The validity of the Nonlinear Schrödinger approximation in higher space dimensions

Von der Fakultät Mathematik und Physik der Universität Stuttgart zur Erlangung der Würde eines Doktors der Naturwissenschaften (Dr. rer. nat.) genehmigte Abhandlung

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Tag der mündlichen Prüfung:	24. November 2014

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2014

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List of Symbols

Spaces

$C^0_{b,u}(\mathbb{R}^2, \mathbb{R})$	space of uniformly continuous and uniformly bounded functions. 13
$H^\theta(m)(\mathbb{R}^2, \mathbb{C})$	weighted H^θ-space. 14
H^θ	Sobolev space of order θ. 14
$L^1(\mathbb{R}^2, \mathbb{R})$	space of Lebesgue integrable functions. 14
$L^1(m)$	weighted L^1-space. 14
L^1_g	weighted L^1-space in Chapter 5. 51
$L^p(\mathbb{R}^2, \mathbb{R})$	space of L^p-functions. 14
W_α	space of analytic functions. 51
W	$\mathcal{F}(L^1)$. 15

Operators

E_n	non-resonant mode filter. 56
$\mathcal{F}$	Fourier transform. 13
$\widehat{E}_r$	resonant mode filter. 56
$\widehat{u}$	Fourier transform of a function u. 13
$e^{\Lambda(\mathbf{k})t}$	semigroup. 22

Greek Symbols

$\varepsilon\Psi$	improved ansatz for cubic nonlinearities. 19
$\Gamma(t)$	free surface. 73
$\Omega(t)$	domain for the liquid. 73
η	free top surface. 73
ν	coefficient. 9
ρ	constant density of the fluid. 74
$\varepsilon\psi_{\mathrm{FWI}}$	FWI ansatz. 79
$\varepsilon\psi_{\mathrm{NLS}}$	NLS ansatz. 17
φ	eigenvector. 73
$\widehat{\alpha}$	coefficient. 20
$\widehat{\vartheta}$	weight function. 38

Roman Symbols

C	generic constant. 13
P_0	projection operator. 42
P_1	projection operator. 42
R_n	non-resonant error. 57
R_r	resonant error. 57
R	error. 21
T	scaled time variable $T = \varepsilon^2 t$. 11
X	scaled space variable $X = \varepsilon x$. 11
Y	scaled space variable $Y = \varepsilon y$. 11
$\mathcal{K}(\mathbf{k}_0)$	set of the resonant wave vectors. 49
$\mathcal{R}_n$	supremum of the non-resonant error. 61
$\mathcal{R}_r$	supremum of the resonant error. 61
$\mathcal{R}_z$	supremum of the error. 61
$\mathbf{Res}(u)$	residual for cubic nonlinearities. 18
$g(\mathbf{k}, t)$	weight function. 51
g	gravitation. 74
p	pressure. 74

Zusammenfassung

Das Ziel dieser Arbeit ist der Nachweis von Approximationssätzen für die Nichtlineare Schrödinger-Approximation in höheren Raumdimensionen für dipersive Systeme. Das Hauptaugenmerk liegt dabei auf Systemen mit resonanten quadratischen Nichtlinearitäten, welche zu einer Explosion der Lösungen vor Ende des Approximationsintervalls führen können. Im Vergleich zu einer Raumdimension ist die Resonanzstruktur im höherdimensionalen Fall deutlich koplizierter. Der Nachweis der Approximationseigenschaft geschieht mittels Normalformtransformationen und zusätzlich mit zeitabhängigen Normen.

Abstract

The goal of the present work is the proof of approximation results for the Nonlinear Schördinger approximation in higher space dimensions for dispersive systems. The focus is on systems with resonant quadratic terms, which can lead to some explosion before the end of the approximation interval. In higher space dimensions the resonance structure is much more complicated than in case of one space dimension. The proof of approximation results is based on normal form transforms and the use of time-dependent norms.

Danksagung

Herrn Prof. Dr. Guido Schneider möchte ich für das hochinteressante Thema und die Bereitstellung des Arbeitsplatzes herzlich danken. Ich danke ihm darüber hinaus für jede hilfreiche Unterstützung und für viele inspirierende Diskussionen. Besonders bedanken will ich mich auch für die Gewährung einer DFG-Stelle über das gesamte Forschungsprojekt, was mir ermöglichte, mich voll und ganz meiner Dissertation zu widmen.

Mein besonderer Dank gilt auch Herrn Dr. Wolf-Patrick Düll für seine immer freundliche, uneingeschränkte und geduldige Bereitschaft, mir sein großes mathematisches Wissen weiterzugeben.

Ebenso möchte ich mich bei Herrn Dr. Dominik Zimmermann für sein jederzeit tatkräftiges und herzliches Entgegenkommen bedanken.

Mein Dank gilt auch meinen Mitdoktoranden Danish Ali Sunny für die vielen interessanten Gespräche und die tolle Unterstützung.

Ein ganz großer Dank geht an meine Eltern und meinen Bruder. Ihre Liebe bedeutet mir unbeschreiblich viel. Herzlichen Dank für jede Unterstützung und den großen Beistand, den sie jederzeit zu geben bereit waren.

Meinem Ehemann Waldemar danke ich aus ganzem Herzen für alles. Für seine Liebe, Ausdauer, Ruhe und Geduld, womit er mir stets zur Seite stand und mich immer wieder aufgemuntert hatte.

1. Introduction

Recently, 3D oscillatory water waves and localized electromagnetic waves in higher dimensional media became a subject of active research in mathematics, cf. [1, 2, 3]. Usually these systems are very complex. However, these original systems can be reduced to systems which are much easier than the original system. Because of the complicated nature of these original systems a Nonlinear Schrödinger (NLS) equation can be derived as an universal amplitude equation to describe the dynamics of these systems. Due to its simple structure, the NLS equation can be analyzed and hence correct predictions about the dynamics of the original system can be made. It is the purpose of the present work to transfer and improve existing approximation results for the NLS approximation in 1D to higher space dimensions.

We start with the 1D nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u - u + u^3, \tag{1.1}$$

with $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, and $t \geq 0$. On the time interval $[0, 1/\varepsilon^2]$ equation (1.1) has $\mathcal{O}(\varepsilon)$-amplitude solutions which are slow spatial and temporal modulations of an underlying spatially and temporarily oscillating wave packet. Such solutions are described by the ansatz

$$\varepsilon\psi_{\text{NLS}}(x,t) = \varepsilon A(X,T)e^{\mathrm{i}(k_0 x + \omega_0 t)} + \text{c.c.}, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, k_0 and ω_0 are the spatial and the temporal wave numbers of the underlying carrier wave $e^{\mathrm{i}(k_0 x + \omega_0 t)}$, $A(X,T) \in \mathbb{C}$ is the complex valued envelope function and c.c. is the complex conjugate. Here $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$ and $c_g = \frac{k_0}{\omega_0}$ is the negative group velocity. Inserting the ansatz (1.2) into (1.1) and equating the coefficients in front of $\varepsilon^{j_1} e^{j_2 \mathrm{i}(k_0 x - \omega_0 t)}$ for $j_1 \in \mathbb{N}$, $j_2 \in \mathbb{Z}$, to zero gives a NLS equation

$$\partial_T A = \mathrm{i}\nu_1 \partial_X^2 A + \mathrm{i}\nu_2 A|A|^2, \tag{1.3}$$

where $T \in \mathbb{R}$, $X \in \mathbb{R}$, $\nu_1, \nu_2 \in \mathbb{R}$, and where the basic temporal and spatial wave numbers ω_0 and k_0 are related via the linear dispersion relation $\omega_0^2 = 1 + k_0^2$ of the underlying dispersive wave system (1.1).

The goal of the present thesis is to establish approximation results like

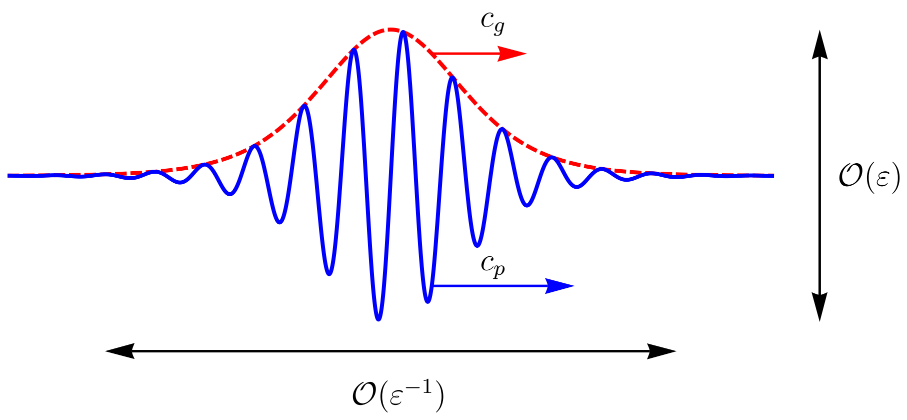


Figure 1.1: The envelope of the oscillating wave packet is described by the amplitude A which solves the NLS equation and advances with the group velocity c_g where the underlying carrier wave advances with the phase velocity $c_p = \omega_0/k_0$.

Approximation Property (APP).
Let $A \in C([0,T_0], H^{\theta_A}(\mathbb{R},\mathbb{C}))$ *be a solution of the* NLS *equation* (1.3) *for a* $\theta_A \geq 0$ *sufficiently big. Then there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there are solutions* u *of* (1.1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |u(x,t) - \varepsilon\psi_{\mathrm{NLS}}(\varepsilon,x,t)| \leq C\varepsilon^{3/2}.$$

This means that the solutions of the original system behave as predicted by the NLS equation on the time interval $[0, T_0/\varepsilon^2]$.

Approximation results: In the 1D case the NLS equation has been derived first in [4] for the water wave problem with infinite depth and no surface tension. The first justification result which shows that the error made by the approximation is small over the natural time scale has been established in [5] for very general quasilinear hyperbolic systems. In case of purely cubic nonlinearities in the original system error estimates can be proven with a simple application of Gronwall's inequality [6]. If semilinear quadratic terms are present in the original system, then they can be removed by a near identity change of variables if the eigenvalues of the linearized problem satisfy a so-called non-resonance condition. Quasilinear quadratic terms have been excluded explicitly in [5]. In the following years there were several efforts to weaken this non-resonance condition from [5] since it is not satisfied for the water wave problem. We refrain from giving a complete overview about existing approximation results at this point and start immediately with the 2D case.

As an example we consider a rotationally symmetric system with two unbounded space directions

$$\partial_t^2 u = \Delta u - u + u^3, \tag{1.4}$$

with $\Delta = \partial_x^2 + \partial_y^2$, $u = u(x,y,t) \in \mathbb{R}$, x, $y \in \mathbb{R}$ and $t \geq 0$. The NLS equation can be derived by making the ansatz

$$\varepsilon\psi_{\mathrm{NLS}}(x,y,t) = \varepsilon A(X,Y,T)e^{\mathrm{i}(k_{01}x+\omega_0 t)} + \text{c.c.}, \tag{1.5}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter, the wave vector $\mathbf{k} = (k_1, k_2)$, the negative group velocity $c_g = \partial_{k_1}\omega|_{\mathbf{k}=\mathbf{k}_0,\omega=\omega_0}$, $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$, $Y = \varepsilon y$, $A(X,Y,T) \in \mathbb{C}$, c.c. the complex conjugate and for notational simplicity we assumed $\mathbf{k}_0 = (k_{01}, 0)$. Herein, $\mathbf{k} = \mathbf{k}_0 \in \mathbb{R}^2$ and $\omega = \omega_0$ satisfy the linear dispersion relation $\omega_0^2 = 1 + |\mathbf{k}_0|^2$.

In lowest order we obtain that A has to satisfy the 2D NLS equation

$$\partial_T A = \mathrm{i}\nu_1\partial_X^2 A + \mathrm{i}\nu_2\partial_Y^2 A + \mathrm{i}\nu_3 A|A|^2, \tag{1.6}$$

with $T \in \mathbb{R}$, X, $Y \in \mathbb{R}$ and real-valued coefficients

$$\begin{aligned}
\nu_1 &= \nu_1(\mathbf{k_0}) = -\frac{1}{2}\partial_{k_1}^2\omega|_{\mathbf{k}=\mathbf{k}_0,\omega=\omega_0},\\
\nu_2 &= \nu_2(\mathbf{k_0}) = -\frac{1}{2}\partial_{k_2}^2\omega|_{\mathbf{k}=\mathbf{k}_0,\omega=\omega_0},\\
\nu_3 &= \nu_3(\mathbf{k_0}),
\end{aligned}$$

and $\mathbf{k}_0 \in \mathbb{R}^2$ and $A(X,Y,T) \in \mathbb{C}$.

It is the goal of the present thesis to transfer and improve the existing approximation results explained above for the 1D case to the 2D case. Very recently N. Totz [7] justified the Davey-Stewartson system [8] for the water wave problem in case of infinite depth without surface tension. In Chapter 2 the approximation result in the case of a cubic nonlinearity is presented. As an example we consider (1.4). The proof of an approximation result is based on a simple application of Gronwall's inequality.

For the quadratic nonlinear wave equation

$$\partial_t^2 u = \Delta u - u + u^2, \tag{1.7}$$

a simple application of Gronwall's inequality only would give estimates on a time interval of length $\mathcal{O}(\varepsilon^{-1})$ instead of $\mathcal{O}(\varepsilon^{-2})$. The approximation result for the NLS equation can be shown with the help of a normal form transform if the linear dispersion relation satisfies a non-resonance condition. This can be found in Chapter 3. It turns out that in these two cases the transfer of the 1D results [6, 5] is rather easy and the proofs follow almost line for line the 1D case.

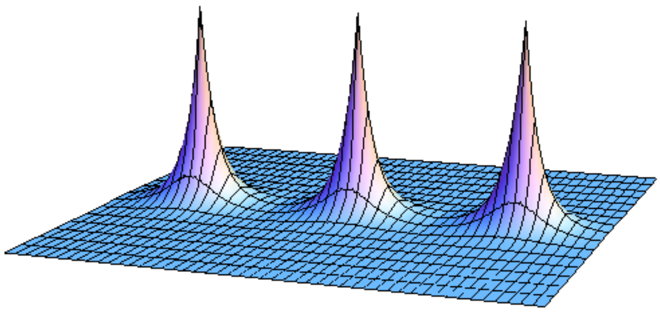


Figure 1.2: The mode concentration of the NLS ansatz with analytic initial conditions in Fourier space.

In Chapter 4 we consider the case of quadratic nonlinearities with a single resonance at $\mathbf{k} = 0$. As an example we consider

$$\partial_t^2 u = \Delta u + \partial_t^2 \Delta u + \Delta(u^2).$$

Again by a normal form transform and by some wave number dependent scaling of the error function we are able to control the dangerous terms that emerge by the additional non-trivial resonance at $\mathbf{k} = \mathbf{k}_0$ which is implied by the resonance at $\mathbf{k} = 0$.

Another case is the influence of a pair of resonances at $\mathbf{k} \neq 0$. We get rid of the difficulty that comes from the quadratic terms in

$$\partial_t^2 u = \Delta u - u - \Delta^2 u + u^2,$$

by using the fact that modes associated to the resonances are exponentially small with respect to ε a priori if analytic initial conditions for the NLS equation are chosen. However, the set of resonant wave vectors and the set of integer multiples of the basic wave vector $\mathbf{k}_0$ must have a positive distance. Hence, we find at integer multiples of the basic wave vector $\mathbf{k}_0$ small peaks of width of $\mathcal{O}(\varepsilon)$ as we can see in Figure 1.2. Our approach follows an idea that has been pointed out already in [9] where a first attempt has been made to weaken the non-resonance condition of [5].

In Chapter 6 we consider the Boussinesq model

$$\partial_t^2 u = \Delta u + \partial_t^2 \Delta u + \Delta(u^2) + \mu \Delta^3 u,$$

where μ is a parameter which can be interpreted as strength of surface tension. The proof of the APP in case of unstable resonances will be a combination of the proof in Chapter 4 for the handling of the trivial resonance at $\mathbf{k} = 0$ and the method introduced in Chapter 5. Obviously, this is an improvement to the method used in [10] since the APP holds if the resonance is unstable for the given system, too. In principal the approach would also apply to the water wave problem with surface tension if the problem with the quasilinear quadratic nonlinearities could be solved.

Finally, in Chapter 7 we prove the approximation result for the four-wave interaction (FWI) system. In their simplest form the equations are given by

$$\partial_T A_j = c_j \cdot \nabla A_j + \sum_{l \in \{1,\dots,4\}} d_{j,l} |A_l|^2 A_j,$$

with group velocity $c_j = \nabla_{\mathbf{k}} \omega|_{\mathbf{k}=\mathbf{k}_j, \omega=\omega_j}$, $j \in \{-4, \dots 4\} \backslash \{0\}$, $\nabla = (\partial_x, \partial_y)^T$ and coefficients $d_{j,l} \in \mathbb{R}$. We will see that the results from Chapter 6 can be applied directly to the problem. Due to the different scaling of the space an even stronger result can be established. The existing literature for the 1D cases will be discussed at the beginning of each chapter.

1.1 Functional analytic setup

We use C for constants which can be chosen independently by the small perturbation parameter $0 < \varepsilon \ll 1$.

In 2D, the coordinates in physical space are denoted by $\mathbf{x} = (x, y)$ and the coordinates in Fourier space by $\mathbf{k} = (k_1, k_2)$. Further we consider functions $u : \mathbb{R}^2 \to \mathbb{R}$. All statements in the following are made for such functions.

The space of uniformly continuous and uniformly bounded functions $u : \mathbb{R}^2 \to \mathbb{R}$ is denoted by $C^0_{b,u}(\mathbb{R}^2, \mathbb{R})$. It is equipped with the norm

$$\|u\|_{C^0_b} = \sup_{\mathbf{x} \in \mathbb{R}^2} |u(\mathbf{x})|.$$

In the proof of the approximation result the Fourier transform is an essential tool. The Fourier transform $\mathcal{F}$ of a function u is denoted by

$$(\mathcal{F}u)[\mathbf{k}] = \widehat{u}(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} u(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}.$$

The inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^{-1}\widehat{u})[\mathbf{x}] = u(\mathbf{x}) = \int_{\mathbb{R}^2} \widehat{u}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}.$$

Multiplication of two functions $(uv)(\mathbf{x}) = u(\mathbf{x})v(\mathbf{x})$ in $\mathbf{x}$-space corresponds in Fourier space to the convolution

$$(\widehat{u} * \widehat{v})(\mathbf{k}) = \int_{\mathbb{R}^2} \widehat{u}(\mathbf{k}-\mathbf{l})\widehat{v}(\mathbf{l})d\mathbf{l}.$$

The Fourier transform of $A(\varepsilon\mathbf{x})e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ is given by

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A(\varepsilon\mathbf{x})e^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}}e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\
=&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A(\varepsilon\mathbf{x})e^{\mathrm{i}(\mathbf{k}_0-\mathbf{k})\mathbf{x}}d\mathbf{x}\\
=&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{1}{\varepsilon^2}A(\mathbf{X})e^{\mathrm{i}(\frac{\mathbf{k}_0-\mathbf{k}}{\varepsilon})\mathbf{X}}d\mathbf{X}\\
=&\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\mathbf{k}-\mathbf{k}_0}{\varepsilon}\right),
\end{aligned}$$

where $\mathbf{X} = (X, Y)$.

The space of n times differentiable functions $u : \mathbb{R}^2 \to \mathbb{R}$ with uniformly continuous and uniformly bounded derivatives $\partial_x^{j_1}\partial_y^{j_2}u$ with $|j_1| + |j_2| = j$ are denoted with $C^n_{b,u}(\mathbb{R}^2, \mathbb{R})$. It is equipped with the norm

$$\|u\|_{C^n_b} = \sum_{j=0}^{n} \sum_{|j_1|+|j_2|=j,\, j_1,j_2\geq 0} \|\partial_x^{j_1}\partial_y^{j_2}u\|_{C^0_b}.$$

The space of Lebesgue integrable functions $u : \mathbb{R}^2 \to \mathbb{R}$ is denoted with $L^1(\mathbb{R}^2, \mathbb{R})$. It is equipped with the norm

$$\|u\|_{L^1} = \int_{\mathbb{R}^2} |u(\mathbf{x})|d\mathbf{x}.$$

For $p \geq 1$ a function u is in $L^p(\mathbb{R}^2, \mathbb{R})$ if $|u|^p \in L^1(\mathbb{R}^2, \mathbb{R})$. The space L^p is equipped with the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^2} |u(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}.$$

The Sobolev space H^θ is equipped with the norm

$$\|u\|_{H^\theta} = \left(\int_{\mathbb{R}^2} |\widehat{u}(\mathbf{k})|^2(1+|\mathbf{k}|^2)^\theta d\mathbf{k}\right)^{1/2}.$$

We define the space $L^1(m)$ by $u \in L^1(m) \Leftrightarrow u\rho^m \in L^1$, where $\rho(\mathbf{k}) = (1+\mathbf{k}^2)^{1/2}$.

Moreover, we use the space $H^\theta(m)(\mathbb{R}^2,\mathbb{C})$ equipped with the norm

$$\|u\|_{H^\theta(m)} = \|u\rho^m\|_{H^\theta}.$$

Fourier transform is an isomorphism between the space $H^\theta(m)(\mathbb{R}^2,\mathbb{C})$ and the space $H^m(\theta)(\mathbb{R}^2,\mathbb{C})$ and a continuous mapping from $L^1(m)$ into C_b^m, i.e.,

$$\|u\|_{C_b^m} \leq C\|\widehat{u}\|_{L^1(m)}.$$

We also use $L^2(m) = H^0(m)$. Due to Sobolev's embedding theorem we have

$$\|u\|_{C_b^m(\mathbb{R}^2,\mathbb{R})} \leq C\|u\|_{H^\theta(\mathbb{R}^2,\mathbb{R})},$$

if $m < \theta - 1$. Moreover, we need the space

$$W = \{u : \mathbb{R}^2 \to \mathbb{R} : \|u\|_W < \infty\},$$

where $\|u\|_W = \|\widehat{u}\|_{L^1}$. We have

$$\|u \cdot v\|_W = \|\widehat{u} * \widehat{v}\|_{L^1} \leq \|\widehat{u}\|_{L^1}\|\widehat{v}\|_{L^1} \leq \|u\|_W\|v\|_W, \tag{1.8}$$

since L^1 is closed under convolution. The elements of W are uniformly continuous and uniformly bounded functions, which additionally satisfy $\lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$ due to Riemann-Lebesgue theorem.

For the functions in Fourier space we find

Lemma 1.1.
There exists a constant C and $m = m_1+m_2$ such that the following holds. For $\theta > 1$ and $m \geq 0$ we have

$$\|uv\|_{H^\theta(m)} \leq C\|u\|_{H^\theta(m_1)}\|v\|_{H^\theta(m_2)}, \tag{1.9}$$

and for $m > 1$ we have

$$\|\widehat{u} * \widehat{v}\|_{L^2(m)} \leq C\|\widehat{u}\|_{L^2(m)}\|\widehat{v}\|_{L^2(m)}. \tag{1.10}$$

Proof. Young's inequality and Sobolev's embedding theorem yield

$$\|uv\|_{H^\theta(m)} \leq C\|u\rho^m v\|_{H^\theta} \leq C\|u\rho^{m_1}\|_{H^\theta}\|v\rho^{m_2}\|_{H^\theta} \leq C\|u\|_{H^\theta(m_1)}\|v\|_{H^\theta(m_2)},$$

and

$$\begin{aligned}\|\widehat{u} * \widehat{v}\|_{L^2(m)} &= \|(\widehat{u} * \widehat{v})\rho^m\|_{L^2} \\ &\leq C(\|(\widehat{u}\rho^m) * \widehat{v}\|_{L^2} + \|\widehat{u} * (\widehat{v}\rho^m)\|_{L^2}) \\ &\leq C(\|\widehat{u}\rho^m\|_{L^2}\|\widehat{v}\|_{L^1} + \|\widehat{u}\|_{L^1}\|\widehat{v}\rho^m\|_{L^2}) \\ &\leq C\|\widehat{u}\|_{L^2(m)}\|\widehat{v}\|_{L^2(m)},\end{aligned}$$

since $\rho^m(\mathbf{k}) \leq C(\rho^m(\mathbf{k} - \mathbf{l}) + \rho^m(\mathbf{l}))$. □

Remark 1.2. We always have $\left\|\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^1} \leq \|\widehat{A}\|_{L^2(2)} = \mathcal{O}(1)$ as $\varepsilon \to 0$.

2. Cubic nonlinearities

In this chapter we transfer the existing approximation results for the NLS approximation from 1D to 2D in case that the original system does not contain quadratic nonlinearities. The case of cubic nonlinearities is simple in the sense that there will be no terms of $\mathcal{O}(\varepsilon)$ in the error equation, which can lead to an explosion on the long time interval $[0, T_0/\varepsilon^2]$ in comparison to the case with quadratic nonlinearities. For the proof that solutions of the original system behave as predicted by the NLS equation the application of Gronwall's inequality is sufficient. Such a result was shown in the 1D case in [6]. The transfer to the 2D case will be straightforward.

2.1 Derivation of the NLS equation and estimates for the residual

We consider in 2D the nonlinear wave equation

$$\partial_t^2 u = \Delta u - u + u^3, \tag{2.1}$$

with $\Delta = \partial_x^2 + \partial_y^2$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $u(x, y, t) \in \mathbb{R}$. The NLS equation describes slow modulations in time and space of a wave train $e^{\mathrm{i}(k_{01}x - \omega_0 t)}$ in (2.1).

The ansatz for the derivation of the NLS equation is given by

$$\varepsilon\psi_{\mathrm{NLS}}(x, y, t) = \varepsilon A(X, Y, T) e^{\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.}, \tag{2.2}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter, $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$, $Y = \varepsilon y$, $A(X, Y, T) \in \mathbb{C}$, and c.c. the complex conjugate. Formally, the approximation is a good approximation if the terms that do not cancel after inserting $\varepsilon\psi_{\mathrm{NLS}}$ into (2.1) are small.

2.1.1 The residual

The terms that do not cancel after inserting the approximation into (2.1) are collected in the residual

$$\mathrm{Res}(u) = -\partial_t^2 u + \partial_x^2 u + \partial_y^2 u - u + u^3.$$

We have $\mathrm{Res}(\varepsilon\psi) = 0$ if $\varepsilon\psi$ is an exact solution of (2.1). Inserting the ansatz (2.2) into equation (2.1) yields

$$\begin{aligned}\mathrm{Res}(\varepsilon\psi_{\mathrm{NLS}}) =& \varepsilon \mathbf{E}(\omega_0^2 - k_{01}^2 - 1)A + \mathrm{c.c.} \\ &+ \varepsilon^2 \mathbf{E}(2\mathrm{i}\omega_0(-c_g)\partial_X A + 2\mathrm{i}k_{01}\partial_X A) + \mathrm{c.c.} \\ &+ \varepsilon^3 \mathbf{E}(-2\mathrm{i}\omega_0 \partial_T A - c_g^2 \partial_X^2 A + \partial_X^2 A + \partial_Y^2 A + 3A|A|^2) + \mathrm{c.c.} \\ &+ \varepsilon^3 \mathbf{E}^3 A^3 + \mathrm{c.c.} \\ &+ \mathcal{O}(\varepsilon^4) + \mathrm{c.c.},\end{aligned}$$

where $\mathbf{E} = e^{\mathrm{i}(k_{01}x - \omega_0 t)}$. We choose $\omega = \omega_0$ and $\mathbf{k} = (k_{01}, 0) \in \mathbb{R}^2$ to satisfy the linear dispersion relation

$$\omega^2 = 1 + |\mathbf{k}|^2, \tag{2.3}$$

c_g to be the linear group velocity

$$c_g = \frac{k_{01}}{\omega_0}, \tag{2.4}$$

and A to satisfy the NLS equation

$$2\mathrm{i}\omega\partial_T A = (1 - c_g^2)\partial_X^2 A + \partial_Y^2 A + 3A|A|^2. \tag{2.5}$$

Then we obtain the residual

$$\mathrm{Res}(\varepsilon\psi) = \varepsilon^3 \mathbf{E}^3 A^3 + \mathcal{O}(\varepsilon^4) + \mathrm{c.c.}.$$

2.1.2 Making the residual small

In order to prove that the NLS equation (2.5) provides a good approximation for solutions of the original system (2.1) the residual should be made smaller by adding higher order terms to the ansatz (2.2).

Therefore, we define

$$\varepsilon\Psi(x, y, t) = \varepsilon\psi_{\mathrm{NLS}}(x, y, t) + \varepsilon^3 A_{31}(x, y, t)\mathbf{E}^3,$$

and find

$$\operatorname{Res}(\varepsilon\Psi) = \varepsilon^3 \mathbf{E}^3 (A^3 + (9\omega^2 - 9k_{01}^2 - 1)A_{31}) + \mathcal{O}(\varepsilon^4) + \text{c.c.}.$$

Choosing $A_{31} = -(9\omega^2 - 9k_{01}^2 - 1)^{-1} A^3$ formally yields

$$\operatorname{Res}(\varepsilon\Psi) = \mathcal{O}(\varepsilon^4).$$

Since $\varepsilon^3 A_{31}(X,Y,T) e^{3\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \ll \mathcal{O}(\varepsilon)$ the approximation $\varepsilon\Psi$ makes the same predictions as $\varepsilon\psi_{\text{NLS}}$ about the behavior of the solutions of the original system.

Remark 2.1. In order to make the residual even smaller we can make the general ansatz

$$\varepsilon\Psi(x,y,t) = \sum_{|m|=1,3,\ldots,2N+1} \sum_{n=1}^{\beta(m)} \varepsilon^{\alpha(m)+n} A_{mn}(X,T)\mathbf{E}^m, \tag{2.6}$$

where $\alpha(m) = ||m| - 1|$ with N and $\beta(m)$ sufficiently large.

Although the residual is small the NLS equation can make wrong predictions about the behavior of the original system. There are counterexamples, cf. [11], showing that the estimates for the residual are a necessary step but not at all sufficient. Therefore, we consider the difference, which is called the error, between the correct solution and the approximation.

2.2 The error estimates

In order to obtain the estimates for the difference between the approximation $\varepsilon\psi$ and true solutions u of the original system we use the variation of constants formula.

We write

$$\partial_t^2 \widehat{u} = -(|\mathbf{k}|^2 + 1)\widehat{u} + \widehat{u} * \widehat{u} * \widehat{u},$$

as a first-order system

$$\begin{aligned} \partial_t \widehat{u} &= -\sqrt{|\mathbf{k}|^2 + 1}\,\widehat{v}, \\ \partial_t \widehat{v} &= \sqrt{|\mathbf{k}|^2 + 1}\,\widehat{u} - \frac{1}{\sqrt{|\mathbf{k}|^2 + 1}}(\widehat{u} * \widehat{u} * \widehat{u}), \end{aligned} \tag{2.7}$$

and abbreviate it as

$$\partial_t \widehat{U} = \mathcal{M}(\mathbf{k})\widehat{U} + \mathcal{N}(\widehat{U}), \tag{2.8}$$

where

$$\mathcal{M}(\mathbf{k}) = \begin{pmatrix} 0 & -\sqrt{|\mathbf{k}|^2+1} \\ \sqrt{|\mathbf{k}|^2+1} & 0 \end{pmatrix},$$

$$\widehat{U} = \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = \begin{pmatrix} \widehat{u} \\ -\frac{1}{\sqrt{|\mathbf{k}|^2+1}}\partial_t \widehat{u} \end{pmatrix},$$

and $\mathbf{k} = (k_1, k_2)$. By the transformation $\widehat{V}(\mathbf{k}) = S(\mathbf{k})\widehat{U}(\mathbf{k})$ with

$$S = \frac{1}{2\mathrm{i}} \begin{pmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{pmatrix},$$

we obtain the diagonalized system

$$\partial_t V = \Lambda V + N(V, V, V), \tag{2.9}$$

where

$$\widehat{\Lambda}(\mathbf{k}) = \begin{pmatrix} \mathrm{i}\sqrt{|\mathbf{k}|^2+1} & 0 \\ 0 & -\mathrm{i}\sqrt{|\mathbf{k}|^2+1} \end{pmatrix},$$

$$\widehat{N}(\widehat{V}_1, \widehat{V}_2, \widehat{V}_3) = -\frac{1}{\sqrt{|\mathbf{k}|^2+1}} S\widetilde{N}(S^{-1}\widehat{V}_1, S^{-1}\widehat{V}_2, S^{-1}\widehat{V}_3),$$

$$\widetilde{N}(\widehat{V}_1, \widehat{V}_2, \widehat{V}_3)(\mathbf{k}) = \begin{pmatrix} 0 \\ ((\widehat{V}_1)_1 * (\widehat{V}_2)_1 * (\widehat{V}_3)_1)(\mathbf{k}) \end{pmatrix},$$

with $\widehat{V}_i = ((\widehat{V}_i)_1, (\widehat{V}_i)_2)$ for $i \in \{1, 2, 3\}$.

