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General Interface Problems—I

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We study transmission problems for elliptic operators of order $2m$ with general boundary and interface conditions, introducing new covering conditions. This allows to prove solvability, regularity and asymptotics of solutions in weighted Sobolev spaces. We give some numerical examples for the location of the singular exponents.

1. Introduction

Boundary value problems in non-smooth domains are extensively studied in the literature; but for the applications interface problems (also called transmission problems) often appear, for instance, in solid mechanics if a body consists of composite materials. Unfortunately, in the literature, transmission problems are only studied in some particular cases (essentially, for second-order operators, see e.g. [3, 10, 17, 12, 8, 24, 19] and the references cited therein). So our main goal is to extend the theory of boundary value problems in non-smooth domains to general interface problems for operators of arbitrary order with non-constant coefficients. We restrict ourselves to domains of the plane $\mathbb{R}^2$ or the space $\mathbb{R}^3$, since these are the realistic domains for the mechanical applications. Obviously, the results given here could be extended to domains of $\mathbb{R}^n$, for arbitrary $n \geqslant 2$.

Let us mention that some mechanical problems lead to transmission problems set on 2-D networks. Such problems were studied in [22, 23] for the Laplace operator and the biharmonic one. In these papers, we actually see that boundary value problems in non-smooth domains and transmission problems are in the same framework of networks. Therefore, we certainly could extend the results stated here to 2-D or 3-D networks.

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Our work is divided in two parts: In Part I, we introduce the class of interface problems in which we shall consider and study the regularity results in weighted Sobolev spaces for homogeneous operators with constant coefficients. We also give some numerical results about the eigenvalue problems. In Part II, we shall extend the previous results to general operators in usual Sobolev spaces. Finally, we shall consider the stabilization procedure which is necessary, when unstable decompositions appear near critical angles of the conical points.

The plan of Part I is the following: In section 2, we define the general class of interface problems which we shall study. Along the external boundary, we impose classical boundary conditions satisfying the so-called Shapiro–Lopatinskii conditions, while on the interfaces we define general transmission conditions and introduce a new covering condition. Roughly speaking, this condition means that at every point of the interfaces we transform locally our interface problem into a system of boundary value problems , which satisfies the Shapiro–Lopatinskii condition. These problems will be called regular elliptic transmission problems.

For the applications, it is important to consider a weak formulation of interface problems. With the help of Green's formula, we describe 'weak' transmission problems, and give a necessary and sufficient condition in order to be regular elliptic ones (based on Agmon's arguments [1, 27]).

In section 3, the Shapiro–Lopatinskii conditions allow us to use Agranovitch–Visik's results to get, as for boundary value problems, the solvability, regularity and asymptotics of solutions in weighted Sobolev spaces.

Using the previous results and a lifting trace theorem we prove in section 4 that the weak solution of a transmission problem for homogeneous operators with constant coefficients admits a decomposition into singular and regular parts in weighted Sobolev spaces. This is made in a more or less usual way, except in dimension 2 and data in H^k, k a positive integer (due to the limit case of the Sobolev imbedding theorem), where we use an interpolation argument (see [22] for a particular situation). Let us also notice that our boundary and interface conditions are not preserved by multiplication by cut-off functions. This leads to some difficulties, which are not usual.

In section 5, we finally give some numerical results about the location of the eigenvalues for some practical transmission problems in dimension 2, for instance, we consider the Laplace operator, the biharmonic operator and the Lamé system with different boundary and interface conditions for variable angles. In a forthcoming work, we shall consider some three-dimensional examples.

2. Formulation of the problem

2.1. *The domains*

We shall consider interface problems in the following two- or three-dimensional domains (which is a realistic case).

In dimension 2, we suppose that Ω is a bounded domain consisting of N parts Ω_i, $i = 1, \ldots, N$, such that

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i.$$

We suppose that the boundary of Ω_i is Lipschitz-continuous and is a smooth curvilinear polygons, i.e.

$$\partial\Omega_i = \bigcup_{q=1}^{Q_i} \bar{\gamma}_{iq},$$

where the sides γ_{iq} are smooth curves. The vertices of Ω_i will simply be the intersection between two consecutive sides.

The compatibility conditions between the Ω_i's are the following ones (see Fig. 1): for all $i, j \in \{1, \ldots, N\}$: $i \neq j$, one of the following holds:

(i) $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$,
(ii) $\bar{\Omega}_i \cap \bar{\Omega}_j$ is a whole common side, denoted by $\bar{\Gamma}_{ij}$ (it is more convenient to suppose that Γ_{ij} is open),
(iii) $\bar{\Omega}_i \cap \bar{\Omega}_j$ is a common vertex.

This definition is in accordance with the notion of two-dimensional topological networks introduced in Ref. [8].

Remark. In the above definition, the case of an interface which is smooth except at a finite number of points is included, introducing then some new artificial interfaces (see Fig. 2).

In dimension 3, for simplicity, we suppose that Ω is the union of two domains Ω_1, Ω_2:

$$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2,$$

where Ω_1 is completely included into Ω. The boundary of Ω_1 (resp. Ω_2) is supposed to be smooth except at one point 0, where Ω_1 (resp. Ω_2) coincides with a smooth cone K_1 (resp. K_2) of $\mathbb{R}^3$ (i.e. the intersection of K_1 (resp. K_2) with the unit sphere is a domain G_1 (resp. G_2) with a smooth boundary) (see Fig. 3).

In this paper, we exclude domains with edges (see Fig. 4); nevertheless, it is possible to consider them using Dauge's techniques of [7], for instance.

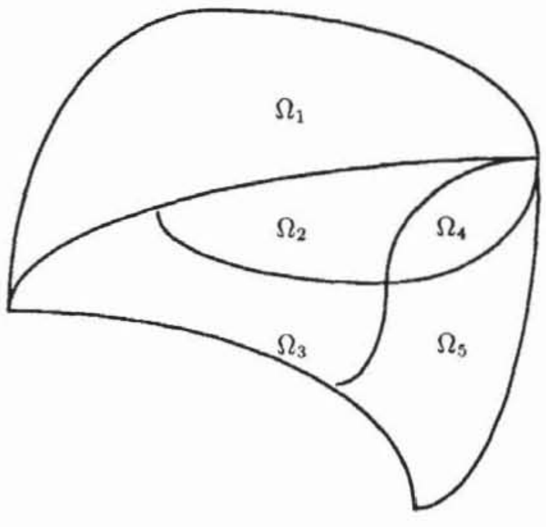

Fig. 1

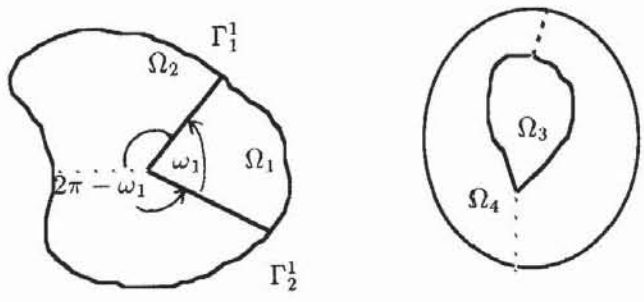

Fig. 2.

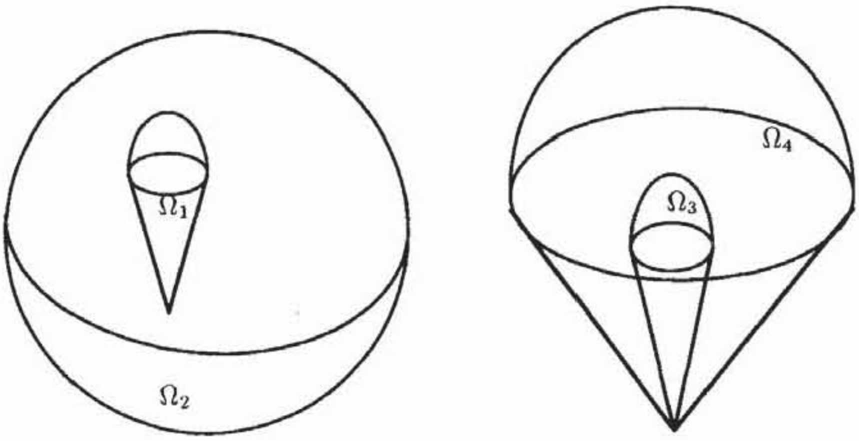

Fig. 3.

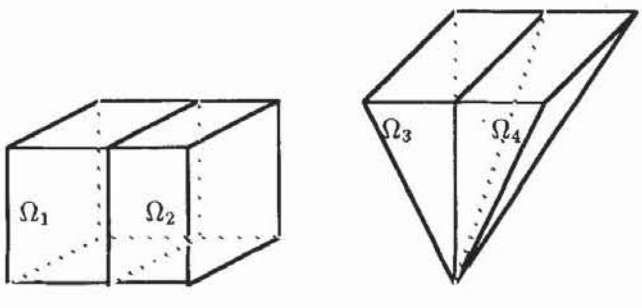

Fig. 4.

Let us introduce some notations, which we shall use in the sequel:

(1) $\mathscr{S}$ will be the set of the vertices of Ω (i.e. the union of all vertices of the Ω_i's); obviously, in dimension 3, it is reduced to one point 0.

(2) For a fixed $S \in \mathscr{S}$, $\mathscr{N}_S$ will be the set of integers i such that the face Ω_i contains S, in other words:

$$\mathscr{N}_S = \{i \in \{1, \ldots, N\}: S \in \bar{\Omega}_i\} \text{ in dimension 2,}$$

$$\mathscr{N}_S = \{1, 2\} \text{ in dimension 3.}$$

(3) The indices i, i' always run from 1 to N in dimension 2 and from 1 to 2 in dimension 3 without any comment.

(4) For a function u defined on Ω, we denote by u_i its restriction to Ω_i.
(5) The dimension of Ω will be denoted by n.
(6) In dimension 3, in order to have coherent notations, we set:

$$\gamma_{11} = \partial\Omega_1 = \text{the boundary of } \Omega_1,$$

$$\partial\Omega_2 = \bar{\gamma}_{21} \cup \bar{\gamma}_{22},$$

where γ_{22} is the external boundary of Ω_2 (or equivalently the boundary of Ω) and γ_{21} is the common boundary between Ω_1 and Ω_2.

We now define some Sobolev spaces in Ω.

Definition 2.1. *Let k be a non-negative integer, p a real number $\geqslant 1$. We set*

$$\mathscr{W}^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon u_i \in W^{k,p}(\Omega_i)\},$$

it is a Banach space with the norm

$$\| u \|_{k,p,\Omega} := \left(\sum_i \| u_i \|^p_{W^{k,p}(\Omega_i)} \right)^{1/p}.$$

For $p = 2$, it is a Hilbert space and we simply write it as $\mathscr{H}^k(\Omega)$.
We now define

$$C^\infty_v(\bar{\Omega}_i) = \{u \in C^\infty(\bar{\Omega}_i) \colon u = 0 \text{ in a neighbourhood of the vertices of } \Omega_i\},$$

$$H^k_v(\Omega_i) = \overline{C^\infty_v(\bar{\Omega}_i)}^{H^k(\Omega_i)},$$

$$\mathscr{H}^k_v(\Omega) = \{u \in \mathscr{H}^k(\Omega) \colon u_i \in H^k_v(\Omega_i)\},$$

this last one is clearly a closed subspace of $\mathscr{H}^k(\Omega)$.

Let us remark that $u \in \mathscr{H}^k_v(\Omega)$ satisfies

$$D^\alpha u_i(S) = 0, \quad \forall \alpha \colon |\alpha| < k - 1, \quad S \in \mathscr{S},\ i \in \mathscr{N}_S. \tag{2.1}$$

Due to [7, Theorem A.7] in dimension 3 and (3.1) of [7] in dimension 2, we have

$$\mathscr{H}^k_v(\Omega) = \{u \in \mathscr{H}^k(\Omega) \text{ satisfying (2.1)}\}.$$

This space $\mathscr{H}^k_v(\Omega)$ is introduced in order to cancel the corner effects in Green's formula, for instance.

2.2. *The operators*

Condition I. Let A_i be an elliptic differential operator of order $2m$ on Ω_i (properly elliptic in dimension 2) given by

$$A_i(x, D) = \sum_{|\alpha| \leqslant 2m} a^i_\alpha(x) D^\alpha, \tag{2.2}$$

with smooth coefficients, i.e.

$$a^i_\alpha := a_\alpha|_{\bar{\Omega}_i} \in C^\infty(\bar{\Omega}_i).$$

We shall consider classical boundary conditions on the external boundary of Ω, i.e. if γ_{iq} is not a common side of two subdomains (in the sequel, we simply write $\gamma_{iq} \in \mathscr{E}$), then we consider m boundary operators B_{iqj}, $j = 0, \ldots, m-1$, defined by

$$B_{iqj}(x, D) = \sum_{|\alpha| \leqslant m_{iqj}} b_{iqj\alpha}(x) D^\alpha, \tag{2.3}$$

with $b_{iqj\alpha} \in C^\infty(\bar{\gamma}_{iq})$, m_{iqj} is the order of the operator B_{iqj}, supposed to be $\leqslant 2m - 1$.

Condition II. As usual, we suppose that the system $\underline{B}_{iq} := \{B_{iqj}\}_{j=0}^{m-1}$ covers the operators A_i on $\gamma_{iq} \in \mathscr{E}$ (see [13, Definition 2.1.4] for the exact terminology).

Conversely, the interface conditions are defined on the 'internal boundary' of Ω, in other words, let

$$\gamma_{iq} = \gamma_{i'q'}$$

be the common side of Ω_i and $\Omega_{i'}$ (in the sequel, we write $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$). Then we suppose given $4m$ operators, B_{iqj} and $B_{i'q'j}$, $j = 0, \ldots, 2m-1$ defined by

$$B_{iqj}(x, D) = \sum_{|\alpha| \leqslant m_{iqj}} b_{iqj\alpha}(x) D^\alpha, \tag{2.4}$$

$$B_{i'q'j'}(x, D) = \sum_{|\alpha| \leqslant m_{iqj}} b_{i'q'j\alpha}(x) D^\alpha, \tag{2.5}$$

where $b_{iqj\alpha}, b_{i'q'j\alpha} \in C^\infty(\bar{\gamma}_{iq})$, m_{iqj} being the order of both operators B_{iqj} and $B_{i'q'j}$.

The covering condition is now expressed as follows: denote by A_i^0 the principal part of A_i on Ω_i:

$$A_i^0(x, D) = \sum_{|\alpha| = 2m} a_\alpha^i(x) D^\alpha.$$

Analogously, B_{iqj}^0 and $B_{i'q'j}^0$ are the principal parts of B_{iqj} and $B_{i'q'j}$.

For every couple of linearly independent vectors (ζ, ζ') in $\mathbb{R}^n$, the ellipticity condition implies that the polynomial (in τ)

$$A_i^0(x, \zeta + \tau\zeta')$$

has m roots with positive imaginary parts denoted by $\tau_{ik}^+(x, \zeta, \zeta')$, $k = 1, \ldots, m$.

Condition III. For all $x \in \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$, all vectors $\zeta \in \mathbb{R}^n$ tangent to γ_{iq} at x and all vectors $\zeta' \in \mathbb{R}^n$ normal to γ_{iq} at x, the rows of the matrix

$$\begin{gathered}((B_{iqj}^0(x, \zeta + \tau\zeta'))_{j=0}^{2m-1} (B_{i'q'j}^0(x, \zeta - \tau\zeta'))_{j=0}^{2m-1}) \\ \times \begin{pmatrix} A_{i'}^0(x, \zeta - \tau\zeta') & 0 \\ 0 & A_i^0(x, \zeta + \tau\zeta') \end{pmatrix}\end{gathered}$$

are linearly independent modulo the polynomial

$$\prod_{k=1}^{m} (\tau - \tau_{ik}^+(x, \zeta, \zeta'))(\tau - \tau_{ik}^+(x, \zeta, -\zeta')).$$

In other words, sending a neighbourhood $\theta_{i'}$ of x in Ω_i into a neighbourhood θ_i of x in Ω_i by a mirror technique (i.e. flatting the boundary and using the change of variable $(x', x_n) \to (x', -x_n)$), this covering condition III says that the system

$$((B_{iqj}(x, D))_{j=0}^{2m-1} (\tilde{B}_{i'q'j}(x, D))_{j=0}^{2m-1})$$

covers the system

$$\begin{pmatrix} A_i(x, D) & 0 \\ 0 & \tilde{A}_{i'}(x, D) \end{pmatrix}$$

in the sense of Douglis–Nirenberg, where $\tilde{B}_{i'q'j}(x, D)$ (resp. $\tilde{A}_{i'}(x, D)$) is the operator obtained from $B_{i'q'j}(x, D)$ (resp. $A_{i'}(x, D)$) by the above change of variables.

