Justification of an Approximation Equation for the Bénard-Marangoni Problem

Von der Fakultät Mathematik und Physik der Universität Stuttgart zur Erlangung der Würde eines Doktors der Naturwissenschaften (Dr. rer. nat.) genehmigte Abhandlung

Vorgelegt von

Dominik Zimmermann

aus Bruchsal

Hauptberichter:	Prof. Dr. Guido Schneider
Mitberichter:	Prof. Dr. Hannes Uecker
	Prof. Dr. Peter Wittwer
Tag der mündlichen Prüfung:	14. Januar 2014

Institut für Analysis, Dynamik und Modellierung der Universität Stuttgart

2014

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Greek Symbols

$\varepsilon\Psi$	improved Ginzburg-Landau ansatz/approximation	[p. 34]
$\varepsilon\Psi_{an}$	Ginzburg-Landau ansatz/approximation	(2.2),[p. 20]
λ_ε	eigenvalue curve	[p. 20]
ρ	weight function	[p. 22]
$\rho_{\varepsilon,\gamma}$	weight function for concentration at γ	[p. 29]
χ_0	indicator function of small $\mathcal{O}(1)$-interval centred at 0	[p. 23]
$\chi_{\pm c}$	indicator function of small $\mathcal{O}(1)$-interval centred at ± 1	[p. 23]
χ_s	$:= 1 - \chi_0 - \chi_c - \chi_{-c}$	[p. 23]
Ψ_0	neutral part of the improved approximation	[p. 25]
$\Psi_{\pm c}$	critical parts of the improved approximation	[p. 25]
Ψ_s	stable part of the improved approximation	[p. 25]

Roman Symbols

C	generic constant	[p. 22]
$g_j(R,\Psi)$	higher order terms in the equation for R_j, $j = 0, c, s$	[p. 25]
$N_j(R,\Psi)$	lowest order terms in the equation for R_j, $j = 0, c, s$	[p. 25]
R	error	[p. 25]
R_0	neutral part of the error	[p. 25]
$R_{\pm c}$	critical parts of the error	[p. 25]
R_s	stable part of the error	[p. 25]
$\mathcal{R}$	sum of the Sobolev norms of the parts of the error	[p. 26]
$\tilde{\mathcal{R}}$	summed error norms respecting concentration	(2.8),[p. 31]
$\mathrm{Res}(\varepsilon\Psi)$	residual	[p. 22]
Res_j	parts of the residual $E_j\mathrm{Res}(\varepsilon\Psi)$, $j = 0, \pm c, s$	[p. 25]
T	scaled time variable $T = \varepsilon^2 t$	[p. 20]
X	scaled space variable $X = \varepsilon x$	[p. 20]

Chapter 3 and Appendix A

Spaces

Operators

Greek Symbols

Roman Symbols

C	generic constant	[p. 22]
$g_j(R, \Psi)$	higher order terms in the equation for R_j, $j = 0, c, s$	[p. 38]
$N_j(R, \Psi)$	lowest order terms in the equation for R_j, $j = 0, c, s$	[p. 38]
R	error	[p. 25]
R_0	neutral part of the error	[p. 25]
$R_{\pm c}$	critical parts of the error	[p. 25]
R_s	stable part of the error	[p. 25]
$\mathbf{R}$	$:= (R_c, R_0, R_s)$	[p. 54]
$\tilde{\mathcal{R}}$	summed error norms respecting concentration	(2.8),[p. 31]
$\mathrm{Res}(\varepsilon\Psi)$	residual	[p. 22]
Res_j	parts of the residual $E_j\mathrm{Res}(\varepsilon\Psi)$, $j = 0, \pm c, s$	[p. 25]
T	scaled time variable $T = \varepsilon^2 t$	[p. 38]
X	scaled space variable $X = \varepsilon x$	[p. 38]

Chapter 4 and Appendix B

Spaces

Operators

$\widehat{(\cdot)}$	Fourier transform	[p. 22]
D	$:= \frac{\mathrm{d}}{\mathrm{d}z}$	[p. 82]
$\mathfrak{e}_1$	operator for homogenising tangential stress boundary condition	[p. 66]
$\mathfrak{e}_1^0$	analogue to $\mathfrak{e}_1$ for usual Sobolev spaces	[p. 73]
$\mathfrak{e}_2$	operator for homogenising cooling boundary condition	[p. 67]
$\mathfrak{e}_2^0$	analogue to $\mathfrak{e}_2$ for usual Sobolev spaces	[p. 73]
E_0	neutral mode filter	[p. 87]
$E_{\pm c}$	critical mode filters	[p. 87]
E_s	stable mode filter	[p. 87]
$\mathscr{E}$	$\mathscr{E}f = \nabla\phi$, where ϕ solves (B.18)	[p. 121]
$\mathcal{F}_x$	Fourier transform w.r.t. x	[p. 61]
G	linear operator for the system for $(\eta, \mathbf{v}, \theta)$	(4.59),[p. 69]
L_B	linear operator for the boundary conditions	[p. 87]
Λ_ε	equal to G, emphasis on ε-dependence	[p. 87]
$\tilde{M}_0^{-1}$	solution operator of System (4.61) in $\mathcal{K}_0^r$-spaces	[p. 71]
$\mathfrak{P}_j^\varepsilon(k)$	spectral projection on $\mathrm{span}(\mathcal{U}^\varepsilon(k,\cdot))$	[p. 87]
$\mathscr{P}$	projection on divergence-free vector fields	[p. 64]

Greek Symbols

$\varepsilon\Psi$	improved Ginzburg-Landau ansatz/approximation	[p. 86]
$\varepsilon\Psi_{an}$	Ginzburg-Landau ansatz/approximation	[p. 85]
η	free top surface	[p. 57]
$\bar{\eta}$	extension of η into Ω	[p. 61]
θ	transformed temperature	[p. 61]
λ_ε	first eigenvalue curve	[p. 82]
φ	$\varphi(u)$ satisfies nonlinear boundary conditions if $u \in \mathcal{H}^m_{(0),00}$	[p. 74]
χ_j	indicator functions of small $\mathcal{O}(1)$-intervals centred at jk_c	[p. 85]
Ψ_0	neutral part of the improved approximation	[p. 88]
$\Psi_{\pm c}$	critical parts of the improved approximation	[p. 88]
Ψ_s	stable part of the improved approximation	[p. 88]
Ω	fixed domain for the liquid	[p. 61]
Ω_t	time-dependent domain for the liquid	[p. 58]

Roman Symbols

B_i	Biot number	[p. 60]
B_o	Bond number	[p. 60]
C	generic constant	[p. 22]
C_r	Crispation number	[p. 60]
$\mathbf{F}_0$, F_j	nonlinearities for the Bénard-Marangoni problem on Ω	[p. 117]
$g^*_\star$	higher order terms in the evolution equation for $R^*_\star$	[p. 88 ff.]
$g^*_{B,\star}$	higher order terms in the boundary conditions for $R^*_\star$	[p. 88 ff.]
J	Jacobian determinant of $\mathcal{T}_t$	[p. 61]
J_{ij}	components of $J_{\mathcal{T}_t}$	[p. 61]
J^{ij}	components of $J^{-1}_{\mathcal{T}_t}$	[p. 61]
$J_{\mathcal{T}_t}$	Jacobian of $\mathcal{T}_t$	[p. 61]
k_c	critical wave number	[p. 82]
K_j	$:= \frac{k-jk_c}{\varepsilon}$	[p. 85]
M_a	Marangoni number	[p. 60]
$\mathfrak{M}_a(k,\mu)$	levels $\mathfrak{M}_a(k,\mu) \equiv M_a$ yield implicit dispersion relation (B.68),	[p. 131]
$N^*_\star$	lowest order terms in the evolution equation for $R^*_\star$	[p. 88 ff.]
$N^*_{B,\star}$	lowest order terms in the boundary conditions for $R^*_\star$	[p. 88 ff.]
p	perturbation of pressure in physical coordinates	[p. 60]
P_r	Prandtl number	[p. 60]
q	transformed pressure	[p. 61]
R	error	[p. 88]
R_0	neutral part of the error	[p. 88]
$R_{\pm c}$	critical parts of the error	[p. 88]
R_s	stable part of the error	[p. 88]
$R_{j,h}$	counterparts of R_j for the homogenised error system	[p. 94]
$\mathbf{R}$	$:= (R_c, R_s, R_0)$	[p. 90]
$\mathbf{R}_h$	$:= (R_{c,h}, R_{s,h}, R_{0,h})$	[p. 94]
$\check{\mathbf{R}}$	local solution for error equations	[p. 90]
$\mathrm{Res}(\varepsilon\Psi)$	residual	[p. 88]
Res_j	parts of the residual $E_j\mathrm{Res}(\varepsilon\Psi)$, $j = 0, \pm c, s$	[p. 88]
T	scaled time variable $T = \varepsilon^2 t$	[p. 38]
T	perturbation of temperature field in physical coordinates	[p. 57]
T_A	atmospheric temperature	[p. 58]
T_B	bottom temperature	[p. 58]
$\mathcal{T}_t$	diffeomorphism between Ω and Ω_t	[p. 61]
$\mathbf{u}$	$:= (u_1, u_2)^\top$, perturbation of velocity in physical coordinates	[p. 57]
$\mathcal{U}^\varepsilon$	(k,ε)-dependent eigenfunctions	[p. 84]
$\mathscr{U}_{\pm 1}$	ε-independent critical eigenfunctions	[p. 84]
$\mathbf{v}$	$:= (v_1, v_2)^\top$, transformed velocity field	[p. 61]
X	scaled space variable $X = \varepsilon x$	[p. 38]

Zusammenfassung

Das Bénard-Marangoni Problem ist ein mathematisches Modell zur Beschreibung temperaturabhängiger Flüssigkeitsströmungen in sehr dünnen Schichten mit einer nach oben freien Oberfläche. Nach unten ist die Flüssigkeit durch eine horizontale Platte mit einer gewissen Temperatur begrenzt. Über der freien Oberfläche befindet sich eine Atmosphäre, deren Temperatur niedriger als die des Bodens ist. Es existiert ein bewegungsfreier Zustand der reinen Wärmeleitung. Dieser Zustand ist stabil, solange der Unterschied zwischen der Temperatur des Bodens und der Atmosphäre hinreichend klein ist. Überschreitet der Temperaturunterschied eine gewisse Schwelle, so stellt sich eine Konvektionsströmung ein. Diese wird in solch dünnen Schichten nicht vornehmlich durch Auftriebskräfte verursacht, sondern durch Differenzen in der Oberflächenspannung verschiedener Bereiche auf der freien Oberfläche. Das Einsetzen der Konvektionsströmung kann als die Ausbreitung eines räumlich periodischen Musters gesehen werden, so dass wir das Bénard-Marangoni Problem als musterbildendes System interpretieren. In der vorliegenden Arbeit beschäftigen wir uns mit dem Verhalten des Systems, wenn der Zustand der reinen Wärmeleitung instabil wird.

Aus den Gleichungen des Bénard-Marangoni Problems leiten wir formal ein Ginzburg-Landau-artiges System von Modulationsgleichungen her, mit dessen Hilfe wir Näherungslösungen für das volle Problem konstruieren. Wir erbringen den Nachweis, dass die so gefundenen Näherungslösungen für lange Zeiten nahe an tatsächlichen Lösungen des Bénard-Marangoni Problems liegen, d.h. wir beweisen einen Approximationssatz für die gefundenen Modulationsgleichungen.

Die Gültigkeit der Ginzburg-Landau-Approximation ist bereits für eine Reihe musterbildender Systeme gezeigt worden. Allerdings liegt im Falle des Bénard-Marangoni Problems eine spektrale Situation vor, die es nicht erlaubt, die existierenden Methoden zum Nachweis der Approximationseigenschaft anzuwenden. Daher entwickeln wir zunächst eine allgemeine Methode zur Handhabung eines solchen Spektrums am Beispiel eines einfachen Spielzeugproblems.

Weiterhin sind die bisher existierenden Methoden auf die Behandlung semilinearer Probleme beschränkt. Die Gleichungen des Bénard-Marangoni Problems hingegen sind quasilinear. Deshalb entwickeln wir wieder anhand einfacher Beispiele eine Methode für den Beweis von Approximationssätzen, wenn die zugrunde liegenden Gleichungen quasilinear sind.

Schließlich übertragen wir die obigen Methoden auf das volle Bénard-Marangoni Problem, für das wir zunächst die lokale Existenz und Eindeutigkeit von Lösungen nachweisen.

Abstract

The Bénard-Marangoni problem is a mathematical model for the description of a temperature dependent fluid flow in very thin liquid layers with a free top surface. The liquid is bounded from below by a horizontal plate of a certain temperature. Above the liquid there is an atmosphere cooler than the bottom plate. There is a purely conducting steady state, where the liquid is at rest. This state is stable as long as the difference between the temperature of the bottom plate and the temperature of the atmosphere is sufficiently small. If the temperature difference surpasses a certain threshold, convection sets in, which is mainly driven by surface tension rather than buoyancy. The onset of convection can be seen as the propagation of a spatially periodic pattern, such that we interpret the Bénard-Marangoni problem as a pattern forming system. In this thesis we are interested in the behaviour of the system when the purely conducting steady state becomes unstable.

From the equations of the Bénard-Marangoni problem we formally derive a Ginzburg-Landau like system of modulation equations, which we use to construct approximate solutions for the full problem. In this thesis we prove an approximation theorem for these modulation equations. That means, we show that the approximate solutions lie close to true solutions of the Bénard-Marangoni problem, at least for a long time.

The validity of the Ginzburg-Landau approximation was already shown for a number of pattern forming systems. In case of the Bénard-Marangoni problem, however, we have a spectral situation that does not allow a direct application of the existing approximation proofs. Hence, we first consider a toy problem exhibiting such a kind of spectrum and develop a method for proving an approximation result in this case.

Furthermore, the existing approximation proofs were restricted to semilinear problems. However, the equations of the Bénard-Marangoni problem are quasilinear. Therefore, we also develop a method for proving approximation results for quasilinear problems.

We then turn back to the Bénard-Marangoni problem. After showing local existence and uniqueness of solutions, we apply our new methods in order to prove the desired approximation result.

Danksagung

Diese Arbeit entstand während meiner Zeit als akademischer Mitarbeiter am Institut für Analysis, Dynamik und Modellierung (IADM) der Universität Stuttgart. An dieser Stelle möchte ich allen Personen danken, die zum Gelingen dieser Arbeit beigetragen haben.

Allen voran gilt mein herzlicher Dank meinem Betreuer, Prof. Dr. Guido Schneider, der mir die Möglichkeit gab, an seinem Lehrstuhl zu promovieren. Neben den sehr fruchtbaren Diskussionen gerade gegen Ende meiner Promotionszeit und der angenehmen Arbeitsatmosphäre rechne ich ihm einen Pep Talk hoch an, den ich zum damaligen Zeitpunkt dringend nötig hatte.

Meiner Schwester Judith Kleffmann und meinem Büronachbarn Dr. Markus Daub danke ich für das Korrekturlesen dieser Arbeit.

Im Namen meiner kleinen Familie geht ein besonderer Dank an Dr. Adriana Lalegname und Dr. Heiko Schulz dafür, dass sie die technischen Voraussetzungen dafür geschaffen haben, dass ich einen guten Teil meiner Arbeit von zu Hause aus erledigen konnte.

Mein Dasein in der badischen Diaspora haben mir die folgenden, teils ehemaligen, aber allesamt schwäbischen lieben Kollegen sehr erleichtert: Frau Stefanie Siegert, mit der ich immer gerne Übungsblätter erstellt habe, Dr. Markus Daub, der im Zweifelsfall als einziger über meine Witze lachte, und Dr. Tobias Häcker, der es im Zweifelsfall vermochte, zu meinen Witzen eine sachliche Frage zu stellen.

Meinen Eltern Maria und Michael Zimmermann danke ich für die jahrelange bedingungslose Unterstützung in allen Belangen.

Der unerschütterliche Glaube meiner Schwester Miriam Zimmermann an ihren großen Bruder stellte eine Motivation dar, für die ich mich nicht genug bedanken kann.

Schließlich möchte noch meine Frau Franziska erwähnen, die stets Verständnis für meine Situation hatte und Rücksicht auf mich nahm. Angesichts ihrer Dreifachbelastung durch Studium, Kindererziehung und Schwangerschaft hätte es eigentlich umgekehrt sein sollen. Ihr gilt mein unendlicher Dank.

Chapter 1

Introduction

The Ginzburg-Landau equation appears as a universal amplitude equation for spatially extended pattern forming systems close to the first instability. We refer to Chapter 2 for a detailed introduction into the subject. In this thesis we extend the existing methods for justifying the Ginzburg-Landau approximation. Motivated by an actual physical problem – namely the Bénard-Marangoni convection – we develop general techniques for handling the following classes of problems.

A Spectral Situation as Depicted in Figure 1.2 due to the Existence of a Conserved Quantity

Chapter 2 is devoted to the study of such problems. To this purpose we introduce a semilinear toy problem of conservation law form. Furthermore, this toy problem possesses the reflection symmetry of the Bénard-Marangoni problem. Up to now, this kind of problems lay outside the scope of existing methods for justifying a Ginzburg-Landau approximation. Existing proofs used the fact that in a spectral situation as in Figure 1.1 the nonlinear interaction of critical modes yields terms that are exponentially damped. A new technique has to be developed in order to control the error, when nonlinear interaction of the critical modes leads to terms that are only diffusively damped. The content of Chapter 2 is already published in a slightly different version in [SZ13].

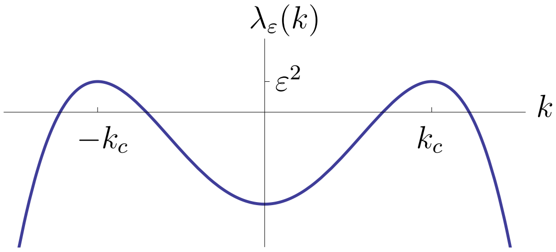


Figure 1.1: Classical situation

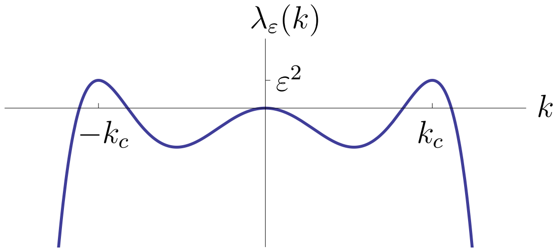


Figure 1.2: New situation

Quasilinear Problems

In Chapter 3 we modify the semilinear toy problem from Chapter 2 by a nonlinear term containing a derivative of highest appearing order, thus making the problem quasilinear. We use and extend ideas from the method of optimal regularity, which can be used to show local existence and uniqueness for a certain class of quasilinear problems, in order to control the error on the correct time scale. We demonstrate the new optimal regularity approach for a most basic example – a quasilinear Swift-Hohenberg model, i.e., when the spectral situation is the classical one from Figure 1.1. Afterwards, we combine this approach with the methods from Chapter 2 and prove an approximation theorem for the quasilinear toy problem.

In Chapter 4 we first prove the local existence and uniqueness of solutions of the Bénard-Marangoni problem. Due to the presence of nonlinear boundary conditions, we then reformulate the problem in a way such that we can apply the newly developed methods from Chapters 2 and 3 in order to prove the desired approximation result.

Chapter 2

A Semilinear Toy Problem[1]

The Ginzburg-Landau equation can be derived via multiple scaling analysis for the Bénard-Marangoni convection problem, which is driven by temperature-dependent surface tension, and which is the subject of our interest. In this chapter we prove estimates between this formal approximation and true solutions of a scalar pattern forming model problem showing the same spectral picture as the Bénard-Marangoni convection problem in case of a thin fluid. The new difficulties come from neutral modes touching the imaginary axis for the wave number $k = 0$ and from identical group velocities at the critical wave number $k = k_c$ and the wave number $k = 0$. The problem is solved by using the reflection symmetry of the system and by using the fact that the modes concentrate at integer multiples of the critical wave number $k = k_c$. In this chapter we present a method that is applicable whenever this kind of instability occurs.

2.1 Introduction

In [HSZ11] we already considered a toy problem that showed an instability as it appears for the Bénard-Marangoni convection problem, see [Tak81a]. We proved that a formally derived Ginzburg-Landau equation makes correct predictions about the dynamics of the original problem close to the first instability. The instability considered in [HSZ11] appears when the height of the fluid is not too small. In case of a thin film of fluid, however, the instability differs qualitatively from the one considered in [HSZ11].

In this chapter we consider a toy problem showing an instability as it appears for the Bénard-Marangoni convection problem in case of a thin film of fluid. Moreover, it possesses the reflection symmetry of the Bénard-Marangoni problem. We

[1]The content of this chapter was already published in a slightly different version in [SZ13]

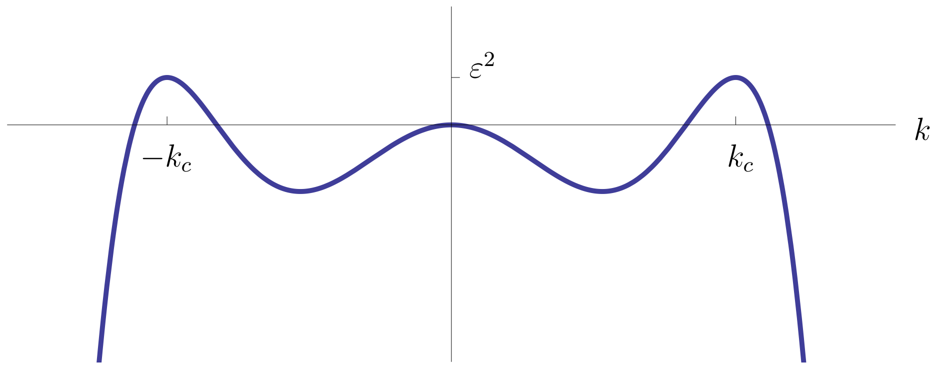


Figure 2.1: The curve of eigenvalues $k \mapsto \lambda_\varepsilon(k)$ for $0 < \varepsilon \ll 1$.

consider

$$\partial_t u = L_\varepsilon(\partial_x)u + \partial_x^2(u^2), \quad \text{with} \quad L_\varepsilon(\partial_x) = \frac{1}{2}(1+\partial_x^2)^2\partial_x^2 + \frac{\varepsilon^2}{2}(\partial_x^6 - 3\partial_x^2), \tag{2.1}$$

and $u(x,t) \in \mathbb{R}$. The solutions of the linearised problem are given by $\mathrm{e}^{\mathrm{i}kx+\lambda_\varepsilon(k)t}$ with

$$\lambda_\varepsilon(k) = -\frac{1}{2}(1-k^2)^2k^2 + \frac{\varepsilon^2}{2}(3k^2 - k^6).$$

The graph of the curve of eigenvalues $k \mapsto \lambda_\varepsilon(k)$ is plotted in Figure 2.1. We see that there exist strictly positive eigenvalues for $\varepsilon > 0$. As necessary for the derivation of the Ginzburg-Landau equation the instability occurs at a non-zero wave number, namely here at the critical wave numbers $\pm k_c = \pm 1$. It has been pointed out by previous work [vH91] that the proof of an approximation theorem is not a trivial task if quadratic terms are present in the original system. The situation complicates even further, since for all values of ε the curve of eigenvalues is touching zero at the wave number $k = 0$. This is an important new difficulty, since existing results for the justification of the Ginzburg-Landau approximation rely on the fact that all non-critical modes are exponentially damped with a uniform rate.

We are interested in the dynamics of (2.1) close to the first instability, i.e., for $0 < \varepsilon \ll 1$. In order to derive the Ginzburg-Landau equation for the description of the bifurcating solutions we make the ansatz $u(x,t) = \varepsilon\Psi_{an}(x,t)$ with

$$\varepsilon\Psi_{an}(x,t) = \varepsilon A_1(X,T)\mathrm{e}^{\mathrm{i}x} + \varepsilon^2 A_2(X,T)\mathrm{e}^{2\mathrm{i}x} + \varepsilon^2 A_0(X,T)/2 + \text{c.c.}, \tag{2.2}$$

where $X = \varepsilon x$ and $T = \varepsilon^2 t$. We find

$$\begin{aligned} \partial_T A_1 &= -\lambda_0''(1)\partial_X^2 A_1/2 + A_1 - 2(A_0A_1 + A_2A_{-1}), \\ \partial_T A_0 &= -\lambda_0''(0)\partial_X^2 A_0/2 + 2\partial_X^2(A_1A_{-1}), \\ 0 &= \lambda_0(2)A_2 - 4(A_1)^2, \end{aligned} \tag{2.3}$$

where $A_{-j} := \overline{A_j}$. Eliminating A_2 in the first equation of (2.3) by using the third equation of (2.3) gives the generalised Ginzburg-Landau system

$$\begin{aligned} \partial_T A_1 &= -\lambda_0''(1)\partial_X^2 A_1/2 + A_1 - 2A_0A_1 - \frac{8}{\lambda_0(2)}A_1|A_1|^2, \\ \partial_T A_0 &= -\lambda_0''(0)\partial_X^2 A_0/2 + 2\partial_X^2(A_1A_{-1}). \end{aligned} \tag{2.4}$$

It is the goal of this chapter to establish the following approximation theorem.

Theorem 2.1.1
Let $m_A \geq 11$ and $(A_1, A_0) \in C([0,T_0], H^{m_A}(\mathbb{R}) \times H^{m_A-1}(\mathbb{R}))$ be a solution of the generalised Ginzburg-Landau system (2.4). Then there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions u of (2.1) satisfying

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |u(x,t) - (\varepsilon A_1(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.})| \leq C\varepsilon^{3/2}.$$

Remark 2.1.2
We would like to remark that the existence of such an approximation result is not obvious. There are counterexamples, cf. [Sch95, GS01, SSZ12], where formally correctly derived modulation equations make wrong predictions about the dynamics of the original system.

$\square$

Remark 2.1.3
The Ginzburg-Landau equation can be derived via multiple-scaling analysis as a universal amplitude equation for the description of bifurcating solutions in spatially extended pattern-forming systems close to the threshold of the first instability for many other systems and other situations. Approximation results in the sense of Theorem 2.1.1 have been shown for instance in [CE90, vH91, KSM92, Sch94a, Sch94c] and recently in [SU07]. It has been explained in [HSZ11] that these proofs do not apply in case of a neutral mode at the wave number $k = 0$. We refer to [HSZ11] for a more complete discussion of the literature. Subsequently we will explain that the ideas of [HSZ11] do not apply to the present situation and why a completely different approach is necessary. The major difference to [HSZ11] is that the advective character of the Bénard-Marangoni problem in case of fluids of finite height no longer exists for thin fluids and can therefore no longer be used. In this chapter we present a method which is applicable whenever this kind of instability occurs.

$\square$

Notation. All functions in the following depend on $x \in \mathbb{R}$ and $t \geq 0$. We denote the Fourier transform $\mathcal{F}$ of a function u w.r.t. the spatial variable x with $\widehat{u}$. We define the norm of the Sobolev space $H^m = H^m(\mathbb{R})$ by $\|u\|_{H^m} = \|\widehat{u}\rho^m\|_{L^2}$ where $\rho(k) = (1+k^2)^{1/2}$ and $L^2 = L^2(\mathbb{R})$. Many possibly different constants are denoted with the same symbol C if they can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$.

2.2 Outline of the Approximation Proof

The error $\varepsilon^\beta R = u - \varepsilon\Psi_{an}$ satisfies

$$\partial_t R = L_\varepsilon(\partial_x)R + 2\varepsilon\partial_x^2(\Psi_{an}R) + \varepsilon^\beta\partial_x^2(R^2) + \varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi_{an}),$$

where the residual $\mathrm{Res}(\varepsilon\Psi_{an})$, which is defined by

$$\mathrm{Res}(u) = -\partial_t u + L_\varepsilon(\partial_x)u + \partial_x^2(u^2),$$

contains all terms that do not cancel after inserting the ansatz $\varepsilon\Psi_{an}$ into (2.1). A direct estimate for all $t \in [0, T_0/\varepsilon^2]$ with the help of Gronwall's inequality is not possible due to the term $2\varepsilon\partial_x^2(\Psi_{an}R)$, which could lead to a growth rate $\mathcal{O}(\mathrm{e}^{r\varepsilon t})$ for some $r > 0$. Unfortunately, such a growth cannot be bounded uniformly for $\varepsilon \to 0$ on the whole time interval $[0, T_0/\varepsilon^2]$. Before we explain how to overcome this problem, we show how to make the residual small.

2.2.1 The Improved Approximation

To any given β and n we can find an approximation $\varepsilon\Psi$ such that

$$\varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi) = \mathcal{O}(\varepsilon^n) \qquad \text{and} \qquad \varepsilon\Psi_{an} - \varepsilon\Psi = \mathcal{O}(\varepsilon^2).$$

We then choose β and n so large that no difficulty will occur in bounding the terms $\varepsilon^\beta\partial_x^2(R^2) + \varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi_{an})$ for all $t \in [0, T_0/\varepsilon^2]$. In Appendix 2.A the modified approximation is constructed and the estimates for the residual are given.

2.2.2 The Mode Filters

We come back to the handling of the term $2\varepsilon\partial_x^2(\Psi R)$. We first recall the 'classical' situation, i.e., the case of no neutral modes at the wave number $k = 0$, cf. [vH91, Sch94a, Sch94c, SU07]. For example, we are in this case if we replace the linear operator $L_\varepsilon(\partial_x)$ in our toy problem (2.1) by

$$L_\varepsilon^{sh}(\partial_x) = (1 + \partial_x^2)^2 + \varepsilon^2.$$

For the treatment of this situation so-called mode filters have been introduced, which are defined as multiplication operators in Fourier space by

$$E_j u = \mathcal{F}^{-1}(\chi_j \hat{u}), \quad j = \pm c, s,$$

where $\chi_j : \mathbb{R} \to \{0,1\}$, with $\chi_{\pm c}(k) = 1 \Leftrightarrow |k \mp 1| \leq 1/30$, and $\chi_s = 1 - \chi_c - \chi_{-c}$. Then we have $\varepsilon E_c \Psi = \mathcal{O}(\varepsilon)$ and $\varepsilon E_s \Psi = \mathcal{O}(\varepsilon^2)$. The error is split into a critical and a non-critical part and is scaled differently, i.e., $\varepsilon^\beta R = \varepsilon^\beta R_c + \varepsilon^\beta R_{-c} + \varepsilon^{\beta+1} R_s$, where $R_{\pm c} = E_{\pm c} R$ and $\varepsilon R_s = E_s R$. They satisfy a system of the form

$$\begin{aligned} \partial_t R_c &= L_\varepsilon^{sh}(\partial_x) R_c + \mathcal{O}(\varepsilon^2 |R_c| + \varepsilon^2 |R_s|) + \mathcal{O}(\varepsilon^2), \\ \partial_t R_s &= L_\varepsilon^{sh}(\partial_x) R_s + \mathcal{O}(|R_c| + \varepsilon |R_s|) + \mathcal{O}(1), \end{aligned}$$

due to the fact that E_c applied to the quadratic interaction of critical modes vanishes, i.e., $E_c(\varepsilon \partial_x^2((E_c\Psi)R_c)) = 0$. Since R_s is exponentially damped by the semigroup generated by $L_\varepsilon^{sh}(\partial_x)$ it is easy to obtain an estimate $R_s = \mathcal{O}(|R_c|) + \mathcal{O}(1)$. Inserting this into the first equation yields

$$\partial_t R_c = L_\varepsilon^{sh}(\partial_x) R_c + \mathcal{O}(\varepsilon^2 |R_c|) + \mathcal{O}(\varepsilon^2).$$

A direct estimate for all $t \in [0, T_0/\varepsilon^2]$ with the help of Gronwall's inequality is now possible and so we obtain $R_c = \mathcal{O}(1)$ and $R_s = \mathcal{O}(1)$ for all $t \in [0, T_0/\varepsilon^2]$.

2.2.3 The Handling of the Neutral Modes

The situation complicates heavily when neutral modes at $k = 0$ are present. The non-critical part R_s of the error is no longer damped if we use the same mode filters as in Section 2.2.2. Hence, the previous arguments no longer apply. As a consequence we introduce mode filters E_j by

$$E_j u = \mathcal{F}^{-1}(\chi_j \hat{u}), \quad j = 0, \pm c, s,$$

where $\chi_j : \mathbb{R} \to \{0,1\}$, with $\chi_0(k) = 1 \Leftrightarrow |k| \leq 1/10$, $\chi_{\pm c}(k) = 1 \Leftrightarrow |k \mp 1| \leq 1/30$, and $\chi_s = 1 - \chi_0 - \chi_c - \chi_{-c}$. According to Appendix 2.A we have

$$\varepsilon E_c \Psi = \mathcal{O}(\varepsilon), \quad \varepsilon E_s \Psi = \mathcal{O}(\varepsilon^2), \text{ and } \quad \varepsilon E_0 \Psi = \mathcal{O}(\varepsilon^2). \tag{2.5}$$

The error is split into a critical, a stable, and a neutral part and is scaled differently, i.e., $\varepsilon^\beta R = \varepsilon^\beta R_c + \varepsilon^\beta R_{-c} + \varepsilon^{\beta+1} R_s + \varepsilon^{\beta+1} R_0$, where $R_{\pm c} = E_{\pm c} R$, $\varepsilon R_s = E_s R$, and $\varepsilon R_0 = E_0 R$. They satisfy a system of the form

$$\begin{aligned} \partial_t R_c &= L_\varepsilon(\partial_x) R_c + \mathcal{O}(\varepsilon^2 |R_c| + \varepsilon^2 |R_s| + \varepsilon^2 |R_0|) + \mathcal{O}(\varepsilon^2), \\ \partial_t R_s &= L_\varepsilon(\partial_x) R_s + \mathcal{O}(|R_c| + \varepsilon |R_s| + \varepsilon |R_0|) + \mathcal{O}(1), \end{aligned}$$

$$\partial_t R_0 \quad = \quad L_\varepsilon(\partial_x) R_0 + \mathcal{O}(|R_c| + \varepsilon |R_s| + \varepsilon |R_0|) + \mathcal{O}(1),$$

again due to the fact that E_c applied to the quadratic interaction of critical modes vanishes, i.e., $E_c(\varepsilon \partial_x^2((E_c \Psi) R_c)) = 0$. The critical and the stable part R_c and R_s can be handled as above. However, a growth proportional to t can occur in the equation for R_0. In order to avoid this secular growth, additional properties of the Marangoni problem, such as the reflection symmetry and the existence of conserved quantities, have to be used.

The Case of Finite Height

In this section we explain how to handle the Bénard-Marangoni problem in a strip of not too small height, i.e., we explain the ideas of [HSZ11]. The Bénard-Marangoni problem has a number of conserved quantities, which imply that with the eigenvalue at the wave number $k = 0$ the nonlinear terms at $k = 0$ vanish as well. As a consequence the nonlinear terms must have a derivative in front. In case of finite height the friction still allows long wave surface perturbations to create two waves: one moving to the right and one to the left with the same velocity $c > 0$. Hence, there are two curves of eigenvalues touching zero, and an appropriate model for the behaviour of the neutral modes is given in lowest order by

$$\begin{aligned}
\partial_t R_{0,+} &= \partial_x^2 R_{0,+} + c \partial_x R_{0,+} + \partial_x((E_c \Psi) R_{-c} + (E_{-c} \Psi) R_c + \ldots), \\
\partial_t R_{0,-} &= \partial_x^2 R_{0,-} - c \partial_x R_{0,-} - \partial_x((E_c \Psi) R_{-c} + (E_{-c} \Psi) R_c + \ldots),
\end{aligned}$$

using the reflection symmetry of the problem. The basic idea to avoid the secular growth is to eliminate the nonlinear terms of order $\mathcal{O}(1)$ by a near identity change of coordinates. The necessary non-resonance condition is satisfied due to $c \neq 0$ and the derivative ∂_x in front of the nonlinear terms, cf. [HSZ11, p.108-110]. After the transform the terms of order $\mathcal{O}(\varepsilon)$ are controlled by using the concentration, resp. smoothing, properties of the linear semigroup. In detail, we use that $\int_1^{T_0/\varepsilon^2} \mathrm{e}^{t \partial_x^2} \partial_x(\ldots) \mathrm{d}t = \mathcal{O}(1/\varepsilon)$, which allows to bound the influence of the $\mathcal{O}(\varepsilon)$-terms by an $\mathcal{O}(1)$-bound on the $\mathcal{O}(1/\varepsilon^2)$ time scale.

The Case of a Thin Film

This section contains a summary of the new ideas which can be used to handle the Bénard-Marangoni problem in case of a thin fluid. In this case the friction is so strong that long wave surface perturbations are no longer able to create left and right moving waves. The perturbations simply vanish diffusively. Hence, using again the conserved quantities and the reflection symmetry of the problem

$$\partial_t R_0 = \partial_x^2 R_0 + \partial_x^2((E_c \Psi) R_{-c} + (E_{-c} \Psi) R_c + \ldots),$$

is an appropriate model for the behaviour of the neutral modes in lowest order, cf. our model (2.1). Since now $c = 0$, no near identity change of coordinates is possible such that we cannot eliminate the terms that are of formal order $\mathcal{O}(1)$. Hence, the first idea of [HSZ11] does not apply. Nonetheless, the second idea seems to be a suitable tool. However, we have $\int_1^{T_0/\varepsilon^2} \mathrm{e}^{t\partial_x^2}\partial_x^2(\ldots)\mathrm{d}t = \mathcal{O}(|\ln\varepsilon|)$, which is unbounded for $\varepsilon \to 0$ and does not allow to control the nonlinear terms of order $\mathcal{O}(1)$. The analysis up to this point is explained in detail in Section 2.3. A way to avoid this unwanted growth is then explained in detail in Section 2.4. The idea is first to consider $R_{\pm c}$ as a one time differentiable function of $X = \varepsilon x$, secondly to split $\partial_x^2 = \partial_x\partial_x$, thirdly to use that then $\partial_x((E_c\Psi)R_{-c} + (E_{-c}\Psi)R_c) = \mathcal{O}(\varepsilon)$, and finally to use that $\int_1^{T_0/\varepsilon^2} \mathrm{e}^{t\partial_x^2}\partial_x(\ldots)\mathrm{d}t = \mathcal{O}(1/\varepsilon)$. This allows to bound the influence of the nonlinear $\mathcal{O}(1)$-terms by an $\mathcal{O}(1)$-bound on the $\mathcal{O}(1/\varepsilon^2)$ time scale. However, we then have to show that the full system respects the fact $R_{\pm c}$ is chosen as a one time differentiable function of $X = \varepsilon x$. As already said, this will be done in Section 2.4.

2.3 The Basic Estimates

In the following, we use the definition of the mode filters as they have been defined in Section 2.2.3 and the improved approximation $\varepsilon\Psi$ as it has been defined in Section 2.2.1. We refer to Lemma 2.A.1 for the precise definition of $\varepsilon\Psi$. In order to use this lemma we assume throughout the next sections that $m_A \geq 11$, $m_A - 10 \geq m \geq 1$, and $(A_1, A_0) \in C([0, T_0], H^{m_A}(\mathbb{R}) \times H^{m_A-1}(\mathbb{R}))$.

We use the abbreviations $\varepsilon\Psi_{\pm c} = E_{\pm c}(\varepsilon\Psi)$, $\varepsilon^2\Psi_j = E_j(\varepsilon\Psi)$ for $j = 0, s$, cf. (2.5), and introduce $R_{\pm c} = E_{\pm c}(R)$, $\varepsilon R_j = E_j(R)$ for $j = 0, s$. Inserting

$$\begin{aligned} u &= \varepsilon\Psi + \varepsilon^\beta R \\ &= \varepsilon\Psi_c + \varepsilon\Psi_{-c} + \varepsilon^2\Psi_s + \varepsilon^2\Psi_0 + \varepsilon^\beta R_c + \varepsilon^\beta R_{-c} + \varepsilon^{\beta+1}R_s + \varepsilon^{\beta+1}R_0, \end{aligned}$$

with $\beta = 5/2$ and $R(\cdot, 0) = 0$ into (2.1) and applying the mode filters yields

$$\begin{aligned} \partial_t R_c &= L_\varepsilon(\partial_x)R_c + \varepsilon^2 N_c(R, \Psi) + \varepsilon^3 g_c(R, \Psi) + \varepsilon^{-5/2}\mathrm{Res}_c, \\ \partial_t R_s &= L_\varepsilon(\partial_x)R_s + N_s(R, \Psi) + \varepsilon g_s(R, \Psi) + \varepsilon^{-7/2}\mathrm{Res}_s, \\ \partial_t R_0 &= L_\varepsilon(\partial_x)R_0 + \partial_x^2(N_0(R, \Psi) + \varepsilon g_0(R, \Psi)) + \varepsilon^{-7/2}\mathrm{Res}_0, \end{aligned}$$

with

$$\begin{aligned} \mathrm{Res}_j &= E_j\mathrm{Res}(\varepsilon\Psi), \\ N_c(R, \Psi) &= 2E_c\partial_x^2(\Psi_c R_0 + \Psi_0 R_c + \Psi_s R_c + \Psi_s R_{-c} + \Psi_{-c}R_s + \Psi_c R_s), \\ N_s(R, \Psi) &= 2E_s\partial_x^2(\Psi_c R_c + \Psi_{-c}R_{-c}), \end{aligned}$$

$$
\begin{aligned}
N_0(R,\Psi) &= 2E_0(\Psi_c R_{-c} + \Psi_{-c} R_c),\\
g_c(R,\Psi) &= 2E_c\partial_x^2(\varepsilon^{1/2}R_cR_0 + \varepsilon^{1/2}R_sR_c + \varepsilon^{1/2}R_sR_{-c} + \Psi_sR_s + R_0\Psi_s + R_s\Psi_0)\\
&\quad + \varepsilon E_c\partial_x^2(\varepsilon^{1/2}R_s^2 + 2\varepsilon^{1/2}R_0R_s),\\
g_s(R,\Psi) &= E_s\partial_x^2[\varepsilon^{1/2}R_c^2 + \varepsilon^{1/2}R_{-c}^2 + 2(\Psi_cR_0 + \Psi_cR_s + \Psi_{-c}R_0 + \Psi_{-c}R_s\\
&\qquad + \Psi_0R_c + \Psi_0R_{-c} + \Psi_sR_c + \Psi_sR_{-c})]\\
&\quad + 2\varepsilon E_s\partial_x^2(\Psi_0R_0 + \Psi_0R_s + \Psi_sR_0 + \Psi_sR_s + \varepsilon^{1/2}R_cR_0\\
&\qquad + \varepsilon^{1/2}R_cR_s + \varepsilon^{1/2}R_{-c}R_0 + \varepsilon^{1/2}R_{-c}R_s)\\
&\quad + \varepsilon^2E_s\partial_x^2(\varepsilon^{1/2}R_0^2 + 2\varepsilon^{1/2}R_0R_s + \varepsilon^{1/2}R_s^2),\\
g_0(R,\Psi) &= 2E_0(\Psi_cR_s + \Psi_{-c}R_s + \Psi_sR_c + \Psi_sR_{-c} + \varepsilon^{1/2}R_cR_{-c})\\
&\quad + 2\varepsilon E_0(\varepsilon^{1/2}R_cR_s + \varepsilon^{1/2}R_{-c}R_s + \Psi_0R_0 + \Psi_0R_s + \Psi_sR_0 + \Psi_sR_s)\\
&\quad + \varepsilon^2E_0(\varepsilon^{1/2}R_0^2 + \varepsilon^{1/2}R_s^2 + 2\varepsilon^{1/2}R_0R_s),
\end{aligned}
$$

where we used that the terms

$$
\begin{aligned}
&E_{\pm c}(\Psi_\alpha R_\beta),\ E_{\pm c}(R_\alpha R_\beta) \text{ for } \alpha,\beta\in\{c,-c\},\\
&E_{\pm c}(\Psi_0R_0),\ E_{\pm c}(R_0^2),\ E_{\pm c}(\Psi_0R_{\mp c}),\ E_s(\Psi_{\pm c}R_{\mp c}),\ E_s(R_{\pm c}R_{\mp c}),\\
&E_0(\Psi_{\pm c}R_{\pm c}),\ E_0(R_{\pm c}^2),\ E_0(\Psi_{\pm c}R_0),\ E_0(\Psi_0R_{\pm c})
\end{aligned}
$$

vanish due to disjoint supports in Fourier space.

Using that H^m is an algebra for $m > 1/2$ and $\|\Psi R\|_{H^m} \le \|\Psi\|_{C_b^m}\|R\|_{H^m} \le C\|R\|_{H^m}$, we have the estimates

$$
\begin{aligned}
\|N_c(R,\Psi)\|_{H^m} &\le C\mathcal{R},\\
\|N_s(R,\Psi)\|_{H^{m-2}} &\le C(\|R_c\|_{H^m} + \|R_{-c}\|_{H^m}),\\
\|N_0(R,\Psi)\|_{H^m} &\le C(\|R_c\|_{H^m} + \|R_{-c}\|_{H^m}),\\
\|g_c(R,\Psi)\|_{H^m} &\le C\mathcal{R} + C\mathcal{R}^2,\\
\|g_s(R,\Psi)\|_{H^{m-2}} &\le C\mathcal{R} + C\mathcal{R}^2,\\
\|g_0(R,\Psi)\|_{H^m} &\le C\mathcal{R} + C\varepsilon^{1/2}\mathcal{R}^2,
\end{aligned} \tag{2.6}
$$

where $\mathcal{R}(t) := \|R_c(t)\|_{H^m} + \|R_{-c}(t)\|_{H^m} + \|R_s(t)\|_{H^m} + \|R_0(t)\|_{H^m}$. Due to compact supports in Fourier space of N_c, N_0, g_c, and g_0, there is no loss of regularity for these terms.

With the help of the variation of constants formula, we write the error equations as

$$
\begin{aligned}
R_c(t) &= \int_0^t S_c(t-\sigma)\big[\varepsilon^2 N_c(R,\Psi) + \varepsilon^3 g_c(R,\Psi) + \varepsilon^{-5/2}\mathrm{Res}_c\big](\sigma)\,\mathrm{d}\sigma, \\
R_s(t) &= \int_0^t S_s(t-\sigma)\big[N_s(R,\Psi) + \varepsilon g_s(R,\Psi) + \varepsilon^{-7/2}\mathrm{Res}_s\big](\sigma)\,\mathrm{d}\sigma, \qquad (2.7)\\
R_0(t) &= \int_0^t S_0(t-\sigma)\big[\partial_x^2(N_0(R,\Psi) + \varepsilon g_0(R,\Psi)) + \varepsilon^{-7/2}\mathrm{Res}_0\big](\sigma)\,\mathrm{d}\sigma
\end{aligned}
$$

where $S(t) = \mathrm{e}^{L_\varepsilon(\partial_x)t}$ and $S_j(t) = S(t)E_j$, $j = 0, \pm c, s$.

The semigroups can be estimated as follows.

Lemma 2.3.1
For all $l \in \mathbb{N}_0$ and $m \geq 0$ there exist constants κ, $C > 0$, such that

$$
\begin{aligned}
&\textbf{i)} \quad \|S_0(t)\partial_x^l u\|_{H^m} \leq C\min\{1, t^{-l/2}\}\|u\|_{H^m}, \\
&\textbf{ii)} \quad \|S_{\pm c}(t)\partial_x^l u\|_{H^m} \leq C\mathrm{e}^{\varepsilon^2 t}\|u\|_{H^m}, \\
&\textbf{iii)} \quad \|S_s(t)\partial_x^l u\|_{H^m} \leq C\mathrm{e}^{-\kappa t}(1 + t^{-l/6})\|u\|_{H^m}.
\end{aligned}
$$

PROOF:

i) We have $(\hat{S}(t)\hat{u})(k) = \mathrm{e}^{\lambda_\varepsilon(k)t}\hat{u}(k)$. For $|k| \leq 1/10$ we have $\lambda_\varepsilon(k) \leq -\sigma k^2$ for a $\sigma > 0$ such that

$$
\begin{aligned}
\|S(t)E_0\partial_x^l u\|_{H^m} &\leq C\|\mathcal{F}(S(t)E_0\partial_x^l u)\rho^m\|_{L^2} \\
&\leq C\|k \mapsto \mathrm{e}^{\lambda_\varepsilon(k)t}\chi_0(k)(\mathrm{i}k)^l\hat{u}(k)\rho(k)^m\|_{L^2} \\
&\leq C\sup_{k\in\mathbb{R}}|\mathrm{e}^{-\sigma k^2 t}\chi_0(k)k^l|\|\hat{u}\rho^m\|_{L^2} \\
&\leq C\sup_{|k|\leq 1/10}|\mathrm{e}^{-\sigma k^2 t}k^l|\|u\|_{H^m} \\
&\leq C\min\{1, t^{-l/2}\}\|u\|_{H^m},
\end{aligned}
$$

where $\rho(k) = (1+k^2)^{1/2}$, since for small t we estimate

$$
\sup_{|k|\leq 1/10}|\mathrm{e}^{-\sigma k^2 t}k^l| \leq (1/10)^l,
$$

and for large t we estimate

$$
\sup_{|k|\leq 1/10}|\mathrm{e}^{-\sigma k^2 t}k^l| \leq \sup_{s\in\mathbb{R}}|\mathrm{e}^{-\sigma s^2}(s/\sqrt{t})^l|.
$$

ii) Since $\lambda_\varepsilon(k) \leq \varepsilon^2$ for all $k \in \mathbb{R}$, we immediately obtain $\|S(t)u\|_{H^m} \leq C\mathrm{e}^{\varepsilon^2 t}\|u\|_{H^m}$. Since $\chi_{\pm c}$ has a compact support, there exists an $\mathfrak{r} > 0$, such that $\operatorname{supp}\chi_{\pm c} \subset B_{\mathfrak{r}}(0)$. This implies

$$\|E_{\pm c}\partial_x^l u\|_{H^m} \leq C \sup_{k\in B_{\mathfrak{r}}(0)} |(\mathrm{i}k)^l|\|u\|_{H^m} \leq C\mathfrak{r}^l\|u\|_{H^m},$$

which together with the first estimate yields the assertion.

iii) We have

$$\|S(t)E_s\partial_x^l u\|_{H^m} \leq C \sup_{k\in\mathbb{R}} \left(\mathrm{e}^{\lambda_\varepsilon(k)t}|k|^l\chi_s(k)\right)\|u\|_{H^m}.$$

Since $\sup\limits_{k\in\operatorname{supp}\chi_s} \lambda_\varepsilon(k) =: -\kappa < 0$ there exists a $K > 0$, such that we have $\lambda_\varepsilon(k) \leq -\kappa - Kk^6$ for all $k \in \operatorname{supp}\chi_s$. It follows

$$\sup_{k\in\mathbb{R}}(\mathrm{e}^{\lambda_\varepsilon(k)t}|k|^l\chi_s(k)) \leq \mathrm{e}^{-\kappa t} \sup_{k\in\operatorname{supp}\chi_s} \left(\mathrm{e}^{-Kk^6 t}|k|^l\right) \leq \mathrm{e}^{-\kappa t}\left(\frac{C}{t^{l/6}} + C\right).$$

∎

Combining the estimates for the nonlinear terms with the estimates for the linear semigroup stated in Lemma 2.3.1, we obtain the following result.

