Structure theory for cellularly stratified diagram algebras

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Contents

Zusammenfassung

Wir untersuchen die Struktur zellulär stratifizierter Algebren A, die Gruppen-algebren von symmetrischen Gruppen oder Hecke Algebren als Teilalgebren B_l von $e_l A e_l$ haben. Schwerpunktmäßig betrachten wir Partitionenalgebren. Wir definieren Analoga $Ae_l \underset{B_l}{\otimes} M^\lambda$ von Permutationsmoduln und ihren unzerlegbaren Summanden, den Young-Moduln $Y(l,\lambda)$, in Anlehnung an die Definition von Permutationsmoduln für Brauer Algebren von Hartmann und Paget, [HP06]. Wie bei Permutationsmoduln von Gruppenalgebren von symmetrischen Gruppen finden wir eine Ordnungsrelation, so dass ein Young-Modul nur dann Summand eines Permutationsmoduls ist, falls er einen größeren Index hat. Des Weiteren geben wir hinreichende Bedingungen an, wann $Ae_l \underset{B_l}{\otimes} M^\lambda$ eine Zellfiltrierung besitzt und wann $Y(l,\lambda)$ relativ projektive Decke des Zellmoduls $Ae_l \underset{e_l A e_l}{\otimes} S_\lambda$, in Bezug auf die Kategorie $\mathcal{F}(\Theta)$ der zellfiltrierten Moduln, ist. Hierdurch ergibt sich ein neuer Beweis für die Ergebnisse von Hartmann und Paget über Permutationsmoduln von Brauer Algebren. Unsere Methode ist jedoch grundsätzlich allgemeiner anwendbar. Zum Beispiel erhalten wir Permutationsmoduln für Partitionenalgebren mit oben genannten Eigenschaften, falls die Charakteristik des zugrunde liegenden Körpers groß genug ist. Insbesondere zeigen wir, dass die Einschränkung von Zellmoduln der Partitionenalgebra zu $k\Sigma_l-\mathrm{mod}$, mit $l \leq r$, dual Specht-filtriert ist, vorausgesetzt es gilt $\mathrm{char} k > \lfloor \frac{r}{3} \rfloor$. Basierend auf den Ergebnissen von Hartmann, Henke, König und Paget, [HHKP10], wissen wir nun, dass Filtrierungsmultiplizitäten wohldefiniert sind und Schur-Weyl Dualität zwischen A und $\mathrm{End}_A(Y)$ herrscht, wobei Y eine Summe von Young-Moduln

(mit Multiplizitäten) ist. Im letzten Abschnitt untersuchen wir duale Zell- und Permutationsmoduln. Die Kategorie $\mathcal{F}_C(\Delta) \cap \mathcal{F}_C(\nabla)$ von Moduln mit Standard- und Kostandardfiltrierungen über einer quasierblichen Algebra C enthält genau so viele unzerlegbare Moduln wie die Anzahl der Standardmoduln. Im Gegensatz dazu enthält die Kategorie $\mathcal{F}_A(\Theta) \cap \mathcal{F}_A(\Xi)$ der zell- und dual zellfiltrierten Moduln über einer zellulär stratifizierten Algebra A normalerweise mehr unzerlegbare Moduln als die Anzahl der Zellmoduln.

Abstract

We study the structure of cellularly stratified algebras A with group algebras of symmetric groups or Hecke algebras as subalgebras B_l of e_lAe_l, focussing on the example of partition algebras. We define analogues $Ae_l \underset{B_l}{\otimes} M^\lambda$ of permutation modules and their indecomposable summands, the Young modules $Y(l,\lambda)$, following the ideas of Hartmann and Paget, [HP06]. As in the case of permutation modules over group algebras of symmetric groups, we see that there is an order on the indices such that a Young module appears as a summand of a permutation module only if it has larger index. We give sufficient conditions for $Ae_l \underset{B_l}{\otimes} M^\lambda$ to have a cell filtration and for $Y(l,\lambda)$ to be the relative projective cover of the cell module $Ae_l \underset{e_lAe_l}{\otimes} S_\lambda$ with respect to the category $\mathcal{F}(\Theta)$ of cell filtered modules. This gives a new proof for the results of Hartmann and Paget on permutation modules for Brauer algebras. Our method works in larger generality. For example, we obtain permutation modules for partition algebras with the above mentioned properties, in case that the characteristic of the ground field is large enough. In particular, we show that the restriction of a cell module of the partition algebra $P_k(r,\delta)$ to $k\Sigma_l-\text{mod}$, with $l \leq r$, admits a dual Specht filtration provided $\text{char} k > \lfloor \frac{r}{3} \rfloor$. Using results of Hartmann, Henke, König and Paget, [HHKP10], we obtain well-defined filtration multiplicities for modules in $\mathcal{F}(\Theta)$ and Schur-Weyl duality between A and $\text{End}_A(Y)$ holds, where Y is a sum of Young modules (with multiplicities). In the final section, we study dual cell and dual permutation modules. The category $\mathcal{F}_C(\Delta) \cap \mathcal{F}_C(\nabla)$ of modules with both standard and costandard filtration over a quasi-hereditary algebra C has as many indecompos-

able modules as the number of standard modules. In comparison, the intersection $\mathcal{F}_A(\Theta) \cap \mathcal{F}_A(\Xi)$ of cell and dual cell filtered modules over a cellularly stratified algebra A contains in general more indecomposable modules than the number of cell modules.

1. Introduction

The class of algebras studied in this thesis are diagram algebras which are cellularly stratified. There is no real definition of the notion *diagram algebra*. When we speak of a diagram algebra, we mean an associative, unital algebra whose basis can be described in terms of diagrams. Usually, a diagram has a set of dots which are connected to each other with respect to some rules. There may be some additional colourings to distinguish dots or connections from each other. Diagram algebras appear in areas like invariant theory, knot theory, algebraic Lie theory and statistical mechanics. In this thesis, we focus on the point of view from representation theory.

Probably the first algebras which have been studied in this context are the *Brauer algebras* $B_k(r, \delta)$, cf. [Kön08]. They were defined by Richard Brauer in [Bra37] to answer the following question: Let n, r be natural numbers, V an n-dimensional vector space and $V^{\otimes r}$ the r-fold tensor space $V \otimes ... \otimes V$. Then Σ_r acts on $V^{\otimes r}$ by place permutations and $GL_n(\mathbb{C})$ acts diagonally. Let $\alpha : \mathbb{C}\Sigma_r \to \mathrm{End}_{\mathbb{C}GL_n(\mathbb{C})}(V^{\otimes r})$ with

$$\alpha(\sigma)(v_1 \otimes ... \otimes v_r) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(r)}$$

and $\beta : \mathbb{C}GL_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}\Sigma_r}(V^{\otimes r})$ with

$$\beta(g)(v_1 \otimes ... \otimes v_r) = (gv_1, ..., gv_r).$$

Then *Schur-Weyl duality* says that α and β are surjective.

Question. What happens if we replace $GL_n(\mathbb{C})$ by its orthogonal subgroup $O_n(\mathbb{C})$ or its symplectic subgroup $Sp_{2n}(\mathbb{C})$?

Answer. We have the following analogues of Schur-Weyl duality:

$$\begin{array}{ccc} \mathbb{C}\Sigma_r & \xleftrightarrow{\text{Schur-Weyl duality}} & GL_n(\mathbb{C}) \\ \cap & & \cup \\ B_{\mathbb{C}}(r,n) & \xleftrightarrow{\text{Schur-Weyl duality}} & O_n(\mathbb{C}) \end{array} \qquad \begin{array}{ccc} \mathbb{C}\Sigma_r & \xleftrightarrow{\text{Schur-Weyl duality}} & GL_n(\mathbb{C}) \\ \cap & & \cup \\ B_{\mathbb{C}}(r,-n) & \xleftrightarrow{\text{Schur-Weyl duality}} & Sp_{2n}(\mathbb{C}) \end{array}$$

The Brauer algebras $B_{\mathbb{C}}(r,n)$ and $B_{\mathbb{C}}(r,-n)$ are defined as algebras with a basis of diagrams with two rows of dots, r dots in each row, where every dot is connected to exactly one other dot. The parameters n and $-n$ play a role in the multiplication of diagrams, which will be explained in Chapter 4. Of course, we can replace $\mathbb{C}$ by any field k, and n by any parameter $\delta \in k$. Schur-Weyl duality has been shown to hold in positive (odd) characteristic by Dipper, Doty and Hu [DDH08] for symplectic groups and by Doty and Hu [DH09] for orthogonal groups.

In 1996, Graham and Lehrer [GL96] defined *cellular algebras* as algebras with a multiplicative property of the basis, with the aim - amongst others - to discuss simple modules of Brauer algebras. König and Xi [KX98] reformulated this definition structurally, and in 2010, Hartmann, Henke, König and Paget [HHKP10] defined *cellularly stratified algebras* as cellular algebras with the following additional structure: As a vector space, the algebra A is made up of layers J_l', where J_l' is an inflation of a cellular algebra B_l along a vector space V_l. Each layer is generated by an idempotent e_l as a two-sided ideal of the quotient algebra A/J_{l-1}, where $J_{l-1} = \bigoplus_{i=1}^{l-1} J'_i$. The product of x from layer n and y from layer m lies in a layer with index at most $\min\{n, m\}$. We concentrate on the case where B_l is the group algebra $k\Sigma_{l'}$ of a symmetric group or its Hecke algebra $\mathcal{H}_{k,q}(\Sigma_{l'})$ for some l' related to l (e.g. $l' = l$ or $l' = 2l$). Examples for such cellularly stratified algebras are non-degenerate Brauer algebras and partition algebras. The Brauer algebra $B_k(r, \delta)$ is degenerate if and only if r is even and $\delta = 0$; the partition algebra $P_k(r, \delta)$ is degenerate if and only if $\delta = 0$.

In the representation theory of the symmetric group, permutation modules $M^\lambda = k\Sigma_l \otimes_{k\Sigma_\lambda} k$ are a powerful tool. They have a unique indecomposable summand, the Young module Y^λ, with a combinatorially defined submodule S^λ indexed by the same partition λ. In case $k = \mathbb{C}$, the set $\{S^\lambda \mid \lambda \text{ partition of } l\}$ is a complete set of simple modules. If $\operatorname{char} k = p > 0$, S^λ is not necessarily simple. However, it has a simple top D^λ if λ is p-regular. The module S^λ is called *Specht module*. The algebra $k\Sigma_l$ is cellular with cell modules $\{S^\lambda \mid \lambda \text{ partition of } l\}$ or $\{S_\lambda \mid \lambda \text{ partition of } l\}$, where S_λ is the dual of S^λ.

Hartmann and Paget [HP06] generalized this theory to Brauer algebras: They defined permutation modules $M(l, \lambda)$ for Brauer algebras such that $M(l, \lambda)$ has a unique indecomposable summand $Y(l, \lambda)$ with a quotient which is isomorphic to a cell module $\Theta(l, \lambda)$. They show that these modules have well-defined cell filtration multiplicities in case $\operatorname{char} k \neq 2$ or 3 and $\delta \neq 0$ if $r = 2$ or 4. A cell filtration is a chain of submodules $M = M_n \supset M_{n-1} \supset \ldots \supset M_1 \supset 0$

such that M_i/M_{i-1} is isomorphic to a cell module for all $i = 1, ..., n$. If the multiplicity of the cell module $\Theta(m,\mu)$ in a cell filtration of M is independent of the filtration, we say that M has well-defined filtration multiplicities. Furthermore, Hartmann and Paget show that all indecomposable summands of $M(l,\lambda)$ are of the form $Y(m,\mu)$ for some m,μ and $Y(m,\mu)$ is relative projective with respect to the category of cell filtered modules.

The aim of this thesis is to generalize this theory further, for arbitrary cellularly stratified algebras with input algebras $B_l = k\Sigma_{l'}$ or $B_l = \mathcal{H}_{k,q}(\Sigma_{l'})$. We construct permutation modules $Ae_l \underset{B_l}{\otimes} M^\lambda$ such that they have a unique summand $Y(l,\lambda)$ with the corresponding cell module $\Theta(l,\lambda) = Ae_l \underset{e_lAe_l}{\otimes} S_\lambda$ as quotient. We find sufficient conditions for a permutation module to have a cell filtration and for a Young module to be relative projective with respect to the category $\mathcal{F}(\Theta)$ of cell filtered modules. In this case, Theorem 13.1 from [HHKP10] applies, and we have Schur-Weyl duality between A and $\mathrm{End}_A(Y)$, where Y is a direct sum of Young modules $Y(l,\lambda)$ with multiplicities; each Young module appears at least once. As an application, we will re-prove the results of Hartmann and Paget for Brauer algebras. To show that the theory developed is truly more general, we will study the partition algebra and we will show that it satisfies the conditions if the characteristic of the field is large enough.

This thesis is organized as follows. In Chapter 2, we define some notation and recall the representation theory of the group algebra of the symmetric group and its Hecke algebra as far as we use it in the later chapters.

Chapter 3 is devoted to cellular algebras. We state the original definition of Graham and Lehrer [GL96] as well as the equivalent version of König and Xi [KX98]. In Section 3.1, we recall the definition of an iterated inflation from [KX99a], which plays an important role in the definition of cellularly stratified algebras in Section 3.2. The concepts of cell filtrations and relative projectivity are explained in Section 3.3, where it is shown that the category of cell filtered modules is extension-closed. Section 3.4 focusses on quasi-hereditary algebras and their relation to cellular algebras. The module category of a quasi-hereditary algebra is a highest weight category. There is a set of standard modules, indexed by weights λ, such that all composition factors have smaller weight index. An important argument for our proofs in Chapter 6 is Theorem 2 from [DR92], which, in our context, relates the set of cell modules to the set of standard modules of a quasi-hereditary algebra.

Brauer algebras are defined in Chapter 4. We explain the cellularly stratified structure and quote the results of Hartmann and Paget [HP06] on permutation modules for Brauer algebras. A complete set of cell modules for the Brauer algebra $B_k(3, \delta)$ can be found in Appendix AI.

In Chapter 5, we define partition algebras $A = P_k(r, \delta)$, which can be seen as a generalization of Brauer algebras. We explain their cellular structure using iterated inflations, as explained by Xi in [Xi99], using a diagrammatic translation of his results. The cellularly stratified structure of A follows with the definition of the idempotents e_l. In Section 5.2, we study the $(k\Sigma_l, k\Sigma_n)$-bimodule $e_lAe_n/e_lJ_{n-1}e_n$ for $n \leq l$. We detect its indecomposable summands, which are induced from exterior tensor products of tensor induced modules and exterior tensor products of Foulkes modules. In particular, we show

Theorem A. *Let* $A = P_k(r,\delta)$*,* $\mathrm{char}k > \lfloor\frac{r-n}{3}\rfloor$*,* $\delta \neq 0$ *and* $X \in k\Sigma_n-$mod*. Then the* $k\Sigma_l$*-module* $e_lAe_n/e_lJ_{n-1}e_n \underset{e_nAe_n}{\otimes} X$ *has a dual Specht filtration. In particular, restrictions of cell modules to* $k\Sigma_l-$mod *are dual Specht filtered.*

Finally, in Chapter 6, we define permutation modules and Young modules for cellularly stratified algebras, using two types of induction and two types of restriction functors. In Section 6.1, we define and examine these functors. In Section 6.2, we show that the set of cell modules forms a standard system if the characteristic of the field is different from 2 and 3, using the corresponding statements of Hemmer and Nakano for Hecke algebras, [HN04]. This allows us to relate the category of cell filtered modules to the category of standard filtered modules of a quasi-hereditary algebra. Furthermore, we show that there is a unique summand of the permutation module $Ae_l \underset{B_l}{\otimes} M^\lambda$ with quotient isomorphic to the cell module $Ae_l \underset{e_lAe_l}{\otimes} S_\lambda$. This generalizes the definition of Young modules for Brauer algebras from [HP06] to arbitrary cellularly stratified algebras with input algebras $k\Sigma_l$ or $\mathcal{H}_l$. The definitions do not depend on A being a diagram algebra. However, to motivate the notion *permutation module*, we use a diagram basis. Section 6.3 contains our main results. Here, we study the structure of the permutation and Young modules. We assume that the cellularly stratified algebra A satisfies for $n \leq l$:

(I) $J_ne_l \simeq J_{n-1}e_l \oplus J_ne_l/J_{n-1}e_l$ as right B_l-modules.

(II) $J_ne_l/J_{n-1}e_l \simeq Ae_n \underset{e_nAe_n}{\otimes} e_nAe_l/e_nJ_{n-1}e_l$ as right B_l-modules.

(III) $e_nAe_l/e_nJ_{n-1}e_l \underset{B_l}{\otimes} M^\lambda \in B_n$ – mod has a dual Specht filtration.

(IV) $e_lAe_n/e_lJ_{n-1}e_n \underset{e_nAe_n}{\otimes} S_\nu \in B_l$ – mod has a dual Specht filtration.

Then we can prove the following theorems.

Theorem B. *Let A be cellularly stratified with input algebras*

$$B_l \simeq \mathcal{H}_{k,q}(\Sigma_{l'}) \hookrightarrow e_lAe_l,$$

where q is an hth root of unity and $h \geq 4$. Let $\operatorname{char} k \neq 2, 3$ *if $q = 1$. Assume that A satisfies the conditions* (I) *to* (IV) *and let λ be a partition of l. Then $Y(l,\lambda)$ is the relative projective cover of the cell module $\Theta(l,\lambda)$ with respect to the category $\mathcal{F}(\Theta)$ of cell filtered modules.*

Theorem C. *Let A and λ be as above. Then there is a decomposition*

$$Ae_l \underset{B_l}{\otimes} M^\lambda = \bigoplus Y(m,\mu)^{a_{m,\mu}}$$

with non-negative integers $a_{m,\mu}$. Moreover, $a_{l,\lambda} = 1$.

These theorems extend the results of [HP06] for Brauer algebras, which we re-prove in Section 6.4.1. In Section 6.4.2, we introduce q-Brauer algebras and show that they fit into our setting. We show that the first two assumptions are satisfied. The assumptions (III) and (IV) are not proven to hold. We are going to study the modules from assumptions (III) and (IV) in future joint work with Dung Tien Nguyen. Section 6.4.3, together with Theorem A, explains that the non-degenerate partition algebra satisfies our assumptions. We get the following theorem.

Theorem D. *If* $\operatorname{char} k > \lfloor \frac{r}{3} \rfloor$ *and* $\delta \neq 0$, *the partition algebra* $P_k(r, \delta)$ *has permutation modules*

$$Ae_l \underset{k\Sigma_l}{\otimes} M^\lambda,$$

which are a direct sum of indecomposable Young modules. The Young modules are the relative projective covers of the cell modules $Ae_l \underset{e_l A e_l}{\otimes} S_\lambda$.

In Section 6.5, we examine what happens under dualisation. We see that the theory of dual cell modules Ξ works as well as the theory for cell modules Θ, with the exception that the notion permutation module makes more sense in the non-dual setting.

The presented proofs for Theorems C and D differ from those given by Hartmann and Paget in [HP06]. We use the more recently discovered structure of cellular stratification to isolate the statements (I),(II),(III) and (IV). We then give a general proof. This gives a recipe of how to proceed if one wants to show that a cellularly stratified algebra has permutation modules, which decompose into a direct sum of relative projective covers of cell modules. Showing that the assumptions (I) to (IV) are satisfied seems in general much easier than proving the theorems directly.

2. Preliminaries

First, we fix some notation. The set of natural numbers $\{1, 2, \ldots\}$ is denoted by $\mathbb{N}$, and $\mathbb{N}_{>n}$ denotes the set of natural numbers strictly greater than n; the field of complex numbers is denoted by $\mathbb{C}$. The union/disjoint union/intersection of sets X and Y is denoted by $X \cup Y / X \coprod Y / X \cap Y$ respectively. The notion *field* refers to a an algebraically closed field k of arbitrary characteristic char$k$, unless stated otherwise. A ring R is a ring with unit, and A is a unital, associative, finite dimensional algebra over a field k. The category of finitely generated left (right) A-modules is denoted by $A - \text{mod}$ ($\text{mod} - A$), and add M denotes the full subcategory of $A - \text{mod}$ containing all direct summands of finite direct sums of copies of M. By dim, we denote the k-dimension of the underlying vector space of a module or algebra. The identity element of a ring R is denoted by 1_R, the identity morphism of an R-module M by id_M. The composition of $f : X \to Y$ and $g : Y \to Z$ is denoted by $g \circ f := gf : X \to Z$ and the kernel of f is denoted by $ker f$.

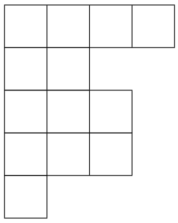
2.1 Partitions, Tableaux and Representation Theory of Symmetric Groups

In this section, we recall the basics of the representation theory of symmetric groups, which we will use in later chapters. All statements can be found in [Jam78], unless stated otherwise.

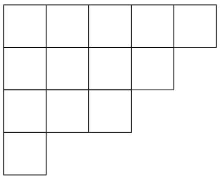
Let $n \in \mathbb{N}$. The *symmetric group* of bijections of the set $\{1, \ldots, n\}$ into itself is denoted by Σ_n, its group algebra by $k\Sigma_n$. More general, if $\mathcal{S}$ is a finite set, we denote by $\Sigma_{\mathcal{S}}$ the group of bijections $\mathcal{S} \xrightarrow{\sim} \mathcal{S}$. The groups $\Sigma_{\mathcal{S}}$ and $\Sigma_{|\mathcal{S}|}$ are isomorphic. A sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ with $\lambda_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, \ldots, t$, such that $\sum_{i=1}^{t} \lambda_i = n$, is a *composition* of n. We write $\lambda \vDash n$. A *partition* λ of n, short $\lambda \vdash n$, with m parts, is a composition with $\lambda_i \geq \lambda_{i+1}$ for all i, and m is the maximal index such that $\lambda_m \neq 0$. Let $p \in \mathbb{N} \cup \{0\}$. A partition λ is called *p-regular* if, for all $m \in \mathbb{N}$, there are at most $p-1$ parts λ_i with $\lambda_i = m$. It is called *p-restricted* if, for all $1 < i \leq n$, the difference $\lambda_i - \lambda_{i-1}$ is strictly smaller than p. If $p = 0$, we say that all partitions are p-restricted. If $\lambda_{i-1} > \lambda_i = \ldots = \lambda_{i+j} > \lambda_{i+j+1}$ holds in a partition λ, we abbreviate $\lambda =: (\ldots, \lambda_{i-1}, \lambda_i^j, \lambda_{i+j+1}, \ldots)$. The *conjugate* of the composition $\lambda \vDash n$ is the partition λ' with $\lambda'_i = |\{\lambda_j \mid \lambda_j \geq i\}|$. There is a partial order $\trianglelefteq$, called *dominance order*, on the set of partitions of n, given by $\lambda \trianglelefteq \mu \Leftrightarrow \sum_{i=1}^{s} \lambda_i \leq \sum_{i=1}^{s} \mu_i$ for all $1 \leq s \leq t$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \vDash n$. A subgroup of Σ_n which is isomorphic to the subgroup $\Sigma_\lambda := \Sigma_{\{1,\ldots,\lambda_1\}} \times \Sigma_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \ldots \times \Sigma_{\{n-\lambda_t+1,\ldots,n\}} \subseteq \Sigma_n$ is called *Young subgroup*. A *Young diagram* of shape λ is a collection of n boxes, arranged in t left-justified rows, with λ_i boxes in the

ith row. For example,



is a Young diagram of shape $\lambda = (4, 2, 3, 3, 1)$. The corresponding partition $(4, 3, 3, 2, 1)$ is 3-regular, but not 2-regular since there are two rows of equal length. The conjugate λ' is depicted by the diagram



which is obtained from the diagram of λ by a reflection along the diagonal, followed by pushing all boxes to the left.

A *Young tableau* of shape λ, or short a λ*-tableau*, is a bijection from the set $\{1, ..., n\}$ to the Young diagram of shape λ. This is denoted by filling each box of the diagram with one of the numbers $1, ..., n$. A λ-tableau is *standard*, if the entries increase along the rows and along the columns. For example,

1	2	4	5
3	6	8	
7			

is a standard $(4, 3, 1)$-tableau. The symmetric group Σ_n acts on a λ-tableau by permuting the entries. The *row stabilizer* of a tableau t is the Young subgroup of Σ_n which permutes the entries within the rows of t. Similarly, the *column stabilizer* is the Young subgroup permuting the entries within the columns. A λ-*tabloid* is an equivalence class of λ-tableaux, where two tableaux are equivalent if and only if their row stabilizers are equal. We write a tabloid like a tableau, where the vertical borders of the boxes are removed. If t is a λ-tableau, we denote the corresponding tabloid by $\bar{t}$. Let C_t be the column stabilizer of a standard tableau t. Then the corresponding λ-*polytabloid* is the signed column sum $e_t := \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma \cdot \bar{t}$.

Example. Let $t = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$. Then $C_t = \Sigma_{\{1,4\}} \times \Sigma_{\{2,5\}} \times \Sigma_{\{3\}}$ and

$$(1,4)\cdot \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 4&2&3\\\hline 1&5\\\cline{1-2}\end{array} \qquad (2,5)\cdot \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 1&5&3\\\hline 4&2\\\cline{1-2}\end{array}.$$

Hence,

$$\begin{aligned} e_t &= \begin{array}{ccc}\hline 1&2&3\\\hline 4&5&\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 4&2&3\\\hline 1&5&\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 1&5&3\\\hline 4&2&\\\cline{1-2}\end{array} + \begin{array}{ccc}\hline 4&5&3\\\hline 1&2&\\\cline{1-2}\end{array} \\ &= \begin{array}{ccc}\hline 1&2&3\\\hline 4&5&\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 2&3&4\\\hline 1&5&\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 1&3&5\\\hline 2&4&\\\cline{1-2}\end{array} + \begin{array}{ccc}\hline 3&4&5\\\hline 1&2&\\\cline{1-2}\end{array}. \end{aligned}$$

Let k be a field of characteristic 0 and λ a partition of n. Then there is a simple $k\Sigma_n$-module S^λ generated by any λ-polytabloid. This module is called *Specht module.*

Theorem 2.1 ([Jam78], Theorem 4.12)**.** *If* $\operatorname{char} k = 0$*, the set* $\{S^\lambda \mid \lambda \vdash n\}$ *of Specht modules is a complete set of simple* $k\Sigma_n$*-modules. Furthermore, the Specht modules are self-dual.*

For an arbitrary field k of positive characteristic $p > 0$, the Specht module

$$S^\lambda := k\Sigma_n \cdot e_t,$$

where t is a standard λ-tableau, is in general not simple anymore. However, for every partition λ of n and every field k, the module $D^\lambda := S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$ is either zero or simple and self-dual, by Theorem 4.9 of [Jam78], where $S^{\lambda^\perp}$ is the orthogonal complement of S^λ with respect to the bilinear form given by $\langle \bar{t_i}, \bar{t_j} \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ on the set of λ-tabloids $\{\bar{t_1}, \bar{t_2}, \ldots\}$.

Theorem 2.2 ([Jam78],Theorem 11.1)**.** *A complete set of non-zero simple modules is given by* $\{D^\lambda \mid \lambda \vdash n \text{ } p\text{-regular}\}$.

Let $\lambda \vdash n$. The $k\Sigma_n$-module whose underlying vector space has a basis consisting of all λ-tabloids is the *permutation module* $M^\lambda = k\Sigma_n \underset{k\Sigma_\lambda}{\otimes} k$. The Specht module S^λ is a submodule of M^λ. The permutation module M^λ has a decomposition into pairwise non-isomorphic, indecomposable modules Y^μ with multiplicity $a_\mu \in \mathbb{N} \cup \{0\}$, with $a_\lambda = 1$,

$$M^\lambda = \bigoplus_{\mu \trianglerighteq \lambda} (Y^\mu)^{a_\mu},$$

by Theorem 3.1 of [Jam83]. The modules Y^μ are called *Young modules*. The Young module Y^λ is the unique summand of M^λ with S^λ as submodule, see for example [Mar93, Definition 4.6.1]. The Young modules are self-dual by part (ii) of [Mar93, Lemma 4.6.2]. Hence, Y^λ is also the unique summand of M^λ such that the *dual Specht module* $S_\lambda := \mathrm{Hom}_k(S^\lambda, k)$, seen as a left module via $(\pi f)(s) := f(\pi^{-1}s)$ for $\pi \in \Sigma_n, f \in S_\lambda$ and $s \in S^\lambda$, is isomorphic to a quotient of Y^λ. By [Jam78, Theorem 8.15], S_λ is isomorphic to $S^{\lambda'} \otimes_k S^{(1^n)}$.

The Littlewood-Richardson rule

The Littlewood-Richardson rule is used to compute the multiplicity a_ν of S^ν as a composition factor of the induced module $\mathbb{C}\Sigma_n \underset{\mathbb{C}\Sigma_{(l,n-l)}}{\otimes} (S^\lambda \boxtimes S^\mu)$, where $0 < l < n, \nu \vdash n, \lambda \vdash l$ and $\mu \vdash n-l$. Here, $\boxtimes$ denotes the exterior tensor product of S^λ and S^μ, i.e. $(\sigma_1, \sigma_2) \cdot (s_1 \boxtimes s_2) := \sigma_1 s_1 \boxtimes \sigma_2 s_2$ for $\sigma_1 \in \Sigma_l, \sigma_2 \in \Sigma_{n-l}, s_1 \in S^\lambda$ and $s_2 \in S^\mu$.

The rule states that $a_\nu = 0$ if there is an index i such that $\lambda_i > \nu_i$. In contrast, if $\lambda_i \leq \nu_i$ for all i, then a_ν is the number of semistandard skew-tableaux of shape $\nu \setminus \lambda$, i.e. we remove the λ_i leftmost boxes from the ith row of the diagram of ν, such that reading the entries from right to left along the rows, we have a sequence of integers i satisfying

- There are μ_i many i's and
- $i > 1$ can only be in position $j \geq 1$ if there are more $i-1$'s than i's in the positions $1, ..., j-1$.

A tableau t is called *semistandard* if its entries are non-decreasing along successive rows and increasing along successive columns. A *skew-tableau*, in comparison to a tableau, does not necessarily have left-aligned boxes.

Example. The multiplicity of $S^{(5,3,1)}$ in $\mathbb{C}\Sigma_9 \underset{\mathbb{C}\Sigma_{(5,4)}}{\otimes} (S^{(3,2)} \boxtimes S^{(3,1)})$ is 2 since there are the three semistandard skew-tableaux

$$\begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & \boxed{1} & & \\ \boxed{2} & & & & \end{array}, \quad \begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & \boxed{2} & & \\ \boxed{1} & & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & & \boxed{1} & \boxed{2} \\ & & \boxed{1} & & \\ \boxed{1} & & & & \end{array}$$

of shape $(5,3,1) \setminus (3,2)$, but only the first two tableaux satisfy the second rule for $\mu = (3,1)$.

The characteristic-free version is due to James and Peel, [JP79]. Let k be any field and let $\beta \vdash m$ with s parts, which are all equal to α_{s+1}, where $\alpha \vdash m+n$. Then $\alpha \setminus \beta$ is made up of two partitions $\lambda \vdash l$ and $\mu \vdash n-l$, and $S^{\alpha \setminus \beta} = k\Sigma_n \underset{k\Sigma_{(l,n-l)}}{\otimes} (S^\lambda \boxtimes S^\mu)$.

Theorem 2.3 ([JP79], Theorem 5.5)**.** *Over any field k, the induced module $S^{\alpha \setminus \beta}$ has a Specht filtration.*

The theorem also gives an explicit description of the multiplicities, which is not relevant for the contents of this thesis, so we omit the statement.

2.2 Representation Theory of Hecke Algebras

In this section, we define the *Hecke algebra* $\mathcal{H}_{k,q}(\Sigma_n)$ over a field k, with parameter $q \in k$, and recall the definition of Specht and permutation modules from [DJ86].

For $\pi \in \Sigma_n$, let $l(\pi) := |\{(i,j) \mid i < j \text{ and } \pi(i) > \pi(j)\}|$ be the *length* of π. If $\pi = \sigma_1\sigma_2\ldots\sigma_s$, where the σ_i are *basic transpositions* of the form $(j, j+1)$, and s is minimal with respect to this property, then $l(\pi) = s$ and $\sigma_1\sigma_2\ldots\sigma_s$ is called *reduced expression* for π.

Definition 2.1. Let k be a field, $q \in k$ and $n \in \mathbb{N}$. The Hecke algebra $\mathcal{H} := \mathcal{H}_n := \mathcal{H}_{k,q}(\Sigma_n)$ is the unital, associative k-algebra with basis $\{T_\pi \mid \pi \in \Sigma_n\}$ subject to the relations

- $T_1 = 1_{\mathcal{H}}$.
- If $\pi = \sigma_1\sigma_2\ldots\sigma_s$ is a reduced expression, then $T_\pi = T_{\sigma_1}T_{\sigma_2}\ldots T_{\sigma_s}$.
- If σ is a basic transposition, then $T_\sigma^2 = q + (q-1)T_\sigma$.

Note that if $q = 1$, then $\mathcal{H}_n \simeq k\Sigma_n$.

Let $\lambda \vdash n$ and let t^λ be the standard tableau of shape λ, where the entries in the ith row are $(\sum_{j=1}^{i-1}\lambda_j) + 1, (\sum_{j=1}^{i-1}\lambda_j) + 2, \ldots, \sum_{j=1}^{i}\lambda_j$. Let ω_λ be the permutation such that $\omega_\lambda t^\lambda$ is the conjugate of $t^{\lambda'}$. Then $\omega_\lambda t^\lambda$ is the λ-tableau with increasing entries along successive columns. In this sense, t^λ and ω_λ can be defined for $\lambda \vDash n$ as well. Let $\mathcal{D}_\lambda := \{\pi \in \Sigma_n \mid \pi t^\lambda \text{ is row standard}\}$. A tableau t is called *row standard*, if the entries $1, ..., n$ increase along the rows of t. In every coset $\pi\Sigma_\lambda = \{\pi\sigma \mid \sigma \in \Sigma_\lambda\}$ of Σ_n/Σ_λ, there is exactly one element ρ such that ρt^λ is row standard. Hence, $\mathcal{D}_\lambda$ is a set of coset representatives of Σ_n/Σ_λ.