In this paper, we study the following interface problem:

$$A_i u_i = f_i \quad \text{in } \Omega_i,\ \forall i = 1, \dots, N, \tag{2.6}$$

$$B_{iqj} u_i = g_{iqj} \quad \text{on } \gamma_{iq},\ \forall j = 0, \dots, m-1,\ \gamma_{iq} \in \mathscr{E}, \tag{2.7}$$

$$B_{iqj} u_j - B_{i'q'j} u_{i'} = g_{ii'j} \quad \text{on } \gamma_{iq} = \gamma_{i'q'},\ \forall j = 0, \dots, 2m-1,\ \gamma_{iq} \in \mathscr{I}. \tag{2.8}$$

Actually, in the forthcoming sections, we shall investigate the solvability, regularity and asymptotic expansion of solution of this problem in weighted Sobolev spaces and in standard ones.

Since conditions I–III guarantee that we can handle transmission problems as elliptic boundary value problems for systems, we make the following definition.

Definition 2.2. *We shall say that problem* (2.6)–(2.8) *is a regular elliptic transmission problem if conditions I–III are satisfied.*

2.3. *Green's formula*

Let A_i be elliptic partial differential operators of order $2m$ satisfying condition I and given in divergence form, i.e. let $a^i_{\alpha\beta} \in C^\infty(\bar{\Omega}_i)$, for $|\alpha|, |\beta| \leqslant m$, be such that A_i given by (2.2) admits also the expression

$$A_i = \sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\beta|} D^\beta (a^i_{\alpha\beta} D^\alpha). \tag{2.9}$$

We now define the sesquilinear form a_i on $H^m(\Omega_i)$ associated with A_i as follows:

$$a_i(u, v) = \sum_{|\alpha|, |\beta| \leqslant m} \int_{\Omega_i} a^i_{\alpha\beta}(x) D^\alpha u D^\beta \bar{v}\, \mathrm{d}x. \tag{2.10}$$

so that for every $u, v \in \mathscr{D}(\Omega_i)$, we have

$$a_i(u, v) = \int_{\Omega_i} A_i u \bar{v}\, \mathrm{d}x. \tag{2.11}$$

On each side γ_{iq} of Ω_i, we fix a Dirichlet system $\{F_{iqj}\}_{j=0}^{m-1}$ of order m (see [13, Definition 2.2.1]) with coefficients in $C^\infty(\bar{\gamma}_{iq})$ and, without loss of generality, we may suppose that the order of F_{iqj} is j.

Lemma 2.3. *For all sides γ_{iq} of Ω_i, there exists a normal system $\{\Phi_{iqj}\}_{j=0}^{m-1}$ with coefficients in $C^\infty(\bar{\gamma}_{iq})$, the order of Φ_{iqj} being $2m-1-j$, such that*

$$\int_{\Omega_i} A_i u \bar{v}\, \mathrm{d}x = a_i(u, v) + \sum_{q=1}^{Q_i} \sum_{j=0}^{m-1} \int_{\gamma_{iq}} \Phi_{iqj} u F_{iqj} \bar{v}\, \mathrm{d}\sigma \tag{2.12}$$

for all $u \in H^{2m}(\Omega_j)$, $v \in H^m_v(\Omega_i)$.

Proof. For $u \in C^\infty(\bar{\Omega}_i)$ and $v \in C^\infty_v(\bar{\Omega}_i)$, (2.12) follows from [13, section 2.2.4] since we may ignore the vertices of Ω_i by rounding the corner singularities outside the support of v. In view of the classical results of [13, section 2.2.4], we immediately see that the Φ_{iqj}'s have coefficients in $C^\infty(\bar{\gamma}_{iq})$ so that (2.12) follows by density. ■

If $\gamma_{iq} \in \mathscr{E}$, we fix a partition of $\{0, 1, \ldots, m-1\}$ in $\mathscr{S}_{iq} \cup \mathscr{T}_{iq}$, while if $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$, we take a partition of $\{0, 1, \ldots, m-1\}$ in $\mathscr{S}^1_{ii'} \cup \mathscr{S}^2_{ii'} \cup \mathscr{T}_{ii'}$. As usual, $\mathscr{S}$ is the set of stable boundary or interface conditions, while $\mathscr{T}$ will be the set of transversal ones (they do not have sense for $u \in \mathscr{H}^m$). We are now able to set

$$V = \{u \in \mathscr{H}^m_v(\Omega) \text{ satisfying (2.13)–(2.15) hereafter}\},$$

$$F_{iqj}u_i = 0 \quad \text{on } \gamma_{iq}, \ \forall \gamma_{iq} \in \mathscr{E}, \ j \in \mathscr{S}_{iq}, \tag{2.13}$$

$$\begin{cases} F_{iqj}u_i = 0 & \text{on } \gamma_{iq}, \\ F_{i'q'j}u_{i'} = 0 & \text{on } \gamma_{i'q'}, \ \forall \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}, \ j \in \mathscr{S}^1_{ii'}, \end{cases} \tag{2.14}$$

$$F_{iqj}u_i = F_{i'q'j}u_{i'} \quad \text{on } \gamma_{iq}, \ \forall \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}, \ j \in \mathscr{S}^2_{ii'}. \tag{2.15}$$

Within this setting, some problems of the form (2.6)–(2.8) admit a weak formulation.

Lemma 2.4. *Define the sesquilinear form $a(\cdot\,,\cdot)$ on V as follows:*

$$a(u, v) = \sum_i a_i(u_i, v_i), \quad \forall u, v \in V. \tag{2.16}$$

Let $f \in L^2(\Omega)$ and let $u \in \mathscr{H}^{2m}_v(\Omega)$ be a solution of

$$a(u, v) = \sum_i \int_{\Omega_i} f_i \bar{v}_i \, dx_i \quad \forall v \in V. \tag{2.17}$$

Then u satisfies (2.6)–(2.8) with $g_{iqj} = 0$, $g_{ii'j} = 0$, when the boundary and interface operators are defined as follows:

(i) *If $\gamma_{iq} \in \mathscr{E}$, then*

$$B_{iqj} = \begin{cases} F_{iqj} & \text{if } j \in \mathscr{S}_{iq}, \\ \Phi_{iqj} & \text{if } j \in \mathscr{T}_{iq}. \end{cases}$$

(ii) *If $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$, then*

$$\begin{cases} B_{iqj} = F_{iqj}, \\ B_{iqj+m} = 0, \\ B_{i'q'j} = 0, \\ B_{i'q'j+m} = F_{i'q'j}, \quad \text{if } j \in \mathscr{S}^1_{ii'}. \end{cases}$$

$$\begin{cases} B_{iqj} = F_{iqj}, \\ B_{iqj+m} = \Phi_{iqj}, \\ B_{i'q'j} = F_{i'q'j} \\ B_{i'q'j+m} = -\Phi_{i'q'j}, \quad \text{if } j \in \mathscr{S}^2_{ii'}. \end{cases}$$

$$\begin{cases} B_{iqj} = \Phi_{iqj}, \\ B_{i'q'j} = 0, \\ B_{iqj+m} = 0, \\ B_{i'q'j+m} = \Phi_{i'q'j}, \quad \text{if } j \in \mathscr{T}_{ii'}. \end{cases}$$

Proof. Applying (2.17) with v such that $v_i \in \mathscr{D}(\Omega_i)$, for all i, we see that u satisfies (2.6). The remainder follows from Green's formula (2.12). ■

An interface problem (2.6)–(2.8), defined by the previous procedure, will be called a weak transmission problem. An interesting question is to know whether it is a regular elliptic one. Following Agmon's ideas [1], we give a necessary and sufficient condition of the form $a(\cdot,\cdot)$ to satisfy the covering conditions of section 2.2.

We first start with a technical result which leads to the sufficiency of the condition. Introduce the Hilbert space

$$Y = \prod_{\gamma_{iq}\in\mathscr{E}} \prod_{j\in\mathscr{S}_{iq}} H^{m-j-1/2}(\gamma_{iq}) \times \prod_{\gamma_{iq}=\gamma_{i'q'}\in\mathscr{I}} \left\{ \prod_{j\in\mathscr{S}^1_{ii'}} (H^{m-j-1/2}(\gamma_{iq}))^2 \times \prod_{j\in\mathscr{S}^2_{ii'}} H^{m-j-1/2}(\gamma_{iq}) \right\}$$

and the trace-transmission operator

$$F : \mathscr{H}^m_v(\Omega) \to Y : u \to \{F_{iqj}u_i\}_{\gamma_{iq}\in\mathscr{E}, j\in\mathscr{S}_{iq}} \times \{(F_{iqj}u_i, F_{i'q'j}u_{i'})\}_{\gamma_{iq}=\gamma_{i'q'}\in\mathscr{I}, j\in\mathscr{S}^1_{ii'}} \times \{F_{iqj}u_i - F_{i'q'j}u_{i'}\}_{\gamma_{iq}=\gamma_{i'q'}\in\mathscr{I}, j\in\mathscr{S}^2_{ii'}}, \tag{2.18}$$

which is a bounded operator due to classical trace theorems.

Proposition 2.5. *Assume that there exists two positive constants C_1, C_2 such that*

$$\|u\|^2_{m,2,\Omega} \leqslant C_1 \operatorname{Re} a(u,u) + C_2(\|u\|^2_{0,2,\Omega} + \|Fu\|^2_Y), \quad \forall u \in \mathscr{H}^m_v(\Omega). \tag{2.19}$$

Then

(i) *for all $\gamma_{iq}\in\mathscr{E}$ and all $x\in\gamma_{iq}$, there exists a positive constant C such that*

$$|u|^2_{H^m(\mathbb{R}^n_+)} \leqslant C \operatorname{Re} \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n_+} a^i_{\alpha\beta}(x)(D^\alpha u)(y) D^\beta \bar{u}(y)\,dy + \sum_{j\in\mathscr{S}_{iq}} \|F^0_{iqj}(x,D)u\|^2_{H^{m-j-1/2}(\mathbb{R}^{n-1})}, \quad \forall u \in H^m(\mathbb{R}^n_+) \tag{2.20}$$

(ii) *for all $\gamma_{iq}=\gamma_{i'q'}\in\mathscr{I}$ and all $x\in\gamma_{iq}$, there exists a positive constant C such that*

$$\begin{aligned} |u_i|^2_{H^m(\mathbb{R}^n_+)} + |u_{i'}|^2_{H^m(\mathbb{R}^n_-)} \leqslant{}& C \operatorname{Re} \sum_{|\alpha|=|\beta|=m} \left\{ \int_{\mathbb{R}^n_+} a^i_{\alpha\beta}(x)(D^\alpha u_i)(y) D^\beta \bar{u}_i(y)\,dy + \int_{\mathbb{R}^n_-} a^{i'}_{\alpha\beta}(x)(D^\alpha u_{i'})(y) D^\beta \bar{u}_{i'}(y)\,dy \right\} \\ &+ \sum_{j\in\mathscr{S}^1_{ii'}} \{\|F^0_{iqj}(x,D)u_i\|^2_{H^{m-j-1/2}(\mathbb{R}^{n-1})} + \|F^0_{i'q'j}(x,D)u_{i'}\|^2_{H^{m-j-1/2}(\mathbb{R}^{n-1})}\} \\ &+ \sum_{j\in\mathscr{S}^2_{ii'}} \|F^0_{iqj}(x,D)u_i - F^0_{i'q'j}(x,D)u_{i'}\|^2_{H^{m-j-1/2}(\mathbb{R}^{n-1})} \end{aligned} \tag{2.21}$$

for all $(u_i,u_{i'})\in H^m(\mathbb{R}^n_+)\times H^m(\mathbb{R}^n_-)$, where $|\cdot|_{H^m(\mathbb{R}^n_+)}$ denotes the semi-norm of $H^m(\mathbb{R}^n_+)$.

Proof. Analogous arguments as in [7, Proposition 8.1]. ■

Theorem 2.6. *The sesquilinear form $a(\cdot\,,\cdot)$ fulfils (2.19) if and only if problem (2.6)–(2.8) associated with $a(\cdot\,,\cdot)$ via Lemma 2.4 is a regular elliptic transmission problem.*

Proof. $\Rightarrow$: For the external boundary, the sufficiency is a direct consequence of [1, Theorems 5.1 and 3.2] since (2.20) is condition (5.6) of [1, Theorem 5.1]. For internal boundary points, it suffices to extend the previous theorems of [1] to a system.

$\Leftarrow$: We follow [27, Theorem 19.3] which holds for boundary value problems. The estimates (2.20) and (2.22) follow using the Fourier transform and the covering conditions. Using a covering of Ω by balls of sufficiently small radius and a perturbation argument, we obtain (2.19).

To finish this section, we shall show that if the range of the operator F, defined by (2.19), is closed, then condition (2.19) is equivalent to the weak coerciveness of $a(\cdot\,,\cdot)$ in V. Let us notice that this range is always closed in dimension 3, while in dimension 2, it is not (see [9, Theorem 1.6.1.5]).

Proposition 2.7. *Assume that the range of $F, R(F)$ is closed in Y. Then (2.19) is equivalent to (2.22);*

$$\|u\|^2_{m,2,\Omega} \leqslant C\,\mathrm{Re}\,a(u,u) + C_0\|u\|^2_{0,2,\Omega}, \quad \forall u \in V \tag{2.22}$$

for some positive constants C, C_0.

Proof. It suffices to show that (2.22) implies (2.19). Since F is a bounded operator, by the closed graph theorem, there exists a positive constant C_4 such that

$$\|\dot{u}\|^2_{\mathscr{H}^m_v(\Omega)/\ker F} \leqslant C_4\|Fu\|_Y, \quad \forall u \in \mathscr{H}^m_v(\Omega), \tag{2.23}$$

as usual $\dot{u}$ denotes the equivalence class of $u \in \mathscr{H}^m_v(\Omega)$ in the quotient space $\mathscr{H}^m_v(\Omega)/\ker F$.

Fix $u \in \mathscr{H}^m_v(\Omega)$, and take an arbitrary v in $\dot{u}$, since $w := u - v$ belongs to V, by (2.22) and the triangular inequality, we get

$$\|u\|^2_{m,2,\Omega} \leqslant 2C\,\mathrm{Re}\,a(w,w) + 2C_0\|w\|^2_{0,2,\Omega} + 2\|v\|^2_{m,2,\Omega}.$$

By the continuity of the form $a(\cdot\,,\cdot)$ and the interpolation inequality

$$ab \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad \forall a, b, \varepsilon > 0,$$

the previous inequality becomes

$$\|u\|^2_{m,2,\Omega} \leqslant C_5\,\mathrm{Re}\,a(u,u) + C_6\{\|u\|^2_{0,2,\Omega} + \|v\|^2_{m,2,\Omega}\} \tag{2.24}$$

for some positive constants C_5 and C_6 independent of u and v. Therefore, taking the infimum over v in (2.24) and using (2.23), we obtain (2.19). ∎

2.4. *Examples*

In the sequel, we shall often illustrate our general statements by some classical examples of regular elliptic transmission problems: the Laplace operator with different media, the plate equation and the Lamé system with 2 different media both in the plane. For simplicity, we shall only give the formulation near a conical point.

Example 1. We consider the Dirichlet problem with q different media ($q \geq 2$). More precisely, for q openings, $\omega_l > 0$ and q positive material constants p_l, $l = 1, \ldots, q$, we set (see Fig. 5)

$$\sigma_0 = 0, \quad \sigma_l = \sum_{j=1}^{l} \omega_j, \ l = 1, \ldots, q,$$

$$C_l = \{re^{i\omega} : r > 0, \sigma_{l-1} < \omega < \sigma_l\}, \quad l = 1, \ldots, q,$$

$$\Gamma_l = \{re^{i\sigma_l} : r > 0\}, \quad \forall l = 0, 1, \ldots, q.$$

We obviously suppose that $\sigma_q \leq 2\pi$ and define

$$\Omega_l = C_l \cap B(0, 1),$$

$$\Gamma_l^1 = \Gamma_l \cap B(0, 1),$$

$$\Gamma_l^2 = \partial\Omega_l \cap \partial B(0, 1).$$

The following interface problem is extensively studied in the literature (see [12, 17, 21, 8] and the references cited therein):

$$p_l \Delta u_l = f_l \quad \text{in } \Omega_l, \ \forall l = 1, \ldots, q, \tag{2.25}$$

$$\gamma_l^1 u_l = \gamma_l^1 u_{l+1} \quad \text{on } \Gamma_l^1, \ \forall l = 1, \ldots, q-1, \tag{2.26}$$

$$p_l \gamma_l^1 \frac{\partial u_l}{\partial \omega} = p_{l+1} \gamma_l^1 \frac{\partial u_{l+1}}{\partial \omega} \quad \text{on } \Gamma_l^1, \ \forall l = 1, \ldots, q-1, \tag{2.27}$$

$$\gamma_0^1 u_1 = 0 \quad \text{on } \Gamma_0^1, \tag{2.28}$$

$$\gamma_q^1 u_q = 0 \quad \text{on } \Gamma_q^1, \tag{2.29}$$

$$\gamma_l^2 u_l = 0 \quad \text{on } \Gamma_l^2, \quad \forall l = 1, \ldots, q, \tag{2.30}$$

where γ_l^i is the trace operator on Γ_l^i.

