G = \sum_{\chi \in \mathcal{X}} a_\chi \chi \qquad \text{for some } a_\chi \in \mathbb{C}.$$

Since the irreducible characters are linearly independent we get $\sigma_\chi = a_\chi \chi$ for all supercharacters χ (where $a_\chi \neq 0$) and $\sigma_M = 0$. Thus we also have $M = \emptyset$.

(ii) $\implies$ (iii) is trivial.

(iii) $\implies$ (i): Let $K \in \mathcal{K}$ be the superclass containing 1. Suppose $g \in K$. Then we have $\chi(g) = \chi(1)$ for all supercharacters χ. For every irreducible character ψ holds $|\psi(g)| \leq \psi(1)$. Hence $\chi(g) = \chi(1)$ implies $\psi(g) = \psi(1)$ for all $\psi \in \operatorname{Irr}(\chi)$. Thus we have $\psi(g) = \psi(1)$ for all irreducible characters of G. But then we also have $\operatorname{reg}_G(g) = \operatorname{reg}_G(1)$, which implies $g = 1$. $\mathrm{qed.}$ |

3.6.3 Remark. *Superclasses are unions of conjugacy classes due to [DI, Theorem 2.2(c)].*

3.6.4 Example. *For every finite group G there exist two 'trivial' supercharacter theories, namely $\big(\operatorname{Irr}(G), \operatorname{ConjCl}(G)\big)$ and $\big(\{\operatorname{reg}_G - \chi_{triv}, \chi_{triv}\}, \{G \smallsetminus \{1\}, \{1\}\}\big)$, where χ_{triv} denotes the trivial character of G and $\operatorname{ConjCl}(G)$ the set of conjugacy classes of G. For the symmetric group $\mathfrak{S}_4$ on 4 letters a nontrivial example is given in Figure 3.20.*

Let us return to the concrete situation. In Definition 3.5.5 we introduced,André-Neto superclasses for $D_n(\mathbf{F}_q)$. In Lemma 3.5.7 we have shown, that AN-superclasses are unions of conjugacy classes.

3.6.5 Notation. *From now on let $G = A_N(\mathbf{F}_q)$ and $U = D_n(\mathbf{F}_q)$.*

	(1^4)	$(2,1^2)$	(4)	(2^2)	$(3,1)$
ψ_1	1	1	1	1	1
ψ_2	1	-1	-1	1	1
ψ_3	3	1	-1	-1	0
ψ_4	3	-1	1	-1	0
ψ_5	2	0	0	2	-1

Character table of the symmetric group on 4 letters.

	(1^4)	$(2,1^2)$	(4)	(2^2)	$(3,1)$
ψ_1	1	1	1	1	1
ψ_2	1	-1	-1	1	1
χ	22	0	0	-2	-2

$\chi = \psi_3(1)\psi_3 + \psi_4(1)\psi_4 + \psi_5(1)\psi_5$

Figure 3.20.: A nontrivial supercharacter theory for $\mathfrak{S}_4$

By $\mathfrak{S}_4$ we denote the symmetric group on 4 letters. We define $\mathcal{X} := \{\psi_1, \psi_2, \chi\}$, where $\chi = \psi_3(1)\psi_3 + \psi_4(1)\psi_4 + \psi_5(1)\psi_5$, and $\mathcal{K} := \{(1^4), (2,1^2) \cup (4), (2^2) \cup (3,1)\}$, where for example $(2,1^2) \cup (4)$ is the union of the conjugacy classes labelled by $(2,1^2)$ and (4). As we can see in the right table $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for $\mathfrak{S}_4$ different from $\big(\mathrm{Irr}(\mathfrak{S}_4), \mathrm{ConjCl}(\mathfrak{S}_4)\big)$ and $\big(\{\, \mathrm{reg}_{\mathfrak{S}_4} - \chi_{\mathrm{triv}}, \chi_{\mathrm{triv}} \,\}, \{\mathfrak{S}_4 \smallsetminus \{1\}, \{1\}\}\big)$.

3.6.6 Reminder. *See Definitions 3.5.5 and 3.5.31.*

(i) The equivalence classes of the equivalence relation $\sim$ on $D_n(\mathbf{F}_q)$, defined by

$$y \sim x \quad :\Longleftrightarrow \quad \text{there exists } g, h \in A_N(\mathbf{F}_q), \text{ such that } y - 1 = g(x-1)h,$$

*are called **André-Neto superclasses** (AN-superclasses for short).*

*(ii) Hook-separated verge patterns are called **André-Neto patterns** (AN-patterns for short).*

If we have André-Neto superclasses we also need André-Neto supercharacters. We follow the definition given in all of the articles [AN1], [AN2] and [AN3]. In Proposition 3.6.16 we will show that we could have defined them as characters of verge modules correpsonding to hook-separated staircase patterns. In view of our theory, this would be the more natural definition, but we want to really show, that the two definitions are equal. To define them we need first a special class of (pattern) subgroups.

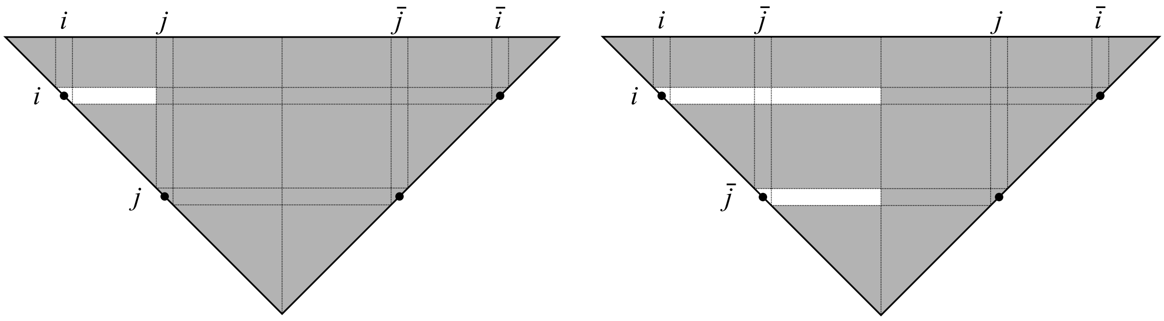
3.6.7 Definition/Lemma (The subgroups $U_{(i,j)}$). *Let $(i,j) \in \triangledown$. Then*

$$J_{(i,j)} := \begin{cases} \triangledown \smallsetminus \{(i,r) \mid i < r < j\} & \text{if } j \le n, \\ \triangledown \smallsetminus \big(\{(i,r) \mid i < r \le n\} \cup \{(\bar{j}, s) \mid \bar{j} < s \le n\}\big) & \text{if } j > n. \end{cases}$$

is a closed subset of $\triangledown$. We define $U_{(i,j)} := U_{J_{(i,j)}}$.[72]

An illustration of $U_{(i,j)}$ is given in Figure 3.21.

[72] c.f. Thm. 1.2.22 for a description of pattern subgroups. Hence $U_{(i,j)} = \{u \in U \mid u_{ir} = 0$ for $i < r < j\}$ if $j \le n$ and $U_{(i,j)} = \{u \in U \mid u_{ir} = 0$ for $i < r \le n$ and $u_{\bar{j}s} = 0$ for $\bar{j} < s \le n\}$ if $j > n$.



Explanation: The figures show ∇. The grey shaded positions form the set $J_{(ij)}$.

Figure 3.21.: Illustrations to Definition 3.6.7

Proof. We use Corollary 1.2.14. Suppose (a, b) and (b, c) are elements of $J_{(i,j)}$. We have to show $(a, c) \in J_{(i,j)}$. We start with case $j > n$. If $a \neq i, \bar{j}$ nothing is to prove. If $a = i$ we have $n < b < c$, hence $(a, c) \in J_{(i,j)}$. If $a = \bar{j}$ we have $n < b < c$, hence also $(a, c) \in J_{(i,j)}$. The case $j \leq n$ is completely analogous. qed.

The pattern subgroups $U_{(i,j)}$ have the following very useful property.

3.6.8 Lemma. *Let* $(i, j) \in \nabla$. *Then we have*

$$(xy)_{ij} = x_{ij} + y_{ij} \qquad \textit{for all } x, y \in U_{(i,j)}.$$

Proof. Let $x, y \in U_{(i,j)}$ be arbitrary.
Case $j \leq n$**:** We have

$$(xy)_{ij} = x_{ij} + y_{ij} + \sum_{i<r<j} \underbrace{x_{ir}}_{=0} y_{rj} = x_{ij} + y_{ij}.$$

Case $j > n$**:** Since $\operatorname{supp}(U_{(i,j)}) \cap \nabla = J_{(i,j)}$, we have $(y^{-1})_{\bar{j}s} = 0$ for $\bar{j} < s \leq n$. Hence

$$(*) \qquad \sum_{n<r<j} x_{ir} y_{rj} \;=\; \sum_{\bar{j}<s<\bar{n}=n+1} x_{i\bar{s}} \underbrace{y_{\bar{s}j}}_{=(y^R)_{\bar{j}s}} \;=\; \sum_{\bar{j}<s\leq n} x_{i\bar{s}} \underbrace{(y^{-1})_{\bar{j}s}}_{=0} \;=\; 0.$$

Using $(*)$ we can finish the proof:

$$(xy)_{ij} = x_{ij} + y_{ij} + \sum_{i<r\leq n} \underbrace{x_{ir}}_{=0} y_{rj} + \sum_{n<r<j} x_{ir} y_{rj} \overset{(*)}{=} x_{ij} + y_{ij}.$$

Hence the result is proven. qed.

This allows us to define linear characters for the groups $U_{(i,j)}$. Of course we can induce these linear characters to U. Before we do this we give characters of verge modules a name. Then we will interpret the induced linear characters of $U_{(i,j)}$ in terms of these verge characters.

3.6.9 Definition/Lemma. *Let A be a staircase pattern. We denote the character of the verge module $\mathbb{C}\mathcal{V}(A)$ by $\Psi_{\mathcal{V}(A)}$ and call it the **verge character** (associated to $\mathcal{V}(A)$). The character values are given by*

$$\Psi_{\mathcal{V}(A)}(u) = \sum_{\substack{B\in\mathcal{V}(A)\\ B.u=B}} \chi_B(u).$$

Proof. This follows directly from Theorem 3.1.14. qed. |

3.6.10 Remark. *From the definition of the verge module to be spanned by all patterns with fixed verge we get immediately:*

$$\dim \mathbb{C}\mathcal{V}(A) = q^{\#\{(i,j)\in\nabla \mid (i,j) \text{ is left of some main condition of } A\}}.$$

3.6.11 Proposition. *Let $(i,j) \in \nabla$ and $a \in \mathbf{F}_q^\times$ be arbitrary. Then Lemma 3.6.8 implies, that $\varphi(u) = \vartheta(au_{ij})$ defines a linear character of $U_{(i,j)}$ and we have*

$$\operatorname{Ind}_{U_{(i,j)}}^U \varphi \;=\; \Psi_{\mathcal{V}(A)}, \qquad \textit{where } A = ae_{ij}.$$

For the proof of this proposition we need the following lemma.