Set

$$
\begin{aligned}
x_\lambda &:= \sum_{\pi\in\Sigma_\lambda} T_\pi \\
y_\lambda &:= \sum_{\pi\in\Sigma_\lambda} (-q)^{-l(\pi)} T_\pi \\
z_\lambda &:= y_{\lambda'} T_{\omega_\lambda} x_\lambda
\end{aligned}
$$

The *permutation module* for the Hecke algebra with respect to the composition $\lambda \vDash n$ is defined as $M^\lambda := M_q^\lambda := \mathcal{H}x_\lambda$. Lemma 3.2 in [DJ86] states that M^λ has a basis $\{T_\rho x_\lambda \mid \rho \in \mathcal{D}_\lambda\}$ and that for

$\sigma = (i, i+1)$, T_σ acts on a basis element $T_\rho x_\lambda$ as follows:

$$T_\sigma T_\rho x_\lambda = \begin{cases} qT_\rho x_\lambda & \text{if } i, i+1 \text{ belong to the same row of } \rho t^\lambda \\ T_{\sigma\rho} x_\lambda & \text{if } i \text{ is in a higher row of } \rho t^\lambda \text{ than } i+1 \\ qT_{\sigma\rho} x_\lambda + (q-1)T_\rho x_\lambda & \text{otherwise.} \end{cases}$$

In particular, multiplication does not necessarily send a basis element to a basis element. However, $\mathcal{D}_\lambda$ is a set of coset representatives of Σ_n/Σ_λ, so $M_q^\lambda \simeq \mathcal{H} \underset{\mathcal{H}_\lambda}{\otimes} k$, where $\mathcal{H}_\lambda$ is the submodule of $\mathcal{H}$ with basis $\{T_\pi \mid \pi \in \Sigma_\lambda\}$. By comparison with the permutation module $k\Sigma_n \underset{k\Sigma_\lambda}{\otimes} k$ of the symmetric group, this motivates the term permutation module.

The *Specht module* with respect to the composition λ is defined as $S^\lambda := S_q^\lambda := \mathcal{H} z_\lambda$. By Corollary 4.2 in [DJ86], S^λ is a non-zero submodule of M^λ.

As in the case of the symmetric group, the quotient $D^\lambda := S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$ is either zero or simple and self-dual if $q \neq 0$ ([DJ86, Theorem 4.9]) and $D^\lambda \neq 0$ if and only if $y_{\lambda'} \mathcal{H} z_\lambda \neq 0$ ([DJ86, Lemma 4.10]). Let h be the minimal natural number such that $\sum_{i=0}^{h-1} q^i = 0$. Then $\{D^\lambda \mid \lambda \vdash n \ h\text{-regular}\}$ is a complete set of simple modules for $\mathcal{H}$. If D^μ is a composition factor of S^λ, then $\mu \trianglerighteq \lambda$, and D^λ occurs exactly once as composition factor of S^λ.

The permutation module M_q^λ has a unique direct summand Y_q^λ, called *Young module*, such that S_q^λ is a submodule of Y_q^λ, by [Mar93, Definition 7.6.2]. Furthermore, all summands of M_q^λ are isomorphic to some Y_q^μ with $\mu \trianglerighteq \lambda$. Again, Y_q^λ is self-dual, [Mar93, Lemma 4.6.2 (ii)]. Therefore, Y_q^λ is the unique summand of M_q^λ which has a quotient isomorphic to the *dual Specht module* $S_\lambda := S_\lambda^q := \mathrm{Hom}_k(S_q^\lambda, k) \simeq$

$S_q^{\lambda'} \otimes_k S_q^{(1^n)}$, cf. [Mat99, Exercise 3.14 (iii)] for the isomorphism.

3. Cellular Algebras

In 1996, Graham and Lehrer [GL96] defined *cellular algebras* as a class of associative algebras with a certain multiplicative property of a basis. This class includes Ariki-Koike Hecke algebras [GL96, Theorem 5.5], in particular group algebras of symmetric groups, as well as many *diagram algebras* - algebras whose basis can be written in diagrammatic form - such as Brauer and Temperley-Lieb algebras [GL96, Theorems 4.10 and 6.7]. Many others, such as the partition algebras [Xi99] and BMW algebras [Xi00], followed. In 1998, König and Xi [KX98] gave an equivalent definition using *cell ideals*. They showed that any cellular algebra is an iterated inflation [KX99a, Theorem 4.1], as will be described in Section 3.1, and studied the relationship between cellular and quasi-hereditary algebras [KX99a, §4] and [KX99b]. In 2010, Hartmann, Henke, König and Paget [HHKP10] defined *cellularly stratified algebras* by adding some extra structure to cellular algebras to get exact functors and stratifications on the level of derived categories. Cellularly stratified algebras will be introduced in Section 3.2. In Section 3.3, we examine the full subcategory of $A-\text{mod}$ consisting of all modules which have a filtration by cell modules. An introduction to quasi-hereditary algebras and their relation to cellular algebras is given in Section 3.4.

Definition 3.1 ([GL96]). Let R be a commutative ring with identity. Let A be an associative algebra with a partially ordered set Λ such that for each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$ and an injective map $C : \prod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to A$ whose image is an R-basis of A. Then A is called *cellular* with cell datum $(\Lambda, M, C, *)$ if

(1) $*$ is an R-linear anti-involution, i.e. $a^{**} = a$ and $(ab)^* = b^* a^*$ for all $a, b \in A$, such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$ for $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, where $C^\lambda_{S,T} := C(S,T) \in A$

(2) for $a \in A, \lambda \in \Lambda$ and $S, T \in M(\lambda)$ we have

$$aC^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \quad (mod A(< \lambda)) \tag{3.1}$$

where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by $\{C^\mu_{S'',T''} \mid \mu < \lambda, S'', T'' \in M(\mu)\}$.

We get a rule for right multiplication by applying $*$ to $a^* C^\lambda_{T,S}$:

$$C^\lambda_{S,T} a = (a^* C^\lambda_{T,S})^* \equiv \sum_{T' \in M(\lambda)} r_{a^*}(T', T) C^\lambda_{S,T'} \quad (mod A(< \lambda)) \tag{3.2}$$

Thus, the R-module $A(\leq \lambda)$ generated by the set $\{C^\mu_{S'',T''} \mid \mu \leq \lambda, S'', T'' \in M(\mu)\}$ is a two-sided ideal of A which is fixed by $*$.

Definition 3.2 ([GL96]). For $\lambda \in \Lambda$, let $W(\lambda)$ be the free R-module with basis $\{C_S \mid S \in M(\lambda)\}$ and left A-action

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S) C_{S'}.$$

Then $W(\lambda)$ is called *cell module.*

The corresponding right module with action $w \cdot a := a^* w$ for $w \in W(\lambda)$, $a \in A$ is denoted by $W(\lambda)^*$.

Let $\lambda \in \Lambda$, $a \in A$ and $S, T, U, V \in M(\lambda)$. By (3.1), the coefficient $r_{C^\lambda_{S,T}}(S, U)$ of $C^\lambda_{S,V}$ in the product $C^\lambda_{S,T} C^\lambda_{U,V}$ is independent of V. On the other hand, by (3.2), the same coefficient $r_{C^\lambda_{V,U}}(V, T)$ is independent of S. In particular, $C^\lambda_{S,T} C^\lambda_{U,V} \equiv r(T, U) C^\lambda_{S,V} \quad (mod A(< \lambda))$ for some $r(T, U) \in R$ which is independent of S and V. So we have a bilinear form defined on basis elements by

$$\begin{aligned} \phi_\lambda : W(\lambda) \times W(\lambda) &\longrightarrow R \\ (C_T, C_U) &\longmapsto r(T, U). \end{aligned}$$

Let

$$\operatorname{rad}(\lambda) := \{x \in W(\lambda) \mid \phi_\lambda(x, y) = 0 \quad \forall y \in W(\lambda)\}$$

and let R be a field. By [GL96, Theorem 3.4], a complete set of simple A-modules is given by $\{L(\lambda) := W(\lambda)/\operatorname{rad}(\lambda) \mid \lambda \in \Lambda, \phi_\lambda \neq 0\}$. By [GL96, Theorem 3.8], the algebra A is semisimple if and only if the non-zero cell modules $W(\lambda)$ are simple and pairwise inequivalent, if and only if $\operatorname{rad}(\lambda) = 0$ for each $\lambda \in \Lambda$.

Lemma 3.3 ([GL96], Lemma 2.2). *There is an isomorphism*

$$W(\lambda) \otimes_R W(\lambda)^* \xrightarrow{\sim} A(\leq \lambda)/A(< \lambda)$$

of (A, A)*-bimodules, sending* $C_S \otimes C_T$ *to* $C^\lambda_{S,T} + A(< \lambda)$.

From now on, let R be a commutative Noetherian integral domain, A an R-algebra and i an anti-involution of A.

Definition 3.4 ([KX98]). A two-sided ideal $J \subseteq A$ is a *cell ideal* if

(1) it is fixed by the anti-involution i and

(2) there is a left ideal $\Delta \subset J$, which is finitely generated and free over R, such that there is an isomorphism $\alpha : J \xrightarrow{\sim} \Delta \otimes_R i(\Delta)$ of (A,A)-bimodules and the diagram

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "\beta"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

commutes, where $\beta(x \otimes y) = i(y) \otimes i(x)$ for $x \in \Delta$, $y \in i(\Delta)$.

Note that, by Lemma 3.3, $A(\leq \lambda)/A(< \lambda)$ is a cell ideal of $A/A(< \lambda)$ with $\Delta = W(\lambda)$.

Definition 3.5 ([KX98]). Let $A = \bigoplus_{j=1}^{r} J'_j$ be an R-module decomposition of the algebra A such that $i(J'_j) = J'_j$ for each j. Setting $J_l := \bigoplus_{j=1}^{l} J'_j$ gives a chain of two-sided ideals $0 \subset J_1 \subset ... \subset J_r = A$. If the quotient $J_j/J_{j-1} = J'_j$ is a cell ideal of A/J_{j-1} for all j, then A is called *cellular*.

The two definitions for cellular algebras are equivalent, as shown in Section 3 of [KX98]. If the ground ring R is a field, then a cell ideal of A is either nilpotent or a heredity ideal [KX98, Proposition 4.1]. Thus there is a close connection to quasi-hereditary algebras, cf. Section 3.4.

Examples for cellular algebras are

- Group algebras $k\Sigma_n$ of symmetric groups, as specialisations of

- Hecke algebras $\mathcal{H}_{k,q}(\Sigma_n)$ of symmetric groups, see [GL96, §5].
- Brauer algebras $B_k(r,\delta)$, defined in Chapter 4.
- Partition algebras $P_k(r,\delta)$, defined in Chapter 5.

Specht modules S^λ, respectively S^λ_q, and dual Specht modules S_λ, respectively S^q_λ, are cell modules for $k\Sigma_n$, respectively for $\mathcal{H}_{k,q}(\Sigma_n)$.

3.1 Inflations

The following construction is due to König and Xi [KX99a].

Let R be a commutative Noetherian integral domain, B an R-algebra with an involution j and V a finitely generated free R-module. Let $\varphi : V \otimes_R V \to B$ be a bilinear form. Then $B \otimes_R V \otimes_R V$, the *inflation of B along V*, is an associative (in general non-unital) algebra with respect to the multiplication defined on basis elements by

$$(a \otimes x \otimes y) \cdot (b \otimes u \otimes v) \coloneqq a\varphi(y,u)b \otimes x \otimes v$$

for all $a,b \in B, x,y,u,v \in V$. If φ satisfies $j(\varphi(x,y)) = \varphi(y,x)$ for all $x,y \in V$, then an involution i on $B \otimes_R V \otimes_R V$ can be defined by setting $i(a \otimes x \otimes y) \coloneqq i(a) \otimes y \otimes x$.

Now, let $B = R \otimes_R V \otimes_R V$ and let C be a unital R-algebra. An algebra structure on $B \oplus C$ can be defined by setting the summands $B \otimes_R B \to C$, $C \otimes_R B \to C$, $B \otimes_R C \to C$ of the multiplication map $(B \oplus C) \otimes_R (B \oplus C) \to B \oplus C$ to be zero, $B \otimes_R B \to B$ and $C \otimes_R C \to C$ to be the multiplication maps of B and C, respectively, and $\beta : B \otimes_R C \to B, \gamma : C \otimes_R B \to B, \delta : C \otimes_R C \to B$ are bilinear maps subject to certain relations ensuring associativity of the multiplication given by

$$(b_1, c_1) \cdot (b_2, c_2) = (b_1 b_2 + \beta(b_1, c_2) + \gamma(c_1, b_2) + \delta(c_1, c_2), c_1 c_2).$$

If there is an element $b \in B$ satisfying

- $\forall c \in C : \delta(1_C, c) + \beta(b, c) = 0 = \delta(c, 1_C) + \gamma(c, b)$
- $\forall d \in B : (b - 1)d = \gamma(1_C, d)$ and $d(b - 1) = \beta(d, 1_C)$

then $(b, 1_C)$ is a unit element of $B \oplus C$.

B is a cell ideal of $B \oplus C$ if

- $\forall c \in C, r \otimes x \otimes y \in B : \gamma(c, r \otimes x \otimes y) \in R \otimes_R V \otimes_R y$
- $\forall c \in C, r \otimes x \otimes y \in B : \beta(r \otimes x \otimes y, c) \in R \otimes_R x \otimes_R V$.

In this case, $B \oplus C$ is called an *inflation of C along B*. Repetition of this process, with some $B' = R' \otimes_{R'} V' \otimes_{R'} V'$ and $C' = B \oplus C$, where C' is an algebra over R', is called *iterated inflation*.

Theorem 3.1 ([KX99a]). *An (iterated) inflation of a cellular algebra is cellular and any cellular algebra is the iterated inflation of finitely many copies of R.*

3.2 Cellularly Stratified Algebras

In this section, we recall the definition and some basic properties of cellularly stratified algebras from [HHKP10].

Let k be a field and let A be an iterated inflation of cellular algebras B_l along vector spaces V_l such that there is a chain of two-sided ideals

$0 \subset J_1 \subset \ldots \subset J_r = A$, which can be refined to a cell chain and such that $J_l/J_{l-1} = B_l \otimes_k V_l \otimes_k V_l$ as a unital algebra.

Definition 3.6 ([HHKP10]). A is called *celullarly stratified* with stratification data $(B_1, V_1, \ldots, B_r, V_r)$ if, in every layer of the inflation structure, there is an idempotent e_l of the form $1_{B_l} \otimes u_l \otimes v_l$ for some $u_l, v_l \in V_l \setminus \{0\}$ such that for $1 \leq m \leq l \leq r$ it is $e_l e_m = e_m = e_m e_l$.

Remark. If A is cellularly stratified and the input algebra B_l is isomorphic to a subalgebra of $e_l A e_l$, then $b \in B_l$ can be seen as an element $b \otimes u_l \otimes v_l \in A$, where u_l and v_l are the vectors from the definition of e_l. In this case, we have

$$be_l = (b \otimes u_l \otimes v_l)(1 \otimes u_l \otimes v_l) = b\varphi(v_l, u_l) \otimes u_l \otimes v_l = b \otimes u_l \otimes v_l = e_l b.$$

$\varphi(v_l, u_l) = 1$ by Remark (b) after Definition 2.1 in [HHKP10].

As an immediate consequence of the definition, we get

Lemma 3.7. *Let A be cellularly stratified and $1 \leq l \leq r$.*

(1) The cell ideal J_l is generated by the idempotent e_l, i.e. $J_l = Ae_lA$.

(2) The quotient algebra A/J_{l-1} is cellularly stratified.

(3) There is an algebra isomorphism $B_l \xrightarrow{\sim} e_l A e_l / e_l J_{l-1} e_l$ given by $1_{B_l} \mapsto e_l$.

(4) If $B_n \subseteq B_{n+1}$ for all $n < l < r$, the algebra $e_l A e_l$ is cellularly stratified with stratification data $(B_1, V_1^l, \ldots, B_l, V_l^l)$, where $V_n^l \subseteq V_n$ is a subspace such that $e_n = 1_{B_n} \otimes u_n \otimes v_n \in B_n \otimes_k V_n^l \otimes_k V_n^l$.

Proof. Parts *(1)* and *(2)* are Lemma 2.2, part *(3)* is Lemma 2.3 in [HHKP10]. Let $A = \bigoplus_{n=1}^{r} B_n \otimes_k V_n \otimes_k V_n$. Then

$$\begin{aligned} e_l A e_l &= e_l(\bigoplus_{n=1}^{r} B_n \otimes_k V_n \otimes_k V_n)e_l \\ &= e_l(\bigoplus_{n=1}^{l} B_n \otimes_k V_n \otimes_k V_n)e_l \\ &\subseteq B_l \oplus (\bigoplus_{n=1}^{l-1} B_n \otimes_k V_n \otimes_k V_n). \end{aligned}$$

Hence, $e_l A e_l = \bigoplus_{n=1}^{l} B_n^l \otimes_k V_n^l \otimes_k V_n^l$ for some $B_n^l \subseteq B_n, V_n^l \subseteq V_n$. But $e_l e_n e_l = e_n = 1_{B_n} \otimes u_n \otimes v_n \in B_n^l \otimes_k V_n^l \otimes_k V_n^l = e_l(B_n \otimes_k V_n \otimes_k V_n)e_l$ for all $n \leq l$. Since $B_n \subseteq B_l$, we have $be_l = e_l b$ for $b \in B_n$ by the remark above, so $b \otimes u_n \otimes v_n = be_n = be_l e_n e_l = e_l b e_n e_l \in B_n^l \otimes_k V_n^l \otimes_k V_n^l$. Hence $B_n^l = B_n$. □

The cellular structure of a cellularly stratified algebra A is induced from the cellular structure of the input algebras B_l. In fact, it was shown in [HHKP10], Lemma 3.4 and Proposition 4.2, that the cell modules of A are of the form $Ae_l/J_{l-1}e_l \underset{e_l A e_l}{\otimes} S_l$, where S_l is a cell module of B_l.

Proposition 3.8 ([HHKP10], Proposition 3.5)**.** *Let A be cellularly stratified. Then $e_l A/e_l J_{l-1}$ is free of rank $\dim(V_l)$ over B_l.*

Corollary 3.9. *As left B_l-module, $e_l A$ equals $e_l J_{l-1} \oplus e_l A/e_l J_{l-1}$.*

3.3 Filtrations and Relative Projectivity

Let A be a cellular algebra and $X \in A-$mod. We say that X is *filtered by cell modules* (or *cell filtered*) if there is a chain of submodules, called *filtration*,

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_1 \supset X_0 = \{0\}$$

such that for all $1 \leq i \leq n$, the quotient X_i/X_{i-1} is isomorphic to a cell module of A. We denote the class of cell filtered modules by $\mathcal{F}(\Delta)$.

When studying modules over group algebras $k\Sigma_n$ of symmetric groups, we usually choose as cell modules the dual Specht modules S_ν, $\nu \vdash n$, defined in Section 2.1. To distinguish between symmetric groups Σ_n and Σ_m, we write $\mathcal{F}_n(S)$ for the class of cell filtered $k\Sigma_n$-modules and $\mathcal{F}_m(S)$ for the class of cell filtered $k\Sigma_m$-modules.

Let $M \in \mathcal{F}(\Delta)$. We say that $\Delta(i)$ appears with *multiplicity* m in a filtration $M = M_n \supset M_{n-1} \supset \ldots \supset M_1 \supset M_0 = \{0\}$ of M if m of the quotients M_j/M_{j-1} are isomorphic to $\Delta(i)$. A module M has *well-defined filtration multiplicities*, if the multiplicity of each cell module $\Delta(i)$ in M is independent of the filtration.

It is a well-known fact that Young modules possess both Specht and dual Specht filtrations. The statement that $Y_q^\lambda \in \mathcal{H}_{k,q}(\Sigma_n)$ has a dual Specht filtration can be found in [Don98, 4.4.(3)]. Using that Y_q^λ is self-dual, we get a Specht filtration as well. We get the statement for $Y^\lambda \in k\Sigma_n -$mod by specializing to $q = 1$.

For $A = \mathcal{H}_{k,q}(\Sigma_n)$, where q is an hth root of unity with $h \geq 4$ (or char$k = p \geq 5$ in case $q = 1$), Hemmer and Nakano have shown in [HN04] that both Specht and dual Specht filtration multiplicities are well-defined, by relating the category of dual Specht filtrations to the category of standard filtered modules over the *q-Schur algebra* $S_q(n) = \mathrm{End}_{\mathcal{H}}(\bigoplus M_q^\lambda)$, where λ runs over all compositions of n.

Lemma 3.10. *The category $\mathcal{F}(\Delta)$ of cell filtered A-modules is* extension-closed*, i.e. if $0 \to X \to Y \to Z \to 0$ is exact with X and Z in $\mathcal{F}(\Delta)$, then $Y \in \mathcal{F}(\Delta)$.*

Proof. Let $X, Z \in \mathcal{F}(\Delta)$ and $0 \to X \to Y \to Z \to 0$ a short exact sequence in $A-\mathrm{mod}$. Let $0 \subset X_1 \subset ... \subset X_{m-1} \subset X_m = X$ be a cell chain for X and let $0 \subset Z_1 \subset ... \subset Z_{n-1} \subset Z_n = Z$ be a cell chain for Z. We proceed by induction on the length n of the cell chain of Z. If $n = 1$, Z is isomorphic to a cell module Δ, and $0 \subset X_1 \subset ... \subset X_{m-1} \subset X_m = X \subset Y$ is a cell chain of Y. Now, let $n > 1$. Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & X \arrow[d, "f"] & & \\
0 \arrow[r] & K_1 \arrow[r] & Y \arrow[r, "hg"] \arrow[d, "g"] & \Delta \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & Z \arrow[r, "h"] \arrow[d] & \Delta \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

By the universal property of the kernel of h, there is a unique map

$K_1 \to Z_{n-1}$ (with kernel K_2) such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_2 \arrow[d] & X \arrow[d, "f"] & & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & Y \arrow[r, "hg"] \arrow[d, "g"] & \Delta \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & Z \arrow[r, "h"] \arrow[d] & \Delta \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

is commutative. The universal property of the kernel of g then gives a unique map $K_2 \to X$ such that

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_2 \arrow[r] \arrow[d] & X \arrow[d, "f"] & & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & Y \arrow[r, "hg"] \arrow[d, "g"] & \Delta \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & Z \arrow[r, "h"] \arrow[d] & \Delta \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

is commutative. The snake lemma[1] shows that $K_1 \to Z_{n-1}$ is surjective and $K_2 \simeq X$. Hence, K_1 is the extension of the cell filtered module Z_{n-1} by the cell filtered module K_2, where Z_{n-1} has a cell

[1]See for example [ASS06, I.5.1].

chain of length $n-1$. By induction, K_1 has a cell filtration. Now the base case for the sequence $0 \to K_1 \to Y \to \Delta \to 0$ shows that $Y \in \mathcal{F}(\Delta)$. $\square$

We conclude this section with the definition of relative projective modules. An explicit construction of such modules for cellularly stratified algebras will be given in Chapter 6.

Definition 3.11 ([HHKP10], Definition 11.2)**.** Let $X, Y \in \mathcal{F}(\Delta)$.

(1) Y is called *relative projective* in $\mathcal{F}(\Delta)$ if $\mathrm{Ext}^1_A(Y, M) = 0$ for all $M \in \mathcal{F}(\Delta)$.

(2) Y is a *relative projective cover* of X if

 (i) Y is relative projective,

 (ii) there is an epimorphism $\rho : Y \twoheadrightarrow X$ with $\ker(\rho) \in \mathcal{F}(\Delta)$ and

 (iii) if Y' is relative projective and ρ' is an epimorphism $Y' \twoheadrightarrow X$ such that $\ker(\rho') \in \mathcal{F}(\Delta)$, then there is a split map $f : Y' \to Y$ such that $\rho' = \rho \circ f$.

The dual of Proposition 4.1.1 in [HN04] shows that, in case $h \geq 4$ and q an hth root of unity, the Young modules Y_q^λ are relative projective in the category of dual Specht filtered modules.

3.4 Quasi-hereditary Algebras and their Relation to Cellular Algebras

In this section, we introduce the notion of quasi-hereditary algebras and explain their connection to cellular algebras. Quasi-hereditary algebras were defined by Cline, Parshall and Scott in [CPS88].

Let k be a field, A a finite-dimensional k-algebra and $\mathrm{rad}(A)$ its Jacobson radical.

Definition 3.12. An ideal $J \subseteq A$ is called *heredity ideal* if

- J is idempotent, i.e. $J = AeA$ for some idempotent $e = e^2 \in A$,
- $J\mathrm{rad}(A)J = 0$ and
- J is projective as left (or right) A-module.

The algebra A is called *quasi-hereditary* if there is a chain

$$0 \subset J_1 \subset \ldots \subset J_s = A$$

of left (or right) ideals, such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1}.

The *heredity chain* in the above definition is reminiscent of Definition 3.5 of cellular algebras. In fact, there is a close connection:

Theorem 3.2 ([KX99b], Theorem 1.1)**.** *A cellular algebra A is quasi-hereditary if and only if A has finite global dimension, if and only if its Cartan determinant equals* 1*.*

Lemma 3.13 ([CPS88], Lemma 3.4 and [CPS89], §1)**.** *Let A be quasi-hereditary. Then the category $A-\mathrm{mod}$ is a* highest-weight category*. This means that A has enough projectives and comes equipped with a set of standard modules $\Delta(\lambda)$. The projective cover $P(\lambda)$ of a simple module $L(\lambda)$ has a filtration by standard modules $\Delta(\mu)$ with $\mu \geq \lambda$ and with $\Delta(\lambda)$ occurring exactly once.*

Let A be quasi-hereditary. Then there is a finite partially ordered set Λ indexing the isomorphism classes of simple modules. The elements $\lambda \in \Lambda$ are called *weights* in analogy with Lie theory. Let $P(\lambda)$ be the projective cover of $L(\lambda)$ and $\Delta(\lambda)$ its maximal factor module with composition factors $L(\mu)$, where $\mu \leq \lambda$. Then $\Delta(\lambda)$ is called *standard module.* The dual notion is the *costandard module* $\nabla(\lambda)$. It is defined as the maximal submodule of the injective envelope $E(\lambda)$ with composition factors $L(\mu)$, $\mu \leq \lambda$.

Remark. The standard modules are called *Weyl modules* in [CPS89], the costandard modules are called *induced modules.*

We will now collect some properties of the category $\mathcal{F}(\Delta)$ of modules with a filtration by standard modules.

Lemma 3.14. *Let A be quasi-hereditary. Then the category $\mathcal{F}(\Delta)$ of standard filtered modules is closed under direct summands.*

Proof. Let $M \in \mathcal{F}(\Delta)$ such that $M = M_1 \oplus M_2$. Let $\mathcal{S}$ be a finite set of A-modules and let $\mathrm{tr}_{\mathcal{S}}M$ denote the *trace* of $\mathcal{S}$ in M, i.e. $\mathrm{tr}_{\mathcal{S}}M$ is the sum of the images of all A-homomorphisms from modules in $\mathcal{S}$ to M. Denote by $\mathcal{P}(\geq \mu)$ and $\mathcal{P}(> \mu)$ respectively the sets $\{P(\lambda) \mid \lambda \geq \mu\}$ and $\{P(\lambda) \mid \lambda > \mu\}$. Dlab and Ringel have shown in [DR92, Lemma 1.4] that M has a standard filtration if and only if $\mathrm{tr}_{\mathcal{P}(\geq\mu)}M/\mathrm{tr}_{\mathcal{P}(>\mu)}M$ is isomorphic to a direct sum of copies of $\Delta(\mu)$ for all μ. Let P be a projective A-module and $f \in \mathrm{Hom}_A(P, M) = \mathrm{Hom}_A(P, M_1) \oplus \mathrm{Hom}_A(P, M_2)$. Then there are $f_1 \in \mathrm{Hom}_A(P, M_1)$ and $f_2 \in \mathrm{Hom}_A(P, M_2)$ such that $f = f_1 + f_2$. On the other hand, $f_1 + 0$ and $0 + f_2$ are homomorphisms from P to M by definition. Hence, $\mathrm{tr}_P M = \mathrm{tr}_P M_1 + \mathrm{tr}_P M_2$. By definition, $\mathrm{tr}_P M_1$ is a submodule of M_1. Therefore,

$$\begin{aligned}\mathrm{tr}_{\mathcal{P}(\geq\mu)}M/\mathrm{tr}_{\mathcal{P}(>\mu)}M &= \mathrm{tr}_{\mathcal{P}(\geq\mu)}(M_1 \oplus M_2)/\mathrm{tr}_{\mathcal{P}(>\mu)}(M_1 \oplus M_2)\\ &= \mathrm{tr}_{\mathcal{P}(\geq\mu)}M_1/\mathrm{tr}_{\mathcal{P}(>\mu)}M_1 \oplus \mathrm{tr}_{\mathcal{P}(\geq\mu)}M_2/\mathrm{tr}_{\mathcal{P}(>\mu)}M_2.\end{aligned}$$

We know that the left hand side is isomorphic to a direct sum of copies of $\Delta(\mu)$. Hence, so is the right hand side. □

Lemma 3.15 ([DR92], Lemma 1.5)**.** *Let Λ be the set of weights of a quasi-hereditary algebra A such that for every $M \in A-\mathrm{mod}$ with top isomorphic to $L(\lambda)$ and socle isomorphic to $L(\mu)$, where λ and μ are incomparable, there is a $\nu \in \Lambda$ such that $\nu > \lambda$ or $\nu > \mu$ and such that $L(\nu)$ appears as a composition factor of M. Then $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms.*

Lemma 3.16. *Let A be quasi-hereditary with weight set Λ as above, e.g. $\Lambda = \{1, ..., n\}$. Then the relative projective A-modules with respect to the category $\mathcal{F}(\Delta)$ are exactly the projective A-modules.*

Proof. Let $M \in \mathcal{F}(\Delta)$ be non-projective and let $P(M)$ be the projective cover of M. Then there is a non-split exact sequence $0 \to K \to P(M) \to M$, where K is the kernel of the map $P(M) \twoheadrightarrow M$. By [DR92, Theorem 1], the left A-module A is in $\mathcal{F}(\Delta)$ and so by Lemma 3.14, the direct summand $P(M)$ of a direct sum of copies of A is in $\mathcal{F}(\Delta)$. By Lemma 3.15, $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms. Therefore, $K \in \mathcal{F}(\Delta)$. In particular, we have $\mathrm{Ext}^1_A(M, K) \neq 0$ for some $K \in \mathcal{F}(\Delta)$. □

Let A be a finite-dimensional k-algebra and Λ a finite partially ordered set. The set $\{\Theta(\lambda) \mid \lambda \in \Lambda\}$ of pairwise non-isomorphic A-modules is called *standard system*, provided

- $\mathrm{End}_A(\Theta(\lambda))$ is a division ring for all $\lambda \in \Lambda$.
- $\mathrm{Hom}_A(\Theta(\lambda), \Theta(\mu)) \neq 0 \Rightarrow \lambda \geq \mu$.
- $\mathrm{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) \neq 0 \Rightarrow \lambda > \mu$.

We will use the following theorem due to Dlab and Ringel, to relate the category of cell filtered modules of a cellular algebra to the category of standard filtered modules of a quasi-hereditary algebra.

Theorem 3.3 ([DR92],Theorem 2). *Let $(\Lambda, \leq)$ be a finite partially ordered set and let A be an algebra with a* standard system $\{\Theta(\lambda) \mid \lambda \in \Lambda\}$. *Then there is a quasi-hereditary algebra C, unique up to Morita equivalence, with standard modules Δ, such that the category $\mathcal{F}_A(\Theta)$ of Θ-filtered A-modules and the category $\mathcal{F}_C(\Delta)$ of Δ-filtered C-modules are equivalent.*

Remark. (a) The ordering conditions in a standard system are opposite to the usual ordering conditions for quasi-hereditary algebras.

(b) Dlab and Ringel used the more general notion of a *standardizable set* to prove the theorem for arbitrary abelian categories. The definition of a standard system comes from [HHKP10], where Theorem 3.3 is applied to module categories of cellularly stratified algebras. For a module category, the notions standard system and standardizable set are equivalent.

Corollary 3.17. *Let A be cellular such that the cell modules form a standard system $\{\Theta(\lambda) \mid \lambda \in \Lambda\}$. Then the category $\mathcal{F}(\Theta)$ is closed under direct summands.*

Proof. The functor establishing the equivalence in Theorem 3.3 is

$$F := \mathrm{Hom}_A(P, -) : \mathcal{F}(\Theta) \xrightarrow{\sim} \mathcal{F}(\Delta)$$

for some $P \in A-\mathrm{mod}$. The functor $\mathrm{Hom}_A(P,-)$ is defined on the whole module category, not just on $\mathcal{F}(\Theta)$, and it is additive. Let $M = M_1 \oplus M_2$ and $M \in \mathcal{F}(\Theta)$. Then $F(M_1) \oplus F(M_2) = F(M) \in \mathcal{F}(\Delta)$. But $\mathcal{F}(\Delta)$ is closed under direct summands by Lemma 3.14, so $F(M_1)$ and $F(M_2)$ are in $\mathcal{F}(\Delta)$. Applying the inverse functor $H : \mathcal{F}(\Delta) \xrightarrow{\sim} \mathcal{F}(\Theta)$ shows that $HF(M_1) \simeq M_1$ and $HF(M_2) \simeq M_2$ are in $\mathcal{F}(\Theta)$. □

Note that the property of being closed under kernels of epimorphisms is not preserved under exact equivalences. In particular, $\mathrm{Ext}^1_A(P, M)$ is in general non-zero for $M \notin \mathcal{F}(\Theta)$.

The following statement examines the intersection of standard and costandard filtered modules.