This transmission problem (2.25)–(2.30) is clearly a regular elliptic one since the sesquilinear form, associated with it, is given by (see [8])

$$a(u, v) = \sum_{l=1}^{q} p_l \int_{\Omega_l} \nabla u_l \cdot \overline{\nabla v_l} \, dx.$$

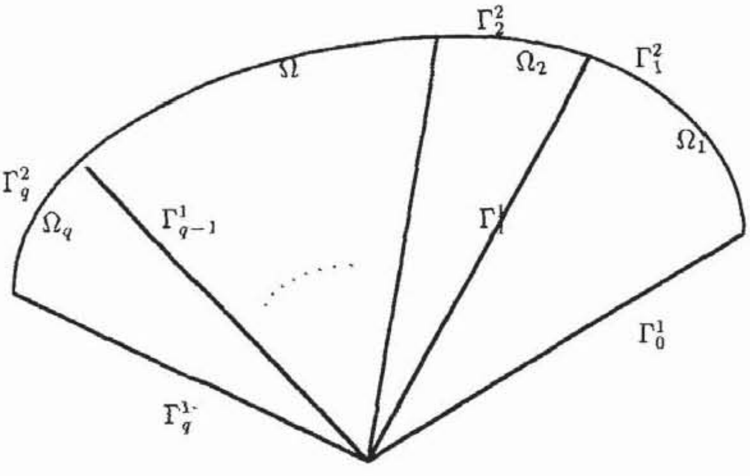

Fig 5.

Therefore, $a(u, u)$ is equivalent to the semi-norm of $\mathscr{H}^1$; so, (2.19) obviously holds.

Some analogous transmission problems where the Laplace operator is replaced by the Helmholtz equation were studied by Meister in [17] and are also included in our framework (see also [6]).

For the three examples below, we shall use the notations of example 1 and only take two media (i.e. $q = 2$).

Example 2. We consider the pure transmission problem for the Laplace operator, i.e. $\sigma_2 = \omega_1 + \omega_2 = 2\pi$ (see also Fig. 2). Then given two material constants p_1 and p_2, we take (counting modulo 2)

$$p_l \Delta u_l = f_l \quad \text{in } \Omega_l, \tag{2.31}$$

$$\gamma_l^1 u_l = \gamma_l^1 u_{l+1} \quad \text{on } \Gamma_l^1, \tag{2.32}$$

$$p_l \gamma_l^1 \frac{\partial u_l}{\partial \omega} - p_{l+1} \gamma_l^1 \frac{\partial u_{l+1}}{\partial \omega} = 0 \quad \text{on } \Gamma_l^1, \tag{2.33}$$

$$\gamma_l^2 u_l = 0 \quad \text{on } \Gamma_l^2, \ \forall l = 1, 2. \tag{2.34}$$

For the same reason as in example 1, it is a regular elliptic transmission problem.

Example 3. For $l = 1, 2$, we fix $E_l > 0$, Young's modulus and $\nu_l \in]0, 1[$, Poisson's ratio of the constitutive material of C_l. We also set

$$\rho_l = E_l/(1 - \nu_l^2),$$

$$M_l u = \rho_l \gamma_1^1 \left(\nu_l \Delta u + (1 - \nu_l) \frac{\partial^2 u}{\partial (\nu^l)^2} \right),$$

$$N_l u = \rho_l \gamma_1^1 \left(\frac{\partial \Delta u}{\partial \nu^l} + (1 - \nu_l) \frac{\partial^3 u}{\partial \nu^l \partial (\tau^l)^2} \right),$$

where $\nu^l = (\nu_1^l, \nu_2^l)$ is the outer normal vector on $\partial\Omega_l$ in Ω_l, while τ^l is the tangent vector to $\partial\Omega_l$, so that (ν^l, τ^l) is a direct basis.

From [23], we know that the following transmission problem is a regular elliptic one:

$$\rho_l \Delta^2 u_l = f_l \quad \text{in } \Omega_l, \ l = 1, 2, \tag{2.35}$$

$$\gamma_0^1 u_1 = \gamma_0^1 \frac{\partial u_1}{\partial \omega} = 0 \quad \text{on } \Gamma_0^1, \tag{2.36}$$

$$\gamma_2^1 u_2 = \gamma_2^1 \frac{\partial u_2}{\partial \omega} = 0 \quad \text{on } \Gamma_2^1, \tag{2.37}$$

$$\gamma_1^1 u_1 = \gamma_1^1 u_2 \quad \text{on } \Gamma_1^1, \tag{2.38}$$

$$\gamma_1^1 \frac{\partial u_1}{\partial \nu^1} = -\gamma_1^1 \frac{\partial u_2}{\partial \nu^2} \quad \text{on } \Gamma_1^1, \tag{2.39}$$

$$\gamma_1^1 M_1 u_1 - \gamma_1^1 M_2 u_2 = 0 \quad \text{on } \Gamma_1^1, \tag{2.40}$$

$$\gamma_1^1 N_1 u_1 + \gamma_1^1 N_2 u_2 = 0 \quad \text{on } \Gamma_1^1, \tag{2.41}$$

$$\gamma_l^2 u_l = \gamma_l^2 \frac{\partial u_l}{\partial \nu^l} = 0 \quad \text{on } \Gamma_l^2, \ \forall l = 1, 2. \tag{2.42}$$

Example 4. For the elasticity system with two different media, we introduce the Lamé constants $\lambda^l, \mu^l, l = 1, 2$ and set

$$\sigma^l_{ij}(\mathbf{u}) = \mu^l\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda^l \operatorname{div}\mathbf{u}\,\delta_{ij}, \ \forall i, j = 1, 2,$$

where $\mathbf{u} = (u_1, u_2)$.

We shall study the eigenvalues of the following transmission problems (2.43), (2.44) and (2.43), (2.45) (see [12], for more details), they are clearly regular elliptic using the so-called Korn's inequality:

$$\begin{cases} \dfrac{\partial}{\partial x_j}(\sigma^l_{ij}(\mathbf{u}^l)) = \underline{f}^l & \text{in } \Omega_l, \quad l = 1, 2, \\ \gamma^1_1\mathbf{u}^1 = \gamma^1_1\mathbf{u}^2 & \text{on } \Gamma^1_1, \\ \gamma^1_1(\sigma^1_{ij}(\mathbf{u}^1)\cdot\nu^1_j) + \gamma^1_1(\sigma^2_{ij}(\mathbf{u}^2)\cdot\nu^2_j) = 0 & \text{on } \Gamma^1_1, \\ \gamma^2_l\mathbf{u}_l = \mathbf{0} & \text{on } \Gamma^2_l, \ \forall l = 1, 2, \end{cases} \tag{2.43}$$

$$\begin{cases} \gamma^1_0\mathbf{u}^1 = \mathbf{0} \quad \text{on } \Gamma^1_0, \\ \gamma^1_2\mathbf{u}^2 = \mathbf{0} \quad \text{on } \Gamma^1_2, \end{cases} \tag{2.44}$$

$$\begin{cases} \gamma^1_0\sigma^1_{ij}(\mathbf{u}^1)\cdot\nu^1_j = \mathbf{0} \quad \text{on } \Gamma^1_0, \\ \gamma^1_2\sigma^2_{ij}(\mathbf{u}^2)\cdot\nu^2_j = \mathbf{0} \quad \text{on } \Gamma^1_2. \end{cases} \tag{2.45}$$

3. Regularity in weighted Sobolev spaces

We have introduced in section 2 the spaces $\mathscr{H}^k_v(\Omega)$ in connection with Green's formula. But the theory of elliptic boundary value problems in domains with conical points is especially well-worked out in the framework of certain weighted Sobolev spaces, introduced by Kondratiev [11]. We now define the corresponding weighted Sobolev spaces for our boundary-transmission problem.

Definition 3.1. *$V^{k,p}_\beta(\Omega_i)$ is the closure of $C^\infty_v(\bar{\Omega}_i)$ with respect to the norm*

$$\| u_i; V^{k,p}_\beta(\Omega_i)\| = \sum_{|\alpha|\leq k} \| r_i^{\beta-k+|\alpha|} D^\alpha u_i\|_{L_p(\Omega_i)}, \tag{3.1}$$

where $r_i = r_i(x) = \operatorname{dist}(x, \mathscr{S}\cap\overline{\Omega_i})$.

$$\mathscr{V}^{k,p}_\beta(\Omega) = \{u \in L^p(\Omega) : u|_{\Omega_i} = u_i \in V^{k,p}_\beta(\Omega_i)\} \tag{3.2}$$

equipped with the norm

$$\| u; \mathscr{V}^{k,p}_\beta(\Omega)\| = \sum_{i=1}^N \| u_i; V^{k,p}_\beta(\Omega_i)\|.$$

The space of traces on the boundary pieces of the subdomains Ω_i, $i = 1, \ldots N$; $q = 1, \ldots, Q_i$, is the quotient space

$$V^{k-(1/p)p}_\beta(\gamma_{iq}) = V^{k,p}_\beta(\Omega_i)/\mathring{V}^{k,p}_\beta(\Omega_i, \gamma_{iq}),$$

where $\mathring{V}^{k,p}_\beta(\Omega_i, \gamma_{iq})$ is the closure of $C^\infty_{\gamma_{iq}}(\Omega_i) = \{u \in C^\infty(\overline{\Omega_i}),\ \operatorname{supp} u \cap \gamma_{iq} = \emptyset\}$ with respect to the norm (3.1).

We now formulate for the regular elliptic transmission problems (2.6)–(2.8), results about the solvability, regularity and asymptotic expansion, which can be proved analogously to the results for pure elliptic boundary value problems. We denote by

$$\mathfrak{U}(x, D_x) = \{A_i(x, D_x), B_{iqj}(x, D_x)\}_{\substack{i=1,\dots,N,\ q=1,\dots,Q_i \\ j=0,\dots,m-1 \text{ if } \gamma_{iq}\in\mathscr{E} \\ j=0,\dots,2m-1 \text{ if } \gamma_{iq}\in\mathscr{I}}} \tag{3.3}$$

the operator, which is generated by problem (2.6)–(2.8) and which maps

$$\mathscr{V}_\beta^{2m+l,p}(\Omega) \text{ into } \mathscr{V}_\beta^{l,p}(\Omega) \times \prod_{\substack{\gamma_{iq}\in\mathscr{E} \\ j=0,\dots,m-1}} V_\beta^{2m+l-m_{iqj}-1/p,p}(\gamma_{iq})$$
$$\times \prod_{\substack{\gamma_{iq}=\gamma_{i'q'}\in\mathscr{I} \\ j=0,\dots,2m-1}} V_\beta^{2m+l-m_{iqj}-1/p,p}(\gamma_{iq}) = Y_\beta^{l,p}(\Omega).$$

Let $S\in\mathscr{S}$. We assume, for simplicity, that the domain Ω_i, for all $i\in\mathscr{N}_S$, coincides in a neighbourhood of S with an infinite cone C_i. Let $C_S = \bigcup_{i\in\mathscr{N}_S} C_i$; all the spaces defined in Ω are defined analogously in C_S (replacing, if necessary, r_i by the distance to S).

First we consider a special boundary-transmission problem in C_S, which is generated by the principal parts of the operators (2.6)–(2.8) with frozen coefficients in S:

$$A_i^S u_i = \sum_{|\alpha|=2m} a_\alpha^i(S) D^\alpha u_i = f_i \quad \text{in } C_i,\ i\in\mathscr{N}_S. \tag{3.4}$$

$$B_{iqj}^S\, u_i = \sum_{|\alpha|=m_{iqj}} b_{iqj\alpha}(S) D^\alpha u_i = g_{iqj} \quad \text{on } \Gamma_{iq}\in\mathscr{E}_S,\ j=0,\dots,m-1. \tag{3.5}$$

$$B_{iqj}^S u_i - B_{i'q'j}^S u_{i'} = \sum_{|\alpha|=m_{iqj}} [b_{iqj\alpha}(S) D^\alpha u_i - b_{i'q'j\alpha}(S) D^\alpha u_{i'}] \tag{3.6}$$
$$= g_{ii'j} \quad \text{on } \Gamma_{iq} = \Gamma_{i'q'}\in\mathscr{I}_S,\ j=0,\dots,2m-1.$$

The notation $\Gamma_{iq}\in\mathscr{E}_S$ or $\Gamma_{iq}\in\mathscr{I}_S$ means Γ_{iq} is a side ($n=2$) or a surface ($n=3$) of the cone C_i and coincides in a neighbourhood of S with $\gamma_{iq}\in\mathscr{E}$ or $\gamma_{iq}\in\mathscr{I}$, respectively. In other words, $\mathscr{E}_S$ is the union of the external pieces of the boundary of C_i and $\mathscr{I}_S$ is the union of the interface pieces of the boundary of C_i for $i\in\mathscr{N}_S$.

Analogous to (3.3), let us denote by $\mathfrak{U}^S(D_x)$ the operator which is generated by (3.4)–(3.6).

Introducing polar co-ordinates $r = r(x) = |x - S|$, $\omega\in S^{n-1}(S)$, where $S^{n-1}(S)$ is a unit sphere around S, and using the Mellin transform

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty r^{-\lambda-1} u(r,\omega)\,dr = \hat{u}(\lambda,\omega),$$

which maps rD_r into the complex parameter λ, we obtain a boundary-transmission problem with parameter λ in the domains $G_i^S = \Omega_i \cap S^{n-1}(S)$, $i\in\mathscr{N}_S$:

$$L_i^S(\omega, D_\omega, \lambda)\hat{u}_i(\lambda,\omega) = \hat{F}_i(\lambda,\omega) \quad \text{for } \omega\in G_i^S, \tag{3.7}$$

$$M_{iqj}^S(\omega, D_\omega, \lambda)\hat{u}_i(\lambda,\omega) = \hat{G}_{iqj}(\lambda,\omega) \quad \text{for } \omega\in\omega_{iq}\in\hat{\mathscr{E}}_S, \tag{3.8}$$

$$M_{iqj}^S(\omega, D_\omega, \lambda)\hat{u}_i(\lambda,\omega) - M_{i'q'j}^S(\omega, D_\omega, \lambda)\hat{u}_i(\lambda,\omega) = \hat{G}_{iqj}(\lambda,\omega)$$
$$\text{for } \omega\in\omega_{iq} = \omega_{i'q'}\in\hat{\mathscr{I}}_S, \tag{3.9}$$

where $\omega_{iq} = \Gamma_{iq} \cap S^{n-1}(S)$. We say $\omega_{iq} \in \hat{\mathscr{E}}_S$, if $\gamma_{iq} \in \mathscr{E}$ and $\omega_{iq} \in \hat{\mathscr{I}}_S$, if $\gamma_{iq} \in \mathscr{I}$. Furthermore, it holds

$$A_i^S(S, D_x) = r^{-2m} L_i^S(\omega, D_\omega, rD_r),$$

$$B_{iqj}^S(S, D_x) = r^{-m_{iqj}} M_{iqj}^S(\omega; D_\omega, rDr).$$

The operator

$$\mathscr{A}_S(\lambda) = \{L_i^S(\omega, D_\omega, \lambda), M_{iqj}^S(\omega, D_\omega, \lambda)\}, \tag{3.10}$$

which is generated by (3.7)–(3.9) and which is defined in $G_S = \bigcup_{i \in N_S} G_i^S$ maps continuously:

$$\mathscr{W}^{2m+l,p}(G_S) \text{ into } \mathscr{W}^{l,p}(G_S) \times \prod_{\substack{\omega_{iq} \in \hat{\mathscr{E}}_S \\ j=0,\dots,m-1}} W^{l+2m-m_{iqj}-1/p,p}(\omega_{iq})$$

$$\times \prod_{\substack{\omega_{iq} = \omega_{i'q'} \in \hat{I}_S \\ j=0,\dots,2m-1}} W^{l+2m-m_{iqj}-1/p,p}(\omega_{iq}).$$

Here it is

$$\mathscr{W}^{2m+l,p}(G_S) = \{u: u_{|G_i^S} \in W^{2m+l,p}(G_i^S), \forall i \in \mathscr{N}_S\}.$$

The generalized eigenvalues of the operator $A_S(\lambda)$ play an important role for the solvability, the regularity and the asymptotic expansion of the solution near S. Let us shortly introduce the necessary standard definitions.

Definition 3.2. *The complex number $\lambda = \lambda_0$ is an eigenvalue of the operator $\mathscr{A}_S(\lambda)$, if there is a non-trivial function $\varphi^{S,\lambda_0,0} \in \mathscr{W}^{2m+l,2}(G_S)$ with $\mathscr{A}_S(\lambda_0)\varphi^{S,\lambda_0,0} = 0$. The function $\varphi^{S,\lambda_0,0}$ is called an eigensolution of $A_S(\lambda)$ for $\lambda = \lambda_0$.*

It follows from the ellipticity of the operator $\mathscr{U}^S(x, D_x)$ in C_S (cf. conditions I–III), the ellipticity of the operator $A_S(\lambda)$ in the sense of Agranovic and Visik [2]. Therefore, $I^{S,\lambda_0} = \dim\ker A_S(\lambda_0)$ is a finite number and in a finite strip $h \leqslant \operatorname{Re}\lambda \leqslant h_1$ there are only a finite number of eigenvalues.