3.6.12 Lemma. *Let $(i,j) \in \nabla$, $a \in \mathbf{F}_q^\times$ and C be a a core pattern with verge ae_{ij}.*[73] *Then $\mathbb{C}\mathcal{O}_U(C)$ is an irreducible module $\mathbb{C}U$-module, whose character is denoted by ψ_C and we have*

$$\varphi | \operatorname{Res}_{U_{(i,j)}}^U \psi_C,$$

where φ is the linear character of $U_{(i,j)}$ defined by $\varphi(u) = \vartheta(au_{ij})$.

Proof. Note, $C = ae_{ij}$ if $j \le n$ and $C = be_{i\bar{j}} + ae_{ij}$ (for some $b \in \mathbf{F}_q$) if $j > n$, since $\operatorname{minor}(C) = \{(i,\bar{j})\}$ and $\operatorname{suppl}(C) = \emptyset$.

Step 1: *We have $B_{ir} = 0$ for every $B \in \mathcal{O}_{U_{(i,j)}}(C) := \{C.u \mid u \in U_{(i,j)}\}$ and $n < r < j$.*

Note, that $n < r < j$ is obviously only possible in case $n < j$. Let $B \in \mathcal{O}_{U_{(i,j)}}(C)$ be arbitrary. Then we have $u \in U$, such that $B = C.u$. Hence

$$B_{ir} = (Cu^{-\top})_{ir} = \sum_s C_{is}(u^{-\top})_{sr} = a(u^{-\top})_{jr} + b(u^{-\top})_{\bar{j}r} \overset{u^{-1}=u^R}{=} au_{\bar{j}\bar{r}} + bu_{j\bar{r}}.$$

Now $n < r < j$ implies $\bar{j} < \bar{r} \le n < j = \bar{\bar{j}}$. Hence we have $(\bar{j},\bar{r}) \in \nabla \setminus J_{(i,j)}$, but then Theorem 1.2.22 implies $u_{\bar{j}\bar{r}} = 0$. In particular we deduce $\bar{r} < j$ from the second

[73] i.e. $C = ae_{ij}$ if $j \le n$ and $C = be_{i\bar{j}} + ae_{ij}$ for some $b \in \mathbf{F}_q$ if $j > n$.

inequality. But this implies $u_{j\bar{r}} = 0$, since u is an upper triangular matrix.

Step 2: *We have $\varphi(u) = \chi_B(u)$ for all $u \in U_{(i,j)}$ and $B \in \mathcal{O}_{U_{(i,j)}}(C)$.*

Clearly $\operatorname{supp}(B) \subseteq i$-th row, due to Corollary 3.1.16. Hence we have

$$\begin{aligned}\kappa\big(B, f(u)\big) &= \sum_{(i,r)\in\nabla} B_{ir} \underbrace{f(u)_{ir}}_{=u_{ir}} = \sum_{i<r<\bar{i}} B_{ir} u_{ir} \overset{(i,j)\in\operatorname{main}(B)}{=} \sum_{i<r\leq j} B_{ir} u_{ir} \\ &\overset{\text{Step 1}}{=} B_{ij}u_{ij} + \sum_{i<r<\min\{j,n+1\}} B_{ir} \underbrace{u_{ir}}_{=0,\text{ since } (i,r)\in\nabla\smallsetminus J_{(i,j)}} = B_{ij}u_{ij} = a u_{ij}.\end{aligned}$$

Applying ϑ on both sides of the equation proves the step.

Finishing the proof: Let us consider the element $x := \sum_{B\in\mathcal{O}_{U_{(i,j)}}(C)}[B]$. We show that $\mathbb{C}x$ is a $\mathbb{C}U_{(i,j)}$-submodule of $\mathbb{C}\mathcal{O}_U(C)$ affording the character φ.

Let $u \in U_{(i,j)}$ be arbitrary. Then we have

$$xu = \sum_{B\in\mathcal{O}_{U_{(i,j)}}(C)} \underbrace{\chi_{B.u}(u)}_{=\varphi(u)}[B.u] = \varphi(u) \sum_{B\in\mathcal{O}_{U_{(i,j)}}(C)} [B.u] = \varphi(u)x.$$

Hence we get $\varphi | \operatorname{Res}^U_{U_{(i,j)}} \psi_C$. It remains to check, that $\mathbb{C}\mathcal{O}_U(C)$ is irreducible, but that follows obviously from Remark 3.3.52. qed. |

Now we can give the

Proof of Proposition 3.6.11. Due to Corollary 3.3.31 we have

$$\Psi_{\mathcal{V}(A)} = \sum_{\substack{C \text{ core pattern} \\ \operatorname{verge}(C)=\operatorname{verge}(A)}} \psi_C,$$

where ψ_C denotes the character of the U-orbit module $\mathbb{C}\mathcal{O}_U(C)$. Note that by Remark 3.3.52 all ψ_C are pairwise orthogonal irreducible characters of U. By Lemma 3.6.12 we have $\varphi | \operatorname{Res}^U_{U_{(i,j)}} \psi_C$ and hence $\psi_C | \operatorname{Ind}^U_{U_{(i,j)}} \varphi$ by Frobenius reciprocity. Since the ψ_C's are pairwise orthogonal, the character $\Psi_{\mathcal{V}(A)} = \sum_C \psi_C$ is a constituent of $\operatorname{Ind}^U_{U_{(i,j)}} \varphi$. We show equality by checking, that the characters $\operatorname{Ind}^U_{U_{(i,j)}} \varphi$ and $\Psi_{\mathcal{V}(A)}$ have the same dimension. Using Remark 3.6.10 we get $\dim(\Psi_{\mathcal{V}(A)}) = q^{\#\{r \mid i<r<j\}}$. On the other hand we have

$$\dim \operatorname{Ind}^U_{U_{(i,j)}} \varphi = \frac{|U|}{|U_{(i,j)}|} = \begin{cases} q^{\#\{r \mid i<r<j\}} & \text{if } j \leq n, \\ q^{\#\{r \mid i<r\leq n\}+\#\{s \mid \bar{j}<s\leq n\}} & \text{if } j > n. \end{cases}$$

But since $\{s \mid \bar{j} < s \leq n\} = \{s \mid n+1 \leq \bar{s} < j\}$ the result follows. qed. |

3.6.13 Definition (André-Neto supercharacters, [AN2, page 1278])**.** *Let A be an André-Neto pattern. Then we define the* ***André-Neto supercharacter*** Ψ_A^{AN} *corresponding to A as*

$$\Psi_A^{\mathrm{AN}} = \prod_{(i,j)\in\mathrm{main}(A)} \mathrm{Ind}_{U_{(i,j)}}^{U} \varphi_{(i,j),A_{ij}}$$

where $\varphi_{(i,j),A_{ij}}$ is the linear character of $U_{(i,j)}$ given by $\varphi_{(i,j),A_{ij}}(u) := \vartheta(A_{ij}u_{ij})$.[74]

Note, that C.A.M André and A.M. Neto use for their definition the notion of 'basic pairs' instead of (what we call) André-Neto patterns, c.f. Remark 3.5.35 to see how to translate between both notions.

3.6.14 Remark. *According to Proposition 3.6.11 we have*

$$\Psi_A^{\mathrm{AN}} \;=\; \textit{character of the } \mathbb{C}U\textit{-module} \bigotimes_{(i,j)\in\mathrm{main}(A)} \mathbb{C}\mathcal{V}(A_{ij}e_{ij}).$$

Is there a nicer description for this tensor product in therms of verge modules? Yes, there is one, and in fact a very simple description. To give the final description in terms of verge modules we need the following Lemma.

3.6.15 Lemma (essentially [Yan2, Corollary 6.2])**.** *Let A and B be pattens, such that there is no row in which lies a nonzero entry of A and one of B.*[75] *Then we have*

$$\mathbb{C}\mathcal{V}(A+B) \cong \mathbb{C}\mathcal{V}(A) \otimes \mathbb{C}\mathcal{V}(B).$$

Reminder: The '+' between $A+B$ is matrix addition.

Proof. Our assumption implies the map

$$`+' :\ \mathcal{V}(A)\times\mathcal{V}(B) \longrightarrow \mathcal{V}(A+B) :\ (C,D) \longmapsto C+D$$

to be bijective. Since $\{[C]\otimes[D] \mid C\in\mathcal{V}(A), D\in\mathcal{V}B\}$ forms a $\mathbb{C}$-basis of $\mathbb{C}\mathcal{V}(A)\otimes\mathbb{C}\mathcal{V}(B)$ we get a $\mathbb{C}$-vectorspace isomorphism

$$t :\ \mathbb{C}\mathcal{V}(A)\otimes\mathbb{C}\mathcal{V}(B) \longrightarrow \mathbb{C}\mathcal{V}(A+B) :\ [C]\otimes[D] \longmapsto [C+D].$$

It remains to check that t is a $\mathbb{C}U$-module homomorphism, but this follows directly, since truncated column operation only affects rows with nonzero entries.[76] qed. |

[74] Lemma 3.6.8 ensures, that $\varphi_{(i,j),A_{ij}}$ is indeed a character. C.A.M. André and A.M. Neto call the characters $\mathrm{Ind}_{U_{(i,j)}}^{U} \varphi_{(i,j),A_{ij}}$ **elementary characters**.

[75] i.e. we have $\mathrm{supp}(A)\cap\mathrm{supp}(B)\cap i$-th row $=\emptyset$ for all i.