Lemma 2.3.2
There exists a constant $C_\Psi > 0$ independent of ε such that

$$\begin{aligned}
\textbf{i)}\quad & \|S_{\pm c}(t-\sigma)(\varepsilon^2 N_{\pm c}(R,\Psi) + \varepsilon^3 g_{\pm c}(R,\Psi))(\sigma)\|_{H^m} \\
&\quad \leq \varepsilon^2 C_\Psi \mathrm{e}^{\varepsilon^2(t-\sigma)}(\mathcal{R}(\sigma) + \varepsilon\mathcal{R}(\sigma)^2), \\
\textbf{ii)}\quad & \|S_s(t-\sigma)(N_s(R,\Psi) + \varepsilon g_s(R,\Psi))(\sigma)\|_{H^m} \\
&\quad \leq C_\Psi \mathrm{e}^{-\kappa(t-\sigma)}\left(1 + (t-\sigma)^{-1/3}\right) \\
&\qquad \times\big(\|R_c(\sigma)\|_{H^m} + \|R_{-c}(\sigma)\|_{H^m} + \varepsilon(\mathcal{R}(\sigma) + \mathcal{R}(\sigma)^2)\big), \\
\textbf{iii)}\quad & \|S_0(t-\sigma)(\partial_x^2 N_0(R,\Psi) + \varepsilon g_0(R,\Psi))(\sigma)\|_{H^m} \\
&\quad \leq C_\Psi \min\{1, (t-\sigma)^{-1}\} \\
&\qquad \times\big(\|R_c(\sigma)\|_{H^m} + \|R_{-c}(\sigma)\|_{H^m} + \varepsilon\mathcal{R}(\sigma) + \varepsilon^{3/2}\mathcal{R}(\sigma)^2\big).
\end{aligned}$$

With the estimate **iii)** from the last lemma we are exactly in the situation described in Section 2.2.3. Executing the integration $\int_1^{T_0/\varepsilon^2}(\ldots)\mathrm{d}\sigma$ in **iii)** would lead to a singularity $\mathcal{O}(|\ln\varepsilon|)$, which is unbounded for $\varepsilon \to 0$. The way to avoid this unwanted growth is explained in the subsequent Section 2.4.

2.4 The Improved Estimates

In order to handle this difficulty, we use that the Fourier transform of the critical part of the error, $\hat{R}_{\pm c}$ is concentrated around $\pm k_c = \pm 1$. The concentration is described by a modification of the norms that we used so far.

Definition 2.4.1
Let $\varepsilon \in (0,1]$, $\rho(k) = (1+k^2)^{1/2}$, *and* $\rho_{\varepsilon,\gamma}(k) := \rho((k-\gamma)/\varepsilon)$. *Then we define for* $1 \le p \le \infty$

$$\|\widehat{u}\|_{L^p(s)} := \|\widehat{u}\rho^s\|_{L^p} \quad \text{and} \quad \|\widehat{u}\|_{L^p_{\varepsilon,\gamma}(s)} := \|\widehat{u}\rho^s_{\varepsilon,\gamma}\|_{L^p}.$$

We write $\widehat{u} \in L^p_{\varepsilon,\gamma}(s)$ *if* $\|\widehat{u}\|_{L^p_{\varepsilon,\gamma}(s)} = \mathcal{O}(1)$ *for* $\varepsilon \to 0$.

We have the following embedding property.

Lemma 2.4.2
For all $p \in [1,\infty)$ *and* $s \ge 0$ *there exists a* $C > 0$ *such that* $\|\widehat{u}\|_{L^p(s)} \le C\|\widehat{u}\|_{L^p_{\varepsilon,\gamma}(s)}$ *and* $\|\widehat{u}\|_{L^p_{\varepsilon,\gamma}(s)} \le C\varepsilon^{-s}\|\widehat{u}\|_{L^p(s)}$ *for all* $\varepsilon \in (0,1]$.

Proof: We have

$$\|\widehat{u}\rho^s\|_{L^p} = \|\widehat{u}\rho^s_{\varepsilon,\gamma} \cdot (\rho/\rho_{\varepsilon,\gamma})^s\|_{L^p} \le \|\rho/\rho_{\varepsilon,\gamma}\|^s_{C^0_b}\|\widehat{u}\rho^s_{\varepsilon,\gamma}\|_{L^p}$$

and

$$\|\widehat{u}\rho^s_{\varepsilon,\gamma}\|_{L^p} = \|\widehat{u}\rho^s \cdot (\rho_{\varepsilon,\gamma}/\rho)^s\|_{L^p} \le \|\rho_{\varepsilon,\gamma}/\rho\|^s_{C^0_b}\|\widehat{u}\rho^s\|_{L^p}.$$

It is easy to see that

$$\|\rho/\rho_{\varepsilon,\gamma}\|_{C^0_b} = \sup_{k\in\mathbb{R}} |\rho(k)/\rho_{\varepsilon,\gamma}(k)| = \mathcal{O}(1),$$

and

$$\|\rho_{\varepsilon,\gamma}/\rho\|_{C^0_b} = \sup_{k\in\mathbb{R}} |\rho_{\varepsilon,\gamma}(k)/\rho(k)| = \mathcal{O}(\varepsilon^{-s}).$$

Therefore, we are done.

∎

The convolution of two concentrated functions yields again a concentrated function. The new point of concentration is given by the sum of the old ones. This statement is made precise in the following lemma.

Lemma 2.4.3
For $s > 1/2$ *there exists a constant* $C > 0$ *such that for all* $\varepsilon \in (0,1]$ *the following holds. For* $\widehat{u} \in L^2_{\varepsilon,\gamma_1}(s)$ *and* $\widehat{v} \in L^2_{\varepsilon,\gamma_2}(s)$ *we have* $\widehat{u} * \widehat{v} \in L^2_{\varepsilon,\gamma_1+\gamma_2}(s)$ *with*

$$\|\widehat{u} * \widehat{v}\|_{L^2_{\varepsilon,\gamma_1+\gamma_2}(s)} \le C\|\widehat{u}\|_{L^2_{\varepsilon,\gamma_1}(s)}\|\widehat{v}\|_{L^2_{\varepsilon,\gamma_2}(s)}.$$

Proof: Using

$$\begin{aligned}\rho_{\varepsilon,\gamma_1+\gamma_2}(k)^s &\leq 2^s\left(\rho\left(\frac{k-m-\gamma_1}{\varepsilon}\right)^s + \rho\left(\frac{m-\gamma_2}{\varepsilon}\right)^s\right)\\ &= 2^s(\rho_{\varepsilon,\gamma_1}(k-m)^s + \rho_{\varepsilon,\gamma_2}(m)^s),\end{aligned}$$

Young's inequality, Sobolev's embedding $L^2(s) \subset L^1$ for $s > 1/2$, and Lemma 2.4.2 yield

$$\begin{aligned}\|(\widehat{u} * \widehat{v})\rho^s_{\varepsilon,\gamma_1+\gamma_2}\|^2_{L^2} &\leq 2^{2s}(\|\widehat{u}\|^2_{L^1}\|\widehat{v}\rho^s_{\varepsilon,\gamma_2}\|^2_{L^2} + \|\widehat{u}\rho^s_{\varepsilon,\gamma_1}\|^2_{L^2}\|\widehat{v}\|^2_{L^1})\\ &\leq 2^{2s}C_s^2(\|\widehat{u}\|^2_{L^2(s)}\|\widehat{v}\rho^s_{\varepsilon,\gamma_2}\|^2_{L^2} + \|\widehat{u}\rho^s_{\varepsilon,\gamma_1}\|^2_{L^2}\|\widehat{v}\|^2_{L^2(s)})\\ &\leq 2^{2s}C_s^2(C_{\gamma_1}+C_{\gamma_2})\|\widehat{u}\|^2_{L^2_{\varepsilon,\gamma_1}(s)}\|\widehat{v}\|^2_{L^2_{\varepsilon,\gamma_2}(s)}.\end{aligned}$$

■

For the control of terms of the form ΨR in weighted spaces we need the following analogue to Young's inequality.

Lemma 2.4.4
For $s > (p-1)/p$ and $p \in [1,\infty)$ there exists a constant $C > 0$ such that for all $\varepsilon \in (0,1]$ the following holds. For $\widehat{u} \in L^1_{\varepsilon,\gamma_1}(s)$ and $\widehat{v} \in L^p_{\varepsilon,\gamma_2}(s)$ we have $\widehat{u} * \widehat{v} \in L^p_{\varepsilon,\gamma_1+\gamma_2}(s)$ with

$$\|\widehat{u} * \widehat{v}\|_{L^p_{\varepsilon,\gamma_1+\gamma_2}(s)} \leq C\|\widehat{u}\|_{L^1_{\varepsilon,\gamma_1}(s)}\|\widehat{v}\|_{L^p_{\varepsilon,\gamma_2}(s)}.$$

Proof: Young's inequality, Sobolev's embedding $L^p(s) \subset L^1$ for $s > (p-1)/p$, and Lemma 2.4.2 yield

$$\begin{aligned}\|\widehat{u} * \widehat{v}\|^p_{L^p_{\varepsilon,\gamma_1+\gamma_2}(s)} &\leq C(\|(\widehat{u}\rho_{\varepsilon,\gamma_1}) * \widehat{v}\|^p_{L^p} + \|\widehat{u} * (\widehat{v}\rho_{\varepsilon,\gamma_2})\|^p_{L^p})\\ &\leq C(\|\widehat{u}\|^p_{L^1_{\varepsilon,\gamma_1}(s)}\|\widehat{v}\|^p_{L^p} + \|\widehat{u}\|^p_{L^1}\|\widehat{v}\|^p_{L^p_{\varepsilon,\gamma_2}(s)})\\ &\leq C\|\widehat{u}\|^p_{L^1_{\varepsilon,\gamma_1}(s)}\|\widehat{v}\|^p_{L^p_{\varepsilon,\gamma_2}(s)}.\end{aligned}$$

■

In order to use this lemma we additionally need

Lemma 2.4.5
Let $m_A \geq 2$. Then there exists a C such that for all $\varepsilon \in (0,1]$ we have

$$\|\widehat{\Psi}_{\pm c}\|_{L^1_{\varepsilon,\pm 1}(1)} \leq C = \mathcal{O}(1).$$

PROOF: The assertion is obvious by the construction of the improved approximation given in Appendix 2.A, the fact that the Fourier transform of $x \mapsto \varepsilon A(\varepsilon x)$ is given by $k \mapsto \widehat{A}(k/\varepsilon)$, and the scaling properties of the L^1-norm.

■

The estimates from (2.6) concerning the nonlinear terms can be transferred to the new situation by using Lemma 2.4.2. All estimates look exactly the same except for the fact that we have to replace $\mathcal{R}$ by $\tilde{\mathcal{R}}$, where

$$\tilde{\mathcal{R}}(t) := \|\widehat{R}_c(t)\|_{L^2_{\varepsilon,1}(1)} + \|\widehat{R}_{-c}(t)\|_{L^2_{\varepsilon,-1}(1)} + \|R_s(t)\|_{H^m} + \|R_0(t)\|_{H^m}, \tag{2.8}$$

that on the right hand side of N_s the term $\|R_c\|_{H^m} + \|R_{-c}\|_{H^m}$ has to be replaced by $\|\widehat{R}_c\|_{L^2_{\varepsilon,1}(1)} + \|\widehat{R}_{-c}\|_{L^2_{\varepsilon,-1}(1)}$. Furthermore, from Lemma 2.4.4 and Lemma 2.4.5 we have the additional estimate

$$\|\widehat{N_0(R,\Psi)}\|_{L^2_{\varepsilon,0}(1)} \leq C(\|\widehat{R}_c\|_{L^2_{\varepsilon,1}(1)} + \|\widehat{R}_{-c}\|_{L^2_{\varepsilon,-1}(1)}).$$

With these estimates at hand we can improve Lemma 2.3.2 **iii)** in the following way.

Lemma 2.4.6
There exists an ε independent constant $C > 0$, such that

$$\begin{aligned}&\|S_0(t-\sigma)\partial_x^2(N_0(R,\Psi) + \varepsilon g_0(R,\Psi))(\sigma)\|_{H^m} \\ &\leq \varepsilon \frac{C}{\sqrt{t-\sigma}}(\tilde{\mathcal{R}}(\sigma) + \varepsilon^{1/2}\tilde{\mathcal{R}}(\sigma)^2).\end{aligned}$$

PROOF: For the first term we obtain

$$\begin{aligned}\|S_0(t-\sigma)\partial_x^2 N_0(R,\Psi)(\sigma)\|_{H^m} &\leq C\min\{1,(t-\sigma)^{-1/2}\}\|\partial_x N_0(R,\Psi)(\sigma)\|_{H^m} \\ &\leq \frac{C}{\sqrt{t-\sigma}}\|k \mapsto \chi_0(k)\mathrm{i}k\widehat{N_0(R,\Psi)}(k,\sigma)\|_{L^2(m)} \\ &\leq \frac{C}{\sqrt{t-\sigma}}\|k \mapsto \chi_0(k)\mathrm{i}k\widehat{N_0(R,\Psi)}(k,\sigma)\|_{L^2} \\ &\leq \frac{C}{\sqrt{t-\sigma}}\|k \mapsto \chi_0(k)\mathrm{i}k\widehat{N_0(R,\Psi)}(k,\sigma)\|_{L^2_{\varepsilon,0}(0)} \\ &\leq \varepsilon\frac{C}{\sqrt{t-\sigma}}\|k \mapsto \widehat{N_0(R,\Psi)}(k,\sigma)\|_{L^2_{\varepsilon,0}(1)} \\ &\leq \varepsilon\frac{C}{\sqrt{t-\sigma}}\big(\|\widehat{R}_c\|_{L^2_{\varepsilon,1}(1)} + \|\widehat{R}_{-c}\|_{L^2_{\varepsilon,-1}(1)}\big).\end{aligned}$$

The other terms can be estimated in exactly the same way with the difference that ∂_x does not gain an ε if applied to g_0 due to the missing concentration.

■

In order to use this estimate we have to control $\|\widehat{R}_{\pm c}(\sigma)\|_{L^2_{\varepsilon,\pm 1}(1)}$. For this purpose we need the following improved estimate for the semigroup.

Lemma 2.4.7
There exists an ε independent constant $C > 0$ such that for all $t \in [0, T_0/\varepsilon^2]$ we have

$$\|k \mapsto \mathrm{e}^{\lambda_\varepsilon(k)t}\chi_c(k)\widehat{u}(k)\|_{L^2_{\varepsilon,1}(1)} \leq \frac{C}{\varepsilon}\frac{1}{\sqrt{t}}\|\widehat{u}\|_{L^2}.$$

PROOF: Since there exists an ε-independent $\alpha > 0$, such that $\lambda_\varepsilon(k) \leq \varepsilon^2 - \alpha(k-1)^2$ for all $|k-1| \leq 1/30$, we have

$$\begin{aligned} \|k \mapsto \mathrm{e}^{\lambda_\varepsilon(k)t}\chi_c(k)\widehat{u}(k)\|^2_{L^2_{\varepsilon,1}(1)} &\leq \int_{\mathbb{R}} \mathrm{e}^{2\left(\varepsilon^2-\alpha(k-1)^2\right)t}\chi_c(k)|\widehat{u}(k)|^2 \left(1+\left(\frac{k-1}{\varepsilon}\right)^2\right) \mathrm{d}k \\ &\leq \sup_{|k-1|\leq 1/30} \left|\mathrm{e}^{2\left(1-\alpha\left(\frac{k-1}{\varepsilon}\right)^2\right)\varepsilon^2 t}\left(1+\left(\frac{k-1}{\varepsilon}\right)^2\right)\right| \|\widehat{u}\|^2_{L^2} \\ &\leq \frac{C}{\varepsilon^2}\frac{1}{t}\|\widehat{u}\|^2_{L^2} \end{aligned}$$

for all $t \in [0, T_0/\varepsilon^2]$.

■

2.5 The Final Estimates

With the previous lemmas we obtain

$$\begin{aligned} \|\widehat{R}_c(t)\|_{L^2_{\varepsilon,1}(1)} &\leq C_{\mathrm{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}}(\tilde{\mathcal{R}}(\sigma) + \varepsilon\tilde{\mathcal{R}}(\sigma)^2)\,\mathrm{d}\sigma, \\ \|R_s(t)\|_{H^m} &\leq C_{\mathrm{Res}} + \int_0^t C\mathrm{e}^{-\kappa(t-\sigma)}\left(1+(t-\sigma)^{-1/3}\right) \\ &\qquad \times(\|\widehat{R}_c(\sigma)\|_{L^2_{\varepsilon,1}(1)} + \|\widehat{R}_{-c}(\sigma)\|_{L^2_{\varepsilon,-1}(1)} + \varepsilon(\tilde{\mathcal{R}}(\sigma) + \tilde{\mathcal{R}}(\sigma)^2)\,\mathrm{d}\sigma, \\ \|R_0(t)\|_{H^m} &\leq C_{\mathrm{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}}\left(\tilde{\mathcal{R}}(\sigma) + \varepsilon^{1/2}\tilde{\mathcal{R}}(\sigma)^2\right)\,\mathrm{d}\sigma, \end{aligned}$$

where $\tilde{\mathcal{R}}$ has been defined in (2.8) and C_{Res} stands for the $\mathcal{O}(1)$ constants that are obtained when integrating the residual terms. Next, we introduce

$$\begin{aligned} q_c(t) &= \sup_{\tau\in[0,t]} \|\widehat{R}_c(\tau)\|_{L^2_{\varepsilon,1}(1)}, \\ q_s(t) &= \sup_{\tau\in[0,t]} \|R_s(\tau)\|_{H^m}, \\ q_0(t) &= \sup_{\tau\in[0,t]} \|R_0(\tau)\|_{H^m}. \end{aligned}$$

We immediately obtain

$$q_s(t) \leq C_{\text{Res}} + Cq(t) + C\varepsilon((q(t)+q_s(t)) + (q(t)+q_s(t))^2),$$

where $q(t) = q_c(t) + q_0(t)$. This yields $q_s(t) \leq C(q(t) + C_{\text{Res}})$ for $C\varepsilon(1 + (q(t) + q_s(t))) \leq 1/2$. As a consequence we have

$$\begin{aligned} q_c(t) &\leq CC_{\text{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}}(q(\sigma) + \varepsilon q(\sigma)^2)\,\mathrm{d}\sigma, \\ q_0(t) &\leq CC_{\text{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}}(q(\sigma) + \varepsilon^{1/2} q(\sigma)^2)\,\mathrm{d}\sigma. \end{aligned}$$

Adding these estimates yields

$$\begin{aligned} q(t) &\leq CC_{\text{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}}(q(\sigma) + \varepsilon^{1/2} q(\sigma)^2)\,\mathrm{d}\sigma \\ &\leq CC_{\text{Res}} + \int_0^t \frac{2C\varepsilon}{\sqrt{t-\sigma}} q(\sigma)\,\mathrm{d}\sigma \end{aligned}$$

if $\varepsilon^{1/2} q(\sigma) \leq 1$. With $T = \varepsilon^2 t$ and $\tilde{q}(T) = q(t)$ we have

$$\tilde{q}(T) \leq CC_{\text{Res}} + \int_0^T \frac{2C}{\sqrt{T-\tilde{\sigma}}} \tilde{q}(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}.$$

Since this equation is independent of ε, Gronwall's inequality immediately yields the existence of a constant $M_q = \mathcal{O}(1)$ such that

$$\sup_{T\in[0,T_0]} \tilde{q}(T) =: M_q < \infty$$

or equivalently

$$\sup_{t\in[0,T_0/\varepsilon^2]} q(t) = M_q < \infty.$$

Then we obtain

$$q_s(t) \leq M_s := C(C_{\text{Res}} + M_q).$$

Choosing $\varepsilon_0 > 0$ so small that $\varepsilon_0^{1/2} M_q \leq 1$ and $C\varepsilon_0(1 + M_q + M_s) \leq 1/2$, we proved error estimates in H^m for every $m \geq 1$. Then Sobolev's embedding theorem yields the sup-estimate as stated in Theorem 2.1.1.

2.A Derivation of the Improved Approximation

It is the goal of this section to prove the following lemma.

Lemma 2.A.1
For $m_A \geq 11$, $m_A - 10 \geq m \geq 1$, and $(A_1, A_0) \in C([0,T_0], H^{m_A}(\mathbb{R}) \times H^{m_A-1}(\mathbb{R}))$ there exists an approximation $\varepsilon\Psi$ with

$$\begin{array}{rl}
\textbf{i)} & \sup\limits_{t\in[0,T_0/\varepsilon^2]} \|E_{\pm c}\mathrm{Res}(\varepsilon\Psi)\|_{H^m} = \mathcal{O}(\varepsilon^{9/2}), \\
\textbf{ii)} & \sup\limits_{t\in[0,T_0/\varepsilon^2]} \|E_{\pm c}\mathrm{Res}(\varepsilon\Psi)\|_{L^2_{\varepsilon,\pm1}(1)} = \mathcal{O}(\varepsilon^{9/2}), \\
\textbf{iii)} & \sup\limits_{t\in[0,T_0/\varepsilon^2]} \|E_{s}\mathrm{Res}(\varepsilon\Psi)\|_{H^m} = \mathcal{O}(\varepsilon^{7/2}), \\
\textbf{iv)} & \sup\limits_{t\in[0,T_0/\varepsilon^2]} \|E_{0}\mathrm{Res}(\varepsilon\Psi)\|_{H^m} = \mathcal{O}(\varepsilon^{11/2}),
\end{array}$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\Psi_{an} - \varepsilon\Psi\|_{H^m} = \mathcal{O}(\varepsilon^{3/2}).$$

PROOF: With the ansatz $\varepsilon\Psi_{an}$, given by (2.2), we have

$$\|E_{\pm c}\mathrm{Res}(\varepsilon\Psi_{an})\|_{H^m} = \mathcal{O}(\varepsilon^{7/2}), \quad \|E_{\pm c}\mathrm{Res}(\varepsilon\Psi_{an})\|_{L^2_{\varepsilon,\pm1}(1)} = \mathcal{O}(\varepsilon^{7/2}),$$
$$\|E_{s}\mathrm{Res}(\varepsilon\Psi_{an})\|_{H^m} = \mathcal{O}(\varepsilon^{5/2}), \quad \|E_{0}\mathrm{Res}(\varepsilon\Psi_{an})\|_{H^m} = \mathcal{O}(\varepsilon^{7/2}).$$

The formal calculation from the Introduction gives these orders at the wave numbers $k = \pm1$ for $E_{\pm c}\mathrm{Res}(\varepsilon\Psi_{an})$, at $k = \pm2$ for $E_s\mathrm{Res}(\varepsilon\Psi_{an})$, and at $k = 0$ for $E_0\mathrm{Res}(\varepsilon\Psi_{an})$. The difference 1/2 in the exponents compared with the formal orders in ε comes from the scaling properties of the L^2-norm. Using the subsequent Remark 2.A.2 shows that the order at $k = 2$ is not influenced by the orders at $k = 0$ and $k = \pm1$, and vice versa.

With these ideas at hand, we can improve the previous ansatz as follows

$$\begin{aligned}
\varepsilon\Psi(x,t) &= \varepsilon A_1(X,T)e^{ix} + \varepsilon^2 A_2(X,T)e^{2ix} + \varepsilon^2 A_0(X,T)/2 + \text{c.c.} \\
&\quad + \varepsilon^2 A_{11}(X,T)e^{ix} + \varepsilon^3 A_{21}(X,T)e^{2ix} + \varepsilon^3 A_{01}(X,T)/2 + \text{c.c.} \\
&\quad + \varepsilon^3 A_3(X,T)e^{3ix} + \varepsilon^4 A_{02}(X,T)/2 + \text{c.c.}
\end{aligned}$$

with $X = \varepsilon x$ and $T = \varepsilon^2 t$. We find

$$\begin{aligned}
\partial_T A_1 &= -\lambda_0''(1)\partial_X^2 A_1/2 + A_1 - 2(A_0A_1 + A_2A_{-1}), \\
\partial_T A_{11} &= -\lambda_0''(1)\partial_X^2 A_{11}/2 + A_{11} \\
&\quad -2(A_{01}A_1 + A_0A_{11} + A_{21}A_{-1} + A_2A_{-11}) - i\lambda_0'''(1)\partial_X^3 A_1/6,
\end{aligned}$$

$$\begin{aligned}
\partial_T A_0 &= -\lambda_0''(0)\partial_X^2 A_0/2 + 2\partial_X^2(A_1 A_{-1}),\\
\partial_T A_{01} &= -\lambda_0''(0)\partial_X^2 A_{01}/2 + 2\partial_X^2(A_{11}A_{-1} + A_1 A_{-11}),\\
\partial_T A_{02} &= -\lambda_0''(0)\partial_X^2 A_{02}/2 + 4\partial_X^2(A_{11}A_{-11}) + \lambda_0''''(0)\partial_X^4 A_0/24,\\
0 &= \lambda_0(2)A_2 - 4A_1^2,\\
0 &= \lambda_0(2)A_{21} - 8A_1A_{11} + i\lambda_0'(2)\partial_X A_2,\\
0 &= \lambda_0(3)A_3 - 18A_1A_2,
\end{aligned}$$

where we used $\lambda_\varepsilon'''(0) = 0$ and $A_{-jl} = \overline{A}_{jl}$. Then at the wave numbers $k = \pm 1$ all terms of order $\mathcal{O}(\varepsilon^4)$, at the wave number $k = 0$ all terms of order $\mathcal{O}(\varepsilon^5)$, and at the wave numbers $k = \pm 2$ and $k = \pm 3$ all terms of order $\mathcal{O}(\varepsilon^3)$ have been cancelled. The subsequent Remark 2.A.2 shows that the residual terms are much smaller outside of these wave numbers such that the formal order, $-1/2$ from the L^2-scaling, gives the correct order. The regularity aspects are discussed in Remark 2.B.1.

■

Remark 2.A.2
In order to make the formal calculations from above rigorous we use

$$\begin{aligned}
\left\|\mu(\cdot - k_1)\varepsilon^{-1}\hat{A}\left(\tfrac{\cdot - k_1}{\varepsilon}\right)\right\|_{L^2(m)} &\leq \varepsilon^{-1}\sup_{k\in\mathbb{R}}\left|\mu(k-k_1)\frac{(1+k^2)^{m/2}}{\left(1+\left(\frac{k-k_1}{\varepsilon}\right)^2\right)^{\ell/2}}\right|\\
&\quad\times\left\|\hat{A}\left(\tfrac{\cdot - k_1}{\varepsilon}\right)\right\|_{L^2_{\varepsilon,k_1}(\ell)}\\
&\leq C\varepsilon^{\ell-m-1/2}\|\hat{A}\|_{L^2(\ell)}
\end{aligned}$$

if $|\mu(k-k_1)| \leq C|k-k_1|^{\ell-m}$, where the loss of $-1/2$ in the exponent comes again from the scaling of the L^2-norm. For example, a direct consequence is

$$\|E_s(A(\varepsilon\cdot)\mathrm{e}^{\mathrm{i}\cdot})\|_{H^m} \leq C\varepsilon^{m_A-m-1/2}\|A\|_{H^{m_A}}$$

such that for instance the order at $k = 2$ is not influenced by the order at $k = 1$, since $m_A - m \geq 10$ has been chosen.

□

2.B Local Existence and Uniqueness of Solutions

Remark 2.B.1
At first sight, the generalised Ginzburg-Landau system (2.4) looks like a quasilinear parabolic system. However, system (2.4) can be handled as a semilinear parabolic

system if $A_1 \in H^{m_A}$ and $A_0 \in H^{m_A-1}$ is chosen. Therefore, we have at least the local existence and uniqueness of solutions in these spaces, cf. [Hen81]. All other approximation equations appearing in Appendix 2.A are either linear algebraic or linear inhomogeneous parabolic PDEs. Hence, the solutions of the other equations exist as long as the solutions of system (2.4) exist. We find $A_2 \in H^{m_A}$, $A_3 \in H^{m_A}$ and $A_{21} \in H^{m_A-1}$. Using the smoothing properties of the linear semigroups we additionally find $A_{11} \in H^{m_A-2}$, $A_{01} \in H^{m_A-3}$, and $A_{02} \in H^{m_A-4}$. Since (2.1) contains 6th order derivatives with A_1 chosen in H^{m_A} we can estimate the residual in H^{m_A-10}.

□

Remark 2.B.2
Our original equation (2.1) is a semilinear parabolic system for which the local existence and uniqueness of solutions can be established in every H^m with $m > 1/2$. The combination of this local existence and uniqueness result with the error estimates gives the long time existence and uniqueness of solutions of (2.1) for all $t \in [0, T_0/\varepsilon^2]$. Hence, for the proof of Theorem 2.1.1 it is sufficient to establish the error estimates in H^m as we did in Section 2.5.

□

Chapter 3

A Quasilinear Toy Problem

Since our considerations are motivated by Bénard-Marangoni convection, we develop a method for justifying the Ginzburg-Landau approximation for quasilinear problems. To this end, in Section 3.1 we modify the semilinear toy problem introduced in Chapter 2 by adding a nonlinear term containing derivatives of highest appearing order. The approximation result is formulated in this section, as well.

In Section 3.2 we point out the differences to the semilinear case and why a new technique is needed to handle the quasilinear case. This technique is based on the method of optimal regularity, which is a tool for proving local existence and uniqueness of solutions for a certain class of quasilinear parabolic problems.

It turns out however, that we have to obtain more detailed information about the size of solutions than are needed for mere local existence and uniqueness results. Therefore, the ideas of the method of optimal regularity have to be modified, this is done in Section 3.3.

We consider a quasilinear Swift-Hohenberg model in Section 3.3.3 in order to demonstrate how to use these new ideas to control the error on the correct time scale.

After we have shown how the Ginzburg-Landau approximation can be justified in this most basic quasilinear case, we finally turn to our original quasilinear toy problem in Section 3.4 and combine the estimates from Chapter 2 with our optimal regularity approach.

3.1 The Approximation Result

This chapter is devoted to the transfer of the methods developed in Chapter 2 to the quasilinear case. Therefore, the toy problem (2.1) is modified by adding $\partial_x^6(u^2)$. We obtain

$$\partial_t u = L_\varepsilon(\partial_x)u + \partial_x^2(u^2) + \partial_x^6(u^2), \tag{3.1}$$

where $L_\varepsilon(\partial_x)$ is the same operator as in (2.1). An existence and uniqueness result for (3.1) can be found in Appendix A.

Making the same ansatz $u(x,t) = \varepsilon\Psi_{an}(x,t)$ as before, namely

$$\varepsilon\Psi_{an}(x,t) = \varepsilon A_1(X,T)\mathrm{e}^{\mathrm{i}x} + \varepsilon^2 A_2(X,T)\mathrm{e}^{2\mathrm{i}x} + \varepsilon^2 A_0(X,T)/2 + \text{c.c.}, \tag{3.2}$$

where $X = \varepsilon x$ and $T = \varepsilon^2 t$, yields

$$\begin{aligned} \partial_T A_1 &= -\lambda_0''(1)\partial_X^2 A_1/2 + A_1 - 4(A_0A_1 + A_2A_{-1}), \\ \partial_T A_0 &= -\lambda_0''(0)\partial_X^2 A_0/2 + 2\partial_X^2(A_1A_{-1}), \\ 0 &= \lambda_0(2)A_2 - 68(A_1)^2. \end{aligned} \tag{3.3}$$

Eliminating A_2 in the first equation of (3.3) by using the third equation of (3.3) gives the generalised Ginzburg-Landau system

$$\begin{aligned} \partial_T A_1 &= -\lambda_0''(1)\partial_X^2 A_1/2 + A_1 - 4A_0A_1 - \frac{272}{\lambda_0(2)}A_1|A_1|^2, \\ \partial_T A_0 &= -\lambda_0''(0)\partial_X^2 A_0/2 + 2\partial_X^2(A_1A_{-1}), \end{aligned} \tag{3.4}$$

where $A_{-j} := \overline{A_j}$. System (3.4) can be handled as a semilinear parabolic system if $A_1 \in H^{m_A}$ and $A_0 \in H^{m_A-1}$ is chosen, see Section 2.B.

The goal of this chapter is to extend the method of the last chapter to quasilinear problems to prove the following approximation theorem.

Theorem 3.1.1
Let $m_A \geq 11$ and $(A_1, A_0) \in C([0,T_0], H^{m_A}(\mathbb{R}) \times H^{m_A-1}(\mathbb{R}))$ be a solution of the generalised Ginzburg-Landau system (3.4)*. Then there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions u of* (3.1) *satisfying*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |u(x,t) - (\varepsilon A_1(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}x} + \text{c.c.})| \leq C\varepsilon^{3/2}.$$

3.2 Differences to the Semilinear Case

We can derive the equations for the error (R_c, R_s, R_0) exactly as in Section 2.3 with the obvious modifications of replacing ∂_x^2 by $(\partial_x^2 + \partial_x^6)$ where necessary. Then we obtain the quasilinear system

$$\partial_t R_c = L_\varepsilon(\partial_x)R_c + \varepsilon^2 N_c(R_c, \Psi_c) + \varepsilon^3 g_c(R, \Psi) + \varepsilon^{-5/2}\text{Res}_c, \tag{3.5}$$

$$\partial_t R_s = L_\varepsilon(\partial_x)R_s + N_s(R_c, \Psi_c) + \varepsilon g_s(R, \Psi) + \varepsilon^{-7/2}\text{Res}_s, \tag{3.6}$$

$$\partial_t R_0 = L_\varepsilon(\partial_x) R_0 + (\partial_x^2 + \partial_x^6)\big(N_0(R_c, \Psi_c) + \varepsilon g_0(R, \Psi)\big) + \varepsilon^{-7/2}\mathrm{Res}_0. \tag{3.7}$$

Even after local existence and uniqueness for the error equations has been established, the methods from Section 2.4 for controlling the error cannot be transferred completely due to the quasilinear nature of system (3.5) – (3.7). For example, if we proceed as in Sections 2.3 and 2.4, we obtain

$$\begin{aligned}
\|\widehat{R}_c(t)\|_{L^2_{\varepsilon,1}(1)} &\leq C_{\mathrm{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}} \big(\tilde{\mathcal{R}}(\sigma) + \varepsilon \tilde{\mathcal{R}}(\sigma)^2\big)\,\mathrm{d}\sigma,\\
\|R_s(t)\|_{H^m} &\leq C_{\mathrm{Res}} + \int_0^t C\mathrm{e}^{-\kappa(t-\sigma)}\left(1 + (t-\sigma)^{-1}\right)\big(\tilde{\mathcal{R}}(\sigma) + \varepsilon \tilde{\mathcal{R}}(\sigma)^2\big)\,\mathrm{d}\sigma,\\
\|R_0(t)\|_{H^m} &\leq C_{\mathrm{Res}} + \int_0^t \frac{C\varepsilon}{\sqrt{t-\sigma}} \big(\tilde{\mathcal{R}}(\sigma) + \varepsilon^{1/2} \tilde{\mathcal{R}}(\sigma)^2\big)\,\mathrm{d}\sigma,
\end{aligned}$$

where $\tilde{R}$ has been defined in (2.8) and C_{Res} stands for the $\mathcal{O}(1)$ constants coming from the residual terms. Unfortunately, the method from Section 2.4 fails due to the nonintegrable singularity in the inequality for R_s.

The idea is to split the integral into parts and to not use the full smoothing of the semigroup near the singularity in the variation of constants formula such that we obtain an estimate of the form

$$\begin{aligned}
\|R_s(t)\|_{H^m} \leq C &\int_0^{t-\delta} \mathrm{e}^{-\kappa(t-\sigma)}\left(1 + (t-\sigma)^{-1}\right)\|\ldots\|_{H^m}\mathrm{d}\sigma\\
&+ C\int_{t-\delta}^t \mathrm{e}^{-\kappa(t-\sigma)}\left(1 + (t-\sigma)^{-1+\alpha/6}\right)\|\ldots\|_{H^{m+\alpha}}\mathrm{d}\sigma,
\end{aligned}$$

with small α, $\delta > 0$. Hence, we avoid the non-integrable singularity but now have to control the $H^{m+\alpha}$-norm of the terms on the right.

The idea to close this gap in regularity is to solve the quasilinear error equations on the time interval $(t-2\delta, t)$ with initial condition $R(t-2\delta) \in H^m$ using a method that guarantees higher regularity of solutions after a short time and that the higher norm of the solution can be bounded by $\|R(t-2\delta)\|_{H^m}$. In Section 3.3 we present such a method and work out the above ideas in detail for a basic example.

3.3 The Optimal Regularity Approach

The variation of constants formula is no longer helpful for solving quasilinear problems. Thus, another approach is needed.

3.3.1 Classical Optimal Regularity Theory

The method of optimal regularity is a means of showing local existence and uniqueness of solutions for the following class of quasilinear parabolic problems, see e.g. [Bea81, Bea84, Uec07]. In the following we give a rough outline of the basic ideas of this method.

Consider the problem of finding solutions to

$$\partial_t u = Lu + N(u), \quad u|_{t=t_0} = u_0, \tag{3.8}$$

where L is an elliptic operator of order $2m$ and $N : H^s \to H^{s-2m}$ is a smooth mapping for all $s \geq 2m$ that is at least quadratic in u.

The above nonlinear problem is related to the following linear inhomogeneous problem of finding a solution u that is H^r w.r.t. the spatial variable to

$$(\partial_t - L)u = f, \tag{3.9}$$

where f is H^{r-2m} in the spatial variable. That means we have to show the existence of the inverse $(\partial_t - L)^{-1}$ that has the optimal gain of $2m$ in spatial regularity. Then we can reformulate (3.8) as a fixed point equation

$$u = (\partial_t - L)^{-1}(N(u)) \tag{3.10}$$

and show the existence and uniqueness of a solution with the help of the contraction mapping principle. In order to show that the right-hand side of (3.10) is a contraction in an appropriate complete metric space, the assumptions that N is at least quadratic in u and that the initial condition u_0 is sufficiently small in an appropriate norm are necessary.

In order to construct the inverse $(\partial_t - L)^{-1}$, we reduce (3.9) to a family of stationary problems with the help of the Laplace transform,

$$(\lambda - L)u = f, \tag{3.11}$$

where $f \in H^{r-2m}$, $\lambda \in \mathbb{C}$. Since we want to replace the differential operator ∂_t by a multiplication operator $\lambda\cdot$ in order to obtain a family of stationary problems, we apply the Laplace transform w.r.t. time. Then we show that this one-parameter family of stationary problems can be solved for all $\lambda \in \mathbb{C}$ with sufficiently large real part and that the corresponding solution u lies in H^r. If the H^r-norm of u fulfils a certain estimate depending on $\|f\|_{H^{r-2m}}$ and $|\lambda|$, we can conclude that the inverse $(\partial_t - L)^{-1}$ exists and that it is a continuous mapping between some appropriate spaces.

Of course, the above description is incomplete and strongly simplified. For example, we have not yet specified the space for the solution u. However, if we keep in mind the general ideas of the method of optimal regularity, we get a clue what properties the space for u should have.

- Since we apply the Laplace transform w.r.t. time, it is advantageous that u is a Sobolev function w.r.t. the time variable t.
- The problem is parabolic, such that one time derivative counts as $2m$ spatial derivatives. Thus, it makes sense to define Sobolev spaces with different regularities in space and time.
- The Laplace transform $\mathcal{L}$, see (A.6), has the property that the initial condition has to be taken into account when derivatives w.r.t. t are transformed, due to
$$\mathcal{L}(\partial_t u)(\cdot,\lambda) = \lambda\mathcal{L}u(\cdot,\lambda) - u_0.$$
Hence, spaces with initial condition $u_0 = 0$ are of particular interest for the reduction of (3.9) to (3.11).

With these considerations in mind, the following definitions make sense.

Definition 3.3.1
Let $H^{r,s}((t_0,t_1)) = L^2((t_0,t_1),H^r(\mathbb{R})) \cap H^s((t_0,t_1),L^2(\mathbb{R}))$, with norm

$$\|u\|_{H^{r,s}((t_0,t_1))} = \left(\int_{t_0}^{t_1} \|u(\cdot,t)\|_{H^r}^2 \,\mathrm{d}t + \|u\|_{H^s((t_0,t_1),L^2)}^2\right)^{\frac{1}{2}}.$$

Moreover, we set

$$H_0^{r,s}((t_0,t_1)) := \{u \in H^{r,s}((t_0,t_1)) \,|\, \partial_t^j u(\cdot,t_0) = 0 \text{ for } 2j < 2s-1\} \tag{3.12}$$

and write $K^r((t_0,t_1);2m) = H^{r,\frac{r}{2m}}((t_0,t_1))$ and $K_0^r((t_0,t_1);2m) = H_0^{r,\frac{r}{2m}}((t_0,t_1))$ to be consistent with the notations in Chapter 4.

The properties of these spaces are collected in Section A.1.2. It is of particular importance that the elements of the space $K_0^r((0,\infty);2m)$ can be completely characterised by their Laplace transform, see Lemma A.1.6. This fact can be used to prove the following result, which relates the non-stationary problem (3.9) to the family of stationary ones (3.11).

Lemma 3.3.2
Let $m \in \mathbb{N}$, $r \geq 2m$ and L a differential operator of order $2m$ with the following properties:

i) *There exists a $\gamma \geq 0$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \gamma$ and all $f \in H^{r-2m}(\mathbb{R})$ there exists a unique solution $u \in H^r(\mathbb{R})$ of*

$$(\lambda - L)u = f.$$

ii) *This solution satisfies the resolvent estimate*

$$\|u\|_{H^r} + |\lambda|^{\frac{r}{2m}} \|u\|_{L^2} \leq C\big(\|f\|_{H^{r-2m}} + |\lambda|^{\frac{r-2m}{2m}} \|f\|_{L^2}\big)$$

with a constant $C > 0$ independent of u and f.

Then for all $r > 2m$, $(r+m)/(2m) \notin \mathbb{N}$, $l > 0$ there exists a constant $C_l > 0$, with $C_l \sim e^{\gamma l}$, such that for all $t_0 < t_1$ with $(t_1 - t_0) < l$ and any $f \in K_0^{r-2m}((t_0, t_1); 2m)$ there exists a unique solution $u \in K_0^r((t_0, t_1); 2m)$ of

$$Mu := (\partial_t - L)u = f,$$

which fulfils $\|u\|_{K^r((t_0,t_1);2m)} \leq C_l \|f\|_{K^{r-2m}((t_0,t_1);2m)}$.

The solution operator from $K_0^{r-2m}((t_0, t_1); 2m)$ to $K_0^r((t_0, t_1); 2m)$ that maps f to u is denoted by M_0^{-1}.

PROOF: See Section A.1.3 for the case $m = 3$. The generalisation for the case of arbitrary m is straightforward.

∎

For the remaining steps of proving the contraction property of the right-hand side of (3.10) and how to incorporate the initial condition u_0, we refer the reader to Section A.1.4.

3.3.2 Modifications for a Different Class of Problems

Essentially, the error equations (3.5) – (3.7) take the form

$$(\partial_t - L)u_1 = \varepsilon N_1(u_1, u_2) + f_1, \tag{3.13}$$

$$(\partial_t - L)u_2 = \mathcal{A}u_1 + \varepsilon N_2(u_1, u_2) + f_2, \tag{3.14}$$

$$(u_1, u_2)|_{t=t_0} = U_0, \tag{3.15}$$

where (3.13) corresponds to (3.5) and (3.14) corresponds to the combination of (3.6) – (3.7). The N_j and $\mathcal{A}$ are nonlinear and linear mappings, respectively, with the maximal loss of $2m$ in spatial regularity.

Unfortunately, the classical method of optimal regularity cannot be applied directly to systems like (3.13) – (3.15) in order to prove local existence and uniqueness, since there are several differences to the type of quasilinear problems considered in Section 3.3.1:

- the nonlinear terms are not necessarily at least quadratic in $u = (u_1, u_2)$,
- additional inhomogeneities f_1 and f_2 are present,

- ultimately, we will need to solve the problem when the initial condition is not necessarily small.

However, for small $\varepsilon > 0$ we expect that the behaviour of system (3.13) – (3.15) is governed by the initial condition U_0 and the inhomogeneities f_1 and f_2. Therefore, as a first step, we extend the fundamental Lemma 3.3.2 for inhomogeneous problems with zero initial conditions to inhomogeneous problems with arbitrary initial conditions.

Throughout this section we assume that $M := (\partial_t - L)$, where the operator L satisfies the conditions of Lemma 3.3.2.

Lemma 3.3.3
Let $2m < r < 3m$, $l > 0$ and $t_0 < t_1$ with $(t_1 - t_0) < l$. There exists a constant $C_l > 0$ such that for any $f \in K^{r-2m}((t_0, t_1); 2m)$ there exists a unique solution $u \in K^r((t_0, t_1); 2m)$ of

$$Mu = f, \quad u|_{t=t_0} = u_0 \in H^{r-m},$$

which fulfils $\|u\|_{K^r((t_0,t_1);2m)} \leq C_l(\|u_0\|_{H^{r-m}} + \|f\|_{K^{r-2m}((t_0,t_1);2m)})$.

Before we present the proof of the lemma, we first explain why we restrict ourselves to the case of low regularities $r \in (2m, 3m)$, because it is indeed possible to prove the result of Lemma 3.3.3 for higher regularities as well.

It is our goal to reduce the above initial value problem for $u \in K^r((t_0, t_1); 2m)$ and $f \in K^{r-2m}((t_0, t_1); 2m)$ to an equivalent problem with $\tilde{u} \in K_0^r((t_0, t_1); 2m)$ and some $\tilde{f} \in K_0^{r-2m}((t_0, t_1); 2m)$ such that we can apply the solution operator M_0^{-1} from Lemma 3.3.2. This means that we have to guarantee that $\partial_t^j \tilde{u}|_{t=t_0} = 0$ for $0 \leq 2j < r/m - 1$ and $\partial_t^n \tilde{f}|_{t=t_0} = 0$ for $0 \leq 2n < r/m - 3$. If $r \in (2m, 3m)$, we only need $\tilde{u}|_{t=t_0} = 0$, since the spaces $K^{r-2m}((t_0, t_1); 2m)$ and $K_0^{r-2m}((t_0, t_1); 2m)$ then coincide, which simplifies the proof significantly. Furthermore, with regard to the proof of Theorem 3.1.1 these low regularities are sufficient. Hence, we refrain from giving the proof for higher regularities.

Proof of Lemma 3.3.3: The trace theorem Lemma A.1.2 guarantees that for any given $u_0 \in H^{r-m}$ we can find a $v \in K^r((t_0, t_1); 2m)$ with $v|_{t=t_0} = u_0$ that satisfies the estimate

$$\|v\|_{K^r((t_0,t_1);2m)} \leq C\|u_0\|_{H^{r-m}}.$$

Since we restricted ourselves to regularities $r \in (2m, 3m)$, we have that the spaces $K^{r-2m}((t_0, t_1); 2m)$ and $K_0^{r-2m}((t_0, t_1); 2m)$ coincide, such that $(f - Mv) \in K_0^{r-2m}((t_0, t_1); 2m)$. If we now set

$$w := M_0^{-1}(f - Mv) \in K_0^r((t_0, t_1); 2m),$$

then $u = v + w \in K^r((t_0,t_1);2m)$ is a solution of

$$Mu = f, \quad u|_{t=t_0} = u_0 \in H^{r-m}$$

with $\|u\|_{K^r((t_0,t_1);2m)} \le C_l\big(\|u_0\|_{H^{r-m}} + \|f\|_{K^{r-2m}((t_0,t_1);2m)}\big)$.