Besides the linearly independent eigensolutions $\varphi^{S,\lambda_0,\mu,0}$, $\mu = 1, \dots, I^{S,\lambda_0}$, there exist $N^{S,\lambda_0,\mu}$ associated eigensolutions $\varphi^{S,\lambda_0,\mu,k}$, $k = 1, \dots, N^{S,\lambda_0,\mu}$ in general. We set $N^{S,\lambda_0,\mu} = 0$, if no associated eigensolution exist. The associated eigensolutions are defined by the following definition.

Definition 3.3. *The system $\{\varphi^{S,\lambda_0,\mu,k}\}_{\substack{\mu=1,\dots,I^{S,\lambda_0} \\ k=0,\dots,N^{S,\lambda_0,\mu}}}$ consists of eigensolutions and associated eigensolutions (usually called system of Jordan chains) if*

$$\sum_{q=0}^{k} \frac{1}{q!} (\partial/\partial\lambda)^q \mathscr{A}_S(\lambda)\varphi^{S,\lambda_0,\mu,k-q}|_{\lambda=\lambda_0} = 0$$

for $k = 0, \dots, N^{S,\lambda_0,\mu}$, $N^{S,\lambda_0,\mu}$ being decreasing with respect to μ.

The following theorems can be proved analogously to the theorems for elliptic boundary value problems (see e.g. [11] for $p = 2$, [15] for $p \neq 2$).

Theorem 3.4. *The operator $\mathfrak{U}(x, D_x)$, defined by (3.3), is a Fredholm operator iff no eigenvalue of $\mathscr{A}^S(\lambda)$ lies on the line $\operatorname{Re}\lambda = -\beta - (n/p) + 2m + l$ for all $S \in \mathscr{S}$.*

Theorem 3.5. *If for all $S \in \mathscr{S}$ no eigenvalue of $\mathscr{A}^S(\lambda)$ is situated in the strip*

$$-\beta - \frac{n}{p} + 2m + l \leqslant \operatorname{Re}\lambda \leqslant -\beta_1 - \frac{n}{p_1} + 2m + l_1,$$

then a solution $u \in \mathscr{V}_\beta^{l+2m,p}(\Omega)$ of (2.6)–(2.8) is contained in $\mathscr{V}_{\beta_1}^{l_1+2m,p_1}(\Omega)$ too, provided

$$f \in \mathscr{V}_\beta^{l,p}(\Omega) \cap \mathscr{V}_{\beta_1}^{l_1,p_1}(\Omega), \quad g_{iqj} \in V_\beta^{l+2m-m_{iqj}-1/p,p}(\gamma_{iq}) \cap V_{\beta_1}^{l_1+2m-m_{iqj}-1/p_1,p_1}(\gamma_{iq})$$

for $\gamma_{iq} \in \mathscr{E}$ and $j = 0, \ldots, m-1$, and $\gamma_{iq} \in \mathscr{I}$ and $j = 0, \ldots, 2m-1$.

Here l and l_1 are non-negative integers and β and β_1 are real numbers.

Theorem 3.6. *Assume that $\mathfrak{U}^S(D_x) = \mathfrak{U}(x, D_x)$ in a neighbourhood of S and the lines*

$$\operatorname{Re}\lambda = 2m + l - \frac{n}{p} - \beta = h, \qquad \operatorname{Re}\lambda = 2m + l_1 - \frac{n}{p_1} - \beta_1 = h_1$$

have no eigenvalue of $\mathscr{A}^S(\lambda)$ with $h < h_1$. Then the following asymptotic expansion of a solution $u \in \mathscr{V}_\beta^{1+2m,p}(\Omega)$ of problem (2.6)–(2.8), with data as in Theorem 3.5, holds near S:

$$u = w + \eta^S \sum_{\lambda,\mu,k}{}^{\cdot} c_{S,\lambda,\mu,k}\,\sigma^{S,\lambda,\mu,k}, \tag{3.11}$$

where $w \in \mathscr{V}_{\beta_1}^{l_1+2m,p_1}(\Omega)$, $c_{S,\lambda,\mu,k} \in \mathbb{C}$, η^S is an appropriate cut-off function such that $\eta^S \equiv 1$ in a neighbourhood of S, the sum extends to all eigenvalues λ of $\mathscr{A}_S(\xi)$ in the strip $\operatorname{Re}\xi \in]h, h_1[$, $\mu = 1, \ldots, I^{S,\lambda}$, $k = 0, \ldots, N^{S,\lambda,\mu}$ and, finally,

$$\sigma^{S,\lambda,\mu,k} = r^\lambda \sum_{q=0}^{k} \frac{(\ln r)^q}{q!} \varphi^{S,\lambda,\mu,k-q}. \tag{3.12}$$

There are formulae for the coefficients $c_{S,\lambda,\mu,k}$ as in [16], this shows that the coefficients depend continuously on the right-hand side.

4. Regularity results in usual Sobolev spaces

We only give these regularity results for weak transmission problems, which are regular and for homogeneous operators with constant coefficients. This means that we fix operators A_i, Dirichlet systems $\{F_{iqj}\}_{j=0}^{m-1}$ on all γ_{iq} satisfying the assumptions of section 2.3. Moreover, we suppose that the corresponding transmission problem (2.6)–(2.8) is a regular elliptic one or equivalently we suppose that the associated sesquilinear form $a(\cdot\,,\cdot)$ satisfies (2.19). We also assume that A_i and F_{iqj} are homogeneous with constant coefficients.

For some technical reasons, we further suppose that the sesquilinear form $a(\cdot\,,\cdot)$ is strictly coercive, i.e. there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u, u) \geqslant \alpha \|u\|_V^2, \quad \forall u \in V. \tag{4.1}$$

For simplicity, in dimension 2, we suppose that the Ω_i's coincide with a wedge in a neighbourhood of each vertex (this was already the case in dimension 3).

Lemma 4.1. *Let $u \in V$ be a solution of problem (2.17) with a right-hand side $f \in L^p(\Omega)$, where $1 < p \leqslant 2$ in dimension 2 and $p = 2$ in dimension 3. Then*

$$u \in \mathscr{V}_\alpha^{2m,p}(\Omega), \tag{4.2}$$

with $\alpha = m + (n/2) - (n/p) + \gamma'$ for all $\gamma' \in]0, 2 - (n/2)[$.

Proof. Classical regularity results for systems (see [20]) and the covering conditions imply that u belongs to $\mathscr{V}^{2m,p}$ far from the vertices of Ω. Therefore, it remains to prove (4.2) near the vertices.

Applying [7, Proposition AA.29] to u_i belonging to $H_v^m(\Omega_i)$, we deduce that

$$u_i \in V_\gamma^{m,2}(\Omega_i), \quad \forall \gamma \in \left]0, 2 - \frac{n}{2}\right[. \tag{4.3}$$

By Hölder's inequality when $p < 2$, (4.3) implies that

$$u_i \in V_{\alpha-2m}^{0,p}(\Omega_i) \tag{4.4}$$

for all α given in the lemma.

We now fix a vertex S of Ω and use spherical co-ordinates (r, ω) centred at S. Let us consider the following dyadic cover of $C_S^{r_0} = \{x \in \Omega: r(x) < 2r_0\}$ for $r_0 > 0$ sufficiently small such that $C_S^{r_0} \subset \Omega$: for all $v \in \mathbb{N}^*$, we set

$$S_v = \left\{ x \in \Omega: \frac{r_0}{v} < r(x) < \frac{2r_0}{v} \right\},$$

$$S'_v = \left\{ x \in \Omega: \frac{r_0}{2v} < r(x) < \frac{3r_0}{v} \right\}.$$

Interior Agmon–Douglis–Niremberg estimates for systems ensure the existence of $C > 0$ such that

$$\| u \|_{2m,p,S_1} \leqslant C\{ \| Au \|_{0,p,S'_1} + \| u \|_{0,p,S'_1} \}.$$

By similarity, we get (since the operators have constant coefficients and are homogeneous)

$$\sum_{|\beta| \leqslant 2m} v^{-p|\beta|} \| D^\beta u \|_{0,p,S_v}^p \leqslant C\{ v^{-2mp} \| Au \|_{0,p,S'_v}^p + \| u \|_{0,p,S'_v}^p \}. \tag{4.5}$$

Multiplying (4.5) by $v^{-\alpha+2m}$, using the fact that $r(x)$ is equivalent to v on S_v and S'_v and summing over $v \in \mathbb{N}^*$, we obtain

$$\| u; \mathscr{V}_\alpha^{2m,p}(C_S^{r_0}) \| \leqslant C\{ \| f \|_{0,p,\Omega} + \| u; \mathscr{V}_{\alpha-2m}^{0,p}(\Omega) \| \}.$$

This proves the lemma, owing to (4.4). ∎

From now on, Φ_S will denote a cut-off function fulfilling $\Phi_S \in D(\mathbb{R}^n)$, $\Phi_S = 1$ in a neighbourhood of the vertex S, and $\Phi_S = 0$ in a neighbourhood of the other vertices of Ω.

Theorem 4.2. *Assume that $k - \gamma \geqslant 0$ and that the line $\operatorname{Re} \zeta = k + 2m - n/p - \gamma$ contains no eigenvalue of $A_S(\zeta)$ for all $S \in \mathscr{S}$. Then for all $f \in \mathscr{V}_\gamma^{k,p}(\Omega)$, with $p \in]1, +\infty[$ in* *dimension* 2 *and* $p = 2$ *in dimension* 3, *there exists a unique solution* $u \in V$ *of the problem*

$$A_i u_i = f_i \quad \text{in } \Omega_i, \quad \forall i = 1, \dots, N, \tag{4.6}$$

$$B_{iqj} u_i = 0 \quad \text{on } \gamma_{iq}, \ \forall j = 0, \dots, m-1, \quad \gamma_{iq} \in \mathscr{E}, \tag{4.7}$$

$$B_{iqj} u_i - B_{i'q'j} u_{i'} = 0 \quad \text{on } \gamma_{iq} = \gamma_{i'q'}, \ \forall j = 0, \dots, 2m-1, \ \gamma_{iq} \in \mathscr{I}. \tag{4.8}$$

This solution admits the following expansion:

$$u = u_0 + \sum_{\substack{S \in \mathscr{S} \\ (\lambda, \mu, k) \in \Lambda_S^1(k, p, \gamma)}} c_{S, \lambda, \mu, k} \Phi_S \sigma^{S, \lambda, \mu, k}, \tag{4.9}$$

where $u_0 \in \mathscr{V}_\gamma^{k+2m, p}(\Omega)$, $c_{S, \lambda, \mu, k} \in \mathbb{C}$ *and we set*

$$\Lambda_S^1(k, p, \gamma) = \Bigg\{ (\lambda, \mu, k) \colon \lambda \text{ is an eigenvalue of } \mathscr{A}_S(\zeta) \text{ such that}$$

$$\operatorname{Re} \lambda \in \left] m - \frac{n}{2}, k + 2m - \frac{n}{p} - \gamma \right[, \ \mu = 1, \dots, I^{S, \lambda}, \ k = 0, \dots, N^{S, \lambda, \mu} \Bigg\}. \tag{4.10}$$

Moreover, there exists a constant $C > 0$ *independent of* u *such that*

$$\| u_0; \mathscr{V}_\gamma^{k+2m, p}(\Omega) \| + \sum_{\substack{s \in \mathscr{S} \\ (\lambda, \mu, k) \in \Lambda_S^1(k, p, \gamma)}} |c_{S, \lambda, \mu, k}| \leqslant C \| f; \mathscr{V}_\gamma^{k, p}(\Omega) \|. \tag{4.11}$$

Proof. For $n = 2$, [22, Lemma 1.2] shows that there exists $r \in]1, 2]$ such that $f \in L^r(\Omega)$, while for $n = 3$, the assumption $k - \gamma \geqslant 0$ directly implies that $f \in L^2(\Omega)$. By Lemma 4.1, u belongs to $\mathscr{V}_\alpha^{2m, r}(\Omega)$, with $\alpha = m + (n/2) - (n/r) + \gamma'$, for all $\gamma' \in]0, 2 - (n/2)[$, where $r = 2$ in dimension 3. We easily check that $\alpha > 0$, which implies that $f \in \mathscr{V}_\alpha^{0, r}(\Omega) \cap \mathscr{V}_\gamma^{k, p}(\Omega)$. Applying the comparison theorem in weighted Sobolev spaces (Theorem 3.6) in each cone C_S to $\Phi_S u \in \mathscr{V}_\alpha^{2m, r}(\Omega)$ and choosing γ' sufficiently close to 0, we conclude. ∎

In the sequel, we shall need the following lifting trace theorem.

Lemma 4.3. *For all* $g_{iqj} \in V_\gamma^{k+2m-m_{iqj}-1/p, p}(\gamma_{iq})$, $j \in \{0, \dots, m-1\}$, $\gamma_{iq} \in \mathscr{E}$, *and all* $g_{ii'j} \in V_\gamma^{k+2m-m_{iqj}-1/p, p}(\gamma_{iq})$, $j \in \{0, \dots, 2m-1\}$, $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$, *there exists* $v \in \mathscr{V}_\gamma^{k+2m, p}(\Omega)$ *fulfilling*

$$B_{iqj} v_i = g_{iqj} \quad \text{on } \gamma_{iq}, \quad \forall j = 0, \dots, m-1, \ \gamma_{iq} \in \mathscr{E}, \tag{4.12}$$

$$B_{iqj} v_i - B_{i'q'j} v_{i'} = g_{ii'j} \quad \text{on } \gamma_{iq}, \ \forall j = 0, \dots, 2m-1, \ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}. \tag{4.13}$$

Proof. We firstly build new functions

$$h_{iqj} \in V_\gamma^{k+2m-j-1/p, p}(\gamma_{iq}), \qquad h_{iqj}^1 \in V_\gamma^{k+j+1-1/p, p}(\gamma_{iq}),$$

$j = 0, 1, \dots, m-1$ such that if $v \in \mathscr{V}_\gamma^{k+2m, p}(\Omega)$ fulfills

$$F_{iqj} v_i = h_{iqj} \quad \text{on } \gamma_{iq}, \tag{4.14}$$

$$\Phi_{iqj} v_i = h_{iqj}^1 \text{ on } \gamma_{iq}, \quad \forall j = 0, 1, \dots, m-1, \tag{4.15}$$

then v satisfies (4.12) and (4.13). The construction is the following.

If $\gamma_{iq} \in \mathscr{E}$, then we set

$$h_{iqj} = g_{iqj}, h^1_{iqj} = 0 \quad \text{if } j \in \mathscr{S}_{iq},$$

$$h_{iqj} = 0, h^1_{iqj} = g_{iqj} \quad \text{if } j \in \mathscr{T}_{iq}.$$

If $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$, then we take

(a) for $j \in \mathscr{S}^1_{ii'}$,

$$h_{iqj} = g_{ii'j}, h_{i'q'j} = -g_{ii'j+m},$$

$$h^1_{iqj} = h^1_{i'q'j} = 0,$$

(b) for $j \in \mathscr{S}^2_{ii'}$,

$$h_{iqj} = q_{ii'j}, \ h_{i'q'j} = 0,$$

$$h^1_{iqj} = g_{ii'j+m}, h^1_{i'q'j} = 0,$$

(c) for $j \in \mathscr{T}_{ii'}$,

$$h_{iqj} = h_{i'q'j} = 0,$$

$$h^1_{iqj} = g_{ii'j}, h^1_{i'q'j} = -g_{ii'j+m}.$$

From the definition of the B's with respect to the F's and Φ's, one easily shows that (4.14), (4.15) imply (4.12), (4.13).