[76] Let $C\in\mathcal{V}(A), D\in\mathcal{V}(B)$, and $u\in U$. Due to Proposition 2.1.29 we have $C.u+D.u=(C+D).u$. Hence

$$\begin{aligned} t\big(([C]\otimes[D])u\big) &= \chi_{C.u}(u)\chi_{D.u}(u)t\big([C.u]\otimes[D.u]\big) = \chi_{C.u+D.u}(u)[C.u+D.u] \\ &= \chi_{(C+D).u}(u)[(C+D).u] = [C+D]u = t\big([C]\otimes[D]\big)u. \end{aligned}$$

3.6.16 Proposition (André-Neto supercharacters as characters of verge modules)**.** *Let A be an André-Neto pattern. Then*

$$\Psi_A^{\mathrm{AN}} \;=\; \Psi_{\mathcal{V}(A)} \;=\; \textit{character of the } \mathbb{C}U\textit{-module } \;\mathbb{C}\mathcal{V}(A).$$

Proof. We have $A = \sum_{(i,j)\in \mathrm{main}(A)} A_{ij}e_{ij}$. Hence the result follows from Lemma 3.6.15 using the point of view from Remark 3.6.14. qed. |

3.6.17 Remark. *Corollary 3.4.14 gives us another interpretation of the André-Neto supercharacters. We have*

$$\Psi_A^{\mathrm{AN}} \;=\; \mathrm{Res}_U^G \,\Psi_A^{\mathrm{AY}},$$

where Ψ_A^{AY} is the character of the module $\mathbb{C}\mathcal{O}_G(A)$. Since A is a staircase pattern this is one of N. Yan's supercharacters, c.f. Theorem 2.2.25 and Notation 2.2.27.

Now we have defined AN-supercharacters and AN-superclasses. C.A.M. André and A.M. Neto proved in their series of papers [AN1], [AN2] and [AN3] that the pair of AN-supercharacters and AN-superclasses form indeed a supercharacter theory in the sense of Definition 3.6.2. We want to re-prove this result using our own theory. The description of AN-supercharacters as characters of verge modules allows us to apply our results.

3.6.18 Proposition. *The verge characters are André-Neto superclass functions.*

Structural proof. From Corollary 3.4.14 we see that the verge characters arise as restrictions of characters of $A_N(\mathbf{F}_q)$-orbit nodules from $A_N(\mathbf{F}_q)$ to $D_n(\mathbf{F}_q)$. On the other hand the AN-superclasses are just the restrictions of the superclasses of $A_N(\mathbf{F}_q)$ to $D_n(\mathbf{F}_q)$.[77] Hence the result follows from Corollary 2.7 of [Yan2]. qed. |

In order to keep this thesis elementary and self contained (as promised) we also give an elementary proof.[78]

Elementary proof. Suppose $x, y \in U$ lie in the same AN-superclass. Then

$$\text{there exist } g, h \in G \text{, such that } \quad y = g(x-1)h + 1 = gxh - gh + 1.$$

Step 1: *For every pattern B we have*

$$B.y = B \quad\Longleftrightarrow\quad (B.h^{-1}).x = (B.h^{-1}).$$

Since

$$\pi(By^\top) \overset{\pi \text{ is } \mathbf{F}_q\text{-linear}}{=} \pi\big(B(gxh)^\top\big) - \pi\big(B(gh)^\top\big) + \pi(B),$$

[77] c.f. Lemma 2.2.28 and the ‘comparison’ part of Definition 2.2.30.

[78] The proof remains true if $\pi: \;V_0 \longrightarrow V_\triangleleft$, $f = \pi|_G$ and $\mathcal{V}(A)$ is substituted by $\mathcal{O}_G(A)$. Hence this in fact the proof that the characters of G-orbit modules of G are superclass functions, where by definition x, y lie in the same superclass if $x-1$ and $y-1$ lie in the same G-G-biorbit. This is not very surprising, as the ‘structural proof’ shows.

we obtain

$$B.y^{-1} = B \quad \Longleftrightarrow \quad B.(gxh)^{-1} = B.(gh)^{-1}.$$

Which is equivalent to the desired result.

Step 2: *For every pattern B we have*

$$\chi_B(y) = \chi_{B.h^{-1}}(x) \qquad \text{if } (B.h^{-1}).x = (B.h^{-1}).$$

Let B satisfy $(B.h^{-1}).x = B.h^{-1}$. We have

(1) $$f(y) = \pi(y) = \pi(gxh) - \pi(gx) + \underbrace{\pi(1)}_{=0} = \underbrace{f(gxh) - f(gh)}_{= f(gx)\circ h + f(h) - f(g)\circ h - f(h)} = \big(f(gx) - f(g)\big) \circ h,$$

(2) $$\begin{aligned}\chi_{B.h^{-1}}(gx) &= \vartheta\kappa\big(B.h^{-1}, f(gx)\big) = \vartheta\kappa\big(B.h^{-1}, f(g)\circ x + f(g)\big)\\ &= \vartheta\kappa\big(\underbrace{(B.h^{-1}).x^{-1}}_{=B.h^{-1}}, f(g)\big)\vartheta\kappa\big(B.h^{-1}, f(x)\big) = \chi_{B.h^{-1}}(g)\chi_{B.h^{-1}}(x).\end{aligned}$$

Using (1) and (2) we get

$$\chi_B(y) = \vartheta\kappa\big(B, f(y)\big) \overset{(1)}{=} \vartheta\kappa\big(B.h^{-1}, f(gx) - f(g)\big) = \chi_{B.h^{-1}}(gx)\chi_{B.h^{-1}}(g)^{-1} \overset{(2)}{=} \chi_{B.h^{-1}}(x).$$

Finishing the proof: We have

$$\Psi_{\mathcal{V}(A)}(y) \;=\; \sum_{\substack{B\in\mathcal{V}(A)\\ B.y=B}} \chi_B(y) \;=\; \sum_{\substack{B\in\mathcal{V}(A)\\ (B.h^{-1}).x=(B.h^{-1})}} \chi_{B.h^{-1}}(x) \;=\; \sum_{\substack{C\in\mathcal{V}(A)\\ C.x=C}} \chi_C(x) \;=\; \Psi_{\mathcal{V}(A)}(x).$$

Hence the result is proven. qed.

3.6.19 Corollary (mainly [AN2, Theorem 4.1]). *The André-Neto supercharacters from an orthogonal basis of the $\mathbb{C}$-vectorspace of AN-superclass functions.*[79]

Proof. Corollaries 3.3.31 and 3.3.42 imply the orthogonality of pairwise different AN-supercharacters. In particular we get, that the AN-supercharacters are linearly independent. Due to Corollary 3.5.34 the $\mathbb{C}$-vector space of AN-superclass functions has dimension equal to the number of André-Neto patterns, which is also equal to he number of André-Neto supercharacters. qed.

Now we can re-prove the main theorem of C.A.M. André and A.M. Neto:

3.6.20 Theorem (Main result from [AN2]). *The André-Neto supercharacters and the André-Neto superclasses form a supercharacter theory.*[80]

[79] We view the space of AN-superclss functions as a subspace of the space of class functions equipped with the usual scalar product.

[80] In the symbols used in Definition/Lemma 3.6.2 we take as $\mathcal{X}$ the set of AN-supercharacters and as $\mathcal{K}$ the set of AN-superclasses.

Proof. Corollary 3.6.19 is in fact a reformulation of the properties of Definition 3.6.1, hence the AN-supercharacters and AN-superclasses form a pre-supercharacter theory for $D_n(\mathbf{F}_q)$. It remains to check, that one of the equivalent statements of Definition/Lemma 3.6.2 holds. Of course $\{1\}$ is an AN-superclass. Hence the result is proven. Alternatively one could also use Corollary* 3.3.19 combined with Corollary 3.3.31.[81] qed. |

We want to remind, that in our theory verge modules are modules, which we build as direct sums of U-orbit modules. Now, that we have established the importance of (a subclass) of verge modules we can use our building of verge modules to *decompose* André-Neto supercharacters into U-orbit modules. And our theory also states, that this decomposition has strong properties.

3.6.21 Theorem (Main Theorem of Thesis). *Let A be an André-Neto pattern. Then the supercharacter Ψ_A^{AN} decomposes according to*

$$\Psi_A^{\mathrm{AN}} = \sum_{\substack{C \text{ core pattern, such that} \\ \mathrm{verge}(C)=\mathrm{verge}(A)}} \psi_C$$

where ψ_C denotes the character of $\mathbb{C}\mathcal{O}_U(C)$.

Furthermore two of these constituents ψ_C, ψ_D are either equal or orthogonal.

Proof. We only need to combine the Corollaries 3.3.31 and 3.3.43. qed. |

We want to round this section off with a discussion of the strengths and weaknesses of our main theorem. Hence we have included a large collection of examples into this thesis, c.f. Appendix A.2 and A.3.

3.6.22 Remark. *To Theorem 3.6.21.*

1. *The decomposition of the André-Neto supercharacters is purely combinatorial. One has just to write down the minor conditions and supplementary conditions for a set of given main conditions. This can be done directly in the picture, c.f. Examples 3.2.31 and A.2.1.*

2. *From the same picture we can read of the dimension of the constituents (which have all equal dimension). We also illustrate this in Examples 3.2.31 and A.2.1.*

3. *If the main conditions are heading from bottom left to upper right[82] as it is the case in Example A.2.1. Then the constituents ψ_C are even pairwise orthogonal irreducible characters.*

[81] by turning the thing on its head we get a new proof for Corollary* 3.3.19 using the properties of a supercharactertheory and the fact, that $\{1\}$ is an AN-superclass.

[82] c.f. Remark 3.3.51 for a more formal description.

4. *The decomposition provided by our main theorem is not always into irreducible pairwise orthogonal characters, c.f. Example A.2.7.*

5. *Corollary 3.3.50 can be seen as a measure for the difficulty of the problem of further decomposing the supercharacters. We understand the cases* $|E(A)| = 0, 1$ *relatively easy (Proposition A.2 is very helpful). The other cases require further investigation.*

6. *If the matrix size is small, then the 'combinatoric collapses', meaning that we can control the decomposition of orbit modules into irreducible characters. We will do this in order count the number conjugacy classes for* $D_4(\mathbf{F}_q)$ *in Appendix A.3.*

What is next? A glimpse into the future

First, it is noteworthy that the 1-cocycle construction of Section 2.1 can also be used in another context: Let G denote $\mathrm{GL}_n(\mathbf{F}_q)$, $P = P_\lambda$ the parabolic subgroup of G corresponding to the partition $\lambda = (m, n-m)$, and U the group of lower triangular matrices over $\mathbf{F}_q$. In her PhD thesis ([Guo1],c.f. also [Guo2]) Q. Guo considers the U-module structure of the permutation module G/P. She first decomposes $\mathrm{Res}_U^G G/P$ into so-called $\mathfrak{t}$-batches $\mathfrak{M}_\mathfrak{t}$ (which are U-modules) using Mackey decomposition. Thereby $\mathfrak{t}$ runs over all possible row standard $(m, n-m)$-tableaux. In a second, but crucial step, she constructs a monomial basis for $\mathrm{Res}_{U_\mathfrak{t}}^U \mathfrak{M}_\mathfrak{t}$, where $U_\mathfrak{t}$ denotes suitably constructed subgroup of U. In a third step then she shows that every $U_\mathfrak{t}$-orbit module of $\mathrm{Res}_{U_\mathfrak{t}}^U \mathfrak{M}_\mathfrak{t}$ is an irreducible U_t-module. In a forth step she shows that all those U_t-orbit modules are in fact a U-modules, thus gaining a complete description of the U-module structure of $\mathrm{Res}_U^G G/P$. Now, it turned out that her second step (the construction of the monomial $U_\mathfrak{t}$-module) can be understood perfectly in the framework of Section 2.1, leading to a more structural understanding of her construction.