We introduce the coefficients $\widehat{\alpha}_{j_1,j_2,j_3,j_4}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l}-\mathbf{m}, \mathbf{m})$ for $j_1, j_2, j_3, j_4 \in \{1, 2\}$ of the trilinear mapping N

$$\begin{aligned}(\widehat{N}(\widehat{V}, \widehat{V}, \widehat{V}))_{j_1} = \int_{\mathbb{R}^4} \sum_{j_2,j_3,j_4 \in \{1,2\}} & \widehat{\alpha}_{j_1,j_2,j_3,j_4}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l}-\mathbf{m}, \mathbf{m}) \widehat{V}_{j_2}(\mathbf{k}-\mathbf{l}) \widehat{V}_{j_3}(\mathbf{l}-\mathbf{m}) \\ & \times \widehat{V}_{j_4}(\mathbf{m}) d\mathbf{l} d\mathbf{m},\end{aligned}$$

such that (2.9) can be written as

$$\partial_t \widehat{V}_1(\mathbf{k}, t) = \mathrm{i}\sqrt{|\mathbf{k}|^2+1}\, \widehat{V}_1(\mathbf{k}, t)$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{1111}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_1(\mathbf{k}-\mathbf{l})\widehat{V}_1(\mathbf{l}-\mathbf{m})\widehat{V}_1(\mathbf{m})d\mathbf{l}d\mathbf{m}\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{1112}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_1(\mathbf{k}-\mathbf{l})\widehat{V}_1(\mathbf{l}-\mathbf{m})\widehat{V}_2(\mathbf{m})d\mathbf{l}d\mathbf{m}\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{1122}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_1(\mathbf{k}-\mathbf{l})\widehat{V}_2(\mathbf{l}-\mathbf{m})\widehat{V}_2(\mathbf{m})d\mathbf{l}d\mathbf{m}\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{1222}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_2(\mathbf{k}-\mathbf{l})\widehat{V}_2(\mathbf{l}-\mathbf{m})\widehat{V}_2(\mathbf{m})d\mathbf{l}d\mathbf{m},\\
\partial_t \widehat{V}_2(\mathbf{k},t) =& -\mathrm{i}\sqrt{|\mathbf{k}|^2+1}\,\widehat{V}_2(\mathbf{k},t)\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{2111}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_1(\mathbf{k}-\mathbf{l})\widehat{V}_1(\mathbf{l}-\mathbf{m})\widehat{V}_1(\mathbf{m})d\mathbf{l}d\mathbf{m}\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{2112}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_1(\mathbf{k}-\mathbf{l})\widehat{V}_1(\mathbf{l}-\mathbf{m})\widehat{V}_2(\mathbf{m})d\mathbf{l}d\mathbf{m}\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{2122}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_1(\mathbf{k}-\mathbf{l})\widehat{V}_2(\mathbf{l}-\mathbf{m})\widehat{V}_2(\mathbf{m})d\mathbf{l}d\mathbf{m}\\
&+\int_{\mathbb{R}^4} \widehat{\alpha}_{2222}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l}-\mathbf{m},\mathbf{m})\widehat{V}_2(\mathbf{k}-\mathbf{l})\widehat{V}_2(\mathbf{l}-\mathbf{m})\widehat{V}_2(\mathbf{m})d\mathbf{l}d\mathbf{m}.
\end{aligned}
$$

In order to prove that the NLS equation is a good approximation we estimate the difference between the correct solution V and its approximation $\varepsilon\Psi$. Therefore, we write a solution V as a sum of the approximation and an error

$$V = \varepsilon\Psi + \varepsilon^\beta R,$$

where $\beta = 2$. Inserting this ansatz into (2.9) we find R to satisfy

$$\partial_t R = \Lambda R + 3\varepsilon^2 N(\Psi,\Psi,R) + 3\varepsilon^{\beta+1}N(\Psi,R,R) + \varepsilon^{2\beta}N(R,R,R) + \varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi),$$

where

$$\mathrm{Res}(\varepsilon\Psi) = -\partial_t(\varepsilon\Psi) + \varepsilon\Lambda\Psi + N(\varepsilon\Psi,\varepsilon\Psi,\varepsilon\Psi).$$

With our previous estimates for the residual we have the following lemma.

Lemma 2.2.
For θ_A sufficiently large the following holds. Let $A \in C([0,T_0], H^{\theta_A}(\mathbb{R}^2,\mathbb{C}))$ be a solution of the NLS *equation* (2.5)*. Then there exist an approximation $\varepsilon\Psi$, constants C_Ψ, C_{Res}, $C_1 > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$
\begin{aligned}
\sup_{t\in[0,T_0/\varepsilon^2]} \|\Psi(t)\|_W \leq& C_\Psi,\\
\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\Psi(t))\|_W \leq& C_{\mathrm{Res}}\varepsilon^4,
\end{aligned}
$$

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\Psi(t) - \varepsilon\Psi_{\mathrm{NLS}}(t)\|_{C_b^0} \leq C_1\varepsilon^3.$$

Remark 2.3. The first estimate in Lemma 2.2 is used for instance for the estimate

$$\|N(\Psi, R)\|_W \leq C\|\Psi\|_W\|R\|_W.$$

Remark 2.4. Note that $\|\Psi\|_{H^\theta} \leq \mathcal{O}(\varepsilon^{-1})$ such that Ψ has to be estimated in the W-norm. Then we have

$$\begin{aligned}\|\Psi\|_W = \left\|\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) + \text{h.o.t.}\right\|_{L^1} &\leq C\left\|\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^1} + \text{h.o.t.}\\ &\leq C\|\widehat{A}\|_{L^2(2)} + \text{h.o.t.} \leq C\|\widehat{A}\|_{H^0(2)} + \text{h.o.t.}.\end{aligned}$$

With the help of the variation of constants formula we write the error equations as

$$\begin{aligned}R(t) = \int_0^t e^{\Lambda(t-\tau)}\big(&3\varepsilon^2 N(\Psi,\Psi,R) + 3\varepsilon^{\beta+1}N(\Psi,R,R)\\ &+ \varepsilon^{2\beta}N(R,R,R) + \varepsilon^{-\beta}\mathrm{Res}(\varepsilon\Psi)\big)(\tau)d\tau,\end{aligned}$$

where $e^{\Lambda(\mathbf{k})t}$ is the semigroup generated by $\Lambda(\mathbf{k})$ which is defined for fixed $\mathbf{k}$ by

$$e^{\Lambda(\mathbf{k})t} = \sum_{n=0}^{\infty}\frac{(\Lambda(\mathbf{k})t)^n}{n!}.$$

Lemma 2.5.
The semigroup is uniformly bounded in W, i.e., there exists a $C > 0$ such that we have $\|e^{\Lambda(\mathbf{k})t}\|_{W\to W} \leq C$.

We obtain the estimates

$$\begin{aligned}\|3\varepsilon^2\Psi^2 R\|_W \leq& 3C\varepsilon^2 C_\Psi^2\|R\|_W,\\ \|3C\varepsilon^{\beta+1}\Psi R^2\|_W \leq& 3C\varepsilon^{\beta+1}C_\Psi\|R\|_W^2,\\ \|3\varepsilon^{2\beta}R^3\|_W \leq& 3C\varepsilon^{2\beta}\|R\|_W^3,\end{aligned}$$

and so

$$\begin{aligned}\|R(t)\|_W \leq \int_0^t \Big(&3C\varepsilon^2 C_\Psi^2\|R(\tau)\|_W + 3C\varepsilon^{\beta+1}C_\Psi\|R(\tau)\|_W^2 + 3C\varepsilon^{2\beta}\|R(\tau)\|_W^3\\ &+ C_{\mathrm{Res}}\varepsilon^2\Big)d\tau.\end{aligned} \tag{2.10}$$

Choosing $3C\varepsilon^{\beta+1}C_\Psi\|R\|_W^2 + 3C\varepsilon^{2\beta}\|R\|_W^3 \leq \varepsilon^2$ we then have

$$\|R(t)\|_W \leq \int_0^t \Big(3C\varepsilon^2 C_\Psi^2 \|R(\tau)\|_W + \varepsilon^2 + C_{\mathrm{Res}}\varepsilon^2\Big) d\tau. \tag{2.11}$$

An application of Gronwall's inequality to (2.11) implies

$$\begin{aligned}\|R(t)\|_W &\leq (1 + C_{\mathrm{Res}})\varepsilon^2 t e^{3CC_\Psi^2\varepsilon^2 t} \\ &\leq (1 + C_{\mathrm{Res}})T_0 e^{3CC_\Psi^2 T_0} =: C_R,\end{aligned}$$

independent of $\varepsilon \in (0, \varepsilon_0)$ for all $t \in [0, T_0/\varepsilon^2]$. We choose $\varepsilon_0 > 0$ so small that

$$3C\varepsilon_0^{\beta-1}C_\Psi C_R^2 + 3C\varepsilon_0^{2\beta-2}C_R^3 \leq 1.$$

We are done and so the dynamics of the NLS equation on a time interval $[0, T_0/\varepsilon^2]$ can really be seen in the original system, too, since for $\varepsilon \to 0$ the error of $\mathcal{O}(\varepsilon^\beta)$ for $\beta = 2$ is much smaller than the solution and the approximation which are both of $\mathcal{O}(\varepsilon)$.

We obtain the following result.

Theorem 2.6.
For all θ_A sufficiently large the following holds. Let $A \in C([0, T_0], H^{\theta_A}(\mathbb{R}^2, \mathbb{C}))$ be a solution of the NLS *equation* (2.5)*. Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions u of the 2D nonlinear wave equation* (2.1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\,\cdot_x,\,\cdot_y, t) - \varepsilon\psi_{\mathrm{NLS}}(\varepsilon,\,\cdot_x,\,\cdot_y, t)\|_W \leq C\varepsilon^2.$$

Since $\|u\|_{C_b^0} \leq \|u\|_W$ we have

Corollary 2.7.
We have

$$\sup_{t\in[0,T_0/\varepsilon^2]}\ \sup_{(x,y)\in\mathbb{R}^2} |u(x, y, t) - \varepsilon\psi_{\mathrm{NLS}}(\varepsilon, x, x, t)| \leq C\varepsilon^2. \tag{2.12}$$

Remark 2.8. The use of the space H^θ instead of W would lead in Lemma 2.2 to $\|\mathrm{Res}(\varepsilon\Psi)\|_{H^\theta} = \mathcal{O}(\varepsilon^3)$ due to the scaling properties of the L^2-norm. Therefore, in order to obtain $\|\mathrm{Res}(\varepsilon\Psi)\|_{H^\theta} = \mathcal{O}(\varepsilon^4)$ additional terms have to be added to Ψ.

3. Quadratic nonlinearities without resonances

As in Chapter 2 we will show that the solutions of the NLS equation predict the behavior of the solutions of the original system correctly in case of quadratic nonlinearities without resonances in 2D. Such an approximation result for 1D can be found for example in [5]. Due to the quadratic nonlinearity there are now terms of $\mathcal{O}(\varepsilon)$ in the error equation. This means that a direct estimate with the help of Gronwall's inequality is not possible. The semigroup generated by all linear terms then increases as $e^{C\varepsilon t}$ which is of $\mathcal{O}(e^{1/\varepsilon})$ for $t = \mathcal{O}(1/\varepsilon^2)$. In order to obtain estimates on the long time interval $[0, T_0/\varepsilon^2]$ the influence of these terms has to be controlled. For many systems they turn out to be oscillatory and can be removed by a normal form transform. After the removal Gronwall's inequality can again be applied. The analysis is carried out for a 2D nonlinear wave equation.

3.1 Derivation of the NLS equation

We consider

$$\partial_t^2 u = \Delta u - u + u^2, \tag{3.1}$$

with $\Delta = \partial_x^2 + \partial_y^2$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $u(x, y, t) \in \mathbb{R}$.

The NLS equation describing slow modulations in time and space of a wave train $e^{\mathrm{i}(k_{01}x - \omega_0 t)}$ in (3.1) can be derived by the ansatz

$$\begin{aligned} \varepsilon\psi_{\mathrm{NLS}}(x, y, t) =& \varepsilon A_1(\varepsilon(x + c_g t), \varepsilon y, \varepsilon^2 t) e^{\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \\ &+ \varepsilon^2 A_2(\varepsilon(x + c_g t), \varepsilon y, \varepsilon^2 t) e^{2\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \\ &+ \varepsilon^2 A_0(\varepsilon(x + c_g t), \varepsilon y, \varepsilon^2 t), \end{aligned} \tag{3.2}$$

with $c_g = \partial_{k_1}\omega|_{\mathbf{k}=\mathbf{k}_0, \omega=\omega_0}$ the negative group velocity, A an amplitude describing the envelope of the wave packet, and c.c. the complex conjugate. Inserting this ansatz into (3.1) and equating the coefficients in front of $\varepsilon^{j_1} e^{j_2 \mathrm{i}(k_{01}x - \omega_0 t)}$ with $j_1 \geq 0$, $j_2 \in \mathbb{Z}$, to zero we obtain with $e^{\mathrm{i}(k_{01}x - \omega_0 t)} =: \mathbf{E}$ for the coefficients at $\varepsilon\mathbf{E}$ the linear dispersion relation

$$\omega^2 = |\mathbf{k}|^2 + 1,$$

with $\omega = \omega_0$ and $\mathbf{k} = (k_{01}, 0) \in \mathbb{R}^2$, at $\varepsilon^2 \mathbf{E}$ the negative group velocity and

$$\begin{aligned}
\varepsilon^3 \mathbf{E} : 0 =& -2\mathrm{i}\omega \partial_T A_1 + (1 - c_g^2)\partial_X^2 A_1 + \partial_Y^2 A_1 + 2A_1 A_0 + 2A_2 A_{-1}, \\
\varepsilon^2 \mathbf{E}^0 : 0 =& - A_0 + 2A_1 A_{-1}, \\
\varepsilon^2 \mathbf{E}^2 : 0 =& (-4\omega_0^2 + 4k_{01}^2 + 1)A_2 + A_1^2,
\end{aligned}$$

where $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$, $Y = \varepsilon y$, $A_0(X, Y, T) \in \mathbb{R}$, $A_i(X, Y, T) \in \mathbb{C}$ for $i \in \{-2, -1, 1, 2\}$.

Hence, we find with $A_0 = 2A_1 A_{-1}$ and $A_2 = A_1^2/(-4\omega_0^2 + 4k_{01}^2 + 1)$ the NLS equation

$$2\mathrm{i}\omega_0 \partial_T A_1 = (1 - c_g^2)\partial_X^2 A_1 + \partial_Y^2 A_1 + \gamma A_1 |A_1|^2, \tag{3.3}$$

with

$$\gamma = 4 + \frac{2}{-4\omega_0^2 + 4k_{01}^2 + 1}.$$

We prove the following approximation result.

Theorem 3.1.
For all θ_A sufficiently large the following holds. Let $A_1 \in C([0, T_0], H^{\theta_A}(\mathbb{R}^2, \mathbb{C}))$ be a solution of the NLS *equation* (3.3). *Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions u of the 2D nonlinear wave equation* (3.1) *with*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \| u(\,\cdot_x, \,\cdot_y, t) - \varepsilon \psi_{\mathrm{NLS}}(\varepsilon, \,\cdot_x, \,\cdot_y, t) \|_W \le C\varepsilon^2.$$

Remark 3.2. Such an approximation result should not be taken for granted. There are counterexamples, cf. [11], where formally correctly derived modulation equations similar to (3.3) make wrong predictions about the dynamics of the original system.

As before the residual

$$\mathrm{Res}(u) = -\partial_t^2 u + \Delta u - u + u^2,$$

can be made smaller by adding higher order terms to the approximation (3.2). Since this goes very similar to the procedure in Chapter 2 we skip the details of this step. We have again

Lemma 3.3.
For θ_A sufficiently large the following holds. Let $A_1 \in C([0, T_0], H^{\theta_A}(\mathbb{R}^2, \mathbb{C}))$ be a solution of the NLS *equation* (3.3). *Then there exist an approximation $\varepsilon\psi$ and*

constants C_{Res}, $C_1 > 0$ *such that for all* $\varepsilon \in (0,1)$ *we have*

$$\sup_{t\in[0,T/\varepsilon^2]} \|\text{Res}(\varepsilon\psi(t))\|_W \leq C_{\text{Res}}\varepsilon^4,$$
$$\sup_{t\in[0,T/\varepsilon^2]} \|\varepsilon\psi(t) - \varepsilon\psi_{\text{NLS}}(t)\|_{C_b^0} \leq C_1\varepsilon^3.$$

As in Chapter 2 we write equation (3.1) as a first-order system

$$\partial_t V = \Lambda V + N(V,V), \tag{3.4}$$

where Λ is a linear mapping and N a bilinear mapping. In detail, in Fourier space we have

$$S = \frac{1}{2\mathrm{i}}\begin{pmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{pmatrix},$$

$$\widehat{\Lambda}(\mathbf{k}) = \begin{pmatrix} \mathrm{i}\sqrt{\mathbf{k}^2+1} & 0 \\ 0 & -\mathrm{i}\sqrt{|\mathbf{k}|^2+1} \end{pmatrix},$$

$$\widehat{N}(\widehat{V}_1,\widehat{V}_2) = -\frac{1}{\sqrt{|\mathbf{k}|^2+1}} S\widetilde{N}(S^{-1}\widehat{V}_1, S^{-1}\widehat{V}_2),$$

$$\widetilde{N}(\widehat{V}_1,\widehat{V}_2)(\mathbf{k}) = \begin{pmatrix} 0 \\ ((\widehat{V}_1)_1 * (\widehat{V}_2)_1)(\mathbf{k}) \end{pmatrix},$$

with $\widehat{V}_i = ((\widehat{V}_i)_1, (\widehat{V}_i)_2)$ for $i \in \{1,2\}$.

We introduce the coefficients $\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})$ of the bilinear mapping N by

$$(\widehat{N}(\widehat{V},\widehat{V})(\mathbf{k}))_{j_1} = \int_{\mathbb{R}^2} \sum_{j_2,j_3\in\{1,2\}} \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\widehat{V}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{V}_{j_3}(\mathbf{l})d\mathbf{l}.$$

We define the new residual as

$$\text{RES}(V) = -\partial_t V + \Lambda V + N(V,V).$$

As a direct consequence of Lemma 3.3 we have

Lemma 3.4.
There exist an approximation $\varepsilon\Psi$ *and a* θ_A *sufficiently large such that the following holds. Let* $A_1 \in C([0,T_0], H^{\theta_A}(\mathbb{R}^2,\mathbb{C}))$ *be a solution of the* NLS *equation* (3.3)*. Then*

there exists a $C_{\mathrm{RES}} > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$\sup_{t\in[0,T/\varepsilon^2]} \|\mathrm{RES}(\varepsilon\Psi(t))\|_W \leq C_{\mathrm{RES}}\varepsilon^3.$$

Again the solution V is split into an approximation $\varepsilon\Psi$ and into an error $\varepsilon^\beta R$ with $\beta > 2$, i.e., $V = \varepsilon\Psi + \varepsilon^\beta R$. Inserting this into (3.4) gives a system for the error

$$\partial_t R = \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon^\beta N(R,R) + \varepsilon^{-\beta}\mathrm{RES}(\varepsilon\Psi). \tag{3.5}$$

The last two terms on the right hand side are of order $\mathcal{O}(\varepsilon^2)$ which can be handled with the help of Gronwall's inequality. But, the term $2\varepsilon N(\Psi, R)$ causes difficulties in estimating the error on the long time interval. Since this term can lead to a growth of the solutions of order $e^{1/\varepsilon}$ on time scales of $\mathcal{O}(1/\varepsilon^2)$ the control over the size of R would be lost on the time scale $[0, T_0/\varepsilon^2]$. However, this term turns out to be oscillatory in time and can be eliminated by averaging or a normal form transform. After eliminating this term the proof for the cubic nonlinearities can be applied again line by line.

3.2 The normal form transform

In order to show that the solutions of the error equation (3.5) remain small over the natural time scale we eliminate the term $2\varepsilon N(\Psi, R)$ via a normal form transform

$$\widetilde{R} = R + \varepsilon B(\Psi, R), \tag{3.6}$$

where

$$(\widehat{B(\Psi, R)})_{j_1}(\mathbf{k}) = \int_{\mathbb{R}^2} \sum_{j_2,j_3\in\{1,2\}} \widehat{b}^{j_1}_{j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})\widehat{\Psi}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l}, \tag{3.7}$$

with $\mathbf{k}$, $\mathbf{l} \in \mathbb{R}^2$ similar to

$$(\widehat{N(\Psi, R)})_{j_1}(\mathbf{k}) = \int_{\mathbb{R}^2} \sum_{j_2,j_3\in\{1,2\}} \widehat{\alpha}^{j_1}_{j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})\widehat{\Psi}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l}, \tag{3.8}$$

$\mathbf{k} = (k_1, k_2)$, $\mathbf{l} = (l_1, l_2)$ and $\widehat{\alpha}^{j_1}_{j_2,j_3}$, $\widehat{b}^{j_1}_{j_2,j_3}$ some coefficients.

We use $\partial_t\psi = \Lambda\Psi + \mathcal{O}(\varepsilon^2)$ to obtain

$$\begin{aligned}\partial_t\widetilde{R} &= \partial_t R + \varepsilon B(\partial_t\Psi, R) + \varepsilon B(\Psi, \partial_t R)\\ &= \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon B(\partial_t\Psi, R) + \varepsilon B(\Psi, \partial_t R) + \varepsilon^2 M\end{aligned}$$

$$
\begin{aligned}
&= \Lambda\widetilde{R} - \varepsilon\Lambda B(\Psi,R) + 2\varepsilon N(\Psi,R) + \varepsilon B(\partial_t\Psi,R) + \varepsilon B(\Psi,\partial_t R) + \varepsilon^2 M \\
&= \Lambda\widetilde{R} - \varepsilon\Lambda B(\Psi,R) + 2\varepsilon N(\Psi,R) + \varepsilon B(\Lambda\Psi,R) + \varepsilon B(\Psi,\Lambda R) + \varepsilon^2 M,
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon^2 M =&\varepsilon B(\Psi, 2\varepsilon N(\Psi,R)) + \mathcal{O}(\varepsilon^3) + \varepsilon^{\beta} N(R,R) + \varepsilon^{-\beta}\mathrm{RES}(\varepsilon\Psi) \\
&+ \varepsilon B(\Psi, \varepsilon^{\beta} N(R,R)) + \varepsilon B(\Psi, \varepsilon^{-\beta}\mathrm{RES}(\varepsilon\Psi)).
\end{aligned}
$$

In order to eliminate the term $2\varepsilon N(\Psi,R)$ we have to find a B such that

$$
-\Lambda B(\Psi,R) + B(\Lambda\Psi,R) + B(\Psi,\Lambda R) + 2N(\Psi,R) = 0.
$$

Since the approximation $\varepsilon\Psi$ is of $\mathcal{O}(\varepsilon)$ only close to the wave vectors $\pm\mathbf{k}_0$. Since all kernels are globally Lipschitz, and also because we have a real-valued problem we write

$$
\widehat{\Psi}(\mathbf{k}) = \frac{1}{\varepsilon^2} a_+\left(\frac{\mathbf{k}-\mathbf{k}_0}{\varepsilon}\right) + \frac{1}{\varepsilon^2} a_-\left(\frac{\mathbf{k}+\mathbf{k}_0}{\varepsilon}\right) + \mathcal{O}(\varepsilon),
$$

where a_d stands for the terms concentrated at $d\mathbf{k}_0$ for $d \in \{+,-\}$.

Using now Lemma 4.8 from Chapter 4 terms (3.7) and (3.8) can be simplified to

$$
\begin{aligned}
(\widehat{N(\Psi,R)})_j(\mathbf{k}) = &\int_{\mathbb{R}^2} \sum_{j_3\in\{1,2\}} \widehat{\alpha}^{j}_{+,j_3}(\mathbf{k},\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0)\frac{1}{\varepsilon^2} a_+\left(\frac{\mathbf{k}-\mathbf{l}-\mathbf{k}_0}{\varepsilon}\right)\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l} \\
&+ \int_{\mathbb{R}^2} \sum_{j_3\in\{1,2\}} \widehat{\alpha}^{j}_{-,j_3}(\mathbf{k},-\mathbf{k}_0,\mathbf{k}+\mathbf{k}_0)\frac{1}{\varepsilon^2} a_-\left(\frac{\mathbf{k}-\mathbf{l}+\mathbf{k}_0}{\varepsilon}\right)\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l} \\
&+ \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

and

$$
\begin{aligned}
(\widehat{B(\Psi,R)})_j(\mathbf{k}) = &\int_{\mathbb{R}^2} \sum_{j_3\in\{1,2\}} \widehat{b}^{j}_{+,j_3}(\mathbf{k},\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0)\frac{1}{\varepsilon^2} a_+\left(\frac{\mathbf{k}-\mathbf{l}-\mathbf{k}_0}{\varepsilon}\right)\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l} \\
&+ \int_{\mathbb{R}^2} \sum_{j_3\in\{1,2\}} \widehat{b}^{j}_{-,j_3}(\mathbf{k},-\mathbf{k}_0,\mathbf{k}+\mathbf{k}_0)\frac{1}{\varepsilon^2} a_-\left(\frac{\mathbf{k}-\mathbf{l}+\mathbf{k}_0}{\varepsilon}\right)\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l} \\
&+ \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

such that we get the relation

$$
\mathrm{i}(\omega_{j_1}(\mathbf{k}) - \omega_+(\mathbf{k}_0) - \omega_{j_3}(\mathbf{k}-\mathbf{k}_0))\widehat{b}^{j_1}_{+,j_3}(\mathbf{k},\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0) = 2\widehat{\alpha}^{j_1}_{+,j_3}(\mathbf{k},\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0), \quad (3.9)
$$

cf. [12]. For the notational simplicity we set $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$.

The expression (3.9) can be solved with respect to $\widehat{b}^{j_1}_{+,j_3}$ if the non-resonance condition

$$\inf_{\mathbf{k}\in\mathbb{R}^2} |\omega(\mathbf{k}) - \omega(\mathbf{k}_0) - \omega(\mathbf{k}-\mathbf{k}_0)| > 0,$$

is satisfied. For finite $\mathbf{k}$ Figure 3.1 shows that for $\omega_\pm(\mathbf{k}) = \pm\sqrt{1+|\mathbf{k}|^2}$ there are no resonances and hence the non-resonance condition is valid. In detail, for $|\mathbf{k}| \to \infty$ the term

$$\omega(\mathbf{k}) - \omega(\mathbf{k}_0) - \omega(\mathbf{k}-\mathbf{k}_0),$$

is bounded away from 0 since

$$\begin{aligned}\omega(\mathbf{k}) - \omega(\mathbf{k}_0) - \omega(\mathbf{k}-\mathbf{k}_0) =& \sqrt{1+|\mathbf{k}|^2} - \sqrt{1+(\mathbf{k}-\mathbf{k}_0)^2} - \sqrt{1+|\mathbf{k}_0|^2} \\ =& \frac{1+|\mathbf{k}|^2 - (1+|\mathbf{k}|^2 - 2\mathbf{k}\mathbf{k}_0 + |\mathbf{k}_0|^2)}{\sqrt{1+|\mathbf{k}|^2} + \sqrt{1+(\mathbf{k}-\mathbf{k}_0)^2}} - \sqrt{1+|\mathbf{k}_0|^2} \\ =& \frac{2\mathbf{k}_0 - \frac{|\mathbf{k}_0|^2}{\mathbf{k}}}{\sqrt{\frac{1}{|\mathbf{k}|^2}+1} + \sqrt{\frac{1}{|\mathbf{k}|^2} + \left(1-\frac{\mathbf{k}_0}{\mathbf{k}}\right)^2}} - \sqrt{1+|\mathbf{k}_0|^2},\end{aligned}$$

and as $|\mathbf{k}| \to \infty$ we get

$$\mathbf{k}_0 - \sqrt{1+|\mathbf{k}_0|^2} = \frac{|\mathbf{k}_0|^2 - 1 - |\mathbf{k}_0|^2}{\mathbf{k}_0 + \sqrt{1+|\mathbf{k}_0|^2}} < 0.$$

Therefore, the non-resonant condition

$$\inf_{\mathbf{k}\in\mathbb{R}^2} |\omega(\mathbf{k}) - \omega(\mathbf{k}_0) - \omega(\mathbf{k}-\mathbf{k}_0)| > 0,$$

is satisfied.

With

$$\begin{aligned}(\widehat{B(\Psi,R)})_j(\mathbf{k}) =& \int_{\mathbb{R}^2} \sum_{j_3\in\{1,2\}} \frac{2\widehat{\alpha}^j_{+,j_3}(\mathbf{k},\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0)}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_1(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0)} a_+\left(\frac{\mathbf{k}-\mathbf{l}-\mathbf{k}_0}{\varepsilon}\right) \\ &\times \widehat{R}_{j_3}(\mathbf{l})d\mathbf{l} + \int_{\mathbb{R}^2} \sum_{j_3\in\{1,2\}} \frac{2\widehat{\alpha}^j_{-,j_3}(\mathbf{k},-\mathbf{k}_0,\mathbf{k}+\mathbf{k}_0)}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_1(-\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}+\mathbf{k}_0)} \\ &\times a_-\left(\frac{\mathbf{k}-\mathbf{l}+\mathbf{k}_0}{\varepsilon}\right)\widehat{R}_{j_3}(\mathbf{l})d\mathbf{l},\end{aligned}$$

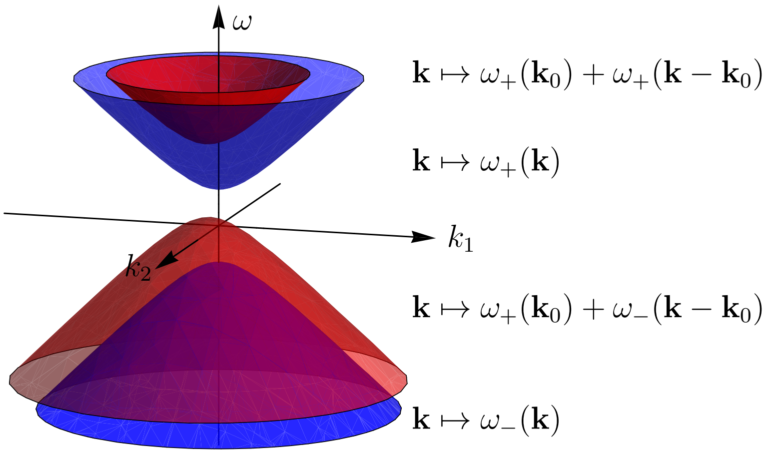


Figure 3.1: Validity of the non-resonance condition. The surfaces of eigenvalues $\omega_\pm(\mathbf{k}) = \pm\sqrt{1 + |\mathbf{k}|^2}$ with $k_{01} = 1$ and $k_{02} = 0$ have no intersections.

all dangerous terms can be eliminated and since

$$\sup_{j_1, j_3 \in \{1,2\}, \mathbf{k}, \mathbf{k}_0 \in \mathbb{R}^2} \left| \frac{\widehat{\alpha}^{j_1}_{+,j_3}(\mathbf{k}, \mathbf{k}_0, \mathbf{k} - \mathbf{k}_0)}{\omega_{j_1}(\mathbf{k}) - \omega_1(\mathbf{k}_0) - \omega_{j_3}(\mathbf{k} - \mathbf{k}_0)} \right| \leq C < \infty,$$

we get

$$\|B(\Psi, R)\|_W \leq C \|\Psi\|_W \|R\|_W. \tag{3.10}$$

Therefore, (3.5) transforms into

$$\partial_t \widetilde{R} = \Lambda \widetilde{R} + \mathcal{O}(\varepsilon^2). \tag{3.11}$$

All terms of $\mathcal{O}(\varepsilon)$ are eliminated and so the rest of the proof for the cubic case from Chapter 2 can be applied.

4. Nonlinearities with a single resonance at k=0

In this chapter we justify the Davey Stewartson (DS) system [8] for a 2D Boussinesq equation

$$\partial_t^2 u = \Delta u + \partial_t^2 \Delta u + \Delta(u^2), \tag{4.1}$$

where $\Delta = \partial_x^2 + \partial_y^2$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ for $u(x, y, t) \in \mathbb{R}$. The 1D version can formally be derived from the 2D water wave problem [13, 14], where the solution u of this equation can be interpreted as the vertical velocity of the fluid at the free surface. This equation possesses a resonance at the wave vector $\mathbf{k} = 0$ which turns out to be trivial. A resonance is called trivial if the nonlinear terms vanish for the resonant wave vector, too. Otherwise it is called non-trivial. This trivial resonance at the wave vector $\mathbf{k} = 0$ always implies another resonance for the wave vector $\mathbf{k} = \mathbf{k}_0$, which is non-trivial for our Boussinesq equation. This situation is shown in Figure 4.2. Hence, the ideas of Chapter 3 for the justification of the NLS equation do not apply. Improving the method by an appropriate wave vector dependent rescaling of the error function R used in [15, 10] the problem can be handled. Additionally a second normal form transform has to be made.