Since the system $\{F_{iqj}, \Phi_{iqj}\}_{j=0}^{m-1}$ is a Dirichlet system on γ_{iq}, using [14, Lemma 3.1] (which also holds in dimension 3) and local charts, problem (4.14), (4.15) has a solution $v_i \in V^{k+2m,p}_\gamma(\Omega_i)$. ■

Corollary 4.4. *Let the assumptions of Theorem* 4.2 *be satisfied. Then for all* $f \in \mathscr{V}^{k,p}_\gamma(\Omega)$, $g_{iqj} \in V^{k+2m-m_{iqj}-1/p,p}_\gamma(\gamma_{iq})$, $j \in \{0, \dots, m-1\}$, $\gamma_{iq} \in \mathscr{E}$, $g_{ii'j} \in V^{k+2m-m_{iqj}-1/p,p}_\gamma(\gamma_{iq})$, $j \in \{0, \dots, 2m-1\}$, $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$, *there exists a unique weak solution* $u \in \mathscr{H}^m_v(\Omega)$ *of problem* (2.6)–(2.8), *in the following sense: for a fixed* $v \in \mathscr{V}^{k+2m,p}_\gamma(\Omega)$ *satisfying* (4.12), (4.13), $\tilde{u} = u - v$ *belongs to* V *and is solution of*

$$a(\tilde{u}, w) = \sum_i \int_{\Omega_i} \tilde{f}_i \bar{w}_i \, dx, \quad \forall w \in V, \tag{4.16}$$

where we set $\tilde{f}_i = f_i - A_i v_i$. *This is equivalent to say that* u *belongs to the affine space* $V_0 := v + V$, *and satisfies*

$$\begin{aligned} a(u, w) = \sum_i \int_{\Omega_i} f_i \bar{w}_i \, dx &- \sum_{\gamma_{iq} \in \mathscr{E}} \sum_{j \in \mathscr{T}_{iq}} \int_{\gamma_{iq}} g_{iqj} F_{iqj} \bar{w}_i \, d\sigma \\ &- \sum_{\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}} \Bigg[\sum_{j \in \mathscr{S}^2_{ii'}} \int_{\gamma_{iq}} g_{ii'j} F_{iqj} \bar{w}_i \, d\sigma \\ &+ \sum_{j \in \mathscr{T}_{ii'}} \int_{\gamma_{iq}} \{ g_{ii'j} F_{iqj} \bar{w}_i + g_{ii'j+m} F_{i'q'j} \bar{w}_{i'} \} \, d\sigma \Bigg], \quad \forall w \in V. \end{aligned} \tag{4.17}$$

Moreover, this solution u *admits the expansion* (4.9), *with the same coefficient* $c_{S,\lambda,\mu,k}$ *and an estimate analogous to* (4.11), *where the right-hand side is replaced by the sum of norms of the data.*

Proof. From Theorem 4.2, it is clear that $\tilde{u}$ solution of (4.16) admits the following expansion:

$$\tilde{u} = \tilde{u}_0 + \sum_{\substack{S \in \mathscr{S} \\ (\lambda,\mu,k) \in \Lambda_S^1(k,p,\gamma)}} c_{S,\lambda,\mu,k}(\tilde{u}) \Phi_S \sigma^{S,\lambda,\mu,k},$$

where $\tilde{u}_0 \in \mathscr{V}_\gamma^{k+2m,p}(\Omega)$. This directly implies the desired expansion (4.9) for u, since the fact that $v \in \mathscr{V}_\gamma^{k+2m,p}(\Omega)$ leads to

$$c_{S,\lambda,\mu,k}(v) = 0.$$

The assumption $k - \gamma \geqslant 0$ and the Sobolev imbedding theorem allow to show that

$$\mathscr{V}_\gamma^{k+2m,p}(\Omega) \subsetneq \mathscr{V}_0^{2m,p}(\Omega) \subsetneq \mathscr{H}_v^m(\Omega)$$

and

$$\mathscr{H}^m(\Omega) \subsetneq L^q(\Omega)$$

for $1/p + 1/q = 1$. These both injections imply that Green's formula (2.12) still holds for $u \in \mathscr{V}_\gamma^{k+2m,p}(\Omega)$ and $v \in \mathscr{H}_v^m(\Omega)$. This permits to transform $a(w, v)$ in (4.16) and to prove the equivalence between (4.16) and (4.17), using (4.12), (4.13) and the definitions of the B_{iqj}'s with respect to the F_{iqj}'s and the Φ_{iqj}'s. ■

5. Numerical examples

As we have seen in section 3, the distribution of the eigenvalues of the operator $A_S(\lambda)$ plays an important role for the solvability, regularity and asymptotic expansion of the solution of our interface problems in weighted Sobolev spaces. Our goal is now to show how to compute the eigenvalues for some examples and in which cases associated eigensolutions exist.

Example 1. We consider again the Dirichlet-interface problem for Poisson's equation in a plane domain of q different materials, which was formulated by the relations (2.25)–(2.30). We introduce for the origin point S the generalized eigenvalue problem for the operator $\mathscr{A}_S(\lambda)$:

$$\frac{\partial^2 \hat{u}_i(\lambda, \omega)}{\partial \omega^2} + \lambda^2 \hat{u}_i(\lambda, \omega) = 0 \quad \text{for } \sigma_{i-1} < \omega < \sigma_i,\ i = 1, \ldots, q, \tag{5.1}$$

$$\hat{u}_1(\lambda, 0) = 0 = \hat{u}_q(\lambda, \sigma_q), \tag{5.2}$$

$$\hat{u}_i(\lambda, \sigma_i) - \hat{u}_{i+1}(\lambda, \sigma_i) = 0, \tag{5.3}$$

$$p_i \frac{\partial \hat{u}_i}{\partial \omega}(\lambda, \sigma_i) - p_{i+1} \frac{\partial \hat{u}_{i+1}}{\partial \omega}(\lambda, \sigma_i) = 0 \quad \text{for } i = 1, \ldots, q-1. \tag{5.4}$$

The solutions of (5.1) have the form

$$\hat{u}_i(\lambda, \omega) = C_{1i} \cos \lambda\omega + C_{2i} \sin \lambda\omega, \quad i = 1, \ldots, q.$$

Inserting these solutions into the boundary conditions (5.2) and the interface conditions (5.3), (5.4), we get non-trivial solutions $\hat{u}_i(\lambda, \omega)$ for such values λ for which the determinant $D_q^{\mathrm{D}}(\lambda)$ vanishes. The zeros of $D_q^{\mathrm{D}}(\lambda)$ are the eigenvalues of $\mathscr{A}_S(\lambda)$. $D_q^{\mathrm{D}}(\lambda)$ is defined by the following recurrence formula:

First step:

$$D_1^{\mathrm{D}}(\lambda) = \sin(\lambda\omega_1),$$

$$D_1^{\mathrm{M}}(\lambda) = p_1 \cos(\lambda\omega_1).$$

Second step:

$$D_2^{\mathrm{D}}(\lambda) = p_2 \cos(\lambda\omega_2)\sin(\lambda\omega_1) + p_1 \sin(\lambda\omega_2)\cos(\lambda\omega_1),$$

$$D_2^{\mathrm{M}}(\lambda) = -\, p_2^2 \sin(\lambda\omega_1)\sin(\lambda\omega_2) + p_1 p_2 \cos(\lambda\omega_1)\cos(\lambda\omega_2).$$

ith step:

$$D_i^{\mathrm{D}}(\lambda) = p_i \cos(\lambda\omega_i) D_{i-1}^{\mathrm{D}}(\lambda) + \sin(\lambda\omega_i) D_{i-1}^{\mathrm{M}}(\lambda),$$

$$D_i^{\mathrm{M}}(\lambda) = -\, p_i^2 \sin(\lambda\omega_i) D_{i-1}^{\mathrm{D}}(\lambda) + p_i \cos(\lambda\omega_i) D_{i-1}^{\mathrm{M}}(\lambda).$$

Let us notice that $D_q^{\mathrm{M}}(\lambda)$ gives the eigenvalues of the transmission problem with mixed boundary condition on the external boundary, i.e. we replace the Dirichlet condition (2.28) on Γ_0^1 by

$$\gamma_0^1 \frac{\partial u_1}{\partial \nu^1} = 0 \quad \text{on } \Gamma_0^1. \tag{5.5}$$

From [21, Theorem 2.2], we know that the eigenvalues of $\mathscr{A}_S(\lambda)$ are real, since they can be defined as eigenvalues of a self-adjoint operator. The eigenvalues are simple, i.e. no associated eigenfunction exists; moreover, for 2 materials, we can prove that the strip $(-1/4, 1/4)$ is free of eigenvalues of $\mathscr{A}_S(\lambda)$ for arbitrary materials and angles $\sigma_2 = \omega_1 + \omega_2$.

Figures 6 and 7 show the distribution of the eigenvalues of $\mathscr{A}_S(\lambda)$ for 2 materials; $p_1 = 0.25$ (10^{11} Nm^{-2}) (glass) and $p_2 = 1.5$ (10^{11} Nm^{-2}) (molybdenum).

Figure 6 describes the eigenvalues for a domain with the reentrant corner $\omega_1 + \omega_2 = 270°$, where ω_1 runs from 0 to 270°; Fig. 7 describes a domain with a slit, namely $\omega_1 + \omega_2 = 360°$, where ω_1 runs from 0° to 360°.

Theorem 3.6 yields the following asymptotic expansion for a solution $u \in \mathscr{V}_1^{2,2}(\Omega)$:

$$\Phi_S u_1 = \Phi_S \sum_{0<\lambda_i<1} c_i r^{\lambda_i} \left[\tan \lambda_i(\omega_1 + \omega_2) \frac{\cos \lambda_i \omega_1}{\sin \lambda_i \omega_1} - 1 \right] \sin \lambda_i \omega + w_1,$$

$$\Phi_S u_2 = \Phi_S \sum_{0<\lambda_i<1} c_i r^{\lambda_i} [-\tan \lambda_i(\omega_1 + \omega_2) \cos \lambda_i \omega + \sin \lambda_i \omega] + w_2,$$

provided $\cos \lambda_i(\omega_1 + \omega_2) \neq 0$, $\sin \lambda_i \omega_1 \neq 0$. The remainder w_i have the property that $\Phi_S w_i \in V_0^{2,2}(\Omega_i)$, $i = 1, 2$. Let us point out that the singular functions are not the same on each face.

The situation for 3 materials $p_1 = 0.25$ (10^{11} Nm^{-2}), $p_2 = 0.5$ (10^{11} Nm^{-2}), $p_3 = 1.5$ (10^{11} Nm^{-2}) is shown in Fig. 8. We have considered a slit domain $\sigma_3 = \omega_1 + \omega_2 + \omega_3 = 360°$, $\omega_2 = 0°$, $\omega_2 = 90°$, $\omega_2 = 180°$ and $\omega_2 = 270°$ are fixed and ω_1 runs from 0° to ω_2.

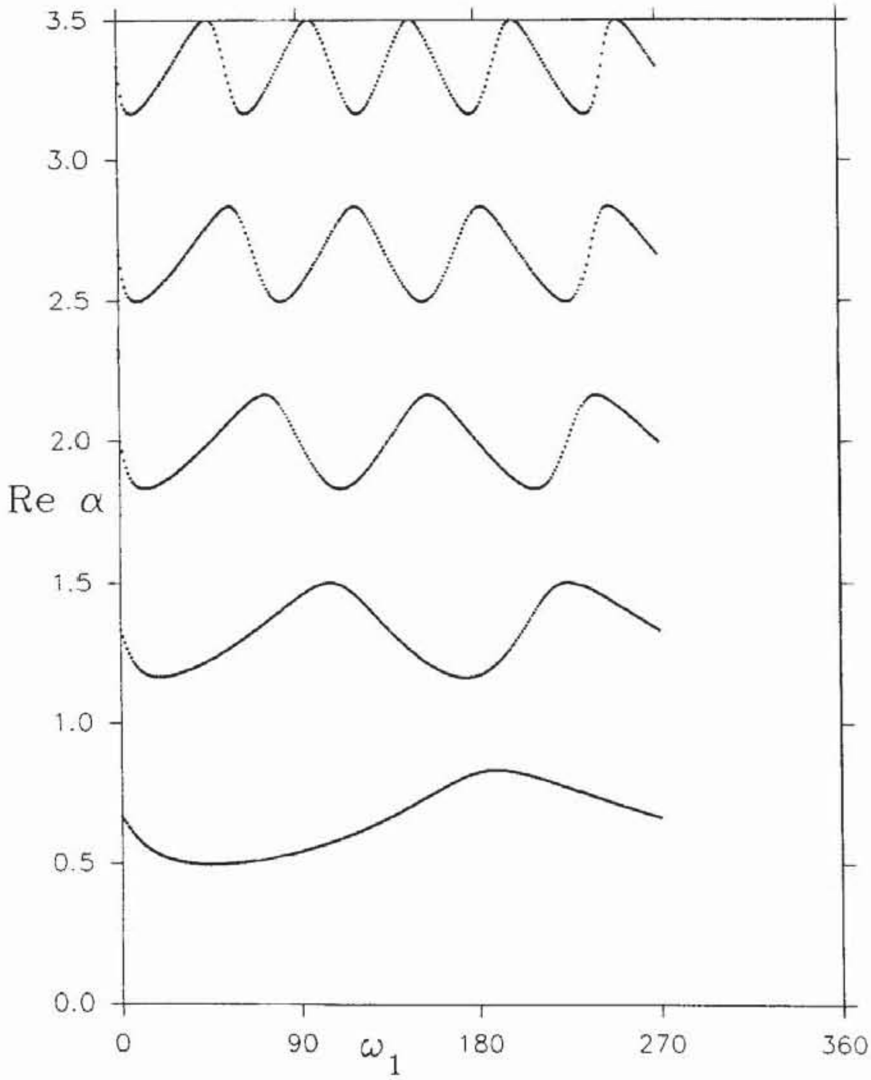

Fig. 6. Dirichlet-interface problem for Poisson's equation, $p_1 = 0.25$, $p_2 = 1.5$, $\omega_1 + \omega_2 = 270°$

Example 2. We consider an inclusion with a conical boundary point in a plane domain of a different material; see (2.31)–(2.34) and Fig. 2. We introduce for the origin S the generalized eigenvalue problem for the operator $\mathscr{A}_S(\lambda)$:

$$\frac{\partial^2 \hat{u}_i(\lambda, \omega)}{\partial \omega^2} + \lambda^2 \hat{u}_i(\lambda, \omega) = 0 \quad \text{for } 0 < \omega < \omega_i,\ i = 1, 2,\ \omega_2 = 2\pi - \omega_1, \tag{5.6}$$

$$\hat{u}_1(\lambda, 0) - \hat{u}_2(\lambda, 0) = 0. \tag{5.7}$$

$$\hat{u}_1(\lambda, \omega_1) - \hat{u}_2(\lambda, 2\pi - \omega_1) = 0, \tag{5.8}$$

$$p_1 \frac{\partial \hat{u}_1}{\partial \omega}(\lambda, 0) + p_2 \frac{\partial \hat{u}_2}{\partial \omega}(\lambda, 0) = 0, \tag{5.9}$$

$$p_1 \frac{\partial \hat{u}_1}{\partial \omega}(\lambda, \omega_1) + p_2 \frac{\partial \hat{u}_2}{\partial \omega}(\lambda, 2\pi - \omega_1) = 0. \tag{5.10}$$

Inserting the solutions $\hat{u}_i(\lambda, \omega) = C_{1i} \cos \lambda\omega + C_{2i} \sin \lambda\omega$, $i = 1, 2$, into the interface

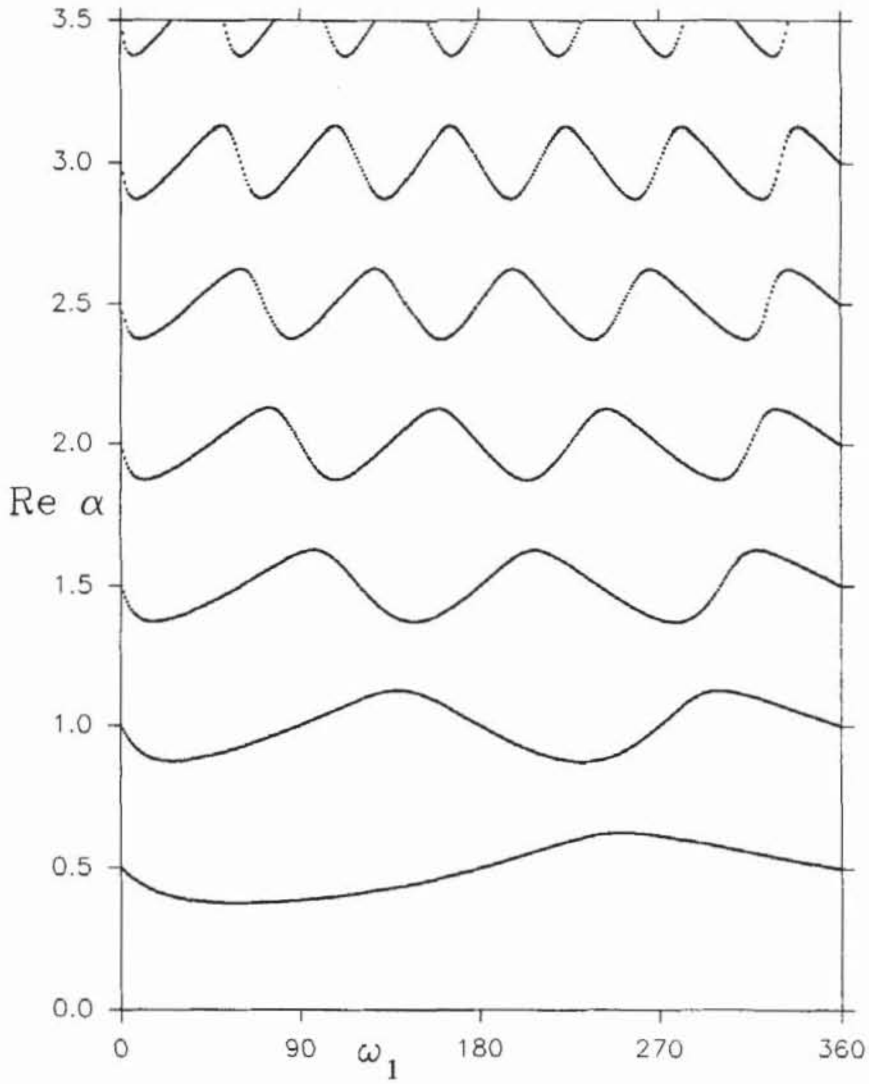

Fig. 7. Dirichlet-interface problem for Poisson's equation, $p_1 = 0.25$, $p_2 = 1.5$, $\omega_1 + \omega_2 = 360°$

conditions (5.7)–(5.10), we get non-trivial solutions $\hat{u}_i(\lambda, \omega)$ for such values λ for which the determinant

$$D^{\mathrm{I}}_{\Delta}(\lambda) = (p_1 - p_2)^2 \sin^2 \lambda(\pi - \omega_1) - (p_1 + p_2)^2 \sin^2 \lambda\pi \tag{5.11}$$

vanishes. The eigenvalues of $\mathscr{A}_S(\lambda)$ are the zeros of $D^{\mathrm{I}}_{\Delta}(\lambda)$ and are real, as in example 1.