While writing this thesis it turned out that most of the results for type D can be generalized to other series of classical groups (more precisely types $B_n, C_n, {}^2A_n, {}^2D_n$.) Together with type D_n from this thesis and type A_n from N. Yan's work this would imply some common theory for all the classical types. Up to date the following is known:

- For the remaining untwisted types the results of this thesis can be completely generalized. The needed combinatorics differs slightly but everything (including the decomposition of André-Neto supercharacters) carries over. This part will be published as soon as possible.

- For the twisted groups up to date there is no non-trivial supercharacter theory known. The methods developed in this thesis can be applied for these types of groups. There is already some 1-cocylce constructed and we have a classification of staircase modules and some further pieces. Everything so far (the results already obtained and the made calculations for the remaining parts, including the superclass side) suggest an André-Neto type supercharacter theory.

Another question is, if the isomoprhism classes of hook-separated staircase modules (recall, two of them are either isomorphic or orthogonal) form the character theoretical side of a supercharacter theory. In order to answer this question one would first have to find a classification of these isomorphism classes. Afterwards it remains to check if there is some suitable superclass-notion. This is yet open.

A. Appendix

A.1. Root subgroup structure of $A_N(\mathbf{F}_q)$

In the proof of Theorem 1.2.22 we need a result of the structure theory of $A_N(\mathbf{F}_q)$ not covered in Chapter 2 of [AB], which we assume the reader to know. Nevertheless it is a very basic fact of the structure of $A_N(\mathbf{F}_q)$. We are confronted with the demands of keeping the text straight and elementary, which contradicts itself a bit. So we decided to keep the next proposition out of the main text. In this appendix we give an elementary proof for the result. By doing this we keep the promise, that the prior knowledge of $A_N(\mathbf{F}_q)$ in [AB, Chapter 2] suffices and all our results can be obtained are completely elementary. Especially, we don't need the general theory of Chevalley groups (of e.g. [Ste]).

A.1.1 Proposition. *For $(i,j) \in \nabla$ and $\lambda \in \mathbf{F}_q$, we define $\tilde{x}_{ij}(\lambda) := 1 + \lambda e_{ij}$. We have*

$$A_N(\mathbf{F}_q) = \Big\{ \prod_{(i,j)\in\nabla} \tilde{x}_{ij}(\lambda_{ij}) \;\Big|\; \lambda_{ij} \in \mathbf{F}_q \Big\},$$

where the product is taken in an arbitrary, but fixed order. In particular, we have

$$\prod_{(i,j)\in\nabla} \tilde{x}_{ij}(\lambda_{ij}) = \prod_{(i,j)\in\nabla} \tilde{x}_{ij}(\mu_{ij}) \qquad \Longleftrightarrow \qquad \lambda_{ij} = \mu_{ij} \text{ for all } (i,j) \in \nabla.$$

Proof. We have $|A_N(\mathbf{F}_q)| = q^{|\nabla|}$. On the other hand the set $\{\prod \tilde{x}_{ij}(\lambda_{ij}) \mid \lambda_{ij} \in \mathbf{F}_q\} \subseteq A_N(\mathbf{F}_q)$ has at most $q^{|\nabla|}$ elements. Therefore, if we can show that any $g \in A_N(\mathbf{F}_q)$ equals some $\prod \tilde{x}_{ij}(\lambda_{ij})$, then we also get the second statement of the proposition.

We denote the order in which the product is taken with $\preccurlyeq$. It remains to show that for every $g_{ab} \in \mathbf{F}_q$, where (a,b) runs through ∇, there exists a solution $(\lambda_{ij})_{(i,j)\in\nabla}$ of the following system of equations:

$$g_{ab} = \Big[\prod_{(i,j)\in\nabla} \tilde{x}_{ij}(\lambda_{ij}) \Big]_{ab} \overset{(*)}{=} \lambda_{ab} \;+ \sum_{\substack{(i_1,j_1)\preccurlyeq(i_2,j_2)\preccurlyeq\cdots\preccurlyeq(i_\ell,j_\ell)\\ a=i_1,j_1=i_2,\ldots,j_{\ell-1}=i_\ell,j_\ell=b}} \lambda_{i_1 j_1} \cdot \ldots \cdot \lambda_{i_\ell j_\ell}.$$

The equation $(*)$ holds, since $e_{ab}e_{cd} = \delta_{bc}e_{ad}$. But, since all $(i_r, j_r) \in \nabla$ satisfy $i_r < j_r$, we get $a = i_1 < i_2 < \ldots i_\ell$ and $b = j_\ell > j_{\ell-1} > \cdots > j_1$. Thus all λ_{i_r,j_r} in the sum have coordinates on the left and below of (a,b). Hence we can solve the system of equations uniquely, by first solving the equations, where $b = a+1$, then the ones, where $b = a+2$, and so on. qed.

A.2. Examples of decompositions of AN-supercharacters into Irreducibles

A.2.1 Example. *Let A be the hook-separated verge pattern of $\in D_7(\mathbf{F}_q)$ given by the following pattern*

$$A = \left(\begin{array}{ccccccc|ccccccc}
\cdot & 0_1 & 0_2 & 0_3 & 0_4 & 0_5 & 0_6 & 0_7 & 0_8 & A_{1,10} & & & & \cdot \\
 & \cdot & 0_9 & 0_{10} & 0_{11} & 0_{12} & 0_{13} & 0_{14} & A_{2,9} & & & & \cdot & \\
 & & \cdot & 0_{15} & 0_{16} & 0_{17} & 0_{18} & A_{3,8} & & & & \cdot & & \\
 & & & \cdot & 0_{19} & 0_{20} & A_{4,7} & & & & \cdot & & & \\
 & & & & \cdot & & & & & \cdot & & & & \\
 & & & & & \cdot & & & \cdot & & & & & \\
 & & & & & & \cdot & \cdot & & & & & &
\end{array}\right).$$

Entries denoted with '$\cdot$', '0_i', 0 or whose positions are left empty denote zero values. The '$\cdot$'s are only there to indicate the $\triangledown$-shape of the pattern, whereas we write the zeros, since we get the dimension of $\mathbb{C}\mathcal{V}(A)$ as $q^{\text{\# of black 0's}}$, hence $\dim \mathbb{C}\mathcal{V}(A) = q^{20}$. Therefore the André-Neto supercharacter corresponding to the verge pattern A has dimension q^{20}. The $|$ is there for indicating the symmetry axis of the shape $\triangledown$ in the pattern A.

Clearly A is hook-separated and we have $\operatorname{main}(A) = \{(1,10),(2,9),(3,8),(4,7)\}$. The first step of our combinatorial algorithm for decomposing a verge pattern into core patterns (and hence decomposing an André-Neto supercharacter into characters of U-orbit modules) is to mirror right main conditions on the symmetry axis $|$. This leads to the minor conditions, which are in our example $\operatorname{minor}(A) = \{(1,5),(2,6),(3,7)\}$. Then write arbitrary elements of the field $\mathbf{F}_q$ at these mirrored positions. We get

$$\left(\begin{array}{ccccccc|ccccccc}
\cdot & 0 & 0 & 0 & C_{1,5} & 0 & 0 & 0 & 0 & A_{1,10} & & & & \cdot \\
 & \cdot & 0 & 0 & 0 & C_{2,6} & 0 & 0 & A_{2,9} & & & & \cdot & \\
 & & \cdot & 0 & 0 & 0 & C_{3,7} & A_{3,8} & & & & \cdot & & \\
 & & & \cdot & 0 & 0 & A_{4,7} & & & & \cdot & & & \\
 & & & & \cdot & & & & & \cdot & & & & \\
 & & & & & \cdot & & & \cdot & & & & & \\
 & & & & & & \cdot & \cdot & & & & & &
\end{array}\right),$$

where $C_{1,5}, C_{2,6}$ and $C_{3,7}$ are arbitrary field elements. It remains to determine the supplementary conditions. This can be obtained from the picture by going through columns containing a minor condition. Say j is such a row, then (i,j) is a supplementary condition if in the i-th row on the right of (i,j) lies a minor condition or a left main condition. Hence for our example $\operatorname{suppl}(A) = \{(2,5),(3,5),(3,6),(4,5),(4,6)\}$. On every matrix position, that is a supplementary condition, we write an arbitrary field element. Having *done this, the patterns*

$$C = \left(\begin{array}{ccccccc|ccccccc}
\cdot & 0_1 & 0_2 & 0_3 & C_{1,5} & 0_4 & 0_5 & 0_6 & 0_7 & A_{1,10} & & & & \cdot \\
 & \cdot & 0_8 & 0_9 & C_{2,5} & C_{2,6} & 0_{10} & 0_{11} & A_{2,9} & & & & \cdot & \\
 & & \cdot & 0_{12} & C_{3,5} & C_{3,6} & C_{3,7} & A_{3,8} & & & & \cdot & & \\
 & & & \cdot & C_{4,5} & C_{4,6} & A_{4,7} & & & & \cdot & & & \\
 & & & & \cdot & & & & & \cdot & & & & \\
 & & & & & \cdot & & & \cdot & & & & & \\
 & & & & & & \cdot & \cdot & & & & & &
\end{array}\right)$$

are the core patterns satisfying $\operatorname{verge}(C) = \operatorname{verge}(A)$, *where* $C_{1,5}$, $C_{2,6}$, $C_{3,7}$, $C_{2,5}$, $C_{3,5}$, $C_{3,6}$, $C_{4,5}$ *and* $C_{4,6}$ *are arbitrary field elements. Hence the formula*

$$\Psi_A^{\mathrm{AN}} = \sum_{\substack{C \text{ core pattern, such that} \\ \operatorname{verge}(C)=\operatorname{verge}(A)}} \psi_C$$

gets a concrete interpretation for the example. By counting the remaining 0*'s we see, that each* ψ_C *has dimension* q^{12} *and Remark 3.3.52 implies the* ψ_C*'s to be pairwise orthogonal and irreducible. Putting this into words, we have decomposed the André-Neto supercharacter* Ψ_A^{AN}, *which is a character of dimension* q^{20} *into* q^8 *many pairwise orthogonal, irreducible characters of dimension* q^{12}.