4.1 Derivation of the NLS equation

In order to derive the DS system we make the ansatz

$$\begin{aligned} \varepsilon\psi_{\mathrm{DS}}(x, y, t) =& \varepsilon A_1(\varepsilon(x + c_g t), \varepsilon y, \varepsilon^2 t) e^{\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \\ &+ \varepsilon^2 A_2(\varepsilon(x + c_g t), \varepsilon y, \varepsilon^2 t) e^{2\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \\ &+ \varepsilon^2 A_0(\varepsilon(x + c_g t), \varepsilon y, \varepsilon^2 t). \end{aligned} \tag{4.2}$$

Inserting the ansatz into the residual

$$\mathrm{Res}(u) = -\partial_t^2 u + \partial_x^2 u + \partial_y^2 u + \partial_t^2 \partial_x^2 u + \partial_t^2 \partial_y^2 u + \partial_x^2(u^2) + \partial_y^2(u^2),$$

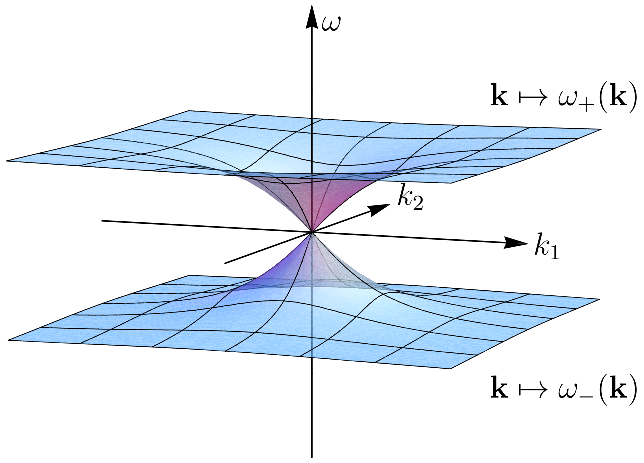


Figure 4.1: The surfaces of eigenvalues $\omega_\pm(\mathbf{k}) = \pm\sqrt{\frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}}$ for the 2D Boussinesq equation.

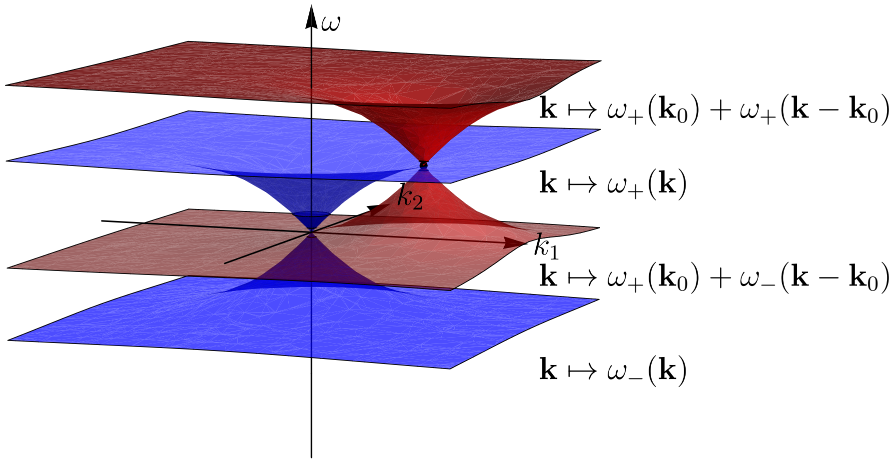


Figure 4.2: Beside the trivial resonance at the wave vector $\mathbf{k} = 0$ there is one more resonance at the wave vector $\mathbf{k} = \mathbf{k}_0$ where here $\mathbf{k}_0 = (1, 0)$ has been chosen.

with $\mathbf{E} = e^{\mathrm{i}(k_{01}x - \omega_0 t)}$ we find for the coefficients

$$
\begin{aligned}
\varepsilon\mathbf{E} : 0 =&(\omega_0^2 - k_{01}^2 + k_{01}^2\omega_0^2)A_1, \\
\varepsilon^2\mathbf{E} : 0 =&(2\mathrm{i}\omega_0 c_g + 2\mathrm{i}k_{01} - 2\mathrm{i}k_{01}\omega_0^2 + 2\mathrm{i}\omega_0 c_g k_{01}^2)\partial_X A_1, \\
\varepsilon^2\mathbf{E}^2 : 0 =&4\omega_0^2 A_2 - 4k_{01}^2 A_2 + 16k_{01}^2\omega_0^2 A_2 - 4k_{01}^2 A_1^2, && (4.3)\\
\varepsilon^3\mathbf{E} : 0 =&2\mathrm{i}\omega_0\partial_T A_1 + (1 - c_g^2)\partial_X^2 A_1 + \partial_Y^2 A_1 - 2k_{01}^2 A_1 A_0 - 2k_{01}^2 A_2 \bar{A}_1, && (4.4)\\
\varepsilon^4\mathbf{E}^0 : 0 =&(\partial_X^2 + \partial_Y^2 - c_g^2\partial_X^2)A_0 + 2(\partial_X^2 + \partial_Y^2)(A_1\bar{A}_1), && (4.5)
\end{aligned}
$$

where $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$, $Y = \varepsilon y$, $A_0(X, Y, T) \in \mathbb{R}$, $A_i(X, Y, T) \in \mathbb{C}$ for $i \in \{-2, -1, 1, 2\}$. From (4.3) we get

$$A_2 = \frac{k_{01}^2}{\omega_0^2 - k_{01}^2 + 4k_{01}^2\omega_0^2} A_1^2.$$

With (4.5) we obtain

$$A_0 = -\frac{2(\partial_X^2 + \partial_Y^2)}{\partial_X^2 + \partial_Y^2 - c_g^2\partial_X^2}|A_1|^2.$$

Inserting this into expression (4.4) we find the DS system

$$
\begin{aligned}
2\mathrm{i}\omega\partial_T A_1 =&(-1 + c_g^2)\partial_X^2 A_1 - \partial_Y^2 A_1 + \gamma_1 A_1|A_1|^2 + \gamma_2 A_1 A_0, \\
A_0 =& -\frac{(2\partial_X^2 + \partial_Y^2)}{\partial_X^2 + \partial_Y^2 - c_g^2\partial_X^2}|A_1|^2,
\end{aligned}
\qquad (4.6)
$$

where

$$\gamma_1 = \frac{2k_{01}^4}{\omega_0^2 - k_{01}^2 + 4k_{01}^2\omega_0^2} \quad \text{and} \quad \gamma_2 = -2k_{01}^2.$$

Herein, $\omega = \omega_0$ and $\mathbf{k} = (k_{01}, 0)$ satisfy the linear dispersion relation

$$\omega^2 = |\mathbf{k}|^2 - \omega^2|\mathbf{k}|^2,$$

where the solutions ω are given by

$$\omega_\pm(\mathbf{k}) = \pm\sqrt{\frac{|\mathbf{k}|^2}{1 + |\mathbf{k}|^2}},$$

and $c_g = \frac{-k_{01} + k_{01}\omega^2}{\omega_0 + \omega_0 k_{01}^2}$ is the linear group velocity. For the derivation, a finite number of discrete non-resonance conditions have to be satisfied. These are

$$\omega_j(m\mathbf{k}_0) \neq m\omega_+(\mathbf{k}_0) \quad \text{and} \quad \omega_j'(0) \neq \omega_+'(\mathbf{k}_0), \qquad \text{(NR)}$$

for all integers m with $|m| \leq M$ defined in Remark 2.1 and $j \in \{+,-\}$. For notational simplicity we set $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$.

We prove the following result.

Theorem 4.1.
For all θ_A sufficiently large the following holds. Let $A_0 \in C([0,T_0], H^{\theta_A}(\mathbb{R}^2,\mathbb{R}))$, $A_1 \in C([0,T_0], H^{\theta_A}(\mathbb{R}^2,\mathbb{C}))$ be solutions of the DS *system* (4.6)*. Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions u of* (4.1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\cdot_x,\cdot_y,t) - \varepsilon\psi_{\mathrm{DS}}(\cdot_x,\cdot_y,t)\|_W \leq C\varepsilon^2.$$

4.2 Proof of Theorem 4.1

The residual can be made arbitrarily small by adding higher order terms to the ansatz, i.e., we have

Lemma 4.2.
There exists an approximation $\varepsilon\psi$ and a θ_A sufficiently large such that the following holds. Let $A_1 \in C([0,T_0], H^{\theta_A}(\mathbb{R}^2,\mathbb{C}))$ be a solution of the DS *system* (4.6)*. Then there exist constants C_ψ, C, $C_{\mathrm{Res}} > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\begin{aligned}
\sup_{t\in[0,T_0/\varepsilon^2]} \|\psi(t)\|_W &\leq C_\psi,\\
\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi(t) - \varepsilon\psi_{\mathrm{DS}}(t)\|_{C_b^0} &\leq C\varepsilon^2,\\
\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\psi(t))\|_W &\leq C_{\mathrm{Res}}\varepsilon^7.
\end{aligned}$$

Remark 4.3. The first estimate in Lemma 4.2 is used for instance for the estimate

$$\|N(\psi,R)\|_W \leq C\|\psi\|_W\|R\|_W.$$

In the proof we used

Lemma 4.4.
For all $\theta \geq 0$, $m \in \mathbb{N}$, $C_1 > 0$ there exist $C > 0$, $C_2 > 0$ such that for all $\varepsilon \in (0,1)$ the following holds. Let $\nu : \mathbb{R} \to \mathbb{C}$ with $\nu(\mathbf{k}) \leq C_1|\mathbf{k}|^m$ and $A \in C([0,T_0], H^{\theta_A}(\mathbb{R}^2,\mathbb{C}))$. Then

$$\|(\nu A)(\varepsilon\,\cdot)\mathbf{E}\|_W \leq C_2\varepsilon^m\|\widehat{A}\|_{L^1(\theta+m)}.$$

Proof. We have

$$\|A(\varepsilon\,\cdot\,)\mathbf{E}\|_W = \left\|\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot\,-\mathbf{k}_0}{\varepsilon}\right)\right\|_{L^1(\theta)} \leq C\|\widehat{A}\|_{L^1(\theta)}.$$

Using $\nu(\mathbf{k}) \leq C|\mathbf{k}|^m$ for a $m \geq 0$ we obtain

$$\begin{aligned}\|(\nu A)(\varepsilon\,\cdot\,)\mathbf{E}\|_W &= \left\|\nu(\,\cdot\,-\mathbf{k}_0)\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot\,-\mathbf{k}_0}{\varepsilon}\right)\right\|_{L^1(\theta)} \\ &\leq \sup_{\mathbf{k}\in\mathbb{R}^2}\left|C_1|\mathbf{k}|^m\varepsilon^m\left(1+\left(\frac{\mathbf{k}-\mathbf{k}_0}{\varepsilon}\right)^2\right)^{-m/2}\right|\|\widehat{A}\|_{L^1(\theta+m)} \\ &\leq C_2\varepsilon^m\|\widehat{A}\|_{L^1(\theta+m)}.\end{aligned}$$

□

We now write the equation (4.1) as a first-order system

$$\partial_t U = \Lambda U + N(U,U), \tag{4.7}$$

where Λ is a skew-symmetric diagonal linear operator and $N(\,\cdot\,,\,\cdot\,) : W \times W \to W$ a symmetric bilinear mapping. In detail, in Fourier space we have

$$\widehat{\Lambda}(\mathbf{k}) = \begin{pmatrix} \mathrm{i}\sqrt{\frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}} & 0 \\ 0 & -\mathrm{i}\sqrt{\frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}} \end{pmatrix},$$

and

$$(\widehat{N}(\widehat{U},\widehat{U})(\mathbf{k}))_{j_1} = \int_{\mathbb{R}^2} \sum_{j_1,j_2,j_3\in\{1,2\}} \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\widehat{U}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{U}_{j_3}(\mathbf{l})d\mathbf{l},$$

with $\widehat{\alpha}_{j_1,j_2,j_3}$ some uniformly bounded coefficients and

$$\widehat{N}(\widehat{U},\widehat{V}) = \sqrt{\frac{|\mathbf{k}|^2}{1+|\mathbf{k}|^2}}S\widetilde{N}(S^{-1}\widehat{U},S^{-1}\widehat{V}),$$

$$\widetilde{N}(\widehat{U},\widehat{V}) = \begin{pmatrix} 0 \\ \widehat{U}_1 * \widehat{V}_1 \end{pmatrix},$$

$$S = \frac{1}{2\mathrm{i}}\begin{pmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{pmatrix},$$

where $\widehat{U} = (\widehat{U}_1, \widehat{U}_2)$.

4.2.1 Estimates for the residual

With $\varepsilon\Psi$ we denote the extended approximation associated to $\varepsilon\psi$, namely

$$\varepsilon\Psi = \sum_{|j|\leq M} \sum_{\beta(j)\leq M} \varepsilon^{\beta(j)}\psi_j, \tag{4.8}$$

where $\beta(j) = 1 + ||j| - 1|$ and $\psi_j = A_j(X, Y, T)e^{j\mathrm{i}(k_{01}x+\omega_0 t)} + \text{c.c.}$ with M sufficiently large.

We define the new residual as

$$\mathrm{RES}(U) = -\partial_t U + \Lambda U + N(U, U).$$

As a direct consequence of Lemma 4.2 we have

Lemma 4.5.
There exists an approximation $\varepsilon\Psi$ and a θ_A sufficiently large such that the following holds. Let $A_1 \in C([0, T_0], H^{\theta_A}(\mathbb{R}^2, \mathbb{C}))$ be a solution of the DS *system* (4.6)*. Then there exists a $C_{\mathrm{RES}} > 0$ such that for all $\varepsilon \in (0, 1)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{RES}(\varepsilon\Psi(t))\|_W \leq C_{\mathrm{RES}}\varepsilon^6.$$

Remark 4.6. We have $\mathcal{O}(\varepsilon^6)$ instead of $\mathcal{O}(\varepsilon^7)$ due to the application of $|\mathbf{k}|^{-1}$ in the definition of $\widehat{N}$ which loses a factor ε^{-1}.

In order to prove Theorem 4.1 we estimate the error. By inserting $U = \varepsilon\Psi + \varepsilon^\beta R$ with $\beta > 2$ sufficiently big into (4.7) we find R to satisfy

$$\partial_t R = \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon^\beta N(R, R) + \varepsilon^{-\beta}\mathrm{RES}(\varepsilon\Psi). \tag{4.9}$$

The last two terms are at least of $\mathcal{O}(\varepsilon^2)$ such that these terms can be bounded by a simple application of Gronwall's inequality. Therefore, it remains to control or eliminate the effects of the term $2\varepsilon N(\Psi, R)$. Usually this term can be removed by a normal form transform provided the original system does not possess non-trivial resonances. At the wave vector $\mathbf{k} = 0$ this term vanishes but a resonance at the wave vector $\mathbf{k} = 0$ always implies another resonance for the wave vector $\mathbf{k} = \mathbf{k}_0$ which is non-trivial for our Boussinesq equation. In order to get rid of this difficulty we define a function ϑ by

$$\widehat{\vartheta}(\mathbf{k}) = \begin{cases} 1, & \text{for } |\mathbf{k}| > |\mathbf{k}_0|/10, \\ \varepsilon + 10(1-\varepsilon)\frac{|\mathbf{k}|}{|\mathbf{k}_0|}, & \text{for } |\mathbf{k}| \leq |\mathbf{k}_0|/10. \end{cases}$$

We then write a solution U as a sum of the approximation and an error

$$U = \varepsilon\Psi + \varepsilon^{\beta}\vartheta R, \tag{4.10}$$

where $\beta = 3$ and ϑR is defined by $\widehat{\vartheta R} = \widehat{\vartheta}\widehat{R}$. Note that $\widehat{\vartheta}(t)\widehat{R}(t)$ is small at the wave vectors close to zero.

Inserting (4.10) into (4.9) we find R to satisfy

$$\partial_t R = \Lambda R + 2\varepsilon\vartheta^{-1}N(\Psi, \vartheta R) + \varepsilon^{\beta}\vartheta^{-1}N(\vartheta R, \vartheta R) + \varepsilon^{-\beta}\vartheta^{-1}\mathrm{RES}(\varepsilon\Psi). \tag{4.11}$$

We now look in detail at the term $\varepsilon\vartheta^{-1}N(\Psi, \vartheta R)$. Its kernel is given in Fourier space by

$$\varepsilon\widehat{\vartheta}^{-1}(\mathbf{k})\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})\widehat{\vartheta}(\mathbf{l}), \tag{4.12}$$

for j_1, j_2, $j_3 \in \{1, 2\}$. The main property we use in the following is the validity of an estimate $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k})| \leq C\min(1, |\mathbf{k}|)$.

Extracting the dangerous terms

The approximation is of the form

$$\varepsilon\Psi = \varepsilon\Psi_c + \varepsilon^2\Psi_s,$$

where $\mathrm{supp}(\widehat{\Psi}_c) = \{\mathbf{k} \in \mathbb{R}^2 : |\mathbf{k} \pm \mathbf{k}_0| \leq \delta\}$ for some small $\delta > 0$ and $\mathrm{supp}(\widehat{\Psi}_s) = \mathbb{R}^2 \setminus \mathrm{supp}(\widehat{\Psi}_c)$. Obviously, we have $\|\Psi_c\|_W = \mathcal{O}(1)$ and $\|\Psi_s\|_W = \mathcal{O}(1)$. In a first step we prove that $\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R)$ is of $\mathcal{O}(\varepsilon)$ and not of $\mathcal{O}(1)$.

Lemma 4.7.
There exists $C_1 > 0$ such that

$$\begin{aligned} \|\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R)\|_W \leq& C_1\varepsilon\|R\|_W, \\ \|\varepsilon\vartheta^{-1}N(\varepsilon\Psi_s, \vartheta R)\|_W \leq& C_1\varepsilon^2\|R\|_W. \end{aligned}$$

Proof. Since $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k})| \leq C\min(1, |\mathbf{k}|)$ for some $C > 0$, both estimates follow from

$$\left|\frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k})}{\widehat{\vartheta}(\mathbf{k})}\right| \leq C\frac{\min(1, |\mathbf{k}|)}{\min(\varepsilon + |\mathbf{k}|, 1)} \leq C = \mathcal{O}(1). \qquad \square$$

Since terms of $\mathcal{O}(\varepsilon^2)$ do not cause intense growth in the error R on the time interval $[0, T_0/\varepsilon^2]$ the term $2\varepsilon^2\vartheta^{-1}N(\Psi_s, \vartheta R) = \mathcal{O}(\varepsilon^2)$ has not to be eliminated. Eventually, it can be handled together with the remaining terms of $\mathcal{O}(\varepsilon^2)$ with the help of

Gronwall's inequality. Thus, R satisfies

$$\begin{aligned}\partial_t R =& \Lambda R + 2\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R) + 2\varepsilon^2\vartheta^{-1}N(\Psi_s, \vartheta R) \\ &+ \varepsilon^{\beta}\vartheta^{-1}N(\vartheta R, \vartheta R) + \varepsilon^{-\beta}\vartheta^{-1}\mathrm{RES}(\varepsilon\Psi).\end{aligned} \tag{4.13}$$

4.2.2 The first normal form transform

In order to control the remaining dangerous terms $\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R)$ we make the normal form transform

$$\widetilde{R} = R + \varepsilon B(\Psi_c, R), \tag{4.14}$$

where B is a bilinear mapping. Using $\partial_t \Psi_c = \Lambda\Psi_c + \mathcal{O}(\varepsilon^2)$ we obtain

$$\begin{aligned}\partial_t \widetilde{R} =& \partial_t R + \varepsilon B(\partial_t\Psi_c, R) + \varepsilon B(\Psi_c, \partial_t R) \\ =& \Lambda R + 2\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R) + 2\varepsilon^2\vartheta^{-1}N(\Psi_s, \vartheta R) + \varepsilon^{\beta}\vartheta^{-1}N(\vartheta R, \vartheta R) \\ &+ \varepsilon^{-\beta}\vartheta^{-1}\mathrm{RES}(\varepsilon\Psi_c) + \varepsilon B(\Lambda\Psi_c, R) + \varepsilon B(\mathcal{O}(\varepsilon^2), R) + \varepsilon B(\Psi_c, \Lambda R) \\ &+ \varepsilon B(\Psi_c, 2\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R)) + \varepsilon B(\Psi_c, \varepsilon^{\beta}\vartheta^{-1}N(\vartheta R, \vartheta R)) \\ &+ \varepsilon B(\Psi_c, \varepsilon^{-\beta}\vartheta^{-1}\mathrm{RES}(\varepsilon\Psi)),\end{aligned} \tag{4.15}$$

where $\Lambda R = \Lambda\widetilde{R} - \varepsilon\Lambda B(\Psi_c, R)$.

In order to eliminate all terms of $\mathcal{O}(\varepsilon)$ we have to choose B in such a way that

$$-\varepsilon\Lambda B(\Psi_c, R) + \varepsilon B(\Lambda\Psi_c, R) + \varepsilon B(\Psi_c, \Lambda R) = -2\varepsilon\vartheta^{-1}N(\Psi_c, \vartheta R). \tag{4.16}$$

In Fourier space the j_1-th component of the right hand side of (4.16) can be written as

$$\vartheta^{-1}(\widehat{N(\Psi_c, \vartheta R)})_{j_1} = \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l}) \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \widehat{R}_{j_3}(\mathbf{l}) d\mathbf{l},$$

where $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})| \leq C|\mathbf{k}|$ and $\mathbf{k}, \mathbf{l} \in \mathbb{R}^2$. By (4.16) we obtain

$$\begin{aligned}(\widehat{B(\Psi_c, R)})_{j_1} =& \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \frac{2\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \\ &\times \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \widehat{R}_{j_3}(\mathbf{l}) d\mathbf{l}.\end{aligned}$$

The normal form transform is well-defined if

$$\left| \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \right| \leq C = \mathcal{O}(1) < \infty, \tag{4.17}$$

is valid. Since (4.12) is of $\mathcal{O}(\varepsilon^2)$ for $|\mathbf{l}| < \varepsilon$ and since the approximation Ψ_c has localized support near $\mathbf{k}-\mathbf{l} = \pm\mathbf{k}_0$ respectively the expression (4.17) only has to be analyzed for $|\mathbf{k} \pm \mathbf{k}_0| > \varepsilon$. In order to express the term of the variable $\mathbf{k}$ alone we use the following Lemma.

Lemma 4.8.
Let $\widehat{K} = \widehat{K}(\mathbf{k}, \mathbf{l}) \in C^2(\mathbb{R}^2, \mathbb{C})$ be globally Lipschitz-continuous. Then there exists a $C \geq 0$, such that

$$\left\| \int \widehat{K}(\,\cdot\,, \mathbf{l}) \frac{1}{\varepsilon^2} \widehat{a}\left(\frac{\cdot - \mathbf{l} - \mathbf{k}_0}{\varepsilon}\right) \widehat{R}(\mathbf{l}) d\mathbf{l} - \int \widehat{K}(\,\cdot\,, \,\cdot - \mathbf{k}_0) \frac{1}{\varepsilon^2} \widehat{a}\left(\frac{\cdot - \mathbf{l} - \mathbf{k}_0}{\varepsilon}\right) \widehat{R}(\mathbf{l}) d\mathbf{l} \right\|_W \leq C\varepsilon \|\widehat{R}\|_W \|\widehat{a}\|_{L^1(m+1)}.$$

Proof. The proof is an adaption of the 1D case [10] to the 2D case. With Young's inequality the right hand side can be estimated by

$$\begin{aligned}
&\left\| \int \left(\widehat{K}(\,\cdot\,, \mathbf{l}) - \widehat{K}(\,\cdot\,, \,\cdot - \mathbf{k}_0)\right) \frac{1}{\varepsilon^2} \widehat{a}\left(\frac{\cdot - \mathbf{l} - \mathbf{k}_0}{\varepsilon}\right) \widehat{R}(\mathbf{l}) d\mathbf{l} \right\|_W \\
&\leq C \left\| \int \left| (\cdot - \mathbf{l} - \mathbf{k}_0) \frac{1}{\varepsilon^2} \widehat{a}\left(\frac{\cdot - \mathbf{l} - \mathbf{k}_0}{\varepsilon}\right) \widehat{R}(\mathbf{l}) \right| d\mathbf{l} \right\|_W \\
&\leq C \|\widehat{R}\|_W \left\| (\cdot - \mathbf{k}_0) \frac{1}{\varepsilon^2} \widehat{a}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) \right\|_{L^1(m)} \\
&\leq C\varepsilon \|\widehat{R}\|_W \left\| \frac{1}{\varepsilon^2} \frac{(\cdot - \mathbf{k}_0)}{\varepsilon} \widehat{a}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) \right\|_{L^1(m)} \\
&\leq C\varepsilon \|\widehat{R}\|_W \, |\widehat{a}|_{L^1(m+1)} \,.
\end{aligned}$$

□

So the condition (4.17) can be weakened to

$$\max_{j_1,j_2,j_3 \in \{1,2\}} \sup_{|\mathbf{k}-\mathbf{k}_0| > \varepsilon} \left| \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}_0, \mathbf{k}-\mathbf{k}_0)}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0)} \frac{\widehat{\vartheta}(\mathbf{k}-\mathbf{k}_0)}{\widehat{\vartheta}(\mathbf{k})} \right| < \infty. \tag{4.18}$$

For the resonances $\mathbf{k} = 0$ and $\mathbf{k} = \mathbf{k}_0$ the denominator of (4.18) becomes zero. Therefore, we have to look in detail at the zeroes $\mathbf{k} = 0$ and $\mathbf{k} = \mathbf{k}_0$.

- $\mathbf{k}$ close to 0:

  We have

  $$|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}_0, \mathbf{k}-\mathbf{k}_0)| \leq C|\mathbf{k}|,$$

  and $\widehat{\vartheta}(\mathbf{k}-\mathbf{k}_0) = \mathcal{O}(1)$. Moreover

  $$|\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0)|$$

$$= |\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}(\omega_{j_3}(-\mathbf{k}_0) + \omega'_{j_3}(-\mathbf{k}_0)\mathbf{k} + \mathcal{O}(\mathbf{k}^2))| \geq C|\mathbf{k}|,$$

due to $|\omega'(0)| > |\omega'(\mathbf{k}_0)|$ and $\omega(\mathbf{k}_0) = -\omega(-\mathbf{k}_0)$.

Thus, the expression (4.18) behaves as $1/\widehat{\vartheta}(\mathbf{k})$ and for $\mathbf{k} \to 0$ it will be of $\mathcal{O}(\varepsilon^{-1})$.

- $\mathbf{k}$ close to $\mathbf{k}_0$:

  We have $\widehat{\vartheta}(\mathbf{k}) = \mathcal{O}(1)$ for $\mathbf{k} \to \mathbf{k}_0$ and as before a bound on the denominator of the form

$$|\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k} - \mathbf{k}_0)| \geq C|\mathbf{k} - \mathbf{k}_0|. \tag{4.19}$$

  The term (4.19) can be balanced by $\widehat{\vartheta}(\mathbf{k} - \mathbf{k}_0)$, which behaves as $\varepsilon + |\mathbf{k} - \mathbf{k}_0|$.

Inverting the first normal form transform

Although $1/\widehat{\vartheta}(\mathbf{k}) = \mathcal{O}(\varepsilon^{-1})$ for $\mathbf{k} \to 0$ an $\mathcal{O}(1)$-bounded normal form transform is possible. In order to show that the transformation is invertible we define projection operators P_0 and P_1 by the Fourier multipliers $P_0 = \chi_{|\mathbf{k}| \leq \delta}$ and $P_1 = 1 - P_0$ for a $\delta > 0$ sufficiently small, but independent of $0 < \varepsilon \ll 1$ and split the error into

$$R = R_0 + R_1,$$

with $R_j = P_j R_j$ for $j = 0, 1$ such that (4.14) is given by

$$\begin{aligned} \widetilde{R}_0 &= R_0 + \varepsilon P_0 B(\Psi_c, R_0) + \varepsilon P_0 B(\Psi_c, R_1), \\ \widetilde{R}_1 &= R_1 + \varepsilon P_1 B(\Psi_c, R_0) + \varepsilon P_1 B(\Psi_c, R_1). \end{aligned} \tag{4.20}$$

Since $P_0(\mathbf{k}) = 0$ for $|\mathbf{k}| > \delta$ we have $P_0 B(\Psi_c, R_0) = 0$ if $\delta > 0$ is sufficiently small, but independent of ε. Hence, (4.20) can be rewritten as

$$R_0 = \widetilde{R}_0 - \varepsilon P_0 B(\Psi_c, R_1), \tag{4.21}$$

$$\begin{aligned} R_1 &= \widetilde{R}_1 - \varepsilon P_1 B(\Psi_c, R_1) - \varepsilon P_1 B(\Psi_c, R_0) \\ &= \widetilde{R}_1 - \varepsilon P_1 B(\Psi_c, R_1) - \varepsilon P_1 B(\Psi_c, \widetilde{R}_0) + \varepsilon^2 P_1 B(\Psi_c, P_0 B(\Psi_c, R_1)). \end{aligned} \tag{4.22}$$

Hence, only $R_1 = R_1(\widetilde{R}_1, \widetilde{R}_0)$ appears implicitly. Since $P_1 B$ is of $\mathcal{O}(1)$, on the right hand side, all terms with R_1 are at least of $\mathcal{O}(\varepsilon)$. Inserting (4.22) into (4.21) we obtain that the term $\varepsilon P_0 B$ is of $\mathcal{O}(\varepsilon)$ and hence the normal form transform can be inverted by a Neumann series.

The terms of $\mathcal{O}(\varepsilon)$ are removed and so the error equation (4.15) transforms into

$$\begin{aligned}\partial_t \widetilde{R} =& \Lambda \widetilde{R} + 2\varepsilon^2 B(\Psi_c, \vartheta^{-1} N(\Psi_c, \vartheta R)) + 2\varepsilon^2 \vartheta^{-1} N(\Psi_s, \vartheta R) \\ &+ \varepsilon B(\mathcal{O}(\varepsilon^2), R) + \varepsilon^{\beta} \vartheta^{-1} N(\vartheta R, \vartheta R) + \varepsilon^{\beta+1} B(\Psi_c, \vartheta^{-1} N(\vartheta R, \vartheta R)) \\ &+ \varepsilon^{-\beta+1} B(\Psi_c, \vartheta^{-1} \mathrm{RES}(\varepsilon \Psi)) + \varepsilon^{-\beta} \vartheta^{-1} \mathrm{RES}(\varepsilon \Psi).\end{aligned} \tag{4.23}$$

We have to consider the term $2\varepsilon^2 B(\Psi_c, \vartheta^{-1} N(\Psi_c, \vartheta R))$ in detail because it can be of $\mathcal{O}(\varepsilon)$ due to the factor ϑ^{-1} which is of $\mathcal{O}(\varepsilon^{-1})$.

Since

$$\widehat{P}_1(\mathbf{k}) \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\alpha}_{j_3,j_4,j_5}(\mathbf{l}, \mathbf{l}-\mathbf{n}, \mathbf{n}) \frac{\widehat{\vartheta}(\mathbf{n})}{\widehat{\vartheta}(\mathbf{l})} = \mathcal{O}(1),$$

it remains to consider

$$\widehat{P}_0(\mathbf{k}) \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\alpha}_{j_3,j_4,j_5}(\mathbf{l}, \mathbf{l}-\mathbf{n}, \mathbf{n}) \frac{\widehat{\vartheta}(\mathbf{n})}{\widehat{\vartheta}(\mathbf{l})}. \tag{4.24}$$

We use again Lemma 4.8 in order to express the kernel

$$\frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\alpha}_{j_3,j_4,j_5}(\mathbf{l}, \mathbf{l}-\mathbf{n}, \mathbf{n}) \frac{\widehat{\vartheta}(\mathbf{n})}{\widehat{\vartheta}(\mathbf{l})},$$

in terms of $\mathbf{k}$ alone. Due to the concentration of Ψ_c at $\pm\mathbf{k}_0$ we write $\widehat{\Psi}_c = \frac{1}{\varepsilon^2} a_+ + \frac{1}{\varepsilon^2} a_-$ where a_d stands for the terms concentrated at $d\mathbf{k}_0$. We consider two cases:

- The combination of $d\mathbf{k}_0$ with $-d\mathbf{k}_0$ for $d \in \{+,-\}$

$$\begin{aligned}\widehat{P}_0(\mathbf{k}) &\frac{\widehat{\alpha}_{j_1,j,j_3}(\mathbf{k}, d\mathbf{k}_0, \mathbf{k}-d\mathbf{k}_0)}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_d(d\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-d\mathbf{k}_0)} \frac{\widehat{\vartheta}(\mathbf{k}-d\mathbf{k}_0)}{\widehat{\vartheta}(\mathbf{k})} \\ &\times \widehat{\alpha}_{-j_3,j,j_5}(\mathbf{k}-d\mathbf{k}_0, -d\mathbf{k}_0, \mathbf{k}) \frac{\widehat{\vartheta}(\mathbf{k})}{\widehat{\vartheta}(\mathbf{k}-d\mathbf{k}_0)}.\end{aligned} \tag{4.25}$$

  We have $|\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_d(d\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-d\mathbf{k}_0)| \geq C|\mathbf{k}|$ which can be balanced by the term $|\widehat{\alpha}_{j_1,j,j_3}(\mathbf{k}, d\mathbf{k}_0, \mathbf{k}-d\mathbf{k}_0)| \leq C|\mathbf{k}|$. The non-resonance condition is satisfied and so we have an $\mathcal{O}(1)$-bound for (4.25).