Figure 9 shows the distribution of the eigenvalues of $\mathscr{A}_S(\lambda)$ if ω_1 runs from 0 to 360°. The number $\lambda = 0$ is an eigenvalue for all angles ω_1. It leads to constant solutions which describe the rigid-body motion.

As Fig. 9 suggests, we can show that the number $\alpha = k, k = 1, 2, \ldots$ is an eigenvalue if $\omega_1 = (\pi/k)i$, $i = 0, 1, \ldots, 2k$. In these cases, $\dim\ker \mathscr{A}_S(k) = 2$, i.e. two linearly independent solutions exist and

$$u_1 = r^k(C_1 \cos k\omega + C_2 \sin k\omega),$$

$$u_2 = r^k\left(- C_1 \frac{p_1}{p_2} \cos k\omega + C_2 \sin k\omega \right),$$

are eigensolutions of $\mathfrak{U}^S(D_x)$, where C_1 and C_2 are arbitrary coefficients.

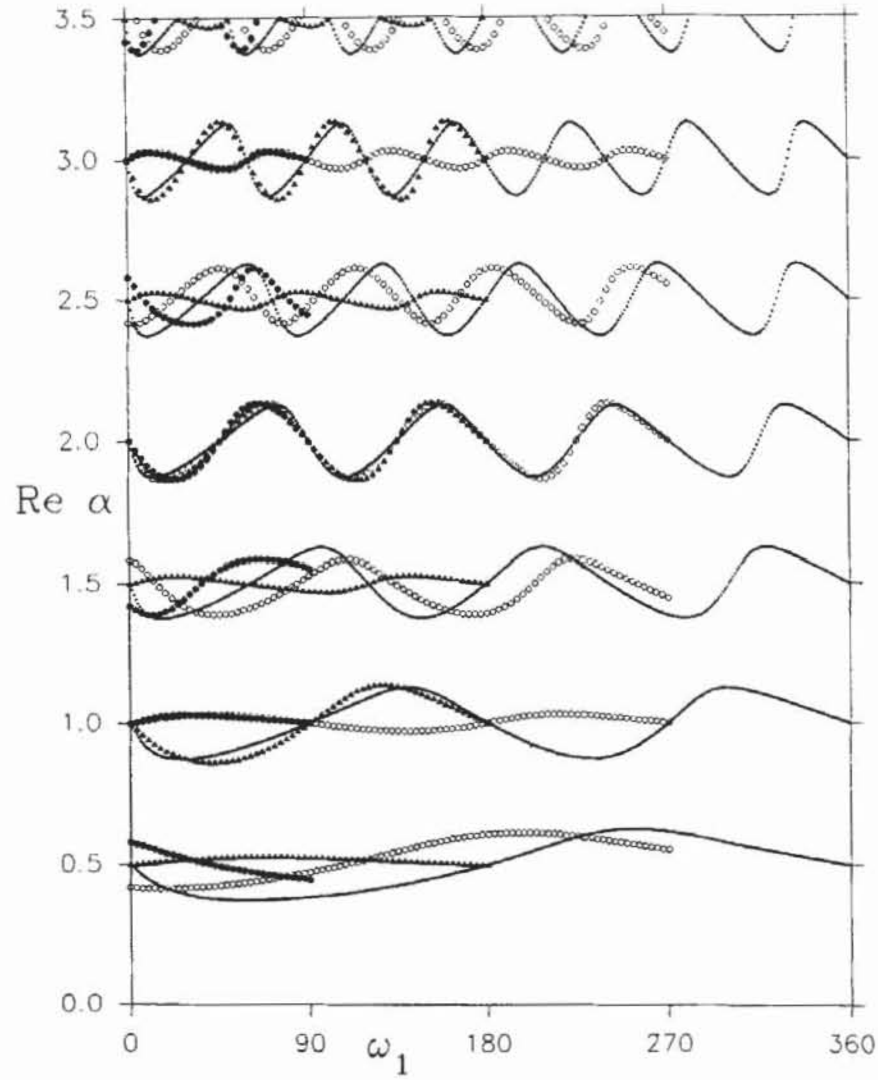

Fig. 8. Dirichlet-interface problem for Poisson's equation, $p_1 = 0.25$, $p_2 = 0.5$, $p_3 = 1.5$, $\omega_1 + \omega_2 + \omega_3 = 360°$, $\omega_2 = 90, 180, 270, 360°$

Theorem 3.6 yields the following asymptotic expansion for a solution $u \in \mathscr{V}_1^{2,2}(\Omega)$ of problem (2.31)–(2.34), provided $\omega_1 \notin \{0, \pi, 2\pi\}$:

$$\Phi_S u_1 = cr^{\lambda_0}\Phi_S\left(-\frac{p_2}{p_1}\cos\lambda_0\omega + \frac{p_2}{p_1}\frac{\sin\lambda_0\omega_1 - \sin\lambda_0(2\pi-\omega_1)}{\cos\lambda_0\omega_1 - \cos\lambda_0(2\pi-\omega_1)}\sin\lambda_0\omega\right) + w_1,$$

$$\Phi_S u_2 = cr^{\lambda_0}\Phi_S\left(\cos\lambda_0\omega + \frac{p_2}{p_1}\frac{\sin\lambda_0\omega_1 - \sin\lambda_0(2\pi-\omega_1)}{\cos\lambda_0\omega_1 - \cos\lambda_0(2\pi-\omega_1)}\sin\lambda_0\omega\right) + w_2.$$

Here $\lambda_0 = \lambda_0(\omega_1)$ is the eigenvalue of $\mathscr{A}_S(\lambda)$ in the strip $0 < \lambda_0 < 1$ and the remainders are so regular that $\Phi_S w_i \in V_0^{2,2}(\Omega_i)$, $i = 1, 2$.

Example 3.1. We investigate the behaviour of the solutions of the plate equation (2.35)–(2.42) near a boundary corner point S, where two different materials meet; Dirichlet conditions are given by (2.36), (2.37) and interface conditions by (2.38)–(2.41). We formulate for the origin S the generalized eigenvalue problem for the

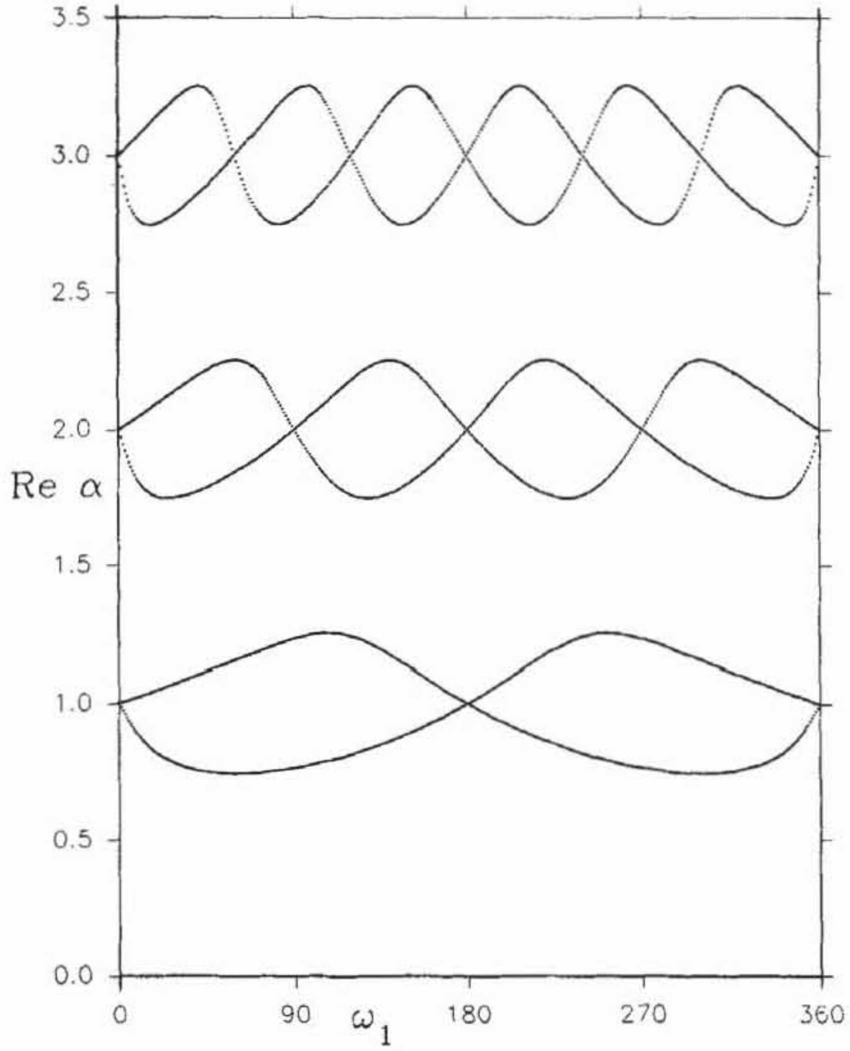

Fig. 9. Inclusion-interface problem for Poisson's equation, $p_1 = 0.25$, $p_2 = 1.5$

corresponding operator $\mathscr{A}_S(\lambda)$:

$$\frac{\partial^4 \hat{u}_i(\lambda, \omega)}{\partial \omega^4} + 2(\lambda^2 - 2\lambda + 2)\frac{\partial^2 \hat{u}_i}{\partial \omega^2}(\lambda, \omega) + \lambda^2(\lambda - 2)^2 \hat{u}_i(\lambda, \omega) = 0$$

$$\text{for } 0 < \omega < \omega_1 \quad \text{if } i = 1, \text{ and } \omega_1 < \omega < \omega_1 + \omega_2 \ \text{ if } i = 2, \tag{5.12}$$

$$\hat{u}_1(\lambda, 0) = \frac{\partial \hat{u}_1}{\partial \omega}(\lambda, 0) = 0, \tag{5.13}$$

$$\hat{u}_2(\lambda, \sigma_2) = \frac{\partial \hat{u}_2}{\partial \omega}(\lambda, \sigma_2) = 0, \tag{5.14}$$

$$\hat{u}_1(\lambda, \omega_1) = \hat{u}_2(\lambda, \omega_1), \tag{5.15}$$

$$\frac{\partial \hat{u}_1}{\partial \omega}(\lambda, \omega_1) = \frac{\partial \hat{u}_2}{\partial \omega}(\lambda, \omega_1), \tag{5.16}$$

$$\rho_1\left[\frac{\partial^2 \hat{u}_1}{\partial\omega^2}(\lambda,\omega_1)+A_1\hat{u}_1(\lambda,\omega_1)\right]=\rho_2\left[\frac{\partial^2 \hat{u}_2}{\partial\omega^2}(\lambda,\omega_1)+A_2\hat{u}_2(\lambda,\omega_1)\right], \quad (5.17)$$

$$\rho_1\left[\frac{\partial^3 \hat{u}_1}{\partial\omega^3}(\lambda,\omega_1)+B_1\frac{\partial \hat{u}_1}{\partial\omega}(\lambda,\omega_1)\right]=\rho_2\left[\frac{\partial^3 \hat{u}_2}{\partial\omega^3}(\lambda,\omega_1)+B_2\frac{\partial \hat{u}_2}{\partial\omega}(\lambda,\omega_1)\right], \quad (5.18)$$

where

$$A_i=\nu_i\lambda^2+(1-\nu_i)\lambda, \qquad B_i=(2-\nu_i)\lambda^2-3(1-\nu_i)\lambda+2(1-\nu_i) \quad i=1,2.$$

For $\lambda\notin\{0,1,2\}$ the general solutions of (5.12) have the form

$$\hat{u}_i(\lambda,\omega)=C_{1i}\sin\lambda\omega+C_{2i}\cos\lambda\omega+C_{3i}\sin(\lambda-2)\omega+C_{4i}\cos(\lambda-2)\omega. \quad (5.19)$$

Inserting these solutions in the boundary conditions (5.13), (5.14) and the interface conditions (5.15)–(5.18), we get 8 equations for the unknowns C_{ji}, $i=1,2$,

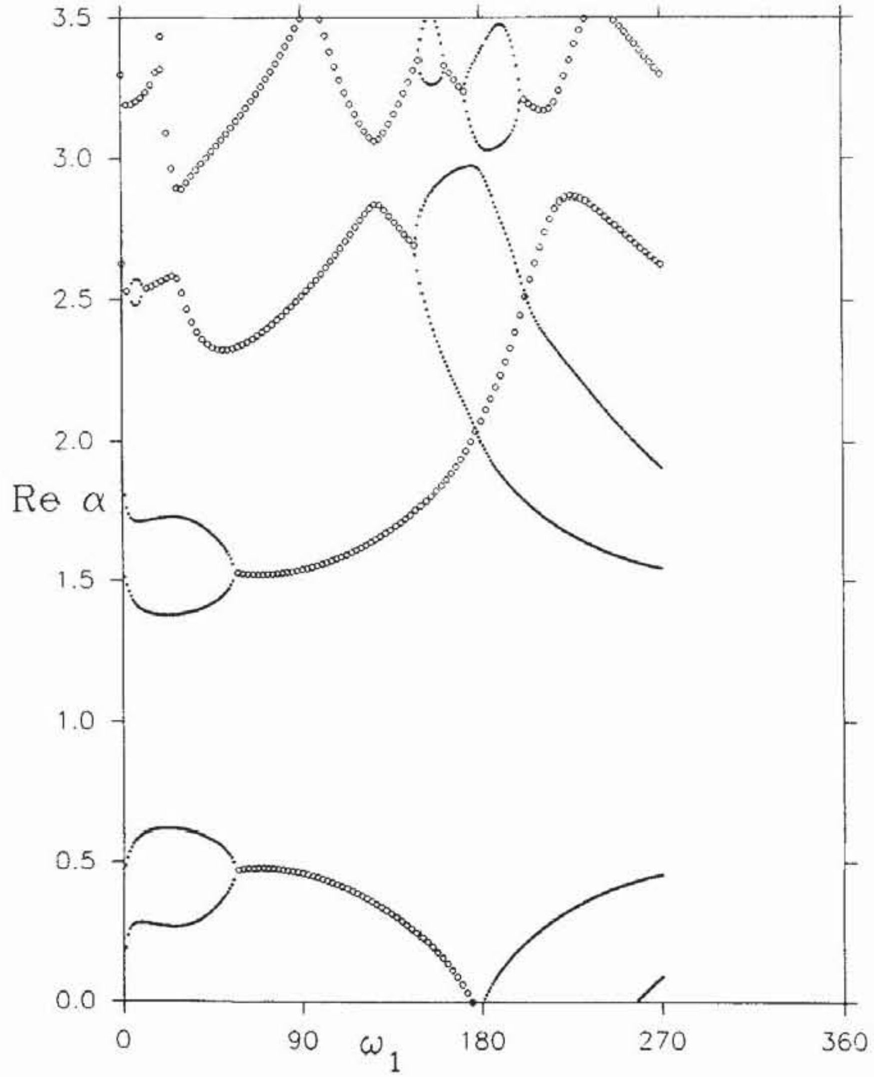

Fig 10. Dirichlet-interface problem for the biharmonic equation, $\nu_1=0.17$, $\nu_2=0.29$, $E_2/E_1=20$, $\omega_1+\omega_2=270°$.

$j = 1, 2, 3, 4$. There are non-trivial solutions $\hat{u}_i(\lambda, \omega)$ if the corresponding determinant $D^D_{\Delta^2}(\lambda)$ vanishes. The zeros of $D^D_{\Delta^2}(x)$ are the eigenvalues of $A_S(\lambda)$. Figures 10 and 11 show the distribution of the eigenvalues for the materials concrete ($\nu_1 = 0.17$) and steel ($\nu_2 = 0.29$); $E_2/E_1 = 20$, for $\omega_1 + \omega_2 = 270°$, ω_1 runs from 0 to 270° and for a slit domain $\omega_1 + \omega_2 = 360°$, ω_1 runs from 0 to 360°. The line $\lambda = 1$ is the symmetry axis. The solid lines indicate real eigenvalues, while the dotted lines indicate the real parts of the pair of the complex eigenvalues.