Of course the decomposition of André-Neto supercharacters is not always as pleasant as in the last example. The example was of course chosen carefully to show the situation where our main theorem provides the most impact. Therefore we give an example, where the decomposition is not into pairwise orthogonal irreducible U-orbit modules. But first we make the following observation.

A.2.2 Proposition. *Let* A *be a hook-separated verge pattern. Let* $C \in \mathcal{V}(A)$ *be a core pattern, such that there exist at least* $q^{|E(A)|}$ *core patterns* $D \in \mathcal{V}(A)$ *satisfying* $\mathbb{C}\mathcal{O}_U(C) \cong \mathbb{C}\mathcal{O}_U(D)$.

Then the U*-orbit module* $\mathbb{C}\mathcal{O}_U(C)$ *is irreducible.*

Proof. Let $\mathcal{C} := \{D \in \mathcal{V}(A) \mid D \text{ core pattern and } \mathbb{C}\mathcal{O}_U(D) \cong \mathbb{C}\mathcal{O}_U(C)\}$. By assumption we have $|\mathcal{C}| \geq q^{|E(A)|}$. Since $\bigoplus_{D \in \mathcal{C}} \mathbb{C}\mathcal{O}_U(D) \leq \mathbb{C}\mathcal{V}(A)$ we get

$$\begin{aligned}
q^{|E(A)|} &\overset{\text{Cor. 3.3.50}}{=} \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C), \mathbb{C}\mathcal{V}(A)\right) \\
&\overset{\text{Cor. 3.3.43}}{=} \sum_{D \in \mathcal{C}} \dim_{\mathbb{C}} \underbrace{\operatorname{Hom}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C), \mathbb{C}\mathcal{O}_U(D)\right)}_{\cong \operatorname{End}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U(C)),\text{ since } \mathbb{C}\mathcal{O}_U(C) \cong \mathbb{C}\mathcal{O}_U(D)} \geq q^{|E(A)|} \dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C)\right).
\end{aligned}$$

But this implies $\dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C)\right) = 1$, hence $\mathbb{C}\mathcal{O}_U(C)$ is irreducible. qed. |

Obviously we can derive from the proof of Proposition A.2.2 also

A.2.3 Corollary. *Let A be a hook-separated verge pattern and $C \in \mathcal{V}(A)$ a core pattern. We define*

$$\mathcal{C}_{\mathcal{V}(A)}(C) := \{D \in \mathcal{V}(A) \mid D \text{ core pattern and } \mathbb{C}\mathcal{O}_U(D) \cong \mathbb{C}\mathcal{O}_U(C)\}.$$

Suppose $|\mathcal{C}_{\mathcal{V}(A)}(C)| = q^a \leq q^{|E(A)|}$ for some positive integer a. Then we have

$$\dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C)\right) = q^{|E(A)|-a}.$$

We can use our observation to decompose the following verge module:

A.2.4 Example. *Let $A = A_{1,5}e_{1,5} + A_{2,10}e_{2,10} + A_{3,7}e_{3,7} + A_{4,6}e_{4,6}$. Clearly A is a hook-separated verge pattern. We determine the core patterns C with verge A as in Example A.2.1:*

$$C = \left(\begin{array}{ccccccc|ccccccc}
\cdot & 0_1 & 0_2 & 0_3 & A_{1,5} & & & & & & & & & \cdot \\
 & \cdot & 0_4 & 0_5 & C_{2,5} & 0_6 & 0_7 & 0_8 & 0_9 & A_{2,10} & & & \cdot & \\
 & & \cdot & 0_{10} & C_{3,5} & 0_{11} & A_{3,7} & & & & & \cdot & & \\
 & & & \cdot & C_{4,5} & A_{4,6} & & & & & \cdot & & & \\
 & & & & \cdot & & & & & \cdot & & & & \\
 & & & & & \cdot & & & \cdot & & & & & \\
 & & & & & & \cdot & \cdot & & & & & &
\end{array}\right),$$

where $C_{2,5}, C_{3,5}$ and $C_{4,5}$ are arbitrary elements of $\mathbf{F}_q$. It is easy to determine from the picture that we have $E(A) = \{(1,2),(1,3),(1,4)\}$. Corollary 3.3.14 implies, that $\mathbb{C}\mathcal{O}_U(C) \cong \mathbb{C}\mathcal{O}_U(A)$, as we can use the first row to delete the entries $C_{2,5}, C_{3,5}$ and $C_{4,5}$. Hence Proposition A.2.2 implies that $\mathbb{C}\mathcal{O}_U(A)$ is irreducible. We summarize our observations:

The André-Neto supercharacter Ψ_A^{AN}, which is a character of dimension q^{14}, decomposes into q^3 many pairwise isomorphic, irreducible characters of dimension q^{11}.

The isomorphisms between the U-orbit modules of the last example are the easiest possible. We can extend our observations to:

A.2.5 Proposition. *Suppose A is a hook-separated verge pattern. Let*

$$\operatorname{del}(A) := \left\{ (i,j) \in \nabla \;\middle|\; \begin{array}{c} (i,j) \text{ is below of some left main condition} \\ \text{and itself a minor or supllementary condition.} \end{array} \right\}$$

denote the set of ***deletable positions***. *Then*

$$\Psi_A^{\mathrm{AN}} = q^{\#\operatorname{del}(A)} \sum_{\substack{\text{core patterns } C \\ \operatorname{verge}(C)=\operatorname{verge}(A) \\ \operatorname{supp}(C)\cap\operatorname{del}(\mathrm{A})=\emptyset}} \psi_C.$$

Hence $\Psi := q^{-\#\,\mathrm{del}(A)}\Psi_A^{\mathrm{AN}}$ *is the character of some module* M *and we have*

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(C)\big) \leq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(C), M\big) = q^{|E(A)|-\#\,\mathrm{del}(A)}$$

for every core pattern C *of verge* A.

Orbit modules are not irreducible in general. For U-orbit modules, having Endomorphism rings of dimension q, we understand the decomposition into irreducible modules:

A.2.6 Proposition. *Suppose* M *is a* $\mathbb{C}U$*-module, such that*

$$\dim_{\mathbb{C}}(M) = q^a \qquad \textit{and} \qquad \dim_{\mathbb{C}}\big(\mathrm{End}_{\mathbb{C}U}(\mathrm{M})\big) = q,$$

where a *is some positive integer. Then* M *decomposes as direct sum of* q *irreducible and pairwise orthogonal* $\mathbb{C}U$*-modules of dimension* q^{a-1}.

Proof. Due to [And8, Theorem 1.3][83] all irreducible $\mathbb{C}U$-modules S have dimension q^b for some positive integer b. The module M is not irreducible, since its endomorphism ring is of dimension q. Thus every irreducible constituent of M has dimension at most q^{a-1}. Hence M has *at least* q irreducible constituents. On the other hand suppose $M = \bigoplus_{i=1}^k S_i$ is the decomposition of M into irreducible constituents. Then

$$q = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}(M) = \sum_{i,j=1}^{k} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}(S_i, S_j) \geq \sum_{i=1}^{k} \underbrace{\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}(S_i)}_{=1} = k.$$

Hence M has *at most* q irreducible constituents, which implies 'M has exactly q irreducible constituents'. Thus we have

$$0 = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}(M) - \#\{\text{irr. const. of } M\} = \sum_{i \neq j} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}(S_i, S_j) \geq 0.$$

And thus the irreducible constituents of M are pairwise orthogonal. qed. |

A.2.7 Example. *Let us consider the hook-separated verge pattern*

$$A = \left(\begin{array}{cccc|cccc} \cdot & 0_1 & A_{1,3} & & & & & \cdot \\ & \cdot & 0_2 & A_{2,4} & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right).$$

Clearly A *is the only core pattern with verge* A, *hence we have* $\mathbb{C}\mathcal{V}(A) = \mathbb{C}\mathcal{O}_U(A)$. *Since* $E(A) = \{(1,2)\}$, *we get*

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(C), \mathbb{C}\mathcal{V}(A)\big) \overset{\text{Cor. 3.3.50}}{=} q^{|E(A)|} = q.$$

Hence $\mathbb{C}\mathcal{O}_U(A)$, *which is of dimension* q^2 *is a direct sum of* q *pairwise orthogonal, irreducible constituents. In particular* $\mathbb{C}\mathcal{O}_U(A)$ *is* ***not*** *irreducible.*

[83] we may apply the Theorem, since we assume p to be odd.

Combining our methods, we also have a bit more complicated situations under control:

A.2.8 Example. *Let us consider the hook-separated verge pattern A and as always determine the core patterns of verge A.*

$$A = \left(\begin{array}{ccccc|ccccc} \cdot & 0_1 & \color{blue}{C_{1,3}} & 0_2 & 0_3 & 0_4 & 0_5 & \color{red}{A_{1,8}} & & \cdot \\ & \cdot & \color{green}{C_{2,3}} & \color{red}{A_{2,4}} & & & & & \cdot & \\ & & \cdot & 0_6 & \color{red}{A_{3,5}} & & & \cdot & & \\ & & & \cdot & \color{blue}{C_{4,5}} & \color{red}{A_{4,6}} & \cdot & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & & & & & \end{array} \right).$$

In this situation $|E(A)| = |\{(2,3),(3,4)\}| = 2$. We want to determine the irreducible constituents. Using Proposition A.2.5 it suffices to consider

$$\left(\begin{array}{ccccc|ccccc} \cdot & 0 & \color{blue}{C_{1,3}} & 0 & 0 & 0 & 0 & \color{red}{A_{1,8}} & & \cdot \\ & \cdot & \color{green}{C_{2,3}} & \color{red}{A_{2,4}} & & & & & \cdot & \\ & & \cdot & 0 & \color{red}{A_{3,5}} & & & \cdot & & \\ & & & \cdot & \text{del.} & \color{red}{A_{4,6}} & \cdot & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & & & & & \end{array} \right),$$

where 'del.' *denotes 0 and indicates, that we do not need to consider that matrix position, even if it is a minor or supplementary condition. Also from the proposition we get*

$$\dim_{CC} \mathrm{End}_{\mathbb{C}U} \left(\mathbb{C}\mathcal{O}_U(C)\right) \leq q^{|E(A)| - \#\,\mathrm{del}} = q^{2-1} = q.$$

On the other hand we know according to Proposition 3.3.49, that the

$$\left\{ \lambda_{x_{23}(\mu)x_{34}(\nu)}|_{\mathbb{C}\mathcal{O}_U(C)} \;\middle|\; \mu,\nu \in \mathbf{F}_q \right\} \quad \textit{form a } \mathbb{C}\textit{-basis of } \mathrm{Hom}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C), \mathbb{C}\mathcal{V}(A)\right).$$

But $x_{2,3}(\mu)C \in \mathbb{C}\mathcal{O}_U(C)$ for every $\mu \in \mathbf{F}_q$. Since $\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C)\right) = q$, we have

$$\left\{ \lambda_{x_{23}(\mu)}|_{\mathbb{C}\mathcal{O}_U(C)} \;\middle|\; \mu \in \mathbf{F}_q \right\} \quad \textit{is a } \mathbb{C}\textit{-basis of } \mathrm{End}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(C)\right).$$

But then we can use Proposition A.2.6 to get, that each $\mathbb{C}\mathcal{O}_U(C)$ decomposes as a direct sum of q pairwise orthogonal, irreducible constituents of dimension q^5. We summarize our results:

The André-Neto supercharacter Ψ_A^{AN}, which is a character of dimension q^9, contains q^3 different types of irreducible characters, each of dimension q^5 and multiplicity q.