Proposition 3.18 ([DR92], Proposition 3.1)**.** *Let A be a quasi-hereditary algebra. Then there are indecomposable modules $T(\lambda)$, $\lambda \in \Lambda$, and exact sequences*

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0$$

and

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0,$$

where $X(\lambda)$ is filtered by standard modules $\Delta(\mu)$ with $\mu < \lambda$ and $Y(\lambda)$ is filtered by costandard modules $\nabla(\mu)$ with $\mu < \lambda$, such that *the module* $T = \bigoplus_{\lambda \in \Lambda} T(\lambda)$ *satisfies*

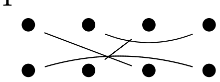
$$\text{add } T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla).$$

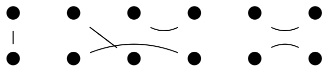
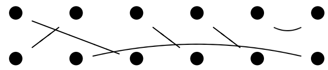
The module T in the above Proposition is a generalised tilting and cotilting module called *characteristic (tilting) module.*

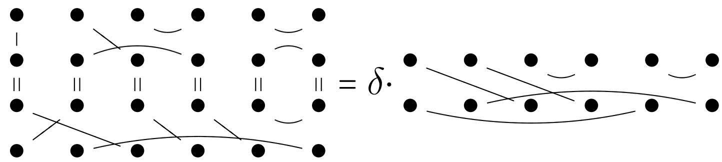
4. Brauer Algebras

The Brauer algebra was first studied by Richard Brauer in [Bra37] to answer the question which algebra replaces $k\Sigma_n$ in the Schur-Weyl duality when we replace GL_n by one of its subgroups O_n or Sp_{2n}, as we explained in the introduction. In this chapter, we define Brauer algebras and state the results of Hartmann and Paget on permutation modules for Brauer algebras, which we will generalize in Chapter 6.

Let k be an algebraically closed field of arbitrary characteristic and let $\delta \in k$. Let $r \in \mathbb{N}$.

The Brauer algebra $B_k(r, \delta)$ is the associative k-algebra whose underlying vector space has a basis consisting of diagrams with two rows of r dots each, where each dot is connected to exactly one other dot. We call a connection between a dot of the top row and a dot of the bottom row a *through string*, a connection between two dots of the same row is called *(horizontal) arc*. We denote an arc from dot n to dot m by (n, m). A Brauer diagram d is uniquely defined by its top row $\mathrm{top}(d)$, its bottom row $\mathrm{bottom}(d)$ and the permutation $\Pi(d)$ induced by the through strings. For example, $d =$  has $\mathrm{top}(d) =$ , $\mathrm{bottom}(d) =$  and $\Pi(d) = (12) \in \Sigma_2$. Multiplication of two diagrams x and y is as follows: Write x on top of y. Identify the ith dot of $\mathrm{bottom}(x)$ with the ith dot of $\mathrm{top}(y)$ via vertical lines. Replace all *circles*, i.e. connections of dots which are not connected to top(x) or bottom(y), by a factor δ. Remove the remaining dots of bottom(x) and top(y). This procedure, without the factor δ, is called *concatenation* of diagrams.

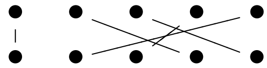
Example. Let $x =$  and $y =$  in $B_k(6, \delta)$. Then

$xy =$ 

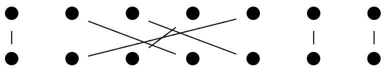
The group algebra of the symmetric group Σ_r is a subalgebra of $B_k(r, \delta)$, where a permutation $\pi \in \Sigma_r$ corresponds to the diagram d, where the ith dot of top(d) is connected to the $\pi(i)$th dot of bottom(d). In particular, there is no horizontal arc in d. For $l < r$, the symmetric group Σ_l can be embedded into the symmetric group Σ_r as usual: In the diagram for the element of Σ_l, add dots $l+1, ..., r$ in both top and bottom row. Connect these dots vertically, i.e. dot m in the top row is connected to the dot m in the bottom row for $m > l$. If $r - l$ is even, we embed the Brauer algebra $B_k(l, \delta)$ into $B_k(r, \delta)$ in a slightly different way: A diagram $d \in B_k(l, \delta)$ becomes a diagram in $B_k(r, \delta)$ by attaching arcs $(l+1, l+2), ..., (r-1, r)$ in both rows. Hence, there are two essentially different ways to embed $k\Sigma_l$ into $B_k(r, \delta)$:

$$\begin{tikzcd}
 & k\Sigma_r \arrow[dr, hook] & \\
k\Sigma_l \arrow[ur, hook] \arrow[dr, hook] & & B_k(r,\delta) \\
 & B_k(l,\delta) \arrow[ur, hook] &
\end{tikzcd}$$

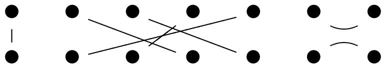
Example. $\pi = (2435) \in \Sigma_5$ embeds into $B_k(5, \delta)$ as

 $\in B_k(5, \delta)$

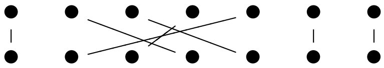
and into Σ_7 as

 $\in \Sigma_7$.

Hence, π corresponds to the diagram

 $\in B_k(5, \delta) \subset B_k(7, \delta)$

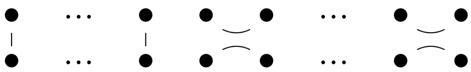
as well as to the diagram

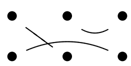
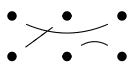
 $\in k\Sigma_7 \subset B_k(7, \delta)$.

Which of the two embeddings is used will be clear from the context.

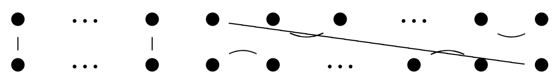
Theorem 4.1 ([KX99b], Theorem 1). *The Brauer algebra $B_k(r, \delta)$ is quasi-hereditary if and only if* $\operatorname{char} k \in \mathbb{N}_{>r} \cup \{0\}$ *and $\delta \neq 0$ if r is even.*

The Brauer algebra is one of the main examples for cellular algebras. In [GL96], the cellularity, with respect to the involution i which turns a diagram upside down[1], is shown. It is also one of the main examples for cellularly stratified algebras in [HHKP10], provided $\delta \neq 0$ if r is even. If r is odd, δ can be chosen arbitrarily. We choose the following (scalar multiples of) diagrams for the idempotents e_l with even difference $r - l$ from Definition 3.6:

$e_l = \frac{1}{\delta^{\frac{r-l}{2}}} \cdot$  if $\delta \neq 0$

[1]For example, $i\Big($  $\Big) =$ .

and

$e_l =$  if $\delta = 0$,

where both diagrams have exactly l through strings. Note that if $\delta = 0$ and r is even, we cannot define an idempotent e_0 in the above manner. Hence, we exclude this case.

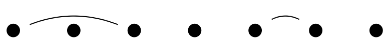
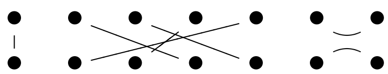
Let V_l be the vector space with basis of *partial diagrams* with $\frac{r-l}{2}$ arcs. A partial diagram is a diagram with only one row of r dots, and each dot is connected to at most one other dot. The cell modules of $B_k(r,\delta)$ are of the form $V_l \underset{k}{\otimes} S_\lambda$ by [HP06]. The module structure is given as follows: Let $a \in A = B_k(r,\delta)$ and $v \otimes s \in V_l \underset{k}{\otimes} S_\lambda$. Then

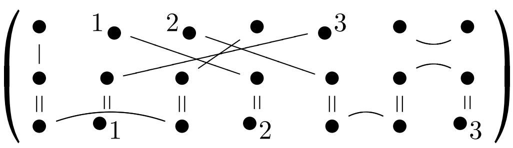
$$a \cdot (v \otimes s) = av \otimes \pi(a,v)s,$$

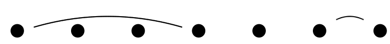
where

- av is the partial diagram obtained by placing a on top of v, identifying bottom(a) with v and following just the lines starting and ending in top(a), multiplied by δ^t, where t is the number of circles. Set $av := 0$ if the diagram is not in V_l.

- $\pi(a,v)(i) = j$ if the ith free dot of av is connected to the jth free dot of v in the diagram with a on top of v. A *free dot* is a dot which does not belong to any horizontal arc.

Example. Let $v =$  $\in V_3$ and $s \in S^\lambda$ for some $\lambda \vdash 3$. Let $a =$  $\in B_k(7,\delta)$. Then

$av = \text{top}$ 

$=$ 

and $\pi(a,v) = (123)$.

By [HHKP10], there is also a set of cell modules of the form $Ae_l \underset{e_l A e_l}{\otimes} S_\lambda$. Here, the module structure can be described much easier as

$$a \cdot (be_l \otimes s) = abe_l \otimes s.$$

We explain in Section 6.4.3 that these two sets of cell modules coincide.

The restriction of a cell module to $k\Sigma_r - \mathrm{mod}$ has a dual Specht filtration by [Pag07, Proposition 8]. In [HP06], Hartmann and Paget defined permutation modules $Ae_l \underset{k\Sigma_l}{\otimes} M^\lambda$. They show that these permutation modules inherit a lot of structure from the permutation modules for the symmetric group: $Ae_l \underset{k\Sigma_l}{\otimes} M^\lambda$ decomposes into so-called Young modules $Y(m,\mu)$, which are defined as the unique indecomposable summand of $Ae_m \underset{k\Sigma_m}{\otimes} M^\mu$ with quotient $V_m \underset{k}{\otimes} S_\mu$ and the $Y(m,\mu)$ are relative projective with respect to the category of cell filtered modules. A Young module $Y(l,\lambda)$ is projective if and only if λ is p-restricted, where $p = \mathrm{char} k \neq 2$ or 3. Furthermore, the filtration multiplicities of cell modules in a cell filtered module are independent of the choice of filtration.

In Chapter 6, we define permutation and Young modules for arbitrary cellularly stratified algebras which satisfy certain conditions. We show that the above mentioned statements hold in this general setting. In Section 6.4.1, we re-prove the statements for Brauer algebras, using the more abstract definition from [HHKP10]. These proofs use less of the combinatorial aspects of Brauer algebras. An example of all cell modules and permutation modules for $B_k(3,\delta)$ can be found in Appendix AI.

5. Partition Algebras

In this chapter, we define the partition algebra $P_k(r,\delta)$, which will provide an example for applications of the results from Chapter 6. It is well-known that $P_k(r,\delta)$ is cellular and that the group algebras $k\Sigma_n$ are subalgebras of $P_k(r,\delta)$ for every $n \leq r$. However, it is not known whether the restrictions of cell modules of $P_k(r,\delta)$ to $k\Sigma_n-\mathrm{mod}$ have dual Specht filtrations, i.e. filtrations such that the subquotients are cell modules of $k\Sigma_n$. We give an affirmative answer to this question in Section 5.2, provided the characteristic of the field is large enough. This is done by decomposing the $(k\Sigma_l, k\Sigma_n)$-bimodule $e_lAe_n/e_lJ_{n-1}e_n$, where $A = P_k(r,\delta)$, e_l, e_n and e_{n-1} are idempotents of the form $\begin{array}{cccc} \bullet & \cdots & \bullet & \bullet - \cdots - \bullet \\ \vdots & & \vdots & \vdots \\ \bullet & \cdots & \bullet & \bullet - \cdots - \bullet \end{array}$, $J_{n-1} = Ae_{n-1}A$ and $n \leq l$. The indecomposable summands U_v, corresponding to equivalence classes of top rows of diagrams in e_lAe_n, are then examined with regard to dual Specht filtrations. In multiple steps, we find tensor factorisations for U_v which we refine until we have a good enough understanding of the factors. These factors include tensor induced modules and Foulkes modules, which are known to have dual Specht filtrations in large characteristic.

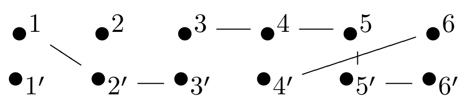
The partition algebra was defined by Martin [Mar94], and independently by Jones [Jon94], as a generalization of the Temperley-Lieb algebra, to study the Potts model in statistical mechanics.

Let k be an algebraically closed field of arbitrary characteristic and let $\delta \in k$. Let $r \in \mathbb{N}$.

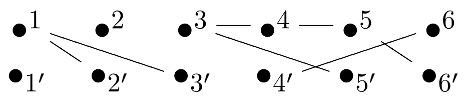
The partition algebra $P_k(r, \delta)$ is the associative k-algebra with basis consisting of set partitions of $\{1, ..., r, 1', ..., r'\}$. A *set partition* of a set X is a collection of pairwise disjoint subsets $X_i \subseteq X$, such that $\coprod X_i = X$. Regarding $P_k(r, \delta)$ as a diagram algebra, this means that the basis consists of diagrams with two rows of r dots each (top row labelled by $1, ..., r$ and bottom row labelled by $1', ..., r'$), where dots which belong to the same part of the partition are connected transitively. Note that this description is not unique. For example, the set partition

$$\{\{1, 2', 3'\}, \{2\}, \{3, 4, 5, 5', 6'\}, \{6, 4'\}, \{1'\}\}$$

corresponds, among others, to the diagram



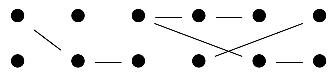
as well as to the diagram



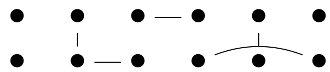
Multiplication is given by concatenation of diagrams (writing one diagram on top of the other), where parts which have no dot in top or bottom row are replaced by δ, like in the case of the Brauer algebra.

We usually omit the labels $1, .., r, 1', ..., r'$.

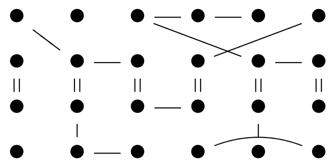
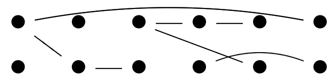
Example 5.1. Let

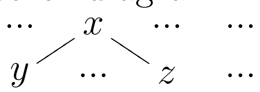
$x =$ 

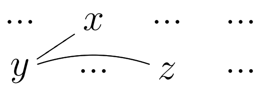
and

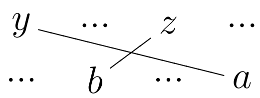
$y =$ 

Then

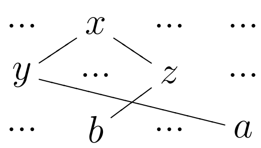
$xy =$  $= \delta \cdot$ 

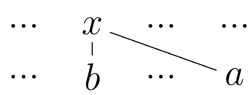
Multiplication is independent of the choice of diagram: Assume that (x, y) and (x, z) are arcs in d, i.e. $d =$ 

, (x, y) and (y, z) are arcs in d', i.e. $d' =$ 

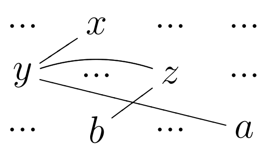
, and d and d' describe the same set partition. Assume further that (y, a) and (z, b) are arcs in e, i.e. $e =$ 

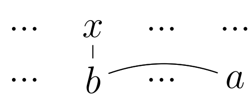
. Then (x, a) and (x, b) are arcs in de and (x, y, a) and (y, z, b) are arcs in $d'e$, so (a, b) is an arc in $d'e$. In particular, both de and $d'e$ connect x, a and b. In diagrammatic terms we have

$de =$ 

$=$ 

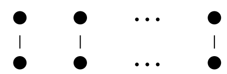
and

$d'e =$ 

$=$ 

$= de.$

Hence, the diagrams de and $d'e$ describe the same set partition and multiplication is independent of the choice of diagram.

We choose to write all diagrams as follows: First, connect dots of the top row belonging to the same part from left to right. Do the same in the bottom row. Parts which contain both top and bottom row dots will be connected via the respective first (=left-most) dots. In the above example, d' is standard notation, while d is not. Parts connecting top and bottom row are often called *propagating parts* in the literature. The number $\#_p(d)$ of propagating parts of a diagram d is called *propagating number*. We call the actual line connecting a top and a bottom row dot *propagating line*. We denote the top row of a diagram d by $\mathrm{top}(d)$, its bottom row by $\mathrm{bottom}(d)$ and the permutation induced by the propagating lines by $\Pi(d)$. Note that multiplication of diagrams cannot increase the propagating number, since a propagating part of $a \cdot b$ connects $\mathrm{top}(a)$ to $\mathrm{bottom}(b)$ via $\mathrm{bottom}(a) = \mathrm{top}(b)$, hence $\#_p(a \cdot b) \leq \min\{\#_p(a), \#_p(b)\}$. The unit element of $P_k(r, \delta)$ is given by the set partition $\{\{1, 1'\}, \{2, 2'\}, ..., \{r, r'\}\} =$ .

A list of all basis diagrams, i.e. diagrams in the vector space basis, of $P_k(3, \delta)$ can be found in Appendix AII.

5.1 Structural Properties

The Brauer algebra $B_k(r, \delta)$ defined in Chapter 4 is a unitary subalgebra of $P_k(r, \delta)$, consisting of set partitions where each part is of size 2. In particular, the group algebra $k\Sigma_r$ is a unitary subalgebra of $P_k(r, \delta)$. Again, we have different embeddings of $k\Sigma_l$ and $B_k(l, \delta)$

in $P_k(r,\delta)$ $(l \leq r)$:

$$\begin{tikzcd}
 & k\Sigma_r \arrow[dr, hook] & & \\
k\Sigma_l \arrow[ur, hook] \arrow[dr, hook] & & B_k(r,\delta) \arrow[dr, hook] & \\
 & B_k(l,\delta) \arrow[ur, hook] \arrow[dr, hook] & & P_k(r,\delta) \\
 & & P_k(l,\delta) \arrow[ur, hook] &
\end{tikzcd}$$

where a smaller partition algebra $P_k(l,\delta)$ is embedded into $P_k(r,\delta)$ by adding dots $l+1,...,r$ in the top row and $(l+1)',...,r'$ in the bottom row, and attaching the new dots of the top and bottom row, respectively, to the lth , respectively l'th , dot of the diagram in $P_k(l,\delta)$.

$k\Sigma_r$ is also a quotient of $P_k(r,\delta)$ by the ideal generated by all diagrams with propagating number at most $r-1$. This is in fact a consequence of the cellularly stratified structure of $P_k(r,\delta)$, which we describe in this section.

Xi showed in [Xi99] that $P_k(r,\delta)$ is cellular by considering $P_k(r,\delta)$ as iterated inflation of group algebras of symmetric groups. In [HHKP10], the partition algebra is one of the main examples for cellularly stratified algebras. We will now explain these structures.

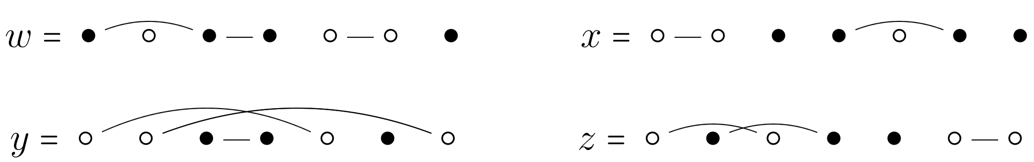
A diagram consisting of one row with r dots and arbitrary connections is called *partial diagram.* We have to distinguish certain parts from others; we say they are *labelled* and write the dots as $\circ$ instead of $\bullet$. We count the labelled parts from left to right, according to the leftmost dot of the part. Let V_n be the vector space with basis all partial diagrams with exactly n labelled parts (and possibly further unlabelled parts). For example,  is a basis element of V_2 in case $r = 7$; the labelled singleton $\circ$ is the first labelled part, the part $\circ - \circ$ is the second. For $x, y \in V_n$, define $x \cdot y$ to be the partial diagram obtained by writing the connecting lines of x and y on the same row of dots. A part is labelled if it contains at least one dot labelled. If it contains labelled dots of x as well as labelled dots of y, it is labelled twice, denoted by $\circledcirc$.

We define a bilinear form $\Phi : V_n \times V_n \to k\Sigma_n$ by

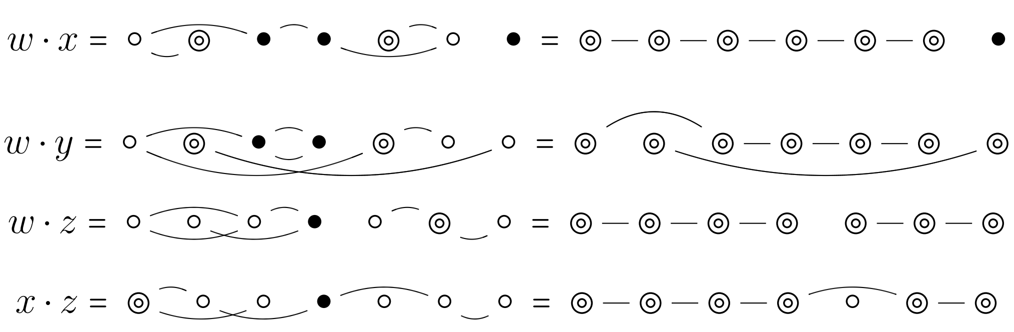
$$\Phi(x,y) = \begin{cases} 0 & \text{if } x \cdot y \text{ has a part which is labelled exactly once} \\ 0 & \text{if two distinct labelled parts of } x \text{ are connected in } x \cdot y \\ 0 & \text{if two distinct labelled parts of } y \text{ are connected in } x \cdot y \\ \delta^{u_{x \cdot y}} \pi & \text{otherwise} \end{cases}$$

where $u_{x \cdot y}$ is the number of unlabelled parts in $x \cdot y$ and $\pi \in \Sigma_n$ is the permutation with $\pi(i) = j$ if the part of $x \cdot y$ containing the ith labelled part of x contains the jth labelled part of y as well. Note that this is well-defined since there are exactly one labelled part of x and one of y in each labelled part of $x \cdot y$ in this case. Note that $\Phi(x, y) \neq 0$ if and only if $x \cdot y$ has exactly n parts which are labelled twice.

Example. Let



To distinguish between connections of x and y, we write the connections coming from x curved upwards and those coming from y curved downwards in the diagram for $x \cdot y$. We have



thus

- $\Phi(w, x) = 0$ since the labelled part contains all labelled parts of w and x.
- $\Phi(w, y) = (1, 2)$ since the first part of w (= the second dot) lies in the same part as the second part of y (= the dots 2 and 7) and vice versa.
- $\Phi(w, z) = 1_{\Sigma_2}$ since the first part of w lies in the same part as the first part of z and the second part of w lies in the same part as the second part of z.
- $\Phi(x, z) = 0$ since the part consisting of the fifth dot is only labelled once.

The vector space V_n and the bilinear form Φ described here are a diagrammatic interpretation of V and Φ defined in Section 4 of [Xi99].

Theorem 5.1 ([Xi99, Theorem 4.1]). *The partition algebra $P_k(r,\delta)$ is cellular as an iterated inflation of the form $\bigoplus_{n=0}^{r} k\Sigma_n \otimes_k V_n \otimes_k V_n$, with respect to the involution i turning a diagram upside down.*

For the cellularly stratified structure, we need the existence of idempotents $e_n = 1_{\Sigma_n} \otimes u_n \otimes v_n$ such that $e_n e_m = e_m = e_m e_n$ for $m \leq n$. Let $\delta \neq 0$ and set

$$e_0 := \frac{1}{\delta} \cdot \begin{array}{c} \bullet^{1} - \bullet - \cdots - \bullet - \bullet^{r} \\ \bullet_{1'} - \bullet - \cdots - \bullet - \bullet_{r'} \end{array}$$

$$e_n := \begin{array}{ccccc} \bullet^{1} & \cdots & \bullet & \bullet^{n} - \cdots - \bullet^{r} \\ | & & | & | \\ \bullet_{1'} & \cdots & \bullet & \bullet_{n'} - \cdots - \bullet_{r'} \end{array}$$

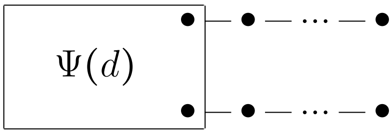
Theorem 5.2 ([HHKP10, Proposition 2.6]). *The partition algebra $P_k(r,\delta)$ is cellularly stratified with stratification data $(k, V_0, k, V_1, k\Sigma_2, V_2, \ldots, k\Sigma_r, V_r)$ and idempotents e_n for all parameters $\delta \in k \setminus \{0\}$.*

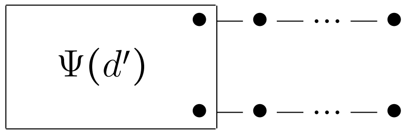
We observe that for $0 \leq n \leq r$, the algebras $e_n P_k(r,\delta) e_n$ and $P_k(n,\delta)$ are isomorphic:

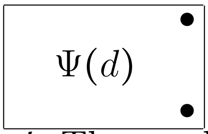
The diagrams in $e_n P_k(r,\delta) e_n$ have arbitrary connections between the dots $1, \ldots, n, 1', \ldots, n'$ and dot n lies in the same part as the dots $n+1, \ldots, r$ while dot n' lies in the same part as the dots $(n+1)', \ldots, r'$. Hence $e_n P_k(r,\delta) e_n$ is the image of the embedding $P_k(n,\delta) \hookrightarrow P_k(r,\delta)$ described in the beginning of this section, which sends a basis of $P_k(n,\delta)$ to a basis of $e_n P_k(r,\delta) e_n$. To show that this map is a proper homomorphism of algebras, we consider the k-linear map

$$\Psi : e_n P_k(r,\delta) e_n \to P_k(n,\delta)$$

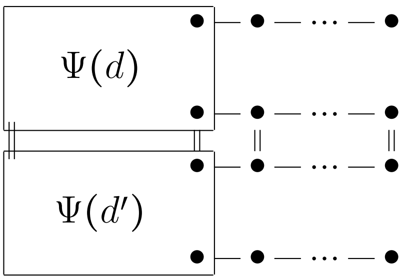
defined on diagrams d of $e_nP_k(r,\delta)e_n$ by removing the dots $n+1$ up to r from top(d) and removing dots $(n+1)'$ up to r' from bottom(d). This is the inverse of the given embedding. It remains to show that Ψ is multiplicative. Let $d, d' \in e_nP_k(r,\delta)e_n$ be diagrams. Then we can visualise d and d' as follows:

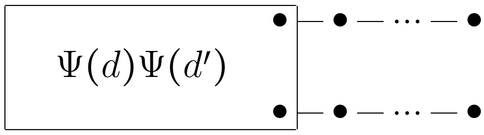
$d =$ 

, $d' =$ 

where 

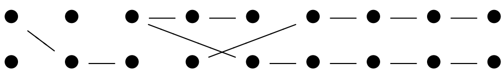
denotes the diagram for $\Psi(d)$, and the dots are n and n'. The product dd' is then given by

$dd' =$ 

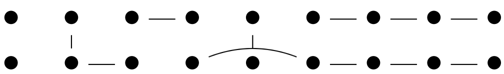
$=$ 

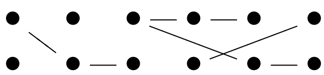
so $\Psi(dd') = \Psi(d)\Psi(d')$.

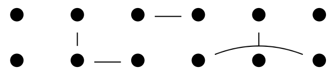
Example 5.2. Let $r = 9$, $n = 6$,

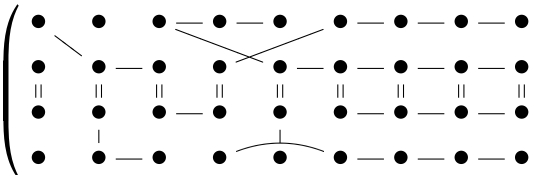
$d =$ 

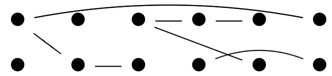
and

$d' =$ 

Then we have $\Psi(d) =$  $= x$ from Example 5.1,

$\Psi(d') =$  $= y$ from Example 5.1 and

$\Psi(dd') = \Psi\Bigg($  $\Bigg)$

$= \delta \cdot$  $= \Psi(d)\Psi(d').$

For the remainder of this chapter, let $A := P_k(r, \delta)$. By abuse of notation, we write $e_n(A/J_{n-1})$ for the (e_nAe_n, A)-bimodule e_nA/e_nJ_{n-1}, where $J_{n-1} = Ae_{n-1}A$ is the ideal generated by all diagrams d with $\#_p(d) \leq n-1$.

5.2 Restriction of Cell Modules

From the cellularly stratified structure of A, we know that the cell modules are of the form $(A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} S_\nu$. Here, we choose as cell modules for $k\Sigma_n$ the dual Specht modules S_ν, which are e_nAe_n-modules via the quotient map $e_nAe_n \twoheadrightarrow k\Sigma_n$. Hence, $(A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} S_\nu \simeq (A/J_{n-1})e_n \underset{k\Sigma_n}{\otimes} S_\nu$. Let $1 \leq l \leq r$. Then a cell module of e_lAe_l is of the form $e_l(A/J_{n-1})e_n \underset{k\Sigma_n}{\otimes} S_\nu$. If $l < n$, then $e_l \in J_{n-1}$, so this is zero. To see what happens when we restrict the cell module to $k\Sigma_l-$mod in case $l \geq n$, we have to study the $(k\Sigma_l, k\Sigma_n)$-bimodule $e_l(A/J_{n-1})e_n$. We start with the easy cases $l = n$ and $l = n+1$. The general analysis for arbitrary l follows in Section 5.2.2.

As we have seen in Lemma 3.7, $e_n(A/J_{n-1})e_n$ is isomorphic to $k\Sigma_n$. Thus, the cell module $e_n(A/J_{n-1})e_n \underset{k\Sigma_n}{\otimes} S_\nu$ is isomorphic to S_ν as left $k\Sigma_n$-modules.

5.2.1 From one layer to the next

To study $e_{n+1}(A/J_{n-1})e_n$, we may assume without loss of generality that $r = n+1$ since $e_{n+1}Ae_{n+1} \simeq P_k(n+1,\delta)$, so that

$$\begin{aligned} e_{n+1}(A/J_{n-1})e_n &\simeq (e_{n+1}Ae_{n+1}/e_{n+1}J_{n-1}e_{n+1})e_n \\ &\simeq (A'/J'_{n-1})e_n, \end{aligned}$$

where $A' := P_k(n+1,\delta)$, $J'_{n-1} := A'e'_{n-1}A'$ and e'_{n-1} is the idempotent of A' generating the layer $n-1$.

The basis of the vector space V_n consists of partial diagrams with n labelled parts, so in case $r = n+1$ this means that there are either n labelled singletons and one unlabelled singleton or there are $n-1$ labelled singletons and one labelled part of size 2. We define an equivalence relation $\sim$ on V_n by setting $v \sim w$ if and only if the sizes and labels of the parts of v and w coincide, i.e. if and only if there is a $\pi \in \Sigma_r$ such that $w = \pi v$, where π permutes the dots of v.

The basis diagrams of $e_{n+1}(A/J_{n-1})e_n$ are of the form

- top row equivalent to

$$v_1 := \circ^1 \quad \cdots \quad \circ^n \quad \bullet^{n+1}$$

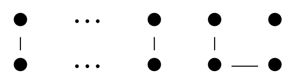
or

$$v_2 := \circ^1 \quad \cdots \quad \circ^{n-1} \quad \circ\!-\!\circ \;,$$

- fixed bottom row v_2 and
- exactly n propagating lines.

Multiplication with $k\Sigma_{n+1}$ from above or $k\Sigma_n$ from below rearranges dots and the attached propagating lines, but does not change the equivalence class of the top row. The bottom row is invariant. Hence, $e_{n+1}(A/J_{n-1})e_n$ has a $(k\Sigma_{n+1}, k\Sigma_n)$-bimodule decomposition $e_{n+1}(A/J_{n-1})e_n = U_{v_1} \oplus U_{v_2}$, where U_{v_i} is the bimodule whose underlying vector space has a basis of diagrams with top row equivalent to v_i.

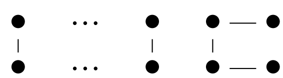
A diagram $b \in U_{v_1}$ is uniquely determined by the permutation of propagating parts $\Pi(b)$ and the location of the unlabelled dot in the top row. Hence, we can write the diagram b as τd, where $\tau \in k\Sigma_{n+1}$ and

$d :=$  $= 1_{k\Sigma_n} \otimes v_1 \otimes v_2.$

This gives an epimorphism $k\Sigma_{n+1} \twoheadrightarrow U_{v_1}$ of left $k\Sigma_{n+1}$-modules. It is an isomorphism since $\dim U_{v_1} = |\Sigma_n| \cdot |\{w \in V_n \mid w \sim v_1\}| \cdot |\{v_2\}| = n! \cdot (n+1) \cdot 1 = \dim k\Sigma_{n+1}$. For $\eta \in \Sigma_n$, it is $\eta d = \eta \otimes v_1 \otimes v_2 = d\eta$, so $\tau d\eta = \tau\eta d$ for $\tau \in \Sigma_{n+1}$, $\eta \in \Sigma_n$ and we have in fact an isomorphism of $(k\Sigma_{n+1}, k\Sigma_n)$-bimodules. This shows

Lemma 5.3. *The summand U_{v_1} is isomorphic to $k\Sigma_{n+1}$ as $(k\Sigma_{n+1}, k\Sigma_n)$-bimodule. In particular, $U_{v_1} \underset{k\Sigma_n}{\otimes} S_\nu \simeq k\Sigma_{n+1} \underset{k\Sigma_n}{\otimes} S_\nu \in \mathcal{F}_{n+1}(S)$ for all $\nu \vdash n$.*

To describe U_{v_2} in a similar manner, we redefine the diagram d as

$d :=$  $= 1_{k\Sigma_n} \otimes v_2 \otimes v_2$

and consider the map

$$\begin{aligned} \psi : k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n &\longrightarrow U_{v_2} \\ \tau \otimes \eta &\longmapsto \tau d\eta, \end{aligned}$$

where Σ_{n-1} acts on Σ_n by the usual subgroup action and $\Sigma_{\{n,n+1\}} \simeq \Sigma_2$ acts trivially. Note that, again, Σ_n commutes with d: $\Sigma_n \cdot d = \Sigma_n \otimes v_2 \otimes v_2 = d \cdot \Sigma_n$.