The uniqueness of this solution follows immediately from the fact that the difference of two solutions $u_{(1)}$ and $u_{(2)}$ lies in $K_0^r((t_0,t_1);2m)$ and fulfils

$$M(u_{(1)} - u_{(2)}) = 0 \quad \Leftrightarrow \quad u_{(1)} - u_{(2)} = M_0^{-1}0 = 0.$$

∎

As a preliminary for the existence and uniqueness result for system (3.13) – (3.15), we need the following modification of the contraction mapping principle for problems that exhibit a Jordan structure like (3.13) – (3.15).

Proposition 3.3.4
Let $\varepsilon_0 > 0$, $(X_j, \|\cdot\|_j)$, $j = 1,2$, Banach spaces and $\mathcal{M}_j \subset X_j$, $j = 1,2$, closed. Furthermore, let $F^\varepsilon : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_1 \times \mathcal{M}_2$ be a one parameter family of mappings with

$$F^\varepsilon(x,y) = \begin{pmatrix} f_1^\varepsilon(x,y) \\ f_2^\varepsilon(x,y) \end{pmatrix},$$

such that for all $\varepsilon \in (0,\varepsilon_0)$ there exist constants $a_1, a_2, b_1, b_2 > 0$ such that for all $x_1, x_2 \in \mathcal{M}_1$ and all $y_1, y_2 \in \mathcal{M}_2$ we have

$$\|f_1^\varepsilon(x_1,y_1) - f_1^\varepsilon(x_2,y_2)\|_1 \le \varepsilon a_1\|x_1 - x_2\|_1 + \varepsilon a_2\|y_1 - y_2\|_2, \tag{3.16}$$
$$\|f_2^\varepsilon(x_1,y_1) - f_2^\varepsilon(x_2,y_2)\|_2 \le b_1\|x_1 - x_2\|_1 + \varepsilon b_2\|y_1 - y_2\|_2. \tag{3.17}$$

Then there is an $\varepsilon_1 \in (0,\varepsilon_0)$ such that F^ε has a unique fixed point in $\mathcal{M}_1 \times \mathcal{M}_2$.

Proof: If $\varepsilon_1 < \frac{1}{b_2}$ we immediately conclude from (3.17) that for every $x_0 \in \mathcal{M}_1$ the mappings $f_2^\varepsilon(x_0,\cdot) : \mathcal{M}_2 \to \mathcal{M}_2$ are contractions for all $\varepsilon \in (0,\varepsilon_1)$ with a uniform Lipschitz constant $q = b_2\varepsilon_1 < 1$. Thus, for every $x_0 \in \mathcal{M}_1$ there exists a unique solution $y^{*,\varepsilon}(x_0) \in \mathcal{M}_2$ of

$$y = f_2^\varepsilon(x_0, y).$$

The mapping $x \mapsto y^{*,\varepsilon}(x)$ is Lipschitz continuous, since for all $x_1, x_2 \in \mathcal{M}_1$ we have

$$\begin{aligned}\|y^{*,\varepsilon}(x_1) - y^{*,\varepsilon}(x_2)\|_2 &= \|f_2^\varepsilon(x_1, y^{*,\varepsilon}(x_1)) - f_2^\varepsilon(x_2, y^{*,\varepsilon}(x_2))\|_2 \\ &\le b_1\|x_1 - x_2\|_1 + b_2\varepsilon\|y^{*,\varepsilon}(x_1) - y^{*,\varepsilon}(x_2)\|_2,\end{aligned}$$

and thus

$$\|y^{*,\varepsilon}(x_1) - y^{*,\varepsilon}(x_2)\|_2 \le \frac{b_1}{1 - b_2\varepsilon}\|x_1 - x_2\|_1.$$

With (3.16) we obtain for all $x_1, x_2 \in \mathcal{M}_1$ that

$$\|f_1^\varepsilon(x_1, y^{*,\varepsilon}(x_1)) - f_1^\varepsilon(x_2, y^{*,\varepsilon}(x_2))\|_1 \le a_1\varepsilon\|x_1 - x_2\|_1 + \frac{a_2 b_1 \varepsilon}{1 - b_2\varepsilon}\|x_1 - x_2\|_1.$$

Hence, the mapping $x \mapsto f_1^\varepsilon(x, y^{*,\varepsilon}(x))$ is a contraction on $\mathcal{M}_1$ for all $\varepsilon \in (0, \varepsilon_1)$, if $0 < \varepsilon_1 < (2(a_1 + b_2) + 4a_2 b_1)^{-1}$.

■

With these preparations at hand, we can now turn to the local existence and uniqueness result for system (3.13) – (3.15). For the same reasons as in Lemma 3.3.3 we restrict ourselves to low regularities.

Lemma 3.3.5
Let $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}_0$, $2m < r < \varrho < 3m$, $l > 0$ and $t_0 < \tilde{t}_0 < t_1$ with $(t_1 - t_0) < l$. Suppose N_1, N_2 are smooth mappings from $(H^s)^{n_1+n_2}$ to $(H^{s-2m})^{n_1}$ and $(H^s)^{n_1+n_2}$ to $(H^{s-2m})^{n_2}$, respectively, for all $s \ge 2m$, with $N_1(0) = 0$ and $N_2(0) = 0$. Furthermore, let $\mathcal{A} \in \mathscr{L}((H^s)^{n_1}, (H^{s-2m})^{n_2})$.

Then there exists a constant $C > 0$ such that the following holds: For all $b > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $F = (f_1, f_2) \in (K^{r-2m}((t_0, t_1); 2m))^{n_1} \times (K^{r-2m}((t_0, t_1); 2m))^{n_2}$, $U_0 = (u_{1,0}, u_{2,0}) \in (H^{r-m})^{n_1} \times (H^{r-m})^{n_2}$ with norms less than b the quasilinear system

$$\begin{aligned} Mu_1 &= \varepsilon N_1(u_1, u_2) + f_1, \\ Mu_2 &= \mathcal{A}u_1 + \varepsilon N_2(u_1, u_2) + f_2, \\ (u_1, u_2)|_{t=t_0} &= U_0 \end{aligned}$$

has a unique solution $U = (u_1, u_2) \in (K^r((t_0, t_1); 2m))^{n_1} \times (K^r((t_0, t_1); 2m))^{n_2}$ with

$$\|U\|_{K^r((t_0,t_1);2m)} \le C\big(\|U_0\|_{H^{r-m}} + \|F\|_{K^{r-2m}((t_0,t_1);2m)}\big). \tag{3.18}$$

If $F \in (K^{\varrho-2m}((t_0, t_1); 2m))^{n_1+n_2}$, we have that the restriction $U|_{(\tilde{t}_0,t_1)\times\mathbb{R}}$ lies in $(K^\varrho((\tilde{t}_0, t_1); 2m))^{n_1+n_2}$ and satisfies

$$\|U\|_{K^\varrho((\tilde{t}_0,t_1);2m)} \le C\big(\|U_0\|_{H^{r-m}} + \|F\|_{K^{\varrho-2m}((t_0,t_1);2m)}\big). \tag{3.19}$$

PROOF:

- **Local existence and uniqueness:**

We apply Lemma 3.3.3 componentwise and obtain a unique solution $\tilde{U} = (\tilde{u}_1, \tilde{u}_2) \in (K^r((t_0, t_1); 2m))^{n_1+n_2}$ of

$$MU = F, \quad U|_{t=t_0} = U_0.$$

Furthermore, there exists a constant $C_l > 0$ such that

$$\|\tilde{U}\|_{K^r((t_0,t_1);2m)} \leq C_l\big(\|U_0\|_{H^{r-m}} + \|F\|_{K^{r-2m}((t_0,t_1);2m)}\big).$$

With the decomposition $U = \tilde{U} + W$, we arrive at the problem of finding solutions $W = (w_1, w_2) \in (K_0^r((t_0, t_1); 2m))^{n_1+n_2}$ of the fixed point system

$$w_1 = M_0^{-1}\big(\varepsilon N_1(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2)\big), \tag{3.20}$$
$$w_2 = M_0^{-1}\big(\mathcal{A}w_1 + \varepsilon N_2(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2) + \mathcal{A}\tilde{u}_1\big). \tag{3.21}$$

We can write (3.20) – (3.21) in the form $W = \Phi^\varepsilon(W)$. In the following, we show that Φ^ε satisfies the conditions in Proposition 3.3.4.

As complete metric spaces we choose $\mathcal{B}_1$ and $\mathcal{B}_2$, where

- $\mathcal{B}_1$ is the closed ball in $(K_0^r((t_0, t_1); 2m))^{n_1}$ centred at 0 with radius $\|U_0\|_{H^{r-m}} + \|F\|_{K^{r-2m}((t_0,t_1);2m)}$,
- and $\mathcal{B}_2$ is the closed ball in $(K^r((t_0, t_1); 2m))^{n_2}$ centred at 0 with radius $(1+C_M C_\mathcal{A}(C_l+1))(\|U_0\|_{H^{r-m}}+\|F\|_{K^{r-2m}((t_0,t_1);2m)})$, where C_M, $C_\mathcal{A}$ are positive constants larger than or equal to the operator norms of M_0^{-1} and $\mathcal{A}$, respectively.

From the assumptions for $N = (N_1, N_2)$ we conclude that for all $\mathfrak{r} > 0$ there exist constants $C_\mathfrak{r}, \tilde{C}_\mathfrak{r} > 0$ such that

$$\|N(V)\|_{K^{r-2m}((t_0,t_1);2m)} \leq C_\mathfrak{r}\|V\|_{K^r((t_0,t_1);2m)}, \tag{3.22}$$
$$\|N(V_1) - N(V_2)\|_{K^{r-2m}((t_0,t_1);2m)} \leq \tilde{C}_\mathfrak{r}\|V_1 - V_2\|_{K^r((t_0,t_1);2m)} \tag{3.23}$$

for all V, V_1, $V_2 \in (K^r((t_0, t_1); 2m))^{n_1+n_2}$ with norms less than or equal to $\mathfrak{r}$.

Here we choose $\mathfrak{r} = 2b(2 + C_l + C_M C_\mathcal{A}(C_l + 1))$ such that

$$\|\tilde{U}\|_{K^r((t_0,t_1);2m)} + \|W\|_{K^r((t_0,t_1);2m)} \leq \mathfrak{r}$$

for all $W \in \mathcal{B}_1 \times \mathcal{B}_2$.

From

$$\|M_0^{-1}(\varepsilon N_1(\tilde{U}+W))\|_{K^r((t_0,t_1);2m)} \le \varepsilon C_M C_{\mathfrak{r}}(C_l+1)\big(\|U_0\|_{H^{r-m}}+\|F\|_{K^{r-2m}((t_0,t_1);2m)}\big),$$

we see that the right-hand side of (3.20) maps $\mathcal{B}_1\times\mathcal{B}_2$ to $\mathcal{B}_1$ for all $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0<(C_M C_{\mathfrak{r}}(C_l+1))^{-1}$. Similarly, we conclude from

$$\begin{aligned}\|M_0^{-1}(\mathcal{A}w_1+&\varepsilon N_2(\tilde{U}+W)+\mathcal{A}u_1)\|_{K^r((t_0,t_1);2m)}\\ &\le C_M C_{\mathcal{A}}(1+C_l)\big(\|U_0\|_{H^{r-m}}+\|F\|_{K^{r-2m}((t_0,t_1);2m)}\big)\\ &\quad+\varepsilon C_M C_{\mathfrak{r}}(C_l+1)\big(\|U_0\|_{H^{r-m}}+\|F\|_{K^{r-2m}((t_0,t_1);2m)}\big),\end{aligned}$$

that the right-hand side of (3.21) maps $\mathcal{B}_1\times\mathcal{B}_2$ to $\mathcal{B}_2$ for all $\varepsilon\in(0,\varepsilon_0)$.

Analogously, with the help of (3.23) we see that Φ^ε fulfils the estimates (3.16) and (3.17) in Proposition 3.3.4 for all $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0>0$ sufficiently small. Therefore, we have a unique solution $W\in(K_0^r((t_0,t_1);2m))^{n_1+n_2}$ of (3.20) – (3.21).

It follows from the estimate for $\tilde{U}$ and the fact that $W\in\mathcal{B}_1\times\mathcal{B}_2$, that the unique solution $U=\tilde{U}+W$ satisfies the estimate (3.18).

- **Higher regularity:**

  Let $U\in(K^r((t_0,t_1);2m))^{n_1+n_2}$ be the unique solution found above. Since $K^r((t_0,t_1);2m)\subset H^{\frac{r}{2m}}((t_0,t_1),L^2(\mathbb{R}))$ and $\frac{r}{2m}>1$, we have by Sobolev's embedding that $U(\cdot,t)\in(L^2(\mathbb{R}))^{n_1+n_2}$ is well defined for every $t\in(t_0,t_1)$. Furthermore, since $K^r((t_0,t_1);2m)\subset L^2((t_0,t_1),H^r(\mathbb{R}))$, we have that $U(\cdot,t)\in(H^r(\mathbb{R}))^{n_1+n_2}$ for almost every $t\in(t_0,t_1)$.

  Now we fix any $\tilde{t}_0\in(t_0,t_1)$. Then it holds

$$\|U\|^2_{K^r((t_0,t_1);2m)}\ge\int_{t_0}^{\tilde{t}_0}\|U(\cdot,t)\|^2_{H^r}\,\mathrm{d}t\ge(\tilde{t}_0-t_0)\inf_{t\in(t_0,\tilde{t}_0)}\|U(\cdot,t)\|^2_{H^r}.$$

  Hence, there must exist a $\tau\in(t_0,\tilde{t}_0)$ such that

$$\|U(\cdot,\tau)\|_{H^r}\le 2(\tilde{t}_0-t_0)^{-1/2}\|U\|_{K^r((t_0,t_1);2m)}.$$

  Since $\varrho<3m$, we have $\varrho-m<2m<r$, such that together with the estimate for U from the local existence and uniqueness part we get

$$\|U(\cdot,\tau)\|_{H^{\varrho-m}}\le C(\tilde{t}_0-t_0)^{-1/2}\big(\|U_0\|_{H^{r-m}}+\|F\|_{K^{r-2m}((t_0,t_1);2m)}\big).\tag{3.24}$$

  Then, if $F\in(K^{\varrho-2m}((t_0,t_1);2m))^{n_1+n_2}$ we can solve

$$M\check{u}_1=\varepsilon N_1(\check{u}_1,\check{u}_2)+f_1,$$

$$\begin{aligned}M\check{u}_2 &= \mathcal{A}\check{u}_1 + \varepsilon N_2(\check{u}_1, \check{u}_2) + f_2,\\ (\check{u}_1, \check{u}_2)|_{t=t_0} &= U(\cdot, \tau)\end{aligned}$$

for $\check{U} = (\check{u}_1, \check{u}_2) \in (K^{\varrho}((\tau, t_1); 2m))^{n_1+n_2}$. The solution $\check{U}$ satisfies the estimate

$$\|\check{U}\|_{K^{\varrho}((\tau,t_1);2m)} \leq C\big(\|U(\cdot,\tau)\|_{H^{\varrho-m}} + \|F\|_{K^{\varrho-2m}((\tau,t_1);2m)}\big). \tag{3.25}$$

It is clear that U and $\check{U}$ coincide on (τ, t_1), such that we can combine (3.24) and (3.25) in order to obtain

$$\|U\|_{K^{\varrho}((\tilde{t}_0,t_1);2m)} \leq C\big(\|U_0\|_{H^{r-m}} + \|F\|_{K^{\varrho-2m}((t_0,t_1);2m)}\big).$$

■

3.3.3 A Basic Example

In this section we demonstrate how to combine the variation of constants formula with optimal regularity results in order to estimate the error in a most basic example. We consider a quasilinear Swift-Hohenberg model, namely

$$\partial_t u = \underbrace{-(1+\partial_x^2)^2 u + \varepsilon^2 u}_{=:L_\varepsilon^{sh}(\partial_x)u} - \partial_x^4(u^3). \tag{3.26}$$

With the ansatz

$$\varepsilon\Psi_{an}(x,t) = \varepsilon A_1(X,T)\mathrm{e}^{\mathrm{i}x} + \varepsilon A_{-1}(X,T)\mathrm{e}^{-\mathrm{i}x},$$

$X = \varepsilon x,\ T = \varepsilon^2 t,\quad \overline{A_1} = A_{-1}$, we obtain that at lowest order A_1 has to fulfil the Ginzburg-Landau equation

$$\partial_T A_1 = 4\partial_X^2 A_1 + A_1 - 3A_1|A_1|^2. \tag{3.27}$$

We want to prove the following approximation theorem.

Theorem 3.3.6
Let $m_A \geq 6$. If $A_1 \in C([0,T_0], H^{m_A})$ is a solution of the Ginzburg-Landau equation (3.27)*, then there exist constants C, $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of* (3.26) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} \big|u(x,t) - \big(\varepsilon A_1(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}x} + \text{c.c.}\big)\big| \leq C\varepsilon^{3/2}.$$

By adding the correction terms $+\varepsilon^3 A_3(X,T)\mathrm{e}^{3\mathrm{i}x}$ + c.c. to the ansatz $\varepsilon\Psi_{an}$, we can achieve that the residual for the new ansatz $\varepsilon\Psi$, given by

$$\mathrm{Res}(\varepsilon\Psi) = -\partial_t \varepsilon\Psi + L_\varepsilon^{sh}(\partial_x)\varepsilon\Psi - \partial_x^4((\varepsilon\Psi)^3),$$

is of formal order $\mathcal{O}(\varepsilon^4)$. We write $u = \varepsilon\Psi + \varepsilon^\beta R$, such that the equation for the error reads

$$\partial_t R = L_\varepsilon^{sh}(\partial_x)R - 3\varepsilon^2\partial_x^4(\Psi^2 R) - 3\varepsilon^{\beta+1}\partial_x^4(\Psi R^2) - \varepsilon^{2\beta}\partial_x^4(R^3) + \varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi). \quad (3.28)$$

In order to obtain the desired approximation result, we have to show that R is $\mathcal{O}(1)$ in $C([0,T_0/\varepsilon^2], H^\mu)$ for some $\mu > \frac{1}{2}$. Due to the scaling property of H^m-norms the residual is then of order $\mathcal{O}(\varepsilon^{7/2})$. Hence, we choose $\beta = \frac{3}{2}$ in order to have $\|\varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi)\|_{H^m} = \mathcal{O}(\varepsilon^2)$.

Outline of the Proof

In the following we sketch the basic ideas for controlling the error on the time interval $[0, T_0/\varepsilon^2]$.

First, we consider the classical case, i.e. if the nonlinearity $-\partial_x^4(u^3)$ in (3.26) is replaced by $-u^3$. Then we can use the variation of constants formula and immediately obtain the estimate

$$\tilde{q}(t) \le C + \int_0^t \varepsilon^2(\tilde{q}(\sigma) + \varepsilon^{1/2}\tilde{q}(\sigma)^2 + \varepsilon\tilde{q}(\sigma)^3)\,\mathrm{d}\sigma,$$

where $\tilde{q}(t) := \sup_{\sigma\in[0,t]} \|R(\sigma)\|_{H^m}$. A simple application of Gronwall's inequality then yields the $\mathcal{O}(1)$-boundedness of the error on the time interval $[0, T_0/\varepsilon^2]$ for all $\varepsilon > 0$ sufficiently small. The same method also works if the nonlinearity is at least cubic and contains only derivatives up to third order. Then we have to use the smoothing property of the semigroup generated by $L_\varepsilon^{sh}(\partial_x)$.

In the quasilinear case however, the method fails, since a direct estimate of the integrand in the variation of constants formula gives rise to a non-integrable singularity $\sim (t-\sigma)^{-1}$.

The most intuitive step to avoid this difficulty is to split the interval $[0,t]$ into two parts $[0, t-\delta] \cup [t-\delta, t]$ with some $\delta \in (0,t)$ and to not use the full smoothing of the semigroup on $[t-\delta, t]$. Then we obtain the estimate

$$\begin{aligned}\|R(t)\|_{H^m} &\le C + \int_0^{t-\delta} \frac{\varepsilon^2}{t-\sigma}\|\dots\|_{H^m}\,\mathrm{d}\sigma + \int_{t-\delta}^t \frac{\varepsilon^2}{(t-\sigma)^{1-\alpha/4}}\|\dots\|_{H^{m+\alpha}}\,\mathrm{d}\sigma \\ &\le C + \frac{1}{\delta}\int_0^t \varepsilon^2\|\dots\|_{H^m}\,\mathrm{d}\sigma + \frac{4}{\alpha}\delta^{\alpha/4}\varepsilon^2\|\dots\|_{L^\infty((t-\delta,t),H^{m+\alpha})}\end{aligned}$$

with some $\alpha > 0$.

Obviously, we cannot achieve a uniform estimate for all $t \in (0, T_0/\varepsilon^2]$, since the prefactor $1/\delta$ becomes unbounded for $t \to 0$. This is easily resolved, however, if we only consider $t \geq t_l$ for some fixed $t_l > 0$ so that we can choose $\delta = t_l/4$. The estimate for R in $C([0, t_l], H^\mu)$, $\mu > 1/2$, follows from a local existence and uniqueness result for the quasilinear error equation with the help of the method of optimal regularity and embedding properties of $H^{r,s}$-spaces.

In order to apply Gronwall's inequality we require estimates for the terms in $\| \ldots \|_{H^{m+\alpha}}$ by some $\| \ldots \|_{H^m}$-terms. Therefore, we use Lemma 3.3.5, which guarantees that we have a unique solution of the error equation in $(t - 2\delta, t)$ and that this solution becomes more regular after a short time. Hence, the higher norms in $(t - \delta, t)$ can be estimated by the low norm of the initial condition at $t - 2\delta$.

For the following computations we choose $t_l = 1$ and $\delta = \frac{1}{4}$.

Remark 3.3.7
Another conceivable procedure for proving an approximation result is trying to avoid the variation of constants formula completely by using ideas from the method of optimal regularity for the whole time interval $[0, T_0/\varepsilon^2]$. Thus, we would have to estimate the error in some Sobolev spaces with different space and time regularity, cf. Definition 3.3.1. Then, using embedding results for theses spaces, we would obtain the desired estimate in the C^0-norm.

We work out this method in detail in Section A.2. However, it turns out that the approximation result obtained this way, Theorem A.2.1, has a considerably weaker statement than Theorem 3.3.6.

□

Error Estimates

Using Lemma 3.3.5 we obtain the following local existence and uniqueness result for the error equation (3.28).

Lemma 3.3.8
Let $4 < r < \varrho < 6$ and $0 \leq t_0 < \tilde{t}_0 < t_1 \leq T_0/\varepsilon^2$ with $(t_1 - t_0) < 1$. Then there exists a $C > 0$ such that the following holds: For all $\mathfrak{r} > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial condition $R|_{t=t_0} \in H^{r-2}$ with $\|R|_{t=t_0}\|_{H^{r-2}} \leq \mathfrak{r}$ the error equation (3.28) has a unique solution $R \in K^r((t_0, t_1); 4)$. This solution satisfies

$$\|R\|_{K^r((t_0,t_1);4)} \leq C\big(\|R|_{t=t_0}\|_{H^{r-2}} + \|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{r-4}((t_0,t_1);4)}\big).$$

Furthermore, the restriction $R|_{(\tilde{t}_0,t_1)\times\mathbb{R}}$ *lies in* $K^{\varrho}((\tilde{t}_0,t_1);4)$ *and satisfies*

$$\|R\|_{K^{\varrho}((\tilde{t}_0,t_1);4)} \leq C\big(\|R|_{t=t_0}\|_{H^{r-2}} + \|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{\varrho-4}((t_0,t_1);4)}\big).$$

PROOF: We can prove resolvent estimates for $L_\varepsilon^{sh}(\partial_x)$ in the same way as in Section A.1.1. That means that there exist $\varepsilon_0 > 0$, $C > 0$ and $\gamma > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, $s \geq 4$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \gamma$, and any $f \in H^{s-4}$ the stationary problem

$$(\lambda - L_\varepsilon^{sh}(\partial_x))u = f$$

has a unique solution $u \in H^s$, which satisfies the resolvent estimate

$$\|u\|_{H^s} + |\lambda|^{s/4}\|u\|_{L^2} \leq C\big(\|f\|_{H^{s-4}} + |\lambda|^{(s-4)/4}\|f\|_{L^2}\big).$$

If $m_A \geq 6$, it is guaranteed that $\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi) \in K^{\varrho-4}((t_0,t_1);4)$ and

$$\|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{\varrho-4}((t_0,t_1);4)} = \mathcal{O}(\varepsilon^2)$$

for $\varrho < 6$. Then we can apply Lemma 3.3.5 to the scalar error equation (3.28), where we set $m = 2$, $n_1 = 1$, $n_2 = 0$,

$$\varepsilon N_1(R) = -3\varepsilon^2\partial_x^4(\Psi^2 R) - 3\varepsilon^{5/2}\partial_x^4(\Psi R^2) - \varepsilon^3\partial_x^4(R^3)$$

and

$$f_1 = \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi).$$

∎

In particular, Lemma 3.3.8 guarantees the existence of a unique solution $R \in K^r((0,1);4)$ for any initial condition $R(0) \in H^{r-2}$. Due to Lemma A.1.3 we have the embedding $K^r((0,1);4) \subset C([0,1], H^\mu)$ with $\mu < r-2$. Hence, it holds

$$\sup_{t\in[0,1]} \|R(t)\|_{H^1} \leq C_{[0,1]}$$

with some constant $C_{[0,1]} > 0$.

Now we turn to the error estimate on the time interval $[1, T_0/\varepsilon^2]$. The sectorial operator $L_\varepsilon^{sh}(\partial_x)$ generates an analytic semigroup $S^{sh}(t)$ with the smoothing property

$$\|S^{sh}(t)u\|_{H^r} \leq C\mathrm{e}^{\varepsilon^2 t}\max\{1, t^{-l/4}\}\|u\|_{H^{r-l}}.$$

For $t \geq \frac{1}{2}$ we reformulate (3.28) with the help of the variation of constants formula as

$$\begin{aligned} R(t) &= S^{sh}(t-\tfrac{1}{2})R(\tfrac{1}{2}) \\ &\quad + \int_{1/2}^t S^{sh}(t-\sigma)(\varepsilon^2 N(\Psi,R) + \varepsilon^{5/2}g(\Psi,R) + \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi))(\sigma)\,\mathrm{d}\sigma, \end{aligned}$$

where

$$N(\Psi, R) = -3\partial_x^4(\Psi^2 R), \quad g(\Psi, R) = -3\partial_x^4(\Psi R^2) - \varepsilon^{1/2}\partial_x^4(R^3).$$

Thus, for $t \geq 1$, $\varrho_1 \in (r, 6)$, we get the estimate

$$\begin{aligned}
&\|R(t)\|_{H^{r-2}} \\
&\leq C\|R(\tfrac{1}{2})\|_{H^{r-2}} + C_{\mathrm{Res}} \\
&\quad + \underbrace{\int_{1/2}^{t-1/4} \frac{CC_\Psi}{t-\sigma}\big(\varepsilon^2\|R(\sigma)\|_{H^{r-2}} + \varepsilon^{5/2}(\|R(\sigma)\|_{H^{r-2}}^2 + \|R(\sigma)\|_{H^{r-2}}^3)\big)\,\mathrm{d}\sigma}_{=:I_1(t)} \\
&\quad + \underbrace{\int_{t-1/4}^{t} \frac{CC_\Psi}{(t-\sigma)^{(1-\frac{\varrho_1-r}{4})}}\big(\varepsilon^2\|R(\sigma)\|_{H^{\varrho_1-2}} + \varepsilon^{5/2}(\|R(\sigma)\|_{H^{\varrho_1-2}}^2 + \|R(\sigma)\|_{H^{\varrho_1-2}}^3)\big)\,\mathrm{d}\sigma}_{=:I_2(t)}\,.
\end{aligned}$$

The first integral I_1 can be estimated as follows:

$$\begin{aligned}
I_1(t) &\leq \int_{1/2}^{t-1/4} 4CC_\Psi\big(\varepsilon^2\|R(\sigma)\|_{H^{r-2}} + \varepsilon^{5/2}(\|R(\sigma)\|_{H^{r-2}}^2 + \|R(\sigma)\|_{H^{r-2}}^3)\big)\,\mathrm{d}\sigma \\
&\leq C\int_{1/2}^{1} \big(\varepsilon^2\|R(\sigma)\|_{H^{r-2}} + \varepsilon^{5/2}(\|R(\sigma)\|_{H^{r-2}}^2 + \|R(\sigma)\|_{H^{r-2}}^3)\big)\,\mathrm{d}\sigma \\
&\quad + C\int_{1}^{t} \big(\varepsilon^2\|R(\sigma)\|_{H^{r-2}} + \varepsilon^{5/2}(\|R(\sigma)\|_{H^{r-2}}^2 + \|R(\sigma)\|_{H^{r-2}}^3)\big)\,\mathrm{d}\sigma \\
&\leq \frac{1}{2}C\varepsilon^2 \sup_{\sigma\in[1/2,1]} \big(\|R(\sigma)\|_{H^{r-2}} + \varepsilon^{1/2}(\|R(\sigma)\|_{H^{r-2}}^2 + \|R(\sigma)\|_{H^{r-2}}^3)\big) \\
&\quad + C\int_{1}^{t} \varepsilon^2(q(\sigma) + \varepsilon^{1/2}(q(\sigma)^2 + q(\sigma)^3))\,\mathrm{d}\sigma,
\end{aligned}$$

where

$$q(t) := \sup_{\sigma\in[1,t]} \|R(\sigma)\|_{H^{r-2}}.$$

It follows from Lemma 3.3.8 that $R|_{(\frac{1}{2},1)\times\mathbb{R}}$ lies in $K^{\varrho_1}((\frac{1}{2},1);4)$. Therefore, we have with Lemma A.1.3 that $K^{\varrho_1}((1/2,1)) \subset C([1/2,1], H^\mu)$ with $\mu < \varrho_1 - 2$. In particular, it holds $R \in C([1/2,1], H^{r-2})$ such that we have

$$\begin{aligned}
\sup_{\sigma\in[1/2,1]} \|R(\sigma)\|_{H^{r-2}} &\leq C\|R\|_{K^{\varrho_1}((1/2,1);4)} \\
&\leq C\big(\|R(0)\|_{H^{r-2}} + \|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{\varrho_1-4}((0,1);4)}\big).
\end{aligned}$$

This implies that there exists a constant $C > 0$ such that

$$I_1(t) \leq C + C \int_1^t \varepsilon^2 (q(\sigma) + \varepsilon^{1/2}(q(\sigma)^2 + q(s)^3))\,\mathrm{d}\sigma.$$

For the estimate of I_2 we immediately obtain

$$I_2(t) \leq C\varepsilon^2 \sup_{\sigma \in [t-\frac{1}{4},t]} \big(\|R(\sigma)\|_{H^{\varrho_1-2}} + \varepsilon^{1/2}(\|R(\sigma)\|^2_{H^{\varrho_1-2}} + \|R(\sigma)\|^3_{H^{\varrho_1-2}})\big).$$

Now, we fix some $\varrho_2 \in (\varrho_1, 6)$. We use the embedding property from Lemma A.1.3 to see that

$$\sup_{\sigma \in [t-\frac{1}{4},t]} \|R(\sigma)\|_{H^{\varrho_1-2}} \leq C\|R\|_{K^{\varrho_2}((t-\frac{1}{4},t);4)}.$$

If $\|R(t-\frac{1}{2})\|_{H^{r-2}} \leq \tilde{\mathfrak{r}}$ for some positive constant $\tilde{\mathfrak{r}}$ and $\varepsilon > 0$ sufficiently small, we can now apply Lemma 3.3.8 to obtain

$$\|R\|_{K^{\varrho_2}((t-\frac{1}{4},t);4)} \leq C(\|R(t-\tfrac{1}{2})\|_{H^{r-2}} + \|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{\varrho_2-4}((t-\frac{1}{2},t);4)}).$$

Finally, it holds

$$\begin{aligned}
\|R(t-\tfrac{1}{2})\|_{H^{r-2}} &\leq \sup_{\sigma \in [\frac{1}{2},t]} \|R(\sigma)\|_{H^{r-2}} \\
&\leq \sup_{\sigma \in [\frac{1}{2},1]} \|R(\sigma)\|_{H^{r-2}} + q(t) \\
&\leq C\|R\|_{K^{\varrho_1}((\frac{1}{2},1);4)} + q(t) \\
&\leq C + q(t),
\end{aligned}$$

and thus

$$I_2(t) \leq C + C\varepsilon^2(q(t) + \varepsilon^{1/2}(q(t)^2 + q(t)^3)).$$

Combining the estimates for I_1 and I_2, we arrive at

$$q(t) \leq C + C\varepsilon^2(q(t) + \varepsilon^{1/2}(q(t)^2 + q(t)^3) + C\int_1^t \varepsilon^2(q(\sigma) + \varepsilon^{1/2}(q(\sigma)^2 + q(\sigma)^3))\,\mathrm{d}\sigma.$$

If $\varepsilon^{1/2}(q(t) + q(t)^2) \leq 1$ and $\varepsilon < \frac{1}{2C+1}$, we have

$$q(t) \leq C + C\int_1^t \varepsilon^2 q(\sigma)\,\mathrm{d}\sigma.$$

Gronwall's inequality then yields $q(t) \leq C\mathrm{e}^{C\varepsilon^2(T_0/\varepsilon^2-1)} \leq C\mathrm{e}^{CT_0} =: M_q$. Thus, we have to choose $\varepsilon_0 > 0$ so small that $\varepsilon_0^{1/2}(M_q + M_q^2) < 1$ and $\varepsilon_0 < \frac{1}{2C+1}$ in order to gain

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |R(x,t)| \leq C(\sup_{t\in[0,1]} \|R(t)\|_{H^1} + \sup_{t\in[1,T_0/\varepsilon^2]} \|R(t)\|_{H^{r-2}})$$

$$\le C(C_{[0,1]} + M_q) = \mathcal{O}(1).$$

Additionally, we have to choose $\varepsilon_0 > 0$ so small that Lemma 3.3.8 holds with $\tilde{\mathfrak{r}} = M_q$. Thus, we can indeed apply the optimal regularity result at arbitrary times in the interval $[0, T_0/\varepsilon^2]$.

3.4 Controlling the Error

With these preparations we now come back to the proof of Theorem 3.1.1. We follow the lines of Section 3.3.3 and start with the local existence and uniqueness result for the system (3.5) – (3.7).

Lemma 3.4.1
Let $6 < r < \varrho < 9$ and $0 \le t_0 < \tilde{t}_0 < t_1 \le T_0/\varepsilon^2$ with $(t_1 - t_0) \le 1$. Then there exists a $C > 0$ such that the following holds: For all $\mathfrak{r} > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial condition $\mathbf{R}|_{t=t_0} = (R_c, R_s, R_0)|_{t=t_0} \in (H^{r-3})^3$ with $\|\mathbf{R}|_{t=t_0}\|_{H^{r-3}} \le \mathfrak{r}$ system (3.5) - (3.7) has a unique solution $\mathbf{R} = (R_c, R_s, R_0) \in (K^r((t_0, t_1); 6))^3$. This solution satisfies

$$\begin{aligned}\|\mathbf{R}\|_{K^r((t_0,t_1);6)} \le C\big(&\|\mathbf{R}|_{t=t_0}\|_{H^{r-3}} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{K^{r-6}((t_0,t_1);6)} \\ &+ \|\varepsilon^{-7/2}\mathrm{Res}_s\|_{K^{r-6}((t_0,t_1);6)} + \|\varepsilon^{-7/2}\mathrm{Res}_0\|_{K^{r-6}((t_0,t_1);6)}\big).\end{aligned}$$

Furthermore, the restriction $\mathbf{R}|_{(\tilde{t}_0,t_1)\times\mathbb{R}}$ lies in $(K^\varrho((\tilde{t}_0, t_1); 6))^3$ and satisfies

$$\begin{aligned}\|\mathbf{R}\|_{K^\varrho((\tilde{t}_0,t_1);6)} \le C\big(&\|\mathbf{R}|_{t=t_0}\|_{H^{r-3}} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{K^{\varrho-6}((t_0,t_1);6)} \\ &+ \|\varepsilon^{-7/2}\mathrm{Res}_s\|_{K^{\varrho-6}((t_0,t_1);6)} + \|\varepsilon^{-7/2}\mathrm{Res}_0\|_{K^{\varrho-6}((t_0,t_1);6)}\big).\end{aligned}$$

PROOF: We see that the error system (3.5) – (3.7) is of the same form as the quasilinear system considered in Lemma 3.3.5, with $n_1 = 1$, $n_2 = 2$, $m = 3$ and

$$\begin{aligned}u_1 &= R_c, \\ u_2 &= (R_s, R_0), \\ \varepsilon N_1(U) &= \varepsilon^2 N_c(R_c, \Psi_c) + \varepsilon^3 g_c(R, \Psi), \\ \varepsilon N_2(U) &= (\varepsilon g_s(R, \Psi), \varepsilon g_0(R, \Psi)), \\ \mathcal{A}u_1 &= (N_s(R_c, \Psi_c), (\partial_x^2 + \partial_x^6)N_0(R_c, \Psi_c)), \\ f_1 &= \varepsilon^{-5/2}\mathrm{Res}_c, \\ f_2 &= (\varepsilon^{-7/2}\mathrm{Res}_s, \varepsilon^{-7/2}\mathrm{Res}_0).\end{aligned}$$

Furthermore, if the approximation is sufficiently regular, we have that

$$\sup_{t_0,t_1\in[0,T_0/\varepsilon^2],\, 0<t_1-t_0\le 1} \|(\varepsilon^{-5/2}\mathrm{Res}_c, \varepsilon^{-7/2}\mathrm{Res}_s, \varepsilon^{-7/2}\mathrm{Res}_0)\|_{K^{\varrho-6}((t_0,t_1);6)} < C_{\mathrm{R}} < \infty,$$

for all $\varrho \in (6,9)$. Now we can apply Lemma 3.3.5 with $b := \max\{\mathfrak{r}, C_{\mathrm{R}}\}$ and immediately obtain the stated result.

∎

Together with the embedding properties of K^r-spaces, Lemma 3.4.1 yields that for all $r \in (6,9)$ we have that $\mathbf{R} \in \big(C([0,1], C(\mathbb{R}))\big)^3$ and $R \in \big(C([\frac{1}{2},1], H^{r-3})\big)^3$. Furthermore, the norms of $\mathbf{R}$ in theses spaces can be estimated by the H^{r-3}-norm of the initial condition and the norm of the residual in $K^{\varrho-2m}((0,1);6)$. From this, the sup-estimate from Theorem 3.1.1 follows for $t \in [0,1]$.

For controlling the error on the time interval $[1, T_0/\varepsilon^2]$, we note that the estimates for R_c and R_0 are exactly as in the semilinear case, cf. Section 2.5, such that we only have to take care of the equation (3.6) for R_s. In analogy to the semilinear case we define

$$\begin{aligned} q_c(t) &= \sup_{\tau\in[1,t]} \|\hat{R}_c(\tau)\|_{L^2_{\varepsilon,1}(1)}, \\ q_s(t) &= \sup_{\tau\in[1,t]} \|R_s(\tau)\|_{H^{r-3}}, \\ q_0(t) &= \sup_{\tau\in[1,t]} \|R_0(\tau)\|_{H^{r-3}}, \\ q(t) &= q_c(t) + q_0(t). \end{aligned}$$

We proceed as in Section 3.3.3 and split the appearing integral in the variation of constants formula into two parts $\int_{1/2}^{t-1/4} \|\dots\|_{H^{r-3}} \mathrm{d}\sigma + \int_{t-1/4}^{t} \|\dots\|_{H^{\varrho_1-3}} \mathrm{d}\sigma$ with some $\varrho_1 \in (r,9)$. This leads to the estimate

$$\begin{aligned} \|R_s(t)\|_{H^{r-3}} \le C &+ C(q(t) + \varepsilon q(t)^2 + \varepsilon q_s(t) + \varepsilon^2 q_s(t)^2) \\ &+ C\Big(\sup_{\sigma\in[t-\frac{1}{4},t]} (\|R_c(\sigma)\|_{H^{\varrho_1-3}} + \varepsilon \|R_c(\sigma)\|_{H^{\varrho_1-3}}) \\ &\qquad + \varepsilon \sup_{\sigma\in[t-\frac{1}{4},t]} (\|R_0(\sigma)\|_{H^{\varrho_1-3}} + \|R_s(\sigma)\|_{H^{\varrho_1-3}}) \\ &\qquad + \varepsilon^2 \sup_{\sigma\in[t-\frac{1}{4},t]} (\|R_0(\sigma)\|_{H^{\varrho_1-3}} + \|R_s(\sigma)\|_{H^{\varrho_1-3}})^2\Big). \end{aligned}$$

Since $\hat{R}_0$ and $\hat{R}_c$ have compact support in Fourier space, we can estimate

$$\|R_0(\sigma)\|_{H^{\varrho_1-3}} \le C\|R_0(\sigma)\|_{H^{r-3}}, \quad \|R_c(\sigma)\|_{H^{\varrho_1-3}} \le C\|\hat{R}_c(\sigma)\|_{L^2_{\varepsilon,1}(1)}$$

such that

$$\begin{aligned} \|R_s(t)\|_{H^{r-3}} \le C + C\big(&q(t) + \varepsilon q(t)^2 + \varepsilon q_s(t) + \varepsilon^2 q_s(t)^2 \\ &+ \varepsilon \sup_{\sigma\in[t-\frac{1}{4},t]} \|R_s(\sigma)\|_{H^{\varrho_1-3}} + \varepsilon^2 \sup_{\sigma\in[t-\frac{1}{4},t]} \|R_s(\sigma)\|^2_{H^{\varrho_1-3}}\big). \end{aligned}$$

Now, we fix a $\varrho_2 \in (\varrho_1, 9)$. Due to the embedding property of K^r-spaces, we have that

$$\sup_{\sigma\in[t-\frac{1}{4},t]} \|R_s(\sigma)\|_{H^{\varrho_1-3}} \leq C\|R_s\|_{K^{\varrho_2}((t-\frac{1}{4},t);6)}.$$

According to Lemma 3.4.1, for any $\mathfrak{r} > 0$ we can find an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we can solve the error system (3.5) – (3.7) on $(t-\frac{1}{2}, t)$ with initial condition $(R_c, R_s, R_0)|_{\tau=t-\frac{1}{2}}$ and have the estimate

$$\begin{aligned}
\|R_s\|_{K^{\varrho_2}((t-\frac{1}{4},t);6)} &\leq \|\mathbf{R}\|_{K^{\varrho_2}((t-\frac{1}{4},t);6)} \\
&\leq C\big(\|\mathbf{R}|_{\tau=t-\frac{1}{2}}\|_{H^{r-3}} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{K^{\varrho_2-6}((t-\frac{1}{2},t);6)} \\
&\qquad + \|\varepsilon^{-7/2}\mathrm{Res}_s\|_{K^{\varrho_2-6}((t-\frac{1}{2},t);6)} + \|\varepsilon^{-7/2}\mathrm{Res}_0\|_{K^{\varrho_2-6}((t-\frac{1}{2},t);6)}\big) \\
&\leq C + C\big(\sup_{\sigma\in[1/2,t]} \|R_c(\sigma)\|_{H^{r-3}} \\
&\qquad + \sup_{\sigma\in[1/2,t]} \|R_s(\sigma)\|_{H^{r-3}} + \sup_{\sigma\in[1/2,t]} \|R_0(\sigma)\|_{H^{r-3}}\big) \\
&\leq C + C(q(t) + q_s(t)).
\end{aligned}$$

Hence, we obtain

$$q_s(t) \leq C + C(q(t) + \varepsilon q(t)^2 + \varepsilon q_s(t) + \varepsilon^2 q_s(t)^2).$$

Then if $2\varepsilon C(1 + q(t) + q_s(t)) \leq 1$ we get $q_s(t) \leq C + Cq(t)$.

The remaining estimates are exactly the same as in Section 2.5 and therefore, we are done.

Chapter 4

The Bénard-Marangoni Problem

Now we turn to the treatment of the Bénard-Marangoni problem. We start with a brief introduction into the physical background and state the governing equations in Section 4.1.

In Section 4.2 we prove local existence and uniqueness of solutions. Up to now, only a local existence and uniqueness result for the Bénard-Marangoni problem was shown by Nishida and Teramoto in [NT07] in case of functions that are periodic w.r.t. the unbounded spatial variable. However, we can extend this result in order to prove local existence and uniqueness for solutions that are Sobolev functions w.r.t. the unbounded spatial variable. Hereby we use and adjust methods developed by Beale in [Bea84].

In Section 4.3 we discuss the spectral situation and find that it is qualitatively of the same form as in the considered toy problems. Then we reformulate the governing equations as an evolutionary system, for which we prove the approximation result.

The problem gains a new quality compared with the toy problems in the previous chapters due to the presence of (nonlinear) boundary conditions. This heavily complicates the proof of local existence and uniqueness of solutions, for instance. However, once we overcome these difficulties for the existence and uniqueness proof, we can use the same ideas to bring the corresponding error system into the same abstract form as in Chapter 3.

4.1 Modelling

The Bénard-Marangoni problem consists in finding the time-dependent velocity field $\mathbf{u} = (u_1, u_2)^\top(x, z, t)$ and the temperature distribution $\mathrm{T} = \mathrm{T}(x, z, t)$ of a liquid bounded from below by a heat insulating rigid bottom at $z = -1$ as well as finding the evolution of the free top surface $\eta = \eta(x, t)$ open to the atmosphere.

In contrast to the Rayleigh-Bénard problem, see [Ray16], any buoyancy effects are neglected. However, a coupling between the evolution of the velocity field and the evolution of the temperature field is still possible due to the fact that surface tension depends linearly on temperature. The Bénard-Marangoni model is considered to be adequate when the liquid layer is very thin (less than 0.1 cm), see [Tak81a], or in microgravity, when buoyancy effects have to be neglected anyway. It is well known, see [Pea58, Tak81a], that if the difference between the temperature of the bottom and the temperature of the atmosphere is large enough, the purely conducting steady state will become unstable and convection will set in forming a pattern of rolls.

In the following, we give a short description of the mechanism leading to convection. Consider a film of liquid on a plate of temperature T_B. Furthermore, the atmosphere above the liquid possesses the temperature $\mathrm{T}_A < \mathrm{T}_B$. Naturally, there will be small perturbations to the free surface, such that it does not remain completely flat, see Figure 4.1a. Thus, some regions of the surface lie closer to the heated bottom than others and therefore are of higher temperature, see Figure 4.1b. Since surface tension decreases with increasing temperature and the liquid tends to minimise the surface areas of higher surface tension, movement of surface particles sets in from warmer to cooler regions. This is known as Marangoni effect, see Figure 4.1c. Due to the continuity of the liquid, particles from the bottom rise up generating hot spots. Thus, the temperature difference at the surface increases as well as the pull tangential to the surface, see Figure 4.1d. Since the surface forces back to the flat state, cooler particles are pushed downwards and a circular movement is generated, see Figure 4.1e.

In order to investigate the onset of convection we study the evolution of perturbations of the purely conducting steady state.

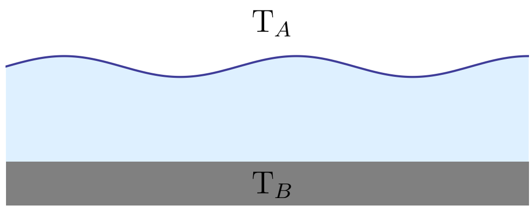
Let x denote the unbounded horizontal spatial variable and z the bounded vertical one. After suitable nondimensionalisation, see [Tak81a], we obtain evolution equations for the free top surface η, as well as for the velocity field $\mathbf{u}$ and the temperature field T in the time-dependent domain

$$\Omega_t := \{(x,z,t) \in \mathbb{R}^3 \,|\, t \geq 0,\ -1 \leq z \leq \eta(x,t)\}.$$

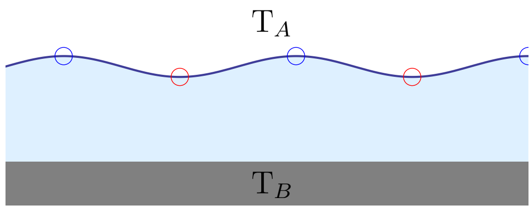
The kinematic boundary condition at the top surface, the evolution of the velocity and the temperature field yield

$$\begin{aligned} \partial_t \eta + u_1 \eta' &= u_2 && \text{at } z = \eta(x,t), && (4.1)\\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla \mathrm{p} + P_r \Delta \mathbf{u} && \text{in } \Omega_t, && (4.2)\\ \partial_t \mathrm{T} + (\mathbf{u}\cdot\nabla)\mathrm{T} &= \Delta \mathrm{T} + u_2 && \text{in } \Omega_t, && (4.3) \end{aligned}$$

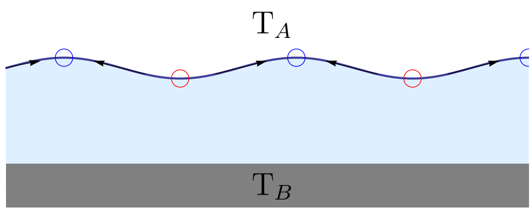
where we used the short-hand notation $\eta' = \partial_x \eta$ and $\eta'' = \partial_x^2 \eta$. The incompress-



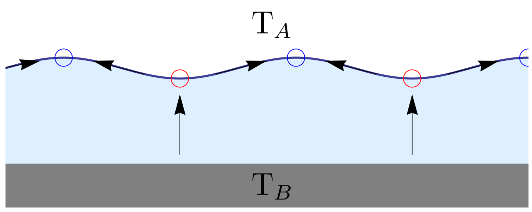
(a) Liquid film open to an atmosphere of temperature T_A, bounded from below by a plate of temperature $T_B > T_A$.