There are some André-Neto supercharacters, where we can do a a complete decomposition into irreducible characters, but where it is not simply obtained by looking at the pattern, as it was thus far.

A.2.9 Example. *We consider*

$$\left(\begin{array}{cccc|cccc} \cdot & 0_1 & C_{1,3} & C_{1,4} & A_{1,5} & & & \cdot \\ & \cdot & C_{2,3} & 0_2 & 0_3 & A_{1,6} & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right).$$

We have $E(A) = \{(1,2)\}$. *A basis of* $\mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(C), \mathbb{C}\mathcal{V}(A)\big)$ *is then given by the elements* $\lambda_{x_{12}(\mu)}|_{\mathbb{C}\mathcal{O}_U(C)}$, *where* μ *runs through* $\mathbf{F}_q$. *Using Proposition A.2, we understand that we will be able to decompose the André-Neto supercharacter corresponding to* A *into irreducibles, as soon as we can answer the question '*$x_{12}(\mu)C \in \mathbb{C}\mathcal{O}_U(C)$*?' for all core patterns* C. *In view of Proposition 3.3.26 the question is equivalent to '*$x_{12}(\mu).C \in \mathcal{O}(C)$*?'. We have*

$$x_{12}(\mu).C = \left(\begin{array}{cccc|cccc} \cdot & 0 & C_{1,3} & C_{1,4} & A_{1,5} & & & \cdot \\ & \cdot & C_{2,3} - \mu C_{1,3} & -\mu C_{1,4} & -\mu A_{1,5} & A_{1,6} & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right).$$

In order to answer the question we delete the entries at positions $(2,4)$ *and* $(2,5)$. *We apply* $x_{34}(\mu {A_{1,6}}^{-1} A_{1,5})$ *from the right and get*

$$\left(\begin{array}{cccc|cccc} \cdot & 0 & C_{1,3} - \mu {A_{1,6}}^{-1} A_{1,5} C_{1,4} & C_{1,4} & A_{1,5} & & & \cdot \\ & \cdot & C_{2,3} - \mu C_{1,3} + \mu^2 {A_{1,6}}^{-1} A_{1,5} C_{1,4} & -\mu C_{1,4} & 0 & A_{1,6} & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right).$$

Then we apply $x_{25}(\mu C_{1,4} {A_{1,6}}^{-1})$ *from the right. This yields*

$$\left(\begin{array}{cccc|cccc} \cdot & 0 & C_{1,3} - 2\mu {A_{1,6}}^{-1} A_{1,5} C_{1,4} & C_{1,4} & A_{1,5} & & & \cdot \\ & \cdot & C_{2,3} - \mu C_{1,3} + \mu^2 {A_{1,6}}^{-1} A_{1,5} C_{1,4} & 0 & 0 & A_{1,6} & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right).$$

Hence $x_{12}(\mu).C \in \mathcal{O}C$ *if and only if*

$$C_{1,3} - 2\mu {A_{1,6}}^{-1} A_{1,5} C_{1,4} = C_{1,3} \qquad \text{and} \qquad C_{2,3} - \mu C_{1,3} + \mu^2 {A_{1,6}}^{-1} A_{1,5} C_{1,4} = C_{2,3}$$

which is, after abbreviating $c := {A_{1,6}}^{-1} A_{1,5} \in \mathbf{F}_q^\times$, *equivalent to*

$$c\mu C_{1,4} = 0 \qquad \text{and} \qquad \mu C_{1,3} = c\mu^2 C_{1,4}.$$

Hence we get for any $\mu \in \mathbf{F}_q^\times$

$$x_{12}(\mu).C \in \mathcal{O}(C) \qquad \text{if and only if} \qquad C_{1,4} = C_{1,3} = 0.$$

Hence we have

$$\dim_{\mathbb{C}} \operatorname{End}\left(\mathbb{C}\mathcal{O}_U(C)\right) = \begin{cases} q & \text{if } C_{1,4} = C_{1,3} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Now Proposition A.2 implies, that the André-Neto supercharacter Ψ_A^{AN} has q^2 distinct irreducible constituents of dimension q^2 (these occur from the situation $C_{1,4} = C_{1,3} = 0$), q^2-1 distinct irreducible constituents of dimension q^3 (these occur from the situation $C_{1,4} \neq 0$ or $C_{1,3} \neq 0$) and no further ones.

For $D_4(\mathbf{F}_q)$ Example A.2.9 is one of two situations, which cannot be analysed by only looking at the pictures. We present the second situation next, since we will need it nevertheless for Appendix A.3 and it is anyway the 'light version' of Example A.2.9.

A.2.10 Example. *With arguments analogue to the ones of Example A.2.9 we can analyse the patterns*

$$\left(\begin{array}{cccc|cccc} \cdot & 0_1 & C_{1,3} & A_{1,4} & & & & \cdot \\ & \cdot & C_{2,3} & 0_2 & 0_3 & A_{1,6} & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right).$$

We have $E(A) = \{(1,2)\}$ and get for any $\mu \in \mathbf{F}_q^\times$

$$x_{12}(\mu).C \in \mathcal{O}(C) \qquad \text{if and only if} \qquad C_{1,3} = 0.$$

Now Proposition A.2 implies, that the André-Neto supercharacter Ψ_A^{AN} has q^2 distinct irreducible constituents of dimension q^2 (these occur from the situation $C_{1,3} = 0$), $q-1$ distinct irreducible constituents of dimension q^3 (these occur from the situation $C_{1,3} \neq 0$) and no further ones.

Until now the endomorphism ring of $\mathbb{C}\mathcal{O}_U(A)$ always had algebra generators of type $\lambda_{x_{ij}}(\lambda)|_{\mathbb{C}\mathcal{O}_U(A)}$. Hence it is tempting to suppose this to be true in general. Unfortunately this is *not* the case as the following example shows.

A.2.11 Example. *Let us consider*

$$A = \left(\begin{array}{cccccc|cccccc} \cdot & 0 & 0 & C_{14} & 0 & A_{16} & & & & & & \cdot \\ & \cdot & 0 & C_{24} & A_{25} & & & & & & \cdot & \\ & & \cdot & C_{34} & 0 & 0 & 0 & 0 & A_{39} & \cdot & & \\ & & & \cdot & & & & & \cdot & & & \\ & & & & \cdot & & & \cdot & & & & \\ & & & & & \cdot & \cdot & & & & & \end{array}\right).$$

Clearly we have $E(A) = \{(1,3),(2,3)\}$. *Now suppose* $C_{14} \neq 0 \neq C_{24}$.

We have $x_{13}(\lambda)x_{23}(\mu).A$

$$= \left(\begin{array}{cccccc|cccccc}
\cdot & 0 & 0 & C_{14} & 0 & A_{16} & & & & & & \cdot \\
 & \cdot & 0 & C_{24} & A_{25} & & & & & & \cdot & \\
 & & \cdot & C_{34} - \lambda C_{14} - \mu C_{24} & -\mu A_{25} & -\lambda A_{16} & 0 & 0 & A_{39} & \cdot & & \\
 & & & \cdot & & & & & \cdot & & & \\
 & & & & \cdot & & & \cdot & & & & \\
 & & & & & \cdot & \cdot & & & & &
\end{array}\right)$$

Applying $x_{48}(\mu A_{39}^{-1} A_{25}) x_{58}(\lambda A_{39}^{-1} A_{16})$ *from the right yields*

$$\left(\begin{array}{cccccc|cccccc}
\cdot & 0 & 0 & C_{14} & 0 & A_{16} & & & & & & \cdot \\
 & \cdot & 0 & C_{24} & A_{25} & & & & & & \cdot & \\
 & & \cdot & C_{34} - \lambda C_{14} - \mu C_{24} & 0 & 0 & 0 & 0 & A_{39} & \cdot & & \\
 & & & \cdot & & & & & \cdot & & & \\
 & & & & \cdot & & & \cdot & & & & \\
 & & & & & \cdot & \cdot & & & & &
\end{array}\right)$$

Hence $x_{13}(\lambda)x_{23}(\mu).A \in \mathcal{O}(A)$ *if and only if* $\lambda C_{14} + \mu C_{24} = 0$. *Thus*

$$\left\{ \lambda_{x_{13}(\lambda)x_{23}(-\lambda C_{24}^{-1}C_{14})}|_{\mathbb{C}\mathcal{O}_U(A)} \;\middle|\; \lambda \in \mathbf{F}_q \right\} \text{ is a } \mathbb{C}\text{-basis of } \operatorname{End}_{\mathbb{C}U}\left(\mathbb{C}\mathcal{O}_U(A)\right).$$

Up to this stage all examples illustrate the possibilities and strengths of our main theorem. Hence it is time to illustrate its limitations. We want to close our collection of examples with the smallest and easiest of examples, which resists our general methods so far:

A.2.12 Example (Prototype of the problematical type)**.** *Let us consider*

$$A = \left(\begin{array}{ccccc|ccccc}
\cdot & 0_1 & A_{1,3} & & & & & & & \cdot \\
 & \cdot & 0_2 & A_{2,4} & & & & & \cdot & \\
 & & \cdot & 0_3 & A_{3,5} & & & \cdot & & \\
 & & & \cdot & & & \cdot & & & \\
 & & & & \cdot & \cdot & & & &
\end{array}\right).$$

We have $E(A) = \{(1,2),(2,3)\}$. *Since* A *is the only core pattern with verge* A, *we have*

$$\dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}U} \mathbb{C}\mathcal{O}_U(A) = q^2.$$

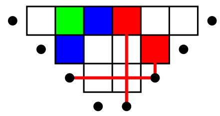
Thus far we have not developed general methods for decomposing U*-orbit modules, whose endomorphism ring has dimension larger than* q. *Hence we can neither decompose* $\mathbb{C}\mathcal{O}_U(A)$ *simply by looking at the pattern, as we did in Examples A.2.1, A.2.4, A.2.7, A.2.8 nor algorithmically as in Examples A.2.9 and A.2.10.*

Using Corollary 3.4.7 we see that our example can be interpreted purely as a problematic case for the series $A_N(\mathbf{F}_q)$. This is no coincidence, as up to date we also have difficulties with that situation for $A_N(\mathbf{F}_q)$, as R. Dipper pointed out to the author. This does not mean, that we cannot decompose this particular $\mathbb{C}\mathcal{O}_U(A)$, but we have no general methods to decompose patterns of this type, i.e. patterns whose main conditions form a line parallel to the diagonal.