Lemma 5.4. *ψ is an isomorphism of $(k\Sigma_{n+1}, k\Sigma_n)$-bimodules.*

Proof. Let $\zeta \in \Sigma_{(n-1,2)}$, i.e. $\zeta = \zeta'\zeta''$ with $\zeta' \in \Sigma_{n-1}, \zeta'' \in \Sigma_{\{n,n+1\}}$. $\Sigma_{\{n,n+1\}}$ acts trivially on d and on Σ_n, so $\zeta d = \zeta' d = d\zeta'$ and $\zeta\eta = \zeta'\eta$ for $\eta \in \Sigma_n$. Hence, $\Psi(\tau\zeta, \eta) = \Psi(\tau, \zeta\eta)$ if Ψ is the map from $k\Sigma_{n+1} \times k\Sigma_n$ to U_{v_2} with $\Psi(\tau, \eta) = \tau d\eta$ for any $\tau \in k\Sigma_{n+1}, \eta \in k\Sigma_n$. This shows that ψ is well-defined.

Let $\eta, \eta' \in \Sigma_n$ and $\tau, \tau' \in \Sigma_{n+1}$. Then

$$\psi(\tau'(\tau \otimes \eta)\eta') = \tau'\tau d\eta\eta' = \tau'\psi(\tau \otimes \eta)\eta',$$

so ψ is a $(k\Sigma_{n+1}, k\Sigma_n)$-bimodule homomorphism.

We check that the inverse of ψ is given by

$$\begin{aligned} \tilde{\psi} : U_{v_2} &\longrightarrow k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n \\ b &\longmapsto \tau \otimes \Pi(\tau d)^{-1}\Pi(b) \quad \text{if } \mathrm{top}(b) = \tau v_2. \end{aligned}$$

Let $\tau_1, \tau_2 \in \Sigma_{n+1}$ such that $\tau_1 v_2 = \tau_2 v_2$. Then there is a $\vartheta \in \Sigma_{(n-1,2)}$, the stabilizer of the partial diagram v_2, such that $\tau_1 = \tau_2\vartheta$. In particular,

$$\begin{aligned} \tau_1 \otimes \Pi(\tau_1 d)^{-1}\Pi(b) &= \tau_2\vartheta \otimes \Pi(\tau_2\vartheta d)^{-1}\Pi(b) \\ &= \tau_2 \otimes \vartheta\Pi(\tau_2\vartheta d)^{-1}\Pi(b) \\ &= \tau_2 \otimes (\Pi(\tau_2\vartheta d)\vartheta^{-1})^{-1}\Pi(b) \\ &= \tau_2 \otimes \Pi(\tau_2\vartheta d\vartheta^{-1})^{-1}\Pi(b) \\ &= \tau_2 \otimes \Pi(\tau_2 d)^{-1}\Pi(b), \end{aligned}$$

so $\tilde{\psi}$ is well-defined.

Let $b \in U_{v_2}$ with $\mathrm{top}(b) = \tau v_2$. Then

$$\psi\tilde{\psi}(b) = \psi(\tau \otimes \Pi(\tau d)^{-1}\Pi(b)) = \tau d \Pi(\tau d)^{-1}\Pi(b) =: c.$$

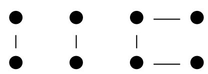
But $\mathrm{top}(c) = \tau v_2 = \mathrm{top}(b)$ and $\Pi(c) = \Pi(\tau d)\Pi(\tau d)^{-1}\Pi(b) = \Pi(b)$, so $c = b$.

Let $\tau \in \Sigma_{n+1}$ and $\eta \in \Sigma_n$. Then

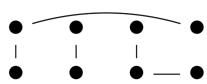
$$\tilde{\psi}\psi(\tau\otimes\eta) = \tilde{\psi}(\tau d\eta) = \tau\otimes\Pi(\tau d)^{-1}\Pi(\tau d\eta) = \tau\otimes\Pi(\tau d)^{-1}\Pi(\tau d)\eta = \tau\otimes\eta.$$

Therefore, $\tilde{\psi}$ is both left and right inverse to ψ, which shows that ψ is an isomorphism. $\square$

Example 5.5. Let $n = 3$. Then

$d =$ 

and $\Sigma_{(n-1,2)} = \Sigma_{\{1,2\}} \times \Sigma_{\{3,4\}}$. Consider the diagram

$b =$  $\in U_{v_2}$.

The diagram b equals $(143)d(13)$ as well as $(132)d(123)$. Lemma 5.4 then says that b corresponds to $(132)\otimes(123) \in k\Sigma_4 \underset{k\Sigma_{(2,2)}}{\otimes} k\Sigma_3$ as well as to $(143)\otimes(13) \in k\Sigma_4 \underset{k\Sigma_{(2,2)}}{\otimes} k\Sigma_3$. But $(143) = (132)(12)(34)$, so $(143)\otimes(13) = (132)(12)(34)\otimes(13) = (132)\otimes(12)(13) = (132)\otimes(123)$.

The tensor product $k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n$ is difficult to understand, so we would like to have another isomorphic description for it. Consider

Σ_n as disjoint union of n right Σ_{n-1}-cosets

$$\Sigma_n = \coprod \Sigma_{n-1}\sigma_i$$

with representatives $\{\sigma_i\}_{i\in\{1,\dots,n\}}$ and define a right $k\Sigma_n$-module structure on

$$\bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_{\{n,n+1\}}}{\otimes} k =: \bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k :$$

Denote the element $(0,\dots,x,\dots,0) \in \bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k$ by $x^{(i)}$ if x sits in the ith summand. For $\tau \in \Sigma_{n+1}$ and $\eta \in \Sigma_n$ with $\sigma_i\eta = \zeta\sigma_j$ for some $\zeta \in \Sigma_{n-1}$ and $j \in \{1,\dots,n\}$, set

$$(\tau\otimes 1)^{(i)} \cdot \eta := (\tau\zeta\otimes 1)^{(j)}.$$

This construction is called *tensor induced module*, cf. [CR81, §13].

Lemma 5.6. *The k-linear map*

$$\begin{aligned} \varphi : k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n &\longrightarrow \bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k \\ \tau\otimes\eta &\longmapsto (\tau\zeta\otimes 1)^{(i)} \qquad \text{if } \eta = \zeta\sigma_i \end{aligned}$$

is an isomorphism of $(k\Sigma_{n+1}, k\Sigma_n)$-bimodules.

Proof. Let $\tau \in \Sigma_{n+1}$ and $\eta = \zeta\sigma_i \in \Sigma_{n-1}\sigma_i \subset \Sigma_n$. Consider the k-linear map $\Phi : k\Sigma_{n+1} \times k\Sigma_n \to \bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k$ given by $\Phi(\tau,\eta) = (\tau\zeta\otimes 1)^{(i)}$ if $\eta = \zeta\sigma_i$. Let $\vartheta \in \Sigma_{n-1}$ and $\vartheta' \in \Sigma_{\{n,n+1\}}$. Then $\Phi(\tau\vartheta\vartheta',\eta) = (\tau\vartheta\vartheta'\zeta\otimes 1)^{(i)} = (\tau\vartheta\zeta\otimes 1)^{(i)}$ since ζ and ϑ' commute and ϑ' can be moved across the tensor, where it acts trivially. On the other hand, we have $\Phi(\tau,\vartheta\vartheta'\eta) = \Phi(\tau,\vartheta\zeta\sigma_i) = (\tau\vartheta\zeta\otimes 1)^{(i)} = \Phi(\tau\vartheta\vartheta',\eta)$. Hence, φ is well-defined.

To check that φ is a homomorphism of $(k\Sigma_{n+1}, k\Sigma_n)$-bimodules, let $\tau, \tau' \in \Sigma_{n+1}$ and $\eta, \eta' \in \Sigma_n$ such that $\eta = \zeta\sigma_i$ and $\sigma_i\eta' = \vartheta\sigma_j$. Then

$$\varphi(\tau'(\tau \otimes \eta)\eta') = \varphi(\tau'\tau \otimes \zeta\vartheta\sigma_j) = (\tau'\tau\zeta\vartheta \otimes 1)^{(j)}$$

and

$$\tau'\varphi(\tau \otimes \eta)\eta' = \tau'(\tau\zeta \otimes 1)^{(i)}\eta' = (\tau'\tau\zeta\vartheta \otimes 1)^{(j)} = \varphi(\tau'(\tau \otimes \eta)\eta').$$

Let $\sum_{i=1}^{n}(\tau_i \otimes a_i)^{(i)}$ be an arbitrary element of $\bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k$. Then φ is surjective, since $\sum_{i=1}^{n}(\tau_i \otimes a_i)^{(i)}$ is the image of $\sum_{i=1}^{n} a_i(\tau_i \otimes \sigma_i) \in k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n$.

The module $k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k$ has a vector space basis indexed by the cosets Σ_{n+1}/Σ_2, so $\dim(\bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k) = n \cdot \frac{(n+1)!}{2}$. We have seen in Lemma 5.4 that $k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n \simeq U_{v_2}$. So $\dim(k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n) = \dim(U_{v_2}) = |\Sigma_n| \cdot |\{w \in V_n \mid w \sim v_2\}| \cdot |\{v_2\}|$. Let $w \in V_n$ be equivalent to v_2. Then there is a $\pi \in \Sigma_{n+1}$ such that $w = \pi v_2$. But $\pi v_2 = v_2$ if and only if $\pi \in \Sigma_{n-1} \times \Sigma_2$, so $|\{w \in V_n \mid w \sim v_2\}| = |\Sigma_{n+1}/(\Sigma_{n-1} \times \Sigma_2)|$. Hence, $\dim(k\Sigma_{n+1} \underset{k\Sigma_{(n-1,2)}}{\otimes} k\Sigma_n) = n! \cdot \frac{(n+1)!}{(n-1)! \cdot 2} \cdot 1 = \dim(\bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k)$. □

Example (5.5 continued)**.** Choose as coset representatives $\sigma_1 := 1$, $\sigma_2 := (13)$ and $\sigma_3 := (132)$. Then $(132) \otimes (123) = (132) \otimes (12)(13)$, so $(132) \otimes (123)$ corresponds, by Lemma 5.6, to the element $((132)(12) \otimes 1)^{(2)} = ((13) \otimes 1)^{(2)}$ in $\bigoplus_{i=1}^{3} k\Sigma_4 \underset{k\Sigma_2}{\otimes} k$. This means that, as diagrams, b equals $(13)d\sigma_2 = (13)d(13)$, which is correct.

Let $X \in k\Sigma_n - \text{mod}$. Then we can write an element

$$(\tau \otimes 1)^{(i)} \otimes x \in (\bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k) \underset{k\Sigma_n}{\otimes} X$$

as $(\tau \otimes 1)^{(1)} \otimes \sigma_i x$, so we get an isomorphism

$$(\bigoplus_{i=1}^{n} k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k) \underset{k\Sigma_n}{\otimes} X \simeq (k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k) \underset{k}{\otimes} (\bigoplus_{i=1}^{n} \sigma_i X)$$

of left $k\Sigma_{n+1}$-modules.

Corollary 5.7. *Let $X \in k\Sigma_n - \text{mod}$. Then the left $k\Sigma_{n+1}$-module $U_{v_2} \underset{k\Sigma_n}{\otimes} X$ is isomorphic to $\dim(\bigoplus_{i=1}^{n} \sigma_i X)$-many copies of $k\Sigma_{n+1} \underset{k\Sigma_2}{\otimes} k = M^{(2,1^{n-1})}$. In particular, $U_{v_2} \underset{k\Sigma_n}{\otimes} X \in \mathcal{F}_{n+1}(S)$.*

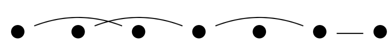
Corollary 5.8. *Let $\nu \vdash n$ and let $s = \dim(\bigoplus_{i=1}^{n} \sigma_i S_\nu)$. Then the restricted cell module $e_{n+1}(A/J_{n-1})e_n \underset{k\Sigma_n}{\otimes} S_\nu$ is isomorphic to $(\bigoplus_{i=1}^{s} M^{(2,1^{n-1})}) \oplus (k\Sigma_{n+1} \underset{k\Sigma_n}{\otimes} S_\nu)$ and admits a filtration by dual Specht modules for $k\Sigma_{n+1}$.*

Proof. As seen in the beginning of this section, $e_{n+1}(A/J_{n-1})e_n = U_{v_1} \oplus U_{v_2}$, so $e_{n+1}(A/J_{n-1})e_n \underset{k\Sigma_n}{\otimes} S_\nu = (U_{v_1} \underset{k\Sigma_n}{\otimes} S_\nu) \oplus (U_{v_2} \underset{k\Sigma_n}{\otimes} S_\nu)$. By Lemma 5.3 and Corollary 5.7, this is isomorphic to $(k\Sigma_{n+1} \underset{k\Sigma_n}{\otimes} S_\nu) \oplus (\bigoplus_{i=1}^{s} M^{(2,1^{n-1})})$, where $s = \dim(\bigoplus_{i=1}^{n} \sigma_i S_\nu)$. □

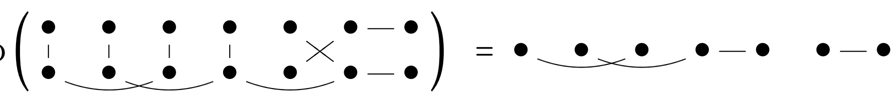
5.2.2 To higher layers

We will now modify the above results to the general case. To visualize the arguments of this section, we refer the reader to the example in Appendix AIII. We could again assume that $r = l$. However, we will explain the slightly more complex case $r \geq l$, to see that it makes a difference which embedding $k\Sigma_l \hookrightarrow P_k(r,\delta)$ we use. Let V_n^l be the subspace of V_n generated by all partial diagrams with n labelled parts, where the last $r-l+1$ dots lie in the same part. We regard $k\Sigma_l$ as the subalgebra of $P_k(r,\delta)$ generated by all diagrams with top and bottom row consisting of $l-1$ labelled dots followed by one labelled part of size $r-l+1$, and l propagating lines connecting the l parts of the top row with the l parts of the bottom row, i.e. we use the embedding $k\Sigma_l \hookrightarrow P_k(l,\delta) \hookrightarrow P_k(r,\delta)$ described in the beginning of Section 5.1. Let $v, w \in V_n^l$. We say that v is equivalent to w, $v \sim w$, if and only if there is a $\pi \in \Sigma_l$ such that $\pi v = w$, where πv is defined as follows: Write the diagram π on top of v and identify bottom(π) with v (forget about the labelling of v). Then πv is the top row of this diagram.

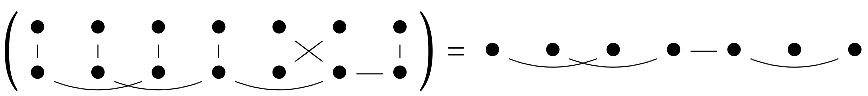
Example 5.9. Let $r = 7, l = 6, \pi = (56) \in \Sigma_6$ and

$v =$ 

Then

$\pi v = \mathrm{top}$ 

Remark. Note that the above defined action of $k\Sigma_l$ on V_n^l ensures that $\pi v \in V_n^l$. If we would use the embedding $k\Sigma_l \hookrightarrow k\Sigma_r \hookrightarrow P_k(r,\delta)$ instead, then $k\Sigma_l$ would act on V_n^l by permuting the dots. With π and v from Example 5.9, we would then have

$\pi v = \mathrm{top}$  $\notin V_n^l$.

Let $v \in V_n^l$ be a partial diagram. Define d_v to be the diagram with $\operatorname{top}(d_v) = v$, $\operatorname{bottom}(d_v) = \operatorname{bottom}(e_n)$ and $\Pi(d_v) = 1_{k\Sigma_n}$. Let $b \in e_l(A/J_{n-1})e_n$ be a diagram with $\operatorname{top}(b) \sim v$. By definition, there is a $\pi \in \Sigma_l$ such that $\operatorname{top}(b) = \pi v$. Then $b = \pi d_v \Pi(\pi d_v)^{-1}\Pi(b)$, i.e. any diagram with top row in the equivalence class of v equals $\tau d_v \eta$ for some $\tau \in \Sigma_l, \eta \in \Sigma_n$. Let U_v be the $(k\Sigma_l, k\Sigma_n)$-bimodule generated by d_v. For $w \in V_n^l$ with $w \nsim v$, we have $U_w \cap U_v = \varnothing$, because a diagram in the intersection would have top row equivalent to v and to w simultaneously. Therefore, every diagram in $e_l(A/J_{n-1})e_n$ lies in exactly one of the U_v's and every diagram of U_v is a diagram in $e_l(A/J_{n-1})e_n$. Hence, the $(k\Sigma_l, k\Sigma_n)$-bimodule $e_l(A/J_{n-1})e_n$ decomposes into a direct sum $\bigoplus_{v \in V_n^l/\sim} U_v$.

Fix a partial diagram $v \in V_n^l$ and set $d := d_v$. Let α_i be the number of labelled parts of size i and β_i the number of unlabelled parts of size i of v, where the last $r - l + 1$ dots count as one dot. Then $\sum_i(\alpha_i \cdot i) + \sum_i(\beta_i \cdot i) = l$ and $\sum_i \alpha_i = n$. Let $\mathcal{S}_i^j$ be the set of dots of v belonging to the jth labelled part of size i and let $\mathcal{T}_i^j$ be the set of dots of v belonging to the jth unlabelled part of size i. Then $\prod_\alpha := \prod_{\substack{i \geq 1 \\ \alpha_i \neq 0}} ((\Sigma_{\mathcal{S}_i^1} \times \ldots \times \Sigma_{\mathcal{S}_i^{\alpha_i}}) \rtimes \Sigma_{\alpha_i})$ is the stabilizer of the labelled parts of v and $\prod_\beta := \prod_{\substack{i \geq 1 \\ \beta_i \neq 0}} ((\Sigma_{\mathcal{T}_i^1} \times \ldots \times \Sigma_{\mathcal{T}_i^{\beta_i}}) \rtimes \Sigma_{\beta_i})$ is the stabilizer of the unlabelled parts of v. In particular, $\prod_\beta$ stabilizes d. Note that $\prod_\alpha \simeq \prod_{\substack{i \geq 1 \\ \alpha_i \neq 0}} (\Sigma_i \wr \Sigma_{\alpha_i})$ and $\prod_\beta \simeq \prod_{\substack{i \geq 1 \\ \beta_i \neq 0}} (\Sigma_i \wr \Sigma_{\beta_i})$.

Example. Let $v = \bullet \quad \circ \quad \bullet - \bullet \quad \circ \quad \bullet \quad \circ - \circ \quad \bullet \quad \circ - \circ$. Then

$$\prod_\alpha = (\Sigma_{\{2,5\}} \times \Sigma_{\{7,8\}} \times \Sigma_{\{10,11\}}) \rtimes \Sigma_3 \simeq \Sigma_2 \wr \Sigma_3$$

and

$$\prod_\beta = \Sigma_{\{1\}} \times ((\Sigma_{\{3,4\}} \times \Sigma_{\{6,9\}}) \rtimes \Sigma_2) \simeq \Sigma_1 \times (\Sigma_2 \wr \Sigma_2).$$

Consider the bimodule $k\Sigma_l \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_n$, where $\prod_\beta$ acts trivially on $k\Sigma_n$ and the action of $\prod_\alpha$ on $k\Sigma_n$ is given by $\zeta \cdot \eta := \Pi(\zeta d)\eta$ for $\zeta \in \prod_\alpha, \eta \in \Sigma_n$. We have $\mathrm{top}(\zeta d) = \mathrm{top}(d)$, so $\zeta d = d\Pi(\zeta d)$ for $\zeta \in \prod_\alpha$.

Lemma 5.10. *The map*

$$\begin{aligned} \psi : k\Sigma_l \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_n &\longrightarrow U_v \\ \tau \otimes \eta &\longmapsto \tau d\eta \end{aligned}$$

is an isomorphism of $(k\Sigma_l, k\Sigma_n)$*-bimodules.*

Proof. Note that Lemma 5.4 is a special case for $l = n+1$.

Let $x \in \prod_\alpha$ and $y \in \prod_\beta$. Then $yd = d$, since y stabilizes the unlabelled parts of v, and $xd = d\Pi(x)$, since x stabilizes the labelled parts of v while it permutes the attached propagating lines. Let $\Psi : k\Sigma_l \times k\Sigma_n \to U_v$ be given by $\Psi(\tau, \eta) = \tau d\eta$. Then $\Psi(\tau(x,y),\eta) = \tau xyd\eta = \tau d\Pi(x)\eta = \tau d\Pi(x)y\eta = \Psi(\tau,(x,y)\eta)$, so ψ is well-defined. It is a bimodule-homomorphism by the same arguments as in Lemma 5.4.

The inverse map is

$$\begin{aligned} \tilde{\psi} : U_v &\longrightarrow k\Sigma_l \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_n \\ b &\longmapsto \tau \otimes \Pi(\tau d)^{-1}\Pi(b) \quad \text{if } \mathrm{top}(b) = \tau v. \end{aligned}$$

We show that $\tilde{\psi}$ is well-defined. If $\mathrm{top}(b) = \tau_1 v = \tau_2 v$, there are

$x \in \prod_\alpha$, $y \in \prod_\beta$ such that $\tau_1 = \tau_2 xy$. Then

$$\begin{aligned}\tau_1 \otimes \Pi(\tau_1 d)^{-1}\Pi(b) &= \tau_2 xy \otimes \Pi(\tau_2 xyd)^{-1}\Pi(b)\\ &= \tau_2 \otimes \Pi(xyd)\Pi(\tau_2 d\Pi(xd))^{-1}\Pi(b)\\ &= \tau_2 \otimes \Pi(xd)\Pi(xd)^{-1}\Pi(\tau_2 d)^{-1}\Pi(b)\\ &= \tau_2 \otimes \Pi(\tau_2 d)^{-1}\Pi(b).\end{aligned}$$

The proof that $\psi\tilde{\psi} = id_{U_v}$ and $\tilde{\psi}\psi = id_{k\Sigma_l \underset{k\Pi_\alpha \times k\Pi_\beta}{\otimes} k\Sigma_n}$ can be copied from the proof of Lemma 5.4. □

Let $l_1 := \sum_i \alpha_i \cdot i$ and $l_2 := \sum_i \beta_i \cdot i$, so $l = l_1 + l_2$. Fix coset representatives $\omega_1, \ldots, \omega_t$ of $k\Sigma_l / k\Sigma_{(l_1,l_2)}$. Denote by $X \boxtimes Y \in k\Sigma_{(l_1,l_2)}-$mod the exterior tensor product of $X \in k\Sigma_{l_1}-$mod and $Y \in k\Sigma_{l_2}-$mod given by

$$(\tau_1, \tau_2) \cdot (x \boxtimes y) = \tau_1 x \boxtimes \tau_2 y$$

for $\tau_1 \in \Sigma_{l_1}, \tau_2 \in \Sigma_{l_2}, x \in X, y \in Y$.

Consider the $(k\Sigma_l, k\Sigma_n)$-bimodule

$$k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\Pi_\beta}{\otimes} k))$$

with right $k\Sigma_n$-module structure given by

$$(\omega \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1))) \cdot \eta' := \omega \otimes ((\tau_1 \otimes \eta\eta') \boxtimes (\tau_2 \otimes 1))$$

for $\omega \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)) \in k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\Pi_\beta}{\otimes} k))$ and $\eta' \in \Sigma_n$.

Lemma 5.11. *The $(k\Sigma_l, k\Sigma_n)$-bimodules $k\Sigma_l \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_n$ and $k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k))$ are isomorphic.*

Proof. Let

$$\begin{aligned} \Theta : k\Sigma_l \times k\Sigma_n &\longrightarrow k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k)) \\ (\tau, \eta) &\longmapsto \omega_i \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)) \qquad \text{if } \tau = \omega_i \tau_1 \tau_2 \end{aligned}$$

where $\tau_1 \in \Sigma_{l_1}$ and $\tau_2 \in \Sigma_{l_2}$, and let $x \in \prod_\alpha, y \in \prod_\beta$. Then $x \in \Sigma_{l_1} \times \{0\} \subset \Sigma_l$, so $\tau_2 x = x\tau_2$ for $\tau_2 \in \Sigma_{l_2}$. Thus, $\Theta(\tau xy, \eta) = \Theta(\omega_i \tau_1 \tau_2 xy, \eta) = \Theta(\omega_i \tau_1 x \tau_2 y, \eta) = \omega_i \otimes ((\tau_1 x \otimes \eta) \boxtimes (\tau_2 y \otimes 1)) = \omega_i \otimes ((\tau_1 \otimes x\eta) \boxtimes (\tau_2 \otimes 1)) = \omega_i \otimes ((\tau_1 \otimes xy\eta) \boxtimes (\tau_2 \otimes 1)) = \Theta(\tau, xy\eta)$. Hence, the map

$$\begin{aligned} \theta : k\Sigma_l \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_n &\longrightarrow k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k)) \\ \tau \otimes \eta &\longmapsto \omega_i \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)) \qquad \text{if } \tau = \omega_i \tau_1 \tau_2 \end{aligned}$$

is well-defined.

Let $\tau' \in \Sigma_l, \eta' \in \Sigma_n$ such that $\tau'\omega_i = \omega_j \tau_1' \tau_2'$. Then

$$\begin{aligned} \theta(\tau'\tau \otimes \eta\eta') &= \theta(\omega_j \tau_1' \tau_2' \tau_1 \tau_2 \otimes \eta\eta') \\ &= \theta(\omega_j \tau_1' \tau_1 \tau_2' \tau_2 \otimes \eta\eta') \\ &= \omega_j \otimes ((\tau_1'\tau_1 \otimes \eta\eta') \boxtimes (\tau_2'\tau_2 \otimes 1)) \end{aligned}$$

and

$$\begin{aligned} \tau'\theta(\tau \otimes \eta)\eta' &= \tau'(\omega_i \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)))\eta' \\ &= \tau'\omega_i \otimes ((\tau_1 \otimes \eta\eta') \boxtimes (\tau_2 \otimes 1)) \\ &= \omega_j \otimes ((\tau_1'\tau_1 \otimes \eta\eta') \boxtimes (\tau_2'\tau_2 \otimes 1)) \end{aligned}$$

so θ is a homomorphism of $(k\Sigma_l, k\Sigma_n)$-bimodules.

The inverse is given by

$$\begin{aligned}\theta^{-1}: k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\Pi_\beta}{\otimes} k)) &\longrightarrow k\Sigma_l \underset{k\Pi_\alpha \times k\Pi_\beta}{\otimes} k\Sigma_n \\ \tau \otimes ((\vartheta \otimes \eta) \boxtimes (\upsilon \otimes a)) &\longmapsto a(\tau\vartheta\upsilon \otimes \eta)\end{aligned}$$

□

The tensor induced module

We want to rewrite $k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n$ as a tensor induced module, like in Section 5.2.1 for $l = l_1 = n+1$. Let α be the composition $(\alpha_1, \alpha_2, ...)$ and $\gamma = (\underbrace{1, ..., 1}_{\alpha_1}, \underbrace{2, ..., 2}_{\alpha_2}, ...) = (1^{\alpha_1}, 2^{\alpha_2}, ...)$. Fix coset representatives $\sigma_1, ..., \sigma_s$ of $\Sigma_\alpha \backslash \Sigma_n$.

Lemma 5.12. *The map*

$$\begin{aligned}\varphi: k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n &\longrightarrow \bigoplus_{i=1}^{s} k\Sigma_{l_1} \underset{k\Sigma_\gamma}{\otimes} k \\ \tau \otimes \eta &\longmapsto (\tau\hat{\zeta} \otimes 1)^{(i)} \qquad \text{if } \eta = \zeta\sigma_i,\end{aligned}$$

where $\zeta = \Pi(\hat{\zeta} d)$ for some $\hat{\zeta} \in \prod_\alpha$, is an isomorphism of $(k\Sigma_{l_1}, k\Sigma_n)$-bimodules. The right $k\Sigma_n$-module structure on $\bigoplus_{i=1}^{s} k\Sigma_{l_1} \underset{k\Sigma_\gamma}{\otimes} k$ is given by

$$(\tau \otimes 1)^{(i)} \cdot \eta := (\tau\vartheta \otimes 1)^{(j)} \text{ if } \sigma_i\eta = \vartheta\sigma_j$$

for $\tau \in \Sigma_{l_1}, \eta \in \Sigma_n$, $\vartheta \in \Sigma_\alpha$. In particular, $k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n \simeq \bigoplus_{i=1}^{s} M^\gamma$ as left $k\Sigma_{l_1}$-modules, so $k\Sigma_{l_1} \underset{k\Pi_\alpha}{\otimes} k\Sigma_n \in \mathcal{F}_{l_1}(S)$.

Proof. Note that the map φ from Lemma 5.6 is a special case of this map for $l_1 = n+1$, where we choose $\hat{\zeta} = \zeta$ for all $\zeta \in \prod_\alpha = k(\Sigma_{n-1} \times \Sigma_2) = k(\Sigma_1 \wr \Sigma_{n-1}) \times k(\Sigma_2 \wr \Sigma_1)$.

Let $\hat{\zeta}, \check{\zeta} \in \prod_\alpha$ such that $\Pi(\hat{\zeta}d) = \Pi(\check{\zeta}d) = \zeta \in \Sigma_\alpha$. Since $\prod_\alpha \times \prod_\beta$ is the stabilizer of $\operatorname{top}(d)$, we have $\operatorname{top}(\hat{\zeta}d) = \operatorname{top}(d) = \operatorname{top}(\check{\zeta}d)$, therefore $\hat{\zeta}d = \check{\zeta}d$. In particular, there is $(\vartheta, \vartheta') \in \Sigma_\gamma \times \prod_\beta$, the stabilizer of d, such that $\check{\zeta} = \hat{\zeta}\vartheta\vartheta'$. But $\check{\zeta} = \hat{\zeta}\vartheta\vartheta'$ and $\hat{\zeta}\vartheta$ are elements of $\prod_\alpha$ while $\vartheta' \in \prod_\beta$, so $\vartheta' = 1$. Hence, φ is independent of the choice of $\hat{\zeta}$.

Let $\Phi : k\Sigma_{l_1} \times k\Sigma_n \to \bigoplus_{i=1}^{s} k\Sigma_{l_1} \underset{k\Sigma_\gamma}{\otimes} k$ with $\Phi(\tau, \eta) = (\tau\hat{\zeta} \otimes 1)^{(i)}$ for $\eta = \zeta\sigma_i$, $\Pi(\hat{\zeta}d) = \zeta$ and let $\xi \in \prod_\alpha$. Then $\Phi(\tau\xi, \eta) = \Phi(\tau\xi, \zeta\sigma_i) = (\tau\xi\hat{\zeta} \otimes 1)^{(i)}$ and $\Phi(\tau, \Pi(\xi d)\eta) = \Phi(\tau, \Pi(\xi d)\zeta\sigma_i) = (\tau\xi\hat{\zeta} \otimes 1)^{(i)}$.

Let $\tau, \tau' \in \Sigma_{l_1}$ and $\eta, \eta' \in \Sigma_n$ such that $\eta = \zeta\sigma_i$ and $\sigma_i\eta' = \zeta'\sigma_j$. Then $\varphi(\tau'\tau \otimes \eta\eta') = \varphi(\tau'\tau \otimes \zeta\zeta'\sigma_j) = (\tau'\tau\tilde{\zeta} \otimes 1)^{(j)}$ where $\Pi(\tilde{\zeta}d) = \zeta\zeta'$. On the other hand,

$$\begin{aligned} \tau'\varphi(\tau \otimes \eta)\eta' &= \tau'(\tau\hat{\zeta} \otimes 1)^{(i)}\eta' \quad &&\text{with } \Pi(\hat{\zeta}d) = \zeta \\ &= (\tau'\tau\hat{\zeta}\check{\zeta} \otimes 1)^{(j)} \quad &&\text{with } \Pi(\check{\zeta}d) = \zeta'. \end{aligned}$$

$\Pi(\hat{\zeta}\check{\zeta}d) = \Pi(\hat{\zeta}d\Pi(\check{\zeta}d)) = \Pi(\hat{\zeta}d)\Pi(\check{\zeta}d) = \zeta\zeta' = \Pi(\tilde{\zeta}d)$, so there is a $\vartheta \in \Sigma_\gamma$ such that $\hat{\zeta}\check{\zeta} = \tilde{\zeta}\vartheta$. Hence $\tau'\tau\hat{\zeta}\check{\zeta} \otimes 1 = \tau'\tau\tilde{\zeta}\vartheta \otimes 1 = \tau'\tau\tilde{\zeta} \otimes 1$ and φ is a homomorphism of $(k\Sigma_{l_1}, k\Sigma_n)$-bimodules.

The inverse is given by

$$\begin{aligned} \bigoplus_{i=1}^{s} k\Sigma_{l_1} \underset{k\Sigma_\gamma}{\otimes} k &\longrightarrow k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n \\ (\tau \otimes 1)^{(i)} &\longmapsto \tau \otimes \sigma_i. \end{aligned}$$

□

The Foulkes module

We will now study the module $k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k$. As before, we can view this as induced from an exterior tensor product.

Lemma 5.13. *Let t be the maximal size of an unlabelled part of v and let $\tilde{\gamma}$ be the composition $(\beta_1, 2\beta_2, ..., t\beta_t)$. There is an isomorphism*

$$k\Sigma_{l_2} \underset{k\Pi_\beta}{\otimes} k \simeq k\Sigma_{l_2} \underset{k\Sigma_{\tilde{\gamma}}}{\otimes} (k \boxtimes (k\Sigma_{2\beta_2} \underset{k(\Sigma_2 \wr \Sigma_{\beta_2})}{\otimes} k) \boxtimes ... \boxtimes (k\Sigma_{t\beta_t} \underset{k(\Sigma_t \wr \Sigma_{\beta_t})}{\otimes} k))$$

of left $k\Sigma_{l_2}$-modules.