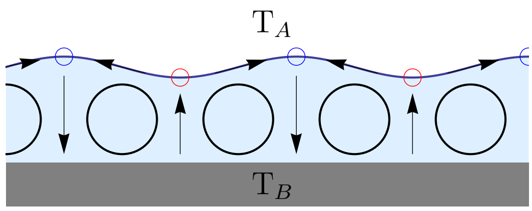
(b) Perturbations of flat surface lead to temperature inhomogeneities; high surface tension at low temperature, low surface tension at high temperature.



(c) Marangoni effect: surface particles move from low to high surface tension.



(d) Heated particles rise up; temperature difference increases.



(e) Cooled particles sink down; circular movement is generated.

Figure 4.1: Onset of Bénard-Marangoni convection.

ibility of the fluid requires that

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega_t. \tag{4.4}$$

The unknown pressure $\mathrm{p} = \mathrm{p}(x, z, t)$ can be expressed in terms of the other unknowns using a projection on the divergence-free vector fields, see Lemma 4.2.7.

We assume that the rigid bottom is impermeable to the liquid and that there is no slip. Furthermore, the temperature of the liquid at the bottom is equal to that of the bottom plate. Hence, we obtain

$$\mathbf{u} = 0 \qquad \text{at } z = -1, \tag{4.5}$$

$$\mathrm{T} = 0 \qquad \text{at } z = -1. \tag{4.6}$$

At the top surface we have the normal stress condition

$$\tfrac{C_r}{P_r}\mathrm{p} - B_o\eta + \eta'' - 2C_r\partial_z u_2 = N_{\mathbf{n}}(\eta, \mathbf{u}, \mathrm{T}) \qquad \text{at } z = \eta(x,t), \tag{4.7}$$

the tangential stress condition, which describes the Marangoni effect,

$$\partial_x u_2 + \partial_z u_1 + M_a(\partial_x \mathrm{T} - \eta') = N_{\mathbf{t}}(\eta, \mathbf{u}, \mathrm{T}) \qquad \text{at } z = \eta(x,t), \tag{4.8}$$

and Newton's law of cooling,

$$\partial_z \mathrm{T} + B_i(\mathrm{T} - \eta) = N_h(\eta, \mathrm{T}) \qquad \text{at } z = \eta(x,t). \tag{4.9}$$

The nonlinear terms in conditions (4.7) – (4.9) read

$$\begin{aligned}
N_{\mathbf{n}}(\eta, \mathbf{u}, \mathrm{T}) &= -M_a C_r(\eta - \mathrm{T})\frac{\eta''}{\sqrt{1+\eta'^2}} \\
&\quad + 2C_r\frac{1}{1+\eta'^2}[\eta'^2\partial_x u_1 - \eta'(\partial_x u_2 + \partial_z u_1)] \\
&\quad - 2C_r\frac{\eta'^2}{1+\eta'^2}\partial_z u_2, \\
N_{\mathbf{t}}(\eta, \mathbf{u}, \mathrm{T}) &= -2(\partial_z u_2 - \partial_x u_1)\eta' + \eta'^2(\partial_x u_2 + \partial_z u_1) \\
&\quad - M_a\eta'\sqrt{1+\eta'^2}\partial_z \mathrm{T} + M_a(1 - \sqrt{1+\eta'^2})(\partial_x \mathrm{T} - \eta'), \\
N_h(\eta, \mathrm{T}) &= (1 - \sqrt{1+\eta'^2})(1 + B_i(\mathrm{T} - \eta)) + \eta'\partial_x \mathrm{T}.
\end{aligned}$$

The appearing dimensionless parameters are the Biot number B_i, the Bond number B_o, the crispation number C_r, the Marangoni number M_a and the Prandtl number P_r, for a definition see [Tak81a]. M_a plays a special role since it is proportional to the temperature difference of the bottom and the atmosphere. Thus, it will be seen as a bifurcation parameter.

4.2 Local Existence and Uniqueness

It is advantageous to transfer system (4.1) – (4.9) to an equivalent one on the fixed flat domain $\Omega := \{(x,z) \in \mathbb{R}^2 \mid -1 \le z \le 0\}$. We follow [Bea84] and introduce for every $t \ge 0$ the transformation $\mathcal{T}_t : \Omega \to \Omega_t$ by

$$\mathcal{T}_t(x,z) = (x, \bar{\eta}(x,z,t) + z(1 + \bar{\eta}(x,z,t))),$$

where

$$\bar{\eta}(x,z,t) = \mathcal{F}_x^{-1}((k \mapsto e^{|k|z}) \cdot \mathcal{F}_x \eta)(x,t).$$

Hereby, $\mathcal{F}_x$ denotes the Fourier transform w.r.t. the horizontal variable x and $\mathcal{F}_x^{-1}$ the inverse transformation. Hence, we have with this definition that $\bar{\eta}|_{z=0} = \eta$.

Now let

$$J_{\mathcal{T}_t} = (J_{ij}) = \begin{pmatrix} 1 & 0 \\ (1+z)\partial_x \bar{\eta} & 1 + \bar{\eta} + (1+z)\partial_z \bar{\eta} \end{pmatrix},$$
$$J_{\mathcal{T}_t}^{-1} = (J^{ij}) = \begin{pmatrix} 1 & 0 \\ -\frac{(1+z)\partial_x \bar{\eta}}{1+\bar{\eta}+(1+z)\partial_z\bar{\eta}} & \frac{1}{1+\bar{\eta}+(1+z)\partial_z\bar{\eta}} \end{pmatrix}$$

denote the Jacobian of $\mathcal{T}_t$, with respect to (x,z), and its inverse. Furthermore, we write

$$J := \det J_{\mathcal{T}_t} = J_{22}$$

for the Jacobian determinant. We introduce the new unknowns

$$\mathrm{q} = \mathrm{p} \circ \mathcal{T}_t, \quad \theta = \mathrm{T} \circ \mathcal{T}_t, \quad \frac{1}{J} J_{\mathcal{T}_t} \mathbf{v} = \mathbf{u} \circ \mathcal{T}_t.$$

The transformation of the velocity field has the advantage that we have $\operatorname{div} \mathbf{u} = 0$ in the coordinates of Ω_t if and only if $\operatorname{div} \mathbf{v} = 0$ in the coordinates of Ω. Furthermore, the kinematic boundary condition (4.1) becomes linear.

This finally leads to the following system of partial differential equations,

$$\partial_t \eta - v_2 = 0 \quad \text{at } z = 0, \tag{4.10}$$
$$\partial_t \mathbf{v} - P_r \Delta \mathbf{v} + \nabla \mathrm{q} = \mathbf{F}_0(\nabla \mathrm{q}, \eta, \mathbf{v}) \quad \text{in } \Omega, \tag{4.11}$$
$$\partial_t \theta - \Delta\theta - v_2 = F_1(\eta, \mathbf{v}, \theta) \quad \text{in } \Omega, \tag{4.12}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \tag{4.13}$$
$$\mathbf{v} = 0 \quad \text{at } z = -1, \tag{4.14}$$
$$\theta = 0 \quad \text{at } z = -1, \tag{4.15}$$
$$\tfrac{C_r}{P_r}\mathrm{q} - B_o \eta + \eta'' - 2C_r \partial_z v_2 = F_2(\eta, \mathbf{v}, \theta) \quad \text{at } z = 0, \tag{4.16}$$
$$\partial_x v_2 + \partial_z v_1 + M_a \partial_x(\theta - \eta) = F_3(\eta, \mathbf{v}, \theta) \quad \text{at } z = 0, \tag{4.17}$$

$$\partial_z\theta + B_i(\theta - \eta) = F_4(\eta, \theta) \qquad \text{at } z = 0, \tag{4.18}$$
$$\eta = \eta_0,\ \mathbf{v} = \mathbf{v}_0,\ \theta = \theta_0 \qquad \text{at } t = 0, \tag{4.19}$$

where $\mathbf{F}_0$, F_j, $j = 1, \dots, 4$, contain terms at least quadratic in the unknowns. For the explicit formulae see Appendix B.1.

Remark 4.2.1
If η is sufficiently regular, the composition with $\mathcal{T}_t$ preserves K^r-spaces as defined in Definition 4.2.2, see [Bea84, Lemma 5.2]. Furthermore, $\mathcal{T}_t$ is a diffeomorphism if η is sufficiently small in norm.

□

4.2.1 Function Spaces

For the treatment of (4.10) – (4.18) we will again use K^r-spaces, cf. Definition 3.3.1. This time, we include the domain for the spatial variable in our notation, since we will have to distinguish between functions in Ω and on $\partial\Omega$.

Definition 4.2.2
Let $\mathcal{M} \subset \mathbb{R}^d$, $-\infty \le t_0 < t_1 \le \infty$, $r, s > 0$, then

$$\begin{aligned} H^{r,s}((t_0,t_1)\times\mathcal{M}) &:= L^2((t_0,t_1), H^r(\mathcal{M})) \cap H^s((t_0,t_1), L^2(\mathcal{M})), \\ K^r((t_0,t_1)\times\mathcal{M}) &:= H^{r,\frac{r}{2}}((t_0,t_1)\times\mathcal{M}), \\ K_0^r((t_0,t_1)\times\mathcal{M}) &:= \{u \in K^r((t_0,t_1)\times\mathcal{M}) \,|\, \partial_t^j u = 0,\ 0 \le 2j < r-1\}. \end{aligned}$$

If $\mathcal{M} = \mathbb{R}$, we simply write $K^r((t_0,t_1)) := K^r((t_0,t_1)\times\mathbb{R})$ and $K_0^r((t_0,t_1)) := K^r((t_0,t_1)\times\mathbb{R})$.

It is important to note that if the domain $\mathcal{M}$ for the spatial variable possesses a non-empty boundary, the traces of $K^r((t_0,t_1)\times\mathcal{M})$ exhibit a more complicated behaviour than in case of $\mathcal{M} = \mathbb{R}^d$.

Lemma 4.2.3
Let $r > 0$ and $u \in K^r((t_0,t_1)\times\mathcal{M})$. Furthermore, let ∂_ν denote the normal derivative on $\partial\mathcal{M}$ oriented towards the interior of $\mathcal{M}$.

Then we have

$$\mathfrak{a}^{(j)} := \partial_\nu^j u|_{[t_0,t_1]\times\partial\mathcal{M}} \in K^{r-j-\frac{1}{2}}((t_0,t_1)\times\partial\mathcal{M}) \qquad \text{for } 0 \le j < r - \frac{1}{2}$$

and

$$\mathfrak{b}^{(k)} := \partial_t^k u|_{\{t_0\}\times\mathcal{M}} \in H^{r-2k-1}(\mathcal{M}) \qquad \text{for } 0 \le k < \frac{r-1}{2}.$$

Let j_r be the largest integer in $[0, r-\frac{1}{2})$ and k_r be the largest integer in $[0,(r-1)/2)$. Then we have that the mapping $\mathfrak{tr}$ from $K^r((t_0,t_1)\times\mathcal{M})$ to $\prod_{j=0}^{j_r} K^{r-j-\frac{1}{2}}((t_0,t_1)\times\partial\mathcal{M})\times\prod_{k=0}^{k_r} H^{r-2k-1}(\mathcal{M})$ with

$$\mathfrak{tr}(u) = \big(\mathfrak{a}^{(0)},\dots,\mathfrak{a}^{(j_r)},\mathfrak{b}^{(0)},\dots,\mathfrak{b}^{(k_r)}\big),$$

is linear and continuous but in general not surjective.

Proof: See [LM72b, Theorem 4.2.1].

■

That the mapping $\mathfrak{tr}$ is in general not surjective can be seen in the following way. If r is large enough, then the spaces $K^r((t_0,t_1)\times\mathcal{M})$ are a subset of the space of continuous functions on $[t_0,t_1]\times\overline{\mathcal{M}}$. So if we prescribe a function $\mathfrak{a}^{(0)}$ on $\{t_0\}\times\mathcal{M}$ and another function $\mathfrak{b}^{(0)}$ on $[t_0,t_1]\times\partial\mathcal{M}$, then there can only be a function $u\in C([t_0,t_1]\times\mathcal{M})$ with $u|_{\{t_0\}\times\mathcal{M}}=\mathfrak{a}^{(0)}$ and $u|_{[t_0,t_1]\times\partial\mathcal{M}}=\mathfrak{b}^{(0)}$, if $\mathfrak{a}^{(0)}$ and $\mathfrak{b}^{(0)}$ coincide on $\{t_0\}\times\partial\mathcal{M}$. It is clear that analogous conditions must hold for derivatives up to a certain order depending on $r>0$.

For notational convenience we formulate the following lemma for the case that $(t_0,t_1)=(0,\infty)$ and $\mathcal{M}=\mathbb{R}^d_+ := \mathbb{R}^{d-1}\times(0,\infty)$.

Lemma 4.2.4
Let $r>\frac{3}{2}$, $r-\frac{1}{2}\notin\mathbb{N}$. Let j_r be the largest integer in $[0,r-\frac{1}{2})$ and k_r be the largest integer in $[0,(r-1)/2)$.

We define $\mathfrak{T}$ as the space of functions $(\mathfrak{a}^{(0)},\dots,\mathfrak{a}^{(j_r)},\mathfrak{b}^{(0)},\dots,\mathfrak{b}^{(k_r)})$ with $\mathfrak{a}^{(j)}\in K^{r-j-\frac{1}{2}}((0,\infty)\times\mathbb{R}^{d-1})$ for $j=0,\dots,j_r$ and $\mathfrak{b}^{(k)}\in H^{r-2k-1}(\mathbb{R}^d_+)$ for $k=0,\dots,k_r$ that satisfy the (local) compatibility conditions

$$\begin{aligned}\partial_t^k\mathfrak{a}^{(j)}(0,x_1,\dots,x_{d-1}) &= \partial_{x_d}^j\mathfrak{b}^{(k)}(x_1,\dots,x_{d-1},0)\\ &\qquad\textit{for all } j,\ k\in\mathbb{N}_0 \textit{ with } j+2k<r-\frac{3}{2}.\end{aligned}\tag{4.20}$$

Then the trace operator $\mathfrak{tr}$ from Lemma 4.2.3 is a surjective mapping from the space $K^r((0,\infty)\times\mathbb{R}^d_+)$ to $\mathfrak{T}$.

Furthermore, there exists a linear continuous extension operator $\mathfrak{ex}:\mathfrak{T}\to K^r((0,\infty)\times\mathbb{R}^d_+)$ with $\mathfrak{tr}\circ\mathfrak{ex}=\mathrm{id}_{\mathfrak{T}}$.

Proof: See [LM72b, Theorem 4.2.3].

■

Remark 4.2.5
If j and k are non-negative integers with $j+2k=r-\frac{3}{2}$, then $\mathfrak{a}^{(j)}$ and $\mathfrak{b}^{(k)}$ have to fulfil quite complicated *global* compatibility conditions, see [LM72b, Section 4.2.5]. We avoid dealing with these unwieldy expressions by requiring that $r-\frac{1}{2}\notin\mathbb{N}$.

□

4.2.2 The Associated Linear Problem

Following the usual steps of the method of optimal regularity, we have to show the existence and uniqueness of solutions $(\eta, \mathbf{v}, \theta, \nabla \mathrm{q}, \mathrm{q}|_{z=0})$, with $\eta \in K_0^{r+1/2}((0,\infty))$, $\mathbf{v} \in (K_0^r((0,\infty)\times\Omega))^2$, $\theta \in K_0^r((0,\infty)\times\Omega)$, $\nabla \mathrm{q} \in (K_0^{r-2}((0,\infty)\times\Omega))^2$, $\mathrm{q}|_{z=0} \in K_0^{r-3/2}((0,\infty))$[1], of

$$
\begin{aligned}
\partial_t \eta - v_2 &= g_0 && \text{at } z=0, && (4.21)\\
\partial_t \mathbf{v} - P_r \Delta \mathbf{v} + \nabla \mathrm{q} &= \mathbf{f}_0 && \text{in } \Omega, && (4.22)\\
\partial_t \theta - \Delta\theta - v_2 &= f_1 && \text{in } \Omega, && (4.23)\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega, && (4.24)\\
\mathbf{v} &= 0 && \text{at } z=-1, && (4.25)\\
\theta &= 0 && \text{at } z=-1, && (4.26)\\
\frac{C_r}{P_r}\mathrm{q} - B_o\eta + \eta'' - 2C_r\partial_z v_2 &= f_2 && \text{at } z=0, && (4.27)\\
\partial_x v_2 + \partial_z v_1 + M_a \partial_x(\theta-\eta) &= f_3 && \text{at } z=0, && (4.28)\\
\partial_z\theta + B_i(\theta-\eta) &= f_4 && \text{at } z=0, && (4.29)
\end{aligned}
$$

with $g_0 \in K_0^{r-1/2}((0,\infty))$, $\mathbf{f}_0 \in (K_0^{r-2}((0,\infty)\times\Omega))^2$, $f_1 \in K_0^{r-2}((0,\infty)\times\Omega)$, f_2, f_3, $f_4 \in K_0^{r-3/2}((0,\infty))$.

Remark 4.2.6
The introduction of the inhomogeneous term g_0 may seem as an unnecessary generalisation, since the right-hand side of (4.10) is equal to 0. But since we want to prove an approximation theorem, we also need a local existence and uniqueness result for the error equations, cf. Section 3.3.2. The error equation corresponding to (4.10) will in general contain residual terms. Thus, it will be necessary to consider the more general form of the linear inhomogeneous problem.

□

To bring the above system in the form of an evolution equation, we have to eliminate q as an unknown in the evolution equation (4.22) as well as in the boundary condition (4.27). This is done by applying a projection $\mathscr{P}$ on the subspace of divergence-free vector fields introduced in [Bea84]. $\mathscr{P}$ is defined as the projection on $L^2(\Omega)$ orthogonal to

$$\{\nabla\phi \,|\, \phi \in H^1(\Omega),\ \phi = 0 \text{ at } z=0\}.$$

[1]Note that $\nabla \mathrm{q} \in (K_0^{r-2}((0,\infty)\times\Omega))^2$, $\mathrm{q}|_{z=0} \in K_0^{r-3/2}((0,\infty))$ does not necessarily imply $\mathrm{q} \in K^{r-1}((0,\infty)\times\Omega)$, see [Bea81].

This corresponds to a projection on the divergence-free vector fields $\mathbf{v} \in H^1(\Omega)$ with $v_2 = 0$ at $z = -1$. The properties of $\mathscr{P}$ are summarised in the following lemma.

Lemma 4.2.7
$\mathscr{P}$ is a bounded operator on $(H^r(\Omega))^2$ and $(K^r((0,\infty)\times\Omega))^2$ for $r \geq 0$. If $\phi \in H^1(\Omega)$, then $\mathscr{P}(\nabla\phi) = \nabla\pi$, where π fulfils

$$\pi = \phi \text{ at } z = 0,\ \Delta\,\pi = 0 \text{ in } \Omega,\ \partial_z\pi = 0 \text{ at } z = -1.$$

Proof: See [Bea81, Lemma 3.1].

■

Applying $\mathscr{P}$ to (4.22) yields

$$\partial_t\mathbf{v} - P_r\mathscr{P}\Delta\mathbf{v} = -\mathscr{P}\nabla\mathrm{q} + \mathscr{P}\mathbf{f}_0,$$

with $\mathbf{v} \in \mathscr{P}(K_0^r((0,\infty)\times\Omega))^2 := L^2((0,\infty),\mathscr{P}(H^r(\Omega))^2) \cap H_0^{r/2}((0,\infty),\mathscr{P}(L^2(\Omega))^2)$.

Thus, $\mathscr{P}\nabla\mathrm{q}$ can be split up as

$$\mathscr{P}\nabla\mathrm{q} = \nabla\mathrm{q}_1 + \nabla\mathrm{q}_2 + \nabla\mathrm{q}_3$$

with

$$\begin{aligned}
&\Delta\mathrm{q}_j = 0 && \text{in } \Omega && \text{for } j = 1,2,3, && (4.30)\\
&\partial_z\mathrm{q}_j = 0 && \text{at } z = -1 && \text{for } j = 1,2,3, && (4.31)\\
&\mathrm{q}_1 = 2P_r\partial_z v_2 && \text{at } z = 0, && && (4.32)\\
&\mathrm{q}_2 = \tfrac{P_rB_o}{C_r}\eta - \eta'' && \text{at } z = 0, && && (4.33)\\
&\mathrm{q}_3 = \tfrac{P_r}{C_r}f_2 && \text{at } z = 0. && && (4.34)
\end{aligned}$$

Then we can write

$$\nabla\mathrm{q}_1 = 2P_r\mathscr{E}(\partial_z v_2|_{z=0}), \quad \nabla\mathrm{q}_2 = \mathscr{E}\left(\tfrac{P_rB_o}{C_r} - \partial_x^2\right)\eta, \quad \nabla\mathrm{q}_3 = \tfrac{P_r}{C_r}\mathscr{E}f_2,$$

where $\mathscr{E}$ is a bounded operator from $H^{r-3/2}$ to $(\mathscr{P}H^{r-2}(\Omega))^2$. Thus, q can be reconstructed as

$$\mathrm{q} = \mathrm{q}_0 + \mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3,$$

where q_0 is defined by

$$\nabla\mathrm{q}_0 = (\mathrm{id} - \mathscr{P})(P_r\Delta\mathbf{v} + \mathbf{f}_0).$$

Furthermore, equations (4.24) and (4.27) drop out such that the system now reads

$$\partial_t\eta - v_2 = g_0 \qquad \text{at } z = 0, \qquad (4.35)$$

$$\begin{aligned}
&\partial_t \mathbf{v} - P_r \mathscr{P} \Delta \mathbf{v} + 2P_r \mathscr{E}(\partial_z v_2|_{z=0}) + \dots \\
&\qquad \dots + \mathscr{E}\left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right)\eta = \mathscr{P}\mathbf{f}_0 - \tfrac{P_r}{C_r}\mathscr{E} f_2 && \text{in } \Omega, && (4.36)\\
&\partial_t \theta - \Delta\theta - v_2 = f_1 && \text{in } \Omega, && (4.37)\\
&\mathbf{v} = 0 && \text{at } z=-1, && (4.38)\\
&\theta = 0 && \text{at } z=-1, && (4.39)\\
&\partial_x v_2 + \partial_z v_1 + M_a \partial_x(\theta - \eta) = f_3 && \text{at } z=0, && (4.40)\\
&\partial_z \theta + B_i(\theta-\eta) = f_4 && \text{at } z=0. && (4.41)
\end{aligned}$$

Next we homogenise boundary conditions (4.40) and (4.41). The homogenisation of (4.40) is done in a similar fashion as in [Bea84].

Lemma 4.2.8
Let $r > 3$, $r - \frac{1}{2} \notin \mathbb{N}$. For any $f \in K_0^{r-3/2}((0,\infty))$ there exists a vector field $\mathbf{v}_f = (v_{f,1}, v_{f,2})^\top \in (K_0^r((0,\infty)\times\Omega))^2$ with the following properties:

$$\begin{aligned}
\operatorname{div} \mathbf{v}_f &= 0 && \textit{in } \Omega,\\
\mathbf{v}_f &= 0 && \textit{at } z=-1,\\
v_{f,2} &= 0 && \textit{at } z=0,\\
\partial_z v_{f,2} &= 0 && \textit{at } z=0,\\
\partial_x v_{f,2} + \partial_z v_{f,1} &= f && \textit{at } z=0.
\end{aligned}$$

The mapping $\mathfrak{e}_1 : f \mapsto \mathbf{v}_f$ is a linear continuous operator from $K_0^{r-3/2}((0,\infty))$ to $(K_0^r((0,\infty)\times\Omega))^2$.

Proof: Let j_r be the largest integer in $[0, r+\frac{1}{2})$ and let k_r be the largest integer in $[0, \frac{r}{2})$. Then set $\mathfrak{b}^{(k)} = 0$ for $k = 0, \dots, k_r$, $\mathfrak{a}^{(j)} = 0$ for $j = 0, 1, 3, \dots, j_r$ and $\mathfrak{a}^{(2)} = f$. Since $f \in K_0^{r-3/2}((0,\infty))$, we have $\partial_t^l f(x,0) = 0$ for $0 \le 2l < r - \frac{5}{2}$. Hence, we have

$$\partial_t^k \mathfrak{a}^{(j)}|_{t=0} = \partial_z^j \mathfrak{b}^{(k)}|_{z=0} \quad \text{for } 0 \le j + 2k \le r - \frac{1}{2}. \tag{4.42}$$

Then we have for $r > 3$, that the $\mathfrak{a}^{(j)}$ and $\mathfrak{b}^{(k)}$ satisfy all the necessary compatibility conditions (4.20) from Lemma 4.2.4 to guarantee the existence of a $w \in K^{r+1}((0,\infty)\times\Omega)$ with

$$\begin{aligned}
&\partial_t^k w(x,z,0) = 0 && \text{for } 0 \le 2k < r,\\
&\partial_z^j w(x,0,t) = 0 && \text{for } j = 0,1,3,\dots,j_r,\\
&\partial_z^2 w(x,0,t) = f(x,t).
\end{aligned}$$

Note that the first condition implies that $w \in K_0^{r+1}((0,\infty)\times\Omega)$.

Let $\psi \in C_0^\infty(\mathbb{R})$ be a cut-off function in z with support near $z = 0$ and set $\tilde{w} = \psi w \in K_0^{r+1}((0,\infty)\times\Omega)$. Now set $\mathbf{v}_f := (\partial_z\tilde{w}, -\partial_x\tilde{w})^\top \in (K_0^r((0,\infty)\times\Omega))^2$. Then it holds

$$\operatorname{div}\mathbf{v}_f = 0 \text{ in } \Omega, \quad \mathbf{v}_f = 0 \text{ at } z = -1.$$

Furthermore, since $\tilde{w}$ and $\partial_z\tilde{w}$ are constantly zero at $z = 0$, we have

$$v_{f,2} = -\partial_x\tilde{w} = 0, \quad \partial v_{f,2} = -\partial_x\partial_z\tilde{w} = 0 \text{ at } z = 0.$$

Finally, it holds

$$\partial_x v_{f,2} + \partial_z v_{0,1} = -\partial_x^2\tilde{w} + \partial_z^2\tilde{w} = f \text{ at } z = 0.$$

That the norm of $\mathbf{v}_f$ can be estimated by the norm of f follows directly from the continuity of the extension operator from Lemma 4.2.4.

∎

As a consequence of Lemma (4.2.4) we get the following result, which we need to homogenise boundary condition (4.41).

Lemma 4.2.9
Let $r > 3$, $r-\frac{1}{2} \notin \mathbb{N}$. For any $f \in K_0^{r-3/2}((0,\infty))$ there exists a $\theta_f \in K_0^r((0,\infty)\times\Omega)$ with the following properties:

$$\begin{aligned} \theta_f &= 0 && \text{at } z = -1, \\ \theta_f &= 0 && \text{at } z = 0, \\ \partial_z\theta_f &= f && \text{at } z = 0. \end{aligned}$$

The mapping $\mathfrak{e}_2 : f \mapsto \theta_f$ is a linear continuous operator from $K_0^{r-3/2}((0,\infty))$ to $K_0^r((0,\infty)\times\Omega)$.

Proof: Let j_r be the largest integer in $[0, r-\frac{1}{2})$ and let k_r be the largest integer in $[0, \frac{r-1}{2})$. Furthermore, we set $\mathfrak{b}^{(k)} = 0$ for $k = 0,\dots,k_r$, $\mathfrak{a}^{(j)} = 0$ for $j = 0, 2, \dots, j_r$ and $\mathfrak{a}^{(1)} = f$. Since with this choice all the necessary compatibility conditions (4.20) are fulfilled, we can immediately apply Lemma 4.2.4 to obtain a $\theta_f \in K_0^r((0,\infty)\times\Omega)$ with the desired properties.

∎

If the unknowns are split up as $\mathbf{v} = \mathbf{v}_h + \mathbf{v}_{f_3}$ and $\theta = \theta_h + \theta_{f_4}$, then system (4.35) - (4.41) reads in terms of the new variables $\mathbf{v}_h$, θ_h

$$\partial_t \eta - v_{h,2} = g_0 \qquad \text{at } z = 0, \tag{4.43}$$

$$\begin{aligned} \partial_t \mathbf{v}_h - P_r \mathscr{P} \Delta \mathbf{v}_h + 2 P_r \mathscr{E}(\partial_z v_{h,2}|_{z=0}) + \cdots \\ \cdots + \mathscr{E}\left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right)\eta = \tilde{\mathbf{f}}_0 \end{aligned} \qquad \text{in } \Omega, \tag{4.44}$$

$$\partial_t \theta_h - \Delta \theta_h - v_{h,2} = \tilde{f}_1 \qquad \text{in } \Omega, \tag{4.45}$$

$$\mathbf{v}_h = 0 \qquad \text{at } z = -1, \tag{4.46}$$

$$\theta_h = 0 \qquad \text{at } z = -1, \tag{4.47}$$

$$\partial_x v_{h,2} + \partial_z v_{h,1} + M_a \partial_x (\theta_h - \eta) = 0 \qquad \text{at } z = 0, \tag{4.48}$$

$$\partial_z \theta_h + B_i(\theta_h - \eta) = 0 \qquad \text{at } z = 0, \tag{4.49}$$

where

$$\tilde{\mathbf{f}}_0 = \mathscr{P}\mathbf{f}_0 - \frac{P_r}{C_r}\mathscr{E} f_2 - \partial_t \mathbf{v}_{f_3} + P_r \mathscr{P} \Delta \mathbf{v}_{f_3}, \tag{4.50}$$

$$\tilde{f}_1 = f_1 - v_{f_3,2} - \partial_t \theta_{f_4} + \Delta \theta_{f_4}. \tag{4.51}$$

Note that the term $2P_r\mathscr{E}(\partial_z v_{f_3,2}|_{z=0})$ vanishes in (4.50) due to the construction of $\mathbf{v}_{f_3}$. The inhomogeneous terms can be estimated by the $K^{r-3/2}((0,\infty))$-norms of f_3 and f_4.

Remark 4.2.10
Note that the expressions for $\tilde{\mathbf{f}}_0$ and $\tilde{f}_1$ in (4.50) and (4.51) both contain time derivatives. Since later in the analysis of the full nonlinear problem the inhomogeneities f_3 and f_4 have to be replaced by nonlinear expressions of the unknowns, the homogenisation of the nonlinear boundary conditions will lead to nonlinear terms containing time derivatives.

□

Finally, a discussion of the space for the unknown free surface η remains. At the beginning of this section it was assumed that $\eta \in K_0^{r+1/2}((0,\infty))$, but there is a natural restriction to η since the physical system is mass preserving, i.e., the volume of the liquid must stay constant. Thus, it is reasonable to demand $\eta \in K_{0,(0)}^{r+1/2}((0,\infty))$, $g_0 \in K_{0,(0)}^{r-1/2}((0,\infty))$, where the spaces $K_{0,(0)}^{s}((0,\infty))$ are defined in the following according to [Bea84].

Definition 4.2.11
Let $L^2_{(0)}$ be the space of all $h \in L^2(\mathbb{R})$ which are finite in the norm

$$\left(\int_{\mathbb{R}} |\hat{h}(k)|^2 \omega(k)^2 \, \mathrm{d}k \right)^{1/2},$$

where $\omega(k) = 1/|k|$ *for* $k \in [-1,1] \setminus \{0\}$, $\omega(k) = 1$ *else. For* $s \geq 0$ *we define*

$$\begin{aligned}
H^s_{(0)} &:= H^s \cap L^2_{(0)},\\
K^s_{(0)}((t_0,t_1)) &:= H^{s/2}((t_0,t_1), L^2_{(0)}) \cap L^2((t_0,t_1), H^s_{(0)}),\\
K^s_{0,(0)}((t_0,t_1)) &:= \{u \in K^s_{(0)}((t_0,t_1)) \,|\, \partial_t^j u = 0,\ 0 \leq 2j < s-1\}.
\end{aligned}$$

The following lemma ensures that $\eta(t,\cdot) \in L^2_{(0)}$ for almost every $t > 0$.

Lemma 4.2.12
If $\mathbf{v} \in \mathscr{P}(H^r(\Omega))^2$ *then* $v_2|_{z=0} \in H^{r-1/2}_{(0)}$. *Furthermore, there exists a constant* C *independent of* $\mathbf{v}$ *such that* $\|v_2|_{z=0}\|_{H^{r-1/2}_{(0)}} \leq C\|\mathbf{v}\|_{(H^r(\Omega))^2}$.

Proof: See [Bea84, Lemma 3.2].

∎

4.2.3 Resolvent Estimates and Main Linear Result

Motivated by the considerations of the last section, the stationary problem for which a resolvent estimate has to be established reads:

$$\lambda\eta - v_2 = g_0 \quad \text{at } z = 0, \tag{4.52}$$

$$\lambda\mathbf{v} - P_r\mathscr{P}\Delta\mathbf{v} + 2P_r\mathscr{E}(\partial_z v_2|_{z=0}) + \mathscr{E}\left(\tfrac{P_rB_o}{C_r} - \partial_x^2\right)\eta = \mathbf{f}_0 \quad \text{in } \Omega, \tag{4.53}$$

$$\lambda\theta - \Delta\theta - v_2 = f_1 \quad \text{in } \Omega, \tag{4.54}$$

$$\mathbf{v} = 0 \quad \text{at } z = -1, \tag{4.55}$$

$$\theta = 0 \quad \text{at } z = -1, \tag{4.56}$$

$$\partial_x v_2 + \partial_z v_1 + M_a\partial_x(\theta - \eta) = 0 \quad \text{at } z = 0, \tag{4.57}$$

$$\partial_z\theta + B_i(\theta - \eta) = 0 \quad \text{at } z = 0, \tag{4.58}$$

for $g_0 \in H^{r-1/2}_{(0)}$, $\mathbf{f}_0 \in \mathscr{P}(H^{r-2}(\Omega))^2$, $f_1 \in H^{r-3/2}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ sufficiently large. We write system (4.52) – (4.58) in the more compact form

$$(\lambda - G)\begin{pmatrix}\eta\\ \mathbf{v}\\ \theta\end{pmatrix} = \begin{pmatrix}g_0\\ \mathbf{f}_0\\ f_1\end{pmatrix} \tag{4.59}$$

where the domain of G is given by

$$\begin{aligned}
D(G) = \{(\eta,\mathbf{v},\theta) \in{}& H^{3/2}_{(0)} \times \mathscr{P}(L^2(\Omega))^2 \times L^2(\Omega),\\
&\eta \in H^{5/2},\ \mathbf{v} \in (H^2(\Omega))^2,\ \theta \in H^2(\Omega),\\
&\mathbf{v} = 0 \text{ at } z = -1,\ \theta = 0 \text{ at } z = -1,\\
&\partial_z\theta + B_i(\theta - \eta) = 0 \text{ at } z = 0,\\
&\partial_x v_2 + \partial_z v_1 + M_a\partial_x(\theta - \eta) = 0 \text{ at } z = 0\}.
\end{aligned}$$

In [NT09], resolvent estimates for the Bénard-Marangoni problem with spatially periodic perturbations have been established. We can use these results to obtain estimates for the Fourier transformed problem for wave numbers k away from zero. For small wave numbers we first show a corresponding estimate for $k = 0$ and apply a perturbation argument afterwards. The details are worked out in Appendix B.

We obtain the following results corresponding to [NT07, Proposition 4.2 & 4.4].

Lemma 4.2.13
There is a $\gamma > 0$ such that, if $\operatorname{Re}\lambda \geq \gamma$, there exists a $C > 0$ and the inverse $(\lambda - G)^{-1}$ in X with $(\lambda - G)^{-1}X = D(G)$ and its operator norm satisfying

$$\|(\lambda - G)^{-1}\|_{X\to X} \leq \frac{C}{|\lambda|},$$

where $X = H^{3/2}_{(0)} \times \mathscr{P}(L^2(\Omega))^2 \times L^2(\Omega)$.

If the data is more regular, the solution gets the higher regularity.

Lemma 4.2.14
Let $r \geq 2$. There is a $\gamma > 0$ such that, if $\operatorname{Re}\lambda \geq \gamma$, there exists a $C > 0$ such that the following holds. Suppose $g_0 \in H^{r-1/2}_{(0)}$, $\mathbf{f}_0 \in \mathscr{P}(H^{r-2}(\Omega))^2$, $f_1 \in H^{r-2}(\Omega)$. Then the solution

$$\begin{pmatrix} \eta \\ \mathbf{v} \\ \theta \end{pmatrix} = (\lambda - G)^{-1} \begin{pmatrix} g_0 \\ \mathbf{f}_0 \\ f_1 \end{pmatrix}$$

satisfies

$$\begin{aligned} \|\mathbf{v}\|_{(H^r(\Omega))^2} + |\lambda|^{r/2}\|\mathbf{v}\|_{(L^2(\Omega))^2} + \|\eta\|_{H^{r+1/2}} + |\lambda|^{r/2+1/4}\|\eta\|_{L^2} \\ + \|\theta\|_{H^r(\Omega)} + |\lambda|^{r/2}\|\theta\|_{L^2(\Omega)} \\ \leq C\big(\|\mathbf{f}_0\|_{(H^{r-2}(\Omega))^2} + |\lambda|^{(r-2)/2}\|\mathbf{f}_0\|_{(L^2(\Omega))^2} \\ + \|f_1\|_{H^{r-2}(\Omega)} + |\lambda|^{(r-2)/2}\|f_1\|_{L^2(\Omega)} \\ + \|g_0\|_{H^{r-1/2}} + |\lambda|^{r/2-1/4}\|g_0\|_{L^2}\big). \end{aligned} \tag{4.60}$$

Remark 4.2.15
By a standard argument, see [Häc10, Lemma 6.25], we can conclude from Lemma 4.2.13 that the operator G is sectorial on X.

□

With the help of the resolvent estimates from Lemma 4.2.14 we can now follow the lines of Section A.1.3 and obtain the local existence and uniqueness result for the non-stationary linear inhomogeneous problem.

For a more compact notation we introduce the following spaces.

Definition 4.2.16
For $s \geq 0$ and $t_0 < t_1$ we set

$$\mathcal{K}^s((t_0,t_1)) := K^{s+1/2}_{(0)}((t_0,t_1)) \times \mathscr{P}(K^s((t_0,t_1)\times\Omega))^2 \times K^s((t_0,t_1)\times\Omega),$$
$$\mathcal{K}^s_0((t_0,t_1)) := K^{s+1/2}_{0,(0)}((t_0,t_1)) \times \mathscr{P}(K^s_0((t_0,t_1)\times\Omega))^2 \times K^s_0((t_0,t_1)\times\Omega),$$

and

$$\tilde{\mathcal{K}}^s((t_0,t_1)) := K^{s+3/2}_{(0)}((t_0,t_1)) \times \mathscr{P}(K^s((t_0,t_1)\times\Omega))^2 \times K^s((t_0,t_1)\times\Omega),$$
$$\tilde{\mathcal{K}}^s_0((t_0,t_1)) := K^{s+3/2}_{0,(0)}((t_0,t_1)) \times \mathscr{P}(K^s_0((t_0,t_1)\times\Omega))^2 \times K^s_0((t_0,t_1)\times\Omega).$$

Then the main linear result is given by the following theorem.

Theorem 4.2.17
Let $r > 3$, $r - \frac{1}{2} \notin \mathbb{N}$ and $(r+1)/2 \notin \mathbb{N}$. Furthermore, let $l > 0$ and $t_0 < t_1$ with $(t_1 - t_0) < l$. Then there exists a constant $C_l > 0$ such that for any $(g_0, \mathbf{f}_0, f_1) \in \tilde{\mathcal{K}}^{r-2}_0((t_0,t_1))$ there exists a unique solution $(\eta, \mathbf{v}, \theta) \in \mathcal{K}^r_0((t_0,t_1))$ of

$$\begin{aligned}(\partial_t - G)\begin{pmatrix}\eta\\ \mathbf{v}\\ \theta\end{pmatrix} &= \begin{pmatrix}g_0\\ \mathbf{f}_0\\ f_1\end{pmatrix},\\ \mathbf{v}|_{z=-1} &= 0,\\ \theta|_{z=-1} &= 0,\\ (\partial_x v_2 + \partial_z v_1 + M_a\partial_x(\theta-\eta))|_{z=0} &= 0,\\ (\partial_z\theta + B_i(\theta-\eta))|_{z=0} &= 0,\end{aligned} \tag{4.61}$$

with

$$\|(\eta,\mathbf{v},\theta)\|_{\mathcal{K}^r((t_0,t_1))} \leq C_l\|(g_0,\mathbf{f}_0,f_1)\|_{\tilde{\mathcal{K}}^{r-2}((t_0,t_1))}.$$

The solution operator from $\tilde{\mathcal{K}}^{r-2}_0((t_0,t_1))$ to $\mathcal{K}^r_0((t_0,t_1))$ that maps $(g_0,\mathbf{f}_0,f_1)$ to $(\eta,\mathbf{v},\theta)$ is denoted by $\tilde{M}_0^{-1}$.

Remark 4.2.18
The restriction that $r - \frac{1}{2} \notin \mathbb{N}$ in Theorem 4.2.17 comes from the conditions in Lemma 4.2.3 and Lemma 4.2.4, cf. Remark 4.2.5. The condition that $(r+1)/2 \notin \mathbb{N}$ comes from Lemma A.1.6.

□

4.2.4 The Full Nonlinear Problem

In the following, we state the local existence and uniqueness result analogous to [NT07, Theorem 6.1] for the full nonlinear problem (4.10) – (4.19). We show how to prove the existence part of the theorem and refer the reader to [Bea84] for the higher regularity result.

Theorem 4.2.19
Let $3 < r < \frac{7}{2}$, $0 < \tilde{t}_0 < t_1$. Then there exist constants C_1, $C_2 > 0$ such that the following holds.

Let $\eta_0 \in H^r_{(0)}$, $\mathbf{v}_0 \in \mathscr{P}(H^{r-1/2}(\Omega))^2$, $\theta_0 \in H^{r-1/2}(\Omega)$ satisfy

$$\|\eta_0\|_{H^r_{(0)}} + \|\mathbf{v}_0\|_{H^{r-1/2}(\Omega)} + \|\theta_0\|_{H^{r-1/2}(\Omega)} \leq C_1$$

as well as the permissibility conditions

$$\operatorname{div} \mathbf{v}_0 = 0 \qquad \text{in } \Omega, \tag{4.62}$$

$$\mathbf{v}_0 = 0 \qquad \text{at } z = -1, \tag{4.63}$$

$$\theta_0 = 0 \qquad \text{at } z = -1, \tag{4.64}$$

$$\partial_z v_{0,1} + \partial_x v_{0,2} + M_a \partial_x (\theta_0 - \eta_0) = F_3(\eta_0, \mathbf{v}_0, \theta_0) \qquad \text{at } z = 0, \tag{4.65}$$

$$\partial_z \theta_0 + B_i(\theta_0 - \eta_0) = F_4(\eta_0, \theta_0) \qquad \text{at } z = 0. \tag{4.66}$$

Then problem (4.10) – (4.19) has a unique solution $(\eta, \mathbf{v}, \theta, \mathrm{q})$ with

$$\eta \in K^{r+1/2}_{(0)}((0,t_1)), \quad \mathbf{v} \in \mathscr{P}(K^r((0,t_1)\times\Omega))^2, \quad \theta \in K^r((0,t_1)\times\Omega),$$
$$\nabla \mathrm{q} \in K^{r-2}((0,t_1)\times\Omega), \quad \mathrm{q} \in K^{r-3/2}((0,t_1))$$

and

$$\begin{aligned} &\|\eta\|_{K^{r+1/2}_{(0)}((0,t_1))} + \|\mathbf{v}\|_{K^r((0,t_1)\times\Omega)} + \|\theta\|_{K^r((0,t_1)\times\Omega)} \\ &\quad + \|\nabla \mathrm{q}\|_{K^{r-2}((0,t_1)\times\Omega)} + \|\mathrm{q}|_{z=0}\|_{K^{r-3/2}((0,t_1))} \leq C_2\big(\|\eta_0\|_{H^r_{(0)}} + \|\mathbf{v}_0\|_{H^{r-1/2}(\Omega)} \\ &\qquad + \|\theta_0\|_{H^{r-1/2}(\Omega)}\big). \end{aligned}$$

Furthermore, the restriction of the solution to the time interval $(\tilde{t}_0, t_1)$ is arbitrarily regular. More precisely, for any $m \in \mathbb{N}$ there exists a $C_3 > 0$ such that

$$\begin{aligned} &\|\eta\|_{K^{r+m+1/2}_{(0)}((\tilde{t}_0,t_1))} \\ &+ \|\mathbf{v}\|_{K^{r+m}((\tilde{t}_0,t_1)\times\Omega)} + \|\theta\|_{K^{r+m}((\tilde{t}_0,t_1)\times\Omega)} \\ &+ \|\nabla \mathrm{q}\|_{K^{r+m-2}((\tilde{t}_0,t_1)\times\Omega)} + \|\mathrm{q}|_{z=0}\|_{K^{r+m-3/2}((\tilde{t}_0,t_1))} \leq C_3\big(\|\eta_0\|_{H^r_{(0)}} + \|\mathbf{v}_0\|_{H^{r-1/2}(\Omega)} \\ &\qquad + \|\theta_0\|_{H^{r-1/2}(\Omega)}\big). \end{aligned}$$

Permissible Initial Conditions

Since the permissibility conditions (4.62) – (4.66) involve some nonlinear partial differential equations, it is not immediately evident that there exist any non-trivial solutions $(\eta_0, \mathbf{v}_0, \theta_0)$. In the following, we present a way of constructing permissible initial conditions from functions satisfying the homogeneous permissibility conditions, i.e., when F_3 and F_4 are set to 0.

We first introduce the following analogue of Lemma 4.2.8 for Sobolev spaces.

Lemma 4.2.20
Let $r \geq 3/2$. Then for any $f \in H^{r-3/2}$ there exists a vector field $\mathbf{v}_f = (v_{f,1}, v_{f,2})^\top \in (H^r(\Omega))^2$ with the following properties:

$$\begin{aligned} \operatorname{div} \mathbf{v}_f &= 0 && \text{in } \Omega, \\ \mathbf{v}_f &= 0 && \text{at } z = -1, \\ v_{f,2} &= 0 && \text{at } z = 0, \\ \partial_z v_{f,2} &= 0 && \text{at } z = 0, \\ \partial_x v_{f,2} + \partial_z v_{f,1} &= f && \text{at } z = 0. \end{aligned}$$

The mapping $\mathfrak{e}_1^0 : f \mapsto \mathbf{v}_f$ is a linear and continuous operator from $H^{r-3/2}$ to $(H^r(\Omega))^2$.

Proof: The proof works in the same way as the proof of Lemma 4.2.8 except for the fact that the construction of the auxiliary function w is much simpler due to the lack of additional compatibility conditions.

Here, we can use the usual trace theorem for Sobolev spaces, see e.g. [Wlo82, Satz 8.6], to guarantee the existence of a $w \in H^{r+1}(\Omega)$ with

$$w = 0, \ \partial_z w = 0, \ \partial_z^2 w = f \text{ at } z = 0.$$

The remainder of the proof is exactly the same as in the proof of Lemma 4.2.8.

∎

In the same way, we obtain a statement analogous to Lemma 4.2.9.

Lemma 4.2.21
Let $r \geq 3/2$. Then for any $f \in H^{r-3/2}$ there exists a $\theta_f \in H^r(\Omega)$ with the following properties:

$$\begin{aligned} \theta_f &= 0 && \text{at } z = -1, \\ \theta_f &= 0 && \text{at } z = 0, \\ \partial_z \theta_f &= f && \text{at } z = 0. \end{aligned}$$

The mapping $\mathfrak{e}_2^0 : f \mapsto \theta_f$ is a linear continuous operator from $H^{r-3/2}$ to $H^r(\Omega)$.

For notational convenience we first introduce the following spaces.

Definition 4.2.22
Let $s \geq 3/2$ and set

$$\begin{aligned}
\mathcal{H}^s_{(0)} &:= H^{s+1/2}_{(0)} \times (H^s(\Omega))^2 \times H^s(\Omega),\\
\mathcal{H}^s_{(0),0} &:= \{(\eta, \mathbf{v}, \theta) \in \mathcal{H}^s_{(0)} \,|\, \operatorname{div} \mathbf{v} = 0 \textit{ in } \Omega;\ \mathbf{v} = 0,\ \theta = 0 \textit{ at } z = -1\},\\
\mathcal{H}^s_{(0),00} &:= \{(\eta, \mathbf{v}, \theta) \in \mathcal{H}^s_{(0),0} \,|\, \partial_z v_1 + \partial_x v_2 + M_a \partial_x(\theta - \eta) = 0,\\
&\qquad\qquad \partial_z \theta + B_i(\theta - \eta) = 0 \textit{ at } z = 0\}.
\end{aligned}$$

We use Lemma 4.2.20 and Lemma 4.2.21 to show that close to any small initial condition satisfying the homogeneous permissibility conditions there always lies an initial condition that satisfies (4.62) – (4.66).