- The combination of $d\mathbf{k}_0$ with $d\mathbf{k}_0$ for $d \in \{+,-\}$

$$
\begin{aligned}
S^{j_1}_{j,j_3,j_5}(\mathbf{k}) = \widehat{P}_0(\mathbf{k}) &\frac{\widehat{\alpha}_{j_1,j,j_3}(\mathbf{k}, d\mathbf{k}_0, \mathbf{k}-d\mathbf{k}_0)}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_d(d\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-d\mathbf{k}_0)} \frac{\widehat{\vartheta}(\mathbf{k}-d\mathbf{k}_0)}{\widehat{\vartheta}(\mathbf{k})} \\
&\times \widehat{\alpha}_{j_3,j,j_5}(\mathbf{k}-d\mathbf{k}_0, d\mathbf{k}_0, \mathbf{k}-2d\mathbf{k}_0) \frac{\widehat{\vartheta}(\mathbf{k}-2d\mathbf{k}_0)}{\widehat{\vartheta}(\mathbf{k}-d\mathbf{k}_0)}.
\end{aligned} \tag{4.26}
$$

The denominator $\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_j(d\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-d\mathbf{k}_0)$ can be balanced by the term $\widehat{\vartheta}(\mathbf{k}-2d\mathbf{k}_0)$ in the numerator. However, there is a term $\vartheta^{-1}(\mathbf{k})$ which is of $\mathcal{O}(\varepsilon^{-1})$ for $\mathbf{k} \to 0$. Thus, we have an $\mathcal{O}(1/\varepsilon)$-bound for (4.26).

Hence, (4.23) simplifies only into

$$
\begin{aligned}
\partial_t \widetilde{R} =& \Lambda \widetilde{R} + 2\varepsilon^2 B(a_+, \vartheta^{-1} N(a_+, \vartheta R)) + 2\varepsilon^2 B(a_-, \vartheta^{-1} N(a_-, \vartheta R)) + \mathcal{O}(\varepsilon^2) \\
&+ \varepsilon B(\mathcal{O}(\varepsilon^2), R) + \varepsilon^{\beta} \vartheta^{-1} N(\vartheta R, \vartheta R) + \varepsilon^{\beta+1} B(\Psi_c, \vartheta^{-1} N(\vartheta R, \vartheta R)) \\
&+ \varepsilon^{-\beta+1} B(\Psi_c, \vartheta^{-1}\mathrm{RES}(\varepsilon\Psi)) + \varepsilon^{-\beta}\vartheta^{-1}\mathrm{RES}(\varepsilon\Psi).
\end{aligned}
$$

4.2.3 The second normal form transform

In order to eliminate the term

$$
2\varepsilon^2 B(a_+, \vartheta^{-1} N(a_+, \vartheta R)) + 2\varepsilon^2 B(a_-, \vartheta^{-1} N(a_-, \vartheta R)), \tag{4.27}
$$

we use a normal form transform

$$
\check{R} = \widetilde{R} + \varepsilon T(a_+, a_+, R) + \varepsilon T(a_-, a_-, R),
$$

where

$$
\begin{aligned}
&T^{j_1}(a_d, a_d, R)(\mathbf{k}) \\
&\quad = \sum_{j_3, j_5 \in \{1,2\}} \int_{\mathbb{R}^4} \frac{\varepsilon S^{j_1}_{j,j_3,j_5}(\mathbf{k})}{\omega_{j_1}(\mathbf{k}) - \omega_d(d\mathbf{k}_0) - \omega_d(d\mathbf{k}_0) - \omega_{j_5}(\mathbf{k}-2d\mathbf{k}_0)} \\
&\qquad\qquad \times a_d(\mathbf{k}-\mathbf{l}) a_d(\mathbf{l}-\mathbf{n}) R_{j_5}(\mathbf{n}) d\mathbf{n} d\mathbf{l},
\end{aligned} \tag{4.28}
$$

with $|S^{j_1}_{j,j_3,j_5}(\mathbf{k})| \leq C$ and $\mathbf{k}$, $\mathbf{l}$, $\mathbf{n} \in \mathbb{R}^2$. The denominator is non zero due to the combination of a_+ with a_+ and a_- with a_-. The transformation works as before and so the term (4.27) can be eliminated with $T_j = \mathcal{O}(1)$ using $\varepsilon S^{j_1}_{j,j_3,j_5}(\mathbf{k}) = \mathcal{O}(1)$.

After the normal form transform we have

$$\begin{aligned}\partial_t \check{R} =& \Lambda \check{R} + 2\varepsilon^2 \vartheta^{-1} N(\Psi_s, \vartheta R) + \varepsilon B(\mathcal{O}(\varepsilon^2), R) + \varepsilon^\beta \vartheta^{-1} N(\vartheta R, \vartheta R) \\ &+ \varepsilon^{\beta+1} B(\Psi_c, \vartheta^{-1} N(\vartheta R, \vartheta R)) + \varepsilon^{-\beta+1} B(\Psi_c, \vartheta^{-1} \mathrm{RES}(\varepsilon\Psi)) \\ &+ \varepsilon^{-\beta} \vartheta^{-1} \mathrm{RES}(\varepsilon\Psi).\end{aligned} \tag{4.29}$$

Substituting R by $\check{R}$ leads to

$$\begin{aligned}\partial_t \check{R} =& \Lambda \check{R} + 2\varepsilon^2 N(\Psi_s, \vartheta \check{R}) + \varepsilon B(\mathcal{O}(\varepsilon^2), \check{R}) + \varepsilon^\beta \vartheta^{-1} N(\vartheta \check{R}, \vartheta \check{R}) \\ &+ \varepsilon^{\beta+1} B(\Psi_c, \vartheta^{-1} N(\vartheta \check{R}, \vartheta \check{R})) + \varepsilon^{-\beta+1} B(\Psi_c, \vartheta^{-1} \mathrm{RES}(\varepsilon\Psi)) \\ &+ \varepsilon^{-\beta} \vartheta^{-1} \mathrm{RES}(\varepsilon\Psi).\end{aligned} \tag{4.30}$$

All terms of $\mathcal{O}(\varepsilon)$ are eliminated. It remains to estimate the error and we will do it by using the variation of constants formula and Gronwall's inequality.

For the terms in (4.30) we find the following estimates

$$\begin{aligned}\|2\varepsilon^2 N(\Psi_s, \vartheta \check{R})\|_W \leq& 2\varepsilon^2 C_{\Psi_s} \|\check{R}\|_W, \\ \|\varepsilon B(\mathcal{O}(\varepsilon^2), \check{R})\|_W \leq& \varepsilon^2 \|\check{R}\|_W, \\ \|\varepsilon^\beta \vartheta^{-1} N(\vartheta \check{R}, \vartheta \check{R})\|_W \leq& \varepsilon^{\beta-1} C \|\check{R}\|_W^2, \\ \|\varepsilon^{\beta+1} B(\Psi_c, \vartheta^{-1} N(\vartheta \check{R}, \vartheta \check{R}))\|_W \leq& \varepsilon^\beta C \|\check{R}\|_W^2.\end{aligned}$$

Choosing

$$\varepsilon^{\beta-1} C \|\check{R}\|_W^2 + \varepsilon^\beta C \|\check{R}\|_W^2 \leq \varepsilon^2, \tag{4.31}$$

we obtain with the help of the variation of constants formula

$$\|\check{R}(t)\|_W \leq \int_0^t \Big(\varepsilon^2 C \|\check{R}(\tau)\|_W + \varepsilon^2 + C_{\mathrm{RES}} \varepsilon^2 \Big) d\tau. \tag{4.32}$$

An application of Gronwall's inequality to (4.32) implies

$$\begin{aligned}\|\check{R}(t)\|_W \leq& (1 + C_{\mathrm{RES}}) \varepsilon^2 t e^{C\varepsilon^2 t} \\ \leq& (1 + C_{\mathrm{RES}}) T_0 e^{CT_0} =: C_R,\end{aligned}$$

independent of $\varepsilon \in (0, \varepsilon_0)$ for all $t \in [0, T_0/\varepsilon^2]$ and $R = \check{R}$. Choosing $\varepsilon_0 > 0$ such that

$$\varepsilon^{\beta-3} C C_R^2 + \varepsilon^{\beta-2} C C_R^2 \leq 1,$$

we have satisfied the condition (4.31) and so proved Theorem 4.1.

□

5. Non-trivial quadratic resonances

In this chapter we prove estimates between the formal approximation, obtained via the NLS equation, and solutions of the system

$$\partial_t^2 u = \Delta u - u - \Delta^2 u + u^2,$$

in case that the original system possesses non-trivial quadratic resonances. The quadratic resonances play a major role, since the modes associated to these resonances can grow to $\mathcal{O}(e^{1/\varepsilon})$ on the long $\mathcal{O}(1/\varepsilon^2)$ time interval. Therefore, it is necessary to control these terms. In order to do so we make use of the fact that in case of analytic initial conditions for the NLS equation the resonances do not matter provided the set of resonant wave vectors and integer multiples of the basic wave vector $\mathbf{k}_0$ have a positive distance. Using spaces with time dependent norms the remaining terms of $\mathcal{O}(\varepsilon)$ can then be controlled by the weight function. See Figure 5.1. This method makes it possible to show that although these resonances

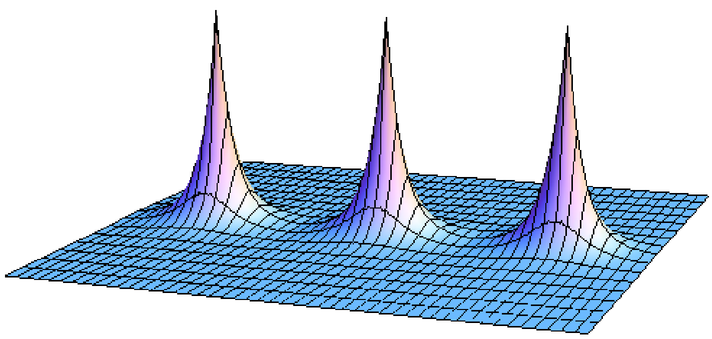


Figure 5.1: The mode concentration of the NLS ansatz in Fourier space.

occur, there are solutions of the original system with initial conditions that behave on time scales of $\mathcal{O}(\varepsilon^{-2})$ as predicted by the NLS equation and do not explode on time scales of $\mathcal{O}(\varepsilon^{-1})$.

In [10] error estimates for the NLS approximation have been shown in case that the basic wave number k_0 of the underlying wave train is resonant and that the associated resonance is stable. The case of unstable resonances remained open in [10]. However, it was pointed out that in this case for resonant wave numbers $k_j = n_j k_0$ with $n_j \in \mathbb{Z}$ and periodic boundary conditions the APP does not hold. This was made rigorous for the water wave problem with surface tension in case of finite depth in [16]. It turns out that quadratic resonances can not be removed by a normal form transform. Our approach follows an idea which has been pointed out in [9] where a first attempt has been made to weaken the non-resonance condition. The assumption of [9] that all resonant wave numbers are bounded away from integer multiples of the basic wave number k_0 can be weakened by the assumption that the resonant wave numbers k_2 and k_3 are bounded away from integer multiples of the basic wave number k_0.

5.1 Derivation of the NLS equation

We consider in 2D a nonlinear wave equation

$$\partial_t^2 u = \Delta u - u - \Delta^2 u + u^2, \tag{5.1}$$

where $\Delta = \partial_x^2 + \partial_y^2$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ for $u(x,y,t) \in \mathbb{R}$. With $X = \varepsilon(x + c_g t)$, $Y = \varepsilon y$ and $T = \varepsilon^2 t$ we make the ansatz

$$\begin{aligned} \varepsilon\psi_{\mathrm{NLS}}(x,y,t) =& \varepsilon A_1(\varepsilon(x+c_g t), \varepsilon y, \varepsilon^2 t) e^{\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \\ &+ \varepsilon^2 A_2(\varepsilon(x+c_g t), \varepsilon y, \varepsilon^2 t) e^{2\mathrm{i}(k_{01}x - \omega_0 t)} + \text{c.c.} \\ &+ \varepsilon^2 A_0(\varepsilon(x+c_g t), \varepsilon y, \varepsilon^2 t), \end{aligned} \tag{5.2}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, $A_0(X,Y,T) \in \mathbb{R}$, $A_j(X,Y,T) \in \mathbb{C}$ for $j \in \{-2,-1,1,2\}$ and c.c. is the complex conjugate. Inserting the ansatz into the residual

$$\mathrm{Res}(u) = -\partial_t^2 u + \partial_x^2 u + \partial_y^2 u - u - 2\partial_x^2\partial_y^2 u - \partial_x^4 u - \partial_y^4 u + u^2,$$

with $\mathbf{E} = e^{\mathrm{i}(k_{01}x - \omega_0 t)}$ we obtain for the coefficients

$$\begin{aligned} \varepsilon\mathbf{E} : 0 =& (\omega_0^2 - 1 - k_{01}^2 - k_{01}^4) A_1, \\ \varepsilon^2\mathbf{E}^0 : 0 =& -A_0 + 2A_1 A_{-1}, \\ \varepsilon^2\mathbf{E} : 0 =& (2\mathrm{i}\omega_0 c_g + 2\mathrm{i}k_{01} + 4\mathrm{i}k_{01}^3)\partial_X A_1, \end{aligned} \tag{5.3}$$

$$\varepsilon^2\mathbf{E}^2 : 0 = (4\omega_0^2 - 4k_{01}^2 - 16k_{01}^4 - 1)A_2 + A_1^2, \tag{5.4}$$

$$\begin{aligned} \varepsilon^3\mathbf{E} : 0 =& 2\mathrm{i}\omega_0\partial_T A_1 + (1 - c_g^2)\partial_X^2 A_1 + \partial_Y^2 A_1 + 6k_{01}^2\partial_X^2 A_1 \\ &+ 2A_1A_0 + 2A_2A_{-1}. \end{aligned} \tag{5.5}$$

From (5.3) and (5.4) we get

$$A_0 = 2A_1A_{-1} \quad \text{and} \quad A_2 = \frac{A_1^2}{-4\omega_0^2 + 4k_{01}^2 - 16k_{01}^2 + 1}.$$

Inserting this into (5.5) a NLS equation

$$2\mathrm{i}\omega_0\partial_T A_1 = (-1 + c_g^2 + 5k_{01}^2)\partial_X^2 A_1 - \partial_Y^2 A_1 + \gamma|A_1|^2 A_1, \tag{5.6}$$

is obtained where

$$\gamma = 4 + \frac{2}{-4\omega_0^2 + 4k_{01}^2 + 16k_{01}^2 + 1}.$$

Herein, $\omega = \omega_0$ and $\mathbf{k} = (k_{01}, 0)$ satisfy the linear dispersion relation

$$\omega^2(\mathbf{k}) = 1 + |\mathbf{k}|^2 + |\mathbf{k}|^4,$$

where the solutions ω are denoted by

$$\omega_\pm(\mathbf{k}) = \pm\sqrt{1 + |\mathbf{k}|^2 + |\mathbf{k}|^4},$$

and $c_g = \frac{-2k_{01} - 4\mathrm{i}k_{01}^3}{2\omega_0}$ is the negative group velocity.

A wave vector $\widetilde{\mathbf{k}}$ is called resonant to the basic wave vector $\mathbf{k}_0$ if

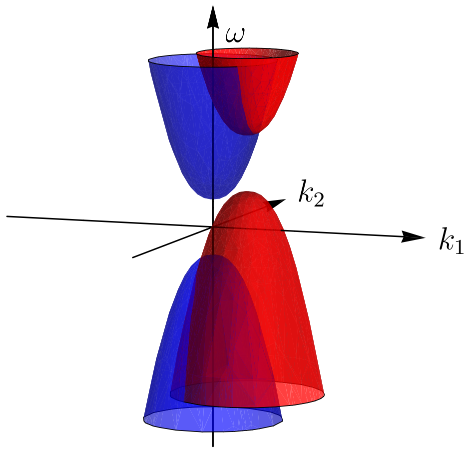
$$\omega_{j_1}(\widetilde{\mathbf{k}}) - \omega_+(\mathbf{k}_0) - \omega_{j_3}(\widetilde{\mathbf{k}} - \mathbf{k}_0) = 0,$$

for some $j_1, j_3 \in \{+, -\}$. We now introduce the set of the resonant wave vectors

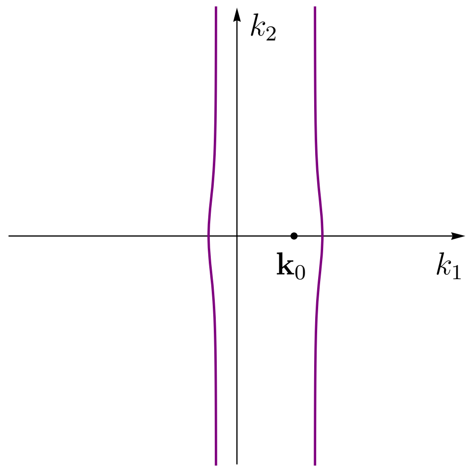
$$\mathcal{K}(\mathbf{k}_0) = \{\, \mathbf{k}_2 \,:\, \exists \mathbf{k}_3 : \mathbf{k}_0 + \mathbf{k}_2 + \mathbf{k}_3 = 0,\ \omega_0 + \omega_2 + \omega_3 = 0\}.$$

In order to see the set of the resonant wave vectors in the (k_1, k_2)-space we first have to look at the surfaces of eigenvalues which are shown in Figure 5.2 (a). The projection of the intersection curves on the (k_1, k_2)-space yields the set of resonant wave vectors, see Figure 5.2 (b). For the wave vector $|\mathbf{k}_0|$ sufficiently big one more intersection occurs, see Figure 5.2 (c). Therefore, in Figure 5.2 (d) a ring of resonances appears. But, this causes no additional problems since the ring of resonances does not intersect the basic wave vector $\mathbf{k}_0$.

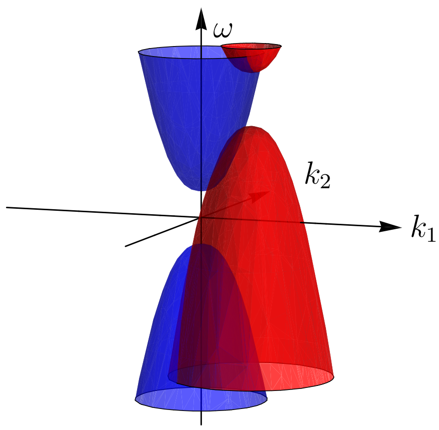
The modes associated to the resonances $\mathbf{k}_2$, $\mathbf{k}_3$ will be called resonant modes. Otherwise they are called non-resonant modes. These resonant modes increase as $e^{\varepsilon t}$ which are of $\mathcal{O}(e^{1/\varepsilon})$ on the long time interval $\mathcal{O}(1/\varepsilon^2)$. Therefore, these modes have to be controlled in order to show that the formal approximation obtained via the NLS approximation $\varepsilon\psi_{\mathrm{NLS}}$ makes correct predictions over the natural time scale. We will solve this problem by using spaces with a time-dependent weight, which will



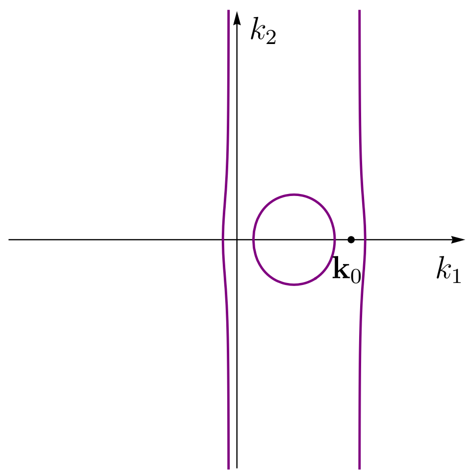
(a) $\mathbf{k}_0 = (1, 0)$. Non-validity of the non-resonance condition. The surfaces of eigenvalues $\omega_\pm(\mathbf{k}) = \pm\sqrt{1 + |\mathbf{k}|^2 + |\mathbf{k}|^4}$ intersect the surfaces $\mathbf{k} \mapsto \omega_\pm(\mathbf{k})$ and $\mathbf{k} \mapsto \omega_+(\mathbf{k}_0) + \omega_\pm(\mathbf{k} - \mathbf{k}_0)$. Intersection points correspond to resonances.



(b) The set $\mathcal{K}(\mathbf{k}_0)$ of wave vectors which are resonant with $\mathbf{k}_0 = (1, 0)$.



(c) $\mathbf{k}_0 = (2, 0)$. Non-validity of the non-resonance condition. The surfaces of eigenvalues $\omega_\pm(\mathbf{k}) = \pm\sqrt{1 + |\mathbf{k}|^2 + |\mathbf{k}|^4}$ intersect the surfaces $\mathbf{k} \mapsto \omega_\pm(\mathbf{k})$ and $\mathbf{k} \mapsto \omega_+(\mathbf{k}_0) + \omega_\pm(\mathbf{k} - \mathbf{k}_0)$. Intersection points correspond to resonances.



(d) The set $\mathcal{K}(\mathbf{k}_0)$ of wave vectors which are resonant with $\mathbf{k}_0 = (2, 0)$. Due to $\mathbf{k}_0 = (2, 0)$ an additional ring of resonances occur.

Figure 5.2: Surfaces of eigenvalues and the set of the resonant wave vectors for the resonant nonlinear wave equation.

be defined below.

5.2 The weighted space

We first define the space

$$W_\alpha = \left\{ u : \mathbb{R}^2 \to \mathbb{C} : \|u\|_{W_\alpha} = \int_{\mathbb{R}^2} |\widehat{u}(\mathbf{k})| e^{\alpha|\mathbf{k}|} d\mathbf{k} < \infty \right\}.$$

The space W_α is closed under point-wise multiplications, i.e.,

$$\|uv\|_{W_\alpha} = \|(\widehat{u} * \widehat{v}) e^{\alpha|\mathbf{k}|}\|_{L^1} \leq C \|\widehat{u} e^{\alpha|\mathbf{k}|}\|_{L^1} \|\widehat{v} e^{\alpha|\mathbf{k}|}\|_{L^1} \leq C \|u\|_{W_\alpha} \|v\|_{W_\alpha},$$

due to Young's inequality for convolutions and the inequality $e^{|\mathbf{k}|} \leq e^{|\mathbf{k}-\mathbf{l}|} e^{|\mathbf{l}|}$. In one space dimension W_α forms a proper subset of the space of functions, that are analytic in a strip in the complex plane symmetric around the real axis equipped with the supremum norm due the Paley-Wiener theorem.

Next, we define the space

$$L^1_g = \left\{ u : \mathbb{R}^2 \to \mathbb{C} : \|u\|_{L^1_g} = \int_{\mathbb{R}^2} |u(\mathbf{k})| g(\mathbf{k}, t) d\mathbf{k} < \infty \right\},$$

where

$$g(\mathbf{k}, t) = \frac{1}{\sup_{m \in \mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon} - \frac{\alpha' \varepsilon t}{T_1}\right)|\mathbf{k} - m\mathbf{k}_0|}|}, \tag{5.7}$$

for $\alpha' < \alpha$ independent of $0 < \varepsilon \ll 1$ and $T_1 = \mathcal{O}(1) < T_0$.

In order to estimate the error, we need the following inequality.

Lemma 5.1.
For all $t \geq 0$ we have

$$\|u * v\|_{L^1_g} \leq \|u\|_{L^1_g} \|v\|_{L^1_g}.$$

Proof. We have

$$\begin{aligned}
\frac{1}{g(\mathbf{k} - \mathbf{l}, t) g(\mathbf{l}, t)} &\leq \sup_{m \in \mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon} - \frac{\alpha' \varepsilon t}{T_1}\right)|\mathbf{k} - \mathbf{l} - m\mathbf{k}_0|}| \sup_{m \in \mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon} - \frac{\alpha' \varepsilon t}{T_1}\right)|\mathbf{l} - m\mathbf{k}_0|}| \\
&\leq \sup_{m \in \mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon} - \frac{\alpha' \varepsilon t}{T_1}\right)|\mathbf{k} - \mathbf{l} + \mathbf{l} - m\mathbf{k}_0|}| \\
&\leq \sup_{m \in \mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon} - \frac{\alpha' \varepsilon t}{T_1}\right)|\mathbf{k} - m\mathbf{k_0}|}| = \frac{1}{g(\mathbf{k}, t)},
\end{aligned}$$

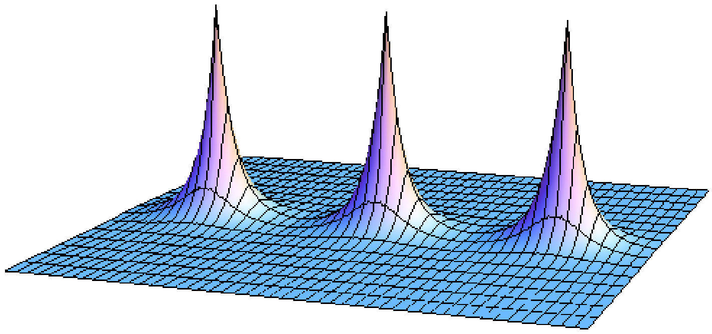


Figure 5.3: Reciprocal weight $g^{-1}(\mathbf{k},t) = \sup_{m\in\mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon}-\frac{\alpha'\varepsilon t}{T_1}\right)|\mathbf{k}-m\mathbf{k}_0|}|$ describing the mode concentration of the NLS ansatz.

and so

$$\begin{aligned}\|u*v\|_{L^1_g} &= \int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} u(\mathbf{k}-\mathbf{l})v(\mathbf{l})d\mathbf{l}\right| g(\mathbf{k})d\mathbf{k}\\ &\le \int_{\mathbb{R}^2}|u(\mathbf{k}-\mathbf{l})|g(\mathbf{k}-\mathbf{l})d\mathbf{k}\int_{\mathbb{R}^2}|v(\mathbf{l})|\,g(\mathbf{l})d\mathbf{l}\\ &= \|u\|_{L^1_g}\|v\|_{L^1_g}. \qquad \square\end{aligned}$$

The analyticity of the initial conditions for the NLS equation implies that the resonant Fourier modes are exponentially small w.r.t. ε if the set of resonant wave vectors and integer multiples of the basic wave vector $\mathbf{k}_0$ have a positive distance, i.e., they are initially of $\mathcal{O}(e^{-\kappa_1/\varepsilon})$ for a $\kappa_1 > 0$ independent of $0 < \varepsilon \ll 1$. Due to the resonance these modes will grow with some rate of $\mathcal{O}(e^{\kappa_2\varepsilon t})$ for a $\kappa_2 > 0$ independent of $0 < \varepsilon \ll 1$. Hence, these modes are less than $\mathcal{O}(\varepsilon^2)$ for all $\varepsilon^2 t \le T_1 = \kappa_1/\kappa_2$. Therefore, the resonances do not matter and so the approximation result can be established. Figure 5.3 shows the mode concentration of the NLS ansatz in Fourier space.

It is the aim to prove the following approximation result.

Theorem 5.2.
Assume $\mathcal{K}(\mathbf{k}_0)\cap(\mathbf{k}_0\mathbb{Z}\times\{0\}) = \emptyset$. Let $\alpha > 0$ and $A_1 \in C([0,T_0],W_\alpha)$ be a solution of the NLS *equation* (5.6)*. Then there exist $\varepsilon_0 > 0$, $T_1 \in (0,T_0]$ and a $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions u of* (5.1) *satisfying*

$$\sup_{t\in[0,T_1/\varepsilon^2]} \|u(\,\cdot\,,\,\cdot\,,t) - \varepsilon\psi_{\mathrm{NLS}}(\,\cdot\,,\,\cdot\,,t)\|_{L^1_{g(t)}} \le C\varepsilon^2,$$

where L^1_g is the weighted space and g the weight function defined in (5.7).

Corollary 5.3.
Assume that the resonant wave vectors are bounded away from integer multiples of the basic wave vector $\mathbf{k}_0$*. Let* A_1 *be a solution of the* NLS *equation* (5.6) *given for* $T \in [0, T_0]$ *whose Fourier transform satisfies*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \int |\widehat{A}_1(\mathbf{K}, T)| e^{\alpha|\mathbf{K}|} d\mathbf{K} < \infty,$$

for some $\alpha > 0$*. Then there exist* $\varepsilon_0 > 0$*,* $T_1 \in (0, T_0]$ *and a* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* u *of* (5.1) *satisfying*

$$\sup_{t\in[0,T_1/\varepsilon^2]} \sup_{(x,y)\in\mathbb{R}^2} |u(x, y, t) - \varepsilon\psi_{\mathrm{NLS}}(x, y, t)| \leq C\varepsilon^2.$$

Remark 5.4. Due to the method in use the error estimates can only be proved on the time interval $[0, T_1/\varepsilon^2]$, but not necessarily for all $t \in [0, T_0/\varepsilon^2]$. Hence, we can only guarantee that parts of the NLS dynamics can be seen in the original system.

The subsequent sections contain the proof of Theorem 5.2

5.3 Higher order approximation and estimates for the residual

We make the extended ansatz

$$\begin{aligned}\varepsilon\psi_4(x, y, t) =& \varepsilon\psi_{\mathrm{NLS}}(x, y, t) + \varepsilon^3 A_3(X, Y, T)\mathbf{E}^3 + \text{c.c.} \\ &+ \varepsilon^3 A_{22}(X, Y, T)\mathbf{E}^2 + \text{c.c.} + \varepsilon^3 A_{02}(X, Y, T).\end{aligned}$$

Choosing

$$\begin{aligned}A_3 &= 2A_1A_2/(-9\omega_0^2 + 9k_{01}^2 + 81k_{01}^4 + 1),\\ A_{22} &= -(c_g + 2\mathrm{i}k_{01} + 8k_{01}^3)\partial_X A_2/(-4\omega_0^2 + 4k_{01}^2 + 16_{01}^4 + 1),\\ A_{02} &= -c_g\partial_X A_0,\end{aligned}$$

we obtain

$$\mathrm{Res}(\varepsilon\psi_4) = \mathcal{O}(\varepsilon^4).$$

Adding more higher order terms to the approximation we finally obtain

$$\mathrm{Res}(\varepsilon\psi_5) = \mathcal{O}(\varepsilon^5).$$

Since all determined equations are linearized NLS equations or linear algebraic equations the approximation exists as long as the solution of the NLS equation exist.