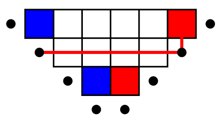
A.3. Counting irreducible characters of $D_4(\mathbf{F}_q)$

In this section we want to apply our instruments to the question analogous to Higman's conjecture for $D_4(\mathbf{F}_q)$. First of all let us make some few remarks about notation. We will denote the patterns more as pictures and less as matrices according to

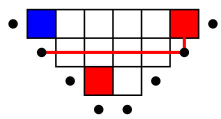
$$\left(\begin{array}{cccc|cccc} \cdot & 0_1 & C_{1,3} & C_{1,4} & A_{1,5} & & & \cdot \\ & \cdot & C_{2,3} & 0_2 & 0_3 & A_{1,6} & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right) =$$



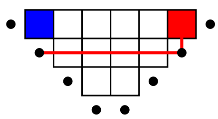
The red lines indicate the lower hooks associated to the main conditions. Hence entries lying on a red line cannot be main conditions of a hook-separated staircase pattern. The first step in counting the irreducible characters for $D_4(\mathbf{F}_q)$ is determining all hook-separated verge modules (or André-Neto supercharacters) and their (graphically obtained) decomposition into hook-separated staircase modules. To ensure, that our list of (graphically decomposed) André-Neto patterns is complete it is best to determine a total ordering on the set of AN-patterns and then to write the list beginning by the 'biggest' pattern following the ordering to the 'smallest' one. We used a lexicographical order to get the following list:



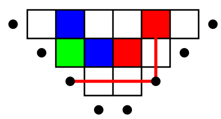
irreducible constituents:
q^2 isom. classes
of dimension q^4
Method: Remark 3.3.52



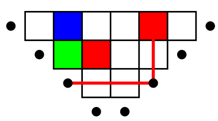
irreducible constituents:
q isom. classes
of dimension q^4
Method: Remark 3.3.52



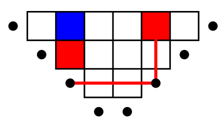
irreducible constituents:
q isom. classes
of dimension q^4
Method: Remark 3.3.52



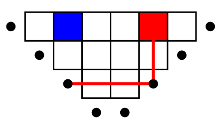
irreducible constituents:
q^3 isom. classes
of dimension q^3
Method: Remark 3.3.52



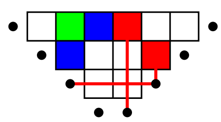
irreducible constituents:
q^2 isom. classes
of dimension q^3
Method: Remark 3.3.52



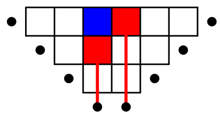
irreducible constituents:
q isom. classes
of dimension q^3
Method: Remark 3.3.52



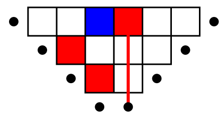
irreducible constituents:
q isom. classes
of dimension q^3
Method: Remark 3.3.52



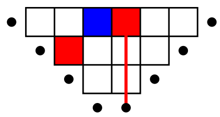
irreducible constituents:
q^2 isom. classes
of dimension q^2 -and-
$q^2 - 1$ isom. classes
of dimension q^3
Method: Example A.2.9



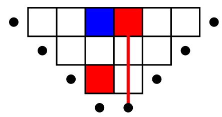
irreducible constituents:
q isom. classes
of dimension q^3
Method: Remark 3.3.52



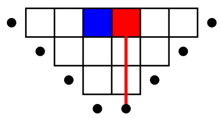
irreducible constituents:
q isom. classes
of dimension q^2
Method: Cor. 3.3.51



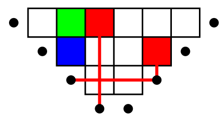
irreducible constituents:
q isom. classes
of dimension q^2
Method: Remark 3.3.52



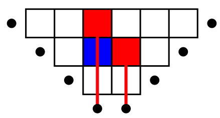
irreducible constituents:
q isom. classes
of dimension q^2
Method: Remark 3.3.52



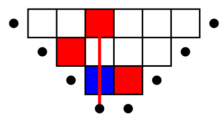
irreducible constituents:
q isom. classes
of dimension q^2
Method: Remark 3.3.52



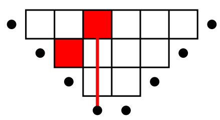
irreducible constituents:
q^2 isom. classes
of dimension q^2 -and-
$q-1$ isom. classes
of dimension q^3
Method: Example A.2.10



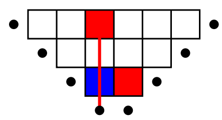
irreducible constituents:
1 isom. class
of dimension q^3
Method: as in A.2.4



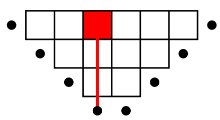
irreducible constituents:
1 isom. class
of dimension q^2
Method: as in A.2.4



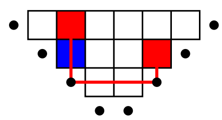
irreducible constituents:
1 isom. class
of dimension q^2
Method: Remark 3.3.52



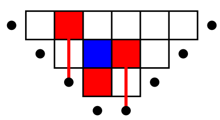
irreducible constituents:
1 isom. class
of dimension q^2
Method: as in A.2.4



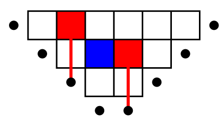
irreducible constituents:
1 isom. class
of dimension q^2
Method: Reamrk 3.3.52



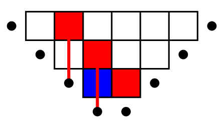
irreducible constituents:
1 isom. classes
of dimension q^3
Method: as in A.2.4



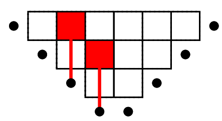
irreducible constituents:
q^2 isom. classes
of dimension q
Method: Prop. A.2.6



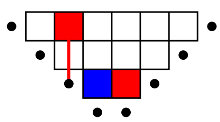
irreducible constituents:
q^2 isom. classes
of dimension q
Method: Prop. A.2.6



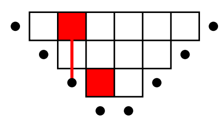
irreducible constituents:
q isom. classes
of dimension q
Method: as in A.2.8



irreducible constituents:
q isom. classes
of dimension q
Method: Prop. A.2.6

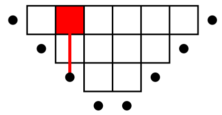


irreducible constituents:
q isom. classes
of dimension q
Method: Cor. 3.3.51

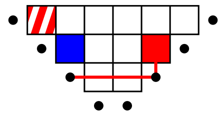


irreducible constituents:
1 isom. class
of dimension q
Method: Cor. 3.3.51

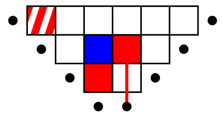
In the next patterns their occur some shaded red positions. These indicate, that at this position can be a main condition or not. Since the shaded positions are not important for the decomposition of the U-orbit modules into irreducible modules we do not distinguish between the different cases there.



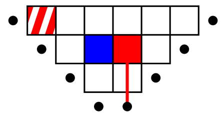
irreducible constituents:
1 isom. class
of dimension q
Method: Remark 3.3.52



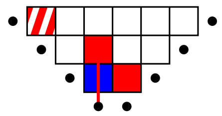
irreducible constituents:
q isom. classes
of dimension q^2
Method: Cor. 3.3.51



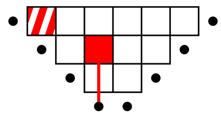
irreducible constituents:
q isom. classes
of dimension q
Method: Cor. 3.3.51



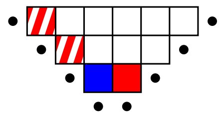
irreducible constituents:
q isom. classes
of dimension q
Method: Cor. 3.3.51



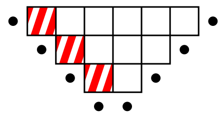
irreducible constituents:
1 isom. class
of dimension q
Method: as in A.2.8



irreducible constituents:
1 isom. class
of dimension q
Method: Cor. 3.3.51



irreducible constituents:
q isom. class
of dimension 1
Method: Cor. 3.3.51



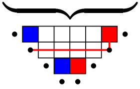
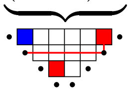
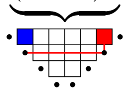
irreducible constituents:
1 isom. class
of dimension 1
Method: Cor. 3.3.51

Alongside each partition we have written the information of the decomposition of the André-Neto supercharacter the picture represents and how the decomposition is obtained. Note, that due to [And8, Theorem 1.3][84] irreducible characters of $D_n(\mathbf{F}_q)$ have dimension q^a for some integer a.

We want to determine the number of irreducible characters of $D_4(\mathbf{F}_q)$ of dimension q^a, denoted $\#\,\mathrm{Irr}_a$ for short. We can do this by first searching the André-Neto supercharaters with irreducible constituents of dimension q^a in our list.