Lemma 4.2.23
There exists an open neighbourhood $\mathcal{N}$ of the origin in $\mathcal{H}^s_{(0),00}$ and a smooth invertible mapping $\varphi : \mathcal{N} \to \mathcal{H}^s_{(0),0}$ with $\varphi(0) = 0$ and $D\varphi|_{U=0} = \mathrm{id}$ with the following property:

Any $V \in \mathcal{H}^s_{(0),0}$ with sufficiently small norm satisfies (4.62) – (4.66) if and only if there is a $U \in \mathcal{N}$ such that $V = \varphi(U)$.

PROOF: We consider the mapping $(\mathrm{id} - \phi) : \mathcal{H}^s_{(0),0} \to \mathcal{H}^s_{(0),0}$, where

$$\phi(V) := (0, \mathfrak{e}^0_1(F_3(V)), \mathfrak{e}^0_2(F_4(V))).$$

Since ϕ is smooth and at least quadratic in V, we have that there exist open neighbourhoods $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{H}^s_{(0)}$ of the origin, such that $(\mathrm{id} - \phi) : \mathcal{N}_1 \to \mathcal{N}_2$ is bijective with a smooth inverse $\varphi := (\mathrm{id} - \phi)^{-1}$. The properties $\varphi(0) = 0$ and $D\varphi|_{U=0} = \mathrm{id}$ follow immediately.

For a more compact notation we introduce $\mathcal{A} \in \mathscr{L}(\mathcal{H}^s_{(0)}, (H^{s-3/2})^2)$ by

$$\mathcal{A}(\eta, \mathbf{v}, \theta) := \begin{pmatrix} (\partial_z v_1 + \partial_x v_2 + M_a \partial_x(\theta - \eta))|_{z=0} \\ (\partial_z \theta + B_i(\theta - \eta))|_{z=0} \end{pmatrix}.$$

Then $V \in \mathcal{N}_1$ satisfies the permissibility conditions (4.62) – (4.66) if and only if $\mathcal{A}V = (F_3, F_4)^\top(V)$. But since we have $\mathcal{A}\phi(V) = (F_3, F_4)^\top(V)$, we obtain

$$\begin{aligned}
V \in \mathcal{N}_1 \text{ satisfies } (4.62) - (4.66) &\Leftrightarrow V \in \mathcal{N}_1 \text{ and } \mathcal{A}(V - \phi(V)) = 0\\
&\Leftrightarrow V \in \mathcal{N}_1 \text{ and } V - \phi(V) \in \mathcal{N}_2 \cap \mathcal{H}^s_{(0),00}\\
&\Leftrightarrow V \in \varphi(\mathcal{N}_2 \cap \mathcal{H}^s_{(0),00}).
\end{aligned}$$

■

Elimination of the Pressure as an Unknown

Now, we bring (4.10) – (4.19) in a form that enables the application of the solution operator $\tilde{M}_0^{-1}$ defined in Theorem 4.2.17. Therefore, we repeat the steps from Section 4.2.2 for the nonlinear problem. Furthermore, to take into account the initial condition.

First, we eliminate the pressure q as an unknown as well as the solenoidal condition by applying the projection $\mathscr{P}$.

The following system can be obtained from (4.10) – (4.18) if we repeat the steps in Section 4.2.2, write $\mathrm{q} = \mathrm{q}_0 + \mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3$ and express q_1, q_2, q_3 in terms of the other unknowns by relations (4.30) – (4.34), where (4.34) is replaced by

$$\mathrm{q}_3|_{z=0} = \frac{P_r}{C_r} F_2(\eta, \mathbf{v}, \theta).$$

Hence, we have $\nabla \mathrm{q}_j = \mathrm{Q}_j(\eta, \mathbf{v}, \theta)$ for $j = 1, 2, 3$, where the Q_j, $j = 1, 2, 3$, are smooth mappings from $\mathcal{K}^r((0, t_1))$ to $\mathscr{P}\big(K^{r-2}((0, t_1) \times \Omega)\big)^2$ with $\mathrm{Q}_j(0) = 0$.

Finally, since $\mathbf{F}_0$ is smooth and at least quadratic, q_0 is uniquely determined by the other unknowns via the implicit equation

$$\nabla q_0 = (\mathrm{id} - \mathscr{P})(P_r \Delta \mathbf{v} + \mathbf{F}_0\big(\nabla q_0 + \textstyle\sum_{j=1}^3 \mathrm{Q}_j(\eta, \mathbf{v}, \theta), \eta, \mathbf{v}\big),$$

if $(\eta, \mathbf{v}, \theta)$ is sufficiently small in norm. Therefore, we can write $\nabla \mathrm{q}_0 = \mathrm{Q}_0(\eta, \mathbf{v}, \theta)$ with a smooth mapping Q_0 from $\mathcal{K}^r((0, t_1))$ to $(\mathrm{id} - \mathscr{P})\big(K^{r-2}((0, t_1))\big)^2$ with $\mathrm{Q}_0(0) = 0$. This leads to

$$
\begin{aligned}
\partial_t \eta - v_2 &= 0 && \text{at } z = 0, && (4.67)\\
\partial_t \mathbf{v} - P_r \mathscr{P} \Delta \mathbf{v} + 2 P_r \mathscr{E}(\partial_z v_2|_{z=0}) + \cdots & && && \\
\cdots + \mathscr{E}\left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right) \eta &= \Phi_0(\eta, \mathbf{v}, \theta) && \text{in } \Omega, && (4.68)\\
\partial_t \theta - \Delta \theta - v_2 &= F_1(\eta, \mathbf{v}, \theta) && \text{in } \Omega, && (4.69)\\
\mathbf{v} &= 0 && \text{at } z = -1, && (4.70)\\
\theta &= 0 && \text{at } z = -1, && (4.71)\\
\partial_x v_2 + \partial_z v_1 + M_a \partial_x(\theta - \eta) &= F_3(\eta, \mathbf{v}, \theta) && \text{at } z = 0, && (4.72)\\
\partial_z \theta + B_i(\theta - \eta) &= F_4(\eta, \theta) && \text{at } z = 0, && (4.73)\\
\eta = \eta_0, \ \mathbf{v} = \mathbf{v}_0, \ \theta &= \theta_0 && \text{at } t = 0, && (4.74)
\end{aligned}
$$

where

$$\Phi_0(\eta, \mathbf{v}, \theta) = \mathscr{P} \mathbf{F}_0\big(\textstyle\sum_{j=0}^3 \mathrm{Q}_j(\eta, \mathbf{v}, \theta), \eta, \mathbf{v}\big) - \tfrac{P_r}{C_r} \mathscr{E} F_2(\eta, \mathbf{v}, \theta).$$

Local Solution

We proceed similar to Section A.1.4 and reduce (4.67) – (4.74) to a problem in $\mathcal{K}_0^r((0,t_1))$ by constructing a local solution $\check{U} \in \mathcal{K}^r((0,\infty))$ with $\check{U}|_{t=0} = U_0$ that satisfies the equations and boundary conditions at $t=0$.

Lemma 4.2.24
Let $3 < r < 7/2$, $b > 0$. Then there exists a constant $C > 0$ such that the following holds.

If $U_0 = (\eta_0, \mathbf{v}_0, \theta_0) \in \mathcal{H}_{(0)}^{r-1/2}$ with $\|U_0\|_{\mathcal{H}_{(0)}^{r-1/2}} \leq b$ fulfils the permissibility conditions (4.62) – (4.66), then there exists a $\check{U} = (\check{\eta}, \check{\mathbf{v}}, \check{\theta}) \in \mathcal{K}_{(0)}^r((0,\infty))$ with $\partial_t\check{\eta} \in K_{(0)}^{r-1/2}((0,\infty))$ and

$$\big[\partial_t\check{\eta} - \check{v}_2|_{z=0}\big]\big|_{t=0} = 0 \qquad at\ z=0, \tag{4.75}$$

$$\begin{aligned}\Big[\partial_t\check{\mathbf{v}} - P_r\mathscr{P}\Delta\check{\mathbf{v}} + 2P_r\mathscr{E}(\partial_z\check{v}_2|_{z=0}) + \cdots \\ \cdots + \mathscr{E}\left(\tfrac{P_rB_o}{C_r} - \partial_x^2\right)\check{\eta}\Big]\Big|_{t=0} &= \Phi_0(\eta_0, \mathbf{v}_0, \theta_0)\end{aligned} \qquad in\ \Omega, \tag{4.76}$$

$$\big[\partial_t\check{\theta} - \Delta\check{\theta} - \check{v}_2\big]\big|_{t=0} = F_1(\eta_0, \mathbf{v}_0, \theta_0) \qquad in\ \Omega, \tag{4.77}$$

$$\check{\mathbf{v}} = 0 \qquad at\ z=-1, \tag{4.78}$$

$$\check{\theta} = 0 \qquad at\ z=-1, \tag{4.79}$$

$$\big[\partial_x\check{v}_2 + \partial_z\check{v}_1 + M_a\partial_x(\check{\theta} - \check{\eta})\big]\big|_{t=0} = F_3(\eta_0, \mathbf{v}_0, \theta_0) \qquad at\ z=0, \tag{4.80}$$

$$\big[\partial_z\check{\theta} + B_i(\check{\theta} - \check{\eta})\big]\big|_{t=0} = F_4(\eta_0, \theta_0) \qquad at\ z=0, \tag{4.81}$$

$$\check{\eta} = \eta_0,\ \check{\mathbf{v}} = \mathbf{v}_0,\ \check{\theta} = \theta_0 \qquad at\ t=0. \tag{4.82}$$

Furthermore, it holds

$$\|\check{U}\|_{\mathcal{K}_{(0)}^r((0,\infty))} \leq C\|U_0\|_{\mathcal{H}_{(0)}^{r-1/2}}.$$

Proof: Since U_0 satisfies the permissibility conditions (4.62) – (4.66), the conditions (4.80) and (4.81) are satisfied automatically if we can find $(\check{\eta}, \check{\mathbf{v}}, \check{\theta}) \in \mathcal{K}_{(0)}^r((0,\infty)\times\Omega)$ fulfilling (4.75) – (4.79) and (4.82).

Since $3 < r < 7/2$, the largest integer in $[0, r-\frac{1}{2})$ is 2 and the largest integer in $[0,(r-1)/2)$ is 1. Furthermore, $j + 2k < r - \frac{3}{2}$ only allows the integer pairs $(j,k) \in \{(0,0),(1,0)\}$.

According to Lemma 4.2.4, we can find a $\check{\theta} \in K^r((0,\infty)\times\Omega)$ with

$$\partial_z^j\check{\theta}|_{z=-1} = \mathfrak{a}^{(j)},\ j=0,1,2, \qquad \partial_t^k\check{\theta}|_{t=0} = \mathfrak{b}^{(k)},\ k=0,1,$$

for any prescribed $\mathfrak{a}^{(j)} \in K^{r-j-\frac{1}{2}}((0,\infty))$, $j = 0,1,2$, and $\mathfrak{b}^{(k)} \in H^{r-2k-1}(\Omega)$, $k = 0,1$ with

$$\mathfrak{a}^{(0)}|_{t=0} = \mathfrak{b}^{(0)}|_{z=-1}, \tag{4.83}$$

$$\mathfrak{a}^{(1)}|_{t=0} = \partial_z \mathfrak{b}^{(0)}|_{z=-1}. \tag{4.84}$$

Conditions (4.82), (4.77) and (4.79) readily give

$$\mathfrak{b}^{(0)} = \theta_0, \quad \mathfrak{b}^{(1)} = \Delta\theta_0 + v_{0,2} + F_1(\eta_0, \mathbf{v}_0, \theta_0), \quad \mathfrak{a}^{(0)} = 0.$$

Then (4.83) holds, since θ_0 satisfies the permissibility condition $\theta_0|_{z=-1} = 0$.

We can use the trace theorem Lemma A.1.2 to construct $\mathfrak{a}^{(1)} \in K^{r-1}((0,\infty))$ with $\mathfrak{a}^{(1)}|_{t=0} = \partial_z\theta_0|_{z=-1}$. Hence, (4.84) holds as well.

Lastly, $\mathfrak{a}^{(2)}$ can be chosen arbitrarily, such that we can set $\mathfrak{a}^{(2)} = 0$. Lemma 4.2.4 guarantees that there exists a constant $C > 0$ such that

$$\|\check{\theta}\|_{K^r((0,\infty)\times\Omega)} \leq C\|U_0\|_{\mathcal{H}^{r-1/2}_{(0)}}.$$

The construction of $\check{\eta}$, $\check{\mathbf{v}}$ with the desired properties can be done in exactly the same way as in [Bea84, Lemma 6.1].

■

Remark 4.2.25
A slight modification of the arguments in [Bea84, Lemma 6.1] yields the existence of a local solution $\check{U}^g$ even when the right-hand side of (4.75) is replaced by an inhomogeneous term $g_0|_{t=0}$ with some $g_0 \in K^{r-1/2}_{(0)}((0,t_1))$. The estimate for $\check{U}^g$ then reads

$$\|\check{U}^g\|_{\mathcal{K}^r((0,\infty))} \leq C\big(\|U_0\|_{\mathcal{H}^{r-1/2}_{(0)}} + \|g_0\|_{K^{r-1/2}_{(0)}((0,t_1))}\big).$$

□

Now, we write the unknown U as the sum $U = \check{U} + U^0$ with $U^0 = (\eta^0, \mathbf{v}^0, \theta^0) \in \mathcal{K}^r_{0,(0)}((0,t_1))$ with some $t_1 > 0$. Furthermore, we interpret $\check{U}$ as a smooth function of the initial condition U_0 according to Lemma 4.2.24 above.

Then we find that U^0 has to satisfy

$$\partial_t\eta^0 - v_2^0 = g_0(U_0) \qquad \text{at } z = 0, \tag{4.85}$$

$$\begin{aligned}\partial_t\mathbf{v}^0 - P_r\mathscr{P}\Delta\mathbf{v}^0 + 2P_r\mathscr{E}(\partial_z v_2^0|_{z=0}) + \cdots \\ \cdots + \mathscr{E}\left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right)\eta^0 = \Phi_0^0(U^0; U_0)\end{aligned} \qquad \text{in } \Omega, \tag{4.86}$$

$$\begin{aligned}
\partial_t\theta^0 - \Delta\theta^0 - v_2^0 &= F_1^0(U^0;U_0) && \text{in } \Omega, && (4.87)\\
\mathbf{v}^0 &= 0 && \text{at } z=-1, && (4.88)\\
\theta^0 &= 0 && \text{at } z=-1, && (4.89)\\
\partial_x v_2^0 + \partial_z v_1^0 + M_a\partial_x(\theta^0-\eta^0) &= F_3^0(U^0;U_0) && \text{at } z=0, && (4.90)\\
\partial_z\theta^0 + B_i(\theta^0-\eta^0) &= F_4^0(U^0;U_0) && \text{at } z=0, && (4.91)
\end{aligned}$$

where

$$\begin{aligned}
g_0(U_0) &= -\partial_t\check{\eta} + \check{v}_2|_{z=0},\\
\Phi_0^0(U^0;U_0) &= \Phi_0(\check{U}+U^0) - \partial_t\check{\mathbf{v}} + P_r\mathscr{P}\Delta\check{\mathbf{v}}\\
&\quad - 2P_r\mathscr{E}(\partial_z\check{v}_2|_{z=0}) - \mathscr{E}\left(\tfrac{P_rB_o}{C_r} - \partial_x^2\right)\check{\eta},\\
F_1^0(U^0;U_0) &= F_1(\check{U}+U^0) - \partial_t\check{\theta} + \Delta\check{\theta} + \check{v}_2,\\
F_3^0(U^0;U_0) &= F_3(\check{U}+U^0) - \partial_x\check{v}_2 - \partial_z\check{v}_1 - M_a\partial_x(\check{\theta}-\check{\eta}),\\
F_4^0(U^0;U_0) &= F_4(\check{\eta}+\eta^0,\check{\theta}+\theta^0) - \partial_z\check{\theta} - B_i(\check{\theta}-\check{\eta}).
\end{aligned}$$

Since $(\eta^0,\mathbf{v}^0,\theta^0)|_{t=0} = (0,0,0)$ for all $U^0 \in \mathcal{K}_{0,(0)}^r((0,t_1))$ and $(\check{\eta},\check{\mathbf{v}},\check{\theta})|_{t=0} = (\eta_0,\mathbf{v}_0,\theta_0)$, we conclude from (4.76), (4.77), (4.80) and (4.81) that the nonlinear terms vanish at $t=0$. Thus, we have

$$\begin{aligned}
g_0(U_0) &\in K_{0,(0)}^{r-1/2}((0,t_1)),\\
\Phi_0^0(U^0;U_0) &\in \mathscr{P}(K_0^{r-2}((0,t_1)\times\Omega))^2,\\
F_1^0(U^0;U_0) &\in K_0^{r-3/2}((0,t_1)\times\Omega),\\
F_j^0(U^0;U_0) &\in K_0^{r-3/2}((0,t_1)) \text{ for } j=3,4.
\end{aligned}$$

Homogenisation of the Boundary Conditions

It is our goal to reduce (4.85) – (4.91) to a system where the nonlinearities in the boundary conditions (4.90) and (4.91) are replaced by zero. Therefore, we want to express U^0 in terms of a variable U_h that satisfies the homogenised boundary conditions and is close to U^0. This can be done in a similar fashion as with the construction of a permissible initial condition in Lemma 4.2.23. Here, we have to use the operators $\mathfrak{e}_1$ and $\mathfrak{e}_2$ from Lemma 4.2.8 and Lemma 4.2.9, respectively. Note that these operators act on $K_0^{r-3/2}((0,\infty))$, which is the reason why we first had to reduce (4.67) – (4.74) to a problem in $\mathcal{K}_0^r((0,t_1))$.

We set

$$\begin{pmatrix}\eta_h\\ \mathbf{v}_h\\ \theta_h\end{pmatrix} := \begin{pmatrix}\eta^0\\ \mathbf{v}^0 - \mathfrak{e}_1(F_3^0(U^0;U_0))\\ \theta^0 - \mathfrak{e}_2(F_4^0(U^0;U_0))\end{pmatrix}. \qquad (4.92)$$

Hence, we see that $U_h = (\eta_h, \mathbf{v}_h, \theta_h)$ satisfies

$$\begin{aligned}\partial_x v_{h,2} + \partial_z v_{h,1} + M_a \partial_x(\theta_h - \eta_h) &= 0, \\ \partial_z \theta_h + B_i(\theta_h - \eta_h) &= 0\end{aligned}$$

at $z = 0$. In order to formulate a system that solely depends on U_h and U_0, we have to solve (4.92) for U^0. We cannot apply the implicit function theorem directly, however, since the set of initial conditions U_0, which satisfy the permissibility conditions (4.62) – (4.66), is no open subset of a Banach space.

But if $\|U_0\|_{\mathcal{H}^{r-1/2}_{(0)}}$ is sufficiently small, we can make use of Lemma 4.2.23. From this, we conclude that the set of permissible initial conditions is given by the range of a smooth invertible mapping $\varphi : \mathcal{N} \to \mathcal{H}^{r-1/2}_{(0)}$, where $\mathcal{N}$ is an open neighbourhood of the origin in $\mathcal{H}^{r-1/2}_{(0),00}$. Hence, we can write $U_0 = \varphi(\bar{U}_0)$, where $\bar{U}_0 \in \mathcal{N}$. Then we can formulate (4.92) as the problem of finding the zeros of the mapping $\mathscr{G} : \mathcal{K}^r_0((0,t_1)) \times \mathcal{K}^r_0((0,t_1)) \times \mathcal{N} \to \mathcal{K}^r_0((0,t_1))$, with

$$\mathscr{G}(U^0, U_h, \bar{U}_0) := U^0 - \begin{pmatrix} 0 \\ \mathfrak{e}_1(F^0_3(U^0; \varphi(\bar{U}_0))) \\ \mathfrak{e}_2(F^0_4(U^0; \varphi(\bar{U}_0))) \end{pmatrix} - U_h.$$

Since $\mathscr{G}(0,0,0) = 0$ and $D_{U^0}\mathscr{G}|_{(U^0,U_h,\bar{U}_0)=(0,0,0)} = \mathrm{id}$, we can solve the above equation for U^0 in a neighbourhood of zero in dependence of U_h and $\bar{U}_0 = \varphi^{-1}(U_0)$ in a unique smooth way.

We can now use this resolution to obtain the following homogenised nonlinear problem:

$$\begin{aligned}
\partial_t \eta_h - v_{h,2} &= g_0(U_0) && \text{at } z = 0, && (4.93)\\
\partial_t \mathbf{v}_h - P_r \mathscr{P} \Delta \mathbf{v}_h + 2P_r \mathscr{E}(\partial_z v_{h,2}|_{z=0}) + \cdots & && && \\
\cdots + \mathscr{E}\left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right)\eta_h &= \Phi_{h,0}(U_h; U_0) && \text{in } \Omega, && (4.94)\\
\partial_t \theta_h - \Delta\theta_h - v_{h,2} &= F_{h,1}(U_h; U_0) && \text{in } \Omega, && (4.95)\\
\mathbf{v}_h &= 0 && \text{at } z = -1, && (4.96)\\
\theta_h &= 0 && \text{at } z = -1, && (4.97)\\
\partial_x v_{h,2} + \partial_z v_{h,1} + M_a \partial_x(\theta_h - \eta_h) &= 0 && \text{at } z = 0, && (4.98)\\
\partial_z \theta_h + B_i(\theta_h - \eta_h) &= 0 && \text{at } z = 0, && (4.99)
\end{aligned}$$

where

$$\Phi_{h,0}(U_h; U_0) = \Phi^0_0(U^0(U_h, U_0); U_0) - \partial_t\big(\mathfrak{e}_1(F^0_3(U^0(U_h, U_0); U_0))\big)$$

$$
\begin{aligned}
&\quad + P_r \mathscr{P} \Delta\big(\mathfrak{e}_1(F_3^0(U^0(U_h, U_0); U_0))\big),\\
F_{h,1}(U_h; U_0) &= F_1^0(U^0(U_h; U_0); U_0) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \mathfrak{e}_1(F_3^0(U^0(U_h, U_0); U_0))\\
&\quad - \partial_t\big(\mathfrak{e}_2(F_4^0(U^0(U_h, U_0); U_0))\big) + \Delta\big(\mathfrak{e}_2(F_4^0(U^0(U_h, U_0); U_0))\big).
\end{aligned}
$$

Fixed Point Argument

Let

$$
\mathscr{K}_0^r((0,t_1)) := \{(\eta_h, \mathbf{v}_h, \theta_h) \in \mathcal{K}_0^r((0,t_1)) \,|\, (4.96) - (4.99) \text{ are satisfied}\}.
$$

Since $(g_0(U_0), \Phi_{h,0}(\cdot; U_0), F_{h,1}(\cdot; U_0))$ maps $\mathcal{K}_0^r((0,t_1))$ into $\tilde{\mathcal{K}}_0^{r-2}((0,t_1))$, we can apply the solution operator $\tilde{M}_0^{-1}$ from Theorem 4.2.17 and formulate (4.93) – (4.99) as a fixed point equation for $U_h \in \mathscr{K}_0^r((0,t_1))$, namely

$$
U_h = \tilde{M}_0^{-1} \begin{pmatrix} g_0(U_0) \\ \Phi_{h,0}(U_h; U_0) \\ F_{h,1}(U_h; U_0) \end{pmatrix}. \tag{4.100}
$$

We can now follow the lines of Section A.1.4 in order to prove that (4.100) has a unique solution. We only have to insert the appropriate norms. For example, we have to replace $\|\cdot\|_{K^r((t_0,t_1);6)}$ by $\|\cdot\|_{\mathcal{K}^r((0,t_1))}$, $\|\cdot\|_{K^{r-6}((t_0,t_1);6)}$ by $\|\cdot\|_{\tilde{\mathcal{K}}^{r-2}((0,t_1))}$ and $\|\cdot\|_{H^{r-3}}$ by $\|\cdot\|_{\mathcal{H}_{(0)}^{r-1/2}}$. The contraction property of the right-hand side of (4.100) then follows in exactly the same way as in Section A.1.4, since the nonlinearity is smooth and the terms containing U_h are at least quadratic. The appearing linear terms on the right-hand side of (4.100) depend solely on U_0.

We can then reverse our steps and thus complete the proof of the existence and uniqueness part of Theorem 4.2.19.

Higher Regularity

Let $3 < r < \varrho < 7/2$ and $\tilde{t}_0 \in (0,t_1)$. For any $U \in \mathcal{K}^r((0,t_1))$ we have due to Sobolev's embedding that $U(\cdot,t)$ is well defined for every fixed $t \in (0,t_1)$. Additionally, it holds $U(\cdot,t) \in \mathcal{H}_{(0)}^r \supset \mathcal{H}_{(0)}^{\varrho-1/2}$ for almost every $t \in (0,t_1)$. We can argue in the same way as in the proof of Lemma 3.3.5 that there exists a constant $C > 0$ and a $\tau \in (0,\tilde{t}_0)$ such that $U(\cdot,\tau) \in \mathcal{H}_{(0)}^{\varrho-1/2}$ with

$$
\|U(\cdot,\tau)\|_{\mathcal{H}_{(0)}^{\varrho-1/2}} \le C\|U_0\|_{\mathcal{H}_{(0)}^{r-1/2}}.
$$

Furthermore, if U is a solution of (4.67) – (4.74), then we have that $U(\cdot,\tau) \in \mathcal{H}_{(0),0}^{\varrho-1/2}$ and that it fulfils (4.72) and (4.73).

Since $3 < \varrho < 7/2$ we conclude that $U(\cdot, \tau)$ is a permissible initial condition for (4.67) – (4.74). Then we can repeat the local existence and uniqueness proof and obtain that the restriction of U to the time interval $(\tilde{t}_0, t_1)$ has to lie in $\mathcal{K}^{\varrho}((\tilde{t}_0, t_1))$ and that it holds

$$\|U\|_{\mathcal{K}^{\varrho}((\tilde{t}_0,t_1))} \leq C\|U_0\|_{\mathcal{H}^{r-1/2}_{(0)}}.$$

Remark 4.2.26
We see from the above considerations that it is readily possible to choose $U(\cdot, \tau) \in \mathcal{H}^r_{(0)}$ with $\|U(\cdot, \tau)\|_{\mathcal{H}^r_{(0)}} \leq C\|U_0\|_{\mathcal{H}^{r-1/2}_{(0)}}$. Hence, this leads us to the assumption that the restriction of the solution U to the time interval $(\tilde{t}_0, t_1)$ lies in $\mathcal{K}^{r+1/2}((\tilde{t}_0, t_1))$. Then one may repeat this argument to achieve any desired regularity.

This is indeed possible. However, for the construction of the local solution $\check{U} = (\check{\eta}, \check{\mathbf{v}}, \check{\theta}) \in \mathcal{K}^{r+m}((\tau, t_1))$ additional permissibility conditions are needed, since (4.76), (4.77), (4.80) and (4.81) have to hold to higher order. For example, if we want to apply the fixed point argument for $r+\frac{1}{2}$, we need that $g_0(U_0) \in K^r_0((0, t_1))$, which means that we require $\partial_t^j g_0(U_0)|_{t=0} = 0$ for $0 \leq j < (r-1)/2$. Hence, the local solution has to fulfil the additional condition $\partial_t\big(\partial_t\check{\eta} - \check{v}_2|_{z=0}\big) = 0$ at $t = 0$.

The problem has been solved for $\check{\eta}$ and $\check{\mathbf{v}}$ in [Bea84, Section 6]. The necessary generalisations for the additional variable $\check{\theta}$ are straightforward but involve considerable notational effort. Hence, we chose to omit these arguments and merely stated the result.

□

4.3 The Ginzburg-Landau Approximation

4.3.1 The Spectral Situation

For the investigation of the spectrum, we linearise (4.67) – (4.73). According to [Tak81a], we can reformulate the linearised system in terms of η, θ and the vertical component v_2 of the transformed velocity field alone, see Section B.3.1 for details. If we make the normal mode ansatz $(\eta, v_2, \theta)(x, z, t) = (Z, W(z), \Theta(z))\mathrm{e}^{\mathrm{i}kx+\mu t}$, we arrive at the eigenvalue problem

$$\begin{aligned}
\mu Z - W &= 0 && \text{at } z = 0,\\
\big(\mathrm{D}^2 - (P_r^{-1}\mu + k^2)\big)\big(\mathrm{D}^2 - k^2\big)W &= 0 && \text{for } z \in (-1, 0),\\
\big(\mathrm{D}^2 - (\mu + k^2)\big)\Theta + W &= 0 && \text{for } z \in (-1, 0),\\
W = \mathrm{D}W = \Theta &= 0 && \text{at } z = -1,\\
(\mu + 3P_r k^2)\mathrm{D}W - P_r\mathrm{D}^3W + \left(\tfrac{P_r B_o}{C_r} + k^2\right)k^2 Z &= 0 && \text{at } z = 0,
\end{aligned}$$

$$\begin{aligned} \mathrm{D}^2 W + k^2 W + M_a k^2 (\Theta - Z) &= 0 \qquad \text{at } z = 0, \\ \mathrm{D}\Theta + B_i(\Theta - Z) &= 0 \qquad \text{at } z = 0, \end{aligned}$$

where $\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}z}$. We obtain the linear dispersion relation in the following way. For any given (k, μ) we compute the Marangoni number M_a so that the above system possesses non-trivial solutions. The parameters B_i, B_o, C_r and P_r will be considered as fixed, such that we obtain the relation

$$M_a = \mathfrak{M}_a(k, \mu), \tag{4.101}$$

with an almost everywhere smooth function $\mathfrak{M}_a$. The explicit expression and the necessary computations are given in Section B.3.2.

Then we can compute the eigenvalue curves $k \mapsto \mu(k)$ for any given Marangoni number numerically by solving the implicit equation (4.101) for $\mu = \mu(k, M_a)$. For $k, \mu \in \mathbb{R}$ some curves are plotted in Figure 4.2. These curves take the same qualitative form for any choice of $3 \leq P_r \leq 10$, $10^{-3} \leq B_i \leq 10^{-1}$, $10^{-3} \leq B_o \leq 1$ and $10^{-6} \leq C_r \leq 10^{-2}$. The parameter regimes for B_i, B_o, C_r are taken from [Tak81a], the choice of P_r comes from the Prandtl number for water. For any given Marangoni number M_a the eigenvalue curve whose real part takes the highest value will be called the first eigenvalue curve $\mu_1 = \mu_1(k, M_a)$.

For $\max_{k\in\mathbb{R}} \mu_1(k, M_a) > 0$, the purely conducting steady state becomes linearly unstable. This happens when the Marangoni number reaches $M_a \approx 80$. Henceforth, we will call this threshold value the critical Marangoni number, which will be denoted by M_a^0.

Furthermore, for small $\varepsilon_0 > 0$, we will denote by M_a^ε the Marangoni number for which $\max_{k\in\mathbb{R}} \mu_1(k, M_a^\varepsilon) = \varepsilon^2$ for $\varepsilon \in [0, \varepsilon_0]$. The positions of the maxima are denoted by $\pm k_c^\varepsilon$ for $\varepsilon > 0$ or simply $\pm k_c$ in case $\varepsilon = 0$. The graph of $k \mapsto \mu_1(k, M_a^\varepsilon)$ qualitatively takes the same form as the eigenvalue curve in Chapters 2 and 3. Therefore, we define $\lambda_\varepsilon := (k \mapsto \mu_1(k, M_a^\varepsilon))$. Then we have

$$M_a^\varepsilon = M_a^0 + \mathcal{O}(\varepsilon^2), \quad k_c^\varepsilon = k_c + \mathcal{O}(\varepsilon^2), \quad \lambda_\varepsilon(k_c) = \varepsilon^2 + \mathcal{O}(\varepsilon^4). \tag{4.102}$$

Remark 4.3.1
In [Tak81a] no regime for the value of the Prandtl number P_r was specified. This is due to the fact that the value of P_r does not affect the critical Marangoni number M_a^0, since it cancels out in the implicit equation $M_a = \mathfrak{M}_a(k, 0)$. However, it has to be taken into account in order to obtain a plot of the eigenvalue curves.

□

Remark 4.3.2
In general, when making a linear stability analysis, we have to consider the case $\mu \in \mathbb{C}$. However, it was argued in [Tak81a] that we only have solutions μ of (4.101)

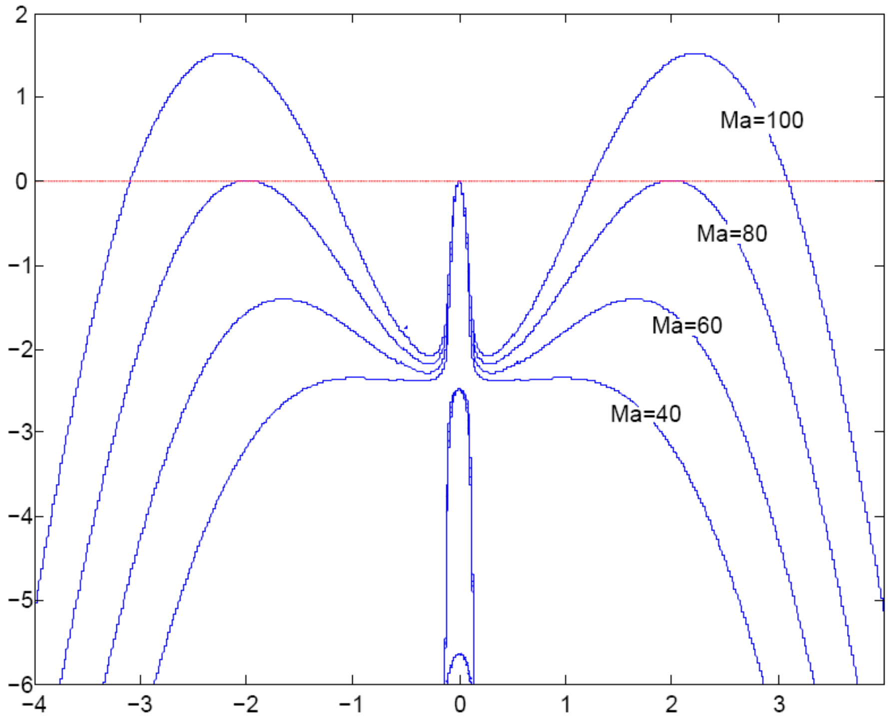


Figure 4.2: Eigenvalue curves for $P_r = 7$, $B_i = 0.01$, $B_o = 0.05$, $C_r = 0.0001$ and several values of M_a.

with positive real part and non-vanishing imaginary part, if $M_a < 0$. Since this is only the case when the bottom plate has lower temperature than the atmosphere, we can restrict ourselves to the case $\mu \in \mathbb{R}$.

□

4.3.2 Approximation Theorem and Ansatz

If we linearise system (4.67) – (4.73) and make the normal mode ansatz

$$(\eta, \mathbf{v}, \theta)(x, z, t) = (Z, V(z), W(z), \Theta(z))\mathrm{e}^{\mathrm{i}kx+\mu t},$$

we have solutions $\mathrm{U}(M_a, k, \mu; \cdot) := (Z, V, W, \Theta)$ of the resulting system given by Z, W, Θ from Section 4.3.1 and V defined by the other unknowns via the solenoidal condition $\operatorname{div} \mathbf{v} = 0$ and boundary condition (4.70). Since the solutions U are only unique up to a constant factor, we make the assumption that the prefactor $c(M_a, k, \mu)$ is chosen in such a way that the mapping $(M_a, k, \mu) \mapsto c(M_a, k, \mu)\mathrm{U}(M_a, k, \mu; \cdot)$ is smooth and that the right-hand side has norm 1 and that the reflection symmetry of the problem is retained via the relation

$$\overline{c(M_a, k, \mu)\mathrm{U}(M_a, k, \mu; \cdot)} = c(M_a, -k, \mu)\mathrm{U}(M_a, -k, \mu; \cdot).$$

With the notations of Section 4.3.1, we set

$$\begin{aligned} \mathcal{U}^\varepsilon(k, z) &:= c(M_a^\varepsilon, k, \lambda_\varepsilon(k))\mathrm{U}(M_a^\varepsilon, k, \lambda_\varepsilon(k); z), \\ \mathscr{U}_{\pm 1}(z) &:= \mathcal{U}^0(\pm k_c, z). \end{aligned}$$

Note that λ_ε is an even function w.r.t. k such that we have $\mathscr{U}_{-1} = \overline{\mathscr{U}_1}$.

The purpose of this section is to prove the following approximation result.

Theorem 4.3.3
Let $m_A \geq 11$ and $(A_1, A_0) \in C([0, T_0], H^{m_A}(\mathbb{R}) \times H^{m_A - 1}_{(0)}(\mathbb{R}))$ be a solution of the following generalised Ginzburg-Landau system

$$\begin{aligned} \partial_T A_1 &= -\lambda_0''(k_c)\partial_X^2 A_1/2 + A_1 + \zeta_{(0,1)}A_0 A_1 + \zeta_{(-1,1,1)}|A_1|^2 A_1, \\ \partial_T A_0 &= -\lambda_0''(0)\partial_X^2 A_0/2 + \zeta_{(-1,1)}\partial_X^2 |A_1|^2, \end{aligned} \tag{4.103}$$

where the coefficients $\zeta_{(0,1)}$, $\zeta_{(-1,1,1)}$, $\zeta_{(-1,1)} \in \mathbb{C}$ are determined by the mapping φ from Lemma 4.2.23 and the nonlinearities Φ_0, F_1, F_3 and F_4 from (4.68), (4.69), (4.72) and (4.73), respectively.

For $3 < r < \frac{7}{2}$ there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions $U = (\eta, \mathbf{v}, \theta) \in \mathcal{K}^r((0, T_0/\varepsilon^2))$ of (4.67) – (4.74) satisfying

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{(x,z) \in \mathbb{R} \times [-1,0]} \left| U(x, z, t) - (\varepsilon A_1(\varepsilon x, \varepsilon^2 t)\mathscr{U}_1(z)\mathrm{e}^{\mathrm{i}k_c x} + \text{c.c.}) \right| \leq C\varepsilon^{3/2}.$$

We denote the approximation given in Theorem 4.3.3 by $\varepsilon\Psi_{an}$. As in Chapters 2 and 3, we prove the approximation theorem for an auxiliary approximation $\varepsilon\Psi$ with

$$\|\varepsilon\Psi_{an} - \varepsilon\Psi\|_{C([0,T_0/\varepsilon^2]\times\overline{\Omega},\mathbb{R})} = \mathcal{O}(\varepsilon^{3/2}).$$

Then the approximation statement from Theorem 4.3.3 follows immediately if we can show that there exist solutions U of (4.67) – (4.74) with

$$\|U - \varepsilon\Psi\|_{C([0,T_0/\varepsilon^2]\times\overline{\Omega},\mathbb{R})} = \mathcal{O}(\varepsilon^{3/2}).$$

As a first step, we start with a preliminary ansatz $\varepsilon\Psi^0 \in C([0,T_0/\varepsilon],\mathcal{H}^s_{(0),00})$ for some sufficiently large s. The construction of $\varepsilon\Psi^0$ is best done in Fourier space.

For a more compact notation we define the following expressions:

- We set $K_j := \dfrac{k - jk_c}{\varepsilon}$ for $j \in \mathbb{Z}$.
- We define χ_j as the indicator function of the compact interval $\mathcal{I}_j := [jk_c - \rho_0, jk_c + \rho_0]$ for some small but fixed $0 < \rho_0 < \frac{k_c}{4}$.
- The expression "+c.c.f." stands for the Fourier transform of the complex conjugate of all terms to the left, i.e., $\hat{u}(k) + \text{c.c.f.} := \hat{u}(k) + \overline{\hat{u}(-k)}$.

Then we define $\varepsilon\Psi^0$ in the following way. We set $\varepsilon\Psi^0 := \mathcal{F}_x^{-1}(\widehat{\varepsilon\Psi^0})$ with

$$\begin{aligned}
\widehat{\varepsilon\Psi^0}(k,z,t) = {}& \varepsilon\varepsilon^{-1}\chi_1(k)\hat{A}_1(K_1,T)\mathcal{U}^\varepsilon(k,z) + \text{c.c.f.} \\
&+ \varepsilon^2\varepsilon^{-1}\chi_1(k)\hat{A}_{11}(K_1,T)\mathcal{U}^\varepsilon(k,z) + \text{c.c.f.} \\
&+ \varepsilon^2\varepsilon^{-1}\chi_0(k)\hat{A}_0(K_0,T)\mathcal{U}^\varepsilon(k,z) + \text{c.c.f.} \\
&+ \varepsilon^3\varepsilon^{-1}\chi_0(k)\hat{A}_{01}(K_0,T)\mathcal{U}^\varepsilon(k,z) + \text{c.c.f.} \\
&+ \varepsilon^4\varepsilon^{-1}\chi_0(k)\hat{A}_{02}(K_0,T)\mathcal{U}^\varepsilon(k,z) + \text{c.c.f.} \qquad (4.104)\\
&+ \varepsilon^2\varepsilon^{-1}\chi_0(k)\hat{\mathscr{A}}_0(K_0,z,T) + \text{c.c.f.} \\
&+ \varepsilon^2\varepsilon^{-1}\chi_2(k)\hat{\mathscr{A}}_2(K_2,z,T) + \text{c.c.f.} \\
&+ \varepsilon^3\varepsilon^{-1}\chi_1(k)\hat{\mathscr{A}}_1(K_1,z,T) + \text{c.c.f.} \\
&+ \varepsilon^3\varepsilon^{-1}\chi_3(k)\hat{\mathscr{A}}_3(K_3,z,T) + \text{c.c.f.},
\end{aligned}$$

where the terms with a factor $\chi_0(k)$ in front have to vanish at $k = 0$, i.e., they have to lie in $C([0,T_0],L^2_{(0)}(\mathbb{R},H^m(-1,0)))$. In analogy to Section 2.A, the $\hat{A}_{01}$, $\hat{A}_{02}$, $\hat{A}_{11}$ again serve the purpose to make the neutral and the critical parts of the residual small. The additional terms $\hat{\mathscr{A}}_j$ are for the handling of the stable part of the residual and have the form

$$\hat{\mathscr{A}}_j(K_j,z,T) = \sum_{l=0}^{N_j} \varepsilon^l \hat{a}_{jl}(K_j,z,T)$$

with some $N_j \in \mathbb{N}$. We will show that all of the higher order correction terms are uniquely determined by $\hat{A}_{\pm 1}$ and $\hat{A}_0$ and lie in some space $C([0,T_0], L^2(m))$.

The ansatz $\varepsilon\Psi$ for which we will prove the approximation result is then defined with the help of the mapping φ given by Lemma 4.2.23, namely

$$\varepsilon\Psi := \varphi(\varepsilon\Psi^0). \tag{4.105}$$

As a consequence, we obtain that $\varepsilon\Psi$ satisfies the nonlinear boundary conditions (4.70) – (4.73) as well as the solenoidal condition.

Next we show that $\varepsilon\Psi_{an}$ is close to $\varepsilon\Psi$ in the following sense.

Lemma 4.3.4

Assume that A_1, A_0, A_{01}, A_{02}, A_{11} and the $\mathscr{A}_j$, $j = 0, \ldots, 3$ in (4.104) *are of order $\mathcal{O}(1)$ in $C([0,T_0], H^{m_A})$. Then there exists a constant $C > 0$ with*

$$\|\varepsilon\Psi_{an} - \varepsilon\Psi\|_{C([0,T_0/\varepsilon^2]\times\overline{\Omega},\mathbb{R})} \leq C\varepsilon^{3/2}.$$

Proof: We begin by showing that

$$\|\varepsilon\Psi_{an} - \varepsilon\Psi^0\|_{C([0,T_0/\varepsilon^2]\times\overline{\Omega},\mathbb{R})} \leq C\varepsilon^{3/2}.$$

Obviously, it suffices to consider the lowest order terms.

Since $\mathcal{U}^\varepsilon(k,\cdot)$ is smooth in k and ε, we have

$$\|z \mapsto \mathscr{U}_1(z) - \mathcal{U}^\varepsilon(k,z)\|_{H^m(-1,0)} \leq C(|k-k_c| + \varepsilon^2).$$

Hence, it holds in Fourier space

$$\begin{aligned}
&\|(k,z) \mapsto \varepsilon\varepsilon^{-1}(\hat{A}_1(K_1,T)\mathscr{U}_1(z) - \chi_1(k)\hat{A}_1(K_1,T)\mathcal{U}^\varepsilon(k,z))(1+k^2)^{\frac{m}{2}}\|^2_{L^2(\mathbb{R},H^m(-1,0))} \\
&\qquad \leq \int_{\mathcal{I}_1} |\hat{A}_1|^2(K_1,T)(1+k^2)^m \|\mathscr{U}_1 - \mathcal{U}^\varepsilon(k,\cdot)\|^2_{H^m(-1,0)}\,\mathrm{d}k \\
&\qquad\quad + \int_{\mathbb{R}\setminus\mathcal{I}_1} |\hat{A}_1|^2(K_1,T)\|\mathscr{U}_1\|^2_{H^m(-1,0)}\,\mathrm{d}k \\
&\qquad \leq (1+(k_c+\rho_0)^2)^m C\varepsilon^2(\rho_0+\varepsilon)^2\|\hat{A}_1(\cdot,T)\|^2_{L^2} \\
&\qquad\quad + C\|\hat{A}_1(\cdot,T)\|^2_{L^2(m)} \sup_{k\in\mathcal{I}_1} \varepsilon^{2m}\left(\frac{1+k^2}{\varepsilon^2+(k-k_c)^2}\right)^m \\
&\qquad = \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^{2m}),
\end{aligned}$$

if $A_1 \in C([0,T_0], H^m)$. Thus, we get with Sobolev's embedding

$$\sup_{t\in[0,T_0]} \sup_{(x,z)\in\mathbb{R}\times\Omega} |(A_1(\varepsilon x, \varepsilon^2 t)\mathscr{U}_1(z)\mathrm{e}^{\mathrm{i}k_c x} + \mathrm{c.c.}) - \varepsilon\Psi^0(x,z,t)| \leq C\varepsilon^{3/2}.$$

From the properties of the mapping φ, we immediately conclude

$$\|\varepsilon\Psi - \varepsilon\Psi^0\|_{C([0,T_0/\varepsilon^2]\times\overline{\Omega},\mathbb{R})} \leq C\varepsilon^{3/2}.$$

■

4.3.3 Error Equations

Now, let $0 \leq \tilde{t}_0 < \tilde{t}_1 \leq t_1$. Let $U \in \mathcal{K}^r((0,t_1))$ be a solution of (4.67) – (4.74). Then we have on the time interval $(\tilde{t}_0, \tilde{t}_1)$ that U solves

$$\begin{aligned}(\partial_t - G)U &= N(U),\\ L_B U &= N_B(U),\\ U|_{t=\tilde{t}_0} &= \tilde{U}_0 \in \mathcal{H}^{r-1/2}_{(0)},\end{aligned}$$

where we used the short-hand notations $N(U)$ for the nonlinear terms in (4.67) – (4.69) and $L_B U = N_B(U)$ for the boundary conditions (4.70) – (4.73). Since $U \in \mathcal{K}^r((0,t_1))$ is a solution of (4.67) – (4.74), we have that

$$L_B \tilde{U}_0 = N_B(\tilde{U}_0).$$

The Mode Filters

In the following, we will write Λ_ε instead of G in order to emphasise the dependence on the parameter $\varepsilon > 0$. We will denote the corresponding operator in Fourier space for fixed k by $\hat{\Lambda}_\varepsilon(k)$.

The operator $\hat{\Lambda}_\varepsilon(k)$ possesses the isolated simple eigenvalue $\lambda_\varepsilon(k)$ with corresponding eigenfunction $\mathcal{U}^\varepsilon(k,\cdot)$. Let Γ_j be a small circle in $\mathbb{C}$ with centre $\lambda_0(jk_c)$, $j = -1, 0, 1$, so that all other spectral values of $\hat{\Lambda}_0(jk_c)$ lie outside of Γ_j. Then for sufficiently small $\varepsilon_0 > 0$, $\rho_0 > 0$ and $|k - jk_c| \leq \rho_0$, we have that $\lambda_\varepsilon(k)$ lies within Γ_j but no other spectral value of $\hat{\Lambda}_\varepsilon(k)$. Thus, for any $k \in [jk_c - \rho_0, jk_c + \rho_0]$ we can define a projection $\mathfrak{P}^\varepsilon_j(k)$ on $\operatorname{span}(\mathcal{U}^\varepsilon(k,\cdot))$ by

$$\mathfrak{P}^\varepsilon_j(k) := \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} (\hat{\Lambda}_\varepsilon(k) - \mu \mathrm{id})^{-1} \, \mathrm{d}\mu.$$

Definition 4.3.5
Let $s > 0$ and $u \in \mathcal{H}^s_{(0)}$. We define the mode filters E_0, $E_{\pm c}$ and E_s by

$$\begin{aligned}E_0 u &:= \mathcal{F}_x^{-1}\big(k \mapsto \chi_0(k)\mathfrak{P}^\varepsilon_0(k)\mathcal{F}_x(u)\big),\\ E_{\pm c} u &:= \mathcal{F}_x^{-1}\big(k \mapsto \chi_{\pm 1}(k)\mathfrak{P}^\varepsilon_1(k)\mathcal{F}_x(u)\big),\\ E_s &:= \mathrm{id} - E_0 - E_c - E_{-c},\end{aligned}$$

where the χ_j are the same indicator functions as in the definition of the preliminary ansatz $\widehat{\varepsilon\Psi^0}$, see (4.104).