Proof. Let $\epsilon_1, ..., \epsilon_u$ be coset representatives of $\Sigma_{l_2}/\Sigma_{\tilde{\gamma}}$ and $\tau = \epsilon_i \tau_2 ... \tau_t$ with $\tau_j \in \Sigma_{\tilde{\gamma}_j}$. Then the assignment

$$\tau \otimes 1 \mapsto \epsilon_i \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes ... \boxtimes (\tau_t \otimes 1))$$

defines the isomorphism, like in Lemma 5.11. □

The module $H^{(a^m)} := k\Sigma_{am} \underset{k(\Sigma_a \wr \Sigma_m)}{\otimes} k$ is called *Foulkes module*. If the characteristic of the field k is strictly greater than m, the Foulkes module is isomorphic to a direct summand of the permutation module $M^{(a^m)}$, as mentioned in [Gia14]. We will give a proof of this statement in Lemma 5.14. In smaller characteristic, this is not true. In general, it is not known whether or not a Foulkes module $H^{(a^m)}$ has a Specht filtration, if $\operatorname{char} k \leq m$ and $a > 3$. The case $a = 2$ in arbitrary characteristic was solved in [Pag07].

Lemma 5.14. *If* $\operatorname{char} k > m$*, the Foulkes module* $H^{(a^m)} = k\Sigma_{am} \underset{k(\Sigma_a \wr \Sigma_m)}{\otimes} k$ *is isomorphic to a direct summand of the permutation module* $M^{(a^m)}$.

Proof. We came across the wreath product $\Sigma_a \wr \Sigma_m$ as the stabilizer of the m (unlabelled) parts of size a of a partial diagram. The Foulkes module $k\Sigma_{am} \underset{k(\Sigma_a \wr \Sigma_m)}{\otimes} k$ has a vector space basis indexed by left cosets

$\Sigma_{am}/(\Sigma_a \wr \Sigma_m)$. Such a coset decides which dots belong to the same part. Thus, $k\Sigma_{am} \underset{k(\Sigma_a \wr \Sigma_m)}{\otimes} k$ has a vector space basis of set partitions of the form

$$\{\{x_1, ..., x_a\}, ..., \{x_{(m-1)a+1}, ..., x_{ma}\}\}$$

with $x_i \in \{1, ..., am\}, x_i \neq x_j$ for $i \neq j$.

Recall from Section 2.1 that the permutation module $M^{(a^m)}$ has a basis of (a^m)-tabloids. Define maps $M^{(a^m)} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} H^{(a^m)}$ with

$$\begin{array}{c}\hline x_1 \quad \cdots \quad x_a \\ \hline \vdots \\ \hline x_{(m-1)a+1} \quad \cdots \quad x_{ma} \\ \hline \end{array} \overset{\Phi}{\longmapsto} \{\{x_1, ..., x_a\}, ..., \{x_{(m-1)a+1}, ..., x_{ma}\}\}$$

$$\frac{1}{m!}\sum_{\sigma \in \Sigma_m} \sigma * \begin{array}{c}\hline x_1 \quad \cdots \quad x_a \\ \hline \vdots \\ \hline x_{(m-1)a+1} \quad \cdots \quad x_{ma} \\ \hline \end{array} \overset{\Psi}{\longleftarrow\!\shortmid} \{\{x_1, ..., x_a\}, ..., \{x_{(m-1)a+1}, ..., x_{ma}\}\}$$

where the $*$-action of Σ_m permutes the rows of a tabloid. Then the $*$ and $\cdot$ actions commute: Let $\sigma \in \Sigma_m, \tau \in \Sigma_{ma}$ and x_i in row k of the tabloid x. If $\tau(i) = j$ and $\sigma(k) = l$, then x_i is in row l of $\sigma * x$, so $x_{\tau(i)}$ is in row l of $\tau \cdot (\sigma * x)$. On the other hand, $x_{\tau(i)}$ is in row k of $\tau \cdot x$ and therefore in row l of $\sigma * (\tau \cdot x)$.

Let $\tau \in \Sigma_{am}$. Then

$$
\begin{aligned}
&\Phi\left(\tau \cdot \begin{array}{ccc}\hline x_1 & \dots & x_a \\ \hline & \vdots & \\ \hline x_{(m-1)a+1} & \dots & x_{ma} \\ \hline\end{array}\right)\\
=&\Phi\left(\begin{array}{ccc}\hline x_{\tau(1)} & \dots & x_{\tau(a)} \\ \hline & \vdots & \\ \hline x_{\tau((m-1)a+1)} & \dots & x_{\tau(ma)} \\ \hline\end{array}\right)\\
=&\{\{x_{\tau(1)}, \dots, x_{\tau(a)}\}, \dots, \{x_{\tau((m-1)a+1)}, \dots, x_{\tau(ma)}\}\}\\
=&\tau \cdot \{\{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\}\}\\
=&\tau\Phi\left(\begin{array}{ccc}\hline x_1 & \dots & x_a \\ \hline & \vdots & \\ \hline x_{(m-1)a+1} & \dots & x_{ma} \\ \hline\end{array}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Psi\left(\tau \cdot \{\{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\}\}\right)\\
=&\Psi\left(\{\{x_{\tau(1)}, \dots, x_{\tau(a)}\}, \dots, \{x_{\tau((m-1)a+1)}, \dots, x_{\tau(ma)}\}\}\right)\\
=&\frac{1}{m!}\sum_{\sigma \in \Sigma_m} \sigma * \begin{array}{ccc}\hline x_{\tau(1)} & \dots & x_{\tau(a)} \\ \hline & \vdots & \\ \hline x_{\tau((m-1)a+1)} & \dots & x_{\tau(ma)} \\ \hline\end{array}\\
=&\frac{1}{m!}\sum_{\sigma \in \Sigma_m} \sigma * \left(\tau \cdot \begin{array}{ccc}\hline x_1 & \dots & x_a \\ \hline & \vdots & \\ \hline x_{(m-1)a+1} & \dots & x_{ma} \\ \hline\end{array}\right)\\
=&\tau \cdot \left(\frac{1}{m!}\sum_{\sigma \in \Sigma_m} \sigma * \begin{array}{ccc}\hline x_1 & \dots & x_a \\ \hline & \vdots & \\ \hline x_{(m-1)a+1} & \dots & x_{ma} \\ \hline\end{array}\right)\\
=&\tau \cdot \Psi\left(\{\{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\}\}\right).
\end{aligned}
$$

Hence, Ψ and Φ are $k\Sigma_{am}$-module homomorphisms. Φ is surjective and $\Phi\Psi$ is the identity on $H^{(a^m)}$, so Φ is a split epimorphism. □

As a direct corollary, we get

Corollary 5.15. *If* $\operatorname{char} k > m$, *the indecomposable direct summands of the Foulkes module* $H^{(a^m)} = k\Sigma_{am} \underset{k(\Sigma_a \wr \Sigma_m)}{\otimes} k$ *are Young modules. In particular,* $H^{(a^m)}$ *is both Specht and dual Specht filtered.*

Corollary 5.16. *If* $\operatorname{char} k > \max\{\beta_i\}$, *then* $k\Sigma_{l_2} \underset{k\Pi_\beta}{\otimes} k \in \mathcal{F}_{l_2}(S)$.

Proof. By Lemma 5.13, $k\Sigma_{l_2} \underset{k\Pi_\beta}{\otimes} k$ is induced from an exterior tensor product of Foulkes modules. Corollary 5.15 shows that the Foulkes modules are dual Specht filtered, provided the characteristic of the field is large enough. The characteristic-free version of the Littlewood-Richardson rule then says that the exterior tensor product of Foulkes modules has a dual Specht filtration. □

Restrictions of Cell Modules have Specht Filtrations

We are now able to put the results about tensor induced and Foulkes modules together to show that the restriction of a cell module to a group algebra of a symmetric group is dual Specht filtered.

The factor $k\Sigma_{2\beta_2} \underset{k(\Sigma_2 \wr \Sigma_{\beta_2})}{\otimes} k$ is the stabilizer of unlabelled parts of size 2. This factor occurs in the Brauer algebra as well, and is dual Specht filtered by [Pag07, Proposition 8]. For $i > 2$, the factor $k\Sigma_{i\beta_i} \underset{k(\Sigma_i \wr \Sigma_{\beta_i})}{\otimes} k$ is dual Specht filtered if $\operatorname{char} k > \beta_i$ by Corollary 5.15. The maximal amount of unlabelled parts of a certain size occurs in the summands U_v, where v consists of n labelled singletons and $\lfloor \frac{r-n}{3} \rfloor$ unlabelled parts of size 3. The remaining $0,1$ or 2 dots form additional unlabelled parts.

Corollary 5.17. *Let* $\operatorname{char} k > \lfloor \frac{r-n}{3} \rfloor$. *Then* $e_l(A/J_{n-1})e_n \in \mathcal{F}_l(S)$.

Proof. By Lemmas 5.10 and 5.11, the summands of $e_l(A/J_{n-1})e_n$ are of the form $k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k))$, where $\prod_\alpha \simeq k\prod(\Sigma_i \wr \Sigma_{\alpha_i})$ and $\prod_\beta \simeq k(\prod \Sigma_i \wr \Sigma_{\beta_i})$, $l_1 = \sum_i \alpha_i \cdot i$ and $l_2 = \sum_i \beta_i \cdot i$. Lemma 5.12 shows that $k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n \in \mathcal{F}_{l_1}(S)$ and by Corollary 5.16, we have $k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k \in \mathcal{F}_{l_2}(S)$ in case $\operatorname{char} k > \max \beta_i$. Since we are looking at the whole bimodule $e_l(A/J_{n-1})e_n$ and not just its summands U_v, we have to consider all possible top rows. By the above arguments, we have that the maximal amount of unlabelled parts of size ≥ 3 is $\lfloor \frac{r-n}{3} \rfloor$. Hence, we have to assume $\operatorname{char} k > \lfloor \frac{r-n}{3} \rfloor$. The characteristic-free version of the Littlewood-Richardson rule then says that the module $k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{k\prod_\alpha}{\otimes} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \underset{k\prod_\beta}{\otimes} k))$ lies in $\mathcal{F}_l(S)$. □

Theorem 5.3 (Theorem A). *Let* $\operatorname{char} k > \lfloor \frac{r-n}{3} \rfloor$ *and* $X \in k\Sigma_n - \operatorname{mod}$. *Then the* $k\Sigma_l$*-module* $e_l(A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} X$ *is in* $\mathcal{F}_l(S)$. *In particular, restrictions of cell modules to* $k\Sigma_l - \operatorname{mod}$ *are dual Specht filtered.*

Proof. By Lemmas 5.10, 5.11, 5.12 and 5.13, $e_l(A/J_{n-1})e_n$ decomposes as $(k\Sigma_l, k\Sigma_n)$-bimodule into a direct sum of modules of the form

$$k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((\bigoplus_{i=1}^{s} k\Sigma_{l_1} \underset{k\Sigma_\gamma}{\otimes} k) \boxtimes (k\Sigma_{l_2} \underset{k\Sigma_{\tilde{\gamma}}}{\otimes} (k \boxtimes (k\Sigma_{\tilde{\gamma}_2} \underset{k(\Sigma_2 \wr \Sigma_{\beta_2})}{\otimes} k) \boxtimes \ldots \boxtimes (k\Sigma_{\tilde{\gamma}_t} \underset{k(\Sigma_t \wr \Sigma_{\beta_t})}{\otimes} k)))).$$

Hence, an element of $e_l(A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} X \simeq e_l(A/J_{n-1})e_n \underset{k\Sigma_n}{\otimes} X$ is of the form

$$\begin{aligned}&\omega \otimes (\sum_{i=1}^{s}(\pi_i \otimes 1)^{(i)} \boxtimes (\upsilon \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \ldots \boxtimes (\tau_t \otimes 1)))) \otimes x\\ =&\omega \otimes (\sum_{i=1}^{s}(\pi_i \otimes 1)^{(1)}\sigma_i \boxtimes (\upsilon \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \ldots \boxtimes (\tau_t \otimes 1)))) \otimes x\\ =&\omega \otimes (\sum_{i=1}^{s}(\pi_i \otimes 1)^{(1)} \boxtimes (\upsilon \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \ldots \boxtimes (\tau_t \otimes 1)))) \otimes \sigma_i x\\ =&\omega \otimes ((\sum_{i=1}^{s}\pi_i \otimes 1)^{(1)} \boxtimes (\upsilon \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \ldots \boxtimes (\tau_t \otimes 1)))) \otimes \sigma_i x\end{aligned}$$

with $\omega \in k\Sigma_l$, $\pi_i \in k\Sigma_{l_1}$, $\nu \in k\Sigma_{l_2}$ and $\tau_i \in k\Sigma_{\tilde{\gamma_i}}$. So the summands of $e_l(A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} X$ are of the form

$$Y := Z \underset{k}{\otimes} (k \underset{k\Sigma_\alpha}{\otimes} k\Sigma_n \underset{k\Sigma_n}{\otimes} X),$$

where Z is the module

$$k\Sigma_l \underset{k\Sigma_{(l_1,l_2)}}{\otimes} ((k\Sigma_{l_1} \underset{\Sigma_\gamma}{\otimes} k) \boxtimes ((k\Sigma_{l_2} \underset{\Sigma_{\tilde{\gamma}}}{\otimes} (k \boxtimes (k\Sigma_{\tilde{\gamma_2}} \underset{k(\Sigma_2 \wr \Sigma_{\beta_2})}{\otimes} k) \boxtimes \ldots \boxtimes (k\Sigma_{\tilde{\gamma_t}} \underset{k(\Sigma_t \wr \Sigma_{\beta_t})}{\otimes} k))))).$$

By the above corollary, we know that Z is dual Specht filtered. Thus, the left $k\Sigma_l$-module $Y = \bigoplus_{i=1}^{h} Z$ is in $\mathcal{F}_l(S)$, where $h = \dim(k \underset{k\Sigma_\alpha}{\otimes} X)$.

□

6. Permutation Modules for Cellularly Stratified Algebras

The aim of this chapter is to generalize the results of Hartmann and Paget for permutation modules of Brauer algebras to cellularly stratified algebras A. We want to generalize the notion of permutation modules, so we assume the input algebras B_l of the cellularly stratified structure to be group algebras of symmetric groups or their Hecke algebras, where we know permutation modules already. In [HP06], the permutation module for the Brauer algebra is defined as $Ae_l \underset{k\Sigma_l}{\otimes} M^\lambda$, a module induced from the permutation module M^λ of the group algebra $k\Sigma_l$. For this tensor product to be well-defined, we need that Ae_l has a right $k\Sigma_l$-module structure. To assure this fact for arbitrary A, we assume that A has subalgebras of the form B_l. We then define two types of induction and two types of restriction functors. The layer induction ind is used to describe cell modules, while the induction Ind is used to define permutation modules. We first examine these functors with regards to properties such as exactness. We proceed, in the following section, with the definition of permutation modules and their indecomposable summands, the Young modules. Some immediate consequences on the summands of a permutation module are explained. In Section 6.3, we investigate the relationship between permutation modules and cell modules. We isolate three properties of A which guarantee that permutation modules have cell filtrations. If, in addition, the cell modules of B_l form a standard system, i.e. if char$k \neq 2, 3$ or $h \geq 4$, where h is the smallest integer such that $\sum_{i=0}^{h-1} q^i = 0$, cf. [HN04], then the direct summands of permutation modules are cell filtered as well. Assuming that, beside the properties we assumed already, the restriction of a cell module of A to B_l − mod has a dual Specht filtration, we show that the Young modules are the relative projective covers of the cell modules, with respect to the category of all cell filtered A-modules. Furthermore, we detect a decomposition of the permutation module $Ae_l \underset{B_l}{\otimes} M^\lambda$ into Young modules with larger or equal index. In Section 6.4, we re-prove the results from [HP06] and show that the non-degenerate partition algebra $P_k(r, \delta)$ satisfies our assumptions if the characteristic of the field is large enough. We give evidence why q-Brauer algebras might have permutation modules with the desired properties. In Section 6.5, we return to the general theory and exhibit what happens under dualisation.

Let A be a cellularly stratified diagram algebra with stratification data

$$(B_1, V_1, ..., B_r, V_r)$$

where B_l is the Hecke algebra $\mathcal{H}_{l'}(q)$ or $k\Sigma_{l'}$ for some $l' \in \mathbb{N}$ related to l. Assume further that B_l is isomorphic to a subalgebra of $e_l A e_l$. By *diagram algebra*, we mean an associative algebra with basis given by diagrams of two rows of dots, which are connected by some rule, and maybe some labelling on the dots or connections. We do not need the assumption that A is a diagram algebra, but for the idea behind the term *(dual) permutation module*, cf. Sections 6.2 and 6.5.

6.1 Functors

As before, we write $e_l(A/J_{l-1})$ as shorthand for e_lA/e_lJ_{l-1}. We define induction and restriction functors

$$\begin{aligned} ind_l : B_l - \text{mod} &\to A - \text{mod} & Ind_l : B_l - \text{mod} &\to A - \text{mod} \\ M &\mapsto Ae_l \underset{e_lAe_l}{\otimes} M & M &\mapsto Ae_l \underset{B_l}{\otimes} M \\ res_l : A - \text{mod} &\to B_l - \text{mod} & Res_l : A - \text{mod} &\to B_l - \text{mod} \\ N &\mapsto e_l(A/J_{l-1}) \underset{A}{\otimes} N & N &\mapsto e_lA \underset{A}{\otimes} N \simeq e_lN \end{aligned}$$

Note that Ae_l has a right B_l-module structure because we assumed B_l to be isomorphic to a subalgebra of e_lAe_l. A B_l-module M has an e_lAe_l-module structure via the quotient map $e_lAe_l \twoheadrightarrow B_l$. The functor ind_l is called *layer induction*, res_l is called *layer restriction.* We say that a module $M \in A - \text{mod}$ *lives in layer* l if J_{l-1} and A/J_l act trivially on M, while the action of J_l is non-trivial.

Properties

For each B_l-module X, we have $X \simeq B_l \underset{B_l}{\otimes} X \simeq B_l \underset{e_lAe_l}{\otimes} X$, where e_lAe_l acts on both X and B_l via the quotient map $e_lAe_l \twoheadrightarrow B_l$. Thus, the layer induction ind_l corresponds to the functor $G_l := Ae_l \underset{e_lAe_l}{\otimes} B_l \underset{e_lAe_l}{\otimes} -$, defined in [HHKP10]. Indeed, we have $G_lX = Ae_l \underset{e_lAe_l}{\otimes} B_l \underset{e_lAe_l}{\otimes} X \simeq Ae_l \underset{e_lAe_l}{\otimes} X = ind_lX$ for all $X \in B_l - \text{mod}$. Hence, we can apply [HHKP10, Lemma 3.4] to get an isomorphism $ind_l X \simeq (A/J_{l-1})e_l \underset{e_l A e_l}{\otimes} X$ of A-modules. We will make extensive use of the isomorphisms $ind_l X \simeq G_l X \simeq (A/J_{l-1})e_l \underset{e_l A e_l}{\otimes} X \simeq (A/J_{l-1})e_l \underset{B_l}{\otimes} X$ without special mention.

Proposition 6.1 ([HHKP10], Propositions 4.1 - 4.3; Corollary 7.4; Propositions 8.1 - 8.2)**.** *The functor ind_l has the following properties:*

(1) It is exact.

(2) It sends cell modules to cell modules.

(3) Each cell module of A is induced from a cell module of one of the B_l.

(4) $\mathrm{Hom}_{B_l}(X, Y) \simeq \mathrm{Hom}_A(ind_l X, ind_l Y)$ *for all* $X, Y \in B_l - \mathrm{mod}$.

(5) $\mathrm{Ext}^i_A(M, N) \simeq \mathrm{Ext}^i_{A/J_l}(M, N)$ *for all* $i > 0$ *and* $M, N \in A/J_l - \mathrm{mod}$.

(6) $\mathrm{Ext}^j_{B_l}(X, Y) \simeq \mathrm{Ext}^j_A(ind_l X, ind_l Y)$ *for all* $j \geq 0$ *and* $X, Y \in B_l - \mathrm{mod}$.

and if $l < m$ then

(7) $\mathrm{Hom}_A(ind_l X, ind_m Y) = 0$ *for all* $X \in B_l - \mathrm{mod}, Y \in B_m - \mathrm{mod}$.

(8) $\mathrm{Ext}^i_A(ind_l X, ind_m Y) = 0$ *for all* $i \geq 1$ *and* $X \in B_l - \mathrm{mod}, Y \in B_m - \mathrm{mod}$.

The induction Ind_l is not exact in general and does not send cell modules to cell modules. However, we will give sufficient conditions for Ind_l to send cell filtered modules to cell filtered modules in Section 6.2. Lemma 6.15 then tells us that, under additional conditions, Ind_l sends relative projective modules to relative projective modules.

The layer restriction res_l is right-exact, but in general not exact.

Lemma 6.2. *res_l is left inverse to ind_l and Ind_l.*

Proof. Let $X \in B_l - \text{mod}$. Then

$$res_l ind_l X = e_l(A/J_{l-1}) \underset{A}{\otimes} Ae_l \underset{e_lAe_l}{\otimes} X \simeq e_l(A/J_{l-1})e_l \underset{e_lAe_l}{\otimes} X \simeq X$$

and

$$res_l Ind_l X = e_l(A/J_{l-1}) \underset{A}{\otimes} Ae_l \underset{B_l}{\otimes} X \simeq e_l(A/J_{l-1})e_l \underset{B_l}{\otimes} X \simeq X.$$

□

Lemma 6.3. *res_l is left adjoint to* $\mathrm{Hom}_{B_l}(e_l(A/J_{l-1}), -)$.

Proof. For $X \in A - \text{mod}$ and $Y \in B_l - \text{mod}$,

$$\mathrm{Hom}_A(X, \mathrm{Hom}_{B_l}(e_l(A/J_{l-1}), Y) \simeq \mathrm{Hom}_{B_l}(e_l(A/J_{l-1}) \underset{A}{\otimes} X, Y)$$

is the usual adjunction isomorphism of $\otimes$ and Hom[1], so res_l is left adjoint to $\mathrm{Hom}_{B_l}(e_l(A/J_{l-1}), -)$.

□

Proposition 6.4. *If X is a cell module of A, then $res_l X$ is a cell module of B_l or zero.*

Proof. Let X be a cell module of A. By Proposition 6.1, part *(2)*, we have $X \simeq ind_n S_\nu$ for some $1 \leq n \leq r$, where S_ν is a dual

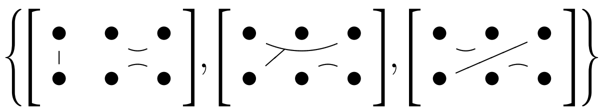
[1]See for example [ASS06, I.2.11].

Specht module in B_n – mod. This implies $res_l X \simeq res_l ind_n S_\nu \simeq e_l(A/J_{l-1}) \underset{A}{\otimes} (A/J_{n-1})e_n \underset{e_nAe_n}{\otimes} S_\nu \simeq e_l(A/J_m)e_n \underset{e_nAe_n}{\otimes} S_\nu$, where $m = \max\{l-1, n-1\}$. If $n < l$, then $e_n \in J_m = J_{l-1}$ and if $n > l$, then $e_l \in J_m = J_{n-1}$. So, in both cases we have $res_l X = 0$. For $n = l$, it is $res_l ind_l S_\nu \simeq S_\nu$ by Lemma 6.2. Thus, the layer restriction of a cell module from the same layer is a cell module, while cell modules from other layers vanish. □

The restriction Res_l is exact, since $e_l A$ is projective as right A-module.

Lemma 6.5. *Res_l is left inverse to ind_l, but in general not to Ind_l.*

Proof. Let $X \in B_l$ – mod. Then $Res_l ind_l X = e_l A \underset{A}{\otimes} Ae_l \underset{e_lAe_l}{\otimes} X \simeq e_lAe_l \underset{e_lAe_l}{\otimes} X \simeq X$ and $Res_l Ind_l X = e_l A \underset{A}{\otimes} Ae_l \underset{B_l}{\otimes} X \simeq e_lAe_l \underset{B_l}{\otimes} X \simeq X \oplus (e_l J_{l-1} e_l \underset{B_l}{\otimes} X)$. If A is the Brauer algebra $B_{\mathbb{C}}(3, \delta)$ with $\delta \neq 0$ and $l = 3$, and X is the trivial $\mathbb{C}\Sigma_3$-module $\mathbb{C}$, then $e_3 J_1 e_3 = J_1 = Ae_1A$, which is depicted in the second table of Appendix AI. The left $\mathbb{C}\Sigma_3$-module $Ae_1A \underset{\mathbb{C}\Sigma_3}{\otimes} \mathbb{C}$ has basis



In particular, $Ae_1A \underset{C\Sigma_3}{\otimes} C$ is non-zero. This shows that, in general, Res_l in not left inverse to Ind_l since Res_l does not remove the layers $1, ..., l-1$ which were added by Ind_l. □

Lemma 6.6. *Res_l is left adjoint to* $\mathrm{Hom}_{B_l}(e_lA, -)$ *and right adjoint to Ind_l.*

Proof. This follows from adjointness of the tensor functor and the Hom-functor and the fact that $\mathrm{Hom}_A(Ae, M) \simeq eM$ for all algebras A with $e \in A$ idempotent and $M \in A-\mathrm{mod}$, cf. [ASS06, I.2.11 and I.4.2].

□

6.2 Young Modules

Let $\Lambda_r := \{(l,\lambda)|0 \le l \le r, \lambda \vdash l'\}$, where l' is the index of the symmetric group related to B_l. We define an order $\prec$ on Λ_r:

$$(n,\nu) \prec (l,\lambda) \Leftrightarrow n \ge l \text{ and if } n = l \text{ then } \nu \lhd \lambda.$$

Let $(l,\lambda) \in \Lambda_r$ and let M^λ be the corresponding permutation module in $B_l-\mathrm{mod}$, defined in Chapter 2. We call the A-module $Ind_l M^\lambda$ *permutation module* for A. Let $d \otimes m$ be a basis element of $Ae_l \underset{B_l}{\otimes} M^\lambda$ and a a basis element of A, i.e. $a, d \in A$ are diagrams and m is a basis element of M^λ. Then $a \cdot (d \otimes m) = ad \otimes m$ and ad is (a δ-multiple of) a diagram in A. If $ad = d'\sigma$ with $d' \in A, \sigma \in B_l$, then $ad \otimes m = d' \otimes \sigma m = d' \otimes m'$, where m' is again a basis element of M^λ, since M^λ is a permutation module for B_l. Thus, the A-action permutes the basis of $Ind_l M^\lambda$, up to scalar multiplicities of the form δ^s.

Proposition 6.7. *$Ind_l M^\lambda$ has a unique direct summand with quotient isomorphic to $ind_l Y^\lambda$.*

Proof. We follow the ideas of the proof of [HP06, Proposition 14].

Recall that the B_l-permutation module M^λ decomposes into a direct sum of indecomposable Young modules Y^μ with multiplicities a_μ, where $a_\lambda = 1$ and $a_\mu \neq 0$ implies $\mu \trianglerighteq \lambda$. Therefore, we have $Ind_l M^\lambda = \bigoplus_{(l,\mu)\in\Lambda_r} (Ind_l Y^\mu)^{a_\mu}$. Let $Ind_l Y^\lambda$ decompose further into a direct sum of indecomposables Y_i for $i = 1, ..., s$.

Claim 1. $Ind_l Y^\lambda$ has a direct summand with quotient isomorphic to $ind_l Y^\lambda$.

Let $\pi_i : Ind_l Y^\lambda \twoheadrightarrow Y_i$ be the projection onto Y_i and $\iota_i : Y_i \hookrightarrow Ind_l Y^\lambda$ the inclusion of Yi. Multiplication with e_l from the left yields the commutative diagram

$$\begin{tikzcd}
Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[r, two heads, "\pi_i"] \arrow[d, "e_l\cdot -"] & Y_i \arrow[r, hook, "\iota_i"] & Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[d, "e_l\cdot -"] \\
e_lAe_l \underset{B_l}{\otimes} Y^\lambda \arrow[rr, "e_l(\iota_i\circ\pi_i)"] & & e_lAe_l \underset{B_l}{\otimes} Y^\lambda
\end{tikzcd}$$

Since $e_lAe_l \simeq B_l \oplus e_lJ_{l-1}e_l$ by Corollary 3.9, the homomorphism $e_l(\iota_i \otimes \pi_i)$ is given by a matrix, where the top left entry is an endomorphism $f_i \in \mathrm{End}_{B_l}(Y^\lambda)$. This gives a commutative diagram

$$\begin{tikzcd}
Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[r, two heads, "\pi_i"] \arrow[d, "e_l\cdot -"] & Y_i \arrow[r, hook, "\iota_i"] & Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[d, "e_l\cdot -"] \\
e_lAe_l \underset{B_l}{\otimes} Y^\lambda \arrow[rr, "e_l(\iota_i\circ\pi_i)"] \arrow[d, "\wr"] & & e_lAe_l \underset{B_l}{\otimes} Y^\lambda \arrow[d, "\wr"] \\
Y^\lambda \oplus (e_lJ_{l-1}e_l \underset{B_l}{\otimes} Y^\lambda) \arrow[rr, "\begin{pmatrix} f_i & * \\ * & * \end{pmatrix}"] & & Y^\lambda \oplus (e_lJ_{l-1}e_l \underset{B_l}{\otimes} Y^\lambda)
\end{tikzcd}$$

Let $a \in A$ such that $e_lae_l = b + j$ with $b \in B_l$, $j \in e_lJ_{l-1}e_l$ and let $y \in Y^\lambda$.

Let $\pi_i(e_l \otimes y) = e_l \otimes x$ + lower terms. Then

$$e_l(\iota_i \circ \pi_i)(e_l a e_l \otimes y) = e_l a e_l \otimes x \; + \text{ lower terms } \in e_l A e_l \underset{B_l}{\otimes} Y^\lambda$$

corresponds to

$$\begin{pmatrix} f_i & * \\ * & * \end{pmatrix} \begin{pmatrix} by \\ j \otimes y' \end{pmatrix} = f_i(by) \; + \text{ lower terms } \in Y^\lambda \oplus (e_l J_{l-1} e_l \underset{B_l}{\otimes} Y^\lambda).$$

In particular, we have $\pi_i(e_l \otimes y) = e_l \otimes f_i(y)$ + lower terms. By *lower terms* we mean terms of the form $je_l \otimes z$ with $j \in J_{l-1}$ and $z \in Y^\lambda$. The identity on $Ind_l Y^\lambda$ is $\sum_{i=1}^{s} \pi_i$. So $e_l \otimes y = \sum_{i=1}^{s} \pi_i(e_l \otimes y) = \sum_{i=1}^{s} e_l \otimes f_i(y)$ + lower terms for any $y \in Y^\lambda$. Hence $\sum_{i=1}^{s} f_i(y) = y$ and $\sum_{i=1}^{s} f_i = id_{Y^\lambda}$.

Let $i \neq j$. Then $\pi_i \iota_j \pi_j = 0$, so for any $y \in Y^\lambda$ it is $0 = \pi_i \iota_j \pi_j (e_l \otimes y) = e_l \otimes f_i f_j(y)$ + lower terms. Therefore, $f_i f_j = 0$. $\mathrm{End}_{B_l}(Y^\lambda)$ is local since Y^λ is finite dimensional and indecomposable[2], so for all i, either f_i or $1 - f_i$ is a unit. To show that at least one f_i is a unit, assume that $f_1, ..., f_{s-1}$ are non-units. Then $\prod_{i=1}^{s-1}(1 - f_i) = 1 - f_1 - ... - f_{s-1} = f_s$ is a unit. We now assume without loss of generality that f_1 is a unit, in particular surjective.

Let

$$\begin{aligned} \varphi : Ind_l Y^\lambda &\longrightarrow ind_l Y^\lambda \\ e_l \otimes y &\longmapsto e_l \otimes y \end{aligned}$$

[2] cf. [ASS06, I.4.8(b)]

and $\varphi' := \varphi \circ \iota_1 \circ \pi_1$ its restriction to Y_1. Then

$$\varphi'(e_l \otimes y) = \varphi(e_l \otimes f_1(y) \ + \ \text{lower terms}) = e_l \otimes f_1(y),$$

since for $j \in J_{l-1}$ and $z \in Y^\lambda$, $\varphi(je_l \otimes z) = je_l \otimes z = 0 \in ind_l Y^\lambda$. Surjectivity of f_1 implies that the A-homomorphism φ' is surjective.

Claim 2. Y_1 is the only summand of $Ind_l Y^\lambda$ with quotient isomorphic to $ind_l Y^\lambda$.

Suppose there is another summand Y_2 of $Ind_l Y^\lambda$ such that there is an epimorphism $\psi : Ind_l Y^\lambda \twoheadrightarrow ind_l Y^\lambda$ with $\psi(Y_2) = ind_l Y^\lambda$ and $\psi(Y_j) = 0$ for all $j \neq 2$. By Lemmas 6.5 and 6.6, ψ is given by $\psi(e_l \otimes y) = e_l \otimes \sigma(y)$ for some $\sigma \in \mathrm{End}_{B_l}(Y^\lambda)$. The surjectivity of ψ gives the existence of a preimage $v = \sum\limits_i a_i e_l \otimes y_i \in Y_2$ of $e_l \otimes y \in ind_l Y^\lambda$. For $j \in J_{l-1}$ and $z \in Y^\lambda$, we have $\psi(je_l \otimes z) = je_l \otimes \sigma(z) = 0$.

As (B_l, B_l)-bimodule, $e_l A e_l$ is isomorphic to $e_l(A/J_{l-1})e_l \oplus e_l J_{l-1} e_l \simeq B_l \oplus e_l J_{l-1} e_l$ by Lemma 3.9, so we can write an element $e_l a e_l$ of $e_l A e_l$ as $b + e_l j e_l$ with $b \in B_l$ and $j \in J_{l-1}$. Then $e_l v = e_l(\sum\limits_i a_i e_l \otimes y_i) = \sum\limits_i e_l a_i e_l \otimes y_i$ $= e_l \otimes w +$ lower terms for some $w \in Y^\lambda$. So ψ sends $e_l v$ to

$$\psi(e_l v) = \psi(e_l \otimes w + \text{lower terms}) = e_l \otimes \sigma(w).$$

On the other hand,

$$\psi(e_l v) = e_l \psi(v) = e_l(e_l \otimes y) = e_l \otimes y,$$

so $\sigma(w) = y \neq 0$, hence $w \neq 0$. But $e_l v \in Y_2$ and

$$\varphi'(e_l v) = \varphi'(e_l \otimes w + \text{lower terms}) = e_l \otimes f_1(w) \neq 0$$

since $w \neq 0$ and f_1 is a unit, in particular injective. Hence $\varphi'(Y_2) \neq 0$, which contradicts the definition of φ'.