And so it follows

Lemma 5.5.
Let $\alpha > 0$ *and* $A_1 \in C([0,T_0], W_\alpha)$ *be a solution of the* NLS *equation* (5.6). *Then there exists an approximation* $\varepsilon\psi$, *an* $\varepsilon_0 > 0$ *and a* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi(\,\cdot\,,t)\|_{L^1_g} \leq C\varepsilon,$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\psi(\,\cdot\,,t))\|_{L^1_g} \leq C\varepsilon^5.$$

Proof. We have for instance

$$\begin{aligned}
\|A(\varepsilon\,\cdot\,)E\|_{L^1_g} &= \left\| \frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) g(\,\cdot\,)\right\|_{L^1} \\
&\leq \sup_{\mathbf{k}\in\mathbb{R}^2}\left|\frac{e^{\alpha|\,\cdot\,-\mathbf{k}_0|/\varepsilon}}{e^{\alpha|\,\cdot\,-\mathbf{k}_0|/\varepsilon}}\right| \left\|\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) e^{\alpha|\,\cdot\,-\mathbf{k}_0|/\varepsilon}\right\|_{L^1} \\
&\leq C\|\widehat{A}(\,\cdot\,)e^{\alpha|\,\cdot\,|}\|_{L^1} = C\|A\|_{W_\alpha},
\end{aligned}$$

which implies the first estimate. For the second estimate we use estimates as follows. For μ with $\mu(\mathbf{k}) \leq C|\mathbf{k}|^m$ and a $m \geq 0$ we obtain

$$\begin{aligned}
\|(\mu A)(\varepsilon\,\cdot\,)E\|_{L^1_g} &= \left\|\mu(\,\cdot\, - \mathbf{k}_0)\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) g(\,\cdot\,)\right\|_{L^1} \\
&\leq \sup_{\mathbf{k}\in\mathbb{R}^2}\left|\mu(\,\cdot\, - \mathbf{k}_0)\frac{e^{\alpha|\,\cdot\,-\mathbf{k}_0|/\varepsilon}}{e^{\alpha|\,\cdot\,-\mathbf{k}_0|/\varepsilon}}\right| \left\|\frac{1}{\varepsilon^2}\widehat{A}\left(\frac{\cdot - \mathbf{k}_0}{\varepsilon}\right) e^{\alpha|\,\cdot\,-\mathbf{k}_0|/\varepsilon}\right\|_{L^1} \\
&\leq C\varepsilon^m\|\widehat{A}(\,\cdot\,)e^{\alpha|\,\cdot\,|}\|_{L^1} = C\varepsilon^m\|A\|_{W_\alpha}.
\end{aligned}$$

Note that the time derivatives can be expressed as spatial derivatives via the right hand side of the NLS equation. □

5.4 Proof of Theorem 5.2

In order to explain the main idea of the proof of Theorem 5.2 we consider equation (5.1) as a first-order system

$$\begin{aligned}\partial_t\widehat{u} &= -\sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4}\,\widehat{v},\\ \partial_t\widehat{v} &= \sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4}\,\widehat{u} - \frac{1}{\sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4}}(\widehat{u}*\widehat{u}).\end{aligned}$$

This is of the form

$$\partial_t\widehat{U} = \mathcal{M}(\mathbf{k})\widehat{U} + \widehat{\mathcal{N}}(\widehat{U}),$$

where

$$\mathcal{M}(\mathbf{k}) = \begin{pmatrix} 0 & -\sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4} \\ \sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4} & 0\end{pmatrix},$$

$$\widehat{U} = \begin{pmatrix}\widehat{u}\\ \widehat{v}\end{pmatrix} = \begin{pmatrix}\widehat{u}\\ -\frac{1}{\sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4}}\partial_t\widehat{u}\end{pmatrix}.$$

The system can be diagonalized via the transform $\widehat{V}(\mathbf{k}) = S(\mathbf{k})\widehat{U}(\mathbf{k})$ with

$$S = \frac{1}{2\mathrm{i}}\begin{pmatrix}\mathrm{i} & -1\\ \mathrm{i} & 1\end{pmatrix},$$

such that we obtain

$$\partial_t V = \Lambda V + N(V,V), \tag{5.8}$$

where

$$\widehat{\Lambda}(\mathbf{k}) = \begin{pmatrix}\mathrm{i}\sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4} & 0\\ 0 & -\mathrm{i}\sqrt{1+|\mathbf{k}|^2+|\mathbf{k}|^4}\end{pmatrix},$$

and

$$(\widehat{N}(\widehat{V},\widehat{V})(\mathbf{k}))_{j_1} = \int_{\mathbb{R}^2}\sum_{j_2,j_3\in\{1,2\}}\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\widehat{V}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{V}_{j_3}(\mathbf{l})d\mathbf{l},$$

with some coefficients $\widehat{\alpha}_{j_1,j_2,j_3}$ of the bilinear mapping N.

The approximation $\varepsilon\psi$ for u can be extended to an approximation of V, namely

$$\varepsilon\Psi = \varepsilon S\begin{pmatrix}\widehat{\Psi}\\ -\mathrm{i}\omega(\mathbf{k})^{-1}\partial_t\widehat{\Psi}\end{pmatrix}.$$

It is obvious that $\varepsilon\Psi$ obeys the estimates from Lemma 7.4, too, where now

$$\mathrm{RES}(V) = -\partial_t V + \Lambda V + N(V, V).$$

As a direct consequence of Lemma 5.5 we have

Lemma 5.6.
Let $\alpha > 0$ and $A_1 \in C([0, T_0], W_\alpha)$ be a solution of the NLS *equation* (5.6)*. Then there exists an approximation $\varepsilon\Psi$, an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{RES}(\varepsilon\Psi(\,\cdot\,, t))\|_{L^1_g} \leq C\varepsilon^5.$$

In order to prove Theorem 5.2 we have to show that the error

$$\varepsilon^\beta R = V - \varepsilon\Psi,$$

is of $\mathcal{O}(\varepsilon^\beta)$ for a $\beta > 2$ and all $t \in [0, T_1/\varepsilon^2]$, i.e., we have to prove that R is of $\mathcal{O}(1)$ for all $t \in [0, T_1/\varepsilon^2]$. We find R to satisfy

$$\partial_t R = \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon^\beta N(R, R) + \varepsilon^{-\beta}\mathrm{RES}(\varepsilon\Psi). \tag{5.9}$$

The error can be shown to be of $\mathcal{O}(1)$ if the term $2\varepsilon N(\Psi, R)$ can be eliminated or controlled. We will solve the problem by separating first the set of the resonant wave vectors from integer multiples of the basic wave vector $\mathbf{k}_0$. Then all non-resonant terms can be removed by a normal form transform. The resonant terms can be controlled with time dependent norms and so Gronwall's inequality can then be applied to obtain the bound of the error.

5.4.1 The error estimates

Now, let a $\delta > 0$ be sufficiently small, but independent of $0 < \varepsilon \ll 1$. In order to extract the resonant modes we define a mode filter on the resonant wave vectors by

$$\widehat{E}_r(\mathbf{k}) = \begin{cases} 1 & \text{for } \mathbf{k} \in U_\delta(\mathcal{K}(\mathbf{k}_0)), \\ 0 & \text{else }, \end{cases}$$

and the one of the non-resonant wave vectors by $\widehat{E}_n = 1 - \widehat{E}_r$.

The approximation function $\varepsilon\Psi$ is split into

$$\varepsilon\Psi = \varepsilon\Psi_{c,+} + \varepsilon\Psi_{c,-} + \varepsilon^2\Psi_s,$$

where $\mathrm{supp}(\widehat{\Psi}_{c,\pm}) = \{\mathbf{k} \in \mathbb{R}^2 : |\mathbf{k} \pm \mathbf{k}_0| \leq \delta\}$ for some small $\delta > 0$ and $\mathrm{supp}(\widehat{\Psi}_s) = \mathbb{R}^2 \setminus \mathrm{supp}(\widehat{\Psi}_{c,\pm})$. We have $\varepsilon E_r\Psi_{c,\pm} = \mathcal{O}(\varepsilon)$ and $\varepsilon^2 E_r\Psi_{c,\pm} = \mathcal{O}(\varepsilon^2)$. We now use the mode filters to separate the error into two parts, namely $R = R_r + R_n$ with $R_r = E_r R_r$ and $R_n = E_n R_n$ to find

$$\begin{aligned}\partial_t R_r =& \Lambda R_r + 2\varepsilon E_r N(\Psi_c, R_n) + 2\varepsilon E_r N(\Psi_c, R_r) + 2\varepsilon^2 E_r N(\Psi_s, R_r) \\ &+ 2\varepsilon^2 E_r N(\Psi_s, R_n) + 2\varepsilon^\beta E_r N(R_r, R_n) + \varepsilon^\beta E_r N(R_r, R_r) \\ &+ \varepsilon^\beta E_r N(R_n, R_n) + \varepsilon^{-\beta} E_r \mathrm{RES}(\varepsilon\Psi), \\ \partial_t R_n =& \Lambda R_n + 2\varepsilon E_n N(\Psi_c, R_n) + 2\varepsilon E_n N(\Psi_c, R_r) + 2\varepsilon^2 E_n N(\Psi_s, R_r) \\ &+ 2\varepsilon^2 E_n N(\Psi_s, R_n) + 2\varepsilon^\beta E_n N(R_r, R_n) + \varepsilon^\beta E_n N(R_r, R_r) \\ &+ \varepsilon^\beta E_n N(R_n, R_n) + \varepsilon^{-\beta} E_n \mathrm{RES}(\varepsilon\Psi),\end{aligned} \tag{5.10}$$

where $\varepsilon\Psi_c = \varepsilon\Psi_{c,+} + \varepsilon\Psi_{c,-}$.

Since we want to eliminate the terms of $\mathcal{O}(\varepsilon)$, we only consider terms on the support of Ψ_c in the error equations for R_r and R_n.

In the following we make a normal form transform in order to eliminate all non-resonant terms to obtain a system of the form

$$\begin{aligned}\partial_t \widetilde{R}_r =& \Lambda \widetilde{R}_r + 2\varepsilon E_r N(\Psi_c, R_r) + \mathcal{O}(\varepsilon^2), \\ \partial_t \widetilde{R}_n =& \Lambda \widetilde{R}_n + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Making now the normal form transform

$$\begin{aligned}\widetilde{R}_r &= R_r + \varepsilon B_{r,n}(\Psi_c, R_n), \\ \widetilde{R}_n &= R_n + \varepsilon B_{n,r}(\Psi_c, R_r) + \varepsilon B_{n,n}(\Psi_c, R_n),\end{aligned} \tag{5.11}$$

where $B_{r,n}$, $B_{n,r}$ and $B_{n,n}$ are smooth bilinear mappings and using $\partial_t \Psi_c = \Lambda\Psi_c + \mathcal{O}(\varepsilon^2)$ we obtain for the first equation in (5.10)

$$\begin{aligned}\partial_t \widetilde{R}_r =& \partial_t R_r + \varepsilon B_{r,n}(\partial_t \Psi_c, R_n) + \varepsilon B_{r,n}(\Psi_c, \partial_t R_n) \\ =& \Lambda \widetilde{R}_r - \Lambda\varepsilon B_{r,n}(\Psi_c, R_n) + \varepsilon B_{r,n}(\Lambda\Psi_c, R_n) + \varepsilon B_{r,n}(\Psi_c, \Lambda R_n) \\ &+ 2\varepsilon E_r N(\Psi_c, R_r) + 2\varepsilon E_r N(\Psi_c, R_n) + \varepsilon^2 Q_r,\end{aligned} \tag{5.12}$$

and for the second equation

$$\begin{aligned}\partial_t \widetilde{R}_n =& \partial_t R_n + \varepsilon B_{n,r}(\partial_t \Psi_c, R_n) + \varepsilon B_{n,r}(\Psi_c, \partial_t R_n) + \varepsilon B_{n,n}(\partial_t \Psi_c, R_n) \\ &+ \varepsilon B_{n,n}(\Psi_c, \partial_t R_n) \\ =& \Lambda \widetilde{R}_n - \Lambda\varepsilon B_{n,r}(\Psi_c, R_r) + \varepsilon B_{n,r}(\Lambda\Psi_c, R_r) + \varepsilon B_{n,r}(\Psi_c, \Lambda R_r) \\ &- \Lambda\varepsilon B_{n,n}(\Psi_c, R_n) + \varepsilon B_{n,n}(\Lambda\Psi_c, R_n) + \varepsilon B_{n,n}(\Psi_c, \Lambda R_n) \\ &+ 2\varepsilon E_n N(\Psi_c, R_r) + 2\varepsilon E_n N(\Psi_c, R_n) + \varepsilon^2 Q_n,\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}\varepsilon^2 Q_r =& 2\varepsilon^2 E_r N(\Psi_s, R_r) + 2\varepsilon^2 E_r N(\Psi_s, R_n) + \varepsilon B_{r,n}(\Psi_c, 2\varepsilon E_r N(\Psi_c, R_r)) \\ &+ \varepsilon B_{r,n}(\Psi_c, 2\varepsilon E_r N(\Psi_c, R_n)) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), R_r) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), R_n) \\ &+ \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(R_r, R_n)) + \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(R_r, R_r)) \\ &+ \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(R_n, R_n) + \varepsilon^{-\beta-1} E_r \mathrm{RES}(\varepsilon\Psi), \\ \varepsilon^2 Q_n =& 2\varepsilon^2 E_n N(\Psi_s, R_r) + 2\varepsilon^2 E_n N(\Psi_s, R_n) + \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon E_n N(\Psi_c, R_r)) \\ &+ \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon E_n N(\Psi_c, R_n)) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), R_r) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), R_n) \\ &+ \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(R_r, R_n)) + \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(R_r, R_r)) \\ &+ \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(R_n, R_n)) + \varepsilon^{-\beta-1} E_n \mathrm{RES}(\varepsilon\Psi),\end{aligned}$$

with $\widetilde{B} = B_{n,r} + B_{n,n}$. In order to eliminate the dangerous terms of $\mathcal{O}(\varepsilon)$ we have to find $B_{r,n}$, $B_{n,r}$ and $B_{n,n}$ such that

$$\begin{aligned}2\varepsilon E_r N(\Psi_c, R_n) =& \Lambda \varepsilon B_{r,n}(\Psi_c, R_n) - \varepsilon B_{r,n}(\Lambda\Psi_c, R_n) - \varepsilon B_{r,n}(\Psi_c, \Lambda R_n), \\ 2\varepsilon E_n N(\Psi_c, R_r) =& \Lambda \varepsilon B_{n,r}(\Psi_c, R_r) - \varepsilon B_{n,r}(\Lambda\Psi_c, R_r) - \varepsilon B_{n,r}(\Psi_c, \Lambda R_r), \\ 2\varepsilon E_n N(\Psi_c, R_n) =& \Lambda \varepsilon B_{n,n}(\Psi_c, R_n) - \varepsilon B_{n,n}(\Lambda\Psi_c, R_n) - \varepsilon B_{n,n}(\Psi_c, \Lambda R_n).\end{aligned} \tag{5.14}$$

In Fourier space the j_1-th component of the nonlinear term $E_z N(\Psi_c, R_z)$ can be written as

$$E_z \widehat{N(\Psi_c, R_z)}_{j_1} = \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \widehat{E}_z(\mathbf{k}) \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l}) \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \widehat{R}_{z,j_3}(\mathbf{l}) d\mathbf{l},$$

where $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})| \leq C$ and $z \in \{r, n\}$. For $B_{n,z}$ we find

$$\widehat{B_{n,z}(\Psi_c, R_z)}_{j_1} = \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \frac{2\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \widehat{R}_{z,j_3}(\mathbf{l}) d\mathbf{l}.$$

We now want to write this expression for the variable $\mathbf{k}$ alone. Using Lemma 4.8 and the fact that Ψ_c is concentrated at $\pm\mathbf{k}_0$ we obtain the condition that the operators $B_{n,z}$, $B_{r,n}$ are well-defined if

$$\max_{j_1, j_1, j_3 \in \{1,2\}} \left| \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}_0, \mathbf{k}-\mathbf{k}_0)}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0)} \right| < \infty. \tag{5.15}$$

Since the denominator of (5.15) becomes zero in case of resonances we consider the non-resonance condition in more detail. In the following we set $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$.

Validity of the non-resonance condition

Terms are resonant to Ψ_c if both indices are r. Since we have at least two indices n, the non-resonance condition is satisfied and so we get

$$\begin{aligned}(r,n,n): &\qquad \mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0) \neq 0,\\ (n,n,r): &\qquad \mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0) \neq 0,\\ (n,n,n): &\qquad \mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}_0) - \mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_0) \neq 0.\end{aligned}$$

Hence, (5.12) and (5.13) simplifies into

$$\begin{aligned}\partial_t \widetilde{R}_r =& \Lambda \widetilde{R}_r + 2\varepsilon E_r N(\Psi_c, R_r) + \varepsilon^2 Q_r,\\ \partial_t \widetilde{R}_n =& \Lambda \widetilde{R}_n + \varepsilon^2 Q_n,\end{aligned}$$

where Q_r, Q_n are defined above.

All non-resonant terms are eliminated. In order to apply Gronwall's inequality we have to get rid of the remaining term $2\varepsilon E_r N(\Psi_c, R_r)$. We can achieve this by using time dependent norms, which take care of the initial mode distribution and damp this term artificially.

For the next step we need the following lemma.

Lemma 5.7.
Let $\alpha > 0$ and $A_1 \in C([0,T_0], W_\alpha)$ be a solution of the NLS *equation* (5.6)*. Then there exists a $C > 0$ such that the following holds. For all $\varepsilon \in (0,1]$ the maps $B^{\pm}_{n,n}(\Psi_{c,\pm}, E_n\,\cdot\,)$, $B^{\pm}_{n,r}(\Psi_{c,\pm}, E_n\,\cdot\,)$ and $B^{\pm}_{r,n}(\Psi_{c,\pm}, E_n\,\cdot\,)$ are bounded linear mappings from L^1_g to L^1_g satisfying*

$$\|B^{\pm}_{n,n}(\Psi_{c,\pm}, E_n R_n)\|_{L^1_g} \leq C\|R_n\|_{L^1_g}.$$

Proof. The proof follows immediately from the first estimate in Lemma 5.5 and then by using Lemma 5.1. □

Since L^1_g is a Banach algebra, Neumann's series allows to invert the near identity change of variables (5.11). Hence, we have

Lemma 5.8.
The transformation

$$\mathcal{T}: \begin{cases} L^1_g \to L^1_g,\\ R \to \widetilde{R},\end{cases}$$

defined through (5.11) *is a small perturbation of identity. The mapping is analytic and for all* $C_1 > 0$ *there exists an* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *the following holds. For all* $\widetilde{R}$ *with* $\|\widetilde{R}\|_{L^1_g} \leq C_1$ *there exists an analytic inverse.*

The final error estimates

Substituting R_r by $\widetilde{R}_r$ and R_n by $\widetilde{R}_n$ and using the variation of constants formula we get

$$\begin{aligned}\widetilde{R}_r = \int_0^t & e^{\Lambda(t-\tau)}\big(2\varepsilon E_r N(\Psi_c, \widetilde{R}_r) + 2\varepsilon^2 E_r N(\Psi_s, \widetilde{R}_r) + 2\varepsilon^2 E_r N(\Psi_s, \widetilde{R}_n) \\ & + \varepsilon B_{r,n}(\Psi_c, 2\varepsilon E_r N(\Psi_c, \widetilde{R}_r)) + \varepsilon B_{r,n}(\Psi_c, 2\varepsilon E_r N(\Psi_c, \widetilde{R}_n)) \\ & + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), \widetilde{R}_r) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), \widetilde{R}_n) + \varepsilon B_{r,n}(\Psi_c, 2\varepsilon^\beta E_r N(\widetilde{R}_r, \widetilde{R}_n)) \\ & + \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(\widetilde{R}_r, \widetilde{R}_r)) + \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(\widetilde{R}_n, \widetilde{R}_n)) \\ & + \varepsilon^{-\beta-1} E_r \mathrm{RES}(\varepsilon\Psi)\big) d\tau,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\widetilde{R}_n = \int_0^t & e^{\Lambda(t-\tau)}\big(2\varepsilon^2 E_n N(\Psi_s, \widetilde{R}_r) + 2\varepsilon^2 E_n N(\Psi_s, \widetilde{R}_n) \\ & + \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon E_n N(\Psi_c, \widetilde{R}_r)) + \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon E_n N(\Psi_c, \widetilde{R}_n)) \\ & + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), \widetilde{R}_r) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), \widetilde{R}_n) + \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon^\beta E_n N(\widetilde{R}_r, \widetilde{R}_n)) \\ & + \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(\widetilde{R}_r, \widetilde{R}_r)) + \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(\widetilde{R}_n, \widetilde{R}_n)) \\ & + \varepsilon^{-\beta-1} E_n \mathrm{RES}(\varepsilon\Psi)\big) d\tau.\end{aligned} \tag{5.17}$$

The remaining terms of the first equation can be estimated as

$$\begin{aligned}\|2\varepsilon^2 E_r N(\Psi_s, \widetilde{R}_r) + \varepsilon B_{r,n}(\Psi_c, 2\varepsilon E_r N(\Psi_c, \widetilde{R}_r)) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), \widetilde{R}_r)\|_{L^1_g} \leq & C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_g}, \\ \|2\varepsilon^2 E_r N(\Psi_s, \widetilde{R}_n) + \varepsilon B_{r,n}(\Psi_c, 2\varepsilon E_r N(\Psi_c, \widetilde{R}_n)) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), \widetilde{R}_n)\|_{L^1_g} \leq & C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_g}, \\ \|\varepsilon B_{r,n}(\Psi_c, 2\varepsilon^\beta E_r N(\widetilde{R}_r, \widetilde{R}_n)) + \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(\widetilde{R}_r, \widetilde{R}_r)) & \\ + \varepsilon B_{r,n}(\Psi_c, \varepsilon^\beta E_r N(\widetilde{R}_n, \widetilde{R}_n))\|_{L^1_g} \leq & \varepsilon^2.\end{aligned}$$

Using these estimates we get for the equation (5.16)

$$\begin{aligned}\|\widetilde{R}_r(t)\|_{L^1_{g(t)}} \leq & \int_0^t \Bigg(\|e^{\Lambda(t-\tau)} 2\varepsilon E_r N(\Psi_c, \widetilde{R}_r)(\tau)\|_{L^1_{g(\tau)}} + C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_{g(\tau)}} + C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_{g(\tau)}} \\ & + \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2 \Bigg) d\tau \\ \leq & \int_0^t \Bigg(\bigg| \int_{\mathbb{R}^2} e^{\Lambda(\mathbf{k})(t-\tau)} 2\varepsilon E_r N(\Psi_c, \widetilde{R}_r)(\tau) g(\mathbf{k}, t) d\mathbf{k} \bigg| + C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_{g(\tau)}}\end{aligned}$$

$$
\begin{aligned}
&+ C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_{g(\tau)}} + \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2\Big) d\tau \\
\leq &\int_0^t \Bigg(\left| \int_{\mathbb{R}^2} e^{\Lambda(\mathbf{k})(t-\tau)} \frac{g(\mathbf{k},t)}{g(\mathbf{k},\tau)} 2\varepsilon E_r N(\Psi_c, \widetilde{R}_r)(\tau) g(\mathbf{k},\tau) d\mathbf{k} \right| \\
&+ C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_{g(\tau)}} + C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_{g(\tau)}} + \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2\Bigg) d\tau \\
\leq &\int_0^t \Bigg(\sup_{\mathbf{k}\in\mathbb{R}^2} \left| e^{\Lambda(\mathbf{k})(t-\tau)} \frac{g(\mathbf{k},t)}{g(\mathbf{k},\tau)} E_r \right| \int_{\mathbb{R}^2} |2\varepsilon N(\Psi_c, \widetilde{R}_r)(\tau)| g(\mathbf{k},\tau) d\mathbf{k} \\
&+ C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_{g(\tau)}} + C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_{g(\tau)}} + \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2\Bigg) d\tau \\
\leq &\int_0^t \Bigg(e^{-\kappa(t-\tau)\varepsilon} 2\varepsilon C \|\widetilde{R}_r(\tau)\|_{L^1_{g(\tau)}} + C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_{g(\tau)}} + C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_{g(\tau)}} \\
&+ \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2\Bigg) d\tau,
\end{aligned}
$$

for a $\kappa > 0$. Introducing the notation $\mathcal{R}_z(t) := \sup_{0\leq\tau\leq t} \|\widetilde{R}_z(\tau)\|_{L^1_{g(\tau)}}$ we find

$$
\begin{aligned}
\mathcal{R}_r(t) \leq &\int_0^t \Big(e^{-\kappa(t-\tau)\varepsilon} 2\varepsilon C \mathcal{R}_r(\tau) + C\varepsilon^2 \mathcal{R}_r(\tau) + C\varepsilon^2 \mathcal{R}_n(\tau) + \varepsilon^2 \\
&+ C_{\mathrm{RES}}\varepsilon^2\Big) d\tau.
\end{aligned} \tag{5.18}
$$

For the terms in (5.17) we obtain the estimates

$$
\begin{aligned}
\|2\varepsilon^2 E_n N(\Psi_s, \widetilde{R}_r) + \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon E_n N(\Psi_c, \widetilde{R}_r)) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), \widetilde{R}_r)\|_{L^1_g} &\leq C\varepsilon^2 \|\widetilde{R}_r\|_{L^1_g}, \\
\|2\varepsilon^2 E_n N(\Psi_s, \widetilde{R}_n) + \varepsilon \widetilde{B}(\Psi_c, 2\varepsilon E_n N(\Psi_c, \widetilde{R}_n)) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), \widetilde{R}_n)\|_{L^1_g} &\leq C\varepsilon^2 \|\widetilde{R}_n\|_{L^1_g}, \\
\|\varepsilon \widetilde{B}(\Psi_c, 2\varepsilon^\beta E_n N(\widetilde{R}_r, \widetilde{R}_n)) + \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(\widetilde{R}_r, \widetilde{R}_r)) & \\
+ \varepsilon \widetilde{B}(\Psi_c, \varepsilon^\beta E_n N(\widetilde{R}_n, \widetilde{R}_n))\|_{L^1_g} &\leq \varepsilon^2.
\end{aligned}
$$

Therefore,

$$
\mathcal{R}_n(t) \leq \int_0^t \Big(C\varepsilon^2 \mathcal{R}_r + C\varepsilon^2 \mathcal{R}_n + \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2 \Big) d\tau. \tag{5.19}
$$

Adding both inequalities (5.18) and (5.19) with $\mathcal{R}(t) = \mathcal{R}_r(t) + \mathcal{R}_n(t)$ gives

$$
\mathcal{R}(t) \leq \int_0^t \Big(e^{-\kappa(t-\tau)\varepsilon} 2\varepsilon C \mathcal{R}_r(\tau) + C\varepsilon^2 \mathcal{R}_r(\tau) + C\varepsilon^2 \mathcal{R}_n(\tau) + \varepsilon^2 + C_{\mathrm{RES}}\varepsilon^2 \Big) d\tau
$$

$$+\int_0^t \Big(C\varepsilon^2\mathcal{R}_r(\tau)+C\varepsilon^2\mathcal{R}_n(\tau)+\varepsilon^2+C_{\mathrm{RES}}\varepsilon^2\Big)d\tau$$
$$\le 2C\kappa^{-1}\mathcal{R}(t)+2\int_0^t \Big(C\varepsilon^2\mathcal{R}(\tau)+\varepsilon^2+C_{\mathrm{RES}}\varepsilon^2\Big)d\tau.$$

For $C\kappa^{-1}<1/4$ which can be achieved by choosing $T_1>0$ sufficiently small, but independent of $0<\varepsilon\ll 1$, we have

$$\mathcal{R}(t)\le 4C\int_0^t \Big(\varepsilon^2\mathcal{R}(\tau)+\varepsilon^2+C_{\mathrm{RES}}\varepsilon^2\Big)d\tau.$$

Applying Gronwall's inequality yields

$$\mathcal{R}(t)\le 4(1+C_{\mathrm{RES}})T_1e^{4CT_1}=:C_{\mathcal{R}},$$

independent of $\varepsilon\in(0,\varepsilon_0)$ where $\varepsilon_0>0$ had to be chosen so small that $\varepsilon^{\beta-1}CC_{\mathcal{R}}^2+\varepsilon^{\beta-1}CC_{\mathcal{R}}C_{\mathcal{R}}+\varepsilon^{\beta-1}CC_{\mathcal{R}}^2\le 1$. Therefore, this system possesses $\mathcal{O}(1)$ bounded solutions for $t\in[0,T_1/\varepsilon^2]$.

6. Resonant Boussinesq model

The method that we used to justify the NLS equation as an approximation equation in case of non-trivial quadratic resonances cannot be applied directly to the problem of a trivial resonance at the wave vector $\mathbf{k} = 0$ as it occurs for the water wave problem. In this situation the resonance at the wave vector $\mathbf{k} = 0$ is an integer multiples of the basic wave vector $\mathbf{k} = \mathbf{k}_0$. Therefore, we combine Chapter 4 and 5 to obtain a DS approximation result for a resonant Boussinesq model. In 1D such a result has been shown in [10] where the NLS approximation is stable in the system for the three wave interaction associated to the resonances. The method developed in Chapter 5 allows us to justify the DS approximation also in case of unstable resonances. However, it turns out that the APP does not hold for every $\mathbf{k}_0$. For $\mathbf{k}_0 = (2.5, 0)$ e.g. the set of the resonances $\mathcal{K}(\mathbf{k}_0)$ is not separated from integer multiples of the basic wave vector $\mathbf{k}_0$, see Figure 6.1 (d). Therefore, Theorem 6.1 can only be proven for $|\mathbf{k}_0|$ sufficiently small.

6.1 The result

We extend the 2D Boussinesq equation from Chapter 4 with a term that can be interpreted as a surface tension term, i.e., we consider

$$\partial_t^2 u = \Delta u + \partial_t^2 \Delta u + \Delta(u^2) + \mu \Delta^3 u. \tag{6.1}$$

The parameter μ can be interpreted as the strength of surface tension. With the ansatz

$$\begin{aligned} \varepsilon\psi_{\mathrm{DS}}(x,y,t) =& \varepsilon A_1(\varepsilon(x+c_g t), \varepsilon y, \varepsilon^2 t)\mathbf{E} + \text{c.c.} \\ &+ \varepsilon^2 A_2(\varepsilon(x+c_g t), \varepsilon y, \varepsilon^2 t)\mathbf{E}^2 + \text{c.c.} \\ &+ \varepsilon^2 A_0(\varepsilon(x+c_g t), \varepsilon y, \varepsilon^2 t), \end{aligned} \tag{6.2}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, $\mathbf{E} = e^{\mathrm{i}(k_{01}x - \omega_0 t)}$, $X = \varepsilon(x + c_g t)$, $Y = \varepsilon y$, $T = \varepsilon^2 t$, $A_0(X,Y,T) \in \mathbb{R}$, $A_j(X,Y,T) \in \mathbb{C}$ for $j \in \{-2,-1,1,2\}$ and c.c.

is the complex conjugate, a DS system

$$\begin{aligned} 2\mathrm{i}\omega\partial_T A_1 =&(-1+c_g^2)\partial_X^2 A_1 - \partial_Y^2 A_1 + \gamma_1 A_1|A_1|^2 + \gamma_2 A_1 A_0, \\ A_0 =& -\frac{(2\partial_X^2+\partial_Y^2)}{\partial_X^2+\partial_Y^2-c_g^2\partial_X^2}|A_1|^2, \end{aligned} \tag{6.3}$$

is obtained where

$$\gamma_1 = \frac{2k_{01}^4}{\omega_0^2 - k_{01}^2 + 4k_{01}^2\omega_0^2} \quad \text{and} \quad \gamma_2 = -2k_{01}^2.$$

See Chapter 4. Herein, $\mathbf{k} = (k_{01}, 0)$ and $\omega = \omega_0$ satisfy the linear dispersion relation

$$\omega^2 = |\mathbf{k}|^2 - \omega^2|\mathbf{k}|^2 + \mu|\mathbf{k}|^6, \tag{6.4}$$

where the solutions ω are denoted with

$$\omega_\pm(\mathbf{k}) = \pm\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}}.$$

For the notational simplicity we set $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$.

We prove the following approximation result.