The Error System

With the help of the mode filters from Definition 4.3.5, we can define different parts of the approximation and the error and scale them in the same way as in Section 2.3. We write

$$\begin{aligned} U &= \varepsilon\Psi + \varepsilon^{5/2}R \\ &= \varepsilon\Psi_c + \varepsilon^2\Psi_s + \varepsilon^2\Psi_0 + \varepsilon^{5/2}R_c + \varepsilon^{7/2}R_s + \varepsilon^{7/2}R_0, \end{aligned}$$

where $\Psi_c = E_c\Psi$, $R_c = E_cR$ and $\varepsilon\Psi_j = E_j\Psi$, $\varepsilon R_j = E_jR$ for $j = 0, s$.

As an example, we derive the error equation for R_c,

$$\begin{aligned} \partial_t R_c &= \Lambda_\varepsilon R_c + \varepsilon^{-5/2}E_cN(\varepsilon\Psi + \varepsilon^{5/2}R) - \varepsilon^{-5/2}(\partial_t - G)\varepsilon\Psi_c, \\ L_BR_c &= \varepsilon^{-5/2}E_c\big(N_B(\varepsilon\Psi + \varepsilon^{5/2}R) - L_B(\varepsilon\Psi)\big), \\ R_c|_{t=\tilde{t}_0} &= \varepsilon^{-5/2}E_c(U - \varepsilon\Psi)|_{t=\tilde{t}_0}, \end{aligned} \tag{4.106}$$

where

$$\mathrm{Res}(\varepsilon\Psi) := -\partial_t\varepsilon\Psi + \Lambda_\varepsilon\varepsilon\Psi + N(\varepsilon\Psi) \tag{4.107}$$

and $\mathrm{Res}_j := E_j\mathrm{Res}(\varepsilon\Psi)$, for $j = 0, \pm c, s$.

If we expand N and N_B in the different parts of the error and the approximation, we can rewrite system (4.106) as

$$\begin{aligned} \partial_t R_c &= \Lambda_\varepsilon R_c + \varepsilon^2N_c(R,\Psi) + \varepsilon^3g_c(R,\Psi) + \varepsilon^{-5/2}\mathrm{Res}_c, \\ L_BR_c &= \varepsilon^2N_{B,c}(R,\Psi) + \varepsilon^3g_{B,c}(R,\Psi), \\ R_c|_{t=\tilde{t}_0} &= \varepsilon^{-5/2}E_c(U - \varepsilon\Psi)|_{t=\tilde{t}_0}. \end{aligned}$$

Note that there is no residual term in the boundary condition, since $L_B\varepsilon\Psi = N_B(\varepsilon\Psi)$.

Similarly, we can derive the following system for all the parts of the error, namely

$$\partial_t R_c = \Lambda_\varepsilon R_c + \varepsilon^2N_c(R,\Psi) + \varepsilon^3g_c(R,\Psi) + \varepsilon^{-5/2}\mathrm{Res}_c, \tag{4.108}$$
$$\partial_t R_s = \Lambda_\varepsilon R_s + N_s(R_c,\Psi_c) + \varepsilon g_s(R,\Psi) + \varepsilon^{-7/2}\mathrm{Res}_s, \tag{4.109}$$
$$\partial_t R_0 = \Lambda_\varepsilon R_0 + \partial_x^2\big(N_0(R_c,\Psi_c) + \varepsilon g_0(R,\Psi)\big) + \varepsilon^{-7/2}\mathrm{Res}_0, \tag{4.110}$$
$$L_BR_c = \varepsilon^2N_{B,c}(R,\Psi) + \varepsilon^3g_{B,c}(R,\Psi), \tag{4.111}$$
$$L_BR_s = N_{B,s}(R_c,\Psi_c) + \varepsilon g_{B,s}(R,\Psi), \tag{4.112}$$
$$L_BR_0 = N_{B,0}(R_c,\Psi_c) + \varepsilon g_{B,0}(R,\Psi), \tag{4.113}$$
$$(R_c, R_s, R_0)|_{t=\tilde{t}_0} = (\tilde{R}_{c,0}, \tilde{R}_{s,0}, \tilde{R}_{0,0}), \tag{4.114}$$

where

$$
\begin{aligned}
\tilde{R}_{c,0} &= \varepsilon^{-5/2} E_c(U - \varepsilon\Psi)|_{t=\tilde{t}_0}, \\
\tilde{R}_{s,0} &= \varepsilon^{-7/2} E_s(U - \varepsilon\Psi)|_{t=\tilde{t}_0}, \\
\tilde{R}_{s,0} &= \varepsilon^{-7/2} E_0(U - \varepsilon\Psi)|_{t=\tilde{t}_0}.
\end{aligned}
$$

Hereby, N_j, $N_{B,j}$, $j = 0, c, s$, contain the terms that are of lowest order in ε and hence are linear in R. In g_j, $g_{B,j}$, $j = 0, c, s$, we collect all other terms that depend explicitly on R. The ∂_x^2 in front of the nonlinear terms in (4.110) results from the reflection symmetry of the underlying problem and the fact that the volume of the liquid is a conserved quantity. We will explain this in more detail in Section 4.3.4.

These terms behave qualitatively the same as the corresponding terms in Section 2.3 in the sense that they satisfy the estimates

$$
\begin{aligned}
\|N_c(R,\Psi)\|_{\mathcal{H}^m_{(0)}} &\leq C\mathcal{R}, \\
\|N_s(R_c,\Psi_c)\|_{\tilde{\mathcal{H}}^{m-2}_{(0)}} &\leq C(\|R_c\|_{\mathcal{H}^m_{(0)}} + \|R_{-c}\|_{\mathcal{H}^m_{(0)}}), \\
\|N_0(R_c,\Psi_c)\|_{\mathcal{H}^m_{(0)}} &\leq C(\|R_c\|_{\mathcal{H}^m_{(0)}} + \|R_{-c}\|_{\mathcal{H}^m_{(0)}}), \\
\|g_c(R,\Psi)\|_{\mathcal{H}^m_{(0)}} &\leq C\mathcal{R} + \mathcal{R}^2 \mathfrak{g}_c(\mathcal{R}), \\
\|g_s(R,\Psi)\|_{\tilde{\mathcal{H}}^{m-2}_{(0)}} &\leq C\mathcal{R} + \mathcal{R}^2 \mathfrak{g}_s(\mathcal{R}), \\
\|g_0(R,\Psi)\|_{\mathcal{H}^m_{(0)}} &\leq C\mathcal{R} + \varepsilon^{1/2}\mathcal{R}^2 \mathfrak{g}_0(\mathcal{R}), \\
\|N_{B,c}(R,\Psi)\|_{\mathscr{H}^m_B} &\leq C\mathcal{R}, \\
\|N_{B,s}(R_c,\Psi_c)\|_{\mathscr{H}^{m-2}_B} &\leq C(\|R_c\|_{\mathcal{H}^m_{(0)}} + \|R_{-c}\|_{\mathcal{H}^m_{(0)}}), \\
\|N_{B,0}(R_c,\Psi_c)\|_{\mathscr{H}^m_B} &\leq C(\|R_c\|_{\mathcal{H}^m_{(0)}} + \|R_{-c}\|_{\mathcal{H}^m_{(0)}}), \\
\|g_{B,c}(R,\Psi)\|_{\mathscr{H}^m_B} &\leq C\mathcal{R} + \mathcal{R}^2 \mathfrak{g}_{B,c}(\mathcal{R}), \\
\|g_{B,s}(R,\Psi)\|_{\mathscr{H}^{m-3/2}_B} &\leq C\mathcal{R} + \mathcal{R}^2 \mathfrak{g}_{B,s}(\mathcal{R}), \\
\|g_{B,0}(R,\Psi)\|_{\mathscr{H}^m_B} &\leq C\mathcal{R} + \varepsilon^{1/2}\mathcal{R}^2 \mathfrak{g}_{B,0}(\mathcal{R}),
\end{aligned}
\tag{4.115}
$$

with the norms of the spaces

$$
\begin{aligned}
\tilde{\mathcal{H}}^m_{(0)} &:= H^{m+3/2}_{(0)} \times \big(H^m(\Omega)\big)^2 \times H^m(\Omega), \\
\mathscr{H}^m_B &:= \{0\}^2 \times H^m \times H^m,
\end{aligned}
$$

where

$$
\mathcal{R}(t) = \sum_{j=0,\pm c,s} \|R_j(t)\|_{\mathcal{H}^m_{(0)}},
$$

and the $\mathfrak{g}_j$, $\mathfrak{g}_{B,j}$, $j = 0, c, s$, are monotonically increasing functions from $[0,\infty)$ to $[0,\infty)$. Due to the compact support in Fourier space there is no loss of regularity in the estimates for $N_{B,j}$, $g_{B,j}$, $j = 0, c$. That there is no loss of regularity in the estimates for N_j, g_j, $j = 0, c$ follows from the compact support in Fourier space and the fact that we project on eigenfunctions that are C^∞ w.r.t. z.

The further handling of the above error equations depends on the method of error control. That means, we have to treat the above system as an equation for $\mathbf{R} = (R_c, R_s, R_0) \in \big(\mathcal{K}^r((\tilde{t}_0, \tilde{t}_1))\big)^3$ with the method of optimal regularity or for $\mathbf{R} \in \big(C([\tilde{t}_0, \tilde{t}_1], \mathcal{H}^{r-1/2}_{(0)})\big)^3$ with the variation of constants formula.

Treatment With Optimal Regularity

First, we construct a local solution $\check{\mathbf{R}} = (\check{R}_c, \check{R}_s, \check{R}_0) \in \big(\mathcal{K}^r((\tilde{t}_0, \infty))\big)^3$ of (4.108) – (4.114). The construction works in the same way as in Lemma 4.2.24, since the initial condition satisfies corresponding permissibility conditions. Due to the appearance of inhomogeneous terms coming from the residual, we have the estimate

$$\begin{aligned} \|\check{\mathbf{R}}\|_{\mathcal{K}^r((\tilde{t}_0,\infty))} \leq C\big(&\|(\tilde{R}_{c,0}, \tilde{R}_{s,0}, \tilde{R}_{0,0})\|_{\mathcal{H}^{r-1/2}_{(0)}} \\ &+ \|(\varepsilon^{-5/2}\mathrm{Res}_c, \varepsilon^{-7/2}\mathrm{Res}_s, \varepsilon^{-7/2}\mathrm{Res}_0)|_{t=\tilde{t}_0}\|_{\mathcal{H}^{r-1/2}_{(0)}}\big), \end{aligned} \tag{4.116}$$

see also Remark 4.2.25. Now, we can reformulate system (4.108) – (4.113) for the new unknown $\mathbf{R}^0 = (R^0_c, R^0_s, R^0_0) \in \big(\mathcal{K}^r_0((\tilde{t}_0, \tilde{t}_1))\big)^3$, where $\mathbf{R} = \check{\mathbf{R}} + \mathbf{R}^0$. Hence, we get

$$\partial_t R^0_c = \Lambda_\varepsilon R^0_c + \varepsilon^2 N^0_c(\mathbf{R}^0, \Psi) + \varepsilon^3 g^0_c(\mathbf{R}^0, \check{\mathbf{R}}, \Psi) + h^0_c(\check{\mathbf{R}}, \Psi) + \varepsilon^{-5/2}\mathrm{Res}_c, \tag{4.117}$$
$$\partial_t R^0_s = \Lambda_\varepsilon R^0_s + N^0_s(R^0_c, \Psi_c) + \varepsilon g^0_s(\mathbf{R}^0, \check{\mathbf{R}}, \Psi) + h^0_s(\check{\mathbf{R}}, \Psi) + \varepsilon^{-7/2}\mathrm{Res}_s, \tag{4.118}$$
$$\partial_t R^0_0 = \Lambda_\varepsilon R^0_0 + N^0_0(R^0_c, \Psi_c) + \varepsilon g^0_0(\mathbf{R}^0, \check{\mathbf{R}}, \Psi) + h^0_0(\check{\mathbf{R}}, \Psi) + \varepsilon^{-7/2}\mathrm{Res}_0, \tag{4.119}$$
$$L_B R^0_c = \varepsilon^2 N^0_{B,c}(\mathbf{R}^0, \Psi) + \varepsilon^3 g^0_{B,c}(\mathbf{R}^0, \check{\mathbf{R}}, \Psi) + h^0_{B,c}(\check{\mathbf{R}}, \Psi), \tag{4.120}$$
$$L_B R^0_s = N^0_{B,s}(R^0_c, \Psi_c) + \varepsilon g^0_{B,s}(\mathbf{R}^0, \check{\mathbf{R}}, \Psi) + h^0_{B,s}(\check{\mathbf{R}}, \Psi), \tag{4.121}$$
$$L_B R^0_0 = N^0_{B,0}(R^0_c, \Psi_c) + \varepsilon g^0_{B,0}(\mathbf{R}^0, \check{\mathbf{R}}, \Psi) + h^0_{B,0}(\check{\mathbf{R}}, \Psi), \tag{4.122}$$

where we collected all the terms that do not depend on $\mathbf{R}^0$ in h^0_j, $h^0_{B,j}$, $j = 0, c, s$. We have chosen not to denote the derivative form of the nonlinearity in the equations for R^0_0 explicitly, since we will not need it for controlling the error on an $\mathcal{O}(1)$ time interval.

Since the right-hand sides of (4.120) – (4.122) vanish at $t = \tilde{t}_0$, we can use the operators $\mathfrak{e}_1$ and $\mathfrak{e}_2$ from Lemma 4.2.8 and Lemma 4.2.9 in order to homogenise the boundary conditions. In particular, we define the linear continuous operator

$\mathfrak{E} : \{0\}^2 \times \big(K_0^{r-3/2}((\tilde{t}_0, \tilde{t}_1))\big)^2 \to \{0\} \times \mathscr{P}\big(K_0^r((\tilde{t}_0, \tilde{t}_1) \times \Omega)\big)^2 \times K_0^r((\tilde{t}_0, \tilde{t}_1) \times \Omega)$ by

$$\mathfrak{E}\begin{pmatrix} 0 \\ 0 \\ f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} 0 \\ \mathfrak{e}_1 f_1 \\ \mathfrak{e}_2 f_2 \end{pmatrix}. \tag{4.123}$$

First, we set

$$R^0_{c,H} := R^0_c - \mathfrak{E}\big(h^0_{B,c}(\check{\mathbf{R}}, \Psi)\big)$$

and find that if we replace R^0_c by $R^0_{c,H} + \mathfrak{E}\big(h^0_{B,c}(\check{\mathbf{R}}, \Psi)\big)$ in (4.117), the resulting evolution equation for $R^0_{c,H}$ is of the same form as (4.117). Furthermore, we have boundary conditions of the form

$$\begin{aligned} L_B R^0_{c,H} &= \varepsilon^2 N^0_{B,c,H}(R^0_{c,H}, R^0_s, R^0_0, \Psi) + \varepsilon^3 g^0_{B,c,H}(R^0_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi) \\ &\quad + \varepsilon^2 h^0_{B,c,H}(\check{\mathbf{R}}, \Psi) \\ &=: \varepsilon^2 F_{c,H}(R^0_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi). \end{aligned}$$

Now, we set

$$R_{c,H} := R^0_{c,H} - \varepsilon^2 \mathfrak{E}\big(F_{c,H}(R^0_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi)\big). \tag{4.124}$$

Hence, we have $L_B R_{c,H} = 0$. In order to derive an evolution equation for $R_{c,H}$, we have to solve equation (4.124) for $R^0_{c,H}$. Suppose that $R_{c,H}$, $R^0_{c,H}$, R^0_s, R^0_0, $\check{\mathbf{R}}$ lie within given bounded sets. Then we are looking for the zeros of a smooth mapping $\mathscr{G}$ with

$$\begin{aligned} \mathscr{G}(R^0_{c,H}, R_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi, \varepsilon) = R^0_{c,H} - R_{c,H} \\ - \varepsilon^2 \mathfrak{E}\big(F_{c,H}(R^0_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi)\big). \end{aligned} \tag{4.125}$$

We see immediately that $\mathscr{G}(R_{c,H}, R_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi, 0) = 0$ for any given $R_{c,H}$, R^0_s, R^0_0, $\check{\mathbf{R}}$, Ψ. Furthermore, it holds

$$D_{R^0_{c,H}} \mathscr{G}(R_{c,H}, R_{c,H}, R^0_s, R^0_0, \check{\mathbf{R}}, \Psi, 0) = \mathrm{id}.$$

Hence, there exists an $\varepsilon_0 > 0$ such that we can solve equation (4.125) for $R^0_{c,H}$ for any $\varepsilon \in (0, \varepsilon_0)$. If we restrict ourselves to arbitrarily large but bounded sets containing the R-terms, the constant $\varepsilon_0 > 0$ can be chosen uniformly for any given $R_{c,H}$, $R^0_{c,H}$, R^0_s, R^0_0, $\check{\mathbf{R}}$. Hence, we have $R^0_{c,H} = R_{c,H} + \mathcal{O}(\varepsilon^2)$.

Therefore, we are able to derive an evolution equation for $R_{c,H}$ which is again of the same form as (4.117).

For the homogenisation of the remaining boundary conditions (4.121) and (4.122) we proceed analogously. For example, if we express R^0_c in terms of $R_{c,H}$ and the other unknowns, we see that (4.121) has the form

$$L_B R^0_s = \varepsilon \tilde{g}^0_{B,s}(R^0_s, R^0_0, \check{\mathbf{R}}, \Psi) + \tilde{h}^0_{B,s}(R_{c,H}, \check{\mathbf{R}}, \Psi).$$

Then we set $R^0_{s,H} := R^0_s - \mathfrak{C}\big(\tilde{h}^0_{B,s}(R_{c,H}, \check{\mathbf{R}}, \Psi)\big)$ and conclude with the same arguments as above that there exists an $R_{s,H} = R^0_{s,H} + \mathcal{O}(\varepsilon)$ with $L_B R_{s,H} = 0$. Repeating the same procedure for R^0_0 yields the following system for $\mathbf{R}_H = (R_{c,H}, R_{s,H}, R_{0,H}) \in \big(\mathscr{K}^r_0((\tilde{t}_0, \tilde{t}_1))\big)^3$,

$$MR_{c,H} = \varepsilon^2 N_{c,H}(\mathbf{R}_H, \Psi) + \varepsilon^3 g_{c,H}(\mathbf{R}_H, \check{\mathbf{R}}, \Psi) + h_{c,H}(\check{\mathbf{R}}, \Psi) + \varepsilon^{-5/2}\mathrm{Res}_c, \quad (4.126)$$

$$MR_{s,H} = N_{s,H}(R_{c,H}, \Psi_c) + \varepsilon g_{s,H}(\mathbf{R}_H, \check{\mathbf{R}}, \Psi) + h_{s,H}(\check{\mathbf{R}}, \Psi) + \varepsilon^{-7/2}\mathrm{Res}_s, \quad (4.127)$$

$$MR_{0,H} = N_{0,H}(R_{c,H}, \Psi_c) + \varepsilon g_{0,H}(\mathbf{R}_H, \check{\mathbf{R}}, \Psi) + h_{0,H}(\check{\mathbf{R}}, \Psi) + \varepsilon^{-7/2}\mathrm{Res}_0, \quad (4.128)$$

where $M := \partial_t - \Lambda_\varepsilon$. Now, we can follow the lines of the proof of Lemma 3.3.5 and conclude that for all $\varepsilon \in (0, \varepsilon_0)$ with sufficiently small $\varepsilon_0 > 0$ there exists a unique solution $\mathbf{R}_H \in \big(\mathscr{K}^r_0((\tilde{t}_0, \tilde{t}_1))\big)^3$ of (4.126) – (4.128) satisfying

$$\|\mathbf{R}_H\|_{\mathcal{K}^r((\tilde{t}_0,\tilde{t}_1))} \leq C\Big(\sum_{j=0,c,s} \|h_{j,H}(\check{\mathbf{R}}, \Psi)\|_{\tilde{\mathcal{K}}^{r-2}((\tilde{t}_0,\tilde{t}_1))} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{\tilde{\mathcal{K}}^{r-2}((\tilde{t}_0,\tilde{t}_1))} + \sum_{j=0,s} \|\varepsilon^{-7/2}\mathrm{Res}_j\|_{\tilde{\mathcal{K}}^{r-2}((\tilde{t}_0,\tilde{t}_1))}\Big).$$

Note that if the residual is sufficiently regular, we can bound $\|\mathrm{Res}_j\|_{\tilde{\mathcal{K}}^{r-2}}$ by $\|\mathrm{Res}_j\|_{C([\tilde{t}_0,\tilde{t}_1],\mathcal{H}^{r-1/2}_{(0)})}$, for $j = 0, c, s$.

Retracing the construction of $\mathbf{R}_H$ from $\mathbf{R}$, we finally obtain the following existence result for (4.108) – (4.114). Additionally a higher regularity result can be obtained in the same way as for (4.67) – (4.74).

Lemma 4.3.6
Let $3 < r < 7/2$ and $0 \leq \tilde{t}_0 < \tau_0 < \tilde{t}_1 \leq T_0/\varepsilon^2$ with $(\tilde{t}_1 - \tilde{t}_0) < 1$. Then there exists a $C > 0$ such that the following holds: For all $\mathfrak{r} > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial condition $\mathbf{R}|_{t=\tilde{t}_0} \in \mathcal{H}^{r-1/2}_{(0),0}$ with $\|\mathbf{R}|_{t=\tilde{t}_0}\|_{\mathcal{H}^{r-1/2}_{(0)}} \leq \mathfrak{r}$ that satisfies (4.111) – (4.113), the error system (4.108) – (4.114) has a unique solution $\mathbf{R} \in \big(\mathcal{K}^r((\tilde{t}_0, \tilde{t}_1))\big)^3$. This solution satisfies

$$\|\mathbf{R}\|_{\mathcal{K}^r((\tilde{t}_0,\tilde{t}_1))} \leq C\Big(\|\mathbf{R}|_{t=\tilde{t}_0}\|_{\mathcal{H}^{r-1/2}_{(0)}} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r-1/2}_{(0)})} + \sum_{j=0,s} \|\varepsilon^{-7/2}\mathrm{Res}_j\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r-1/2}_{(0)})}\Big).$$

Furthermore, the restriction $\mathbf{R}|_{(\tau_0,\tilde{t}_1)}$ lies in $\big(\mathcal{K}^{r+1}((\tau_0, \tilde{t}_1))\big)^3$ and satisfies

$$\|\mathbf{R}\|_{\mathcal{K}^{r+1}((\tau_0,\tilde{t}_1))} \leq C\Big(\|\mathbf{R}|_{t=\tilde{t}_0}\|_{\mathcal{H}^{r-1/2}_{(0)}} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})} + \sum_{j=0,s} \|\varepsilon^{-7/2}\mathrm{Res}_j\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})}\Big).$$

Remark 4.3.7
It may not be immediately clear why we need a higher regularity result that is stronger than the corresponding ones in Lemma 3.3.8 and Lemma 3.4.1. Keep in mind that for controlling the error it is our goal to use the combined optimal regularity and variation of constants method developed in Chapter 3. For example, we needed an estimate for $\|R(\frac{1}{2})\|_{H^{r-2}}$ for the treatment of the quasilinear Swift-Hohenberg model. In case of the Bénard-Marangoni problem this corresponds to an estimate of $\|\mathbf{R}(\frac{1}{2})\|_{\mathcal{H}_{(0)}^{r-1/2}}$. Due to Lemma A.1.3 and Sobolev's embedding, we then have

$$\|\mathbf{R}(\tfrac{1}{2})\|_{\mathcal{H}_{(0)}^{r-1/2}} \leq C\|\mathbf{R}\|_{\mathcal{K}^{\varrho+1/2}((\frac{1}{4},1))}$$

for any $\varrho > r$. In Lemma 4.3.6, we chose $\varrho = r + \frac{1}{2}$ out of convenience.

$\square$

Treatment With the Variation of Constants Formula

Now, suppose $\tilde{t}_0 > 0$ so that the solution U of (4.67) – (4.74) is of any desired regularity. If the ansatz $\varepsilon\Psi$ is regular enough, the error must be of the same regularity as the ansatz, such that in particular $\mathbf{R} \in \big(C([\tilde{t}_0, \tilde{t}_1], \mathcal{H}_{(0)}^{r-1/2})\big)^3$.

As with the optimal regularity treatment of the error equations, we want to homogenise the boundary conditions (4.111) – (4.113). Then we can formulate the error system for a new variable $\mathbf{R}_h \in \big(C([\tilde{t}_0, \tilde{t}_1], \mathcal{H}_{(0),00}^{r-1/2})\big)^3$. Since the operator $\Lambda_\varepsilon = G$ is sectorial, we can then apply the variation of constants formula.

Note that $\mathbf{R}$ is now continuous in time with values in $\mathcal{H}_{(0)}^{r-1/2}$. This allows for a pointwise homogenisation of the boundary conditions for any fixed t. Hence, we can use the operators $\mathfrak{e}_1^0$ and $\mathfrak{e}_2^0$ from Lemma 4.2.20 and Lemma 4.2.21 instead of $\mathfrak{e}_1$ and $\mathfrak{e}_2$. This has the advantage that we do not have to construct a local solution. In analogy to (4.123) we define $\mathfrak{E}^0 : \mathscr{H}_B^{r-3/2} \to \{0\} \times \mathscr{P}\big(H^r(\Omega)\big)^2 \times H^r(\Omega)$ by

$$\mathfrak{E}^0 \begin{pmatrix} 0 \\ 0 \\ f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} 0 \\ \mathfrak{e}_1^0 f_1 \\ \mathfrak{e}_2^0 f_2 \end{pmatrix}.$$

Then we introduce $\mathbf{R}_h = (R_{c,h}, R_{s,h}, R_{0,h}) \in \big(C([\tilde{t}_0, \tilde{t}_1], \mathcal{H}_{(0),00}^{r-1/2})\big)^3$ by

$$\begin{aligned} R_{c,h} &:= R_c - \varepsilon^2 \mathfrak{E}^0\big(N_{B,c}(R,\Psi) + \varepsilon g_{B,c}(R,\Psi)\big), \\ R_{s,h} &:= R_s - \mathfrak{E}^0\big(N_{B,s}(R_c,\Psi_c) + \varepsilon g_{B,s}(R,\Psi)\big), \\ R_{0,h} &:= R_0 - \mathfrak{E}^0\big(N_{B,0}(R_c,\Psi_c) + \varepsilon g_{B,0}(R,\Psi)\big). \end{aligned}$$

Again, we use the implicit function theorem to show that $\|\mathbf{R}_h\|_{C([\tilde{t}_0,\tilde{t}_1],\mathcal{H}^{r-1/2}_{(0)})} = \mathcal{O}(1)$ if and only if $\|\mathbf{R}\|_{C([\tilde{t}_0,\tilde{t}_1],\mathcal{H}^{r-1/2}_{(0)})} = \mathcal{O}(1)$. Furthermore, we obtain a system for $\mathbf{R}_h$ of the form

$$\begin{aligned}\partial_t R_{c,h} = \Lambda_\varepsilon R_{c,h} &+ \varepsilon^2 N^1_{c,h}(\mathbf{R},\Psi) + \varepsilon^3 g^1_{c,h}(\mathbf{R},\Psi)\\ &+ \varepsilon^2 N^2_{c,h}(\partial_t\mathbf{R},\mathbf{R},\Psi) + \varepsilon^3 g^2_{c,h}(\partial_t\mathbf{R},\mathbf{R},\Psi) + \varepsilon^{-5/2}\mathrm{Res}_c,\end{aligned} \tag{4.129}$$

$$\begin{aligned}\partial_t R_{s,h} = \Lambda_\varepsilon R_{s,h} &+ N^1_{s,h}(R_{c,h},\Psi_c) + \varepsilon g^1_{s,h}(\mathbf{R},\Psi)\\ &+ N^2_{s,h}(\partial_t R_{c,h},\Psi) + \varepsilon g^2_{s,h}(\partial_t\mathbf{R},\mathbf{R},\Psi) + \varepsilon^{-7/2}\mathrm{Res}_s,\end{aligned} \tag{4.130}$$

$$\begin{aligned}\partial_t R_{0,h} = \Lambda_\varepsilon R_{0,h} &+ \partial_x^2\big(N^1_{0,h}(R_{c,h},\Psi_c) + \varepsilon g^1_{0,h}(\mathbf{R},\Psi)\big)\\ &+ \partial_x^2\big(N^2_{0,h}(\partial_t R_{c,h},\Psi) + \varepsilon g^2_{0,h}(\partial_t\mathbf{R},\mathbf{R},\Psi)\big) + \varepsilon^{-7/2}\mathrm{Res}_0,\end{aligned} \tag{4.131}$$

$$\mathbf{R}_h|_{t=\tilde{t}_0} = \mathbf{R}_{h,0}. \tag{4.132}$$

In order to apply the variation of constants formula we have to eliminate the terms containing time derivatives on the right-hand sides of system (4.129) – (4.131). For small $\varepsilon > 0$ we can solve (4.129) for $\partial_t R_{c,h}$, i.e., we can express $R_{c,h}$ in terms of $\mathbf{R}_h$, $\partial_t R_{s,h}$ and $\partial_t R_{0,h}$, where terms depending on $R_{s,h}$ or $R_{0,h}$ have at least a prefactor ε^2. Then we can eliminate $\partial_t R_{c,h}$ in (4.130) and (4.131) with the help of this resolution. We see that all terms containing time derivatives on the right-hand side of the resulting equations have at least a prefactor ε. Hence, we can solve for $\partial_t R_{s,h}$ and at last for $\partial_t R_{0,h}$ such that we get the system

$$\partial_t R_{c,h} = \Lambda_\varepsilon R_{c,h} + \varepsilon^2 N_{c,h}(\mathbf{R}_h,\Psi) + \varepsilon^3 g_{c,h}(\mathbf{R}_h,\Psi) + \varepsilon^{-5/2}\mathrm{Res}_c, \tag{4.133}$$

$$\partial_t R_{s,h} = \Lambda_\varepsilon R_{s,h} + N_{s,h}(R_{c,h},\Psi_c) + \varepsilon g_{s,h}(\mathbf{R}_h,\Psi) + \varepsilon^{-7/2}\mathrm{Res}_s, \tag{4.134}$$

$$\partial_t R_{0,h} = \Lambda_\varepsilon R_{0,h} + \partial_x^2\big(N_{0,h}(R_{c,h},\Psi_c) + \varepsilon g_{0,h}(\mathbf{R}_h,\Psi)\big) + \varepsilon^{-7/2}\mathrm{Res}_0, \tag{4.135}$$

$$\mathbf{R}_h|_{t=\tilde{t}_0} = \mathbf{R}_{h,0}, \tag{4.136}$$

where the terms $N_{j,h}$, $g_{j,h}$, $j = 0, c, s$, satisfy estimates analogous to those in (4.115).

4.3.4 Making the Residual Small

In order to apply the methods of Chapter 2 and Chapter 3 for controlling the error, we need

$$\begin{aligned}\|\mathrm{Res}_c\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})} &= \mathcal{O}(\varepsilon^{9/2}),\\ \|\mathrm{Res}_s\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})} &= \mathcal{O}(\varepsilon^{7/2}),\\ \|\mathrm{Res}_0\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})} &= \mathcal{O}(\varepsilon^{11/2}).\end{aligned} \tag{4.137}$$

Due to the properties of the mapping φ, we have that

$$\begin{aligned}\mathrm{Res}(\varepsilon\Psi) &= -\partial_t\varphi(\varepsilon\Psi^0) + \Lambda_\varepsilon\varphi(\varepsilon\Psi^0) + N(\varphi(\varepsilon\Psi^0)) \\ &= -\partial_t\varepsilon\Psi^0 + \Lambda_\varepsilon\varepsilon\Psi^0 + N_\varphi(\varepsilon\Psi^0),\end{aligned}$$

where N_φ is at least quadratic in $\varepsilon\Psi^0$. Since we defined $\varepsilon\Psi^0$ with the help of its Fourier transform $\widehat{\varepsilon\Psi^0}$, it is obvious that the following considerations are best done for the Fourier transformed residual

$$\widehat{\mathrm{Res}}_\Psi = -\partial_t\widehat{\varepsilon\Psi^0} + \hat{\Lambda}_\varepsilon(\cdot)\widehat{\varepsilon\Psi^0} + \hat{N}_\varphi(\widehat{\varepsilon\Psi^0}).$$

We start with the elimination of the $\mathcal{O}(\varepsilon^2)$-terms in $\widehat{\mathrm{Res}_c^+}$, which will lead to the first equation of the generalised Ginzburg-Landau system (4.103). Note that the nonlinearity N_φ consists of sums of products of the unknowns composed with linear operators that do not act on the wave number k. Thus, the $\mathcal{O}(\varepsilon^2)$ nonlinear terms that are concentrated at k_c are generated by

- quadratic interactions of terms concentrated at 0 and k_c,
- quadratic interactions of terms concentrated at $2k_c$ and $-k_c$,
- cubic interactions of terms concentrated at k_c, k_c and $-k_c$.

Thus, we have at lowest order in Res_c^+ an equation of the form

$$\begin{aligned}\varepsilon^2\partial_T\hat{A}_1(K_1,T)\mathcal{U}^\varepsilon(k,z)& \\ = \lambda_\varepsilon(k)&\hat{A}_1(K_1,T)\mathcal{U}^\varepsilon(k,z) \\ &+ \varepsilon^2(\mathfrak{P}_1^\varepsilon(k)\alpha_{0,1}^\varepsilon(k,\cdot))(z)(\hat{A}_0 * \hat{A}_1)(K_1,T) \\ &+ \varepsilon^2(\mathfrak{P}_1^\varepsilon(k)\alpha_{2,-1}^\varepsilon(k,\cdot)(\hat{a}_{20} * \hat{A}_1)(K_1,T))(z) \\ &+ \varepsilon^2(\mathfrak{P}_1^\varepsilon(k)\alpha_{1,1,-1}^\varepsilon(k,\cdot))(z)(\hat{A}_1 * \hat{A}_1 * \hat{A}_{-1})(K_1,T)\end{aligned} \qquad (4.138)$$

for $k \in [k_c - \rho_0, k_c + \rho_0]$. The $\alpha_{\dots}^\varepsilon$ are smooth functions of k and z, determined by the nonlinear interactions.

Now, we want to eliminate the term $\hat{a}_{20}$ in equation (4.138). To this end, we investigate the lowest order terms concentrated at $2k_c$ and show how these can be expressed in terms of $\hat{A}_1$. The lowest order terms concentrated at $2k_c$ are generated by quadratic interactions of lowest order terms concentrated at k_c. Thus, these terms have support in $[2k_c - 2\rho_0, 2k_c + 2\rho_0]$ and take the form

$$\varepsilon n_{1,1}(k,z)(\hat{A}_1 * \hat{A}_1)(K_2,T),$$

with a term $n_{1,1}$ determined by the Fourier transform of the nonlinearity. For $k \in [2k_c - \rho_0, 2k_c + \rho_0]$ we define $\tilde{a}_{2,0}(k, \cdot)$ as the unique solution of the elliptic problem

$$\begin{aligned} \hat{\Lambda}_\varepsilon(k)\tilde{a}_{2,0}(k,\cdot) &= -n_{1,1}(k,\cdot) \quad \text{on} \quad (-1,0), \\ \widehat{\mathrm{div}}(k)\tilde{a}_{2,0}(k,\cdot) &= 0 \qquad\qquad\quad \text{on} \quad (-1,0), \\ \hat{L}_B(k)\tilde{a}_{2,0}(k,\cdot) &= 0, \end{aligned} \tag{4.139}$$

where $\hat{L}_B(k)$ denotes the Fourier transform of the linear part of the boundary conditions (4.70) – (4.73) at k and $\widehat{\mathrm{div}}(k)$ is the Fourier transformed divergence operator. Then we set

$$\begin{aligned} \hat{a}_{2,0}(K_2, z, T) &= \tilde{a}_{2,0}(2k_c + \varepsilon K_2, z)(\hat{A}_1 * \hat{A}_1)(K_2, T) \\ &= \tilde{a}_{2,0}(2k_c, z)(\hat{A}_1 * \hat{A}_1)(K_2, T) + \mathcal{O}(\varepsilon). \end{aligned}$$

If we plug this into (4.138) and expand in K_1 and ε at $(k_c, 0)$, we obtain at $\varepsilon^2\mathcal{U}^0(k_c, z)$

$$\partial_T \hat{A}_1 = \frac{|\lambda_0''(k_c)|}{2} K_1^2 \hat{A}_1 + \hat{A}_1 + \zeta_{0,1}\hat{A}_0 * \hat{A}_1 + \zeta_{1,1,-1}\hat{A}_1 * \hat{A}_1 * \hat{A}_{-1}. \tag{4.140}$$

Now, we turn to the lowest order terms in $\widehat{\mathrm{Res}}_0$.

Formally, the lowest order nonlinear terms concentrated at zero are $\mathcal{O}(\varepsilon)$ and are produced by quadratic interactions of $\mathcal{O}(1)$-terms concentrated at k_c and $-k_c$. These terms take the form

$$(\alpha_{-1,1}(k,z) + \alpha_{1,-1}(k,z))(\hat{A}_1 * \hat{A}_{-1})(K_0, T).$$

If we apply the projection $\mathfrak{P}_0^\varepsilon(k)$, we get an expression like

$$\zeta_{-1,1}^\varepsilon(k)\mathcal{U}^\varepsilon(k,z)(\hat{A}_1 * \hat{A}_{-1})(K_0, T).$$

We know from (B.55) that the eigenfunction $\mathcal{U}^\varepsilon(0,\cdot)$ is a multiple of

$$z \mapsto (1 + B_i, 0, 0, B_i(z+1)).$$

Furthermore, the nonlinearity maps the η-component to $L^2_{(0)}$. This can only be the case when the coefficient $\zeta_0^\varepsilon(k)$ in front of $\hat{A}_1 * \hat{A}_{-1}$ vanishes at $k = 0$. Due to the reflection symmetry of the underlying problem, we have that $\zeta_0^\varepsilon(k) = \tilde{\zeta}_0^\varepsilon k^2 + \mathcal{O}(k^4)$ for $k \to 0$.

With the same argument we conclude that no terms coming from quadratic interaction of $\mathcal{O}(\varepsilon)$-terms concentrated at $\pm k_c$ or $\pm 2k_c$ yield any $\mathcal{O}(\varepsilon^3)$-terms concentrated at zero.

Thus, we find

$$\partial_T \hat{A}_0 = \frac{|\lambda_0''(0)|}{2} K_0^2 \hat{A}_0 + \zeta_{-1,1} K_0^2 \hat{A}_1 * \hat{A}_{-1}. \tag{4.141}$$

The remaining correction terms are then determined in an analogous fashion as in Appendix 2.A, with the appearance of some elliptic equations for the lowest orders of the $\mathscr{A}_j$ similar to (4.139). This implies that the norms of the correction terms can all be estimated by the norms of A_1 and A_0. Hence, they are $\mathcal{O}(1)$ in $C([0, T_0], H^m)$.

Then if the regularity of A_1 and A_0 is large enough, we obtain the desired estimates (4.137) for the different parts of the residual.

Remark 4.3.8
The equations (4.140) and (4.141) then constitute the Fourier transformed generalised Ginzburg-Landau system from Theorem 4.3.3. Local existence and uniqueness of solutions to this system in $H^{m_A} \times H^{m_A-1}$ has already been shown in Remark 2.B.1. That we also have local existence and uniqueness of solutions in the more restrictive phase space $H^{m_A} \times H^{m_A-1}_{(0)}$ is an easy consequence of the conservation law form of (4.141).

□

4.3.5 Controlling the Error

Note that with the approximation $\varepsilon\Psi$ given by (4.105) , $U_0 = \varepsilon\Psi|_{t=0}$ is a permissible initial condition for (4.67) – (4.74). Thus, there exists an exact solution U such that $(\varepsilon\Psi - U)|_{t=0} = 0$. Hence, we can use Lemma 4.3.6 in order to show that $\|\mathbf{R}\|_{\mathcal{K}^r((0,1))} = \mathcal{O}(1)$. By Sobolev's embedding we then get the desired estimate in the supremum norm on the time interval $[0, 1]$.

For $t \geq \frac{1}{2}$ we apply the variation of constants formula to (4.133) – (4.136) and obtain

$$\begin{aligned} R_{c,h}(t) = S_c(t - \tfrac{1}{2}) R_{c,h}(\tfrac{1}{2}) + \int_{1/2}^t & S_c(t-\sigma)[\varepsilon^2 N_{c,h}(\mathbf{R}_h, \Psi) \\ & + \varepsilon^3 g_{c,h}(\mathbf{R}_h, \Psi) + \varepsilon^{-5/2} \mathrm{Res}_c](\sigma)\, \mathrm{d}\sigma, \end{aligned} \tag{4.142}$$

$$\begin{aligned} R_{s,h}(t) = S_s(t - \tfrac{1}{2}) R_{s,h}(\tfrac{1}{2}) + \int_{1/2}^t & S_s(t-\sigma)[N_{s,h}(R_{c,h}, \Psi_c) \\ & + \varepsilon g_{s,h}(\mathbf{R}_h, \Psi) + \varepsilon^{-7/2} \mathrm{Res}_s](\sigma)\, \mathrm{d}\sigma, \end{aligned} \tag{4.143}$$

$$\begin{aligned} R_{0,h}(t) = S_0(t - \tfrac{1}{2}) R_{0,h}(\tfrac{1}{2}) + \int_{1/2}^t & S_0(t-\sigma)[\partial_x^2 \big(N_{0,h}(R_{c,h}, \Psi) \\ & + \varepsilon g_{0,h}(\mathbf{R}_h, \Psi)\big) + \varepsilon^{-7/2} \mathrm{Res}_0](\sigma)\, \mathrm{d}\sigma, \end{aligned} \tag{4.144}$$

where $S_j(t) = \mathrm{e}^{\Lambda_\varepsilon t} E_j$, $j = 0, c, s$. Note that we have $\|\mathbf{R}(\frac{1}{2})\|_{\mathcal{H}^{r-1/2}_{(0)}} = \mathcal{O}(1)$ due to Remark 4.3.7, and thus $\|\mathbf{R}_h(\frac{1}{2})\|_{\mathcal{H}^{r-1/2}_{(0)}} = \mathcal{O}(1)$.

Similar as in Section 3.4, we set

$$\begin{aligned}
q_c(t) &= \sup_{\tau\in[1,t]} \|\mathcal{F}_x R_{c,h}(\tau)\|_{\mathcal{H}^{r-1/2}_{\varepsilon,k_c}(1)}, \\
q_s(t) &= \sup_{\tau\in[1,t]} \|R_{s,h}(\tau)\|_{\mathcal{H}^{r-1/2}_{(0)}}, \\
q_0(t) &= \sup_{\tau\in[1,t]} \|R_{0,h}(\tau)\|_{\mathcal{H}^{r-1/2}_{(0)}}, \\
q(t) &= q_c(t) + q_0(t),
\end{aligned}$$

where we introduced the spaces

$$\begin{aligned}
\mathcal{H}^m_{\varepsilon,\pm k_c}(1) := L^2_{\varepsilon,\pm k_c}(1) \times \big(L^2_{\varepsilon,\pm k_c}(1)(\mathbb{R}, H^m(-1,0))\big)^2 \\
\times L^2_{\varepsilon,\pm k_c}(1)(\mathbb{R}, H^m(-1,0)),
\end{aligned}$$

to account for the concentration of Fourier modes.

The estimates from Lemma 2.3.1, Lemma 2.3.2, Lemma 2.4.6 and Lemma 2.4.7 hold in an analogous way for the semigroups S_j, $j = 0, c, s$. Then we can proceed in the same way as in Section 3.4 and obtain an estimate

$$\begin{aligned}
\|R_{s,h}(t)\|_{\mathcal{H}^{r-1/2}_{(0)}} \le C + C\big(&q(t) + \varepsilon q(t)^2 \mathfrak{m}(q(t)) + \varepsilon q_s(t) + \varepsilon^2 q(t)^2 \mathfrak{m}(q(t)) \\
&+ \varepsilon \mathcal{Q}_s(t) + \varepsilon^2 \mathcal{Q}_s(t)^2 \mathfrak{m}(\mathcal{Q}_s(t))\big),
\end{aligned}$$

where $\mathfrak{m} : [0,\infty) \to [0,\infty)$ is a monotonically increasing function and

$$\mathcal{Q}_s(t) := \sup_{\sigma\in[t-\frac{1}{4},t]} \|R_{s,h}(\sigma)\|_{\mathcal{H}^{r-1/4}_{(0)}}.$$

We can now estimate $\mathcal{Q}_s(t)$ in terms of $q(t)$ and $q_s(t)$ by

$$\begin{aligned}
\mathcal{Q}_s(t) &\le C \sup_{\sigma\in[t-\frac{1}{4},t]} \|R_s(\sigma)\|_{\mathcal{H}^{r-1/4}_{(0)}} \\
&\le C\|\mathbf{R}\|_{\mathcal{K}^{r+1}((t-\frac{1}{4},t))} \\
&\le C\Big(\|\mathbf{R}|_{\tau=t-\frac{1}{2}}\|_{\mathcal{H}^{r-1/2}} + \|\varepsilon^{-5/2}\mathrm{Res}_c\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})} \\
&\qquad + \sum_{j=0,s} \|\varepsilon^{-7/2}\mathrm{Res}_j\|_{C([0,T_0/\varepsilon^2],\mathcal{H}^{r+1/2}_{(0)})}\Big) \\
&\le C + C \sup_{\sigma\in[1/2,t]} \|\mathbf{R}(\sigma)\|_{\mathcal{H}^{r-1/2}_{(0)}}
\end{aligned}$$

$$\leq C + C \sup_{\sigma \in [1/2,t]} \|\mathbf{R}_h(\sigma)\|_{\mathcal{H}^{r-1/2}_{(0)}}$$
$$\leq C + C(q(t) + q_s(t)).$$

Hence, we have

$$q_s(t) \leq C + C\big(q(t) + \varepsilon q(t)^2 \mathfrak{m}(C + q(t) q_s(t)) \\ + \varepsilon q_s(t) + \varepsilon q_s(t)^2 \mathfrak{m}(C + q(t) + q_s(t))\big).$$

Thus, we can conclude that $q_s(t) \leq C + Cq(t)$ if $C\varepsilon < \frac{1}{2}$ and $\varepsilon(q_s(t) + q(t))\mathfrak{m}(C + q(t) + q_s(t)) < 1$.

The remaining estimates for $q(t)$ work exactly the same as in Section 2.5 and therefore, we have shown that $\|\mathbf{R}_h\|_{C([1/2,T_0/\varepsilon^2],\mathcal{H}^{r-1/2}_{(0)})} = \mathcal{O}(1)$, which is equivalent to $\|\mathbf{R}\|_{C([1/2,T_0/\varepsilon^2],\mathcal{H}^{r-1/2}_{(0)})} = \mathcal{O}(1)$. Together with the estimate for $\mathbf{R}$ on the time interval $[0, 1]$ we have thus proved the approximation result, Theorem 4.3.3.

Appendix A

Supplements to Chapter 3

A.1 The Method of Optimal Regularity

In this section, we recapitulate the method of optimal regularity as a means of proving local existence and uniqueness of solutions for a certain class of quasilinear parabolic problems. As an example we use the quasilinear toy problem from Chapter 3, i.e.,

$$\partial_t u = L_\varepsilon(\partial_x)u + N(u), \quad u|_{t=0} = u_0, \tag{A.1}$$

where

$$\begin{aligned} L_\varepsilon(\partial_x) &:= \frac{1}{2}(1+\partial_x^2)^2\partial_x^2 + \frac{\varepsilon^2}{2}(\partial_x^6 - 3\partial_x^2), \\ N(u) &:= \partial_x^2(u^2) + \partial_x^6(u^2). \end{aligned}$$

A.1.1 Stationary Problem and Resolvent Estimates

Let $x \in \mathbb{R}$, $t \geq 0$ and $f \in H^r(\mathbb{R})$, for $r \geq 0$. Consider the stationary problem

$$(\lambda - L_\varepsilon(\partial_x))u = f. \tag{A.2}$$

With the help of Fourier transform we can easily prove an existence and uniqueness result for all $\lambda \in \mathbb{C}$ with sufficiently large real part. It is of particular importance that the solution u lies in H^{r+6}. Hence, it has the optimal gain in regularity. In order to obtain estimates for the non-stationary problem, we also need an estimate for the L^2-norm of the solution in dependence of $|\lambda|$.