There are some angles $\omega_1 = \omega_{1,0}$, where $D^D_{\Delta^2}(\lambda(\omega_{1,0})) = \mathrm{d}/\mathrm{d}\lambda D^D_{\Delta^2}(\lambda(\omega_{1,0})) = 0$. They indicate such graph-points in Figs. 10 and 11 where the dotted curve pieces of the real parts of the complex eigenvalues start or end on the solid lines of the real eigenvalues. We call these points branching points of the graph (see [5]). They indicate that associated eigensolutions exist. Moreover, there are angles ω_1, where dotted or solid lines cross. In these cases there exist $j = j_1 + 2j_2$ linearly independent eigensolutions, where j_1 is the number of the crossing solid lines, j_2 is the number of the crossing dotted lines and no associated eigensolution exists. Let us remark that the above considerations can be proved analogously as in [26, 4].

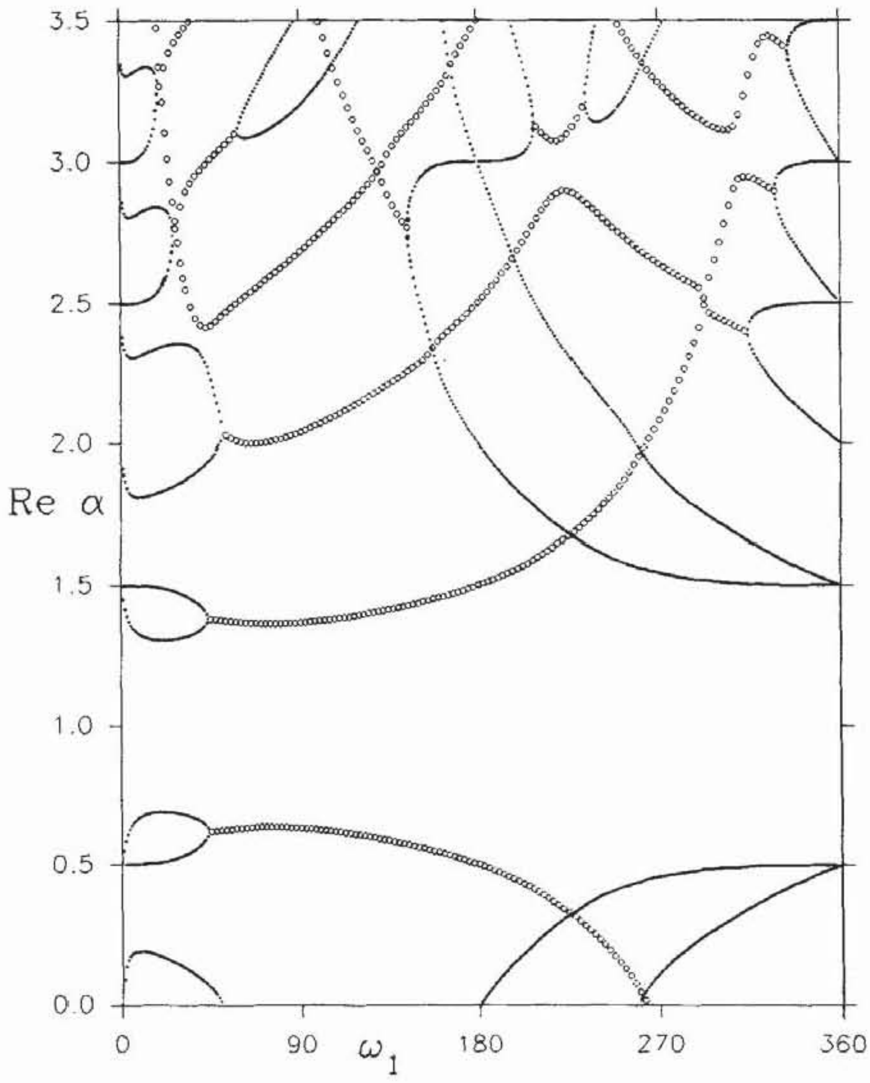

Fig 11. Dirichlet-interface problem for the biharmonic equation, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $E_2/E_1 = 20$, $\omega_1 + \omega_2 = 360°$

Example 3.2. We consider an inclusion with a conical boundary point in a plate (see Fig. 2). We introduce for the origin S a generalized eigenvalue problem for the operator $A_S(\lambda)$ analogously to (5.12)–(5.18).

Figure 12 describes the distribution of the eigenvalues for the material concrete ($\nu_1 = 0.17$) and steel ($\nu_2 = 0.29$); $E_2/E_1 = 20$. The angle ω_1 runs from 0 to 360°. The numbers $\lambda = 0$, 1 and 2 are eigenvalues, for every angle ω_1.

The interpretation of the graph leads to analogous results as in example 3.1.

Example 4. We deal with the Dirichlet interface, the Neumann interface and the pure transmission problem (inclusion) for the linear elasticity system in a plane domain, consisting of two different media. In all cases, we have chosen Poisson's ratios $\nu_1 = 0.17$ and $\nu_2 = 0.29$ and the shear modulus $\mu_2/\mu_1 = 4$.

Let $S \in \mathscr{S}$ be the origin. As usual, we introduce local spherical basic vectors

$$\mathbf{e}_r = \begin{pmatrix} \cos\omega \\ \sin\omega \end{pmatrix}, \qquad \mathbf{e}_\omega = \begin{pmatrix} -\sin\omega \\ \cos\omega \end{pmatrix}$$

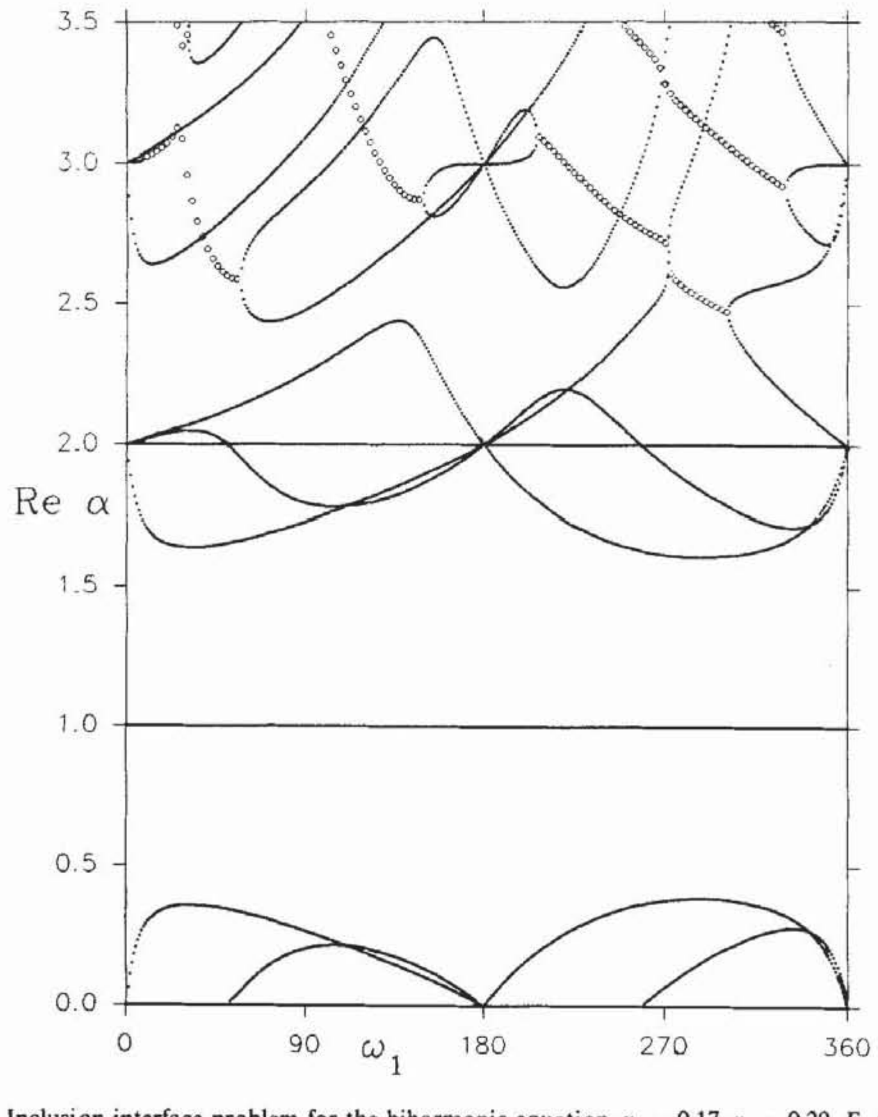

Fig. 12. Inclusion-interface problem for the biharmonic equation, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $E_2/E_1 = 20$

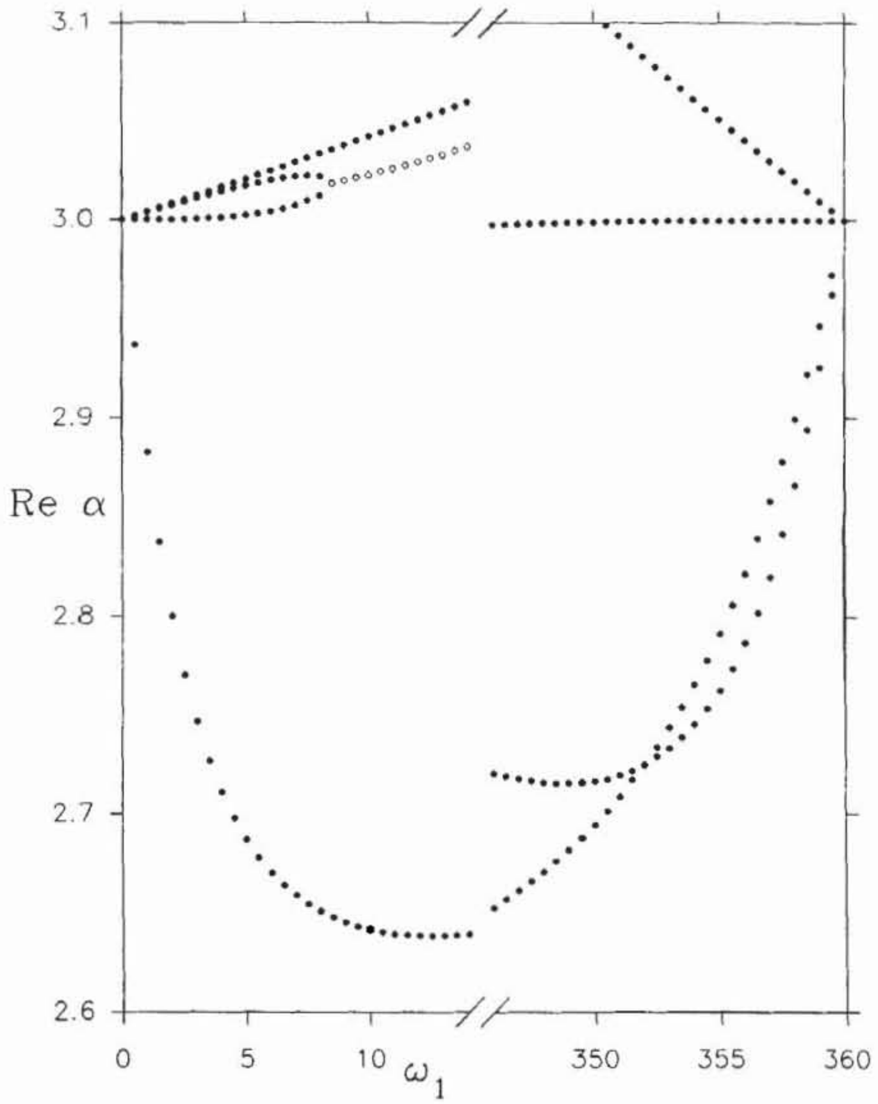

Fig. 13. Window of Fig. 12

and write the displacement vectors $\mathbf{u}_i$ as $\mathbf{u}_i = u_{r,i}(r, \omega)\mathbf{e}_r + u_{\omega,i}(r, \omega)\mathbf{e}_\omega$, or shortly

$$\mathbf{u}_i = \begin{pmatrix} u_{r,i} \\ u_{\omega,i} \end{pmatrix}.$$

The generalized eigenvalue problems for the corresponding operators $A_S(\lambda)$ have the following form.

Differential equations:

$$C_i \frac{\partial^2 \hat{u}_{r,i}(\lambda, \omega)}{\partial \omega^2} + (\lambda^2 - 1)\hat{u}_{r,i}(\lambda, \omega) + [(\lambda - 1) - C_i(\lambda + 1)] \frac{\partial \hat{u}_{\omega,i}(\lambda, \omega)}{\partial \omega} = 0, \tag{5.20}$$

$$\frac{\partial^2 \hat{u}_{\omega,i}(\lambda, \omega)}{\partial \omega^2} + C_i(\lambda^2 - 1)\hat{u}_{\omega,i}(\lambda, \omega) + [\lambda + 1 - C_i(\lambda - 1)] \frac{\partial \hat{u}_{r,i}(\lambda, \omega)}{\partial \omega} = 0 \tag{5.21}$$

for $0 < \omega < \omega_1$ if $i = 1$ and $\omega_1 < \omega < \omega_1 + \omega_2$ if $i = 2$. The constants $C_i = (1 - 2\nu_i/2(1 - \nu_i))$ depend on the medias.

Dirichlet conditions:

$$\hat{\mathbf{u}}_i(\lambda, \omega) = \mathbf{0} \quad \text{for } \omega = 0 \quad \text{if } i = 1 \text{ and for } \omega = \omega_1 + \omega_2 \quad \text{if } i = 2. \tag{5.22}$$

Neumann conditions:

$$\hat{\sigma}[\hat{\mathbf{u}}_i] \cdot \mathbf{v} = \begin{pmatrix} \hat{\sigma}_{r\omega, i}(\lambda, \omega) \\ \hat{\sigma}_{\omega\omega, i}(\lambda, \omega) \end{pmatrix} = \mathbf{0} \tag{5.23}$$

for $\omega = 0$ if $i = 1$ and for $\omega = \omega_1 + \omega_2$ if $i = 2$, where

$$\hat{\sigma}_{r\omega, i}(\lambda, \omega) = \mu_i \left[\frac{\partial \hat{u}_{r,i}(\lambda, \omega)}{\partial \omega} + (\lambda - 1)\hat{u}_{\omega, i}(\lambda, \omega) \right],$$

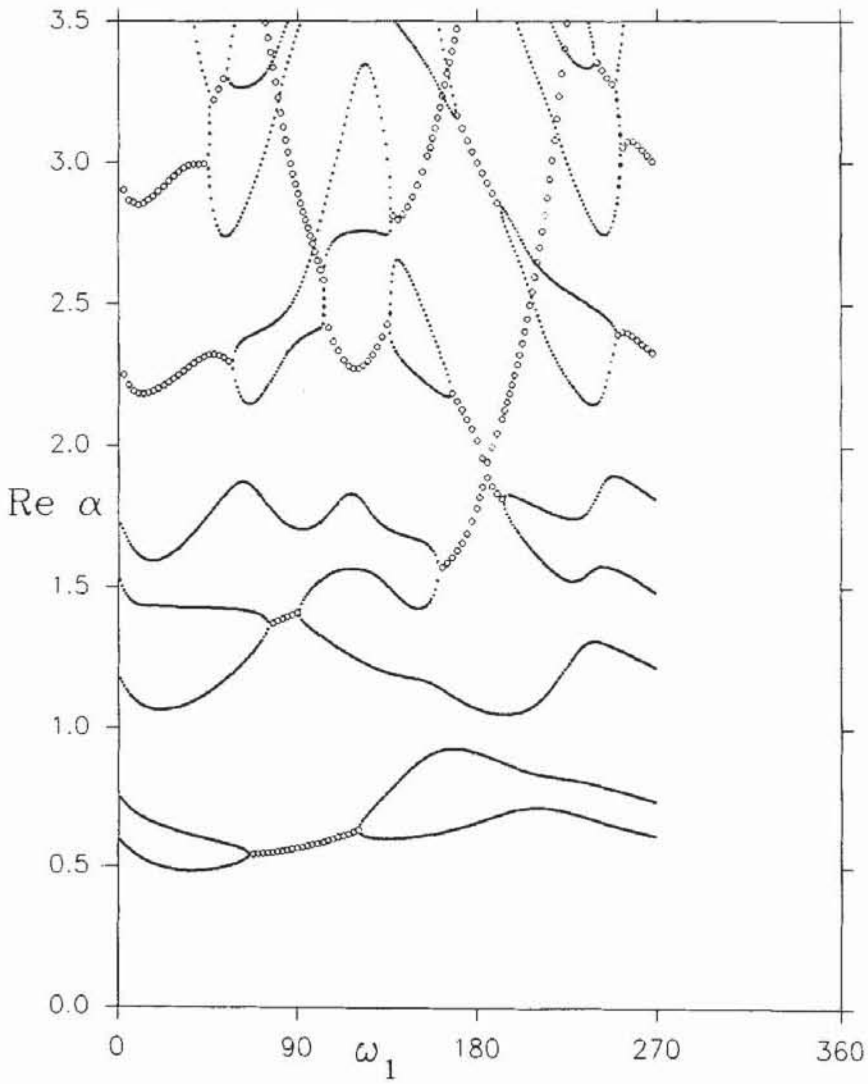

Fig. 14. Dirichlet-interface problem for Lamé's equations, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $\mu_2/\mu_1 = 4$, $\omega_1 + \omega_2 = 270°$

$$\hat{\sigma}_{\omega\omega,i}(\lambda,\omega) = 2\mu_i\left[\frac{\partial \hat{u}_{\omega,i}(\lambda,\omega)}{\partial\omega} + \hat{u}_{r,i}(\lambda,\omega)\right] + \lambda_i\left[(\lambda+1)\hat{u}_{r,i}(\lambda,\omega) + \frac{\partial \hat{u}_{\omega,i}(\lambda,\omega)}{\partial\omega}\right],$$

where μ_i and λ_i are the corresponding Lamé coefficients.