Claim 3. For $\mu \neq \lambda$, there is no summand of $Ind_l Y^\mu$ with quotient $ind_l Y^\lambda$.

Assume there is a direct summand Y^μ of M^λ with $\mu \rhd \lambda$ such that $ind_l Y^\lambda$ is a quotient of $Ind_l Y^\mu$. An arbitrary homomorphism $\Phi : Ind_l Y^\mu \to ind_l Y^\lambda$ is given by $\Phi(e_l \otimes y) = e_l \otimes \varphi(y)$ for some $\varphi \in \mathrm{Hom}_{B_l}(Y^\mu, Y^\lambda)$, again by Lemmas 6.5 and 6.6. Φ is surjective only if φ is surjective.

The rest of the proof can be copied from [HP06] in case $B_l = k\Sigma_l$. We give here a similar proof for Hecke algebras $\mathcal{H}_l$, inspired by the one for group algebras of symmetric groups:

Suppose there is an epimorphism $\varphi : Y^\mu \twoheadrightarrow Y^\lambda$, which we extend to an epimorphism $\hat{\varphi} : M^\mu \to Y^\lambda$ such that $\hat{\varphi}$ is zero on all summands other than Y^μ, i.e. $\hat{\varphi}$ is the projection from M^μ onto the direct summand Y^μ, followed by the map φ. Recall that $M^\mu = \mathcal{H}_l x_\mu$ and $\mathcal{H}_l$ is generated by elements T_π. By [DJ86, Lemma 4.1], $y_{\lambda'} T_\pi x_\mu \neq 0$ implies $\lambda = \lambda'' \unrhd \mu$. So for $\mu \rhd \lambda$, we have $y_{\lambda'} M^\mu = 0$. Then $0 = \hat{\varphi}(0) = \hat{\varphi}(y_{\lambda'} M^\mu) = y_{\lambda'} \hat{\varphi}(M^\mu) = y_{\lambda'} Y^\lambda$. But $y_{\lambda'} Y^\lambda$ contains the generator $y_{\lambda'} T_{w_\lambda} x_\lambda = z_\lambda$ of S^λ, in particular $y_{\lambda'} Y^\lambda \neq 0$. □

Definition 6.8. We denote the unique summand of $Ind_l Y^\lambda$ with quotient $ind_l Y^\lambda$ constructed above by $Y(l, \lambda)$ and call it *Young module* for A with respect to $(l, \lambda) \in \Lambda_r$.

We will now collect conditions for a Young module $Y(m, \mu)$ to appear as a summand of $Ind_l M^\lambda$. They generalize the conditions from [HP06, Lemmas 17 and 18] for $A = B_k(r, \delta)$.

Lemma 6.9. *If $(l, \lambda), (m, \mu) \in \Lambda_r$ with $l < m$, then $Y(m, \mu)$ does not appear as a summand of $Ind_l M^\lambda$.*

Proof. Ind_l is left adjoint to Res_l by Lemma 6.6, so

$$\begin{aligned}\mathrm{Hom}_A(Ind_l M^\lambda, ind_m Y^\mu) &\simeq \mathrm{Hom}_{B_l}(M^\lambda, Res_l ind_m Y^\mu)\\ &\simeq \mathrm{Hom}_{B_l}(M^\lambda, e_l(A/J_{m-1})e_m \underset{e_m A e_m}{\otimes} Y^\mu).\end{aligned}$$

For $l < m$, $e_l \in J_{m-1}$, so $Res_l ind_m Y^\mu = 0$. Thus, there cannot be a (split) map

$$Ind_l M^\lambda \to Y(m, \mu)$$

since it would extend to a map $Ind_l M^\lambda \to ind_m Y^\mu$. □

Lemma 6.10. *If $(l, \lambda), (l, \kappa) \in \Lambda_r$, then $Y(l, \lambda)$ occurs as a direct summand of $Ind_l M^\kappa$ if and only if Y^λ is a direct summand of M^κ. This can only occur if $\lambda \trianglerighteq \kappa$.*

Proof. If Y^λ is a direct summand of M^κ, then $Y(l, \lambda)$, as a direct summand of $Ind_l Y^\lambda$, is a direct summand of $Ind_l M^\kappa$.

If $Y(l, \lambda)$ is a direct summand of $Ind_l M^\kappa$ and $M^\kappa = \bigoplus (Y^\mu)^{a_\mu}$, then $Y(l, \lambda)$ is a summand of $Ind_l Y^\mu$ for some μ. Suppose $\mu \neq \lambda$. By Proposition 6.7, there is an epimorphism $Y(l, \lambda) \twoheadrightarrow ind_l Y^\lambda$, which gives rise to an epimorphism $\Phi : Ind_l Y^\mu \twoheadrightarrow ind_l Y^\lambda$ factoring through $Y(l, \lambda)$. By Lemmas 6.5 and 6.6, $\Phi \in \mathrm{Hom}_A(Ind_l Y^\mu, ind_l Y^\lambda)$ corresponds to some epimorphism φ in $\mathrm{Hom}_{B_l}(Y^\mu, Y^\lambda)$. By precomposing $id_{(A/J_{l-1})e_l} \otimes \varphi$ with the epimorphisms

$$Ind_l Y^\mu \twoheadrightarrow Y(l, \mu) \twoheadrightarrow ind_l Y^\mu$$

from Proposition 6.7, we get a surjection $Ind_l Y^\mu \twoheadrightarrow ind_l Y^\lambda$, which does not exist by Claim 3 of Proposition 6.7. Hence, $\mu = \lambda$ and Y^λ is a direct summand of M^κ. $\square$

Corollary 6.11. *If* $(l,\lambda) \neq (l,\kappa)$*, then* $Y(l,\lambda) \not\simeq Y(l,\kappa)$*.*

Proof. Let $(l,\lambda) \neq (l,\kappa)$. Then $Y^\lambda \not\simeq Y^\kappa$, see for example [Mar93], Section 7.6, so $Ind_l Y^\lambda \not\simeq Ind_l Y^\kappa$ since otherwise $res_l Ind_l Y^\lambda \simeq Y^\lambda$ would be isomorphic to $res_l Ind_l Y^\kappa \simeq Y^\kappa$. Assume that $Y(l,\lambda) \simeq Y(l,\kappa)$. Then $Y(l,\kappa)$ is a direct summand of $Ind_l M^\lambda$ and by Lemma 6.10, Y^κ is a direct summand of M^λ. So $Ind_l Y^\kappa$ is a summand of $Ind_l M^\lambda$ and has a summand $Y(l,\kappa)$ with quotient $ind_l Y^\kappa$. But $Y(l,\kappa)$ is isomorphic to $Y(l,\lambda)$ with quotient $ind_l Y^\lambda$, so $Ind_l Y^\kappa$ has a direct summand with quotient isomorphic to $ind_l Y^\lambda$ and $\kappa \neq \lambda$. This contradicts Claim 3 from Proposition 6.7. $\square$

6.3 Cell Filtrations, Relative Projective Covers and Schur-Weyl duality

The Young modules $Y(l,\lambda)$ are direct summands of $Ind_l M^\lambda$ by definition. In this section, we will show that the indecomposable direct summands of permutation modules are exactly the Young modules, as in the symmetric group case. We will give a necessary condition for a Young module $Y(m,\mu)$ to appear in the decomposition of a permutation module $Ind_l M^\lambda$. The results will generalize the results on Brauer algebras stated in [HP06].

We will also show that the permutation modules for our cellularly stratified algebra A admit a cell filtration and are relative projective in the subcategory $\mathcal{F}(\Theta)$ of cell filtered A-modules. The Young module Y^λ will be shown to be the relative projective cover of the dual

Specht module S_λ. Similarly, the Young module $Y(l,\lambda) \in A-\mathrm{mod}$ will be shown to be the relative projective cover of the cell module $\Theta(l,\lambda) := ind_l S_\lambda$.

Lemma 6.12. *Let $h \geq 4$ and let q be an hth root of unity if the input algebras B_l are Hecke algebras $\mathcal{H}_{k,q}(\Sigma_l)$. Let $\mathrm{char}k = p \neq 2,3$ if the input algebras are group algebras of symmetric groups. Then the cell modules $\Theta(l,\lambda)$ of A form a standard system with respect to the order $\prec$.*

Even though it is part of Theorem 10.2 in [HHKP10], we give a proof below, to train working with cell modules and the statements from Proposition 6.1.

Proof. Hemmer and Nakano have shown in [HN04, Proposition 4.2.1] that for $h \geq 4$ and $\mu \not\triangleright \lambda$, there is no non-trivial extension of S^λ by S^μ, i.e. $\mathrm{Ext}^1_A(S^\mu, S^\lambda) = 0$. They recall from James ([Jam78, Corollary 13.17], for symmetric groups) and Mathas ([Mat99, Exercise 4.11], for Hecke algebras) that for $h \geq 3$, the endomorphism ring of a Specht module is isomorphic to the ground field and if in addition $\mu \not\trianglerighteq \lambda$, then $\mathrm{Hom}_{B_l}(S^\mu, S^\lambda) = 0$. It follows that the dual Specht modules form a standard system with respect to the dominance order if $h \geq 4$ (or $\mathrm{char}k \geq 5$).

By Proposition 6.1, part *(4)*, we have $\mathrm{End}_A(\Theta(l,\lambda)) \simeq k$ for all $(l,\lambda) \in \Lambda_r$ and $\mathrm{Hom}_A(\Theta(l,\lambda), \Theta(l,\mu)) = 0$ for $\mu \not\triangleright \lambda$. It follows from part *(7)* of the same proposition that $\mathrm{Hom}_A(\Theta(l,\lambda), \Theta(m,\mu)) = 0$ if $(m,\mu) \not\prec (l,\lambda)$. If, in addition, $(m,\mu) \neq (l,\lambda)$ then $\mathrm{Ext}^1_A(\Theta(l,\lambda), \Theta(m,\mu)) = 0$ for all $i \geq 1$ by Proposition 6.1, parts *(6)* and *(8)*. □

From now on, assume that char$k \neq 2,3$ (or $h \geq 4$) and A satisfies for $n < l$:

(I) $J_n e_l \simeq J_{n-1} e_l \oplus (J_n/J_{n-1}) e_l$ as right B_l-modules.

(II) $(J_n/J_{n-1}) e_l \simeq ind_n(e_n(A/J_{n-1})e_l)$ as right B_l-modules.

(III) Layer restriction of a permutation module is dual Specht filtered:

$$res_n Ind_l M^\lambda \simeq e_n(A/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda \in \mathcal{F}_n(S)$$

(IV) Restriction of a cell module $ind_n S_\nu$ is dual Specht filtered:

$$Res_l ind_n S_\nu \simeq e_l(A/J_{n-1})e_n \underset{B_n}{\otimes} S_\nu \in \mathcal{F}_l(S)$$

Remark. (a) Instead of (II), we can assume

(II′) $(J_n/J_{n-1}) e_l \simeq B_n \otimes V_n \otimes V_n^l$ as right B_l-modules.

By Lemma 3.7, part *(4)*, the algebra $e_l A e_l$ is cellularly stratified with idempotents $e_n = 1_{B_n} \otimes u_n \otimes v_n \in B_n \otimes_k V_n^l \otimes_k V_n^l \subseteq B_n \otimes_k V_n \otimes_k V_n$. Then $e_n(A/J_{n-1})e_l = e_n(e_l A e_l / e_l J_{n-1} e_l)$ is free of rank $\dim V_n^l$ over B_n by Proposition 3.8 and

$$ind_n(e_n(A/J_{n-1})e_l) \simeq (A/J_{n-1})e_n \underset{B_n}{\otimes} e_n(A/J_{n-1})e_l \simeq \bigoplus_{i=1}^{\dim V_n^l} (A/J_{n-1})e_l$$

as left A-modules. Hence, $\dim(ind_n(e_n(A/J_{n-1})e_l)) =$

$\dim((A/J_{n-1})e_n) \cdot \dim V_n^l = \dim B_n \cdot \dim V_n \cdot \dim V_n^l$ since $(A/J_{n-1})e_n$ is free of rank $\dim V_n$ over B_n.

The multiplication map

$$(A/J_{n-1})e_n \underset{B_n}{\otimes} e_n(A/J_{n-1})e_l \longrightarrow (J_n/J_{n-1})e_l$$
$$(a + J_{n-1})e_n \otimes e_n(b + J_{n-1})e_l \longmapsto (ae_nb + J_{n-1})e_l$$

is an epimorphism of (A, B_l)-bimodules and $\dim(ind_n(e_n(A/J_{n-1})e_l)) = \dim V_n^l \cdot \dim V_n \cdot \dim B_n = \dim((J_n/J_{n-1})e_l)$, so (II) is satisfied.

(b) Assumption (IV) implies that for any $X \in \mathcal{F}_n(S)$, $Res_l ind_n X \in \mathcal{F}_l(S)$: $ind_n X$ has a cell filtration by Proposition 6.1, part *(2)*, and Res_l is exact. Hence, $Res_l ind_n X$ has a filtration by modules of the form $Res_l ind_n S_\nu \in \mathcal{F}_l(S)$. The statement follows since $\mathcal{F}_l(S)$ is extension-closed.

Lemma 6.13. *Assume that A satisfies* (I),(II) *and* (III). *Then the permutation module $Ind_l M^\lambda$ has a filtration by cell modules.*

Proof. $A = J_r \supset J_{r-1} \supset \dots \supset J_1 \supset J_0 = 0$ is a filtration of A with quotients isomorphic to $B_n \otimes_k V_n \otimes_k V_n$, i.e. we have short exact sequences

$$0 \to J_{n-1} \to J_n \to J_n/J_{n-1} \to 0$$

of (A, A)-bimodules for $1 \leq n \leq r$. Application of the exact restriction functor $- \underset{A}{\otimes} Ae_l$ gives exact sequences

$$0 \to J_{n-1}e_l \to J_n e_l \to (J_n/J_{n-1})e_l \to 0$$

of $(A, e_l A e_l)$-bimodules for $n \le l$, which are split exact as sequences of right B_l-modules by assumption (I). Hence, we get exact sequences

$$0 \to J_{n-1}e_l \underset{B_l}{\otimes} M^\lambda \to J_n e_l \underset{B_l}{\otimes} M^\lambda \to (J_n/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda \to 0$$

of left A-modules, which give rise to a filtration

$$Ae_l \underset{B_l}{\otimes} M^\lambda \supset J_{l-1}e_l \underset{B_l}{\otimes} M^\lambda \supset \dots \supset J_1 e_l \underset{B_l}{\otimes} M^\lambda \supset 0$$

of $Ind_l M^\lambda$ with quotients $M^n(l,\lambda) := (J_n/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda$. Assumption (II) gives

$$\begin{aligned} M^n(l,\lambda) &\simeq ind_n(e_n(A/J_{n-1})e_l) \underset{B_l}{\otimes} M^\lambda \\ &\simeq ind_n(e_n(A/J_{n-1})e_l \underset{B_l}{\otimes} M^\lambda) \\ &\simeq ind_n(res_n Ind_l M^\lambda). \end{aligned}$$

By assumption (III), $res_n Ind_l M^\lambda \in \mathcal{F}_n(S)$. The functor ind_n is exact and sends cell modules to cell modules by Proposition 6.1, hence $M^n(l,\lambda) \simeq ind_n(res_n Ind_l M^\lambda) \in \mathcal{F}(\Theta)$ and therefore $Ind_l M^\lambda \in \mathcal{F}(\Theta)$. □

Corollary 6.14. *If, in addition,* $\mathrm{char} k \neq 2, 3$ *or* $h \ge 4$*, then the direct summands of* $Ind_l M^\lambda$ *have cell filtrations.*

Proof. By Lemma 6.12, the cell modules of A form a standard system if $h \ge 4$. Therefore, $\mathcal{F}(\Theta)$ is closed under direct summands by Corollary 3.17. □

An important property of permutation modules M^λ for B_l is their relative projectivity with respect to dual Specht filtered modules, as shown by Hemmer and Nakano in [HN04, Proposition 4.1.1]. This property is translated to the permutation modules $Ind_l M^\lambda$ of A. Furthermore, the Young modules are relative projective covers of the cell modules.

Lemma 6.15. *Assume that A satisfies* (IV). *Then the permutation module $Ind_l M^\lambda$, and all its direct summands, are relative projective in $\mathcal{F}(\Theta)$.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $A-$mod with $X \in \mathcal{F}(\Theta)$. Then $0 \to e_l X \to e_l Y \to e_l Z \to 0$ is exact in B_l-mod. X is filtered by cell modules $ind_n S_\nu$ for various $0 \le n \le r$, $\nu \vdash n$, and Res_l is exact, so $e_l X = Res_l X$ is filtered by modules of the form $Res_l ind_n S_\nu$. For $l < n$ this is zero since $e_l \in J_{n-1}$. For $l = n$, $Res_l ind_n S_\nu \simeq S_\nu$ by Lemma 6.5, and for $l > n$, assumption (IV) says that $Res_l ind_n S_\nu$ lies in $\mathcal{F}_l(S)$. Therefore $e_l X \in \mathcal{F}_l(S)$. M^λ is relative projective in $\mathcal{F}_l(S)$, so $\mathrm{Ext}^1_{B_l}(M^\lambda, e_l X) = 0$. Thus, the sequence

$$0 \to \mathrm{Hom}_{B_l}(M^\lambda, e_l X) \to \mathrm{Hom}_{B_l}(M^\lambda, e_l Y) \to \mathrm{Hom}_{B_l}(M^\lambda, e_l Z) \to 0$$

is exact. For any $N \in A-$mod, we have $\mathrm{Hom}_{B_l}(M^\lambda, e_l N) \simeq \mathrm{Hom}_A(Ind_l M^\lambda, N)$ by Lemma 6.6, so the sequence

$$0 \to \mathrm{Hom}_A(Ind_l M^\lambda, X) \to \mathrm{Hom}_A(Ind_l M^\lambda, Y) \to \mathrm{Hom}_A(Ind_l M^\lambda, Z) \to 0$$

is exact.

For $Z = Ind_l M^\lambda$, we get a split map $Ind_l M^\lambda \to Y$ from the surjectivity of the map $\mathrm{Hom}_A(Ind_l M^\lambda, Y) \twoheadrightarrow \mathrm{Hom}_A(Ind_l M^\lambda, Ind_l M^\lambda)$:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Ind_l M^\lambda \arrow[r] \arrow[d, equal] & 0 \\ & & & Ind_l M^\lambda \arrow[ul, dashed, "\exists"] & \end{tikzcd}$$

so $\mathrm{Ext}^1_A(Ind_l M^\lambda, X) = 0$ for $X \in \mathcal{F}(\Theta)$.

If Z is a direct summand of $Ind_l M^\lambda$ with $\pi : Ind_l M^\lambda \to Z$ the projection to Z and $\iota : Z \to Ind_l M^\lambda$ the inclusion of Z, then surjectivity of the map $\mathrm{Hom}_A(Ind_l M^\lambda, Y) \twoheadrightarrow \mathrm{Hom}_A(Ind_l M^\lambda, Z)$ gives the existence of a map $f : Ind_l M^\lambda \to Y$ such that $\pi = gf$:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] \arrow[d, hook, bend left, "\iota"] & 0 \\ & & & Ind_l M^\lambda \arrow[ul, dashed, "\exists f"] \arrow[u, "\pi"] & \end{tikzcd}$$

But $\pi\iota = id_Z$, so $gf\iota = id_Z$ and $f\iota$ is right inverse to g. Therefore, the sequence splits and $\mathrm{Ext}^1_A(Z, X) = 0$. □

Theorem 6.1 (Theorem B)**.** *Let A be cellularly stratified with input algebras*

$$B_l = \mathcal{H}_{k,q}(\Sigma_{l'}) \hookrightarrow e_l A e_l$$

for some $l' \le r$, where q is an hth root of unity and $h \ge 4$. Let $\mathrm{char} k \ne 2, 3$ *if $q = 1$. Assume that A satisfies the conditions* (I) *to* (IV) *and let $(l, \lambda) \in \Lambda_r$. Then $Y(l, \lambda)$ is the relative projective cover of $\Theta(l, \lambda)$ with respect to the category $\mathcal{F}(\Theta)$ of cell filtered modules.*

Proof. $Y(l, \lambda)$ lies in $\mathcal{F}(\Theta)$ by Corollary 6.14 and is relative projective by Lemma 6.15. It remains to show:

- There is an epimorphism $\Psi : Y(l,\lambda) \twoheadrightarrow \Theta(l,\lambda)$ with $ker(\Psi) \in \mathcal{F}(\Theta)$.

- For any other relative projective Y' with epimorphism $\Psi' : Y' \twoheadrightarrow \Theta(l,\lambda)$ such that $ker(\Psi') \in \mathcal{F}(\Theta)$, there is a split map $f \in \mathrm{Hom}_A(Y', Y(l,\lambda))$ such that $\Psi' = \Psi \circ f$.

The B_l-module Y^λ has a dual Specht filtration with top quotient S_λ, so $ker(Y^\lambda \twoheadrightarrow S_\lambda) \in \mathcal{F}_l(S)$. The functor ind_l is exact and sends dual Specht modules to cell modules, so the kernel of the epimorphism $\psi : ind_l Y^\lambda \twoheadrightarrow ind_l S_\lambda$ has a cell filtration.

Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & ker\phi \arrow[d] & & \\
0 \arrow[r] & ker\Psi \arrow[r] & Y(l,\lambda) \arrow[r, "\Psi"] \arrow[d, "\phi"] & ind_l S_\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & ker\psi \arrow[r] & ind_l Y^\lambda \arrow[r, "\psi"] \arrow[d] & ind_l S_\lambda \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

with $ker\psi, Y(l,\lambda), ind_l Y^\lambda$ and $ind_l S_\lambda$ in $\mathcal{F}(\Theta)$. By the arguments used in the proof of Lemma 3.10, we get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, "\sim"] \arrow[d] & ker\phi \arrow[d] & & \\
0 \arrow[r] & ker\Psi \arrow[r] \arrow[d] & Y(l,\lambda) \arrow[r, "\Psi"] \arrow[d, "\phi"] & ind_l S_\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & ker\psi \arrow[r] \arrow[d] & ind_l Y^\lambda \arrow[r, "\psi"] \arrow[d] & ind_l S_\lambda \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Thus, we have a short exact sequence

$$0 \to ker\phi \to ker\Psi \to ker\psi \to 0.$$

If we can show that $ker\phi \in \mathcal{F}(\Theta)$, then $ker\Psi \in \mathcal{F}(\Theta)$ since $\mathcal{F}(\Theta)$ is extension-closed by Lemma 3.10. We recall from the proof of Proposition 6.7, Claim 1, that $Y(l,\lambda)$ is the unique direct summand of $Ind_l Y^\lambda$ with quotient isomorphic to $ind_l Y^\lambda$ and that the map ϕ equals $\varphi \circ \iota$, where ι is the inclusion of $Y(l,\lambda)$ in $Ind_l Y^\lambda$ and $\varphi : Ind_l Y^\lambda \to ind_l Y^\lambda$ is given by $\varphi(ae_l \otimes y) = (ae_l + J_{l-1}) \otimes y$. Furthermore, we know that there is a unit $f \in \mathrm{End}_{B_l}(Y^\lambda)$ such that $\pi(ae_l \otimes y) = ae_l \otimes f(y)$ + lower layers, where π is the projection

$Ind_l Y^\lambda \twoheadrightarrow Y(l,\lambda)$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & ker(\varphi\iota) \arrow[r, "g"] \arrow[d, dashed, "\tilde{h}"] & Y(l,\lambda) \arrow[r, "\varphi\iota"] \arrow[d, hook, "\iota"] & (A/J_{l-1})e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & ker\varphi \arrow[r, "\tilde{g}"] & Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[r, "\varphi"] & (A/J_{l-1})e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{h} : ker(\varphi\iota) \to ker\varphi$ is defined by sending $x \in ker(\varphi\iota)$ to the unique $\tilde{g}$-preimage of $\iota g(x)$. Next, we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & ker(\varphi\iota) \arrow[r, "g"] & Y(l,\lambda) \arrow[r, "\varphi\iota"] & (A/J_{l-1})e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r] & 0 \\
0 \arrow[r] & ker\varphi \arrow[r, "\tilde{g}"] & Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[r, "\varphi"] \arrow[u, two heads, "\pi"] & (A/J_{l-1})e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r] \arrow[u, "\wr"', "id\otimes f"] & 0
\end{tikzcd}$$

and check commutativity. Let $ae_l \otimes y \in Ae_l \underset{B_l}{\otimes} Y^\lambda$. Then

$$\begin{aligned}
\varphi\iota\pi(ae_l \otimes y) &= \varphi\iota(ae_l \otimes f(y) \; + \text{ lower layers}) \\
&= (ae_l + J_{l-1}) \otimes f(y) \\
&= (id \otimes f)\varphi(ae_l \otimes y).
\end{aligned}$$

Therefore, the diagram commutes and we can define $h : ker\varphi \to ker(\varphi\iota)$ as the homomorphism sending $x' \in ker\varphi$ to the unique g-preimage of $\pi\tilde{g}(x')$.

$$\begin{tikzcd}
0 \arrow[r] & ker(\varphi\iota) \arrow[r, "g"] & Y(l,\lambda) \arrow[r, "\varphi\iota"] & (A/J_{l-1})e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r] & 0 \\
0 \arrow[r] & ker\varphi \arrow[r, "\tilde{g}"] \arrow[u, dashed, "h"] & Ae_l \underset{B_l}{\otimes} Y^\lambda \arrow[r, "\varphi"] \arrow[u, two heads, "\pi"] & (A/J_{l-1})e_l \underset{B_l}{\otimes} Y^\lambda \arrow[r] \arrow[u, "\wr"', "id\otimes f"] & 0
\end{tikzcd}$$

We thus have relations $\pi\iota = id_{Y(l,\lambda)}$, $\tilde{g}\tilde{h} = \iota g$ and $gh = \pi\tilde{g}$, yielding

$$g h\tilde{h} = \pi\tilde{g}\tilde{h} = \pi\iota g = id_{Y(l,\lambda)} g = g = g\, id_{ker(\varphi\iota)}.$$

g is a monomorphism, so $h\tilde{h} = id_{ker(\varphi\iota)}$. The snake lemma shows that $\tilde{h}$ is injective and h is surjective, so h and $\tilde{h}$ are a split morphisms. In particular, $ker\phi = ker(\varphi\iota)$ is a direct summand of $ker\varphi = J_{l-1}e_l \underset{B_l}{\otimes} Y^\lambda$, which in turn is a direct summand of $J_{l-1}e_l \underset{B_l}{\otimes} M^\lambda$. It follows from the proof of Lemma 6.13 that $J_{l-1}e_l \underset{B_l}{\otimes} M^\lambda$ and all its direct summands, in particular $ker\phi$, have a cell filtration. This shows that the kernel of the epimorphism $\Psi : Y(l,\lambda) \twoheadrightarrow ind_l S_\lambda$ is in $\mathcal{F}(\Theta)$ since $\mathcal{F}(\Theta)$ is extension-closed.

To show that $Y(l,\lambda)$ satisfies the universal property of a relative projective cover, let Y' be relative projective in $\mathcal{F}(\Theta)$ with $\Psi' : Y' \twoheadrightarrow ind_l S_\lambda$ such that $ker\Psi' \in \mathcal{F}(\Theta)$. Consider the short exact sequence $0 \to ker\Psi \to Y(l,\lambda) \to ind_l S_\lambda \to 0$ of cell filtered modules. Relative projectivity of Y' implies exactness of the sequence

$$0 \to \mathrm{Hom}_A(Y', ker\Psi) \to \mathrm{Hom}_A(Y', Y(l,\lambda)) \to \mathrm{Hom}_A(Y', ind_l S_\lambda) \to 0.$$

Thus, $\Psi' \in \mathrm{Hom}_A(Y', ind_l S_\lambda)$ has a preimage $f \in \mathrm{Hom}_A(Y', Y(l,\lambda))$ such that $\Psi' = \Psi \circ f$.

On the other hand, we can apply $\mathrm{Hom}_A(Y(l,\lambda), -)$ to the short exact sequence

$$0 \to ker\Psi' \to Y' \to ind_l S_\lambda \to 0.$$

Then every homomorphism $Y(l,\lambda) \to ind_l S_\lambda$ factors through Y'.

Therefore, f is a split map and $Y(l,\lambda)$ a direct summand of Y'. □

Corollary 6.16 ([HHKP10], Corollary 12.4). *If B_l is a group algebra of a symmetric group $\Sigma_{l'}$ for some $l' \leq r$ and* $\operatorname{char} k = p$, *and A satisfies* (I) *to* (IV), *then $Y(l,\lambda)$ is projective if and only if λ is p-restricted.*

In [HHKP10], the Young modules $Y_{pr}(l,\lambda)$ of a cellularly stratified algebra A are defined as the relative projective covers of the cell modules $\Theta(l,\lambda)$, in the case where the cell modules of the input algebras B_l form standard systems. Since we assumed B_l to be $k\Sigma_n$ or $\mathcal{H}_{k,q}(\Sigma_n)$ and $\operatorname{char} k \neq 2,3$, respectively $h \geq 4$, we are in this situation. Therefore, we have the following corollary of Theorem 6.1.

Corollary 6.17. *The Young modules $Y_{pr}(l,\lambda)$, defined abstractly in [HHKP10], coincide with the explicitly defined Young modules $Y(l,\lambda)$ of this thesis.*

In particular, we are in the situation of Theorem 13.1 from [HHKP10]. This shows

Theorem 6.2. *Let A be cellularly stratified with input algebras*

$$B_l = \mathcal{H}_{l'} = \mathcal{H}_{k,q}(\Sigma_{l'}) \hookrightarrow e_l A e_l$$

for some $l' \leq r$, where q is an hth root of unity and $h \geq 4$. Let $\operatorname{char} k \neq 2,3$ *if $q = 1$. Assume that A satisfies the conditions* (I) *to* (IV) *and let $(l,\lambda) \in \Lambda_r$. Then the following holds.*

(1) Let $M \in \mathcal{F}(\Theta)$. Then M has well-defined filtration multiplicities.

(2) *The category* $\mathcal{F}_A(\Theta)$ *of cell filtered* A*-modules is equivalent, as exact categories, to the category* $\mathcal{F}_{\mathrm{End}_A(Y)}(\Delta)$ *of standard filtered modules over the quasi-hereditary algebra* $\mathrm{End}_A(Y)$*, where*

$$Y = \bigoplus_{(l,\lambda)\in\Lambda_r} Y(l,\lambda)^{n_{l,\lambda}}$$

and $n_{l,\lambda} = \begin{cases} \dim L(l,\lambda) & \text{if there is a simple module } L(l,\lambda) \\ 1 & \text{otherwise.} \end{cases}$

(3) *There is a Schur-Weyl duality between* A *and* $\mathrm{End}_A(Y)$*. In particular, we have* $A = \mathrm{End}_{\mathrm{End}_A(Y)}(Y)$*.*

Remark. The multiplicities $n_{l,\lambda}$ of the Young modules $Y(l,\lambda)$ in Y are chosen to be minimal such that all Young modules appear at least once and such that the projective Young modules appear as often as they appear in A, i.e. such that there is a $D \in A-\mathrm{mod}$ with $Y = A \oplus D$.

Using Theorem 6.1, we are now able to show

Theorem 6.3 (Theorem C)**.** *Let* A *be cellularly stratified with input algebras*

$$B_l = \mathcal{H}_{l'} = \mathcal{H}_{k,q}(\Sigma_{l'}) \hookrightarrow e_l A e_l$$

for some $l' \leq r$*, where* q *is an* hth *root of unity and* $h \geq 4$*. Let* $\mathrm{char} k \neq 2,3$ *if* $q = 1$*. Assume that* A *satisfies the conditions* (I) *to* (IV) *and let* $(l,\lambda) \in \Lambda_r$*. Then there is a decomposition*

$$Ind_l M^{\lambda} = \bigoplus_{(m,\mu)\geq(l,\lambda)} Y(m,\mu)^{a_{m,\mu}}$$

with non-negative integers $a_{m,\mu}$*. Moreover,* $a_{l,\lambda} = 1$*.*

Proof. By Lemma 6.12, the set $\{\Theta(l,\lambda)\}$ forms a standard system. Theorem 6.2 says that there is a quasi-hereditary algebra $C = \mathrm{End}_A(Y)$ such that the categories $\mathcal{F}_A(\Theta)$ of cell filtered A-modules and $\mathcal{F}_C(\Delta)$ of standard filtered C-modules are equivalent. To proof this equivalence, Dlab and Ringel show that there is a one-to-one correspondence between the modules in the standard system $\{\Theta\}$, and the indecomposable relative projective modules in $\mathcal{F}(\Theta)$, which are the relative projective covers of the modules $\Theta(l,\lambda)$. Since C is quasi-hereditary, the relative projective C-modules are exactly the projective modules by Lemma 3.16. By Theorem 6.1, the relative projective covers of the cell modules $\Theta(l,\lambda) = ind_l S_\lambda$ are the Young modules $Y(l,\lambda)$.