Theorem 6.1.
Assume $\mathcal{K}(\mathbf{k}_0) \cap (\mathbf{k}_0\mathbb{Z} \times \{0\}) = \{0, \mathbf{k}_0\}$. Let $\alpha > 0$ and A_0, $A_1 \in C([0,T_0], W_\alpha)$ be solutions of the DS *system* (6.3)*. Then there exists $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions u of* (6.1) *satisfying*

$$\sup_{t\in[0,T_1/\varepsilon^2]} \|u(\,\cdot\,,\,\cdot\,,t) - \varepsilon\psi_{\mathrm{DS}}(\,\cdot\,,\,\cdot\,,t)\|_{L_g^1(t)} \le C\varepsilon^2,$$

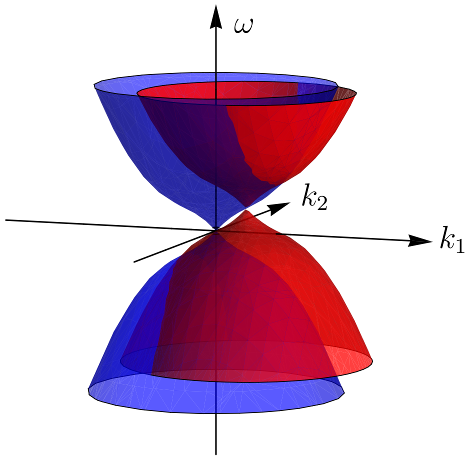
where L_g^1 is the weighted space and g the weight function as defined in (5.7)*.*

Corollary 6.2.
Assume that except of the intersection at 0 *and $\mathbf{k}_0$ the resonant wave vectors are bounded away from integer multiples of the basic wave vector $\mathbf{k}_0$. Let A_1 be a solution of the* DS *system* (6.3) *given for $T \in [0, T_0]$ whose Fourier transform satisfies*

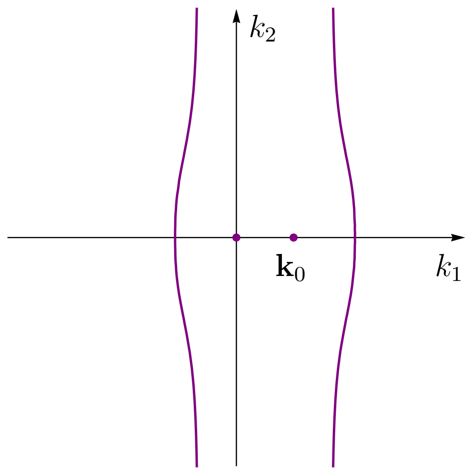
$$\sup_{t\in[0,T_0/\varepsilon^2]} \int |\widehat{A}_1(\mathbf{K},T)| e^{\alpha|\mathbf{K}|} d\mathbf{K} < \infty,$$

for an $\alpha > 0$. Then there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions u of (6.1) *which satisfy*

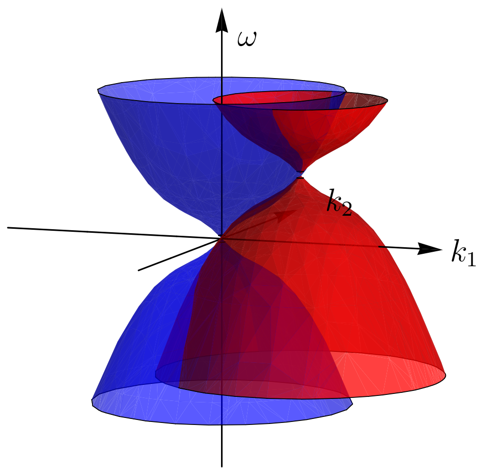
$$\sup_{t\in[0,T_1/\varepsilon^2]} \sup_{(x,y)\in\mathbb{R}^2} |u(x,y,t) - \varepsilon\psi_{\mathrm{DS}}(x,y,t)| \le C\varepsilon^2.$$



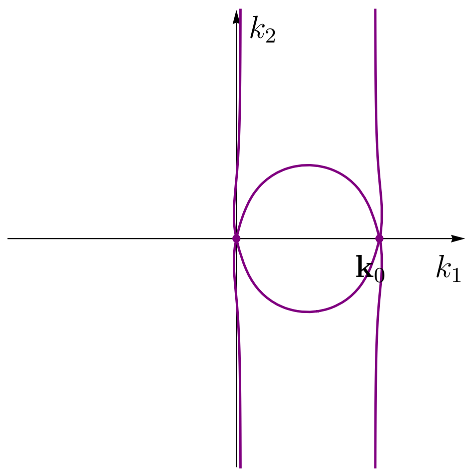
(a) $\mathbf{k}_0 = (1,0)$. The surface of eigenvalues $\omega_\pm(|\mathbf{k}|) = \pm\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}}$ with $\mu = \frac{1}{10}$ for (6.1) and the surfaces $\mathbf{k} \mapsto \omega_\pm(\mathbf{k})$ and $\mathbf{k} \mapsto \omega_+(\mathbf{k}_0) + \omega_\pm(\mathbf{k} - \mathbf{k}_0)$. Intersection points correspond to resonances.



(b) $\mathbf{k}_0 = (1,0)$ and $\mu = \frac{1}{10}$. The set of wave vectors: $\mathcal{K}(\mathbf{k}_0) = \{\mathbf{k}_2 : \exists\, \mathbf{k}_3 : \mathbf{k}_0 + \mathbf{k}_2 + \mathbf{k}_3 = 0,\ \omega_0 + \omega_2 + \omega_3 = 0\}$.



(c) $\mathbf{k}_0 = (2.5,0)$. The surface of eigenvalues $\omega_\pm(|\mathbf{k}|) = \pm\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}}$ with $\mu = \frac{1}{10}$ for (6.1) and the surfaces $\mathbf{k} \mapsto \omega_\pm(\mathbf{k})$ and $\mathbf{k} \mapsto \omega_+(\mathbf{k}_0)+\omega_\pm(\mathbf{k}-\mathbf{k}_0)$. Intersection points correspond to resonances.



(d) $\mathbf{k}_0 = (2.5,0)$ and $\mu = \frac{1}{10}$. The set $\mathcal{K}(\mathbf{k}_0)$ of wave vectors intersect the set of integer multiples of the basic wave vector $\mathbf{k}_0$.

Figure 6.1: Surfaces of eigenvalues and the set of the resonant wave vectors for the resonant Boussinesq model.

Remark 6.3. In Figure 6.1 (b) for the wave vector $\mathbf{k}_0 = (1,0)$ the set of resonances $\mathcal{K}(\mathbf{k}_0)$ is separated from integer multiples of the basic wave vector $\mathbf{k}_0$. However, for $\mathbf{k}_0 = (2.5,0)$ there is an additional ring or resonances which intersect the wave vector $\mathbf{k}_0$, see Figure 6.1 (d). Therefore, Theorem 6.1 can not be proven for $|\mathbf{k}_0|$ sufficiently big.

Remark 6.4. Due to the method in use the error estimates can only be proved on the time interval $[0, T_1/\varepsilon^2]$, but not necessarily for all $t \in [0, T_0/\varepsilon^2]$. Hence, we can only guarantee that parts of the DS dynamics can be seen in the original system.

As before, we can add higher order terms to the ansatz (6.2) in order to construct an approximation, which makes the residual terms sufficiently small for our purpose. And so we obtain with

$$\varepsilon\psi_4(x,y,t) = \varepsilon\psi_{\mathrm{NLS}}(x,y,t) + \varepsilon^3 A_3(X,Y,T)\mathbf{E}^3 + \text{c.c.} + \varepsilon^3 A_{22}(X,Y,T)\mathbf{E}^2 + \text{c.c.},$$

where

$$\begin{aligned} A_3 &= 2k_{01}^2 A_1 A_2/(-\omega_0^2 + k_{01}^2 - 9k_{01}^2\omega_0^2 + 81\omega_0^6), \\ A_{22} &= -(c_g \mathrm{i}\omega_0 + \mathrm{i}k_{01} - 16\mathrm{i}k_{01}\omega_0^2 - 2\mathrm{i}k_{01}^2\omega_0)\partial_X A_2/(\omega_0^2 - k_{01}^2 - 4k_{01}^2\omega_0^2 - 8k_{01}^6), \end{aligned}$$

that $\mathrm{Res}(\varepsilon\psi_4) = \mathcal{O}(\varepsilon^4)$. By adding more terms as above it follows

Lemma 6.5.
Let $\alpha > 0$ and $A_1 \in C([0,T_0], W_\alpha)$ be a solution of the DS *system* (6.3). *Then there exist an approximation $\varepsilon\psi$, an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi(\,\cdot\,,t)\|_{L^1_g} \le C\varepsilon,$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\psi(\,\cdot\,,t))\|_{L^1_g} \le C\varepsilon^7.$$

6.2 Proof of Theorem 6.1

As in Chapter 4 we write the Boussinesq equation (6.1) as a first order system

$$\partial_t U = \Lambda U + N(U,U), \tag{6.5}$$

where Λ is a skew-symmetric diagonal linear operator and $N(\cdot\,,\cdot)$ a symmetric bilinear mapping. In detail, in Fourier space we have

$$\widehat{\Lambda}(\mathbf{k}) = \begin{pmatrix} \mathrm{i}\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}} & 0 \\ 0 & -\mathrm{i}\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}} \end{pmatrix},$$

and

$$(\widehat{N}(\widehat{U},\widehat{U})(\mathbf{k}))_{j_1} = \int_{\mathbb{R}^2} \sum_{j_2,j_3\in\{1,2\}} \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\widehat{U}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{U}_{j_3}(\mathbf{l})d\mathbf{l},$$

with $\widehat{\alpha}_{j_1,j_2,j_3}$ some uniformly bounded coefficients and

$$\widehat{N}(\widehat{U},\widehat{V}) = \sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}}S\widetilde{N}(S^{-1}\widehat{U},S^{-1}\widehat{V}),$$

$$\widetilde{N}(\widehat{U},\widehat{V}) = \begin{pmatrix} 0 \\ \widehat{U}_1 * \widehat{V}_1 \end{pmatrix},$$

$$S = \frac{1}{2\mathrm{i}}\begin{pmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{pmatrix},$$

where $\widehat{U} = (\widehat{U}_1,\widehat{U}_2)$.

Further we define the new residual as

$$\mathrm{RES}(U) = -\partial_t U + \Lambda U + N(U,U).$$

We proceed as in Chapter 4 in order to estimate the error made by the approximation $\varepsilon\Psi$ in (4.8) which denotes the extended approximation associated to $\varepsilon\psi$. Hence, we make the ansatz

$$U = \varepsilon\Psi + \varepsilon^\beta\vartheta R,$$

with $\beta = 3$ and where

$$\widehat{\vartheta}(\mathbf{k}) = \begin{cases} 1, & \text{for } |\mathbf{k}| > |\mathbf{k}_0|/10, \\ \varepsilon + 10(1-\varepsilon)\frac{|\mathbf{k}|}{|\mathbf{k}_0|}, & \text{for } |\mathbf{k}| \le |\mathbf{k}_0|/10. \end{cases}$$

Inserting the ansatz into (6.5) we obtain

$$\partial_t R = \Lambda R + 2\varepsilon\vartheta^{-1}N(\Psi,\vartheta R) + \varepsilon^\beta\vartheta^{-1}N(\vartheta R,\vartheta R) + \varepsilon^{-\beta}\vartheta^{-1}\mathrm{RES}(\varepsilon\Psi). \tag{6.6}$$

Since the approximation $\varepsilon\Psi$ is only of order ε in two δ neighborhoods of $\pm\mathbf{k}_0$ with

$\delta > 0$ small, but independent of $0 < \varepsilon \ll 1$, we split the approximation $\varepsilon\Psi$ into

$$\varepsilon\Psi = \varepsilon\Psi_{c,+} + \varepsilon\Psi_{c,-} + \varepsilon^2\Psi_s,$$

where $\mathrm{supp}(\widehat{\Psi}_{c,\pm}) = \{\mathbf{k} \in \mathbb{R}^2 : |\mathbf{k} \pm \mathbf{k}_0| \leq \delta\}$ for some small $\delta > 0$ and $\mathrm{supp}(\widehat{\Psi}_s) = \mathbb{R}^2 \setminus \mathrm{supp}(\widehat{\Psi}_{c,\pm})$. We find R to satisfy

$$\partial_t R = \Lambda R + 2\varepsilon\vartheta^{-1} N(\Psi_c, \vartheta R) + \mathcal{O}(\varepsilon^2). \tag{6.7}$$

We now remove all non-resonant terms with a normal form transform as we did in Chapter 5. Then the resonant terms can be controlled by time dependent norms. For this purposes we use a mode filter

$$\widehat{E}_r(\mathbf{k}) = \begin{cases} 1 & \text{for } \mathbf{k} \in U_\delta(\widehat{\mathcal{K}}(\mathbf{k}_0)), \\ 0 & \text{else }, \end{cases}$$

for a $\delta > 0$ sufficiently small, but independent of $0 < \varepsilon \ll 1$, and the one of the non-resonant modes by $\widehat{E}_n = 1 - \widehat{E}_r$ with

$$\widehat{\mathcal{K}}(\mathbf{k}_0) = \mathcal{K}(\mathbf{k}_0) \setminus \{0, \mathbf{k}_0\},$$

in order to separate the error in two parts, namely $R = R_r + R_n$ with $R_r = E_r R_r$ and $R_n = E_n R_n$.

We now make the normal form transform

$$\begin{aligned} \partial_t \widetilde{R}_r =& \Lambda R_r + 2\varepsilon B_{r,n}(\Psi_c, R_n), \\ \partial_t \widetilde{R}_n =& \Lambda R_n + 2\varepsilon B_{n,r}(\Psi_c, R_r) + 2\varepsilon B_{n,n}(\Psi_c, R_n), \end{aligned}$$

with $B_{r,n}$, $B_{n,r}$ and $B_{n,n}$ smooth bilinear mappings. Using $\partial_t\Psi_c = \Lambda\Psi_c + \mathcal{O}(\varepsilon^2)$ we obtain

$$\begin{aligned} \partial_t \widetilde{R}_r =& \Lambda \widetilde{R}_r - \Lambda\varepsilon B_{r,n}(\Psi_c, R_n) + \varepsilon B_{r,n}(\Lambda\Psi_c, R_n) + \varepsilon B_{r,n}(\Psi_c, \Lambda R_n) \\ &+ 2\varepsilon E_r \vartheta^{-1} N(\Psi_c, \vartheta R_r) + 2\varepsilon E_r \vartheta^{-1} N(\Psi_c, \vartheta R_n) + E_r \varepsilon^2 G_r, \\ \partial_t \widetilde{R}_n =& \Lambda \widetilde{R}_n - \Lambda\varepsilon B_{n,r}(\Psi_c, R_r) + \varepsilon B_{n,r}(\Lambda\Psi_c, R_r) + \varepsilon B_{n,r}(\Psi_c, \Lambda R_r) \\ &- \Lambda\varepsilon B_{n,n}(\Psi_c, R_n) + \varepsilon B_{n,n}(\Lambda\Psi_c, R_n) + \varepsilon B_{n,n}(\Psi_c, \Lambda R_n) \\ &+ 2\varepsilon E_n \vartheta^{-1} N(\Psi_c, \vartheta R_r) + 2\varepsilon E_n \vartheta^{-1} N(\Psi_c, \vartheta R_n) + E_n \varepsilon^2 G_n, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned}\varepsilon^2 G_r =& 2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta^{-1}(\Psi_s, \vartheta R_r)) + 2\varepsilon^2 B_{r,n} N(\Psi_c, \vartheta^{-1} N(\Psi_s, \vartheta R_n)) \\ &+ 2\varepsilon^2 \vartheta^{-1} N(\Psi_s, \vartheta R_r) + 2\varepsilon^2 \vartheta^{-1} N(\Psi_s, \vartheta R_n) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), R_r) \\ &+ \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), R_n) + \varepsilon^\beta \vartheta^{-1} B_{r,n}(\vartheta R_r, \vartheta R_r) + \varepsilon^\beta \vartheta^{-1} B_{r,n}(\vartheta R_n, \vartheta R_n) \\ &+ \varepsilon^\beta \vartheta^{-1} B_{r,n}(\vartheta R_r, \vartheta R_n) + \mathcal{O}(\varepsilon^{\beta-1}), \\ \varepsilon^2 G_n =& 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta^{-1} N(\Psi_s, \vartheta R_r)) + 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta^{-1} N(\Psi_s, \vartheta R_n)) \\ &+ 2\varepsilon^2 \vartheta^{-1} N(\Psi_s, \vartheta R_r) + 2\varepsilon^2 \vartheta^{-1} N(\Psi_s, \vartheta R_n) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), R_r) \\ &+ \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), R_n) + \varepsilon^\beta \vartheta^{-1} \widetilde{B}(\vartheta R_r, \vartheta R_r) + \varepsilon^\beta \vartheta^{-1} \widetilde{B}(\vartheta R_n, \vartheta R_n) \\ &+ \varepsilon^\beta \vartheta^{-1} \widetilde{B}(\vartheta R_r, \vartheta R_n) + \mathcal{O}(\varepsilon^{\beta-1}),\end{aligned}$$

with $\widetilde{B} = B_{n,r} + B_{n,n}$.

In Fourier space the j_1-th component of the nonlinear term $E_n \vartheta^{-1} N(\Psi_{c,\pm}, \vartheta R_n)$ can be written as

$$\begin{aligned}(E_n \vartheta^{-1} \widehat{N(\Psi_c}, \vartheta R_n))_{j_1}(\mathbf{k}) = \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \widehat{E}_n(\mathbf{k}) \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l}) \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \\ \times \widehat{R}_{n,j_3}(\mathbf{l}) d\mathbf{l},\end{aligned}$$

where $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})| \leq C < \infty$ and $\mathbf{k}, \mathbf{l} \in \mathbb{R}^2$. Choosing

$$\begin{aligned}(B_{n,z}\widehat{(\Psi_c, R_z)})_{j_1} =& \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \frac{2\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \\ & \times \widehat{R}_{z,j_3}(\mathbf{l}) d\mathbf{l}, \\ (B_{r,n}\widehat{(\Psi_c, R_n)})_{j_1} =& \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \frac{2\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}(\mathbf{l})}{\widehat{\vartheta}(\mathbf{k})} \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \\ & \times \widehat{R}_{n,j_3}(\mathbf{l}) d\mathbf{l},\end{aligned}$$

with $z \in \{r, n\}$ all non-resonant terms can be removed. By the choice of ϑ the nontrivial resonance at $\mathbf{k} = \mathbf{k}_0$ can be removed, too. However, for the resonance at $\mathbf{k} \to 0$ additional terms of $\mathcal{O}(\varepsilon)$ appear.

Hence, the error equation (6.8) only simplifies into

$$\begin{aligned}\partial_t \widetilde{R}_r =& \Lambda \widetilde{R}_r + 2\varepsilon E_r \widetilde{N}(\Psi_c, R_r) + 2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta^{-1} N(\Psi_c, \vartheta R_r) \\ &+ 2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta^{-1} N(\Psi_c, \vartheta R_n) + E_r \varepsilon^2 G_r, \\ \partial_t \widetilde{R}_n =& \Lambda \widetilde{R}_n + 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta^{-1} N(\Psi_c, \vartheta R_r) + 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta^{-1} N(\Psi_c, \vartheta R_n) \\ &+ E_n \varepsilon^2 G_n,\end{aligned} \tag{6.9}$$

where

$$(\widehat{\widetilde{N}(\Psi_c, R_r)})_{j_1} = \sum_{j_2, j_3 \in \{1,2\}} \int_{\mathbb{R}^2} \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l}) \widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \widehat{R}_{r,j_3}(\mathbf{l}) d\mathbf{l},$$

and G_r, G_n defined above.

Applying the second normal form transform

$$\begin{aligned}\check{R}_r =& \widetilde{R}_r + \varepsilon \bar{B}_{r,n}(\Psi_{c,+}, \Psi_{c,+}, R_n) + \varepsilon \bar{B}_{r,n}(\Psi_{c,-}, \Psi_{c,-}, R_n), \\ \check{R}_n =& \widetilde{R}_n + \varepsilon \bar{B}_{n,n}(\Psi_{c,+}, \Psi_{c,+}, R_n) + \varepsilon \bar{B}_{n,r}(\Psi_{c,+}, \Psi_{c,+}, R_r) \\ &+ \varepsilon \bar{B}_{n,n}(\Psi_{c,-}, \Psi_{c,-}, R_n) + \bar{B}_{n,r}(\Psi_{c,-}, \Psi_{c,-}, R_r),\end{aligned}$$

where

$$\begin{aligned}&\bar{B}^{j_1}_{r,n}(\Psi_{c,d}, \Psi_{c,d}, R_n)(\mathbf{k}) \\ &\quad = \sum_{j_3, j_5 \in \{1,2\}} \int_{\mathbb{R}^4} \frac{\varepsilon S^{j_1}_{j,j_3,j_5}(\mathbf{k})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_j(d\mathbf{k}_0) - \mathrm{i}\omega_j(d\mathbf{k}_0) - \mathrm{i}\omega_{j_5}(\mathbf{k} - 2d\mathbf{k}_0)} \\ &\qquad \times \Psi_{c,d}(\mathbf{k}-\mathbf{l}) \Psi_{c,d}(\mathbf{l}-\mathbf{n}) R_{n,j_5}(\mathbf{n}) d\mathbf{n} d\mathbf{l},\end{aligned}$$

$$\begin{aligned}&\bar{B}^{j_1}_{n,z}(\Psi_{c,d}, \Psi_{c,d}, R_z)(\mathbf{k}) \\ &\quad = \sum_{j_3, j_5 \in \{1,2\}} \int_{\mathbb{R}^4} \frac{\varepsilon S^{j_1}_{j,j_3,j_5}(\mathbf{k})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_j(d\mathbf{k}_0) - \mathrm{i}\omega_j(d\mathbf{k}_0) - \mathrm{i}\omega_{j_5}(\mathbf{k} - 2d\mathbf{k}_0)} \\ &\qquad \times \Psi_{c,d}(\mathbf{k}-\mathbf{l}) \Psi_{c,d}(\mathbf{l}-\mathbf{n}) R_{z,j_5}(\mathbf{n}) d\mathbf{n} d\mathbf{l},\end{aligned}$$

with $|S^{j_1}_{j,j_3,j_5}(\mathbf{k})| \leq C$, $z \in \{r, n\}$, $d \in \{+, -\}$ and $\mathbf{k}$, $\mathbf{l}$, $\mathbf{n} \in \mathbb{R}^2$ terms of $\mathcal{O}(\varepsilon^2)$ can be removed, since $B_{r,n}$ and $\widetilde{B}$ can be of $\mathcal{O}(\varepsilon^{-1})$ at the wave vector $\mathbf{k} = 0$. The transformation works as before and so the term of $\mathcal{O}(\varepsilon^2)$ can be eliminated with $\bar{B}^j_{r,n} = \mathcal{O}(1)$ and $\bar{B}^j_{n,z} = \mathcal{O}(1)$ using $\varepsilon S = \mathcal{O}(1)$.

Hence, (6.9) simplifies further into

$$\begin{aligned}\partial_t\check{R}_r =&\Lambda\check{R}_r + 2\varepsilon E_r\widetilde{N}(\Psi_c, R_r) + E_r\varepsilon^2 G_r,\\ \partial_t\check{R}_n =&\Lambda\check{R}_n + E_n\varepsilon^2 G_n.\end{aligned}\tag{6.10}$$

Substituting R by $\check{R}$ yields

$$\begin{aligned}\partial_t\check{R}_r =&\Lambda\check{R}_r + 2\varepsilon E_r\widetilde{N}(\Psi_c, \check{R}_r) + E_r\varepsilon^2 \check{G}_r,\\ \partial_t\check{R}_n =&\Lambda\check{R}_n + E_n\varepsilon^2 \check{G}_n,\end{aligned}\tag{6.11}$$

where

$$\begin{aligned}\varepsilon^2\check{G}_r =&2\varepsilon^2\vartheta^{-1}N(\Psi_s,\vartheta\check{R}_r) + 2\varepsilon^2\vartheta^{-1}N(\Psi_s,\vartheta\check{R}_n) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2),\check{R}_r)\\ &+\varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2),\check{R}_n) + \varepsilon^\beta\vartheta^{-1}B_{r,n}(\vartheta\check{R}_r,\vartheta\check{R}_r) + \varepsilon^\beta\vartheta^{-1}B_{r,n}(\vartheta\check{R}_n,\vartheta\check{R}_n)\\ &+\varepsilon^\beta\vartheta^{-1}B_{r,n}(\vartheta\check{R}_r,\vartheta\check{R}_n) + \mathcal{O}(\varepsilon^{\beta-1}),\\ \varepsilon^2\check{G}_n =&2\varepsilon^2\vartheta^{-1}N(\Psi_s,\vartheta\check{R}_r) + 2\varepsilon^2\vartheta^{-1}N(\Psi_s,\vartheta\check{R}_n) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2),\check{R}_r)\\ &+\varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2),\check{R}_n) + \varepsilon^\beta\vartheta^{-1}\widetilde{B}(\vartheta\check{R}_r,\vartheta\check{R}_r) + \varepsilon^\beta\vartheta^{-1}\widetilde{B}(\vartheta\check{R}_n,\vartheta\check{R}_n)\\ &+\varepsilon^\beta\vartheta^{-1}\widetilde{B}(\vartheta\check{R}_r,\vartheta\check{R}_n) + \mathcal{O}(\varepsilon^{\beta-1}).\end{aligned}$$

with $\widetilde{B} = B_{n,r} + B_{n,n}$. It remains to eliminate the term $2\varepsilon E_r\widetilde{N}(\Psi_c,\check{R}_r)$. In order to do so we proceed as in Chapter 5.

In the following we need

Lemma 6.6.
Let $\|\check{R}\|_{L^1_g} \leq Z$. There exist constants C_1, C_3 independent of Z and $\varepsilon \in (0,1]$ and a function $C_2(Z)$ independent of $\varepsilon \in (0,1]$ such that

$$\|\varepsilon^2\check{G}\|_{L^1_g} \leq C_1\varepsilon^2\|\check{R}\|_{L^1_g} + C_2(Z)\varepsilon^3\|\check{R}\|_{L^1_g} + C_3\varepsilon^2.$$

Using now the variation of constants formula we obtain

$$\check{R}_r(t) = \int_0^t e^{\Lambda(t-\tau)}\big(2\varepsilon E_r\widetilde{N}(\Psi_c,\check{R}_r) + \mathcal{O}(\|\varepsilon^2\check{G}_r(\tau)\|_{L^1_g})\big)d\tau,\tag{6.12}$$

$$\check{R}_n(t) = \int_0^t e^{\Lambda(t-\tau)}\mathcal{O}(\|\varepsilon^2\check{G}_n(\tau)\|_{L^1_g})d\tau,\tag{6.13}$$

where

$$\begin{aligned}\varepsilon^2\|\check{G}_r\|_{L^1_g} \leq& C_1\varepsilon^2\|\check{R}_r\|_{L^1_g} + C_1\varepsilon^2\|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3\|\check{R}_r\|_{L^1_g}\|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3\|\check{R}_r\|^2_{L^1_g}\\ &+ C_2(Z)\varepsilon^3\|\check{R}_n\|^2_{L^1_g} + C_3\varepsilon^2,\end{aligned}$$

$$\varepsilon^2\|\check{G}_n\|_{L^1_g} \leq C_1\varepsilon^2\|\check{R}_r\|_{L^1_g} + C_1\varepsilon^2\|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3\|\check{R}_r\|_{L^1_g}\|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3\|\check{R}_r\|^2_{L^1_g} + C_2(Z)\varepsilon^3\|\check{R}_n\|^2_{L^1_g} + C_3\varepsilon^2.$$

We get for $\int_0^t e^{\Lambda(t-\tau)}2\varepsilon E_r\widetilde{N}(\Psi_c,\check{R}_r)d\tau$ in (6.12)

$$\begin{aligned}
&\|2\varepsilon\int_0^t e^{\Lambda(t-\tau)}E_r\widetilde{N}(\Psi_c,\check{R}_r)(\tau)d\tau\|_{L^1_{g(t)}} \\
\leq &\int_{\mathbb{R}^2} 2\varepsilon\int_0^t \left|e^{\Lambda(t-\tau)}E_r\widetilde{N}(\Psi_c,\check{R}_r)(\tau)\right| d\tau g(\mathbf{k},t)d\mathbf{k} \\
\leq &\int_{\mathbb{R}^2} 2\varepsilon\int_0^t \left|e^{\Lambda(t-\tau)}\frac{g(\mathbf{k},t)}{g(\mathbf{k},\tau)}E_r\widetilde{N}(\Psi_c,\check{R}_r)(\tau)\right| g(\mathbf{k},\tau)d\tau d\mathbf{k} \\
\leq &2\varepsilon\int_0^t \left|\int_{\mathbb{R}^2} e^{\Lambda(t-\tau)}\frac{g(\mathbf{k},t)}{g(\mathbf{k},\tau)}E_r\widetilde{N}(\Psi_c,\check{R}_r)(\tau)\right| g(\mathbf{k},\tau)d\mathbf{k}d\tau \\
\leq &2\varepsilon\int_0^t \sup_{\mathbf{k}\in\mathbb{R}^2}\left|e^{\Lambda(t-\tau)}\frac{g(\mathbf{k},t)}{g(\mathbf{k},\tau)}E_r\right|\int_{\mathbb{R}^2}\left|\widetilde{N}(\Psi_c,\check{R}_r)(\tau)\right| g(\mathbf{k},\tau)d\mathbf{k}d\tau \\
\leq &2C\varepsilon\int_0^t e^{-\kappa(t-\tau)\varepsilon}\|\Psi_c\|_{L^1_{g(\tau)}}\|\check{R}_r(\tau)\|_{L^1_{g(\tau)}}d\tau,
\end{aligned}$$

for a $\kappa > 0$. Introducing the notation $\mathcal{R}_z(t) := \sup_{0\leq\tau\leq t}\|\check{R}_z(\tau)\|_{L^1_{g(\tau)}}$ for $z \in \{r,n\}$ and $\mathcal{R}(t) = \mathcal{R}_r(t) + \mathcal{R}_n(t)$ we find

$$\begin{aligned}
\mathcal{R}_r(t) \leq &2C\varepsilon\int_0^t e^{-\kappa(t-\tau)\varepsilon}\mathcal{R}_r(\tau)d\tau + C\int_0^t \Big(C_1\varepsilon^2\mathcal{R}(\tau) + C_2(Z)\varepsilon^3\mathcal{R}(\tau) \\
&+ C_3\varepsilon^2\Big)d\tau.
\end{aligned} \tag{6.14}$$

For the terms in (6.13) we obtain

$$\mathcal{R}_n(t) \leq C\int_0^t \Big(C_1\varepsilon^2\mathcal{R}(\tau) + C_2(Z)\varepsilon^3\mathcal{R}(\tau) + C_3\varepsilon^2\Big)d\tau. \tag{6.15}$$

Adding both inequalities (6.14), (6.15) and choosing $C_2(M)\varepsilon\mathcal{R}(\tau) \leq 1$ gives

$$\begin{aligned}
\mathcal{R}(t) \leq &2C\varepsilon\int_0^t e^{-\kappa(t-\tau)\varepsilon}\mathcal{R}_r(\tau)d\tau + 2C\int_0^t \Big((C_1+1)\varepsilon^2\mathcal{R}(\tau) + C_3\varepsilon^2\Big)d\tau \\
&\leq 2C\kappa^{-1}\mathcal{R}(t) + 2C\int_0^t \Big((C_1+1)\varepsilon^2\mathcal{R}(\tau) + C_3\varepsilon^2\Big)d\tau.
\end{aligned}$$

For $C\kappa^{-1} < 1/4$ which can be achieved by choosing $T_1 > 0$ sufficiently small, but independent of $0 < \varepsilon \ll 1$, we have

$$\mathcal{R}(t) \leq 4C \int_0^t \Big((C_1+1)\varepsilon^2 \mathcal{R}(\tau) + C_3\varepsilon^2 \Big) d\tau.$$

Applying Gronwall's inequality yields

$$\mathcal{R}(t) \leq 4C_3 T_1 e^{4C(1+C_{\mathrm{RES}})T_1} =: C_{\mathcal{R}},$$

independent of $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 > 0$ had to be chosen so small that $C_2(C_{\mathcal{R}})\varepsilon C_{\mathcal{R}} \leq 1$. Therefore, this system possesses $\mathcal{O}(1)$ bounded solutions for $t \in [0, T_1/\varepsilon^2]$. The APP holds and so there are solutions u of the 2D resonant Boussinesq equation (6.1) that behave for all $t \in [0, T_1/\varepsilon^2]$ as predicted by the DS system although resonances occur.

6.3 Consequences for the water wave problem

The justification of the NLS approximation with quasilinear quadratic terms remains unsolved so far in general. The reason for this is that normal form transforms in quasilinear systems lead to a loss of regularity and therefore local existence and uniqueness of solutions in general can no longer be proven. Only for special problems approximation properties could be established. In [17] the Miura transformation has been used to justify the NLS approximation for the KdV equation. In [18] numerical evidence has been given that the NLS approximation makes correct predictions for a quasilinear wave equation. Associated with the normal form transform comes a loss of regularity if quasilinear quadratic terms are eliminated.

In the following we briefly consider the well known 2D water wave problem and then the 3D water wave problem [21, 7].

6.3.1 2D water wave problem

The 2D water wave problem consists in finding the irrotational flow of an incompressible fluid filling the domain $\Omega(t)$ between the bottom $\{(x_1, -1) : x_1 \in \mathbb{R}\}$ and the free surface $\Gamma(t) = \{(x_1, \eta(x_1, t)) : x_1 \in \mathbb{R}\}$, which is bounded in the x_2-direction. Under these assumptions the problem is completely determined by the evolution of the free surface $\Gamma(t)$.