Lemma A.1.1
Let $r \geq 0$, $\operatorname{Re}\lambda \geq 1$ and $\varepsilon_0 \leq \frac{1}{2}$. Then there exists a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $f \in H^r(\mathbb{R})$ problem (A.2) *has a unique solution $u \in H^{r+6}(\mathbb{R})$ with*

$$\|u\|_{H^{r+6}} + |\lambda|^{(r+6)/6}\|u\|_{L^2} \leq C\big(\|f\|_{H^r} + |\lambda|^{r/6}\|f\|_{L^2}\big). \tag{A.3}$$

Proof: Let $\lambda_\varepsilon(k) = -\frac{1}{2}(1-k^2)^2k^2 + \frac{\varepsilon^2}{2}(3k^2-k^6)$. Then (A.2) reads in Fourier space

$$(\lambda - \lambda_\varepsilon(k))\hat{u}(k) = \hat{f}(k) \text{ for almost all } k \in \mathbb{R}.$$

We find $\sup_{k\in\mathbb{R}} \lambda_\varepsilon(k) = \varepsilon^2$ such that for $0 < \varepsilon < 1/2$ and $\operatorname{Re}\lambda > 1$, we have $|\lambda - \lambda_\varepsilon(k)| \geq 3/4$. Hence, there is a unique solution $u \in L^2$ of (A.2), given by

$$u = \mathcal{F}^{-1}\left(k \mapsto \frac{1}{\lambda - \lambda_\varepsilon(k)}\hat{f}(k)\right).$$

If $f \in H^r$, then $u \in H^{r+6}$, since with $\rho(k) = (1+k^2)^{1/2}$ we have

$$\begin{aligned}
\|u\|_{H^{r+6}} &\leq C\|\hat{u}\rho^{r+6}\|_{L^2} \\
&\leq C\|k \mapsto \frac{(1+k^2)^{(r+6)/2}}{\lambda - \lambda_\varepsilon(k)}\hat{f}(k)\|_{L^2} \\
&\leq C\|\hat{f}\rho^r\|_{L^2} \cdot \underbrace{\sup_{k\in\mathbb{R}} \frac{(1+k^2)^3}{|\lambda - \lambda_\varepsilon(k)|}}_{<\infty} \\
&\leq C\|f\|_{H^r}.
\end{aligned}$$

With this estimate at hand, we immediately obtain the estimate (A.3) if we can show

$$\|u\|_{L^2} \leq \frac{C}{|\lambda|}\|f\|_{L^2}. \tag{A.4}$$

For $0 < \varepsilon < \frac{1}{2}$ and $\operatorname{Re}\lambda \geq 1$ we have $\lambda_\varepsilon(k) \leq \frac{1}{2}\operatorname{Re}\lambda$ for all $k \in \mathbb{R}$. Hence, it holds

$$\begin{aligned}
|\lambda - \lambda_\varepsilon(k)|^2 &= (\operatorname{Im}\lambda)^2 + (\operatorname{Re}\lambda - \lambda_\varepsilon(k))^2 \\
&\geq (\operatorname{Im}\lambda)^2 + \frac{1}{4}(\operatorname{Re}\lambda)^2 \\
&\geq \frac{1}{4}|\lambda|^2,
\end{aligned}$$

which implies (A.4).

■

A.1.2 Function Spaces

In the following, we collect several properties of the spaces $H^{r,s}((t_0,t_1))$ and $H_0^{r,s}((t_0,t_1))$, see Definition 3.3.1. Since we are dealing with parabolic problems, we are especially interested in the spaces $K^r((t_0,t_1);2m) := H^{r,\frac{r}{2m}}((t_0,t_1))$ and

$K_0^r((t_0,t_1);2m) := H_0^{r,\frac{r}{2m}}((t_0,t_1))$, where $2m$ is the order of some elliptic linear operator.

We start with a trace theorem for $H^{r,s}$-spaces.

Lemma A.1.2
Let $-\infty < t_0 < t_1 \leq \infty$ and $u \in H^{r,s}((t_0,t_1))$, $r \geq 0$, $s > 1/2$. Then for every non-negative integer $j < s - 1/2$ there exists the trace

$$\partial_t^j u(\cdot, t_0) \in H^{p_j}, \text{ where } p_j = \frac{r}{s}\left(s - j - \frac{1}{2}\right).$$

The mappings $H^{r,s}((t_0,t_1)) \to H^{p_j}(\mathbb{R}) : u \mapsto \partial_t^j u$ are continuous. Furthermore, the mapping $u \mapsto (\partial_t^j u(\cdot,t_0))_{0\leq j<s-1/2}$ from $H^{r,s}((t_0,t_1))$ to $\prod_{0\leq j<s-1/2} H^{p_j}$ is surjective. There exists a continuous extension operator from $\prod_{0\leq j<s-1/2} H^{p_j}$ to $H^{r,s}((t_0,t_1))$, i.e., the trace operator possesses a continuous right inverse.

PROOF: See [LM72a, Theorem 4.2].

■

In particular, if $s = \frac{r}{2m}$, Lemma A.1.2 holds for $r > m$ and all non-negative $j < \frac{r-m}{2m}$ with $p_j = r - 2mj - m$.

We recall the following interpolation property of $H^{r,s}$-spaces.

Lemma A.1.3
Let $r, s \geq 0$ and $\vartheta \in (0,1)$. Then $H^{r,s}(\mathbb{R})$ can be continuously embedded into $H^{\vartheta s}(\mathbb{R}, H^{(1-\vartheta)r})$.

PROOF: See [Häc10, Lemma 6.11].

■

We can use this property to identify the spaces of weak space and time derivatives of $H^{r,s}$-functions.

Lemma A.1.4
Let $r, s \geq 0$ and $l, j \in \mathbb{N}_0$ with $\frac{r}{s}j + l \leq r$. Then, if $u \in H^{r,s}((t_0,t_1))$ we have

$$\partial_t^j \partial_x^l u \in H^{r-\frac{r}{s}j-l,s-\frac{s}{r}l-j}((t_0,t_1)).$$

In particular, if $s = \frac{r}{2m}$, we have for $2mj + l \leq r$ that

$$\partial_t^j \partial_x^l u \in H^{r-2mj-l,\frac{r-2mj-l}{2m}}((t_0,t_1)).$$

Proof: First note that if $u \in H^{\mathfrak{s}}((t_0,t_1),H^{\mathfrak{r}})$, $\mathfrak{r},\mathfrak{s} \geq 0$, we have that for $l \leq \mathfrak{r}$, $j \leq \mathfrak{s}$ it holds

$$\partial_t^j \partial_x^l u \in H^{\mathfrak{s}-j}((t_0,t_1),H^{\mathfrak{r}-l}), \tag{A.5}$$

see [Häc10, Lemma 6.4]. Now, let

$$u \in H^{r,s}((t_0,t_1)) = H^s((t_0,t_1),L^2) \cap L^2((t_0,t_1),H^r).$$

We conclude immediately from (A.5) that $\partial_x^l u \in L^2((t_0,t_1),H^{r-l})$. Setting $\vartheta = \frac{r-l}{r}$, we get with the help of Lemma A.1.3 that $u \in H^{s-\frac{s}{r}l}((t_0,t_1),H^l)$ and therefore

$$\partial_x^l u \in L^2((t_0,t_1),H^{r-l}) \cap H^{s-\frac{s}{r}l}((t_0,t_1),L^2) = H^{r-l,s-\frac{s}{r}l}((t_0,t_1)).$$

Similarly, we get $\partial_t^j u \in H^{s-j}((t_0,t_1),L^2)$ and $u \in H^j((t_0,t_1),H^{r-\frac{r}{s}j})$ – set $\vartheta = \frac{j}{s}$ in Lemma A.1.3 – such that

$$\partial_t^j u \in H^{s-j}((t_0,t_1),L^2) \cap L^2((t_0,t_1),H^{r-\frac{r}{s}j}) = H^{r-\frac{r}{s}j,s-j}((t_0,t_1)).$$

Combining these results shows the assertion of the lemma.

■

An important fact is that the elements of the particular space $H_0^{r,s}((0,\infty))$ can be completely characterised by their Laplace transform with respect to t. We define the Laplace transform $\mathcal{L}$ by

$$\mathcal{L}u(\cdot,\tau) := \frac{1}{\sqrt{2\pi}} \int_0^\infty u(\cdot,t)\mathrm{e}^{-t\tau}\,\mathrm{d}t, \quad \tau \in \mathbb{C}. \tag{A.6}$$

First we note that Laplace transform w.r.t. t conserves the spatial regularity.

Lemma A.1.5
Let $r \geq 0$. If $u \in L^2((0,\infty),H^r)$, then

$$\sup_{\tau_{\mathrm{r}} \geq 0} \int_{\mathbb{R}} \|\mathcal{L}u(\cdot,\tau_{\mathrm{r}} + \mathrm{i}\tau_{\mathrm{i}})\|_{H^r}^2 \leq C\|u\|_{L^2((0,\infty),H^r)}^2.$$

In particular, $\mathcal{L}u(\tau,\cdot) \in H^r$ for almost every τ with $\operatorname{Re}\tau \geq 0$.

Proof: See [Häc10, Lemma 6.8].

■

Furthermore, we can characterise the elements of $H_0^{r,s}((0,\infty))$ with the help of the Paley-Wiener theorem.

Lemma A.1.6
Let $r \geq 0$, $s + \frac{1}{2} \notin \mathbb{N}$. Then $u \in H_0^{r,s}((0,\infty))$ if and only if the Laplace transform $\mathcal{L}u$ has the following properties:

i) $\tau \mapsto \mathcal{L}u(x,\tau)$ *is holomorphic in the half-plane* $\operatorname{Re}\tau > 0$ *for almost every* $x \in \mathbb{R}$.

ii) $\sup_{\tau_\mathrm{r}>0} \int_{\mathbb{R}} |\mathcal{L}u(x, \tau_\mathrm{r} + \mathrm{i}\tau_\mathrm{i})|^2 \mathrm{d}\tau_\mathrm{i} < \infty$ *for almost every* $x \in \mathbb{R}$.

iii) $\left(\int_{\mathbb{R}} (\|\mathcal{L}u(\cdot, \mathrm{i}\tau)\|_{H^r}^2 + |\tau|^{2s} \|\mathcal{L}u(\cdot, \mathrm{i}\tau)\|_{L^2}^2)\, \mathrm{d}\tau \right)^{1/2} < \infty.$

The left-hand side in **iii)** *defines a norm, which is equivalent to* $\|\cdot\|_{H^{r,s}((0,\infty))}$.

Proof: See [Häc10, Theorem 6.15].

■

For the handling of nonlinearities we need that the spaces $K^r((t_0,t_1);2m) = H^{r,\frac{r}{2m}}((t_0,t_1))$ are closed under multiplication.

Lemma A.1.7
Let $r_1 > (2m+1)/2$, $r_1 \geq r_2 \geq 0$. If $u \in K^{r_1}((t_0,t_1);2m)$ and $v \in K^{r_2}((t_0,t_1);2m)$, then $uv \in K^{r_2}((t_0,t_1);2m)$ and there exists a constant $C > 0$ such that

$$\|uv\|_{K^{r_2}((t_0,t_1);2m)} \leq C \|u\|_{K^{r_1}((t_0,t_1);2m)} \|v\|_{K^{r_2}((t_0,t_1);2m)}.$$

Analogously, if $u \in K_0^{r_1}((t_0,t_1);2m)$ and $v \in K_0^{r_2}((t_0,t_1);2m)$, then we have that $uv \in K_0^{r_2}((t_0,t_1);2m)$.

Proof: The assertion of the lemma follows by a slight modification of the arguments in the proof of [Bea84, Lemma 5.1].

■

Remark A.1.8
If the spatial variable is in $\mathbb{R}^d$, then the condition for r_1 in Lemma A.1.7 must read $r_1 > (2m+d)/2$.

□

A.1.3 Non-Stationary Linear Inhomogeneous Problem

Despite the simplicity of our considered toy problem (A.1), the arguments in Section A.1.4 and Section A.1.5 can be generalised very easily to comprise a wide class of similar problems. In fact, the following arguments are almost identical to those used in [Häc10, Section 6.2.2] although the problem considered there was more complicated in several aspects.

Let $r > 6$ and $(r+3)/6 \notin \mathbb{N}$. Consider the problem of finding $u \in K_0^r((t_0, t_1); 6)$ so that

$$Mu := (\partial_t - L)u = f, \quad u|_{t=t_0} = 0, \tag{A.7}$$

holds with $L = L_\varepsilon(\partial_x^2)$ and $f \in K_0^{r-6}((t_0, t_1); 6)$. Due to [LM72a, Theorem 2.2], f can be extended to $K_0^{r-6}((t_0, \infty); 2m)$ in a bounded manner. For any $\gamma \geq 0$ we define

$$U(x,t) := \mathrm{e}^{-\gamma t} u(x, t+t_0) \text{ and } F(x,t) := \mathrm{e}^{-\gamma t} f(x, t+t_0).$$

Thus, if $u \in K_0^r((t_0, \infty); 6)$, $f \in K_0^{r-6}((t_0, \infty); 6)$, then $U \in K_0^r((0, \infty); 6)$ and $F \in K_0^{r-6}((0, \infty); 6)$. Thus, solving (A.7) is equivalent to solving

$$(\partial_t + \gamma - L)U = F, \quad U|_{t=0} = 0. \tag{A.8}$$

After Laplace transform in t the problem reads

$$(\tau + \gamma - L)\mathcal{L}U = \mathcal{L}F, \tag{A.9}$$

since $U|_{t=0} = 0$, Since the Laplace transform with respect to t conserves spatial regularity, see Lemma A.1.5, $\mathcal{L}F(\cdot, \tau)$ lies in H^{r-6} for almost every τ with $\operatorname{Re} \tau \geq 0$. If γ is chosen to be large enough - larger than one in the special case considered here - then there exists a constant $C > 0$ such that for almost every τ with $\operatorname{Re} \tau \geq 0$ there exists a unique solution $\mathcal{L}U(\cdot, \tau)$ of (A.9) which satisfies the resolvent estimate

$$\|\mathcal{L}U\|_{H^r} + |\tau + \gamma|^{r/6} \|\mathcal{L}U\|_{L^2} \leq C(\|\mathcal{L}F\|_{H^{r-6}} + |\tau + \gamma|^{(r-6)/6} \|\mathcal{L}F\|_{L^2}),$$

cf. Lemma A.1.1. Since $\operatorname{Re} \tau \geq 0$ and $\gamma \geq 0$, it holds that $|\tau + \gamma| \geq |\tau|$. Furthermore,

$$\begin{aligned} |\tau + \gamma|^{(r-6)/6} \|\mathcal{L}F\|_{L^2} &\leq C_1(|\tau|^{(r-6)/6} \|\mathcal{L}F\|_{L^2} + \gamma^{(r-6)/6} \|\mathcal{L}F\|_{L^2}) \\ &\leq C_2(\|\mathcal{L}F\|_{H^{r-6}} + |\tau|^{(r-6)/6} \|\mathcal{L}F\|_{L^2}) \end{aligned}$$

such that there exists a constant $C > 0$ with

$$\|\mathcal{L}U\|_{H^r} + |\tau|^{r/6} \|\mathcal{L}U\|_{L^2} \leq C(\|\mathcal{L}F\|_{H^{r-6}} + |\tau|^{(r-6)/6} \|\mathcal{L}F\|_{L^2}). \tag{A.10}$$

In order to show $U \in K_0^r((0,\infty);6)$, it has to be verified that $\mathcal{L}U$ fulfils conditions **i)** – **iii)** in Lemma A.1.6. Condition **iii)** follows immediately from (A.10) and the fact that $F \in K_0^{r-6}((0,\infty);6)$. Condition **i)** follows from the analyticity of the resolvent function $\tau \mapsto (\tau + \gamma - L)^{-1}$ and the analyticity of $\tau \mapsto \mathcal{L}F(x,\tau)$ for almost every x. Finally, condition **ii)** follows from

$$\begin{aligned}\sup_{\tau_{\mathrm{r}}>0} \int_{\mathbb{R}} |\mathcal{L}U(x,\tau_{\mathrm{r}}+\mathrm{i}\tau_{\mathrm{i}})|^2 \,\mathrm{d}\tau_{\mathrm{i}} &\leq C \sup_{\tau_{\mathrm{r}}>0} \int_{\mathbb{R}} \|\mathcal{L}U(\cdot,\tau_{\mathrm{r}}+\mathrm{i}\tau_{\mathrm{i}})\|_{H^6}^2 \,\mathrm{d}\tau_{\mathrm{i}} \\ &\leq C \sup_{\tau_{\mathrm{r}}>0} \int_{\mathbb{R}} \|\mathcal{L}F(\cdot,\tau_{\mathrm{r}}+\mathrm{i}\tau_{\mathrm{i}})\|_{L^2}^2 \,\mathrm{d}\tau_{\mathrm{i}} < \infty,\end{aligned}$$

where we used Sobolev's embedding and the resolvent estimate from Lemma A.1.1; the fact that the expression is finite follows with the help of Lemma A.1.5.

Transferring the result back to the initial problem (A.7) we obtain the following local existence and uniqueness theorem.

Theorem A.1.9
Let $r > 6$ and $(r+3)/6 \notin \mathbb{N}_0$. Then for any $f \in K_0^{r-6}((t_0,t_1);6)$ problem (A.7) *has a unique solution $u \in K_0^r((t_0,t_1);6)$. Furthermore, the norm of the solution can be estimated by the norm of the data, i.e., there exists a constant $C > 0$ such that*

$$\|u\|_{K^r((t_0,t_1);6)} \leq C\|f\|_{K^{r-6}((t_0,t_1);6)}.$$

Remark A.1.10
Although U lies in $K_0^r((0,\infty);6)$ the existence and uniqueness result is only local. This is due to the fact that after the backwards transform, $u(x,t) = \mathrm{e}^{\gamma t}U(x,t-t_0)$, u does not necessarily lie in $K_0^r((0,\infty);6)$ even if U did so, because of the exponentially growing prefactor $\mathrm{e}^{\gamma t}$. Therefore, it would be necessary to show that γ can be chosen to be zero, in order to achieve a global existence result.

□

A.1.4 Solvability of the Full Nonlinear Problem

Let $r \geq 6$, $(r+3)/6 \notin \mathbb{N}_0$. After the considerations of the last section, it is clear that the solutions of the full quasilinear problem (A.1) are sought for in $K^r((t_0,t_1);6)$. From Lemma A.1.2 it follows that the trace $u|_{t=t_0}$ has to lie in H^{r-3}. The full nonlinear problem now reads

$$Mu = N(u), \quad u(t_0,\cdot) = u_0, \tag{A.11}$$

where $M = \partial_t - L_\varepsilon(\partial_x)$, $N(u) = (\partial_x^2 + \partial_x^6)(u^2)$, $t_0 = 0$.

Local Solution

Let M_0^{-1} denote the solution operator from Theorem A.1.9. Since u and $N(u)$ do not necessarily have zero initial values, we have in general that $u \notin K_0^r((t_0,t_1);6)$ and $N(u) \notin K_0^{r-6}((t_0,t_1);6)$. Hence, we cannot apply M_0^{-1} directly to (A.11).

To overcome this difficulty, we write u as the sum $u = v + w$, where $w \in K^r((t_0,t_1);6)$ has to solve

$$Mw = N(v+w) - Mv. \tag{A.12}$$

If we choose $v \in K^r((t_0,t_1);6)$ in an appropriate way so that the right-hand side of (A.12) lies in $K_0^{r-6}((t_0,t_1);6)$ for $w \in K_0^r((t_0,t_1);6)$, we can apply the solution operator M_0^{-1} and thus reformulate (A.12) as a fixed point equation in $K_0^r((t_0,t_1);6)$.

In order to have $(N(v+w)-Mv) \in K_0^{r-6}((t_0,t_1);6)$ for any $w \in K_0^r((t_0,t_1);6)$, we need that $\partial_t^j(N(v+w)-Mv)|_{t=t_0} = 0$ for all $0 \le 6j < r-9$. This is equivalent to

$$\partial_t^j v|_{t=t_0} \stackrel{!}{=} \partial_t^{j-1}(N(v+w)+Lv)|_{t=t_0} \quad \text{for } 0 \le 6j < r-3. \tag{A.13}$$

Remark A.1.11
A function v that fulfils (A.13) is also called a “local solution“, see [Bea84].

□

With the help of the chain rule, we can write (A.13) as

$$\partial_t^j v|_{t=t_0} \stackrel{!}{=} \mathscr{F}_j(v|_{t=t_0}, \dots, \partial_t^{j-1} v|_{t=t_0}) \quad \text{for } 0 \le 6j < r-3,$$

with smooth mappings $\mathscr{F}_j : \prod_{\nu=0}^{j-1} H^{r-6\nu-3} \to H^{r-6j-3}$. Note, that the $\mathscr{F}_j$ are independent of $w \in K_0^r((t_0,t_1);6)$, since the formally appearing terms $\partial_t^\nu w|_{t=t_0}$, $0 \le \nu \le j-1$, are all equal to zero.

The mappings $\mathscr{F}_j$ are at least linear in $(v_0,\dots,v_{j-1})$. Therefore, we have that for all $\mathfrak{r} > 0$ there exists a constant $C_j > 0$ such that

$$\|\mathscr{F}_j(v_0,\dots,v_{j-1})\|_{H^{r-6j-3}} \le C_j(\|v_0\|_{H^{r-3}} + \dots + \|v_{j-1}\|_{H^{r-6j+3}}) \tag{A.14}$$

for all v_ν, $0 \le \nu \le j-1$, with $\|v_\nu\|_{H^{r-6\nu-3}} \le \mathfrak{r}$.

Now, we define $v^{(j)} \in H^{r-6j-3}$, $0 \le 6j < r-3$, inductively via

$$\begin{aligned} v^{(0)} &:= u_0, \\ v^{(j)} &:= \mathscr{F}_j(v^{(0)}, \dots, v^{(j-1)}). \end{aligned}$$

From (A.14) it follows that for all $\mathfrak{r} > 0$ there exists a constant $C > 0$ such that $\|v^{(j)}\|_{H^{r-6j-3}} \leq C\|u_0\|_{H^{r-3}}$, $0 \leq 6j < r-3$, for all initial conditions u_0 with $\|u_0\|_{H^{r-3}} \leq \mathfrak{r}$. The continuous extension operator from Lemma A.1.2 guarantees that there exists a $v \in K^r((t_0,t_1);6)$ with $\partial_t^j v|_{t=t_0} = v^{(j)}$ and

$$\|v\|_{K^r((t_0,t_1);6)} \leq C\|u_0\|_{H^{r-3}}. \tag{A.15}$$

Fixed Point Argument

Now, we can formulate (A.12) as a fixed point equation for $w \in K_0^r((t_0,t_1);6)$, namely

$$w = M_0^{-1}(N(v+w) - Mv). \tag{A.16}$$

We show that the right-hand side of (A.16) defines a contraction on a small ball in $K_0^r((t_0,t_1);6)$ if the initial condition u_0 is sufficiently small in H^{r-3}.

Let $0 < \delta \ll 1$ and $\|u_0\|_{H^{r-3}} < \delta^2$. If $\|w\|_{K^r((t_0,t_1);6)}$, then

$$\begin{aligned}
&\|M_0^{-1}(N(v+w) - Mv)\|_{K^r((t_0,t_1);6)} \\
&\qquad \leq C\|N(v+w) - Mv\|_{K^{r-6}((t_0,t_1);6)} \\
&\qquad \leq C(\|v\|^2_{K^r((t_0,t_1);6)} + \|w\|^2_{K^r((t_0,t_1);6)} + \|v\|_{K^r((t_0,t_1);6)}) \\
&\qquad \leq C(\|u_0\|^2_{H^{r-3}} + \|w\|^2_{K^r((t_0,t_1);6)} + \|u_0\|_{H^{r-3}}) \\
&\qquad \leq C\delta^2 < \delta,
\end{aligned}$$

if δ is chosen small enough. Thus, the right-hand side of (A.16) maps $B_\delta(0) \subset K_0^r((t_0,t_1);6)$ into itself. If $w_1, w_2 \in B_\delta(0)$,

$$\begin{aligned}
&\|M_0^{-1}(N(v+w_1) - N(v+w_2))\|_{K^r((t_0,t_1);6)} \\
&\leq C\|N(v+w_1) - N(v+w_2)\|_{K^{r-6}((t_0,t_1);6)} \\
&\leq C\|w_1 - w_2\|_{K^r((t_0,t_1);6)}(\underbrace{\|v\|_{K^r((t_0,t_1);6)}}_{<C\delta^2} + \underbrace{\|w_1\|_{K^r((t_0,t_1);6)} + \|w_2\|_{K^r((t_0,t_1);6)}}_{\leq 2\delta}) \\
&\leq \frac{1}{2}\|w_1 - w_2\|_{K^r((t_0,t_1);6)}
\end{aligned}$$

if δ is chosen to be small enough, since the terms in the nonlinearity are at least quadratic in the unknowns and pure v-terms are cancelled. Thus, by the contraction mapping principle, there exists a unique solution $w \in K_0^r((t_0,t_1);6)$ of (A.16).

Then we have that $u = v + w \in K^r((t_0,t_1);6)$ is a solution of (A.11). It remains to show that u is the only solution of (A.11), since the choice of v was not unique.

Let $u_1, u_2 \in K^r((t_0,t_1);6)$ be solutions of (A.11). Then $Mu_1 - Mu_2 = N(u_1) - N(u_2)$. Since $N(u_1) - N(u_2) \in K_0^{r-6}((t_0,t_1);6)$ and $u_1 - u_2 \in K_0^r((t_0,t_1);6)$ this is equivalent to

$$u_1 - u_2 = M_0^{-1}(N(u_1) - N(u_2)).$$

Now suppose that $u_1 \neq u_2$. Then we can define $\tilde{t}_0 := \sup\{\theta \in [t_0, t_1] \,|\, u_1(t) = u_2(t)$ for all $t \in [t_0, \theta]\}$. Without loss of generality we can set $\tilde{t}_0 = t_0$. Since the mappings $t \mapsto \|u_j\|_{K^r((t_0,t);6)}$, $j = 1, 2$, are continuous, we can choose $\tilde{t}_1 \in (t_0, t_1)$ so small that

$$\begin{aligned}\|u_1 - u_2\|_{K^r((t_0,\tilde{t}_1);6)} &\leq C\|N(u_1) - N(u_2)\|_{K^{r-6}((t_0,\tilde{t}_1);6)} \\ &\leq C\|u_1 - u_2\|_{K^r((t_0,\tilde{t}_1);6)}(\|u_1\|_{K^r((t_0,\tilde{t}_1);6)} + \|u_2\|_{K^r((t_0,\tilde{t}_1);6)}) \\ &\leq \frac{1}{2}\|u_1 - u_2\|_{K^r((t_0,\tilde{t}_1);6)},\end{aligned}$$

since the nonlinearity N is at least quadratic. Thus, we must have $u_1(t) = u_2(t)$ for all $t \in (t_0, \tilde{t}_1)$ such that $t_0 = \tilde{t}_0 = \tilde{t}_1$, which contradicts the assumption.

Thus, we have proved the following existence and uniqueness theorem.

Theorem A.1.12
Let $r > 6$, $(r+3)/6 \notin \mathbb{N}_0$, $t_0 < t_1 \in \mathbb{R}$. Then there exist constants $C_1, C_2 > 0$ such that the following holds. If

$$\|u_0\|_{H^{r-3}} \leq C_1,$$

then there exists a unique solution $u \in K^r((t_0, t_1); 6)$ of (A.11) with

$$\|u\|_{K^r((t_0,t_1);6)} \leq C_2\|u_0\|_{H^{r-3}}.$$

A.1.5 Higher Regularity of Solutions

The solution of (A.11) becomes arbitrarily smooth after any short time. The underlying idea is simple: Since $u \in K^r((t_0, t_1); 6) \subset C((t_0, t_1), L^2)$, $u(\cdot, t) \in L^2$ is well defined for any fixed time $t \in (t_0, t_1)$. Furthermore, since $u \in K^r((t_0, t_1); 6) \subset L^2((t_0, t_1), H^r)$, it must hold $u(\cdot, t) \in H^r$ for almost every $t \in (t_0, t_1)$. If we choose a $\tau \in (t_0, t_1)$ such that $u(\cdot, \tau) \in H^r$, we can use $u(\cdot, \tau)$ as new initial condition and repeat the arguments of the last section to obtain a solution $u \in K^{r+3}((\tau, t_1); 6)$.

Now, we fix any $\tilde{t}_0 \in (t_0, t_1)$. Then we have

$$\|u\|^2_{K^r((t_0,t_1);6)} \geq \int_{t_0}^{\tilde{t}_0} \|u(\cdot, t)\|^2_{H^r} \,\mathrm{d}t \geq (\tilde{t}_0 - t_0) \inf_{t\in(t_0,\tilde{t}_0)} \|u(\cdot, t)\|^2_{H^r}.$$

Hence, there must exist a $\tau \in (t_0, \tilde{t}_0)$ such that

$$\|u(\cdot, \tau)\|_{H^r} \leq 2(\tilde{t}_0 - t_0)^{-1/2}\|u\|_{K^r((t_0,t_1);6)}.$$

Together with the estimate for u from Theorem A.1.12, we get

$$\|u(\cdot, \tau)\|_{H^r} \leq C(\tilde{t}_0 - t_0)^{-1/2}\|u_0\|_{H^{r-m}}. \tag{A.17}$$

Then we can use Theorem A.1.12 to solve

$$M\tilde{u} = N(\tilde{u}), \quad \tilde{u}|_{t=\tau} = u(\tau,\cdot) \in H^r$$

for $\tilde{u} \in K^{r+3}((\tau,t_1);6)$. The solution $\tilde{u}$ satisfies the estimate

$$\|\tilde{u}\|_{K^{r+3}((\tau,t_1);6)} \leq C\|u(\cdot,\tau)\|_{H^r}. \tag{A.18}$$

It is clear that u and $\tilde{u}$ coincide on (τ,t_1). Hence, we can combine (A.17) and (A.18) and obtain

$$\|u\|_{K^{r+3}((\tilde{t}_0,t_1);6)} \leq C\|u_0\|_{H^{r-m}}.$$

■

Then, by iterating this process we obtain the following result.

Theorem A.1.13
Under the assumptions of Theorem A.1.12 for any $\tilde{t}_0 \in (t_0,t_1)$, $l \in \mathbb{N}$, the following holds. If C_1 is chosen small enough, the unique solution u of (A.11) *lies in $K^{r+l}((\tilde{t}_0,t_1);6)$, i.e., there exists a constant $C = C(l,\tilde{t}_0)$ such that*

$$\|u\|_{K^{r+l}((\tilde{t}_0,t_1);6)} \leq C\|u_0\|_{H^{r-3}}.$$

Remark A.1.14
It is obvious that the methods of Sections A.1.3, A.1.4 and A.1.5 are largely independent of the particular form of the underlying problem. Essentially, it is sufficient to show a resolvent estimate like in Lemma A.1.1. Then the rest follows almost automatically.

□

A.2 An Alternate Approximation Proof

In this section, we present an alternative method for proving an approximation result for the quasilinear Swift-Hohenberg model from Section 3.3.3.

Again, consider

$$\partial_t u = L_\varepsilon^{sh}(\partial_x)u - \partial_x^4(u^3) \tag{A.19}$$

with $L_\varepsilon^{sh}(\partial_x)u = -(1+\partial_x^2)^2u + \varepsilon^2 u$ and the Ginzburg-Landau equation

$$\partial_T A_1 = 4\partial_X^2 A_1 + A_1 - 3A_1|A_1|^2. \tag{A.20}$$

We prove the following result.

Theorem A.2.1
Let $m_A \geq 13$. Then there exist constants C, α, $\varepsilon_0 > 0$ such that the following holds. If $A_1 \in C([0,T_0], H^{m_A}(\mathbb{R}))$ is a solution of (A.20) with $\|A_1\|_{C([0,T_0],H^{m_A})} \leq \alpha$, then there exist solutions u of (A.19) with

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} \left|u(x,t) - \left(\varepsilon A_1(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.}\right)\right| \leq C\varepsilon^{3/2}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Remark A.2.2
As mentioned in Remark 3.3.7, the statement of Theorem A.2.1 is non-trivial but rather weak. The restriction to small solutions of the Ginzburg-Landau equation may exclude interesting dynamics.

That the statement of Theorem A.2.1 is much weaker than that of Theorem 3.1.1 is owed to the different method for the error estimates. The error will be estimated in $K^r((0,T_0/\varepsilon^2);4)$ such that we have an integral norm w.r.t. time. Then, even if the supremum norm w.r.t. time stays small, the integral norm may become large. For the same reason we also have to make the residual smaller. As a consequence, we have to add more higher order correction terms compared with the proof of Theorem 3.1.1. Hence, we also need that the ansatz is more regular.

□

As in Section 3.3.3, we write the exact solution u as the sum of an approximation and an error

$$u = \varepsilon\Psi + \varepsilon^{3/2}R.$$

Then we obtain the same error equation as in Section 3.3.3, namely

$$\partial_t R = L_\varepsilon^{sh}(\partial_x)R - 3\varepsilon^2\partial_x^4(\Psi^2 R) - 3\varepsilon^{5/2}\partial_x^4(\Psi R^2) - \varepsilon^3\partial_x^4(R^3) + \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi). \quad \text{(A.21)}$$

However, we will use a different method than in Section 3.3.3 to show that the error is of order $\mathcal{O}(1)$. Rather than combining optimal regularity results on $\mathcal{O}(1)$-time intervals and the variation of constants formula, we bound the error on the whole time interval $[0, T_0/\varepsilon^2]$ using optimal regularity results. We begin with a more detailed study of the operator $L_\varepsilon^{sh}(\partial_x)$.

A.2.1 Improved Resolvent Estimates

Consider the stationary problem

$$(\lambda - L_\varepsilon^{sh}(\partial_x))u = f \in L^2, \quad \text{(A.22)}$$

where $\lambda \in \mathbb{C}$. In order to control the error on an $\mathcal{O}(1/\varepsilon^2)$ time interval, we have to show that (A.22) has a unique solution for all λ with $\mathrm{Re}\,\lambda > \gamma_\varepsilon$ with $\gamma_\varepsilon = \mathcal{O}(\varepsilon^2)$. The reason for this will become clear in the next section.

Lemma A.2.3
Let $r \geq 0$ and $\varepsilon_0 < \frac{1}{2}$. Then there exists a constant $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 2\varepsilon^2$, and any $f \in H^r(\mathbb{R})$ the stationary problem (A.22) has a unique solution $u \in H^{r+4}(\mathbb{R})$ with

$$\|u\|_{H^{r+4}} + |\lambda|^{(r+4)/4}\|u\|_{L^2} \leq C\big(\varepsilon^{-2}\|f\|_{H^r} + |\lambda|^{r/4}\|f\|_{L^2}\big).$$

PROOF: We proceed as in the proof of Lemma A.1.1 and solve the problem in Fourier space such that

$$\hat{u} = \frac{1}{\lambda - \lambda_\varepsilon^{sh}}\hat{f},$$

where $\lambda_\varepsilon^{sh}(k) = -(1-k^2)^2 + \varepsilon^2$. Again, with $\operatorname{Re}\lambda \geq 2\varepsilon^2$ we have $\lambda_\varepsilon^{sh}(k) < \frac{1}{2}\operatorname{Re}\lambda$. Thus, we conclude that

$$\|u\|_{L^2} \leq \frac{C}{|\lambda|}\|f\|_{L^2}.$$

The estimate for the H^{r+4}-norm of u, however, depends on ε, since for $\lambda = 2\varepsilon^2$ we have

$$\sup_{k\in\mathbb{R}} \frac{(1+k^2)^2}{|2\varepsilon^2 - \lambda_\varepsilon^{sh}(k)|} \leq \frac{4}{\varepsilon^2}.$$

Hence, we have $\|u\|_{H^{r+4}} \leq \frac{C}{\varepsilon^2}\|f\|_{H^r}$.

■

A.2.2 Non-Stationary Linear Inhomogeneous Problem

Exactly as in Section A.1.3, we prove that

$$Mu = (\partial_t - L_\varepsilon^{sh}(\partial_x))u = f \in K_0^{r-4}((0, T_0/\varepsilon^2); 4) \tag{A.23}$$

has a unique solution $u \in K_0^r((0, T_0/\varepsilon^2); 4)$.

We first reduce the above problem to one on $K_0^r((0,\infty); 4)$ and $K_0^{r-4}((0,\infty); 4)$, respectively, by defining

$$U(x,t) = \mathrm{e}^{-2\varepsilon^2 t}u(x,t), \quad F(x,t) = \mathrm{e}^{-2\varepsilon^2 t}f(x,t).$$

Then we conclude with the resolvent estimate from Lemma A.2.3 that there exists a unique solution $U \in K_0^r((0,\infty); 4)$ of

$$(\partial_t - 2\varepsilon^2 - L_\varepsilon^{sh}(\partial_x))U = F \in K_0^{r-4}((0,\infty); 4)$$

with $\|U\|_{K^r((0,\infty);4)} \leq \frac{C}{\varepsilon^2}\|F\|_{K^{r-4}((0,\infty);4)}$.

Ultimately, we obtain a unique solution $u \in K_0^r((0,T_0/\varepsilon^2);4)$ of (A.23) with

$$\begin{aligned}\|u\|_{K^r((0,T_0/\varepsilon^2);4)} &= \|(t,x) \mapsto \mathrm{e}^{2\varepsilon^2 t} U(t,x)\|_{K^r((0,T_0/\varepsilon^2);4)} \\ &\le C\mathrm{e}^{2T_0}\|U\|_{K^r((0,\infty);4)} \\ &\le \frac{C}{\varepsilon^2}\|F\|_{K^{r-4}((0,\infty);4)} \\ &\le \frac{C_M}{\varepsilon^2}\|f\|_{K^{r-4}((0,T_0/\varepsilon^2);4)}\end{aligned}$$

with a constant $C_M > 0$ independent of u, f and ε.

This last estimate makes clear why we needed that the solution exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge \gamma_\varepsilon$, where $\gamma_\varepsilon = \mathcal{O}(\varepsilon^2)$. Otherwise we would have had a prefactor in the estimate that grew exponentially with the length of the time interval.

We denote the solution operator of (A.23) by M_0^{-1}.

A.2.3 Error Estimates

We demand that $R|_{t=0} = 0$. Furthermore, let $4 < r < 6$. Then we are looking for solutions $R \in K_0^r((0,T_0/\varepsilon^2);4)$ of

$$MR = -3\varepsilon^2\partial_x^4(\Psi^2 R) - 3\varepsilon^{5/2}\partial_x^4(\Psi R^2) - \varepsilon^3\partial_x^4(R^3) + \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi).$$

Since $K_0^{r-4}((0,T_0/\varepsilon^2);4) = K^{r-4}((0,T_0/\varepsilon^2);4)$ for $4 < r < 6$, we can apply M_0^{-1} and obtain the equivalent problem

$$R = M_0^{-1}(-3\varepsilon^2\partial_x^4(\Psi^2 R) - 3\varepsilon^{5/2}\partial_x^4(\Psi R^2) - \varepsilon^3\partial_x^4(R^3) + \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi) =: F_\Psi(R).$$

Now, suppose that the ansatz $\varepsilon\Psi$ is chosen in a way that there exists a constant $C_{\mathrm{Res}} > 0$ with

$$\|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{r-4}((0,T_0/\varepsilon^2);4)} \le C_{\mathrm{Res}}\varepsilon^2$$

for all $\varepsilon > 0$ less than a sufficiently small $\varepsilon_0 > 0$, see Remark A.2.4.

As a complete metric space we consider the Ball $\mathcal{B} \subset K_0^r((0,T_0/\varepsilon^2);4)$ centred at zero with radius $2C_M C_{\mathrm{Res}}$. Then, for all $R \in \mathcal{B}$, we have

$$\begin{aligned}\|F_\Psi(R)\|_{K^r((0,T_0/\varepsilon^2);4)} &\le 3C_M\|\Psi\|_{C_b^6}^2\|R\|_{K^r((0,T_0/\varepsilon^2);4)} \\ &\quad + 3C_M\varepsilon^{1/2}\|\Psi\|_{C_b^6}\|R\|_{K^r((0,T_0/\varepsilon^2);4)}^2 \\ &\quad + C_M\varepsilon\|R\|_{K^r((0,T_0/\varepsilon^2);4)}^3 + C_M C_{\mathrm{Res}} \\ &\le \big(6C_M\|\Psi\|_{C_b^6}^2 + 12C_M\|\Psi\|_{C_b^6}\varepsilon^{1/2}C_M C_{\mathrm{Res}} \\ &\qquad\qquad + 4C_M\varepsilon(C_M C_{\mathrm{Res}})^2 + 1\big)C_M C_{\mathrm{Res}}.\end{aligned}$$

Hence, F_Ψ maps $\mathcal{B}$ into itself, provided $\|\Psi\|_{C_b^6}$ and $\varepsilon_0 > 0$ are chosen sufficiently small. For example, such a suitable choice would be

$$\|\Psi\|_{C_b^6} < \min\left\{1, (12C_M)^{-1/2}\right\}, \quad \varepsilon_0 < (16C_M^3 C_{\mathrm{Res}}^2)^{-1} \min\left\{(144C_M)^{-1}, 1\right\}.$$

That F_Ψ is a contraction in $\mathcal{B}$ for all $\varepsilon \in (0, \varepsilon_0)$ follows from

$$\begin{aligned}\|F_\Psi(R_1) - F_\Psi(R_2)\|_{K^r((0,T_0/\varepsilon^2);4)} \leq{} & 3C_M\|\Psi\|_{C_b^6}^2 \|R_1 - R_2\|_{K^r((0,T_0/\varepsilon^2);4)} \\ & + \mathcal{O}\big(\varepsilon^{1/2}\|R_1 - R_2\|_{K^r((0,T_0/\varepsilon^2);4)}\big)\end{aligned}$$

if we choose $\|\Psi\|_{C_b^6}$ as above and ε_0 sufficiently small.

Remark A.2.4
If the residual is of formal order $\mathcal{O}(\varepsilon^j)$, then the scaling property of Sobolev spaces yields

$$\|\mathrm{Res}(\varepsilon\Psi)\|_{C([0,T_0/\varepsilon^2],H^{r-4})} = \mathcal{O}(\varepsilon^{j-1/2}),$$

cf. Section 2.A. However, since we have to estimate the residual in $K^{r-4}((0, T_0/\varepsilon^2))$, we also have to take into account the scaling w.r.t. t. Hence, the correct order of the residual in $K^{r-4}((0, T_0/\varepsilon^2))$ is given by the formal order $-3/2$. Thus, we have to make the residual of formal order $\mathcal{O}(\varepsilon^5)$, if we want $\|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{r-4}((0,T_0/\varepsilon);4)} = \mathcal{O}(\varepsilon^2)$.

Hence, we consider the following ansatz

$$\begin{aligned}\varepsilon\Psi(x,t) = {} & \varepsilon A_1(X,T)\mathrm{e}^{\mathrm{i}x} + \varepsilon A_{-1}(X,T)\mathrm{e}^{-\mathrm{i}x} \\ & + \varepsilon^2 A_{11}(X,T)\mathrm{e}^{\mathrm{i}x} + \varepsilon^2 A_{-11}(X,T)\mathrm{e}^{-\mathrm{i}x} \\ & + \varepsilon^3 A_3(X,T)\mathrm{e}^{3\mathrm{i}x} + \varepsilon^3 A_{-3}(X,T)\mathrm{e}^{-3\mathrm{i}x} \\ & + \varepsilon^4 A_{31}(X,T)\mathrm{e}^{3\mathrm{i}x} + \varepsilon^4 A_{-31}(X,T)\mathrm{e}^{-3\mathrm{i}x},\end{aligned}$$

where $A_{-j} = \overline{A}_j$, $A_{-ij} = \overline{A}_{ij}$, $X = \varepsilon x$ and $T = \varepsilon^2 t$. The residual $\mathrm{Res}(\varepsilon\Psi) = -\partial_t(\varepsilon\Psi) + L_\varepsilon^{sh}(\partial_x)(\varepsilon\Psi) - \partial_x^4(\varepsilon\Psi)^3$ is then of formal order $\mathcal{O}(\varepsilon^5)$ if the following equations are satisfied:

$$\partial_T A_1 = 4\partial_X^2 A_1 + A_1 - 3A_1|A_1|^2, \tag{A.24}$$

$$\begin{aligned}\partial_T A_{11} = {} & 4\partial_X^2 A_{11} + A_{11} - 4\mathrm{i}\partial_X^3 A_1 \\ & + 6A_{11}|A_1|^2 + 3A_{-11}A_1^2 + 12\mathrm{i}\partial_X(A_1|A_1|^2),\end{aligned} \tag{A.25}$$

$$0 = -64A_3 - 81A_1^3, \tag{A.26}$$

$$0 = -64A_{31} + 96\mathrm{i}\partial_X A_3 - 243A_{11}A_1^2 + 108\mathrm{i}\partial_X(A_1^3), \tag{A.27}$$

where the A_{-j}, A_{-ij} have to fulfil the corresponding complex conjugate equations.

Thus, all terms are uniquely determined by A_1. If $m_A \geq 13$ and $A_1 \in C([0, T_0], H^{m_A})$ is a solution of (A.24), we can estimate $\|\Psi\|_{C_b^6}$ by the norm of

A_1 in $C([0,T_0],H^{m_A})$. Furthermore, we have that $\mathrm{Res}(\varepsilon\Psi) \in K^{r-4}((0,T_0/\varepsilon^2;4)$ with

$$\|\varepsilon^{-3/2}\mathrm{Res}(\varepsilon\Psi)\|_{K^{r-4}((0,T_0/\varepsilon^2);4)} = \mathcal{O}(\varepsilon^2).$$

□

Remark A.2.5
The choice of $\Psi \in C_b^6$ is not optimal, since we do not need the boundedness of time derivatives up to sixth order. However, this choice only affects the regularity condition for the solution of the Ginzburg-Landau equation in the approximation result, Theorem A.2.1, in the sense that m_A must be chosen sufficiently large. The main drawback of Theorem A.2.1, i.e., the requirement that A is small, cannot be eliminated just by using a more sophisticated choice of the space for Ψ. Hence, we use C_b^6 out of convenience.

□

Appendix B

Supplements to Chapter 4

B.1 Nonlinearities in the Flat Domain

Below, we list the explicit expressions of the nonlinearities $\mathbf{F}_0$, F_1, F_2, F_3, F_4 in system (4.10) – (4.18). In order to use Einstein's summation convention for a more compact notation, we follow [NT09] and write $\xi_1 = x$ and $\xi_2 = z$. The following terms are almost the same as in [NT09] but are listed for the sake of completeness.

We have $\mathbf{F}_0(\nabla \mathrm{q}, \eta, \mathbf{v}) = (F_{0,1}, F_{0,2})(\nabla \mathrm{q}, \eta, \mathbf{v})$ with

$$
\begin{aligned}
F_{0,i}(\nabla q, \eta, \mathbf{v}) = &-JJ^{ik}\partial_t \left(\frac{J_{kj}}{J}\right) v_j + \frac{1}{J}(1+\xi_2)\partial_t\bar{\eta}\partial_{\xi_2} v_i \\
&+ (1+\xi_2)J^{ik}\partial_t\bar{\eta}\partial_{\xi_2}\left(\frac{J_{kj}}{J}\right) v_j \\
&- J^{ic} v_m \partial_{\xi_m}\left(\frac{J_{cl}}{J}\right) v_l - \frac{1}{J} v_m \partial_{\xi_m} v_i \\
&+ C_P J J^{ic} J^{kj} (\partial_{\xi_k} J^{mj}) \partial_{\xi_m} \left(\frac{J_{cl}}{J}\right) v_l \\
&+ C_P J^{kj} (\partial_{\xi_k} J^{mj}) \partial_{\xi_m} v_i \\
&+ C_P J J^{ic} J^{kj} J^{mj} \partial_{\xi_k}\partial_{\xi_m}\left(\frac{J_{cl}}{J}\right) v_l \\
&+ 2C_P J J^{ic} J^{kj} J^{mj} \partial_{\xi_k}\left(\frac{J_{cl}}{J}\right)\partial_{\xi_m} v_l \\
&+ C_P (J^{kj}J^{mj} - \delta_{ki}\delta_{mi})\partial_{\xi_k}\partial_{\xi_m} v_i \\
&+ (\delta_{ik} - JJ^{ic}J^{kc})\partial_{\xi_k}\mathrm{q},
\end{aligned}
\tag{B.1}
$$

for $i = 1, 2$ where δ_{ij} denotes the Kronecker delta.