The permutation module $Ind_l M^\lambda$ is relative projective in $\mathcal{F}(\Theta)$, so its image under the equivalence $\mathcal{F}(\Theta) \xrightarrow{\sim} \mathcal{F}(\Delta)$ is a projective C-module P. Let $P = \bigoplus_{(n,\nu)\in\Lambda_r} P(n,\nu)^{a_{n,\nu}}$ be a decomposition of P into indecomposable modules. Sending $P(n,\nu)$ back to $\mathcal{F}(\Theta)$ through the equivalence, its image must be an indecomposable relative projective module $Y(m,\mu)$. Thus, $Ind_l M^\lambda = \bigoplus_{(m,\mu)\in\Lambda_r} Y(m,\mu)^{a_{m,\mu}}$ for some non-negative integers $a_{m,\mu}$. $a_{l,\lambda} = 1$ by definition of $Y(l,\lambda)$. Lemmas 6.9 and 6.10 show that we only have to sum over those Young modules $Y(m,\mu)$ with $(m,\mu) \geq (l,\lambda)$. □

6.4 Applications

In [HHKP10], there are three main examples of cellularly stratified algebras: Brauer algebras, partition algebras and BMW algebras, a deformation of Brauer algebras. The work in this chapter is inspired by the work on Brauer algebras by Hartmann and Paget, [HP06], combined with the structural properties of cellular stratification in [HHKP10] and some additional input from Henke and Koenig, [HK12], which is partly based on those two articles as well. We will recover the results for Brauer algebras in Section 6.4.1 and show in Section 6.4.3 that the results hold for partition algebras as well. The theory does not apply for BMW algebras, since we need the cellular algebras $B_l = \mathcal{H}_{l'}$ to be subalgebras. However, the q-Brauer algebras, defined by Wenzl in [Wen12], are another deformation of Brauer algebras, which fit into this setting. They are cellularly stratified as shown by Nguyen in his PhD thesis [Ngu13] and contain the Hecke algebra as subalgebra. There is no diagrammatic basis known for the q-Brauer algebra. We only assumed the algebra to be a diagram algebra to motivate the term permutation module.

Conjecture A. *Let q be an hth root of unity, where $h \geq 4$. Then the q-Brauer algebra A has permutation modules $Ae_l \underset{\mathcal{H}_l}{\otimes} M^{\lambda}$, whose indecomposable direct summands are the relative projective covers of the cell modules (with respect to the category of cell filtered modules).*

To prove this conjecture, one has to show that the q-Brauer algebra satisfies the assumptions I to IV made in the previous section. We define the q-Brauer algebra in Section 6.4.2 and show that it fits into our setting and satisfies (I) and (II′).

6.4.1 Brauer Algebras

In this section, we re-prove the results on permutation modules from Hartmann and Paget, [HP06], stated in chapter 4. Our proofs use less combinatorics specific to the Brauer algebra. Instead, they use the structural properties of cellularly stratified algebras, which have been introduced after Hartmann and Paget's work. The Brauer algebra $B_k(r,\delta)$ is cellularly stratified with stratification data

$$(k\Sigma_t, V_t, k\Sigma_{t+2}, V_{t+2}, ..., k\Sigma_{r-2}, V_{r-2}, k\Sigma_r, V_r),$$

where $t = 0$ if r is even and $t = 1$ if r is odd, and V_l are the vector spaces with basis consisting of partial diagrams with exactly $\frac{r-l}{2}$ horizontal arcs, as defined in Chapter 4. The idempotents e_l were defined in Chapter 4 as well. Recall that $J_l = Ae_lA$.

First, we show that our cell modules, permutation modules and Young modules agree with those in [HP06], up to dualisation

$$D : B_k(r,\delta) - \text{mod} \longrightarrow B_k(r,\delta)^{op} - \text{mod} \simeq \text{mod} - B_k(r,\delta),$$

sending left modules to right modules.

Proposition 6.18. *For any* $X \in k\Sigma_l - \text{mod}$*, there is a* $B_k(r,\delta)$*-isomorphism* $ind_l X \simeq V_l \underset{k}{\otimes} X$*.*

Proof. Let $X \in B_l - \text{mod}$ and consider the map

$$\begin{aligned} \varphi : V_l \otimes_k X &\longrightarrow (A/J_{l-2})e_l \underset{B_l}{\otimes} X \\ v \otimes x &\longmapsto (e^v + J_{l-2}) \otimes x, \end{aligned}$$

where e^v is the diagram in $J_n e_l \smallsetminus J_{l-2} e_l$ with $\text{top}(e^v) = v$ and noncrossing through strings.

For arbitrary $(ae_l + J_{l-2}) \otimes x \in (A/J_{l-2})e_l \underset{B_l}{\otimes} X$, with $ae_l + J_{l-2}$ corresponding to $b \otimes w \otimes v_l$ under the isomorphism $A/J_{l-2} \simeq B_l \otimes_k V_l \otimes_k V_l$, we have $\varphi(w \otimes bx) = (e^w + J_{l-2}) \otimes bx = (e^w b + J_{l-2}) \otimes x = (ae_l + J_{l-2}) \otimes x$, so φ is surjective. By Proposition 3.8, it is $\dim((A/J_{l-2})e_l \underset{B_l}{\otimes} X) = \dim(B_l^{\dim V_l} \underset{B_l}{\otimes} X) = \dim V_l \cdot \dim X = \dim(V_l \otimes_k X)$. Hence, φ is bijective. To see that φ is an isomorphism, we have to check that it is A-linear. Let $a \in A$ and $v \otimes x \in V_l \otimes_k X$. Then

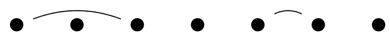
$$\begin{aligned}\varphi(a(v \otimes x)) &= \varphi(av \otimes \pi(a, v)x)\\ &= (e^{av} + J_{l-2}) \otimes \pi(a, v)x\\ &= (e^{av}\pi(a, v) + J_{l-2}) \otimes x\end{aligned}$$

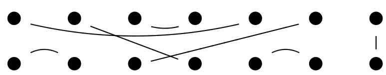
and

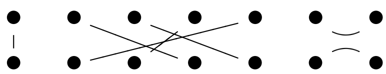
$$\begin{aligned}a\varphi(v \otimes x) &= a((e^v + J_{l-2}) \otimes x)\\ &= (ae^v + J_{l-2}) \otimes x.\end{aligned}$$

If ae^v has $t < l$ through strings, then $ae^v \in J_{l-2}$, so $a\varphi(v \otimes x) = (ae^v + J_{l-2}) \otimes x = 0$. $ae^v \in J_{l-2}$ implies that av has more than $\frac{r-l}{2}$ arcs, so $\varphi(a(v \otimes x)) = \varphi(av \otimes \pi(a, v)x) = \varphi(0) = 0$. If ae^v has $t \geq l$ through strings, then $ae^v \in J_l \smallsetminus J_{l-2}$, so $t-l$ of the free dots of $\mathrm{top}(a)$ are bound by horizontal arcs in ae^v. The remaining l free dots of $\mathrm{top}(a)$ remain end points of through strings. Hence, the permutation of the through strings of ae^v is $\pi(a, v)$. This shows $a\varphi(v \otimes x) = (ae^v + J_{l-2}) \otimes x = (e^{av}\pi(a, v) + J_{l-2}) \otimes x = \varphi(a(v \otimes x))$. □

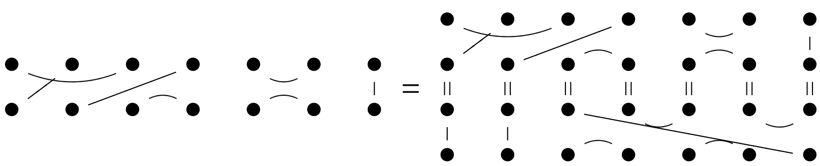
We visualize the above arguments in the following example.

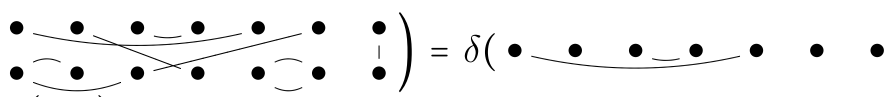
Example. Let $\delta = 0$, $v =$  $\in V_3$ and

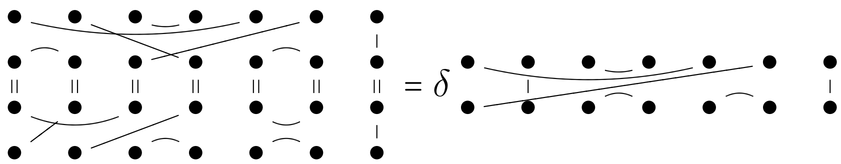
$a =$  $\in J_3$,

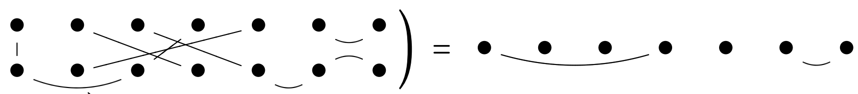
$b =$  $\in A \smallsetminus J_3$.

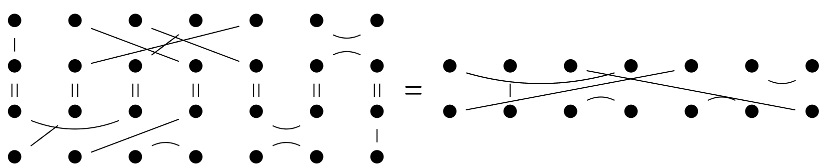
Then

$e^v =$  $= e^v e_l \in Ae_l$,

$av = \mathrm{top}\Big($  $\Big) = \delta(\ \ ) = 0$, $\pi(a, v) = (1, 2)$ and

$ae^v =$  $= 0$.

$bv = \mathrm{top}\Big($  $\neq 0$, $\pi(b, v) = (1, 2, 3)$ and

$be^v =$  $\neq 0$.

Corollary 6.19. *The cell, Young and permutation modules defined here coincide with those defined in [HP06].*

Proving that $B_k(r, \delta)$ satisfies the assumptions (I) to (IV) gives another proof for Proposition 14, Theorem 21, Propositions 23 and 24 of [HP06].This was done in [HK12] and [Pag07].

Let $n \leq l$. Since $J_n e_l$ is generated by the diagrams with at least $\frac{r-n}{2}$ horizontal arcs and $\frac{r-l}{2}$ of these arcs in the bottom row are pushed to the right, the right action of $k\Sigma_l$ on a diagram does not change the amount of arcs, so assumption (I) is satisfied. Assumption (II) is Lemma 4.3 from [HK12]. By Lemma 4.2 of the same article, $e_n(J_n/J_{n-2})e_l \simeq k \underset{H\times k\Sigma_n}{\otimes} k\Sigma_l$, where $H := k(C_2 \wr \Sigma_{\frac{l-n}{2}})$. The left $k\Sigma_n$-module

$$res_n Ind_l M^\lambda = e_n(J_n/J_{n-2})e_l \underset{k\Sigma_l}{\otimes} M^\lambda \simeq k \underset{H\times k\Sigma_n}{\otimes} k\Sigma_l \underset{k\Sigma_l}{\otimes} M^\lambda \simeq k \underset{H\times k\Sigma_n}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$$

is equal to a direct sum of $k\Sigma_n$-permutation modules M^ν by Lemma 4.5 of [HK12]. Therefore, $res_n Ind_l M^\lambda \in \mathcal{F}_n(S)$.

The restriction of a cell module $ind_n S_\nu$ to $k\Sigma_l - \text{mod}$, with $l \geq n$, is dual Specht filtered by Proposition 8 of [Pag07].

Theorem 6.4. *The Brauer algebra $B_k(r,\delta)$, with $\delta \neq 0$ if r is even, has permutation modules $Ind_l M^\lambda$, which are a direct sum of indecomposable Young modules. The Young modules are the relative projective covers of the cell modules $ind_l S_\lambda$.*

6.4.2 q-Brauer Algebras

The theory established in Section 6.3 holds for cellularly stratified algebras with input algebras being Hecke algebras $\mathcal{H}_{k,1}(\Sigma_l)$. These Hecke algebras can be regarded as q-deformations of group algebras of symmetric groups. Hence, it is natural to ask if the theory holds for BMW algebras, a well-known q-deformation of Brauer algebras.

Unfortunately, this is not the case since the Hecke algebra $\mathcal{H}$ is a quotient of the cellularly stratified BMW algebra, but it is not a subalgebra. However, there is another q-deformation of Brauer algebras which have Hecke algebras as subalgebras: The q-Brauer algebras defined by Wenzl, [Wen12]. In this section, we show that the q-Brauer algebra fits into our setting and satisfies the first two assumptions (I) and (II).

We define the q-Brauer algebra in the setting where it is shown to be cellularly stratified.

Definition 6.20. Let k be a field, $q \in k \setminus \{0\}$ and $N \in \mathbb{Z} \setminus 0$. Set $[N] := 1 + q + ... + q^{N-1}$. The *$q$-Brauer algebra* $B_{k,q}(r, N)$ is the associative algebra generated by elements $g_1, ..., g_{r-1}, e$ such that

(H) The elements $g_1, ..., g_{r-1}$ satisfy the relations of the Hecke algebra, i.e.

  (H_1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $1 \leq i \leq r-1$

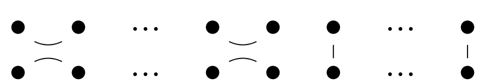
  (H_2) $g_i g_j = g_j g_i$ for $|i-j| > 1$

(E_1) $e^2 = [N]e$

(E_2) $eg_1 = g_1 e = qe$, $eg_2 e = q^N e$, $eg_2^{-1} e = q^{-1} e$ and $eg_i = g_i e$ for $i > 2$

(E_3) Set $\hat{e}_2 := eg_2 g_3 g_1^{-1} g_2^{-1} e$. Then $\hat{e}_2 = g_2 g_3 g_1^{-1} g_2^{-1} \hat{e}_2 = \hat{e}_2 g_2 g_3 g_1^{-1} g_2^{-1}$.

Let $A = B_k(r, N)$ be the classical Brauer algebra and denote the q-Brauer algebra $B_{k,q}(r, N)$ by A^q. Denote by e_n the idempotent

$\frac{1}{N^{\frac{r-n}{2}}}$  $\in A$

with $r - 2n$ through strings and let $J_n = Ae_nA$.

The q-Brauer algebra has a basis indexed by Brauer diagrams, cf. [Ngu13, Theorem 3.1.4]: Let d be a basis diagram of the Brauer algebra with $\frac{r-n}{2}$ horizontal arcs in each row. Then $d = e^{\mathrm{top}(d)}\pi e_n i(e^{\mathrm{bottom}(d)})$, where e^v is the diagram with $\mathrm{top}(e^v) = v$, $\mathrm{bottom}(e^v) = \mathrm{bottom}(e_n)$ and $\Pi(e^v) = 1_{\Sigma_{r-2n}}$ defined in Section 6.4.1, and $\pi = \Pi(d) \in \Sigma_{r-2n}$. In particular, there are unique coset representatives $\omega_1 \in \Sigma_n/\mathrm{stab}_1$ and $\omega_2 \in \mathrm{stab}_2\backslash\Sigma_n$ respectively, such that $\omega_1 e_n = e^{\mathrm{top}(d)}$ and $e_n\omega_2 = i(e^{\mathrm{bottom}(d)})$, where stab_1 is the stabilizer of $\mathrm{top}(d)$ and stab_2 is the stabilizer of $\mathrm{bottom}(d)$. If $\tau = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_l}$ is a reduced expression for $\tau \in \Sigma_r$, then $g_\tau := g_{i_1}g_{i_2}...g_{i_l}$. A basis of the q-Brauer algebra A^q is then given by

$$\{g_d := g_{\omega_1}\hat{e}_n g_\pi g_{\omega_2} \mid d \text{ basis diagram of } A\}$$

where $\hat{e}_n$ is the inductively defined element

$$\hat{e}_n := eg_2...g_{2n-1}g_1^{-1}...g_{2n-2}^{-1}\hat{e}_{n-1}$$

with $\hat{e}_1 := e$. Let I_n be the ideal generated by the set $\{g_d \mid d \in J_{r-2n}\}$. Then $I_n = \sum_{i=n}^{\lfloor \frac{r}{2}\rfloor} \mathcal{H}\hat{e}_i\mathcal{H}$ by [Ngu13, Lemma 2.2.4]. Note that we work here with the reverse order of ideals: $I_n \subset I_{n-1}$, while $J_n \subset J_{n+1}$.

Theorem 6.5 ([Ngu13], Theorems 7.1.7 and 7.1.8)**.** *The q-Brauer algebra A^q, as defined above, is cellularly stratified with stratification data*

$$(\mathcal{H}, k, \mathcal{H}_{3,r}, U_1, ..., \mathcal{H}_{2\lfloor\frac{r}{2}\rfloor-1,r}, U_{\lfloor\frac{r}{2}\rfloor-1}, k, U_{\lfloor\frac{r}{2}\rfloor})$$

where $\mathcal{H}_{2n+1,r} \subseteq \mathcal{H}$ is generated by $\{g_{2n+1}, g_{2n+1}, ..., g_{r-1}\}$ and U_n is the vector space with basis $\{g_d \mid d = e^v$ for some $v \in V_{r-2n}\}$.

By definition, $\mathcal{H} = \mathcal{H}_r$ is a subalgebra of A^q. Therefore, $\mathcal{H}_{2l+1,r} \subset \mathcal{H}$ is a subalgebra of A^q for every $l \leq r$, so we have an embedding $\mathcal{H}_{2l+1,r} \hookrightarrow \hat{e}_l A^q \hat{e}_l$ given by the assignment $h \mapsto \hat{e}_l h \hat{e}_l$.

Thus, the q-Brauer algebra fits into our setting, and we have to check the assumptions (I) to (IV) to prove Conjecture A.

Lemma 6.21. *For $n \geq l$, we have $I_n \hat{e}_l \simeq I_{n+1}\hat{e}_l \oplus (I_n/I_{n+1})\hat{e}_l$ as right $\mathcal{H}_{2l+1,r}$-modules.*

Proof. Since $I_n = \sum_{i=n}^{\lfloor \frac{r}{2} \rfloor} \mathcal{H}\hat{e}_i\mathcal{H}$ by [Ngu13, Lemma 2.2.4], we have $I_n/I_{n+1} \simeq \mathcal{H}\hat{e}_n\mathcal{H}$, so $I_n\hat{e}_l \simeq I_{n+1}\hat{e}_l \oplus (I_n/I_{n+1})\hat{e}_l$. This shows (I). □

Lemma 6.22. *For $n \geq l$, we have $(I_n/I_{n+1})\hat{e}_l \simeq \mathcal{H}_{2n+1,r} \otimes_k U_n \otimes U_n^l$ as right $\mathcal{H}_{2l+1,r}$-modules.*

Proof. Let $n \geq l$ and let $g_d := g_{\omega_1} g_\pi \hat{e}_n g_{\omega_2}$ be a basis element in $I_n \setminus I_{n+1}$, i.e. $\omega_1, \omega_2 \in \Sigma_r$ and $\pi \in \mathcal{H}_{2n+1,r}$. Then $g_d\hat{e}_l \equiv g_{\omega_1} g_\pi \hat{e}_n g_\tau \pmod{I_{n+1}}$ for some $g_\tau \in \mathcal{H}_{2l+1,r}$ by [Wen12, Lemma 3.4], which corresponds to a Brauer diagram c with $\mathrm{top}(c) = \omega_1 \mathrm{top}(e_n)$, $\Pi(c) = \pi$ and $\mathrm{bottom}(c) = \mathrm{bottom}(e_n)\tau$. The action of $\mathcal{H}_{2l+1,r}$ on $\hat{e}_n$ equals the action of $\Sigma_{\{2l+1,\ldots,r\}}$ on e_n, i.e. only the last $r-2l$ dots are permuted. In particular, $\mathrm{bottom}(c) \in V_{r-2n}^{r-2l}$, where V_{r-2n}^{r-2l} is the vector space with basis consisting of partial diagrams with n arcs, where the first $2n-2l$ dots have fixed arcs $(1,2),(3,4),\ldots,(2n-2l-1,2n-2l)$. Hence, $g_d\hat{e}_l + I_{n+1}$ corresponds to an element in $\mathcal{H}_{2n+1,r} \otimes_k U_n \otimes U_n^l$, where U_n^l has a basis $\{g_d \mid d = e^v \text{ for some } v \in V_{r-2n}^{r-2l}\}$. This shows that (II′) is satisfied, which implies (II) by Remark *(a)* following the assumptions. □

It remains to show:

(QIII) For $n \geq l$, we have $\hat{e}_n(I_n/I_{n+1})\hat{e}_l \underset{\mathcal{H}_{2l+1,r}}{\otimes} M_q^\lambda \in \mathcal{F}_{2n+1,r}(S^q)$

(QIV) For $n \geq l$, we have $\hat{e}_l(I_n/I_{n+1})\hat{e}_n \underset{\mathcal{H}_{2n+1,r}}{\otimes} S^q_\nu \in \mathcal{F}_{2l+1,r}(S^q)$

where $\mathcal{F}_{m,r}(S^q)$ is the category of dual Specht filtered $\mathcal{H}_{m,r}$-modules.

In forthcoming joint work with Dung Tien Nguyen, we are going to study the modules in (QIII) and (QIV) with the aim of proving Conjecture A.

6.4.3 Partition Algebras

As we have seen in Chapter 5, $k\Sigma_r$ is a subalgebra of $P_k(r,\delta)$ and $P_k(l,\delta) \simeq e_l P_k(r,\delta) e_l$ for all $0 \leq l \leq r$, so $k\Sigma_l$ is always a subalgebra of the cellularly stratified algebra $e_l P_k(r,\delta) e_l$. For $n \leq l$, the basis diagrams of $(J_n/J_{n-1})e_l$ have exactly n propagating parts and the last $r-l+1$ dots of the bottom row belong to the same part. Hence, $(J_n/J_{n-1})e_l \simeq k\Sigma_n \otimes V_n \otimes V_n^l$, where V_n^l is the subspace of V_n defined in Section 5.2.2, and assumption (II′) is satisfied. The right $k\Sigma_l$-action does not change the propagating number of a diagram $d \in Ae_l$ since all propagating parts of d end in one of the leftmost l dots of bottom(d), which are permuted by Σ_l. This shows that assumption (I) is always satisfied. Assumption (IV) holds by Theorem 5.3 in case $\mathrm{char} k > \lfloor \frac{r}{3} \rfloor$. If we can show that layer restriction of a permutation module has a dual Specht filtration, Theorems 6.1, 6.2 and 6.3 hold for $A = P_k(r,\delta)$ with $\mathrm{char} k > \lfloor \frac{r}{3} \rfloor$.

Layer restriction of a permutation module

We keep the notation from Chapter 5. Let $e_n(A/J_{n-1})e_l = \bigoplus_{v \in V_n^l/\sim} U^v$, where U^v is generated by $d := d^v := i(d_v)$, i.e. U^v has a basis consisting of diagrams $i(b)$, where b is a diagram in the basis of U_v

defined in Section 5.2.2, i.e. $i(b)$ has a fixed top row and a bottom row equivalent to v. Fix a bottom row configuration $v \in V_n^l/_\sim$ with α_i labelled parts of size i and β_i unlabelled parts of size i. Let $\prod_\alpha \simeq \prod_i (\Sigma_i \wr \Sigma_{\alpha_i})$ be the stabilizer of the labelled parts of v and let $\prod_\beta \simeq \prod_i (\Sigma_i \wr \Sigma_{\beta_i})$ be the stabilizer of the unlabelled parts. Then $U^v \simeq k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l$ by Lemma 5.10. We want to understand the summands $k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$ of $res_n Ind_l M^\lambda$ for $\lambda = (\lambda_1, ..., \lambda_m) \vdash l$. Call the set of dots $\{\lambda_{k-1} + 1, ..., \lambda_k\}$ in $w \in V_n^l$ the λ-*block* λ_k. Fix double coset representatives $\pi_1, ..., \pi_q$ of $(\prod_\alpha \times \prod_\beta)\backslash\Sigma_l/\Sigma_\lambda$ and set

$$\nu^i_{j,\lambda_k} = |\{\circ \mid \circ \text{ is the first dot of a labelled part in } v\pi_i \text{ of size } j \text{ in block } \lambda_k\}|$$

where the last $r - l + 1$ dots count as one dot. Let $t = \max\{j\}$ and set

$$\nu^i := (\nu^i_{1,\lambda_1}, ..., \nu^i_{1,\lambda_m}, \nu^i_{2,\lambda_1}, ..., \nu^i_{2,\lambda_m}, ..., \nu^i_{t,\lambda_1}, ..., \nu^i_{t,\lambda_m}).$$

Then ν^i is a composition of n.

Example 6.23. Let $r = 10, l = 8, n = 6, v = \circ \quad \circ \quad \circ \quad \circ - \circ \quad \circ - \circ \quad \circ - \circ - \circ$ and $\lambda = (4, 3, 1)$. Then $\alpha = (4, 2)$ and β is the empty partition. The double coset representatives are calculated by the computer algebra system GAP.

i	π_i	$\upsilon\pi_i$	ν^i
1	1	○ ○ ○ ○–○ ○–○ ○–○–○	$(3,0,1,1,1,0)$
2	$(7,8)$	○ ○ ○ ○–○ ○ ○ ○–○–○	$(3,1,0,1,1,0)$
3	$(5,8,7,6)$	○ ○ ○ ○ ○–○ ○ ○–○–○	$(3,1,0,1,1,0)$
4	$(4,5,6,7,8)$	○ ○ ○ ○ ○–○ ○–○–○–○	$(4,0,0,2,0,0)$
5	$(3,5,4)$	○ ○ ○–○ ○ ○–○ ○–○–○	$(2,1,1,1,0,0)$
6	$(3,5,6,4)$	○ ○ ○ ○ ○ ○ ○ ○–○–○	$(2,1,1,2,0,0)$
7	$(3,5,4)(7,8)$	○ ○ ○–○ ○ ○ ○ ○–○–○	$(2,2,0,1,1,0)$
8	$(3,5,6,4)(7,8)$	○ ○ ○ ○ ○ ○ ○ ○–○–○	$(2,2,0,2,0,0)$
9	$(2,5,3,6,4)$	○ ○–○ ○ ○ ○ ○ ○–○–○	$(1,2,1,2,0,0)$
10	$(2,5,3,6,4)(7,8)$	○ ○–○ ○ ○ ○ ○ ○–○–○	$(1,3,0,2,0,0)$
11	$(1,5,2,6,3,7,4)$	○–○ ○–○ ○ ○ ○ ○–○–○	$(0,3,1,2,0,0)$

Note that the first m digits are a composition of α_1, the second m digits are a composition of α_2, and so on.

Lemma 6.24. *The map*

$$\begin{aligned} \psi : k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k &\longrightarrow \bigoplus_{i=1}^{q} (k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k) \\ \eta \otimes \tau \otimes 1 &\longmapsto (\eta\Pi(d\zeta) \otimes 1)^{(i)} \quad \textit{if } \tau = \zeta\pi_i\vartheta \end{aligned}$$

for some $\zeta \in \prod_\alpha \times \prod_\beta, \vartheta \in \Sigma_\lambda$ *is an isomorphism of left* $k\Sigma_n$*-modules.*

Proof. Let Ψ be the map from $k\Sigma_n \times k\Sigma_l \times k$ to $\bigoplus_{i=1}^{q} k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k$ with $\Psi(\eta,\tau,1) = (\eta\Pi(d\zeta) \otimes 1)^{(i)}$ for $\tau = \zeta\pi_i\vartheta$. Let $\eta \in \Sigma_n, \tau \in \Sigma_l$. Let $\xi \in \prod_\alpha \times \prod_\beta$ and $\sigma \in \Sigma_\lambda$ such that $\xi\tau\sigma = \zeta\pi_i\vartheta$ for some $\zeta \in \prod_\alpha \times \prod_\beta, \vartheta \in \Sigma_\lambda$ and $i \in \{1,...,q\}$. Then $\Psi(\eta,\xi\tau\sigma,1) = (\eta\Pi(d\zeta) \otimes 1)^{(i)}$ and $\Psi(\eta\Pi(d\xi),\tau,1) = (\eta\Pi(d\xi)\Pi(d\xi^{-1}\zeta) \otimes 1)^{(i)} = (\eta\Pi(d\xi\xi^{-1}\zeta) \otimes 1)^{(i)} = (\eta\Pi(d\zeta) \otimes 1)^{(i)}$ since $\tau = \xi^{-1}\zeta\pi_i\vartheta\sigma^{-1}$. This shows that ψ is well-defined. ψ is $k\Sigma_n$-linear and a generator $(\eta \otimes 1)^{(i)}$ of $\bigoplus_{i=1}^{q} k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k$ has preimage $\eta \otimes \pi_i \otimes 1$. Hence, ψ is surjective. We will now show that it is injective as well.

Let $\sum_j a_j(\eta_j \otimes \tau_j \otimes 1) \in k\Sigma_n \underset{k\prod_\alpha \times k\prod_\beta}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k$ with $a_j \in k, \eta_j \in \Sigma_n$ and $\tau_j \in \Sigma_l$ such that $\psi(\sum_j a_j(\eta_j \otimes \tau_j \otimes 1)) = 0$. Let $\tau_j = \zeta_j \pi_{i_j} \vartheta_j$ with $\zeta_j \in \prod_\alpha \times \prod_\beta, \vartheta_j \in \Sigma_\lambda$. Then $\sum_j a_j(\eta_j \Pi(d\zeta_j) \otimes 1)^{(i_j)} = 0$. In particular, $\sum_{t=1}^{s}(\eta_{j_t}\Pi(d\zeta_{j_t}) \otimes a_{j_t}) = 0$, where we sum over all indices j_t such that $i_{j_t} = k$ for a fixed $k \in \{1, ..., q\}$. Now $\sum_{t=1}^{s}(\eta_{j_t}\Pi(d\zeta_{j_t}) \otimes a_{j_t}) = (\sum_{t=1}^{s} a_{j_t}\eta_{j_t}\Pi(d\zeta_{j_t})) \otimes 1$, so $\sum_{t=1}^{s} a_{j_t}\eta_{j_t}\Pi(d\zeta_{j_t}) = 0$. Then

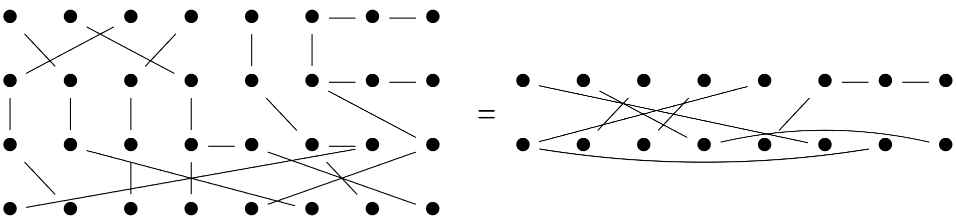
$$\begin{aligned}
\sum_{t=1}^{s} a_{j_t}(\eta_{j_t} \otimes \tau_{j_t} \otimes 1) &= \sum_{t=1}^{s} a_{j_t}(\eta_{j_t} \otimes \zeta_{j_t}\pi_{i_{j_t}}\vartheta_{j_t} \otimes 1) \\
&= \sum_{t=1}^{s} a_{j_t}(\eta_{j_t} \otimes \zeta_{j_t}\pi_{i_{j_t}} \otimes 1) \\
&= (\sum_{t=1}^{s} a_{j_t}\eta_{j_t}\Pi(d\zeta_{j_t})) \otimes \pi_k \otimes 1 = 0.
\end{aligned}$$

Repeat this for every $k \in \{1, ..., q\}$. Then $\sum_j a_j(\eta_j \otimes \tau_j \otimes 1) = 0$ and ψ is injective. □

The following example gives a diagrammatic interpretation of the above Lemma.

Example (6.23 continued)**.** Let l, n, v, λ be as in Example 6.23 and let $r = l = 8$. Let $\tau = (1267)(58) = (283)(4657)\pi_8(123)(67)$ and

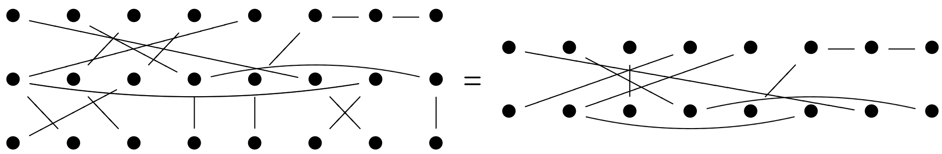
$\eta = (1243)$. Then $\eta \otimes \tau$ corresponds to the diagram

$\eta d\tau =$ 

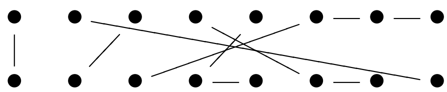
which has bottom row

 $= \upsilon\pi_8(123)(67)$.

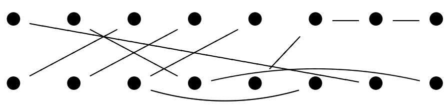
By tensoring over Σ_λ, we identify the diagrams a and b if and only if there is a $\vartheta \in \Sigma_\lambda$ such that $a = b\vartheta$. Hence, $\eta \otimes \tau \otimes 1 \in k\Sigma_6 \underset{k\Pi_\alpha}{\otimes} k\Sigma_8 \underset{k\Sigma_\lambda}{\otimes} k$ corresponds to the diagram $\eta d\tau$ as well as to the diagram $\eta d\tau(123)(67) =$



The map ψ sends $\eta \otimes \tau \otimes 1$ to $(\eta\Pi(d(283)(4657)) \otimes 1)^{(8)}$.

$d(283)(4657) =$ 

so $\Pi(d(283)(4675)) = (263)(45)$ and thus $\psi(\eta \otimes \tau \otimes 1) = (\eta(263)(45) \otimes 1)^{(8)} = ((163)(254) \otimes 1)^{(8)}$. This corresponds to the diagram

$(163)(254)d\pi_8 =$ 

which equals $\eta d\tau(132)(67)$.