Then to each type of AN pattern we determine number of AN patterns of that type, which is $(q-1)^{\#\text{red boxes}} q^{\#\text{shaded red boxes}}$. Since we already know how many isom. classes of irreducible modules are in each AN-supercharacter and due to the orthogonality of the AN-supercharacters we get $\#\,\mathrm{Irr}_a$ as in for example

$$\#\,\mathrm{Irr}_4 = \underbrace{(q-1)^2q^2}+\underbrace{(q-1)^2q}+\underbrace{(q-1)q} = q^4 - q^3 = \underbrace{v^4+3v^3+3v^2+v}_{\text{with } v:=q-1}$$

[84] we may apply the Theorem, since we assume p to be odd.

Performing this for all possible character degrees and setting $v := q - 1$, we get

$$\begin{aligned}
\#\,\mathrm{Irr}_4 &= q^4 - q^3 &&= v^4 + 3v^3 + 3v^2 + v\\
\#\,\mathrm{Irr}_3 &= q^5 - 3q^2 + 2q &&= v^5 + 5v^4 + 10v^3 + 7v^2 + v\\
\#\,\mathrm{Irr}_2 &= 3q^4 - 3q^3 &&= 3v^4 + 9v^3 + 9v^2 + 3v\\
\#\,\mathrm{Irr}_1 &= q^5 - q^2 &&= v^5 + 5v^4 + 10v^3 + 9v^2 + 3v\\
\#\,\mathrm{Irr}_0 &= q^4 &&= v^4 + 4v^3 + 6v^2 + 4v + 1
\end{aligned}$$

Clearly the number of irreducible characters is equal to the number of conjugacy classes is equal to $\sum_a \#\,\mathrm{Irr}_a$. We have

$$\#\,\mathrm{Irr} = \#\,\mathrm{ccl} \quad = 2q^5 + 5q^4 - 4q^3 - 4q^2 + 2q \quad = 2v^5 + 15v^4 + 36v^3 + 34v^2 + 12v + 1.$$

Hence we have shown that the number of conjugacy classes of $D_4(\mathbf{F}_q)$ is a polynomial in q, which is the analogous question to $D_n(\mathbf{F}_q)$ to Higman's conjecture for $A_n(\mathbf{F}_q)$. Note that our polynomial equals the one obtained by S.M. Goodwin and G. Röhrle using computer methods, c.f. Table 1 of [GR].

For the group $A_n(\mathbf{F}_q)$ there are other famous conjectures besides Higman's conjecture, most notably

A.3.1 Conjecture (G.I. Lehrer, Conjecture 6.3(ii) of [Leh]). *The number of irreducible representations of degree q^c is an integer polynomial in q.*

and its stronger version

A.3.2 Conjecture (I.M. Isaacs, Conjecture B of [Isa]). *The number of irreducible representations of degree q^c is an integer polynomial in $q-1$ with non-negative coefficients.*

We can ask the analogous questions for $D_4(\mathbf{F}_q)$ to G.I. Lehrer's and I.M. Isaacs conjectures for $A_n(\mathbf{F}_q)$. Both of the statements are true for $D_4(\mathbf{F}_q)$ as our example shows. These are results not included in S.M. Goodwin's and G. Röhrle's work, since their approach is not to count irreducible characters but to count conjugacy classes. But these results are exactly those of F. Himstedt, T. Le and K. Magaard in [HLM], who are working with a character theoretical approach.

In fact we also have answered the corresponding questions for $D_3(\mathbf{F}_q)$ (using the last 7 pictures of the case $D_4(\mathbf{F}_q)$) and $D_2(\mathbf{F}_q)$ (using the last two pictures of the case $D_4(\mathbf{F}_q)$). But these two cases are not of specific interest, since $D_3(\mathbf{F}_q) \cong A_4(\mathbf{F}_q)$ and $D_2(\mathbf{F}_q) \cong A_2(\mathbf{F}_q) \times A_2(\mathbf{F}_q)$.

Nomenclature

$.$	as in $A.g$, usually a group operation of $D_n(\mathbf{F}_q)$, page 58
$.$	as in $g.A$, usually truncated row operation for $D_n(\mathbf{F}_q)$, page 78
$[A]$	basis element of the group algebra corresponding to pattern A, page 58
$\bar{i}$	the image of the positive integer i under the mirror map, page 18
$/$	set of matrix positions, page 19
$\mathbb{C}\mathcal{V}(A)$	the verge module of A, page 89
$\mathbb{C}^\times$	multiplicative group of complex number field
$\mathcal{V}(A)$	the set of patterns with verge A, page 88
$\operatorname{core}(A)$	the the core of the pattern A, page 65
δ_{ab}	Kronecker delta
$\diagdown$	set of matrix positions, page 19
$\mathbf{F}_q$	finite field with q elements of characteristic p
$\mathbf{F}_q^\times$	multiplicative group of $\mathbf{F}_q$
$\mathbb{C}\mathcal{O}_G(A)$	G-orbit module associated to A, page 45
κ	with the exception of Section 2.1, κ denotes the trace form, page 47
λ_a	$\mathbb{C}U$-right endomorphism of $\mathbb{C}U$ given by left multiplication with a, page 77
$\lambda_g \mid_{\mathbb{C}\mathcal{O}_U(A)}$	restriction of the endomorphism λ_g to the U-orbit module $\mathbb{C}\mathcal{O}_U(A)$, page 77
$\operatorname{l.main}(A)$	the set of left main conditions of the pattern A, page 61
$\triangledown$	set of matrix positions, page 19
$\mathcal{B}^{\mathrm{End}}_{\mathcal{O}_U(A)}$	$\mathbb{C}$-basis of $\operatorname{End}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A)\big)$, page 101
$\mathcal{B}^{\mathrm{Hom}}_{\mathcal{V}(A)}$	$\mathbb{C}$-basis of $\operatorname{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big)$, page 99

$\mathrm{main}(A)$	the set of main conditions of the pattern A, page 61
$\mathrm{minor}(A)$	the set of minor conditions of the pattern A, page 62
$\circ$	as in $f(x) \circ g$, usually denotes a special group operation of $D_n(\mathbf{F}_q)$, page 57
sc-verge	superclass verge, page 110
$\mathrm{sc}(A)$	set of superclass conditions of a pattern $A \in V_{◹}$, page 109
$\mathrm{supp}(A)$	support of a matrix A, page 30
$\overline{\mathrm{supp}}(g)$	essential support of the group element g, page 30
$\overline{\mathrm{supp}}(H)$	support of a subgroup $H \leq A_N(\mathbf{F}_q)$, page 30
$\Phi^+(D_n)$	positive roots of the root system of type D_n, page 26
ψ_A	usually denotes the character of $\mathbb{C}\mathcal{O}_U(A)$
Ψ_A^{AN}	denotes an André-Neto supercharacter, page 127
$\mathrm{r.main}(A)$	the set of right main conditions of the pattern A, page 61
◁	set of matrix positions, page 19
⩤	set of matrix positions, page 19
⋰◸	set of matrix positions, page 19
□	set of matrix positions, page 19
$\mathrm{suppl}(A)$	the set of supplementary conditions of the pattern A, page 65
⩒(A)	the flock of hooks associated to A, page 65
◸	set of matrix positions, page 19
$\mathbb{C}\mathcal{O}_U(A)$	U-orbit module associated to A, page 45
▽	set of matrix positions, page 19
◹	set of matrix positions, page 19
$\Psi_{\mathcal{V}(A)}$	denotes the character of $\mathbb{C}\mathcal{V}(A)$, page 125
$\mathrm{verge}(A)$	the verge of the pattern A, page 62
A^R	right transpose of the matrix u, page 19

$A_n(\mathbf{F}_q)_J$	pattern subgroup of $A_n(\mathbf{F}_q)$, page 33
$A_N(\mathbf{F}_q)$	the unipotent linear group, page 18
C_u^{AN}	André-Neto superclass, page 108
C_g^{AY}	André-Yan superclass, page 108
$D_n(\mathbf{F}_q)$	the uniipotent orthogonal group of Lie type D, page 18
$D_n(\mathbf{F}_q)_J$	pattern subgroup of $D_n(\mathbf{F}_q)$ defined by the closed subset J of $\bigtriangledown$, page 28
$E(A)$	$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}\big(\mathbb{C}\mathcal{O}_U(A), \mathbb{C}\mathcal{V}(A)\big) = q^{\lvert E(A)\rvert}$, page 99
e_{ij}	the matrix with entry 1 at the (i,j)-th position and 0 entries otherwise
f	usually denotes a 1-cocycle, c.f. Definition 2.1.6 for the general definition, and Propositions 2.2.3 and 3.1.12 for important examples
G	usually denotes the group $A_N(\mathbf{F}_q)$
H_i	i-th hook of $\bigtriangledown$, page 85
$H^{\ell}_{(i,j)}$	the lower hook associated to (i,j), page 65
$H^{u}_{(i,j)}$	upper hook associated to (i,j), page 115
$I(A)$	usually the closed subset of $\bigtriangledown(A)$ defining the verge subgroup of $\mathrm{verge}(A)$, page 89
J	usually a closed subset of $\bigtriangledown$, page 27
$J(A)$	closed subset of $\bigtriangledown$, page 71
$L^{\ell}_{(i,j)}$	lower leg associated to (i,j), page 115
$L^{u}_{(i,j)}$	upper leg associated to (i,j), page 115
N	equal to $2n$, it usually denotes the matrix size of the groups $D_n(\mathbf{F}_q)$ and $A_N(\mathbf{F}_q)$
n	a positive integer, it usually denotes the Lie rank of the group $D_n(\mathbf{F}_q)$ or the matrix size of the group $A_n(\mathbf{F}_q)$
p	an odd prime, the characteristic of the finite field $\mathbf{F}_q$
$R(A)$	system of left coset representatives of $U^{\mathrm{Stab}}_{\mathcal{V}(A)}$ in $U_{\mathcal{V}(A)}$, page 99

$T(A)$	c.f. Definition/Lemma 3.3.44
U	usually denotes the group $D_n(\mathbf{F}_q)$
$U_{\mathcal{V}(A)}^{\mathrm{Stab}}$	point stabilizer of truncated row operation on $\mathcal{V}(A)$, page 97
$U_{\mathcal{V}(A)}$	the verge subgroup of a staircase pattern A, page 89
$V^{\perp}$	orthogonal space of V with respect to κ
V_J	vector space associated to a set J, page 46
$V_J^{\perp}$	orthogonal space of V_J with respect to κ
$x_{ij}(\lambda)$	equal to $1 + \lambda e_{ij} - \lambda e_{\overline{j}\overline{i}}$ with $\lambda \in \mathbf{F}_q$, element of a root subgroup of $D_n(\mathbf{F}_q)$, page 26

Index

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