In case of finite depth and no surface tension a NLS equation

$$\partial_T A = \mathrm{i}\nu_1 \partial_X^2 A + \mathrm{i}\nu_2 A|A|^2, \tag{6.16}$$

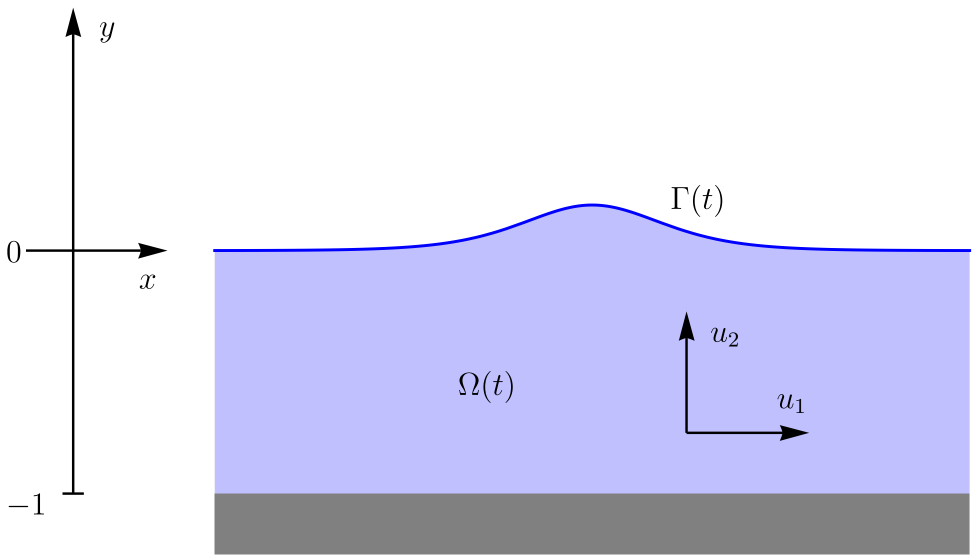


Figure 6.2: The 2D water wave problem.

with coefficients $\nu_j = \nu_j(k_0) \in \mathbb{R}$ can be derived by making the ansatz

$$\begin{pmatrix} \eta \\ u_1 \end{pmatrix} \approx \varepsilon \psi_{\mathrm{NLS}}(x,t) = \varepsilon A(X,T) e^{\mathrm{i}(k_0 x - \omega_0 t)} \varphi(k_0) + \mathrm{c.c.}, \tag{6.17}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, $T \in \mathbb{R}$, $X \in \mathbb{R}$ and $A(X,T) \in \mathbb{C}$. Here $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$, $\varphi(k_0)$ is an explicitly computable two dimensional complex-valued eigenvector and $c_g = \partial_k \omega|_{k=k_0,\omega=\omega_0}$ is the negative group velocity, cf. [22]. The spatial wave number $k = k_0$ and the temporal wave number $\omega = \omega_0$ are related through the linear dispersion relation

$$\omega^2 = k \tanh k, \tag{6.18}$$

of the water wave problem, cf. [23].

6.3.2 3D water wave problem

The Eulerian formulation for the water wave problem

The velocity field $u(x, y, \mathrm{z}) = (u_1, u_2, u_3)$ in $\Omega(t)$ is governed by Euler's equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + g \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$
$$\operatorname{div} u = 0,$$

where p is the pressure, g is the gravitation. We write the velocity field as a gradient of the potential $\varphi : \Omega(t) \to \mathbb{R}$, i.e., $u = \nabla\varphi$.

From Euler's equations it follows that

$$\partial_t \nabla\varphi + \frac{1}{2}\nabla|\nabla\varphi|^2 = -\frac{1}{\rho}\nabla p - g\nabla \mathrm{z},$$

where ρ is the constant density of the fluid.

By integrating this equation we obtain the dynamic boundary condition

$$\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 = -\frac{1}{\rho}p - g\mathrm{z} + \mu\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right),$$

where $\mu\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)$ is the surface tension term. At the top boundary without loss of generality we have $p = 0$ which implies that φ satisfies the equation

$$\begin{aligned}\partial_t\varphi = &-\frac{1}{2}((\partial_x\varphi)^2 + (\partial_y\varphi)^2 + (\partial_\mathrm{z}\varphi)^2) - g\eta \\ &+ \mu\partial_x\left(\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2+(\partial_y\eta)^2}}\right) + \mu\partial_y\left(\frac{\partial_y\eta}{\sqrt{1+(\partial_x\eta)^2+(\partial_y\eta)^2}}\right),\end{aligned} \tag{6.19}$$

for $(x, y, \mathrm{z}) \in \Gamma(t)$ and $g = 1$. Differentiating $\mathrm{z} = \eta(x, y, t)$ and using

$$(1, \partial_t x, \partial_t y, \partial_t \mathrm{z}) = (1, u_1(t), u_2(t), u_3(t)) = (1, \nabla\varphi),$$

we obtain

$$\partial_t\eta = -\partial_\mathrm{z}\varphi - (\partial_x\varphi)(\partial_x\eta) - (\partial_y\varphi)(\partial_y\eta), \tag{6.20}$$

for $(x, y, \mathrm{z}) \in \Gamma(t)$ and $\rho = 1$. With

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in} \quad \Omega(t), \\ u_3|_{x_3=-1} &= \partial_{\mathrm{z}}\varphi|_{\mathrm{z}=-1} = 0, \end{aligned} \tag{6.21}$$

we obtain with (6.19)-(6.21) a closed system. Thus, the dynamics of the system can be completely described by the evolution of the free surface Γ, which is governed by the equations

$$\begin{aligned} \partial_t \eta =& - u_1(\partial_x \eta) - u_2(\partial_y \eta), \\ \partial_t \varphi =& - \frac{1}{2}(u_1^2 + u_2^2) - g\eta + \mu \partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2 + (\partial_y \eta)^2}} \right) \\ &+ \mu \partial_y \left(\frac{\partial_y \eta}{\sqrt{1 + (\partial_x \eta)^2 + (\partial_y \eta)^2}} \right), \end{aligned} \tag{6.22}$$

with initial conditions $\mathrm{z}|_{t=0} = \varphi_1$ and $u_1|_{t=0} = \varphi_2$, cf. [24].

With the ansatz

$$\varepsilon\psi_{\mathrm{DS}}(x, y, t) = \varepsilon A(\varepsilon(x \pm c_g t), \varepsilon y, \varepsilon^2 t) e^{i(k_{01}x \pm \omega_0 t)} \varphi(k_{01}) + \text{c.c.}, \tag{6.23}$$

where $\varphi(k_{01}) \in \mathbb{C}^3$, the DS system (4.6), can be derived. The linear dispersion relation for this problem is given by

$$\omega_+(\mathbf{k}) = -\omega_-(\mathbf{k}) = \sqrt{(g|\mathbf{k}| + \mu|\mathbf{k}|^3)\tanh(h|\mathbf{k}|)},$$

where $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2$.

□

It is the aim of future research to prove the following approximation result.

Claim 6.7.
For all θ_A sufficiently large the following holds. Let $A_0 \in C([0, T_0], H^{\theta_A}(\mathbb{R}^2, \mathbb{R}))$, $A_1 \in C([0, T_0], H^{\theta_A}(\mathbb{R}^2, \mathbb{C}))$ be solutions of the DS *system* (4.6). *Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions of* (6.22) *with*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{(x,y) \in \mathbb{R}^2} \left| \begin{pmatrix} \eta \\ u \end{pmatrix} (x, y, t) - \varepsilon\psi_{\mathrm{DS}}(x, y, t) \right| \leq C\varepsilon^\beta.$$

Figure 6.3 shows that for the water wave problem in case of small surface tension non-trivial resonances also occur. Eliminating terms of $\mathcal{O}(\varepsilon)$ in the error equation with a normal form transform would lead to a loss of regularity and so the local existence and uniqueness theorems for the water wave problem are no longer appli-

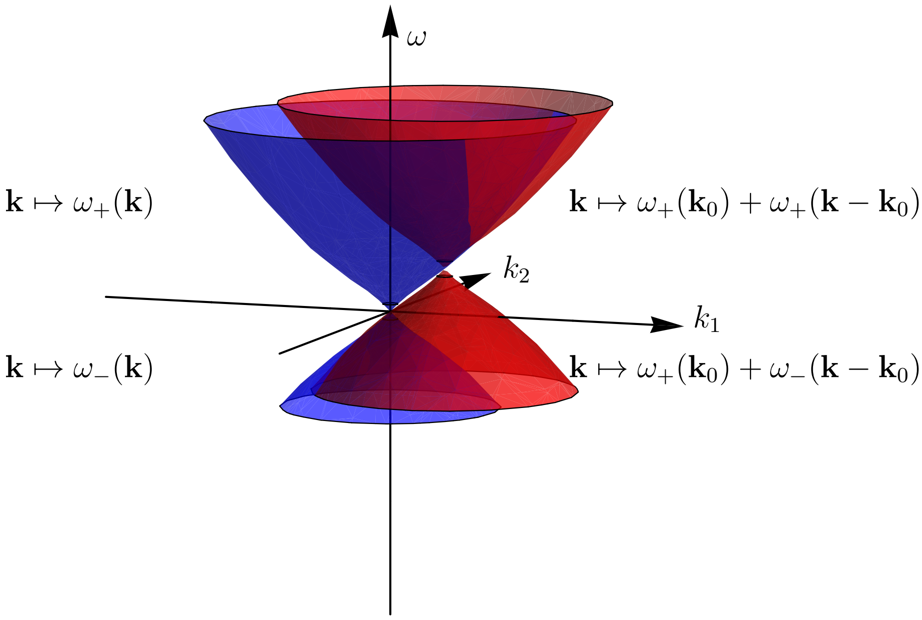


Figure 6.3: The surfaces of eigenvalues $\omega = \sqrt{(g|\mathbf{k}| + \mu|\mathbf{k}|^3)\tanh(h|\mathbf{k}|)}$ with $g = h = 1$, $\mu = 0.1$ and the surfaces $\mathbf{k} \mapsto \omega_\pm(\mathbf{k})$ and $\mathbf{k} \mapsto \omega_+(\mathbf{k}_0) + \omega_\pm(\mathbf{k} - \mathbf{k}_0)$.

cable. Because after the normal form transform there is a loss of a derivative and the Eulerian formulation already loses a derivative as we can see in (6.22). Therefore, the consequences for the water wave problem are that the result in case of our approach does not apply for the DS system.

7. The four-wave interaction system

The resonant four-wave interaction (FWI) system describes the nonlinear interaction of four resonant wave packets. It can be derived as an amplitude equation for the 2D or 3D water wave problem, cf. [25]. There are a number of physical situations where the FWI-system is expected to provide a good description. Examples are pattern formation in vertically oscillated convection [26], multi-wave nonlinear couplings in elastic structures [27], nonlinear optical waves [28], or the four-wave interactions in plasmas [29]. In previous chapters we showed that over the natural time scale there are solutions of the Boussinesq model that behave as predicted by the DS system. The purpose of this chapter is to explain how the justification of the DS system can be transferred to the justification of the equations for the resonant FWI system in 2D. The method used for proving the validity of the FWI as a good approximation is very similar to the one used in Chapter 6. We will explain the method in more detail. Due to the spatial scaling $X = \varepsilon^2 x$ and $Y = \varepsilon^2 y$, there is no restriction on the approximation time T_1 as in the chapters before. Due to the spatial scaling the peaks decay initially as $e^{-\alpha|\mathbf{k}-\mathbf{k}_j|/\varepsilon^2}$. As a consequence the modes associated to the resonant wave vectors are initially exponentially small w.r.t. ε^2 instead of ε as for the NLS approximation resp. the DS system if the set of the resonant wave vectors and integer linear combinations of the basic wave vectors $\mathbf{k}_j$ have a positive distance.

7.1 The result

We will explain under similar assumptions as in Chapter 6 that the FWI system makes correct predictions on the natural time scale of the approximation although quadratic resonances are present in the original system. The equations for the FWI system are given by

$$\begin{aligned}
\partial_T A_1 &= c_1 \cdot \nabla A_1 + \sum_{l \in \{1,\dots,4\}} d_{1,l} |A_l|^2 A_1, \\
\partial_T A_2 &= c_2 \cdot \nabla A_2 + \sum_{l \in \{1,\dots,4\}} d_{2,l} |A_l|^2 A_2, \\
\partial_T A_3 &= c_3 \cdot \nabla A_3 + \sum_{l \in \{1,\dots,4\}} d_{3,l} |A_l|^2 A_3, \\
\partial_T A_4 &= c_4 \cdot \nabla A_4 + \sum_{l \in \{1,\dots,4\}} d_{4,l} |A_l|^2 A_4,
\end{aligned}$$

with group velocities $c_j = \nabla_{\mathbf{k}} \omega|_{\mathbf{k}=\mathbf{k},\omega=\omega_j}$, $\nabla = (\partial_x, \partial_y)^T$ and coefficients $d_{j,l} \in \mathbb{R}$. In case that the four spatial wave vectors $\mathbf{k}_j$ with associated temporal wave numbers ω_j satisfy

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0, \qquad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0,$$

we obtain the resonant FWI system. The equations for the resonant FWI system are given by

$$\begin{aligned}
\partial_T A_1 &= c_1 \cdot \nabla A_1 + \sum_{l \in \{1,\dots,4\}} d_{1,l} |A_l|^2 A_1 + d_1 \overline{A_2 A_3 A_4}, \\
\partial_T A_2 &= c_2 \cdot \nabla A_2 + \sum_{l \in \{1,\dots,4\}} d_{2,l} |A_l|^2 A_2 + d_2 \overline{A_1 A_3 A_4}, \\
\partial_T A_3 &= c_3 \cdot \nabla A_3 + \sum_{l \in \{1,\dots,4\}} d_{3,l} |A_l|^2 A_3 + d_3 \overline{A_1 A_2 A_4}, \\
\partial_T A_4 &= c_4 \cdot \nabla A_4 + \sum_{l \in \{1,\dots,4\}} d_{4,l} |A_l|^2 A_4 + d_4 \overline{A_1 A_2 A_3},
\end{aligned}$$

with additional coefficients $d_j \in \mathbb{R}$.

In order to establish the APP on the natural time scale of $\mathcal{O}(1/\varepsilon^2)$ we have to show that the error is of $\mathcal{O}(1)$. It turns out that the modes associated to the resonant wave vector are initially small w.r.t. ε^2 instead of ε as for the DS system in Chapter 6 if the set of resonant wave vectors and integer linear combinations of the basic wave vector $\mathbf{k}_j$ have a positive distance, i.e., these modes are initially or shortly after $t = 0$ of $\mathcal{O}(e^{-\kappa_1/\varepsilon^2 t})$ for a $\kappa_1 > 0$ independent of $0 < \varepsilon \ll 1$. Due to the resonances these modes will grow with some rate of $\mathcal{O}(e^{\kappa_2 \varepsilon t})$ for a $\kappa_2 > 0$ independently of $0 < \varepsilon \ll 1$. Hence, these modes are less than $\mathcal{O}(\varepsilon^2)$ for all $t \in [0, T_0/\varepsilon^2]$.

7.1.1 Derivation of the FWI system and estimates for the residual

We consider again a resonant 2D Boussinesq equation

$$\partial_t^2 u = \Delta u + \partial_t^2 \Delta u + \Delta(u^2) + \mu \Delta^3 u, \tag{7.1}$$

where $\Delta = \partial_x^2 + \partial_y^2$ and $\mu\Delta^3 u$ models the surface tension.

The ansatz for the resonant wave packets $e^{i(\mathbf{k}_j \cdot \mathbf{x} - \omega_j t)}$ with $\mathbf{k}_j = (k_{j_1}, k_{j_2}) \in \mathbb{R}^2$, $\omega \in \mathbb{R}$, $\mathbf{x} = (x, y)^T$ is given by

$$\begin{aligned}\varepsilon\psi_{\mathrm{FWI}}(x,y,t) = &\sum_{j\in\{-4,\dots,4\}\setminus\{0\}} \varepsilon A_j(X,Y,T) e^{\mathrm{i}(\mathbf{k}_j \cdot \mathbf{x} - \omega_j t)} \\ &+ \sum_{j,n\in\{-4,\dots,4\}\setminus\{0\}} \varepsilon^2 A_{jn}(X,Y,T) e^{\mathrm{i}(\mathbf{k}_j \cdot \mathbf{x} - \omega_j t)} e^{\mathrm{i}(\mathbf{k}_n \cdot \mathbf{x} - \omega_n t)},\end{aligned} \tag{7.2}$$

where $X = \varepsilon^2 x$, $Y = \varepsilon^2 y$ and $T = \varepsilon^2 t$. Inserting this ansatz into the residual with $e^{\mathrm{i}(\mathbf{k}_j \cdot \mathbf{x} - \omega_j t)} = \mathbf{E}_j$ and $e^{\mathrm{i}(\mathbf{k}_n \cdot \mathbf{x} - \omega_n t)} = \mathbf{E}_n$ we find for the coefficients of $\varepsilon\mathbf{E}_j$, $\varepsilon^2\mathbf{E}_j\mathbf{E}_n$, and $\varepsilon^3\mathbf{E}_j$ that

$$\begin{aligned}0 = &\sum_{j\in\{-4,\dots,4\}\setminus\{0\}} \Big(\omega_j^2 - k_{j_1}^2 - k_{j_2}^2 + (k_{j_1}^2 + k_{j_2}^2)\omega_j^2 - \mu(k_{j_1}^6 + k_{j_2}^6 + 3k_{j_1}^4 k_{j_2}^2 \\ &+ 3k_{j_1}^2 k_{j_2}^4)\Big) A_j,\end{aligned} \tag{7.3}$$

$$\begin{aligned}0 = &\sum_{j,n\in\{-4,\dots,4\}\setminus\{0\}} \Big((-3\mu(k_{j_2} + k_{n_2})^2 (k_{j_1} + k_{n_1})^4 - 3\mu(k_{j_1} + k_{n_1})^2 (k_{j_2} + k_{n_2})^4 \\ &+ (\omega_j + \omega_n)^2 - (k_{j_1}^2 + k_{n_1}^2) + (k_{j_1}^2 + k_{n_1}^2)(\omega_j + \omega_n)^2 - (k_{j_2}^2 + k_{n_2}^2) \\ &+ (k_{j_2}^2 + k_{n_2}^2)(\omega_j + \omega_n)^2 - \mu(k_{j_1} + k_{n_1})^6 - \mu(k_{j_2} + k_{n_2})^6) A_{jn} \\ &- 4(k_{j_1}^2 + k_{j_2}^2) A_j^2\Big),\end{aligned} \tag{7.4}$$

$$\begin{aligned}0 = &\sum_{j\in\{-4,\dots,4\}\setminus\{0\}} \Big((2\mathrm{i}\omega_j + 2\mathrm{i}k_{j_1}^2\omega_j + 2\mathrm{i}k_{j_2}^2\omega_j)\partial_T A_j + (2\mathrm{i}k_{j_1} + 5\mu\mathrm{i}k_{j_1}^5 \\ &- 2\mathrm{i}k_{j_1}\omega_j^2)\partial_X A_j + (2\mathrm{i}k_{j_2} + 5\mu\mathrm{i}k_{j_2}^5 - 2\mathrm{i}k_{j_2}\omega_j^2)\partial_Y A_j + 6\mu\mathrm{i}k_{j_1}^4 k_{j_2}\partial_Y A_j \\ &+ 12\mu\mathrm{i}k_{j_1}^3 k_{j_2}^2 \partial_X A_j + 6\mu\mathrm{i}k_{j_1} k_{j_2}^4 \partial_X A_j + 12\mu\mathrm{i}k_{j_1}^2 k_{j_2}^3 \partial_Y A_j - 2(k_{j_1}^2 + k_{j_2}^2) \\ &\times \sum_{\mathbf{k}_m + \mathbf{k}_n + \mathbf{k}_l = \mathbf{k}_j; m,n,l\in\{-4,\dots,4\}\setminus\{0\}} A_{mn} A_l\Big),\end{aligned} \tag{7.5}$$

where $T = \varepsilon^2 t$, $X = \varepsilon^2 x$, $Y = \varepsilon^2 y$, $A_j(X,Y,T) \in \mathbb{C}$, $A_{jn}(X,Y,T) \in \mathbb{C}$. We obtain from (7.3) the linear dispersion relation

$$\omega_j^2 = |\mathbf{k}_j|^2 - \omega_j^2 |\mathbf{k}_j|^2 + \mu |\mathbf{k}_j|^6.$$

From (7.5) we get the FWI system

$$\begin{aligned} &2\mathrm{i}\omega_j(1 + k_{j_1}^2 + k_{j_2}^2)\partial_T A_j \\ &= -(2\mathrm{i}k_{j_1} + 5\mu\mathrm{i}k_{j_1}^5 - 2\mathrm{i}k_{j_1}\omega_j^2 + 12\mu\mathrm{i}k_{j_1}^3 k_{j_2}^2 + 6\mu\mathrm{i}k_{j_1}k_{j_2}^4)\partial_X A_j - (2\mathrm{i}k_{j_2} \\ &+ 5\mu\mathrm{i}k_{j_2}^5 - 2\mathrm{i}k_{j_2}\omega_j^2 + 6\mu\mathrm{i}k_{j_1}^4 k_{j_2} + 12\mu\mathrm{i}k_{j_1}^2 k_{j_2}^3)\partial_Y A_j + \sum_{l\in\{1,\dots,4\}} d_{jl} A_j |A_l|^2, \end{aligned} \tag{7.6}$$

with coefficients $d_{jl} \in \mathbb{R}$. For the notational simplicity we set $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$.

The goal is to show that the formal approximation obtained via the FWI approximation $\varepsilon\psi_{\mathrm{FWI}}$ makes correct predictions over the natural time scale for analytic solutions of the FWI system although unstable resonances are present in the problem.

We define the set of resonant wave vectors

$$\mathcal{K}(\mathbf{k}_j) = \{\widetilde{\mathbf{k}} : \exists \widetilde{\widetilde{\mathbf{k}}} : \mathbf{k}_j + \widetilde{\mathbf{k}} + \widetilde{\widetilde{\mathbf{k}}} = 0,\, \omega_j + \widetilde{\omega} + \widetilde{\widetilde{\omega}} = 0\}, \tag{7.7}$$

and

$$\widehat{\mathcal{K}}(\mathbf{k}_j) = \mathcal{K}(\mathbf{k}_j) \setminus \{0, \mathbf{k}_j\}. \tag{7.8}$$

It is the aim to prove the following result.

Theorem 7.1.
For $\mathbf{k}_j$ *satisfying* $\mathrm{dist}((\cup_{j\in\{1,\dots,4\}}\widehat{\mathcal{K}}(\mathbf{k}_j)), \mathbf{k}_1\mathbb{Z} + \dots + \mathbf{k}_4\mathbb{Z}) > 0$ *the following holds. Let* $\alpha > 0$ *and* $A_1, \dots, A_4 \in C([0, T_0], W_\alpha)$ *satisfy the* FWI *system* (7.6)*. Then there exist* $\varepsilon_0 > 0$ *and a* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* u *of* (7.1) *which satisfy*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\,\cdot\,,\,\cdot\,,t) - \varepsilon\psi_{\mathrm{FWI}}(\,\cdot\,,\,\cdot\,,t)\|_{L^1_{g(t)}} \le C\varepsilon^2,$$

where $L^1_{g(t)}$ *is the weighted space defined in Chapter 5 and* g *is the weight function*

$$g(\mathbf{k}, t) = \frac{1}{\sup_{m_1,\dots,m_n\in\mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon^2} - \frac{\alpha' t}{T_0}\right)|\mathbf{k} - (m_1\mathbf{k}_1 + \dots + m_n\mathbf{k}_n)|}|}, \tag{7.9}$$

for $\alpha' < \alpha$ *independent of* $0 < \varepsilon \ll 1$.

A direct consequence which avoids our previous definitions is as follows.

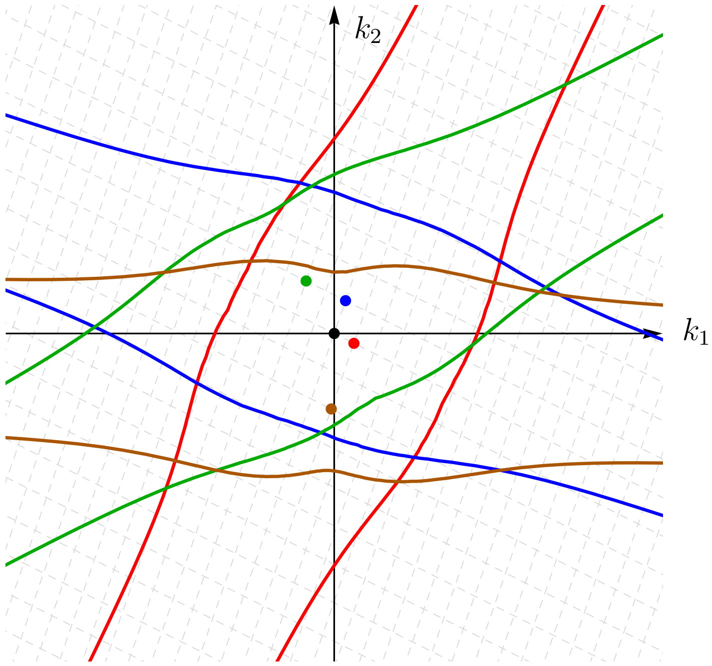


Figure 7.1: The set of the wave vectors $\mathcal{K}(\mathbf{k}_1) \cup ... \cup \mathcal{K}(\mathbf{k}_4)$ with $\mathcal{K}(\mathbf{k}_j) = \{\widetilde{\mathbf{k}} | \, \exists \widetilde{\widetilde{\mathbf{k}}} : \mathbf{k}_j + \widetilde{\mathbf{k}} + \widetilde{\widetilde{\mathbf{k}}} = 0, \, \omega_j + \widetilde{\omega} + \widetilde{\widetilde{\omega}} = 0\}$ which are resonant with every other resonant wave vector.

Corollary 7.2.
Assume that except for $0, \mathbf{k}_1, ..., \mathbf{k}_4$ *the resonant wave vectors are bounded away from integer linear combinations of the basic wave vectors* $\mathbf{k}_j$*. Let* A_j *for* $j \in \{1,2,3,4\}$ *be given for* $T \in [0, T_0]$*, satisfy the* FWI *system* (7.6)*, and whose Fourier transform satisfy*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \int |\widehat{A}_j(\mathbf{K}, T)| e^{\alpha |\mathbf{K}|} d\mathbf{K} < \infty,$$

for some $\alpha > 0$*. Then there exist* $\varepsilon_0 > 0$*, and a* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* u *of* (7.1) *which satisfy*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{(x,y) \in \mathbb{R}^2} |u(x,y,t) - \varepsilon \psi_{\mathrm{FWI}}(x,y,t)| \leq C \varepsilon^2.$$

Remark 7.3. The condition $\mathrm{dist}((\cup_{j \in \{1,...,4\}} \widehat{\mathcal{K}}(\mathbf{k}_j)), \mathbf{k}_1 \mathbb{Z} + ... + \mathbf{k}_4 \mathbb{Z}) > 0$ is a serious restriction on the wave vectors $\mathbf{k}_j$. This means that there exists a two-dimensional lattice $M_\sharp = \{\iota \in \check{\mathbf{k}} \mathbb{Z} + \widehat{\mathbf{k}} \mathbb{Z}\}$ spanned by the two vectors $\check{\mathbf{k}} \in \mathbb{R}^2$ and $\widehat{\mathbf{k}} \in \mathbb{R}^2$ such that $\mathbf{k}_j \in M_\sharp$ for $j \in \{1,2,3,4\}$.

7.1.2 The improved approximation

With the approximation $\varepsilon\psi_{\mathrm{FWI}}$ from above we still have $\mathrm{Res}(\varepsilon\psi_{\mathrm{FWI}}) = \mathcal{O}(\varepsilon^3)$ which can be improved to $\mathrm{Res}(\varepsilon\psi_4) = \mathcal{O}(\varepsilon^4)$ by adding higher order terms to the ansatz $\varepsilon\psi_{\mathrm{FWI}}$. We make the extended ansatz

$$\varepsilon\psi_4(x,y,t) = \varepsilon\psi_{\mathrm{FWI}}(x,y,t) + \sum_{\mathbf{k}_j+\mathbf{k}_m+\mathbf{k}_n \notin \{\mathbf{k}_{\mathrm{w}} | \mathrm{w}\in\{-4,\dots,4\}\setminus\{0\}\}} \varepsilon^3 A_{jmn}(X,Y,T)\mathbf{E}_j\mathbf{E}_m\mathbf{E}_n.$$

Choosing

$$\begin{aligned} A_{jmn} = {}& 2(2k_{j_1}+2k_{j_2}+k_{m_1}+k_{m_2})^2 A_j A_{jm} \Big/ \Big((\omega_j+\omega_m+\omega_n)^2 - (k_{j_1}+k_{m_1}+k_{n_1})^2 \\ & - (k_{j_2}+k_{m_2}+k_{n_2})^2 + (k_{j_1}+k_{m_1}+k_{n_1})^2(\omega_j+\omega_m+\omega_n)^2 + (k_{j_2}+k_{m_2}+k_{n_2})^2 \\ & \times (\omega_j+\omega_m+\omega_n)^2 - 4\mu|\mathbf{k}_j+\mathbf{k}_n+\mathbf{k}_m|^6 \Big), \end{aligned}$$

we obtain $\mathrm{Res}(\varepsilon\psi_4) = \mathcal{O}(\varepsilon^4)$. Adding more higher order terms to the approximation we obtain $\mathrm{Res}(\varepsilon\psi_5) = \mathcal{O}(\varepsilon^5)$.

And so it follows

Lemma 7.4.
Let $\alpha > 0$ and $A_1, \dots, A_4 \in C([0,T_0], W_\alpha)$ satisfy the FWI *system* (7.6). *Then there exist an approximation $\varepsilon\psi$, an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi(\,\cdot\,,t)\|_{L^1_g} \leq C\varepsilon,$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\psi(\,\cdot\,,t))\|_{L^1_g} \leq C\varepsilon^5.$$

Proof. The construction of the approximation goes almost line for line as the construction in Chapter 5 only the spatial scaling is different. □

7.2 The error estimates

As in Chapter 6 we write (7.1) as a first-order system

$$\partial_t U = \Lambda U + N(U,U), \tag{7.10}$$

where

$$\widehat{\Lambda}(\mathbf{k}) = \begin{pmatrix} i\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}} & 0 \\ 0 & -i\sqrt{\frac{|\mathbf{k}|^2+\mu|\mathbf{k}|^6}{1+|\mathbf{k}|^2}} \end{pmatrix},$$

and

$$(\widehat{N(U,U)}(\mathbf{k}))_{j_1} = \int_{\mathbb{R}^2} \sum_{j_2,j_3\in\{1,2\}} \widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\widehat{U}_{j_2}(\mathbf{k}-\mathbf{l})\widehat{U}_{j_3}(\mathbf{l})d\mathbf{l},$$

with some coefficients $\widehat{\alpha}_{j_1,j_2,j_3}$ of the bilinear mapping N.

We define the new residual as

$$\mathrm{RES}(U) = -\partial_t U + \Lambda U + N(U,U).$$

Writing the solution as $U = \varepsilon\Psi + \varepsilon^\beta R$, where $\beta > 2$ we find R to satisfy

$$\partial_t R = \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon^\beta N(R,R) + \varepsilon^{-\beta}\mathrm{RES}(\varepsilon\Psi).$$

As a direct consequence of Lemma 7.4 we have the following result.

Lemma 7.5.
Let $\alpha > 0$ and $A_1, ..., A_4 \in C([0,T_0], W_\alpha)$ be solutions of the FWI *system* (7.6). *Then there exist $\varepsilon_0 > 0$, an approximation $\varepsilon\psi$ and a $C > 0$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{RES}(\varepsilon\psi(\,\cdot\,,t))\|_{L^1_g} \leq C\varepsilon^5.$$

In order to obtain estimates on the long time scale $\mathcal{O}(1/\varepsilon^2)$ with the help of Gronwall's inequality, we have to control the term $2\varepsilon N(\Psi, R)$. These modes are oscillatory and can be removed by a near identity change of variables, except those at the resonant wave vectors. These resonant modes can be handled with time dependent norms. Obviously, we have the same situation as in Chapter 6 since the problem also contains a resonance at the wave vector $\mathbf{k} = 0$ which is trivial but implies other resonances for the wave vectors $\mathbf{k} = \mathbf{k}_j$ that are non-trivial. We will solve this problem by introducing a $\mathbf{k}$-dependent scaling of an error function, followed by normal form transforms.

7.2.1 Extracting the dangerous terms I

Since the approximation $\varepsilon\Psi$ is only of order ε in δ neighborhoods of $\pm\mathbf{k}_j$ with $\delta > 0$ small, but independent of $0 < \varepsilon \ll 1$, we split the approximation $\varepsilon\Psi$ into

$$\varepsilon\Psi = \varepsilon\Psi_{c,+} + \varepsilon\Psi_{c,-} + \varepsilon^2\Psi_s,$$

where the support of $\varepsilon\Psi_{c,\pm}$ in Fourier space is located in the above mentioned δ neighborhoods of $\pm\mathbf{k}_j$.