Corollary 6.25. *$res_n Ind_l M^\lambda \simeq \bigoplus_{v \in V_n^l/\sim} \bigoplus_{i=1}^{q(v)} M^{\nu^i}$ is cell filtered as a left $k\Sigma_n$-module.*

Proof. This is a straightforward calculation, using the decomposition of $e_n(A/J_{n-1})e_l$ into direct summands U^v:

$$\begin{aligned}
res_n Ind_l M^\lambda &\simeq e_n(A/J_{n-1})e_l \underset{k\Sigma_l}{\otimes} M^\lambda \\
&\simeq e_n(A/J_{n-1})e_l \underset{k\Sigma_\lambda}{\otimes} k \\
&\simeq \Big(\bigoplus_{v \in V_n^l/\sim} U^v\Big) \underset{k\Sigma_\lambda}{\otimes} k \\
&\simeq \Big(\bigoplus_{v \in V_n^l/\sim} k\Sigma_n \underset{k\Pi_\alpha \times k\Pi_\beta}{\otimes} k\Sigma_l\Big) \underset{k\Sigma_\lambda}{\otimes} k \qquad \text{by Lemma 5.10} \\
&\simeq \bigoplus_{v \in V_n^l/\sim} \Big(k\Sigma_n \underset{k\Pi_\alpha \times k\Pi_\beta}{\otimes} k\Sigma_l \underset{k\Sigma_\lambda}{\otimes} k\Big) \\
&\simeq \bigoplus_{v \in V_n^l/\sim} \Big(\bigoplus_{i=1}^{q} (k\Sigma_n \underset{k\Sigma_{\nu^i}}{\otimes} k\Big) \qquad \text{by Lemma 6.24.}
\end{aligned}$$

□

In particular, we have shown

Theorem 6.6 (Theorem D)**.** *If* char$k > \lfloor \frac{r}{3} \rfloor$*, the partition algebra $P_k(r,\delta)$ has permutation modules $Ind_l M^\lambda$, which are a direct sum of indecomposable Young modules. The Young modules are the relative projective covers of the cell modules $ind_l S_\lambda$.*

6.5 Dual Statements

In this section, we study the behaviour of our modules under dualisation. We will get the same results as before. However, we will argue why it is more natural to study permutation modules instead of their duals.

Let A be cellularly stratified with $B_l = \mathcal{H}_{k,q}(\Sigma_{l'}) \hookrightarrow e_l A e_l$ for all $1 \leq l \leq r$. Composition of the standard duality

$$\begin{aligned} D: A-\mathrm{mod} &\longrightarrow \mathrm{mod}-A \\ X \quad &\longmapsto \mathrm{Hom}_k(X,k) \end{aligned}$$

with the anti-involution i of A yields a duality

$$\begin{aligned} Di: A-\mathrm{mod} &\longrightarrow A-\mathrm{mod} \\ X \quad &\longmapsto \mathrm{Hom}_k(i(X),k), \end{aligned}$$

where $i(X)$ is isomorphic to X as vector space, and A acts from the right on $i(X)$ via

$$\begin{aligned} i(X) \times A &\longrightarrow i(X) \\ (x,a) &\longmapsto i(a) \cdot x. \end{aligned}$$

Let $J' = Ae'A$, where e' is one of the idempotents from Definition 3.6, and let J be a cell ideal of A/J'. Then $J = AeA$ for some idempotent e by Lemma 3.7 and $J = i(J) = i(AeA) = i(A)i(e)i(A) \simeq Ai(e)A$. Note that, in general, $i(e) \neq e$.

Lemma 6.26. *Let $X \in B-\mathrm{mod}$. Then $i(Ae \underset{B}{\otimes} X) = i(X) \underset{B}{\otimes} i(e)A$.*

Proof. The vector space isomorphism $f : i(X) \underset{B}{\otimes} i(e)A \xrightarrow{\sim} Ae \underset{B}{\otimes} X$ sends $x \otimes i(e)a$ to $i(a)e \otimes x$. To show that this agrees with the tensor product, take $\sigma \in B$ such that $i(e)a = i(e)\sigma a' = \sigma i(e)a'$. Then $x \otimes i(e)a = x \otimes \sigma i(e)a' = i(\sigma)x \otimes i(e)a'$, which is sent to $i(a')e \otimes i(\sigma)x = i(a')ei(\sigma) \otimes x = i(\sigma i(e)a') \otimes x = i(i(e)a) \otimes x = i(a)e \otimes x$, the image of $x \otimes i(e)a$.

Now, let $x \in X$ and $a, a' \in A$. The right action on $i(Ae \underset{B}{\otimes} X$ is defined as $(ae \otimes x)a' = i(a')ae \otimes x$. The right action on $i(X) \underset{B}{\otimes} i(e)A$ is defined as $(x \otimes i(e)a)a' = x \otimes i(e)aa'$. So $f((x \otimes i(e)a)a') = f(x \otimes i(e)aa') = i(aa')e \otimes x = i(a')i(a)e \otimes x = i(a')f(x \otimes i(e)a) = f(x \otimes i(e)a)a'$ and f is an isomorphism of right A-modules. $\square$

Let $(l, \lambda) \in \Lambda_r$ and denote the cell module of A corresponding to (l, λ) by $\Theta(l, \lambda)$. The *dual cell module* is defined as

$$\begin{aligned} Di(ind_l S_\lambda) &= \mathrm{Hom}_k(i(S_\lambda) \underset{B_l}{\otimes} i(e_l)(A/J_{l-1}), k) \\ &\simeq \mathrm{Hom}_{B_l}(i(e_l)(A/J_{l-1}), \mathrm{Hom}_k(i(S_\lambda), k)) \\ &= \mathrm{Hom}_{B_l}(i(e_l)(A/J_{l-1}), S^\lambda) =: \Xi(l, \lambda). \end{aligned}$$

Similarly, we get a *dual permutation module*

$$Di(Ind_l M^\lambda) \simeq \mathrm{Hom}_{B_l}(i(e_l)A, DiM^\lambda) \simeq \mathrm{Hom}_{B_l}(i(e_l)A, M^\lambda).$$

In Section 6.2, we have seen that the A-action permutes the diagram basis of $Ind_l M^\lambda$, up to scalar multiples. We will now examine the A-action on the dual permutation module: We say a partial diagram v is *contained* in a partial diagram w, denoted by $v \subseteq w$, if each part of v is contained in a unique part of w. Note that this assignment is not necessarily injective; multiple parts of v can be contained in the same part of w. For example, if $A = P_k(5, \delta)$, then $\bullet \quad \bullet - \bullet \quad \bullet - \bullet \subseteq \bullet \quad \bullet - \bullet - \bullet - \bullet$.

Let $a \in A$ and let $\{m_1, .., m_t\}$ be a permutation basis of $i(M^\lambda)$. Then there is a basis $\{m_i \otimes d_j\}$ of $i(M^\lambda) \underset{B_l}{\otimes} i(e_l)A$ with dual basis $\varphi_{i,j}$ such that

$$\varphi_{i,j}(n \otimes c) = \begin{cases} 1 & \text{if } n \otimes c = m_i \otimes d_j \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a \cdot \varphi_{i,j}(n \otimes c) = \varphi_{i,j}(n \otimes ca) = \begin{cases} 1 & \text{if } n \otimes ca = m_i \otimes d_j \\ 0 & \text{otherwise.} \end{cases}$$

If $\text{bottom}(a) \notin \text{bottom}(d_j)$, then $n \otimes ca \neq m_i \otimes d_j$ for all $n \otimes c \in i(M^\lambda) \underset{B_l}{\otimes} i(e_l)A$. Therefore, most of the products $a \cdot \varphi_{i,j}$ are zero. However, if $\text{bottom}(a) \subseteq \text{bottom}(d_j)$, then there is a unique basis element $m_k \otimes d_l$ such that $m_k \otimes d_l a = \delta^s m_i \otimes d_j$ for some $s \in \mathbb{N} \cup \{0\}$ since $\{m_i \otimes d_j\}$ is a permutation basis of $i(M^\lambda) \underset{B_l}{\otimes} i(e_l)A$ up to scalars. Hence,

$$a \cdot \varphi_{i,j}(n \otimes c) = \begin{cases} \delta^s & \text{if } n \otimes c = m_k \otimes d_l \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $a \cdot \varphi_{i,j} = \delta^s \varphi_{k,l}$. We see that the dual of a permutation module has a basis whose elements are sent to scalar multiples of basis elements or to zero by the multiplication map

$$A \times DiInd_l M^\lambda \to DiInd_l M^\lambda.$$

If $\delta \neq 0$, the permutation module $Ind_l M^\lambda$ is closer to having a proper permutation basis than its dual $\text{Hom}_{B_l}(i(e_l)A, M^\lambda)$.

We defined the Young module $Y(l, \lambda)$ as the unique summand of

$Ind_l M^\lambda$ with quotient $ind_l S_\lambda$. The *dual Young module* $W(l,\lambda)$ is the unique summand of $\mathrm{Hom}_{B_l}(i(e_l)A, M^\lambda)$ such that $\mathrm{Hom}_{B_l}(i(e_l)(A/J_{l-1}), S^\lambda)$ is a submodule of $W(l,\lambda)$. Young modules for $k\Sigma_l$ are self-dual, cf. Chapter 2. In general, this is not the case for arbitrary A, see for example the remark at the end of Section 12 in [HHKP10], i.e. $W(l,\lambda)$ is in general non-isomorphic to $Y(l,\lambda)$.

The duality Di sends relative projective modules with respect to $\mathcal{F}(\Theta)$ to relative injective modules with respect to $\mathcal{F}(\Xi)$, i.e. modules E such that $\mathrm{Ext}^1_A(X,E) = 0$ for all $X \in \mathcal{F}(\Xi)$: Let P be relative projective in $\mathcal{F}(\Theta)$ and

$$(*) : 0 \to Di(P) \to Y \to X \to 0$$

a short exact sequence with $X \in \mathcal{F}(\Xi)$. Then

$$0 \to Di(X) \to Di(Y) \to P \to 0$$

is exact, thus $Di(Y) \simeq Di(X) \oplus P$. But then $Y \simeq Di(Di(X) \oplus P) \simeq X \oplus Di(P)$, so $(*)$ splits. Since $X \in \mathcal{F}(\Xi)$ can be chosen arbitrarily, $Di(P)$ is relative injective with respect to $\mathcal{F}(\Xi)$.

Set $Y := \bigoplus_{(l,\lambda)\in\Lambda_r} Y(l,\lambda)^{n_{l,\lambda}}$, where $n_{l,\lambda} = \dim L(l,\lambda)$ if there is such a simple module $L(l,\lambda)$ with index (l,λ), and $n_{l,\lambda} = 1$ otherwise. Then $Y = A \oplus D$ for some $D \in A-\mathrm{mod}$. Let f be the projection $Y \twoheadrightarrow A$. Then there exists a quasi-hereditary algebra $C := \mathrm{End}_A(Y)$ with standard modules Δ such that the categories $\mathcal{F}_A(\Theta)$ and $\mathcal{F}_C(\Delta)$ are equivalent as exact categories by Theorem 3.3. This equivalence is given by

$$F := \mathrm{Hom}_A(Y, -) : A - \mathrm{mod} \to \mathrm{End}_A(Y) - \mathrm{mod}$$

and

$$H := f \cdot - : \mathrm{End}_A(Y) - \mathrm{mod} \to A - \mathrm{mod}$$

See [DR92] and [HHKP10] for more details. By Proposition 3.18, there are indecomposable C-modules $T(l,\lambda)$ such that there are exact sequences

$$0 \to \Delta(l,\lambda) \to T(l,\lambda) \to X(l,\lambda) \to 0$$

and

$$0 \to Z(l,\lambda) \to T(l,\lambda) \to \nabla(l,\lambda) \to 0,$$

with

$$X(l,\lambda) \in \mathcal{F}_C(\{\Delta(m,\mu) \mid (m,\mu) \prec (l,\lambda)\})$$

and

$$Z(l,\lambda) \in \mathcal{F}_C(\{\nabla(m,\mu) \mid (m,\mu) \prec (l,\lambda)\})$$

and add $T = \mathcal{F}_C(\Delta) \cap \mathcal{F}_C(\nabla)$, where $T = \bigoplus_{(l,\lambda)\in\Lambda_r} T(l,\lambda)$. Applying H to these sequences yields indecomposable modules $HT(l,\lambda)$ with exact sequences

$$0 \to X'(l,\lambda) \to HT(l,\lambda) \to \Theta(l,\lambda) \to 0$$

and

$$0 \to \Xi(l,\lambda) \to HT(l,\lambda) \to Z'(l,\lambda) \to 0,$$

such that

$$X'(l,\lambda) \in \mathcal{F}_A(\{\Theta(m,\mu) \mid (m,\mu) \prec (l,\lambda)\})$$

and

$$Z(l,\lambda) \in \mathcal{F}_A(\{\Xi(m,\mu) \mid (m,\mu) \prec (l,\lambda)\}).$$

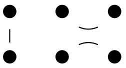
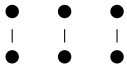
So add $HT \subseteq \mathcal{F}_A(\Theta) \cap \mathcal{F}_A(\Xi)$. To test if the converse inclusion holds, let $M \in \mathcal{F}_A(\Theta) \cap \mathcal{F}_A(\Xi)$. Then $F(M) \in \mathcal{F}_C(\Delta)$ but we do not know whether $F(M) \in \mathcal{F}_C(\nabla)$ since F may not be an equivalence on $\mathcal{F}(\nabla)$. In fact, Paget found a counterexample to Hemmer's conjecture [Hem06, Conjecture 5.1] that all irreducible, self-dual $k\Sigma_r$-modules with Specht and dual Specht filtration are signed Young modules $Y^\lambda \otimes_k S^{(1^r)}$: By [Pag07], the irreducible summands of the module $k\Sigma_{2n} \underset{k(\Sigma_2 \wr \Sigma_n)}{\otimes} k$ have both Specht and dual Specht filtrations, but are not signed Young modules. Since all signed Young modules are both Specht and dual Specht filtered, this shows that, in general, there are more than $|\Lambda_r|$ indecomposable modules in $\mathcal{F}(\Theta) \cap \mathcal{F}(\Xi)$.

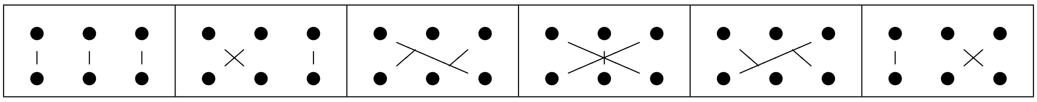
Fact. The category $\mathcal{F}_A(\Theta) \cap \mathcal{F}_A(\Xi)$ of A-modules with cell and dual cell filtrations contains in general more indecomposable modules than the category $\mathcal{F}_C(\Delta) \cap \mathcal{F}_C(\nabla)$ of C-modules with standard and costandard filtrations, where C is Morita equivalent to the quasi-hereditary algebra $\mathrm{End}_A(Y)$.

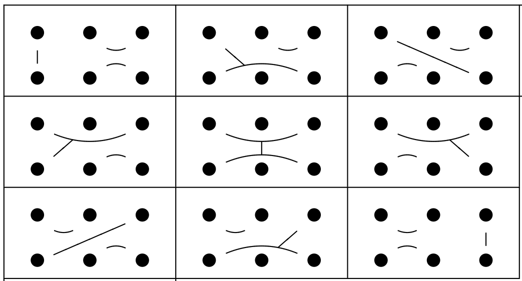
This shows that there is no analogue of Proposition 3.18 for cellularly stratified algebras, i.e. there are no indecomposable modules $T(l,\lambda)$ such that the category $\mathcal{F}(\Theta) \cap \mathcal{F}(\Xi)$ of modules with both cell and dual cell filtrations is equivalent to add $\bigoplus_{(l,\lambda)\in\Lambda_r} T(l,\lambda)$.

Appendices

AI The Brauer Algebra $B_k(3,\delta)$

Let A be the Brauer Algebra $B_k(3,\delta)$, with $\delta \neq 0$, where $\mathrm{char} k \neq 2,3$. Then the idempotents for the cellularly stratified structure are $e_1 :=$  and $e_3 := 1_A =$ . The basis of A consists of the following diagrams:

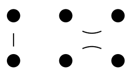




$\in e_1A$

$\in Ae_1$

The diagrams in the top table are those from layer $A/Ae_1A \simeq k\Sigma_3$, the lower table contains the diagrams from layer $J := Ae_1A$.

To find the cell chain for A, we start with the chain $0 \subset J \subset A$. The cell modules are of the form $Ae_l \underset{e_lAe_l}{\otimes} S_\lambda$, with $l \in \{1,3\}$ and $\lambda \vdash l$. There is only one partition (1) of $l = 1$, and $e_1Ae_1 \simeq k$ since  is the only diagram in the intersection of e_1A and Ae_1. $S^{(1)}$ has basis $\overline{\underline{1}}$, so $S_{(1)} \simeq k$. Hence, $\Delta(1) = Ae_1 \underset{e_1Ae_1}{\otimes} S_{(1)} \simeq Ae_1 \underset{k}{\otimes} k \simeq Ae_1$ and $i(\Delta(1)) = i(Ae_1) = e_1A$. J is a cell ideal, since $J = Ae_1A = Ae_1 \underset{e_1Ae_1}{\otimes} e_1A = Ae_1 \underset{k}{\otimes} e_1A = \Delta(1) \underset{k}{\otimes} i(\Delta(1))$.

A/J is not a cell ideal over itself. We know that $A/J \simeq k\Sigma_3$, so we have three cell modules $S_{(3)} \simeq Ae_3 \underset{e_3Ae_3}{\otimes} S_{(3)}$, $S_{(2,1)} \simeq Ae_3 \underset{e_3Ae_3}{\otimes} S_{(2,1)}$ and $S_{(1^3)} \simeq Ae_3 \underset{e_3Ae_3}{\otimes} S_{(1^3)}$ in this layer. By [Jam78, Corollary 17.14], the cell chain of $k\Sigma_3 = M^{(1^3)}$ is independent of the ground field. It is given by

$$0 \subset S_{(1^3)} \subset \underbrace{\begin{matrix} S_{(2,1)} & S_{(2,1)} \\ \multicolumn{2}{c}{S_{(1^3)}} \end{matrix}}_{=\mathrm{rad}(k\Sigma_3)} \subset \underbrace{\begin{matrix} \multicolumn{2}{c}{S_{(3)}} \\ S_{(2,1)} & S_{(2,1)} \\ \multicolumn{2}{c}{S_{(1^3)}} \end{matrix}}_{=k\Sigma_3}.$$

Therefore, the cell chain of A is given by

$$0 \subset Ae_1A \subset A \underset{A}{\otimes} S_{(1^3)} \subset A \underset{A}{\otimes} \mathrm{rad}(k\Sigma_3) \subset A \underset{A}{\otimes} k\Sigma_3.$$

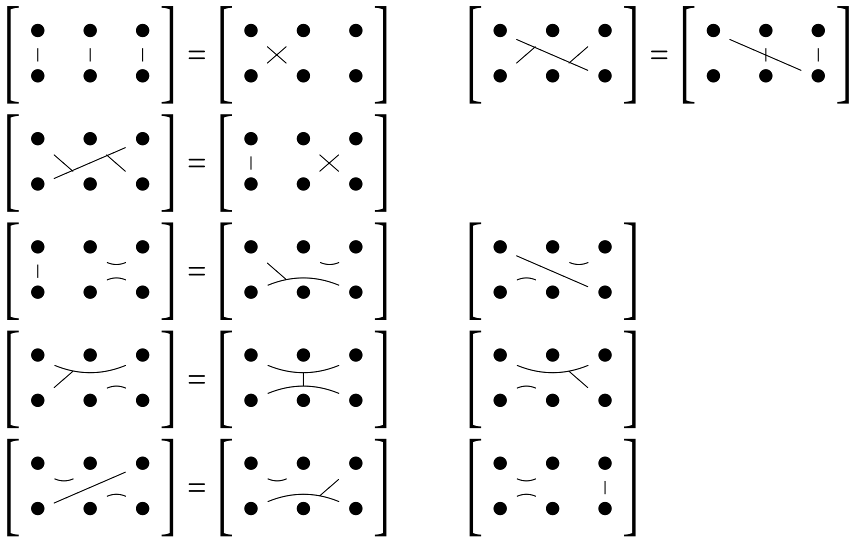
We will now take a look at the permutation modules of A. In the lowest layer, we have the module $Ind_1 M^{(1)} = Ae_1 \underset{k\Sigma_1}{\otimes} k\Sigma_1 \underset{k\Sigma_1}{\otimes} k \simeq Ae_1$. This is the cell module $ind_1 S_{(1)}$.

In the top layer, we have three permutation modules $Ind_3 M^{(3)}$, $Ind_3 M^{(2,1)}$ and $Ind_3 M^{(1^3)}$. By definition, $Ind_3 M^{(3)} = A \underset{k\Sigma_3}{\otimes} k$. The basis consists of equivalence classes of diagrams; diagrams a and $b \in A$ are equivalent if and only if there is a $\pi \in \Sigma_3$ such that $a\pi = b$. That is, the diagrams in the same row of the table for Ae_1A are in the same equivalence class, and the table for $k\Sigma_3$ is another one.

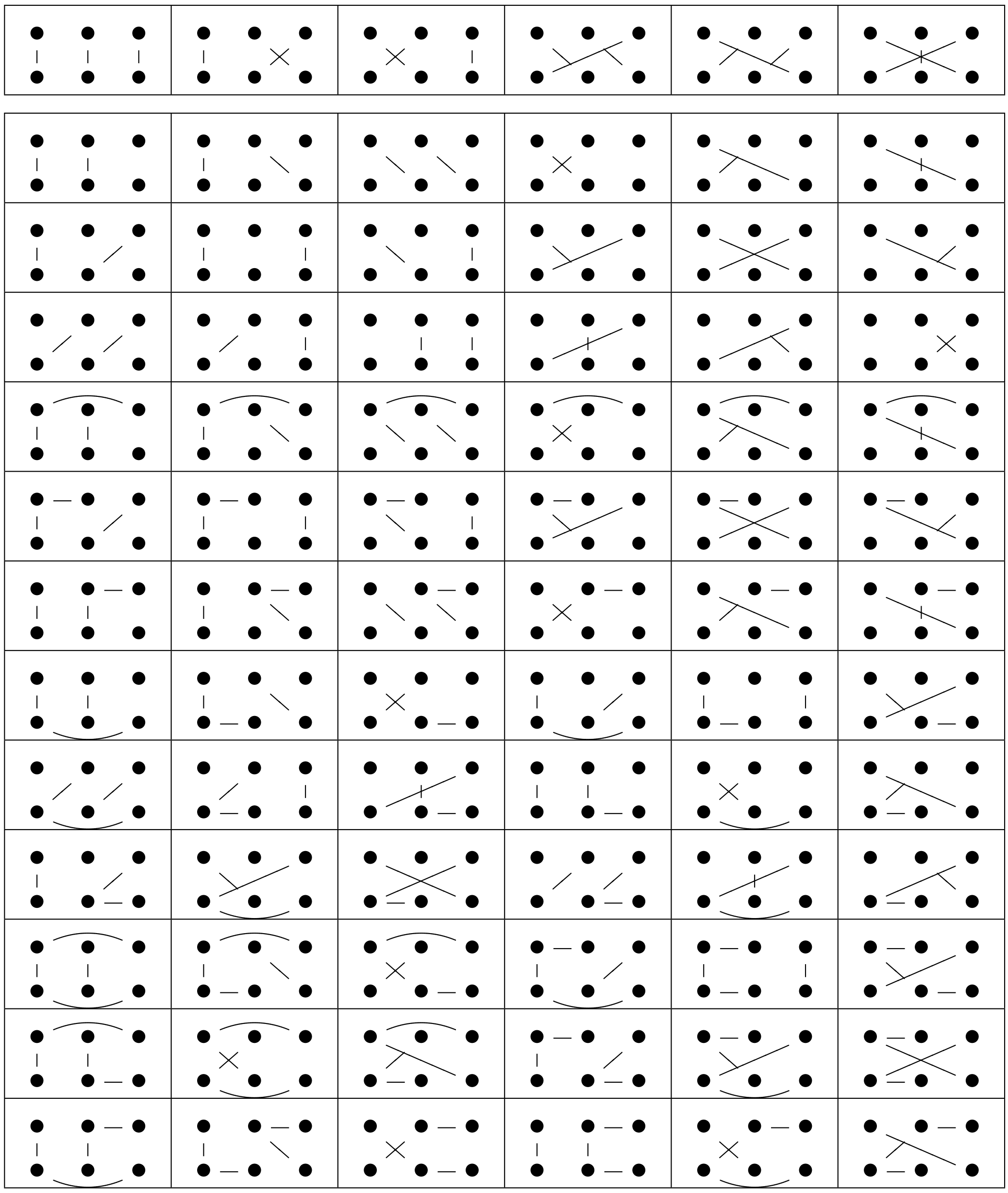
The permutation module $Ind_3 M^{(1^3)}$ is equivalent to A:

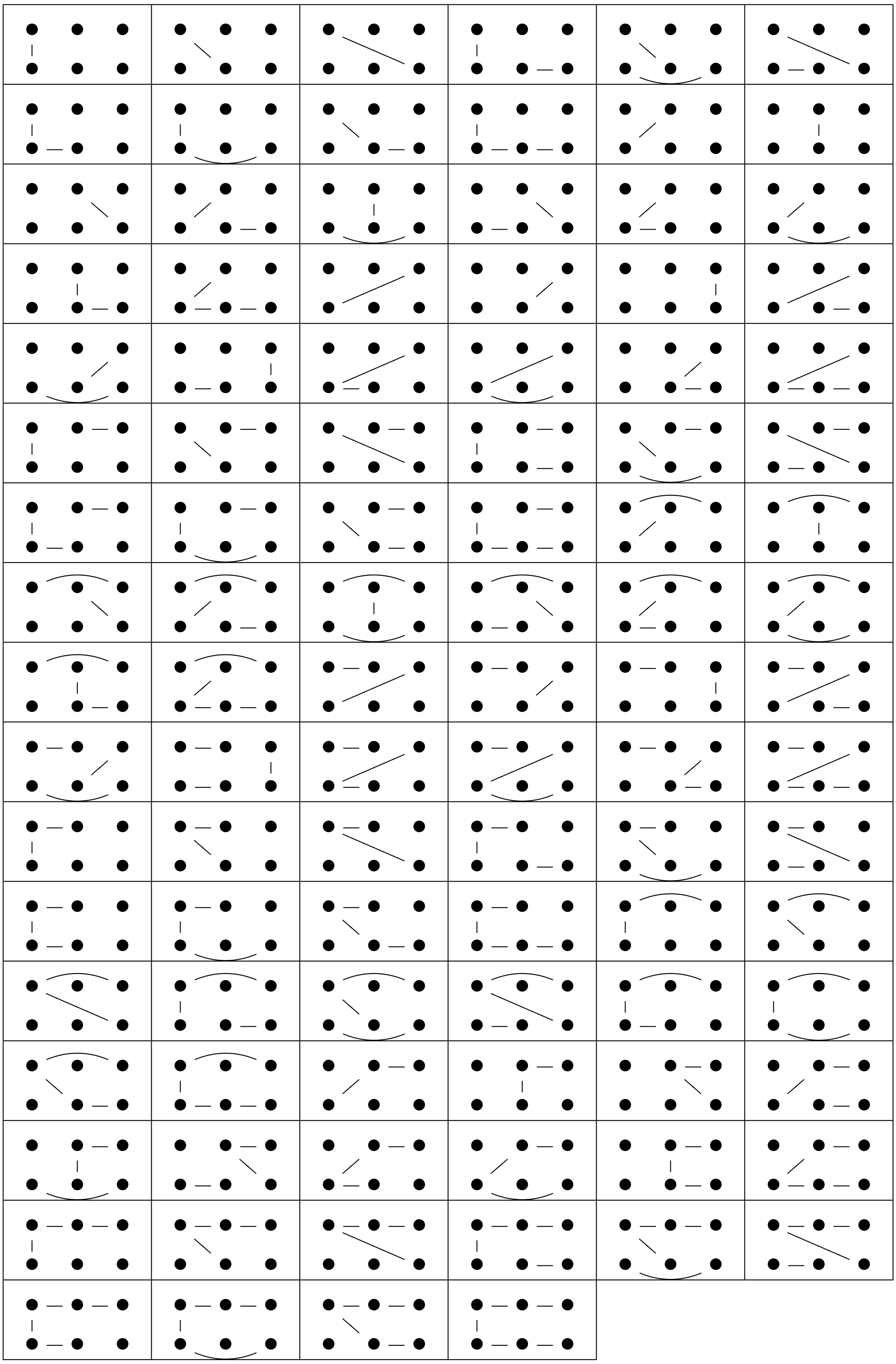
$$Ind_3 M^{(1^3)} = Ae_3 \underset{k\Sigma_3}{\otimes} k\Sigma_3 \underset{k\Sigma_{(1^3)}}{\otimes} k \simeq Ae_3 \underset{k}{\otimes} k \simeq A.$$

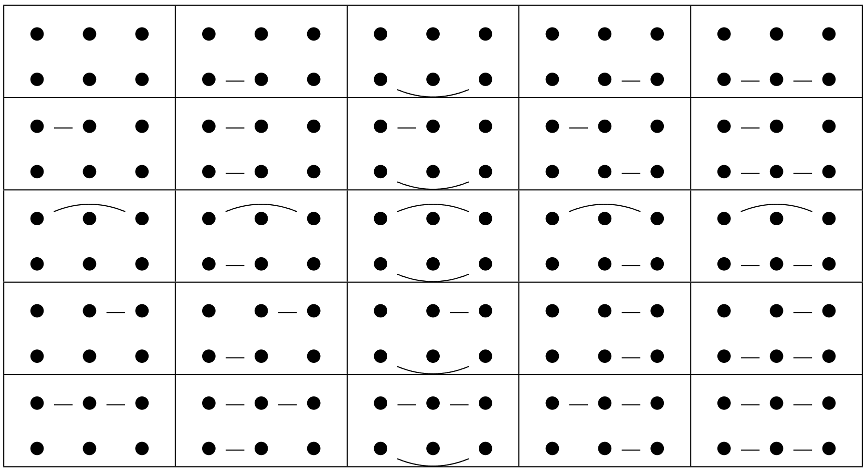
$Ind_3 M^{(2,1)}$ is the 9-dimensional module with vector space basis



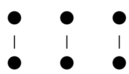
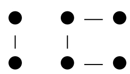
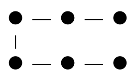
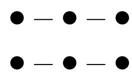
AII Basis of $P_k(3, \delta)$







Let $A = P_k(3,\delta)$, $\delta \neq 0$ and

$e_3 :=$ , $e_2 :=$ , $e_1 :=$ , $e_0 := \dfrac{1}{\delta}$ .

Set $J_i = Ae_iA$.

The first table shows the basis of $A \setminus J_2$, the second $J_2 \setminus J_1$, the third $J_1 \setminus J_0$ and the last table shows the basis of J_0. Rows 6 and 12 of the second table are the basis diagrams of $e_2(J_2/J_1)$.

The generator of the summand U_{v_1} of $(J_2/J_1)e_2$ is the 4th element of row 8, the generator of U_{v_2} is the 4th element of row 12.

The generators of the summands U_v of $(A/J_0)e_1$ are the respective 4th

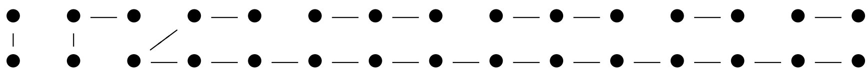
elements of rows $2, 9, 14$ and 19 of the third table.

AIII Example for Section 5.2.2

A step-by-step analysis of the summand U_v with

$$v = \circ \quad \circ - \circ \quad \circ - \circ \quad \bullet - \bullet - \bullet \quad \bullet - \bullet - \bullet \quad \bullet - \bullet \quad \bullet - \bullet$$

and

$d_v =$ 

We have $\alpha = (1,2)$ and $\beta = (0,2,3)$, so $l_1 = 1\cdot 1 + 2\cdot 2 = 5$ and $l_2 = 0{\cdot}1{+}2{\cdot}2{+}3{\cdot}3 = 13$ and $k\prod_\alpha = k \times k(\Sigma_2 \wr \Sigma_2)$, $k\prod_\beta = k(\Sigma_2\wr\Sigma_2)\times k(\Sigma_3\wr\Sigma_3)$. By Lemmas 5.10 and 5.11,

$$U_v \simeq k\Sigma_{18} \underset{k\Sigma_{(5,13)}}{\otimes} \big((k\Sigma_5 \underset{k\prod_\alpha}{\otimes} k\Sigma_3) \boxtimes (k\Sigma_{13} \underset{k\prod_\beta}{\otimes} k)\big).$$

Now $\gamma = (1, 2^2)$, so

$$\begin{aligned} k\Sigma_5 \underset{k\prod_\alpha}{\otimes} k\Sigma_3 &\simeq \bigoplus_{i=1}^{3} k\Sigma_5 \underset{k\Sigma_{(1,2^2)}}{\otimes} k \\ &= \bigoplus_{i=1}^{3} M^{(1,2^2)} \end{aligned}$$

by Lemma 5.12. For the other factor, we see that $\tilde{\gamma} = (0,4,9)$, so

$$\begin{aligned} k\Sigma_{13} \underset{k\prod_\beta}{\otimes} k &\simeq k\Sigma_{13} \underset{k\Sigma_{(4,9)}}{\otimes} \big((k\Sigma_4 \underset{k(\Sigma_2\wr\Sigma_2)}{\otimes} k) \boxtimes (k\Sigma_9 \underset{k(\Sigma_3\wr\Sigma_3)}{\otimes} k)\big) \\ &= k\Sigma_{13} \underset{k\Sigma_{(4,9)}}{\otimes} (H^{(2^2)} \boxtimes H^{(3^3)}). \end{aligned}$$

In conclusion, we have

$$\begin{aligned} U_v &\simeq k\Sigma_{18} \underset{k\Sigma_{5,13}}{\otimes} \Big(\big(\bigoplus_{i=1}^{3} M^{(1,2^2)} \big) \boxtimes \big(k\Sigma_{13} \underset{k\Sigma_{(4,9)}}{\otimes} \big(H^{(2^2)} \boxtimes H^{(3^3)} \big) \big) \Big) \\ &\simeq k\Sigma_{18} \underset{k\Sigma_{(5,4,9)}}{\otimes} \big(\bigoplus_{i=1}^{3} M^{(2^2,1)} \boxtimes H^{(2^2)} \boxtimes H^{(3^3)} \big). \end{aligned}$$

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