Interface conditions:

$$\hat{\mathbf{u}}_1(\lambda,\omega_1) - \hat{\mathbf{u}}_2(\lambda,\omega_1) = \mathbf{0}, \tag{5.24}$$

$$\hat{\sigma}[\hat{\mathbf{u}}_1]\cdot\mathbf{v} - \hat{\sigma}[\hat{\mathbf{u}}_2]\cdot\mathbf{v} = \mathbf{0} \text{ for } \omega = \omega_1, \tag{5.25}$$

and additionally, if we consider an inclusion-interface problem:

$$\hat{\mathbf{u}}_1(\lambda,0) - \hat{\mathbf{u}}_2(\lambda,2\pi) = \mathbf{0}, \tag{5.26}$$

$$\hat{\sigma}[u_1]\cdot\mathbf{v}(0) - \hat{\sigma}[\hat{u}_2]\cdot\mathbf{v}(2\pi) = \mathbf{0}. \tag{5.27}$$

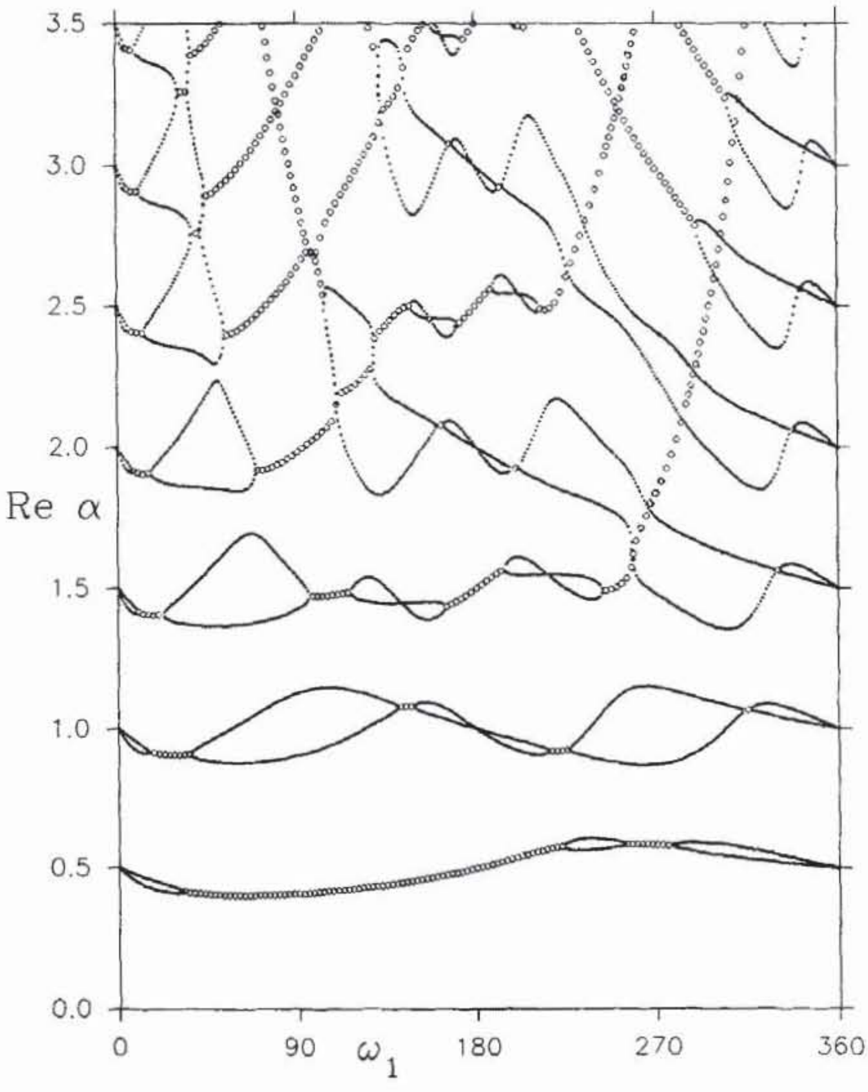

Fig. 15. Dirichlet-interface problem for Lamé's equations, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $\mu_2/\mu_1 = 4$, $\omega_1 + \omega_2 = 360°$

The general solutions of (5.20), (5.21) have for $\lambda \neq 0$ and $3 - \lambda - 4\nu_i \neq 0$ the form

$$\hat{\mathbf{u}}_i(\lambda, \omega) = C_{1i}\begin{pmatrix} 2\lambda\mu_i \sin(1+\lambda)\omega \\ 2\lambda\mu_i \cos(1+\lambda)\omega \end{pmatrix} + C_{2i}\begin{pmatrix} 2\lambda\mu_i \cos(1+\lambda)\omega \\ -2\lambda\mu_i \sin(1+\lambda)\omega \end{pmatrix}$$
$$+ C_{3i}\begin{pmatrix} \mu_i(1-\lambda)(1-D_i)\sin(1-\lambda)\omega \\ \mu_i(1+\lambda)(1-D_i)\cos(1-\lambda)\omega \end{pmatrix}$$
$$+ C_{4i}\begin{pmatrix} \mu_i(1-\lambda)(1-D_i)\cos(1-\lambda)\omega \\ -\mu_i(1+\lambda)(1-D_i)\sin(1-\lambda)\omega \end{pmatrix},$$

where $D_i = (3 + \lambda - 4v_i)/(3 - \lambda - 4v_i)$.

We insert $\hat{\mathbf{u}}_i(\lambda, \omega)$ in the corresponding boundary and interface conditions (5.22)–(5.27) and get 8 equations for the unknowns C_{ji}, $j = 1, \ldots, 4$, $i = 1, 2$, respec-

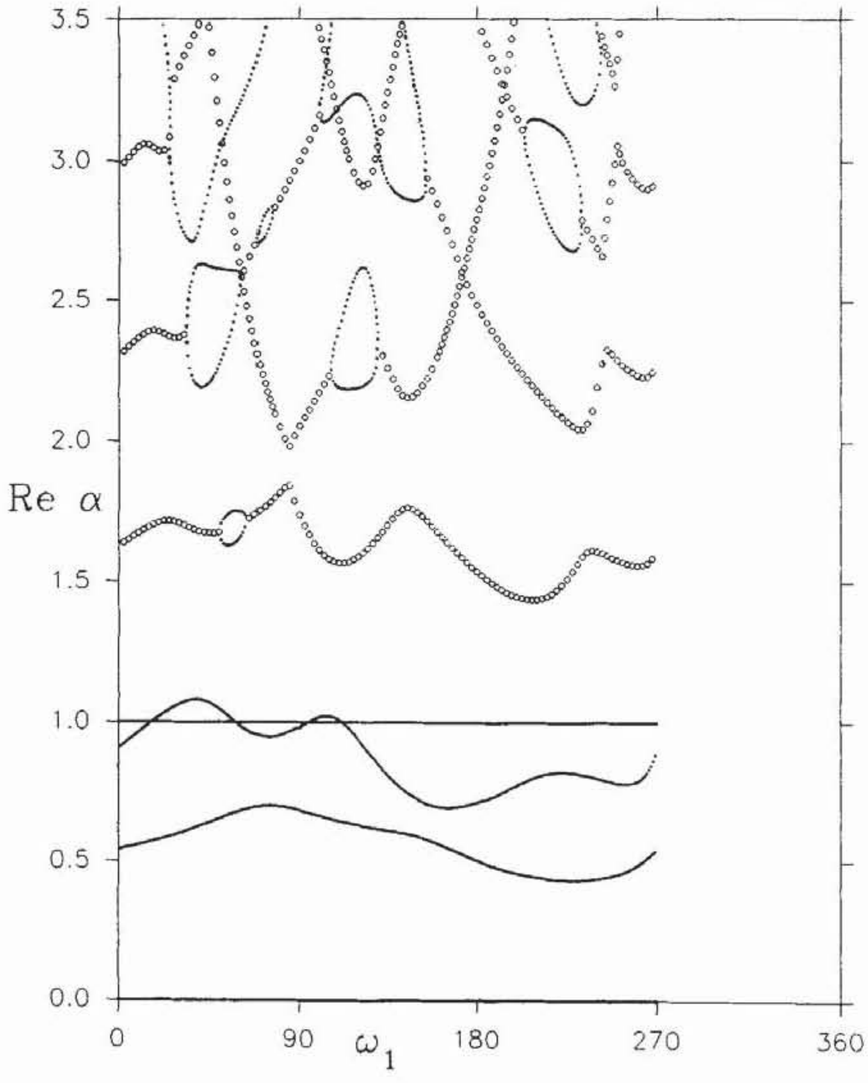

Fig. 16. Neumann-interface problem for Lamé's equations, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $\mu_2/\mu_1 = 4$, $\omega_1 + \omega_2 = 270°$

tively. There are non-trivial solutions, if the corresponding determinants $D_L^D(\lambda)$, $D_L^N(\lambda)$, $D_L^T(\lambda)$ vanish. Figures 14 and 15 show the distribution of the zeros of $D_L^D(\lambda)$, that means the distribution of the eigenvalues of the corresponding operator $A_S(\lambda)$ in a domain with a reentrant right angle and in a slit-domain.

Figures 16 and 17 describe the zeros of $D_L^N(\lambda)$ in the same domains. Let us remark that the numbers $\lambda = 0$ and $\lambda = 1$ are eigenvalues for every angle $\omega = \omega_1$.

Figure 18 gives the zeros of $D_L^T(\lambda)$, that means the eigenvalues of the inclusion interface problem. The numbers $\lambda = 0$ and $\lambda = 1$ are again eigenvalues for every angle $\omega = \omega_1$. The meaning of the solid and dotted lines is the same as in the former examples. The interpretation of branching points or points, where cross dotted and solid lines leads to analogous results as in example 3.1.

Let us remark that some numerical calculations can also be found in [25].

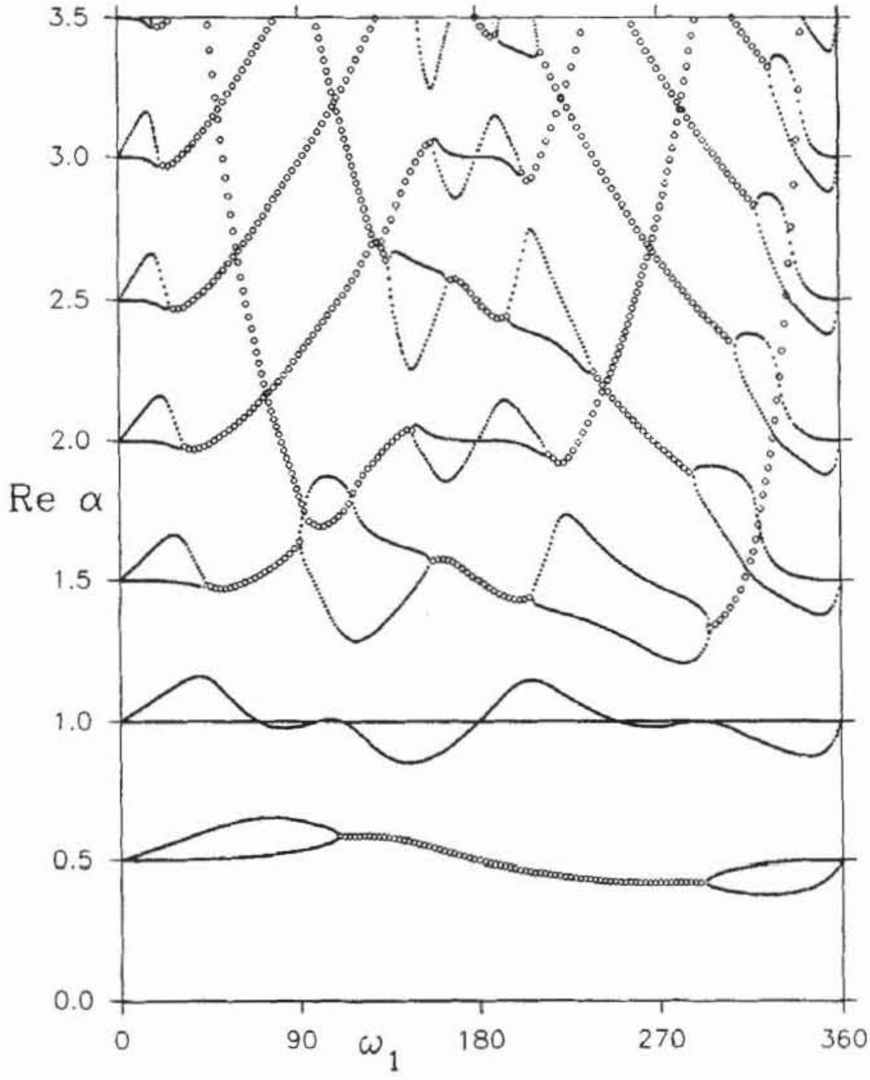

Fig. 17. Neumann-interface problem for Lamé's equations, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $\mu_2/\mu_1 = 4$, $\omega_1 + \omega_2 = 360°$

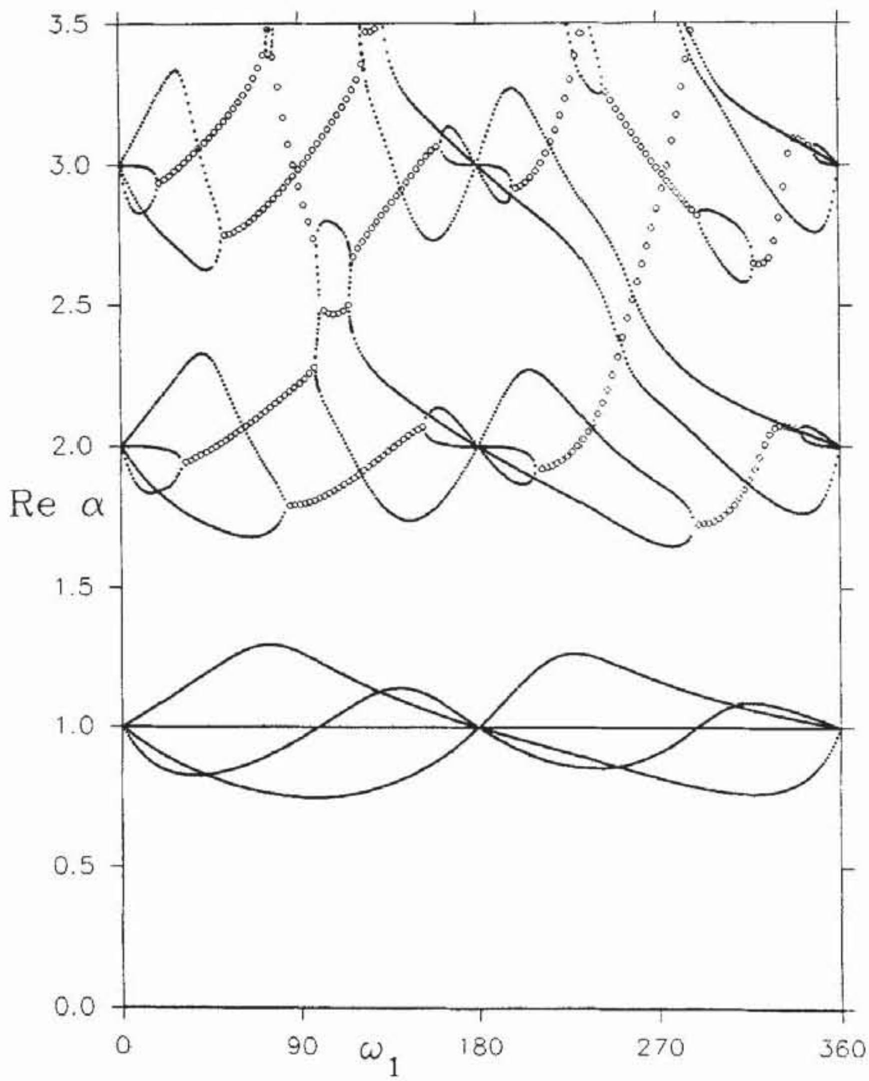

Fig. 18. Inclusion-interface problem for Lamé's equations, $\nu_1 = 0.17$, $\nu_2 = 0.29$, $\mu_2/\mu_1 = 4$

Acknowledgements

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