We introduce a function

$$\widehat{\vartheta}_j(\mathbf{k}) = \begin{cases} 1, & \text{for } |\mathbf{k}| > |\mathbf{k}_j|/10, \\ \varepsilon + 10(1-\varepsilon)\frac{|\mathbf{k}|}{|\mathbf{k}_j|}, & \text{for } |\mathbf{k}| \leq |\mathbf{k}_j|/10, \end{cases}$$

and set $U = \varepsilon\Psi_c + \varepsilon^\beta \vartheta_j R$ with $\beta = 3$ to obtain

$$\partial_t R = \varLambda R + 2\varepsilon\vartheta_j^{-1} N(\Psi_c, \vartheta R) + \varepsilon^\beta \vartheta_j^{-1} N(\vartheta_j R, \vartheta_j R) + \varepsilon^{-\beta}\vartheta_j^{-1}\mathrm{RES}(\varepsilon\Psi). \quad (7.11)$$

The kernel of $2\varepsilon\vartheta_j^{-1} N(\Psi_c, \vartheta_j R)$ is given in Fourier space by

$$\varepsilon\widehat{\vartheta}_j^{-1}(\mathbf{k})\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})\widehat{\vartheta}_j(\mathbf{l}), \quad (7.12)$$

for j_1, j_2, $j_3 \in \{1, 2\}$ with an estimate $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k})| \leq C|\mathbf{k}|$.

Since (7.12) is of $\mathcal{O}(\varepsilon)$. We have

$$\varepsilon\vartheta_j^{-1} N(\Psi_c, \vartheta_j R) = \mathcal{O}(\varepsilon), \qquad \text{and} \qquad \varepsilon^2\vartheta_j^{-1} N(\Psi_s, \vartheta_j R) = \mathcal{O}(\varepsilon^2).$$

Hence, we only have to eliminate the term $\varepsilon\vartheta_j^{-1} N(\Psi_c, \vartheta_j R) = \mathcal{O}(\varepsilon)$ since terms of $\mathcal{O}(\varepsilon^2)$ do not cause intense growth in the error R on the time interval $[0, T_0/\varepsilon^2]$. At the end the term $2\varepsilon^2\vartheta_j^{-1} N(\Psi_s, \vartheta_j R) = \mathcal{O}(\varepsilon^2)$ has to be handled with the remaining terms of $\mathcal{O}(\varepsilon^2)$ with the help of Gronwall's inequality.

Therefore, R satisfies

$$\partial_t R = \varLambda R + 2\varepsilon\vartheta_j^{-1} N(\Psi_c, \vartheta_j R) + \mathcal{O}(\varepsilon^2). \quad (7.13)$$

The first normal form transform

The goal is to remove all non-resonant terms with a normal form transform $\widetilde{R} = R + \varepsilon B(\Psi_c, R)$ with B a smooth bilinear mapping. Terms are resonant to Ψ_c if both indices are r.

In order to extract the resonant modes we use the mode filter of the resonant wave vector

$$\widehat{E}_r(\mathbf{k}) = \begin{cases} 1 & \text{for } \mathbf{k} \in U_\delta(\widehat{\mathcal{K}}(\mathbf{k}_j)), \\ 0 & \text{else }, \end{cases}$$

for a $\delta > 0$ sufficiently small, but independent of $0 < \varepsilon \ll 1$, and the one of the non-resonant modes $\widehat{E}_n = 1 - \widehat{E}_r$. With these mode filters we split the error into two parts, namely $R = R_r + R_n$, with $R_r = E_r R_r$ and $R_n = E_n R_n$.

We now apply the normal form transform

$$\widetilde{R}_r = R_r + \varepsilon B_{r,n}(\Psi_c, R_n),$$

$$\widetilde{R}_n = R_n + \varepsilon B_{n,r}(\Psi_c, R_r) + \varepsilon B_{n,n}(\Psi_c, R_n),$$

with $B_{r,n}$, $B_{n,r}$ and $B_{n,n}$ as smooth bilinear mappings.

Using $\partial_t \Psi_c = \Lambda \Psi_c + \mathcal{O}(\varepsilon^2)$ we find

$$\begin{aligned}
\partial_t \widetilde{R}_r =& \Lambda \widetilde{R}_r - \Lambda \varepsilon B_{r,n}(\Psi_c, R_n) + \varepsilon B_{r,n}(\Lambda \Psi_c, R_n) + \varepsilon B_{r,n}(\Psi_c, \Lambda R_n) \\
&+ 2\varepsilon E_r \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_r) + 2\varepsilon E_r \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_n) + E_r \varepsilon^2 H_r, \\
\partial_t \widetilde{R}_n =& \Lambda \widetilde{R}_n - \Lambda \varepsilon B_{n,r}(\Psi_c, R_r) + \varepsilon B_{n,r}(\Lambda \Psi_c, R_r) + \varepsilon B_{n,r}(\Psi_c, \Lambda R_r) \\
&- \Lambda \varepsilon B_{n,n}(\Psi_c, R_n) + \varepsilon B_{n,n}(\Lambda \Psi_c, R_n) + \varepsilon B_{n,n}(\Psi_c, \Lambda R_n) \\
&+ 2\varepsilon E_n \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_r) + 2\varepsilon E_n \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_n) + E_n \varepsilon^2 H_n,
\end{aligned} \tag{7.14}$$

where

$$\begin{aligned}
\varepsilon^2 H_r =& 2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_r)) + 2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_n)) \\
&+ 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_r) + 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_n) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), R_r) \\
&+ \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), R_n) + \varepsilon^\beta \vartheta_j^{-1} B_{r,n}(\vartheta_j R_r, \vartheta_j R_r) + \varepsilon^\beta \vartheta_j^{-1} B_{r,n}(\vartheta_j R_n, \vartheta_j R_n) \\
&+ \varepsilon^\beta \vartheta_j^{-1} B_{r,n}(\vartheta_j R_r, \vartheta_j R_n) + \mathcal{O}(\varepsilon^{\beta-1}), \\
\varepsilon^2 H_n =& 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_r)) + 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_n)) \\
&+ 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_r) + 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j R_n) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), R_r) \\
&+ \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), R_n) + \varepsilon^\beta \vartheta_j^{-1} \widetilde{B}(\vartheta_j R_r, \vartheta_j R_r) + \varepsilon^\beta \vartheta_j^{-1} \widetilde{B}(\vartheta_j R_n, \vartheta_j R_n) \\
&+ \varepsilon^\beta \vartheta_j^{-1} \widetilde{B}(\vartheta_j R_r, \vartheta_j R_n) + \mathcal{O}(\varepsilon^{\beta-1}),
\end{aligned}$$

with $\widetilde{B} = B_{n,r} + B_{n,n}$ if we choose

$$B_{r,n} = B_{r,n}^+ + B_{r,n}^-, \quad B_{n,r} = B_{n,r}^+ + B_{n,r}^- \quad \text{and} \quad B_{n,n} = B_{n,n}^+ + B_{n,n}^-,$$

to satisfy

$$\begin{aligned}
0 =& \Lambda \varepsilon B_{r,n}^\pm(\Psi_{c,\pm}, R_n) - \varepsilon B_{r,n}^\pm(\Lambda \Psi_{c,\pm}, R_n) - \varepsilon B_{r,n}^\pm(\Psi_{c,\pm}, \Lambda R_n) \\
&- 2\varepsilon E_r \vartheta_j^{-1} N(\Psi_{c,\pm}, \vartheta_j R_n), \\
0 =& \Lambda \varepsilon B_{n,r}^\pm(\Psi_{c,\pm}, R_r) - \varepsilon B_{n,r}^\pm(\Lambda \Psi_{c,\pm}, R_r) - \varepsilon B_{n,r}^\pm(\Psi_{c,\pm}, \Lambda R_r) \\
&- 2\varepsilon E_n \vartheta_j^{-1} N(\Psi_{c,\pm}, \vartheta_j R_r), \\
0 =& \Lambda \varepsilon B_{n,n}^\pm(\Psi_{c,\pm}, R_n) - \varepsilon B_{n,n}^\pm(\Lambda \Psi_{c,\pm}, R_n) - \varepsilon B_{n,n}^\pm(\Psi_{c,\pm}, \Lambda R_n) \\
&- 2\varepsilon E_n \vartheta_j^{-1} N(\Psi_{c,\pm}, \vartheta R_n).
\end{aligned} \tag{7.15}$$

In Fourier space the j_1-th component of the nonlinear term $E_n \vartheta_j^{-1} N(\Psi_{c,\pm}, \vartheta_j R_n)$

can be written as

$$E_n\vartheta_j^{-1}(\widehat{N(\Psi_c,\vartheta_j R_n)})_{j_1}(\mathbf{k}) = \sum_{j_2,j_3\in\{1,2\}}\int_{\mathbb{R}^2}\widehat{E}_n(\mathbf{k})\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\frac{\widehat{\vartheta}_j(\mathbf{l})}{\widehat{\vartheta}_j(\mathbf{k})}\widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l}) \times \widehat{R}_{n,j_3}(\mathbf{l})d\mathbf{l},$$

where $|\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})| \leq C|\mathbf{k}| < \infty$ and $\mathbf{k},\mathbf{l}\in\mathbb{R}^2$. For $B_{n,n}$ we write

$$(\widehat{B_{n,n}(\Psi_c,R_n)})_{j_1} = \sum_{j_2,j_3\in\{1,2\}}\int_{\mathbb{R}^2}\frac{2\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k})-\mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l})-\mathrm{i}\omega_{j_3}(\mathbf{l})}\widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l})\widehat{R}_{n,j_3}(\mathbf{l})d\mathbf{l}.$$

For wave vectors $\mathbf{k}$ in the support of $\widehat{E}_n$, for wave vectors $\mathbf{k}-\mathbf{l}$ in the support of $\widehat{\Psi}_c$, and wave vectors $\mathbf{l}$ in the support of $\widehat{R}_n$ the denominator is bounded away from zero if $\delta > 0$ is chosen sufficiently small due to the definition of sets $\mathcal{K}(\pm\mathbf{k}_j)$ of resonant wave vectors. The same is true for the other five $B^{\pm}_{j_1,j_2}$.

The equations in (7.15) are satisfied if

$$\left|\frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k})-\mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l})-\mathrm{i}\omega_{j_3}(\mathbf{l})}\frac{\widehat{\vartheta}_j(\mathbf{l})}{\widehat{\vartheta}_j(\mathbf{k})}\right| \leq C = \mathcal{O}(1) < \infty. \tag{7.16}$$

In order to express the terms of the variable $\mathbf{k}$ alone we use the Lemma 4.8 such that the expression (7.16) can be written as

$$\left|\frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}_j,\mathbf{k}-\mathbf{k}_j)}{\mathrm{i}\omega_{j_1}(\mathbf{k})-\mathrm{i}\omega_{j_2}(\mathbf{k}_j)-\mathrm{i}\omega_{j_3}(\mathbf{k}-\mathbf{k}_j)}\frac{\widehat{\vartheta}_j(\mathbf{k}-\mathbf{k}_j)}{\widehat{\vartheta}_j(\mathbf{k})}\right| < C. \tag{7.17}$$

By the choice of ϑ_j the nontrivial resonances at $\mathbf{k} = \mathbf{k}_j$ are removed. For the resonance at $\mathbf{k}\to 0$ the expression (7.17) will be of $\mathcal{O}(\varepsilon^{-1})$, see Chapter 4.

Hence, (7.14) simplifies further into

$$\begin{aligned}\partial_t\widetilde{R}_r =& \Lambda\widetilde{R}_r + 2\varepsilon E_r\widetilde{N}(\Psi_c,R_r) + 2\varepsilon^2 B_{r,n}(\Psi_c,\vartheta_j^{-1}N(\Psi_c,\vartheta_j R_r) \\ &+ 2\varepsilon^2 B_{r,n}(\Psi_c,\vartheta_j^{-1}N(\Psi_c,\vartheta_j R_n) + E_r\varepsilon^2 H_r, \\ \partial_t\widetilde{R}_n =& \Lambda\widetilde{R}_n + 2\varepsilon^2\widetilde{B}(\Psi_c,\vartheta_j^{-1}N(\Psi_c,\vartheta_j R_r) + 2\varepsilon^2\widetilde{B}(\Psi_c,\vartheta_j^{-1}N(\Psi_c,\vartheta_j R_n) \\ &+ E_n\varepsilon^2 H_n,\end{aligned} \tag{7.18}$$

where

$$(\widehat{\widetilde{N}(\Psi_c,R_r)})_{j_1} = \sum_{j_2,j_3\in\{1,2\}}\int_{\mathbb{R}^2}\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k},\mathbf{k}-\mathbf{l},\mathbf{l})\widehat{\Psi}_{c,j_2}(\mathbf{k}-\mathbf{l})\widehat{R}_{r,j_3}(\mathbf{l})d\mathbf{l},$$

and H_r, H_n are defined above.

The term $2\varepsilon E_r \widetilde{N}(\Psi_c, R_r)$ comes from the additional resonances $\mathbf{k}_j$, $\widetilde{\mathbf{k}}_j$, $\widetilde{\widetilde{\mathbf{k}}}_j$ which will be removed in the second part. However, the terms

$$\begin{aligned}&2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_r)),\ 2\varepsilon^2 B_{r,n}(\Psi_c, \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_n)),\\ &2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_r)),\ 2\varepsilon^2 \widetilde{B}(\Psi_c, \vartheta_j^{-1} N(\Psi_c, \vartheta_j R_n)),\end{aligned}$$

with $\widetilde{B} = B_{n,r} + B_{n,n}$ can be of $\mathcal{O}(\varepsilon)$ for the wave vectors close to $\mathbf{k} = 0$. This is owned to ϑ_j^{-1}, which is of $\mathcal{O}(\varepsilon^{-1})$. For wave vectors $\mathbf{k}$ close to $\mathbf{k}_j$ these terms are of $\mathcal{O}(\varepsilon^2)$.

Since

$$\widehat{P}_1(\mathbf{k}) \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}_j(\mathbf{l})}{\widehat{\vartheta}_j(\mathbf{k})} \widehat{\alpha}_{j_3,j_4,j_5}(\mathbf{l}, \mathbf{l}-\mathbf{n}, \mathbf{n}) \frac{\widehat{\vartheta}_j(\mathbf{n})}{\widehat{\vartheta}_j(\mathbf{l})} = \mathcal{O}(1),$$

it remains to consider

$$\begin{aligned}S^{j_1}_{j,j_3,j_5}(\mathbf{k}) =& \widehat{P}_0(\mathbf{k}) \frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}_j(\mathbf{l})}{\widehat{\vartheta}_j(\mathbf{k})} \widehat{\alpha}_{j_3,j_4,j_5}(\mathbf{l}, \mathbf{l}-\mathbf{n}, \mathbf{n})\\ &\times \frac{\widehat{\vartheta}_j(\mathbf{n})}{\widehat{\vartheta}_j(\mathbf{l})}.\end{aligned} \tag{7.19}$$

We use again Lemma 4.8 in order to express the kernel

$$\frac{\widehat{\alpha}_{j_1,j_2,j_3}(\mathbf{k}, \mathbf{k}-\mathbf{l}, \mathbf{l})}{\mathrm{i}\omega_{j_1}(\mathbf{k}) - \mathrm{i}\omega_{j_2}(\mathbf{k}-\mathbf{l}) - \mathrm{i}\omega_{j_3}(\mathbf{l})} \frac{\widehat{\vartheta}_j(\mathbf{l})}{\widehat{\vartheta}_j(\mathbf{k})} \widehat{\alpha}_{j_3,j_4,j_5}(\mathbf{l}, \mathbf{l}-\mathbf{n}, \mathbf{n}) \frac{\widehat{\vartheta}_j(\mathbf{n})}{\widehat{\vartheta}_j(\mathbf{l})},$$

in terms of $\mathbf{k}$ alone.

We split $\widehat{\Psi}_c$ in two parts, namely $\widehat{\Psi}_c = \frac{1}{\varepsilon^2}\widehat{\Psi}_{c,+} + \frac{1}{\varepsilon^2}\widehat{\Psi}_{c,-}$ with $\widehat{\Psi}_{c,+}$ concentrated at $\mathbf{k}_j$ and $\widehat{\Psi}_{c,-}$ concentrated at $-\mathbf{k}_j$. The combination of $d\mathbf{k}_j$ with $-d\mathbf{k}_j$ for $d \in \{+,-\}$ leads to an $\mathcal{O}(1)$-bound for (7.19). The combination of $d\mathbf{k}_j$ with $d\mathbf{k}_j$ for $d \in \{+,-\}$ leads to an $\mathcal{O}(1/\varepsilon)$-bound, see Chapter 4.

Hence, (7.18) simplifies only into

$$\begin{aligned}\partial_t \widetilde{R}_r =& \Lambda \widetilde{R}_r + 2\varepsilon E_r \widetilde{N}(\Psi_c, R_r) + 2\varepsilon^2 B_{r,n}(\Psi_{c,+}, \vartheta_j^{-1} N(\Psi_{c,+}, \vartheta_j R_r)) \\ &+ 2\varepsilon^2 B_{r,n}(\Psi_{c,+}, \vartheta_j^{-1} N(\Psi_{c,+}, \vartheta_j R_n)) + 2\varepsilon^2 B_{r,n}(\Psi_{c,-}, \vartheta_j^{-1} \\ &\times N(\Psi_{c,-}, \vartheta_j R_r)) + 2\varepsilon^2 B_{r,n}(\Psi_{c,-}, \vartheta_j^{-1} N(\Psi_{c,-}, \vartheta_j R_n)) + E_r \varepsilon^2 H_r, \\ \partial_t \widetilde{R}_n =& \Lambda \widetilde{R} + 2\varepsilon^2 \widetilde{B}(\Psi_{c,+}, \vartheta_j^{-1} N(\Psi_{c,+}, \vartheta_j R_r)) + 2\varepsilon^2 \widetilde{B}(\Psi_{c,+}, \vartheta_j^{-1} \\ &\times N(\Psi_{c,+}, \vartheta_j R_n)) + 2\varepsilon^2 \widetilde{B}(\Psi_{c,-}, \vartheta_j^{-1} N(\Psi_{c,-}, \vartheta_j R_r)) \\ &+ 2\varepsilon^2 \widetilde{B}(\Psi_{c,-}, \vartheta_j^{-1} N(\Psi_{c,-}, \vartheta_j R_n)) + E_n \varepsilon^2 H_n,\end{aligned} \tag{7.20}$$

where H_r and N_n defined above.

The second normal form transform

In order to eliminate terms of $\mathcal{O}(\varepsilon^2)$ in (7.20) we use again a normal form transform

$$\begin{aligned}\check{R}_r =& \widetilde{R}_r + \varepsilon \bar{B}_{r,n}(\Psi_{c,+}, \Psi_{c,+}, R_n) + \varepsilon \bar{B}_{r,n}(\Psi_{c,-}, \Psi_{c,-}, R_n), \\ \check{R}_n =& \widetilde{R}_n + \varepsilon \bar{B}_{n,n}(\Psi_{c,+}, \Psi_{c,+}, R_n) + \varepsilon \bar{B}_{n,r}(\Psi_{c,+}, \Psi_{c,+}, R_r) \\ &+ \varepsilon \bar{B}_{n,n}(\Psi_{c,-}, \Psi_{c,-}, R_n) + \bar{B}_{n,r}(\Psi_{c,-}, \Psi_{c,-}, R_r),\end{aligned}$$

where

$$\begin{aligned}&\bar{B}^{j_1}_{r,n}(\Psi_{c,d}, \Psi_{c,d}, R_n)(\mathbf{k}) \\ &\quad = \sum_{j_3, j_5 \in \{1,2\}} \int_{\mathbb{R}^4} \frac{\varepsilon S^{j_1}_{j,j_3,j_5}(\mathbf{k})}{\omega_{j_1}(\mathbf{k}) - \omega_d(d\mathbf{k}_j) - \omega_d(d\mathbf{k}_j) - \omega_{j_5}(\mathbf{k} - 2d\mathbf{k}_j)} \\ &\qquad \times \Psi_{c,d}(\mathbf{k}-\mathbf{l}) \Psi_{c,d}(\mathbf{l}-\mathbf{n}) R_{n,j_5}(\mathbf{n}) d\mathbf{n} d\mathbf{l},\end{aligned}$$

$$\begin{aligned}&\bar{B}^{j_1}_{n,z}(\Psi_{c,d}, \Psi_{c,d}, R_z)(\mathbf{k}) \\ &\quad = \sum_{j_3, j_5 \in \{1,2\}} \int_{\mathbb{R}^4} \frac{\varepsilon S^{j_1}_{j,j_3,j_5}(\mathbf{k})}{\omega_{j_1}(\mathbf{k}) - \omega_d(d\mathbf{k}_j) - \omega_d(d\mathbf{k}_j) - \omega_{j_5}(\mathbf{k} - 2d\mathbf{k}_j)} \\ &\qquad \times \Psi_{c,d}(\mathbf{k}-\mathbf{l}) \Psi_{c,d}(\mathbf{l}-\mathbf{n}) R_{z,j_5}(\mathbf{n}) d\mathbf{n} d\mathbf{l},\end{aligned}$$

with $|S^{j_1}_{j,j_3,j_5}(\mathbf{k})| \leq C$, $z \in \{r, n\}$, $d \in \{+, -\}$ and $\mathbf{k}$, $\mathbf{l}$, $\mathbf{n} \in \mathbb{R}^2$. The denominator is non zero due to the combination of $\Psi_{c,+}$ with $\Psi_{c,+}$ and $\Psi_{c,-}$ with $\Psi_{c,-}$. The transformation works as before and so the term of $\mathcal{O}(\varepsilon^2)$ can be eliminated with $\bar{B}^j_{r,n} = \mathcal{O}(1)$ and $\bar{B}^j_{n,z} = \mathcal{O}(1)$ using $\varepsilon S = \mathcal{O}(1)$.

After the normal form transform we have

$$\begin{aligned}\partial_t \check{R}_r =& \Lambda \check{R}_r + 2\varepsilon E_r \widetilde{N}(\Psi_c, R_r) + E_r \varepsilon^2 H_r,\\ \partial_t \check{R}_n =& \Lambda \check{R}_n + E_n \varepsilon^2 H_n.\end{aligned} \tag{7.21}$$

Substituting R by $\check{R}$ yields

$$\begin{aligned}\partial_t \check{R}_r =& \Lambda \check{R}_r + 2\varepsilon E_r \widetilde{N}(\Psi_c, \check{R}_r) + E_r \varepsilon^2 \check{H}_r,\\ \partial_t \check{R}_n =& \Lambda \check{R}_n + E_n \varepsilon^2 \check{H}_n,\end{aligned} \tag{7.22}$$

where

$$\begin{aligned}\varepsilon^2 \check{H}_r =& 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j \check{R}_r) + 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j \check{R}_n) + \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), \check{R}_r)\\ &+ \varepsilon B_{r,n}(\mathcal{O}(\varepsilon^2), \check{R}_n) + \varepsilon^\beta \vartheta_j^{-1} B_{r,n}(\vartheta_j \check{R}_r, \vartheta_j \check{R}_r) + \varepsilon^\beta \vartheta_j^{-1} B_{r,n}(\vartheta_j \check{R}_n, \vartheta_j \check{R}_n)\\ &+ \varepsilon^\beta \vartheta_j^{-1} B_{r,n}(\vartheta_j \check{R}_r, \vartheta_j \check{R}_n) + \mathcal{O}(\varepsilon^{\beta-1}),\\ \varepsilon^2 \check{H}_n =& 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j \check{R}_r) + 2\varepsilon^2 \vartheta_j^{-1} N(\Psi_s, \vartheta_j \check{R}_n) + \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), \check{R}_r)\\ &+ \varepsilon \widetilde{B}(\mathcal{O}(\varepsilon^2), \check{R}_n) + \varepsilon^\beta \vartheta_j^{-1} \widetilde{B}(\vartheta_j \check{R}_r, \vartheta_j \check{R}_r) + \varepsilon^\beta \vartheta_j^{-1} \widetilde{B}(\vartheta_j \check{R}_n, \vartheta_j \check{R}_n)\\ &+ \varepsilon^\beta \vartheta_j^{-1} \widetilde{B}(\vartheta_j \check{R}_r, \vartheta_j \check{R}_n) + \mathcal{O}(\varepsilon^{\beta-1}),\end{aligned}$$

with $\widetilde{B} = B_{n,r} + B_{n,n}$. It remains to eliminate the term $2\varepsilon E_r \widetilde{N}(\Psi_c, R_r)$. In order to do so we proceed as in Chapter 5.

7.2.2 Extracting the dangerous terms II

We need the following lemmas.

Lemma 7.6.
Let $\alpha > 0$ and $A_1, ..., A_4 \in C([0, T_0], W_\alpha)$ be solutions of the FWI *system* (7.6). *Then there exist a $C > 0$ such that the following holds. For all $\varepsilon \in (0, 1]$ the maps $B^{\pm}_{n,n}(\Psi_{c,\pm}, E_n \cdot)$, $B^{\pm}_{n,r}(\Psi_{c,\pm}, E_r \cdot)$ and $B^{\pm}_{r,n}(\Psi_{c,\pm}, E_n \cdot)$ are bounded linear mappings from L^1_g to L^1_g satisfying*

$$\|B^{\pm}_{n,n}(\Psi_{c,\pm}, E_n \check{R}_n)\|_{L^1_g} \leq C \|\check{R}_n\|_{L^1_g}.$$

Lemma 7.7.
Let $\|\check{R}\|_{L^1_g} \leq Z$. There exist constants C_1, C_3 independent of Z and $\varepsilon \in (0, 1]$ and a function $C_2(Z)$ independent of $\varepsilon \in (0, 1]$ such that

$$\|\varepsilon^2 \check{H}\|_{L^1_g} \leq C_1 \varepsilon^2 \|\check{R}\|_{L^1_g} + C_2(Z) \varepsilon^3 \|\check{R}\|_{L^1_g} + C_3 \varepsilon^2.$$

We now use the variation of constants formula to obtain

$$\begin{aligned} \check{R}_r(t) &= \int_0^t e^{\Lambda(t-\tau)} \big(2\varepsilon E_r \widetilde{N}(\Psi_c, \check{R}_r) + \mathcal{O}(\|\varepsilon^2 \check{H}_r(\tau)\|_{L^1_g})\big) d\tau, \\ \check{R}_n(t) &= \int_0^t e^{\Lambda(t-\tau)} \mathcal{O}(\|\varepsilon^2 \check{H}_n(\tau)\|_{L^1_g}) d\tau, \end{aligned} \tag{7.23}$$

where

$$\begin{aligned} \varepsilon^2 \|\check{H}_r\|_{L^1_g} \leq & C_1 \varepsilon^2 \|\check{R}_r\|_{L^1_g} + C_1 \varepsilon^2 \|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3 \|\check{R}_r\|_{L^1_g} \|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3 \|\check{R}_r\|^2_{L^1_g} \\ & + C_2(Z)\varepsilon^3 \|\check{R}_n\|^2_{L^1_g} + C_3 \varepsilon^2, \\ \varepsilon^2 \|\check{H}_n\|_{L^1_g} \leq & C_1 \varepsilon^2 \|\check{R}_r\|_{L^1_g} + C_1 \varepsilon^2 \|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3 \|\check{R}_r\|_{L^1_g} \|\check{R}_n\|_{L^1_g} + C_2(Z)\varepsilon^3 \|\check{R}_r\|^2_{L^1_g} \\ & + C_2(Z)\varepsilon^3 \|\check{R}_n\|^2_{L^1_g} + C_3 \varepsilon^2. \end{aligned}$$

It remains to estimate the term $2\varepsilon \int_0^t e^{\Lambda(t-\tau)} E_r \widetilde{N}(\Psi_c, \check{R}_r)(\tau) d\tau$ in L^1_g-norm. We find

$$\begin{aligned} & \|2\varepsilon \int_0^t e^{\Lambda(t-\tau)} E_r \widetilde{N}(\Psi_c, \check{R}_r)(\tau) d\tau \|_{L^1_{g(t)}} \\ \leq & \int_{\mathbb{R}^2} 2\varepsilon \int_0^t \left| e^{\Lambda(t-\tau)} E_r \widetilde{N}(\Psi_c, \check{R}_r)(\tau) \right| d\tau g(\mathbf{k}, t) d\mathbf{k} \\ \leq & \int_{\mathbb{R}^2} 2\varepsilon \int_0^t \left| e^{\Lambda(t-\tau)} \frac{g(\mathbf{k}, t)}{g(\mathbf{k}, \tau)} E_r \widetilde{N}(\Psi_c, \check{R}_r)(\tau) \right| g(\mathbf{k}, \tau) d\tau d\mathbf{k} \\ \leq & 2\varepsilon \int_0^t \left| \int_{\mathbb{R}^2} e^{\Lambda(t-\tau)} \frac{g(\mathbf{k}, t)}{g(\mathbf{k}, \tau)} E_r \widetilde{N}(\Psi_c, \check{R}_r)(\tau) \right| g(\mathbf{k}, \tau) d\mathbf{k} d\tau \\ \leq & 2\varepsilon \int_0^t \sup_{\mathbf{k} \in \mathbb{R}^2} \left| e^{\Lambda(t-\tau)} \frac{g(\mathbf{k}, t)}{g(\mathbf{k}, \tau)} E_r \right| \int_{\mathbb{R}^2} \left| \widetilde{N}(\Psi_c, \check{R}_r)(\tau) \right| g(\mathbf{k}, \tau) d\mathbf{k} d\tau \\ \leq & 2C\varepsilon \int_0^t e^{-\kappa(t-\tau)} \|\Psi_c\|_{L^1_{g(\tau)}} \|\check{R}_r(\tau)\|_{L^1_{g(\tau)}} d\tau, \end{aligned}$$

for a $\kappa > 0$. Due to the spatial scaling the weight function is

$$g(\mathbf{k}, t) = \frac{1}{\sup_{m_1, \ldots, m_n \in \mathbb{Z}} |e^{-\left(\frac{\alpha'}{\varepsilon^2} - \frac{\alpha' t}{T_0}\right) |\mathbf{k} - (m_1 \mathbf{k}_1 + \ldots + m_n \mathbf{k}_n)|}|}, \tag{7.24}$$

Therefore, $\kappa(t-\tau)$ is of $\mathcal{O}(1)$ instead of $\mathcal{O}(\varepsilon)$ as we have in Chapter 6. Introducing the notation $\mathcal{R}_z(t) := \sup_{0 \leq \tau \leq t} \|\check{R}_z(\tau)\|_{L^1_g(\tau)}$ for $z \in \{r, n\}$ and $\mathcal{R}(t) = \mathcal{R}_n(t) + \mathcal{R}_r(t)$

gives

$$\begin{aligned}\mathcal{R}_n(t) \leq& C\int_0^t \Big(C_1\varepsilon^2\mathcal{R}(\tau) + C_2(M)\varepsilon^3\mathcal{R}(\tau)^2 + C_3\varepsilon^2\Big)d\tau,\\ \mathcal{R}_r(t) \leq& 2C\int_0^t e^{-\kappa(t-\tau)}\mathcal{R}_r(\tau)d\tau + C\int_0^t \Big(C_1\varepsilon^2\mathcal{R}(\tau) + C_2(M)\varepsilon^3\mathcal{R}(\tau)^2 + C_3\varepsilon^2\Big)d\tau.\end{aligned}$$

Adding both inequalities and choosing $C_2(M)\varepsilon\mathcal{R}(\tau) \leq 1$ yields

$$\begin{aligned}\mathcal{R}(t) \leq& 2C\int_0^t \Big((C_1+1)\varepsilon^2\mathcal{R}(\tau) + C_3\varepsilon^2\Big)d\tau + 2C\varepsilon\int_0^t e^{-\kappa(t-\tau)}\varepsilon^2\mathcal{R}(\tau)d\tau,\\ \leq& 2C\int_0^t \Big((C_1+1)\varepsilon^2\mathcal{R}(\tau) + C_3\varepsilon^2\Big)d\tau + 2C\varepsilon\kappa^{-1}\mathcal{R}(\tau).\end{aligned}$$

The last term on the right-hand side can be made small by choosing $\varepsilon > 0$ sufficiently small without further restriction on time. We have

$$\mathcal{R}(t) \leq C\int_0^t \Big((C_1+1)\varepsilon^2\mathcal{R}(\tau) + C_3\varepsilon^2\Big)d\tau.$$

Applying Gronwall's inequality yields

$$\mathcal{R}(t) \leq C_3 T_0 e^{C(C_1+1)T_0} =: C_{\mathcal{R}},$$

independent of $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 > 0$ had to be chosen so small that $C_2(C_{\mathcal{R}})\varepsilon C_{\mathcal{R}} \leq 1$.

Finally, we showed that in case of an approximation by the FWI system the same approach works as in Chapter 6 and due to the different spatial scaling an even stronger result can be established.

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