Furthermore, we have

$$\begin{aligned}
F_1(\eta, \mathbf{v}, \theta) &= \frac{1}{J}(1+\xi_2)\partial_t\bar{\eta}\partial_{\xi_2}\theta - \frac{1}{J}v_k\partial_{\xi_k}\theta + J^{ki}(\partial_{\xi_k}J^{ji})\partial_{\xi_j}\theta \\
&\quad + (J^{ki}J^{ji} - \delta_{ki}\delta_{ji})\partial_{\xi_k}\partial_{\xi_j}\theta + \left(\frac{J_{2k}}{J} - \delta_{2k}\right)v_k,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
F_2(\eta, \mathbf{v}, \theta) &= 2C_r\frac{1}{J}\partial_{\xi_2}\left(\frac{J_{2k}}{J}\right)v_k + 2C_r\left(\frac{1}{J}\frac{J^{2k}}{J} - \delta_{2k}\right)\partial_{\xi_2}v_k \\
&\quad - \left(\frac{1}{\sqrt{1+(\partial_{\xi_1}\eta)^2}^3} - 1\right)\partial_{\xi_1}^2\eta + M_{a,\varepsilon}C_r(\eta+\theta)\frac{\partial_{\xi_1}^2\eta}{\sqrt{1+(\partial_{\xi_1}\eta)^2}^3} \\
&\quad + C_r\frac{2}{1+(\partial_{\xi_1}\eta)^2}\Bigg[(\partial_{\xi_1}\eta)^2J^{l1}\partial_{\xi_l}\left(\frac{J_{1k}}{J}v_k\right) \\
&\qquad - (\partial_{\xi_1}\eta)\left(J^{l1}\partial_{\xi_l}\left(\frac{J_{2k}}{J}v_k\right) + J^{l2}\partial_{\xi_l}\left(\frac{J_{1k}}{J}v_k\right)\right)\Bigg] \\
&\quad + 2C_r\left(\frac{1}{1+(\partial_{\xi_1}\eta)^2} - 1\right)J^{l2}\partial_{\xi_l}\left(\frac{J_{2k}}{J}v_k\right),
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
F_3(\eta, \mathbf{v}, \theta) &= -J^{l1}\partial_{\xi_l}\left(\frac{J_{2k}}{J}\right)v_k + \left(J^{l1}\frac{J_{2k}}{J} - \delta_{l1}\delta_{k2}\right)\partial_{\xi_l}v_k \\
&\quad - J^{l2}\partial_{\xi_l}\left(\frac{J_{1k}}{J}\right)v_k + \left(J^{l2}\frac{J_{1k}}{J} - \delta_{l2}\delta_{k1}\right)\partial_{\xi_l}v_k \\
&\quad - M_a(J^{l1} - \delta_{l1})\partial_{\xi_l}\theta \\
&\quad + 2\left[J^{l1}\partial_{\xi_l}\left(\frac{J_{1k}}{J}v_k\right) - J^{l2}\partial_{\xi_l}\left(\frac{J_{2k}}{J}v_k\right)\right]\partial_{\xi_1}\eta \\
&\quad + (\partial_{\xi_1}\eta)^2\left[J^{l1}\partial_{\xi_l}\left(\frac{J_{2k}}{J}v_k\right) + J^{l2}\partial_{\xi_l}\left(\frac{J_{1k}}{J}v_k\right)\right] \\
&\quad - M_a\left(\sqrt{1+(\partial_{\xi_1}\eta)^2} - 1\right)(J^{l1}\partial_{\xi_l}\theta - \partial_{\xi_1}\eta) \\
&\quad - M_a\sqrt{1+(\partial_{\xi_1}\eta)^2}(\partial_{\xi_1}\eta)(J^{l2}\partial_{\xi_l}\theta),
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
F_4(\eta, \theta) &= -(J^{l2} - \delta_{l2})\partial_{\xi_l}\theta + \left(\sqrt{1+(\partial_{\xi_1}\eta)^2} - 1\right)(1 + C_B(\theta - \eta)) \\
&\quad + (\partial_{\xi_1}\eta)(J^{l1}\partial_{\xi_l}\theta).
\end{aligned} \tag{B.5}$$

Remark B.1.1
Note that the nonlinearities $\mathbf{F}_0$ and F_1 contain time derivatives ∂_t. However, these are only applied to J, J_{ij} and $\bar{\eta}$. Hence, all terms containing time derivatives can

be expressed in terms of $\partial_t \eta$. Finally, we can use (4.10) in order to eliminate any appearing ∂_t.

$\square$

B.2 Resolvent Estimates on the Real Line

In this section, we prove the resolvent estimates from Lemma 4.2.13 and 4.2.14. We obtain the desired estimates by considering the problem in Fourier space. Hence, we have to show that the corresponding estimates in Fourier space hold with a constant independent of the wave number $k \in \mathbb{R}$.

In [NT07], resolvent estimates for the Bénard-Marangoni Problem with periodic boundary conditions w.r.t. $x \in \mathbb{R}$ were proved. We can use these results to obtain uniform estimates for wave numbers outside a neighbourhood of zero.

The estimates for $k \approx 0$ are then obtained by solving the problem in Fourier space for $k = 0$ and performing a perturbation argument.

B.2.1 Estimates for Large Wave Numbers

We consider the Fourier transformed problem (4.52) - (4.58)

$$\lambda \hat{\eta} - \hat{v}_2 = \hat{g}_0 \qquad \text{at } z = 0, \tag{B.6}$$

$$\lambda \hat{\mathbf{v}} + P_r \hat{\mathscr{P}}_k (k^2 - \partial_z^2) \hat{\mathbf{v}} + 2 P_r \hat{\mathscr{E}}_k (\partial_z \hat{v}_2|_{z=0}) + \dots + \hat{\mathscr{E}}_k \left(\tfrac{P_r B_o}{C_r} + k^2 \right) \hat{\eta} = \hat{\mathbf{f}}_0 \qquad \text{for } z \in (-1, 0), \tag{B.7}$$

$$\lambda \hat{\theta} + (k^2 - \partial_z^2) \hat{\theta} - \hat{v}_2 = \hat{f}_1 \qquad \text{for } z \in (-1, 0), \tag{B.8}$$

$$\hat{\mathbf{v}} = 0 \qquad \text{at } z = -1, \tag{B.9}$$

$$\hat{\theta} = 0 \qquad \text{at } z = -1, \tag{B.10}$$

$$\mathrm{i} k \hat{v}_2 + \partial_z \hat{v}_1 + \mathrm{i} k M_a (\hat{\theta} - \hat{\eta}) = 0 \qquad \text{at } z = 0, \tag{B.11}$$

$$\partial_z \hat{\theta} + B_i (\hat{\theta} - \hat{\eta}) = 0 \qquad \text{at } z = 0, \tag{B.12}$$

with $\hat{\mathbf{f}}_0 \in \hat{\mathscr{P}}_k (L^2(-1,0))^2$ and $\hat{f}_1 \in L^2(-1,0)$, where $\hat{\mathscr{P}}_k$ and $\hat{\mathscr{E}}_k$ are the corresponding operators in Fourier space.

For any $k \in \mathbb{R}$ there exists an $a > 0$ such that $k \in a\mathbb{Z}$. We define the torus $\mathbb{T}_a := \mathbb{R} / \frac{2\pi}{a} \mathbb{Z}$ and $\Omega_a := \mathbb{T}_a \times (-1, 0)$. Then we can define operators $\mathscr{P}_a$, $\mathscr{E}_a$ and G_a exactly in the same way as $\mathscr{P}$, $\mathscr{E}$ and G if we replace Ω by Ω_a and $H_{(0)}^{r-1/2}$ by $H_{(0)}^{r-1/2}(\mathbb{T}_a)$, the space of all $g_0 \in H^{r-1/2}(\mathbb{T}_a)$ with zero mean value. This enables the formulation of a problem on Ω_a analogous to (4.59), namely

$$(\lambda - G_a)\begin{pmatrix}\eta\\ \mathbf{v}\\ \theta\end{pmatrix} = \begin{pmatrix}g_0\\ \mathbf{f}_0\\ f_1\end{pmatrix} \tag{B.13}$$

with $D(G_a)$ defined in the obvious way. In [NT07], it was proved that (B.13) has a unique solution and that resolvent estimates analogous to those in Lemmas 4.2.13 and 4.2.14 hold.

Now, let $k \neq 0$ and set $g_0(x) = \hat{g}_0(k)\mathrm{e}^{\mathrm{i}kx}$, $\mathbf{f}_0(x,z) = \hat{\mathbf{f}}_0(k,z)\mathrm{e}^{\mathrm{i}kx}$ and $f_1(x,z) = \hat{f}_1(k,z)\mathrm{e}^{\mathrm{i}kx}$. If we choose $a \in (0,1]$ so that $k \in a\mathbb{Z}\setminus\{0\}$, we can use the results of [NT07] to obtain a unique solution $(x,z) \mapsto (\hat{\eta}(k), \hat{\mathbf{v}}(k,z), \hat{\theta}(k,z))^\top \mathrm{e}^{\mathrm{i}kx}$. Thus, we can conclude that the Fourier transformed system (B.6) – (B.12) has the unique solution $(\hat{\eta}, \hat{\mathbf{v}}, \hat{\theta})^\top$, which satisfies the estimate

$$\begin{aligned}
&(1+|k|)^{r+1/2}|\hat{\eta}(k)| + |\lambda|^{r/2+1/4}|\hat{\eta}(k)|\\
&+ \|\hat{\mathbf{v}}(k,\cdot)\|_{H^r} + |k|^r\|\hat{\mathbf{v}}(k,\cdot)\|_{L^2} + |\lambda|^{r/2}\|\hat{\mathbf{v}}(k,\cdot)\|_{L^2}\\
&+ \|\hat{\theta}(k,\cdot)\|_{H^r} + |k|^r\|\hat{\theta}(k,\cdot)\|_{L^2} + |\lambda|^{r/2}\|\hat{\theta}(k,\cdot)\|_{L^2}\\
&\le C_a\big((1+|k|)^{r-1/2}|\hat{g}_0(k)| + |\lambda|^{r/2-1/4}|\hat{g}_0(k)|\\
&\qquad + \|\hat{\mathbf{f}}_0(k,\cdot)\|_{H^{r-2}} + |k|^{r-2}\|\hat{\mathbf{f}}_0(k,\cdot)\|_{L^2} + |\lambda|^{(r-2)/2}\|\hat{\mathbf{f}}_0(k,\cdot)\|_{L^2}\\
&\qquad + \|\hat{f}_1(k,\cdot)\|_{H^{r-2}} + |k|^{r-2}\|\hat{f}_1(k\cdot)\|_{L^2} + |\lambda|^{(r-2)/2}\|\hat{f}_1(k,\cdot)\|_{L^2}\big),
\end{aligned} \tag{B.14}$$

with a constant $C_a > 0$ depending on a. Furthermore, if we restrict ourselves to wave numbers k with $|k| \ge k_0$ for some $k_0 > 0$, the constant in estimate (B.14) can be chosen uniformly in k. We can simply set $C_0 := \sup_{a\in[k_0,1]} C_a$.

B.2.2 Estimates for Wave Numbers Close to Zero

In order to handle the wave numbers $k \in (-k_0, k_0)$, a perturbation argument has to be applied, which we explain next.

The outline of the argument is simple. First, the left-hand side of (B.6) - (B.8) is interpreted as $\mathcal{A}_{k,\lambda}(\hat{\eta}, \hat{\mathbf{v}}, \hat{\theta})^\top$. Then the operator $\mathcal{A}_{k,\lambda}$ is split into parts independent of k and a rest term by

$$\mathcal{A}_{k,\lambda} = \mathcal{A}_{0,\lambda} + A_{k,\lambda} = (I + A_{k,\lambda}\mathcal{A}_{0,\lambda}^{-1})\mathcal{A}_{0,\lambda}.$$

If one can show that the norm of the operator $A_{k,\lambda}\mathcal{A}_{0,\lambda}^{-1}$ is less than $1/2$ for all λ with sufficiently large real part and any $k \in (-k_0, k_0)$ for some $k_0 > 0$, then, with the help of the Neumann series, properties like invertibility and a resolvent estimate for $\mathcal{A}_{0,\lambda}$ should carry over to $\mathcal{A}_{k,\lambda}$. However, here this argument cannot be applied directly, since the operators $\mathcal{A}_{k,\lambda}$ and $\mathcal{A}_{0,\lambda}$ do not have the same domains due to boundary condition (B.11), which also depends on k. More precisely, if for $s \geq 0$ we write

$$\hat{X}_k^s := \mathbb{C} \times \hat{\mathscr{P}}_k(H^s(-1,0))^2 \times H^s(-1,0),$$

then the domain of each operator $\mathcal{A}_{k,\lambda}$ is given by

$$\begin{aligned} D(\mathcal{A}_{k,\lambda}) = \big\{ &(\hat{\eta}, \hat{\mathbf{v}}, \hat{\theta}) \in \hat{X}_k^2 \mid \\ &\hat{\mathbf{v}} = 0,\ \hat{\theta} = 0 \text{ at } z = -1, \\ &\mathrm{i}k\hat{v}_2 + \partial_z v_1 + \mathrm{i}kM_a(\hat{\theta} - \hat{\eta}) = 0,\ \partial_z\hat{\theta} + B_i(\hat{\theta} - \hat{\eta}) = 0 \text{ at } z = 0 \big\}. \end{aligned}$$

Before we reformulate system (B.6) - (B.12) in a way suitable for a perturbation analysis, it is useful to investigate the operators $\hat{\mathscr{P}}_k$ and $\hat{\mathscr{E}}_k$.

According to [Bea81, proof of Lemma 3.1], the projection $\mathscr{P}$ is given by

$$\mathscr{P}\mathbf{u} = \mathbf{u} - \nabla\psi, \tag{B.15}$$

where ψ is the unique solution of

$$\begin{aligned} \psi &= 0 && \text{at } z = 0, \\ \Delta\psi &= \operatorname{div}\mathbf{u} && \text{in } \Omega, \\ \partial_z\psi &= u_2 && \text{at } z = -1. \end{aligned} \tag{B.16}$$

Then we have $\hat{\mathscr{P}}_k\hat{\mathbf{u}}(k,z) = \hat{\mathbf{u}}(k,z) - (\mathrm{i}k\hat{\psi}, \partial_z\hat{\psi})^\top$, where $\hat{\psi}$ solves

$$\begin{aligned} \hat{\psi} &= 0 && \text{at } z = 0, \\ (-k^2 + \partial_z^2)\hat{\psi} &= \mathrm{i}k\hat{u}_1 + \partial_z\hat{u}_2 && \text{for } z \in (-1,0), \\ \partial_z\hat{\psi} &= \partial_z\hat{u}_2 && \text{at } z = -1. \end{aligned}$$

Since

$$\mathscr{E}f = \nabla\phi, \tag{B.17}$$

where ϕ is the solution of

$$\begin{aligned} \phi &= f && \text{at } z = 0, \\ \Delta\phi &= 0 && \text{in } \Omega, \\ \partial_z\phi &= 0 && \text{at } z = -1, \end{aligned} \tag{B.18}$$

we have $\hat{\mathscr{E}}_k\hat{f} = \begin{pmatrix} \mathrm{i}k\frac{\hat{f}}{2}\cos(kz) \\ k\frac{\hat{f}}{2}\sinh(kz) \end{pmatrix}$.

The Case $k = 0$

From the above considerations it is obvious that

$$\hat{\mathscr{P}}_0 \hat{\mathbf{u}} = \begin{pmatrix} \hat{u}_1 \\ 0 \end{pmatrix}, \quad \hat{\mathscr{E}}_0 \hat{f} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{B.19}$$

This vastly simplifies system (B.6) - (B.12) for $k = 0$, and we obtain

$$\begin{aligned} \lambda\hat{\eta} &= \hat{g}_0 && \text{at } z = 0, \\ \lambda\hat{v}_1 - P_r\partial_z^2\hat{v}_1 &= \hat{f}_{0,1} && \text{for } z \in (-1,0), \\ \lambda\hat{\theta} - \partial_z^2\hat{\theta} &= \hat{f}_1 && \text{for } z \in (-1,0), \\ \hat{v}_1 &= 0 && \text{at } z = -1, \\ \hat{\theta} &= 0 && \text{at } z = -1, \\ \partial_z\hat{v}_1 &= 0 && \text{at } z = 0, \\ \partial_z\hat{\theta} + B_i(\hat{\theta} - \hat{\eta}) &= 0 && \text{at } z = 0, \end{aligned} \tag{B.20}$$

for $(\hat{\eta}, \hat{\mathbf{v}}, \hat{\theta}) \in \hat{X}_0^2$ (note that this implies $\hat{v}_2 = 0$). Then $\hat{\eta}$ is just given by $\frac{\hat{g}_0}{\lambda}$. The remaining equations for $\hat{\mathbf{v}}$ and $\hat{\theta}$ are decoupled, and it follows by standard elliptic theory that a unique weak solution exists. Since this is a system of ordinary differential equations it is easily verified that for $r \in [0,2]$ we have

$$\|\hat{\mathbf{v}}\|_{H^r} \leq C|\lambda|^{(r-2)/2}\|\hat{\mathbf{f}}_0\|_{L^2}, \tag{B.21}$$

$$\|\hat{\theta}\|_{H^r} \leq C|\lambda|^{(r-2)/2}(\|\hat{f}_1\|_{L^2} + |\hat{g}_0|), \tag{B.22}$$

and for any $r > 2$

$$\|\hat{\mathbf{v}}\|_{H^r} \leq C(\|\hat{\mathbf{f}}_0\|_{H^{r-2}} + |\lambda|^{(r-2)/2}\|\hat{\mathbf{f}}_0\|_{L^2}), \tag{B.23}$$

$$\|\hat{\theta}\|_{H^r} \leq C(\|\hat{f}_1\|_{H^{r-2}} + |\lambda|^{(r-2)/2}\|\hat{f}_1\|_{L^2} + |\hat{g}_0| + |\lambda|^{(r-2)/2}|\hat{g}_0|), \tag{B.24}$$

which implies a resolvent estimate like (B.14) with k set to zero for any $r \geq 2$.

Perturbation Analysis

Now let $k \approx 0$. We split the unknown velocity field into parts $\hat{\mathbf{v}} = \mathbf{w} + \varpi$ with $\mathbf{w} = (w,0)^\top \in \hat{\mathscr{P}}_0(H^2(-1,0))^2$, where w satisfies $w|_{z=-1} = \partial_z w|_{z=0} = 0$. Then $\varpi = \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}$ must be chosen such that

$$\mathrm{i}k\varpi_1 + \partial_z\varpi_2 = -\mathrm{i}kw \qquad \text{for } z \in (-1,0), \tag{B.25}$$

$$\varpi_1 = 0 \qquad \text{at } z = -1, \tag{B.26}$$

$$\varpi_2 = 0 \qquad \text{at } z=-1, \tag{B.27}$$
$$\partial_z \varpi_1 + \mathrm{i}k\varpi_2 = -\mathrm{i}kM_a(\hat{\theta}-\hat{\eta}) \qquad \text{at } z=0, \tag{B.28}$$

hold. Then (B.7) can be written as

$$\lambda \mathbf{w} - P_r \hat{\mathscr{P}}_0 \partial_z^2 \mathbf{w} + \mathrm{R}(k,\lambda,\mathbf{w},\varpi) = \hat{\mathbf{f}}_0, \tag{B.29}$$

where

$$\begin{aligned}\mathrm{R}(k,\lambda,\mathbf{w},\varpi) = {} & P_r(\hat{\mathscr{P}}_0 - \hat{\mathscr{P}}_k)\partial_z^2\mathbf{w} + k^2 P_r \hat{\mathscr{P}}_k(\mathbf{w}-\varpi) - P_r\partial_z^2\varpi \\ & + \lambda\varpi + 2P_r\hat{\mathscr{E}}_k(\partial_z\varpi_2|_{z=0}) + k^2\left(\tfrac{P_rB_o}{C_r}+k^2\right)\hat{\eta}.\end{aligned} \tag{B.30}$$

The left-hand side and the right-hand side of (B.7) both lie in $\hat{\mathscr{P}}_k(L^2(-1,0))^2$. It is important to note that for any $\mathbf{u}=(u_1,u_2)\in\hat{\mathscr{P}}_k(L^2(-1,0))^2$ the second component u_2 is uniquely determined by u_1. If $\mathbf{u}\in\hat{\mathscr{P}}_k(H^r(-1,0))^2$ for $r\geq 1$, then this implies

$$\mathrm{i}ku_1 + \partial_z u_2 = 0 \text{ for } z\in(-1,0), \quad u_2 = 0 \text{ at } z=-1.$$

Hence, we have

$$u_2(z) = -\mathrm{i}k\int_{-1}^{z} u_1(y)\,\mathrm{d}y,$$

which extends to a bounded operator, defined for any $\mathbf{u}\in\hat{\mathscr{P}}_k(L^2(-1,0))^2$. This means that if the first components on the left-hand side and on the right-hand side of (B.29) are equal, then so must be the second components. Therefore, solving (B.29) is equivalent to solving its projection on $\hat{\mathscr{P}}_0(L^2(-1,0))^2$. We get

$$\lambda\hat{\eta} \qquad -\varpi_2 = \hat{g}_0 \qquad \text{at } z=0, \tag{B.31}$$
$$\lambda w - P_r\partial_z^2 w \qquad +(\mathrm{R}(k,\lambda,\mathbf{w},\varpi))_1 = \hat{f}_{0,1} \qquad \text{for } z\in(-1,0), \tag{B.32}$$
$$\lambda\hat{\theta} - \partial_z^2\hat{\theta} \qquad +k^2\hat{\theta} - \varpi_2 = \hat{f}_1 \qquad \text{for } z\in(-1,0), \tag{B.33}$$
$$w = 0 \qquad \text{at } z=-1, \tag{B.34}$$
$$\hat{\theta} = 0 \qquad \text{at } z=-1, \tag{B.35}$$
$$\partial_z\hat{v}_1 = 0 \qquad \text{at } z=0, \tag{B.36}$$
$$\partial_z\hat{\theta} + B_i(\hat{\theta}-\hat{\eta}) = 0 \qquad \text{at } z=0, \tag{B.37}$$

where the unknowns are sought for in $\hat{X}_0^2$. The terms in the second column of (B.31) – (B.33) then define the perturbation $A_{k,\lambda}$ of the operator $\mathcal{A}_{0,\lambda}$. It remains to choose a ϖ satisfying (B.25) – (B.28) such that $\|A_{k,\lambda}\mathcal{A}_{0,\lambda}^{-1}\|_{\hat{X}_0^0\to\hat{X}_0^0}\leq\frac{1}{2}$ uniformly in λ with $\operatorname{Re}\lambda>\gamma$ for all k with $|k|\leq k_0$.

For $|\lambda| \geq 2$ we make the ansatz

$$\varpi_1(k,\lambda,z) = \alpha(k,\lambda)(z+1)^{n_\lambda},$$

where n_λ is the largest integer such that $n_\lambda \leq |\lambda|^{1/2}$. Thus, (B.26) is fulfilled independently of the choice of $\alpha(k,\lambda) \in \mathbb{R}$.

Remark B.2.1
The choice of the exponent n_λ is crucial in order to guarantee that the estimate of $\|A_{k,\lambda}\mathcal{A}_{0,\lambda}^{-1}\|_{\hat{X}_0^0 \to \hat{X}_0^0}$ is independent of λ. It reflects the parabolic character of the system in the sense that any derivative with respect to z generates a prefactor proportional to $|\lambda|^{1/2}$, which eventually corresponds to the ratio of one time derivative to two spatial derivatives. The choice of $n_\lambda \in \mathbb{N}$ guarantees arbitrary smoothness with respect to z enabling us to easily prove estimates for higher regularities as well.

□

Conditions (B.25) and (B.27) determine ϖ_2 uniquely in terms of ϖ_1 and w by

$$\varpi_2(k,\lambda,z) = -\mathrm{i}k \int_{-1}^{z} (\varpi_1(k,\lambda,y) + w(y))\,\mathrm{d}y.$$

Together with (B.28) this yields

$$\alpha(k,\lambda) = \mathrm{i}k\left(n_\lambda + \frac{k^2}{1+n_\lambda}\right)^{-1}\left(M_a(\hat{\eta} - \hat{\theta}(1)) + \mathrm{i}k\int_{-1}^{0} w(y)\,\mathrm{d}y\right), \tag{B.38}$$

and thus,

$$\varpi_2(k,\lambda,z) = -\mathrm{i}k\frac{\alpha(k,\lambda)}{1+n_\lambda} z^{n_\lambda+1} - \mathrm{i}k\int_{-1}^{z} w(y)\,\mathrm{d}y. \tag{B.39}$$

With the estimate

$$|\alpha(k,\lambda)| \leq C|k||\lambda|^{-1/2}(|\hat{\eta}| + \|\hat{\theta}\|_{H^1} + \|w\|_{L^2}),$$

we obtain subsequently

$$\begin{aligned}
|\varpi_2(k,\lambda,0)| &\leq C|k|(|\lambda|^{-1}(|\hat{\eta}| + \|\hat{\theta}\|_{H^1}) + \|\mathbf{w}\|_{L^2}),\\
\|\varpi_2(k,\lambda,\cdot)\|_{L^2} &\leq C|k|(|\lambda|^{-1}(|\hat{\eta}| + \|\hat{\theta}\|_{H^1}) + \|\mathbf{w}\|_{L^2}),\\
|\partial_z\varpi_2(k,\lambda,0)| &\leq C|k|(|\lambda|^{-1/2}(|\hat{\eta}| + \|\hat{\theta}\|_{H^1}) + \|\mathbf{w}\|_{L^2}),\\
\|\partial_z^2\varpi(k,\lambda,\cdot)\|_{L^2} &\leq C|k|(\|\mathbf{w}\|_{H^1} + |\lambda|^{1/2}(|\hat{\eta}| + \|\hat{\theta}\|_{H^1} + \|\mathbf{w}\|_{L^2})),\\
\|\lambda\varpi(k,\lambda,\cdot)\|_{L^2} &\leq C|k||\lambda|^{1/2}(|\hat{\eta}| + \|\hat{\theta}\|_{H^1} + |\lambda|^{1/2}\|\mathbf{w}\|_{L^2}).
\end{aligned}$$

In order to estimate the remaining terms in $\mathrm{R}(k, \lambda, \mathbf{w}, \varpi)$ in the L^2-norm, we use that for small $|k|$ we have for any $s \geq 0$

$$\begin{aligned} \|\hat{\mathscr{E}}_k \hat{f}\|_{H^s} &\leq C|k||\hat{f}|, \\ \|\hat{\mathscr{P}}_k \hat{\mathbf{u}}\|_{H^s} &\leq C\|\hat{\mathbf{u}}\|_{H^s}, \\ \|(\hat{\mathscr{P}}_0 - \hat{\mathscr{P}}_k)\hat{\mathbf{u}}\|_{H^{s+1}} &\leq C|k|\|\hat{\mathbf{u}}\|_{H^s}, \end{aligned} \tag{B.40}$$

where the estimate for $\hat{\mathscr{E}}_k$ follows easily from the explicit form given above. The estimates for $\hat{\mathscr{P}}_k$ follow from a standard perturbation argument for $|k|$ sufficiently small. Thus, we have shown

$$\|A_{k,\lambda} \begin{pmatrix} \hat{\eta} \\ \mathbf{w} \\ \hat{\theta} \end{pmatrix} \|_{\hat{X}_0^0} \leq C|k|(|\lambda|^{1/2}|\hat{\eta}| + |\lambda|^{1/2}\|\hat{\theta}\|_{H^1} + \|\mathbf{w}\|_{H^2} + |\lambda|\|\mathbf{w}\|_{L^2}).$$

Together with $|\hat{\eta}| = \frac{|\hat{g}_0|}{|\lambda|}$ and the estimates (B.21) – (B.22), we obtain for $|\lambda| \geq 2$

$$\|A_{k,\lambda}\mathcal{A}_{0,\lambda}^{-1} \begin{pmatrix} \hat{g}_0 \\ \hat{\mathscr{P}}_0\hat{\mathbf{f}}_0 \\ \hat{f}_1 \end{pmatrix} \|_{\hat{X}_0^0} \leq C|k|(|\hat{g}_0| + \|\hat{\mathscr{P}}_0\hat{\mathbf{f}}_0\|_{L^2} + \|\hat{f}_1\|_{L^2}).$$

If $k_0 < \frac{1}{2C}$, we can use the Neumann series to show that the operator $(I + A_{k,\lambda}\mathcal{A}_{0,\lambda}^{-1})$ has a bounded inverse with norm less than 2. Hence, the resolvent estimates (B.21) and (B.22) carry over from $\mathcal{A}_{0,\lambda}$ to $\mathcal{A}_{k,\lambda}$, and we have for $r \in [0, 2]$

$$\| \begin{pmatrix} \hat{\eta} \\ \mathbf{w} \\ \hat{\theta} \end{pmatrix} \|_{\hat{X}_0^r} \leq C|\lambda|^{(r-2)/2}(|\hat{g}_0| + \|\hat{\mathscr{P}}_0\hat{\mathbf{f}}_0\|_{L^2} + \|\hat{f}_1\|_{L^2}).$$

Thus, we have shown a resolvent estimate for all $k \in (-k_0, k_0)$. Combining this with the results of [NT07] we have concluded the proof of Lemma 4.2.13.

Resolvent estimates for $r > 2$ follow inductively. For example it holds $\|\hat{\theta}\|_{H^r} \leq C(\|\hat{\theta}\|_{H^1} + \|\partial_z^2\hat{\theta}\|_{H^{r-2}})$. With the help of (B.33) we can express $\partial_z^2\hat{\theta}$ in terms of $\hat{f}_1$, ϖ_2 and lower order derivatives of $\hat{\theta}$ such that we arrive at

$$\|\hat{\theta}\|_{H^r} \leq C(\|\hat{\theta}\|_{H^1} + |\lambda|\|\hat{\theta}\|_{H^{r-2}} + \|\hat{f}_1\|_{H^{r-2}} + \|\varpi_2\|_{H^{r-2}}).$$

Then we can apply the induction hypothesis to estimate $|\lambda|\|\hat{\theta}\|_{H^{r-2}}$ in terms of the inhomogeneities. To obtain the correct regularities we then use the interpolation inequality

$$|\lambda|\|f\|_{H^{r-4}} \leq C(\|f\|_{H^{r-2}} + |\lambda|^{(r-2)/2}\|f\|_{L^2}).$$

The estimate for ϖ_2 can again be computed explicitly.

The remaining estimates for $\hat{\eta}$ and $\mathbf{w}$ follow the same scheme as for $\hat{\theta}$. Thus, we have shown Lemma 4.2.14.

B.3 Investigation of the Spectrum

In this section, we investigate the linearisation of system (4.67) – (4.73). According to [Tak81a], it is advantageous to consider a reduced system, where the horizontal component v_1 of the transformed velocity field is eliminated. If we make a normal mode ansatz, the general solution of the reduced system can be computed explicitly. With the help of this general solution, we can then determine the spectrum of (4.67) – (4.73) in dependence of the Marangoni number M_a.

Most of this analysis was already carried out by Takashima, see [Tak81a, Tak81b]. But since not all information we need about the form of the eigenfunctions or the eigenvalue curves can be found in the references cited above, we fill in the details here.

B.3.1 Derivation of the Reduced Linear System

We start with the linearisation of system (4.67) – (4.73), namely

$$\partial_t \eta - v_2 = 0 \qquad \text{at } z = 0, \tag{B.41}$$

$$\begin{aligned} \partial_t \mathbf{v} - P_r \mathscr{P} \Delta \mathbf{v} + 2 P_r \mathscr{E}(\partial_z v_2|_{z=0}) + \cdots \\ \cdots + \mathscr{E}\left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right) \eta = 0 \end{aligned} \qquad \text{in } \Omega, \tag{B.42}$$

$$\partial_t \theta - \Delta \theta - v_2 = 0 \qquad \text{in } \Omega, \tag{B.43}$$

$$\mathbf{v} = 0 \qquad \text{at } z = -1, \tag{B.44}$$

$$\theta = 0 \qquad \text{at } z = -1, \tag{B.45}$$

$$\partial_x v_2 + \partial_z v_1 + M_a \partial_x(\theta - \eta) = 0 \qquad \text{at } z = 0, \tag{B.46}$$

$$\partial_z \theta + B_i(\theta - \eta) = 0 \qquad \text{at } z = 0. \tag{B.47}$$

We want to reformulate the system (B.41) – (B.47) for the unknowns η, v_2 and θ alone.

From the definitions of the operators $\mathscr{P}$ and $\mathscr{E}$, given by (B.15), (B.16) and (B.17), (B.18) respectively, we conclude that

$$\begin{aligned} \Delta \mathscr{P} \mathbf{u} &= \Delta \mathbf{u} \ \text{ if } \operatorname{div} \mathbf{u} = 0, \\ \mathscr{P}\mathbf{u}|_{z=0} \cdot (1,0)^\top &= \mathbf{u}|_{z=0} \cdot (1,0)^\top, \\ \Delta \mathscr{E} f &= 0, \\ \mathscr{E} f|_{z=0} \cdot (1,0)^\top &= \partial_x f. \end{aligned}$$

Hence, if we apply Δ to the second component of (B.42) and restrict the first component of (B.42) to $z = 0$ and apply ∂_x we get

$$\begin{aligned} (\partial_t - P_r\Delta)\Delta\, v_2 &= 0 && \text{in } \Omega, \\ (\partial_t - P_r\Delta)\partial_x v_1 + 2P_r\partial_x^2\partial_z v_2 + \left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right)\partial_x^2\eta &= 0 && \text{at } z = 0. \end{aligned}$$

Furthermore, we apply ∂_x to the remaining boundary conditions containing v_1. Afterwards, we use the solenoidal condition to replace $\partial_x v_1$ by $-\partial_z v_2$. Hence, (B.41) – (B.47) becomes

$$\begin{aligned} \partial_t\eta - v_2 &= 0 && \text{at } z = 0, \\ (\partial_t - P_r\Delta)\Delta\, v_2 &= 0 && \text{in } \Omega, \\ (\partial_t - \Delta)\theta - v_2 &= 0 && \text{in } \Omega, \\ v_2 = \partial_z v_2 = \theta &= 0 && \text{at } z = -1, \\ (\partial_t - 3P_r\partial_x^2 - P_r\partial_z^2)\partial_z v_2 - \left(\tfrac{P_r B_o}{C_r} - \partial_x^2\right)\partial_x^2\eta &= 0 && \text{at } z = 0, \\ \partial_x^2 v_2 - \partial_z^2 v_2 + M_a\partial_x^2(\theta - \eta) &= 0 && \text{at } z = 0, \\ \partial_z\theta + B_i(\theta - \eta) &= 0 && \text{at } z = 0. \end{aligned} \tag{B.48}$$

B.3.2 Linear Dispersion Relation

In order to determine the linear dispersion relation $k \mapsto \mu(k)$ in dependence of M_a, we make the normal mode ansatz

$$(\eta, v_1, v_2, \theta)(x, z, t) = (Z, V(z), W(z), \Theta(z))\mathrm{e}^{\mathrm{i}kx+\mu t}.$$

For $k \neq 0$ it is sufficient to consider the reduced system (B.48), since then V can be recovered uniquely from W by the solenoidal condition,

$$V(z) = \frac{\mathrm{i}}{k}\frac{\mathrm{d}}{\mathrm{d}z}W(z).$$

For $k = 0$ we have to study the full system (B.41) – (B.47).

Notation. In the following, we use the short-hand notation $\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}z}$ from [Tak81a].

The Case $k = 0$

In case of $k = 0$, the solenoidal condition yields $\mathrm{D}W(z) = 0$ for $z \in (-1, 0)$. Together with $W(-1) = 0$ from (B.44), we have that $W \equiv 0$ in $(-1, 0)$. Using (B.19) we arrive at the following system for (Z, V, Θ)

$$\mu Z = 0 \qquad \text{at } z = 0, \tag{B.49}$$

$$(\mathrm{D}^2 - P_r^{-1}\mu)V = 0 \qquad \text{for } z \in (-1, 0) \tag{B.50}$$

$$(\mathrm{D}^2 - \mu)\Theta = 0 \qquad \text{for } z \in (-1, 0), \tag{B.51}$$

$$V = \Theta = 0 \qquad \text{at } z = -1, \tag{B.52}$$
$$\mathrm{D}V = 0 \qquad \text{at } z = 0, \tag{B.53}$$
$$\mathrm{D}\Theta + B_i(\Theta - Z) = 0 \qquad \text{at } z = 0. \tag{B.54}$$

For the following calculations we distinguish two cases:

- $\mu = 0$:

  Then $Z = a_1 \in \mathbb{R}$ can be chosen arbitrarily. From

  $$\mathrm{D}^2 V = 0 \text{ for } z \in (-1, 0), \quad V(-1) = \mathrm{D}V(0) = 0,$$

  we conclude that $V \equiv 0$. The remaining conditions

  $$\mathrm{D}^2\Theta = 0 \text{ for } z \in (-1, 0), \quad \Theta(-1) = 0, \quad \mathrm{D}\Theta(0) + B_i(\Theta(0) - a_1)$$

  yield $\Theta(z) = a_1 \frac{B_i}{1+B_i}(z+1)$. Hence, $\mu = 0$ is an eigenvalue with eigenspace

  $$\mathcal{E}_{0,0} := \operatorname{span}\left\{ z \mapsto (1 + B_i, 0, 0, B_i(z+1))^\top \right\}. \tag{B.55}$$

- $\mu \neq 0$:

  Then (B.49) gives $Z = 0$. The general solutions for V and Θ are given by

  $$\begin{aligned} V(z) &= a_1 \mathrm{e}^{\omega_V z} + a_2 \mathrm{e}^{-\omega_V z}, \\ \omega_V &:= \sqrt{P_r^{-1}\mu}, \\ \Theta(z) &= b_1 \mathrm{e}^{\omega_\Theta z} + b_2 \mathrm{e}^{-\omega_\Theta z}, \\ \omega_\Theta &:= \sqrt{\mu}. \end{aligned}$$

  Using boundary conditions (B.52) – (B.54), we see that there can only exist non-trivial solutions of (B.49) – (B.54) if and only if either

  $$\cosh(\omega_V) = 0 \qquad \text{or} \qquad \cosh(\omega_\Theta) + B_i \omega_\Theta^{-1} \sinh(\omega_\Theta) = 0.$$

  It is clear that the above equations can only be satisfied if $\mu < 0$. In particular, we get the countably many eigenvalues $\mu_{V,j} = -P_r(2j+1)^2\frac{\pi^2}{4}$ with corresponding eigenspaces

  $$\mathcal{E}_{0,\mu_{V,j}} := \operatorname{span}\left\{ z \mapsto \left(0, \cos\left(\sqrt{-\mu_{V,j} P_r^{-1}} z\right), 0, 0\right)^\top \right\}$$

  as well as the eigenspaces

  $$\mathcal{E}_{0,\mu_{\Theta,j}} := \operatorname{span}\left\{ z \mapsto \left(0, 0, 0, \sin\left(\sqrt{-\mu_{\Theta,j}}(1+z)\right)\right)^\top \right\},$$

  where $\mu_{\Theta,j} \approx -(2j+1)^2\frac{\pi^2}{4}$ for small values of $B_i > 0$.

The Case $k \neq 0$

As already mentioned, in case of $k \neq 0$ it suffices to consider the reduced problem (B.48). With the ansatz $(\eta, v_2, \theta)(x,z,t) = (Z, W(z), \Theta(z))\mathrm{e}^{\mathrm{i}kx+\mu t}$ we obtain

$$
\begin{aligned}
\mu Z - W &= 0 && \text{at } z = 0, && \text{(B.56)}\\
\big(\mathrm{D}^2 - (P_r^{-1}\mu + k^2)\big)\big(\mathrm{D}^2 - k^2\big)W &= 0 && \text{for } z \in (-1,0), && \text{(B.57)}\\
\big(\mathrm{D}^2 - (\mu + k^2)\big)\Theta &= -W && \text{for } z \in (-1,0), && \text{(B.58)}\\
W = \mathrm{D}W = \Theta &= 0 && \text{at } z = -1, && \text{(B.59)}\\
(\mu + 3P_r k^2)\mathrm{D}W - P_r\mathrm{D}^3W + \left(\tfrac{P_r B_o}{C_r} + k^2\right)k^2 Z &= 0 && \text{at } z = 0, && \text{(B.60)}\\
\mathrm{D}^2W + k^2W + M_a k^2(\Theta - Z) &= 0 && \text{at } z = 0, && \text{(B.61)}\\
\mathrm{D}\Theta + B_i(\Theta - Z) &= 0 && \text{at } z = 0. && \text{(B.62)}
\end{aligned}
$$

It is our goal to determine how M_a has to be chosen for any given pair (k, μ) so that (B.56) – (B.62) possesses non-trivial solutions. Then we have a relation $M_a = \mathfrak{M}_a(k,\mu)$, which can be solved for μ, so that we obtain spectral curves $k \mapsto \mu(k; M_a)$ for any fixed M_a. Therefore, we compute the solutions of (B.56) – (B.62) in dependence of $k \neq 0$ and μ.

We see that W takes the general form

$$
W(z) = a_1\mathrm{e}^{kz} + a_2\mathrm{e}^{-kz} + a_3\mathrm{e}^{\omega_W z} + a_4\mathrm{e}^{-\omega_W z} \tag{B.63}
$$

with $\omega_W = \sqrt{P_r^{-1}\mu + k^2}$ except for the cases $\mu = 0$ and $\mu = -P_r k^2$.

The general solution of (B.58) is given by the sum of Θ_0 and Θ_p, where Θ_p is a particular solution of (B.58) and Θ_0 solves (B.58) with the right-hand side set to zero. Except for the case $\mu = -k^2$, we have that Θ_0 takes the form

$$
\Theta_0(z) = a_5\mathrm{e}^{\omega_\Theta z} + a_6\mathrm{e}^{-\omega_\Theta z} \tag{B.64}
$$

with $\omega_\Theta = \sqrt{\mu + k^2}$. We further assume that $P_r \neq 1$, so that the terms in W are not in resonance with the terms in Θ_0. Hence, Θ_p takes the form

$$
\Theta_p(z) = \alpha_1\mathrm{e}^{kz} + \alpha_2\mathrm{e}^{-kz} + \alpha_3\mathrm{e}^{\omega_W z} + \alpha_4\mathrm{e}^{-\omega_W z}. \tag{B.65}
$$

If we plug this ansatz into (B.58), we obtain

$$
\alpha_{1/2} = \mu^{-1}a_{1/2}, \quad \alpha_{3/4} = P_r\mu^{-1}(P_r - 1)^{-1}a_{3/4}.
$$

From (B.56) we have

$$
Z = \mu^{-1}W(0) = \mu^{-1}(a_1 + a_2 + a_3 + a_4),
$$

With the help of boundary conditions (B.59), (B.60) and (B.62), we see that the vector of coefficients $\mathbf{a} = (a_1, \dots a_6)^\top$ has to satisfy

$$\mathfrak{B}(k,\mu)\mathbf{a} = 0,$$

with $\mathfrak{B}(k,\mu) \in \mathbb{C}^{5\times 6}$ specified below. Let $\mathbf{b}_j$ denote the jth column of $\mathfrak{B}(k,\mu)$. Then we have

$$\begin{aligned}
\mathbf{b}_1^\top &= \left(\mathrm{e}^{-k}, k\mathrm{e}^{-k}, \mu k + 2P_r k^3 + \left(\tfrac{P_r B_o}{C_r} + k^2\right)\tfrac{k^2}{\mu}, \tfrac{1}{\mu}\mathrm{e}^{-k}, \tfrac{k}{\mu}\right),\\
\mathbf{b}_2^\top &= \left(\mathrm{e}^{k}, -k\mathrm{e}^{k}, -\mu k - 2P_r k^3 + \left(\tfrac{P_r B_o}{C_r} + k^2\right)\tfrac{k^2}{\mu}, \tfrac{1}{\mu}\mathrm{e}^{k}, -\tfrac{k}{\mu}\right),\\
\mathbf{b}_3^\top &= \left(\mathrm{e}^{-\omega_W}, \omega_W\mathrm{e}^{-\omega_W}, 2P_r k^2\omega_W + \left(\tfrac{P_r B_o}{C_r} + k^2\right)\tfrac{k^2}{\mu}, \tfrac{P_r}{(P_r-1)\mu}\mathrm{e}^{-\omega_W}, \tfrac{B_i + P_r\omega_W}{(P_r-1)\mu}\right),\\
\mathbf{b}_4^\top &= \left(\mathrm{e}^{\omega_W}, -\omega_W\mathrm{e}^{\omega_W}, -2P_r k^2\omega_W + \left(\tfrac{P_r B_o}{C_r} + k^2\right)\tfrac{k^2}{\mu}, \tfrac{P_r}{(P_r-1)\mu}\mathrm{e}^{\omega_W}, \tfrac{B_i - P_r\omega_W}{(P_r-1)\mu}\right),\\
\mathbf{b}_5^\top &= \left(0, 0, 0, \mathrm{e}^{-\omega_\Theta}, B_i + \omega_\Theta\right),\\
\mathbf{b}_6^\top &= \left(0, 0, 0, \mathrm{e}^{\omega_\Theta}, B_i - \omega_\Theta\right).
\end{aligned}$$

In the following, we assume that we are in the generic case that the matrix $\mathfrak{B}(k,\mu)$ has full rank. Then we can find a $j_0 \in \{1,\dots,6\}$ such that the set

$$\{\mathbf{b}_\ell \,|\, \ell = 1,\dots,6,\ \ell \neq j_0\}$$

is linearly independent. Let $\mathbf{e}_j$ denote the jth unit vector in $\mathbb{C}^6$. Then the matrix

$$\mathfrak{B}_{+j_0}(k,\mu) := \begin{pmatrix} \mathfrak{B}(k,\mu) \\ \mathbf{e}_{j_0}^\top \end{pmatrix} \in \mathbb{C}^{6\times 6}$$

is invertible, and we have

$$\mathfrak{B}_{+j_0}(k,\mu)\mathbf{a} = a_{j_0}\mathbf{e}_6.$$

Hence, we can express all the coefficients as multiples of a_{j_0} via

$$a_\ell = \underbrace{\left(\left(\mathfrak{B}_{+j_0}(k,\mu)\right)^{-1}\mathbf{e}_6\right)\cdot \mathbf{e}_\ell}_{=:\,\beta_{\ell,j_0}(k,\mu)}\, a_{j_0} \tag{B.66}$$

for $\ell = 1,\dots,6$. It is immediately evident that for any index j_1 with the same property as j_0 it holds

$$\beta_{\ell,j_1} = \beta_{\ell,j_0}\cdot\beta_{j_0,j_1}, \tag{B.67}$$

for $\ell = 1,\dots,6$.

Now, we use boundary condition (B.61) in order to choose M_a in dependence of (k, μ) so that there exist non-trivial solutions. Hence, we have the relation

$$\begin{aligned} M_a &\overset{!}{=} \frac{Z - \Theta(0)}{\mathrm{D}^2 W(0) + k^2 W(0)} \\ &= \frac{a_3 + a_4 + (P_r - 1)\mu(a_5 + a_6)}{\mu(P_r - 1)\big[2k^2(a_1 + a_2) + (\omega_W^2 + k^2)(a_3 + a_4)\big]}. \end{aligned}$$

We now express the a_ℓ in terms of a_{j_0} using (B.66). Then the a_{j_0} cancel in the expression for M_a, and we get

$$M_a = \underbrace{\frac{\beta_{3,j_0}(k,\mu) + \beta_{4,j_0}(k,\mu) + (P_r - 1)\mu(\beta_{5,j_0} + \beta_{6,j_0})(k,\mu)}{\mu(P_r - 1)\big[2k^2(\beta_{1,j_0} + \beta_{2,j_0})(k,\mu) + (\omega_W^2 + k^2)(\beta_{3,j_0} + \beta_{4,j_0})(k,\mu)\big]}}_{=: \mathfrak{M}_a(k,\mu)}. \tag{B.68}$$

Note that due to (B.67) the mapping $\mathfrak{M}_a$ is independent of the choice of j_0.

So far we excluded the cases $\mu = 0$, $\mu = -k^2$ and $\mu = -P_r k^2$ since the general solutions for W and Θ take different forms than those given by (B.63), (B.64) and (B.65) due to the existence of resonances. However, it is not necessary to consider these cases separately and derive a condition for M_a in the fashion of (B.68). Instead, we can obtain the necessary information by taking the limits

$$\lim_{\mu \to 0} \mathfrak{M}_a(k,\mu), \qquad \lim_{\mu \to -k^2} \mathfrak{M}_a(k,\mu), \qquad \lim_{\mu \to -P_r k^2} \mathfrak{M}_a(k,\mu),$$

provided they exist.

This can be seen in the following way. Let

$$\mathbf{x} = (x_1, \ldots, x_7)^\top := (Z, W, \mathrm{D}W, \mathrm{D}^2 W, \mathrm{D}^3 W, \Theta, \mathrm{D}\Theta)^\top.$$

Furthermore, we extend system (B.56) – (B.62) by

$$\mathrm{D}Z = 0, \qquad \text{for } z \in (-1, 0). \tag{B.69}$$

Then we can reformulate (B.69), (B.57), (B.58) as a first order system

$$\mathrm{D}\mathbf{x} = \mathbf{A}(k,\mu)\mathbf{x}$$

with a smooth mapping $\mathbf{A} : \mathbb{R} \times \mathbb{C} \to \mathbb{C}^{7\times 7}$. The fundamental matrix solution of this system is given by

$$\mathbf{X}(z; k, \mu) = \exp(\mathbf{A}(k,\mu)z),$$

which is again smooth in k and μ. We write boundary conditions (B.56), (B.60) – (B.62) as

$$\mathbf{B}_0(k, \mu, M_a)\mathbf{x}(0) = 0,$$

where $\mathbf{B}_0 : \mathbb{R} \times \mathbb{C} \times \mathbb{R} \to \mathbb{C}^{4\times 7}$ is smooth. Similarly, we write

$$\mathbf{B}_{-1}(k, \mu)\mathbf{x}(-1) = 0$$

with a smooth mapping $\mathbf{B}_{-1} : \mathbb{R} \times \mathbb{C} \to \mathbb{C}^{3\times 7}$ for boundary conditions (B.59). With this notation, system (B.56) – (B.62) has non-trivial solutions if and only if

$$\delta(k, \mu, M_a) := \det \begin{pmatrix} \mathbf{B}_0(k, \mu, M_a) \\ \mathbf{B}_{-1}(k, \mu)\mathbf{X}(-1, k, \mu) \end{pmatrix} = 0.$$

The mapping δ is smooth in k and μ and since M_a only appears in a single row of $\mathbf{B}_0(k, \mu, M_a)$ we have that δ is affine w.r.t. M_a. Thus, for almost every (k, μ) we can solve the above equation for M_a in a unique smooth way. For $k \neq 0$, $\mu \notin \{0, -k^2, -P_r k^2\}$ this resolution coincides with $\mathfrak{M}_a$, given by (B.68), and is the smooth extension of $\mathfrak{M}_a$ in the other cases.

Hence, for a numerical computation of the eigenvalue curves it is sufficient to compute the level sets of $\mathfrak{M}_a$, given by (B.68).

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