Interactions between universal localisations, ring epimorphisms and tilting modules

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Vorgelegt von
Frederik Marks
aus Köln

Hauptberichter: Prof. Dr. Steffen König
Mitberichter: Apl. Prof. Dr. Wolfgang Rump
2. Mitberichter: Prof. Dr. Idun Reiten

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Institut für Algebra und Zahlentheorie der Universität Stuttgart

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Wie ist das klein, womit wir ringen,
was mit uns ringt, wie ist das groß;
ließen wir, ähnlicher den Dingen,
uns so vom großen Sturm bezwingen, –
wir würden weit und namenlos.
Was wir besiegen, ist das Kleine,
und der Erfolg selbst macht uns klein.
Das Ewige und Ungemeine
will nicht von uns gebogen sein.

(R.M.Rilke, Der Schauende, 1901)

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Contents

Zusammenfassung

Die vorliegende Arbeit untersucht das Zusammenspiel von universellen Lokalisierungen, Ringepimorphismen und (verallgemeinerten) Kippmoduln. Sie verfolgt dabei das Ziel, vielfältige neue Zusammenhänge und Anwendungen dieser Konzepte in der Darstellungstheorie aufzuzeigen.

Universelle Lokalisierungen, im Sinne von Cohn und Schofield, finden seit kurzem Eingang in die Kipptheorie, einen grundlegenden Zweig der Darstellungstheorie. Sie werden genutzt, um Kippmoduln über bestimmten Ringen zu klassifizieren und sie liefern interessante Zerlegungen derivierter Modulkategorien. Aber weder die grundsätzlichen Eigenschaften dieser Lokalisierungen noch das Wesen der vielfältigen Zusammenhänge zur Kipptheorie sind ausreichend verstanden. Zufriedenstellende Antworten sind nur in Einzelfällen bekannt.

Ringepimorphismen ermöglichen es, universelle Lokalisierungen besser zu verstehen. Dabei handelt es sich um Epimorphismen in der Kategorie aller Ringe, die bestimmte abelsche Unterkategorien einer Modulkategorie klassifizieren. Universelle Lokalisierungen liefern stets Ringepimorphismen. Aber es ist bisher unklar, welche Ringepimorphismen man auf diesem Weg erhält.

Die Kapitel 2-4 dieser Arbeit beschäftigen sich mit eben dieser Frage im Kontext endlicher Lokalisierungen über beliebigen Ringen und ebenso in Bezug auf endlich dimensionale Algebren. Insbesondere für selbstinjektive Algebren ist eine Klassifikation relevanter Ringepimorphismen möglich. Zudem zeigt sich ein direkter Zusammenhang zwischen universellen Lokalisierungen und Kippmoduln. Explizite Entsprechungen werden für einige Klassen von Algebren bewiesen.

Die Kapitel 5-7 dieser Arbeit widmen sich dem neuen Konzept der *silting* Moduln und deren Beziehung zu Lokalisierungen. Zunächst wird die allgemeine Theorie dieser Moduln über einem beliebigen Ring entwickelt. *Silting* Moduln verallgemeinern sowohl Kippmoduln als auch τ-Kippmoduln über endlich dimensionalen Algebren und sie parametrisieren unterschiedliche Strukturen in der derivierten Modulkategorie. Über erblichen Ringen klassifizieren minimale *silting* Moduln alle universellen Lokalisierungen und im Allgemeinen kann jeder Lokalisierung ein *silting* Objekt zugeordnet werden. Damit liefert diese neue Theorie einen angemessenen Rahmen, um universelle Lokalisierungen zu studieren.

Abstract

The aim of this thesis is to study the interaction between universal localisations, ring epimorphisms and (generalised) tilting modules. We show that these concepts are intrinsically connected and that they provide various new applications to representation theory.

Universal localisations, as defined by Cohn and Schofield, have recently proven to be useful in tilting theory, a fundamental branch of representation theory. In fact, universal localisations were used to classify tilting modules over some rings and they provide interesting decompositions of the derived module category. However, both the structural properties of universal localisations as well as the nature of the various connections to tilting theory are far from being understood. Satisfying answers are only known in special cases.

One way to approach localisations is via ring epimorphisms. These are epimorphisms in the category of all rings that are relevant to study certain abelian subcategories of a given module category. Even though it is well-known that universal localisations yield ring epimorphisms, the question of which epimorphisms arise from universal localisations is still widely open.

Chapters 2-4 of this thesis provide some answers to the latter question, on the one hand, by looking at finite localisations over any ring and, on the other hand, by focusing on finite dimensional algebras. In particular, over self-injective algebras a classification of certain ring epimorphisms is accessible. We further focus on correspondences between universal localisations and tilting objects. Explicit bijections are established for certain classes of finite dimensional algebras.

Chapters 5-7 of this thesis are dedicated to the new concept of silting modules and its relation to localisations. We begin by developing a general theory of silting modules over any ring. These modules generalise tilting modules as well as support τ-tilting modules over a finite dimensional algebra and they turn out to parametrise diverse structures in the derived module category. Subsequently, we show that minimal silting modules classify all universal localisations over a hereditary ring. Also, in the general setup, we can associate a silting object to every localisation. Thus, silting theory provides an adequate setup to study universal localisations.

Introduction

This thesis focuses on the interplay of universal localisations, ring epimorphisms and (generalised) tilting modules. We show that these concepts are intrinsically connected and, as a consequence, we obtain various new applications to representation theory.

Localisation theory has its origin in commutative algebra with the classical concept of quotient fields obtained by formally adding denominators to a given integral domain. The example to keep in mind, is the process of constructing the rational numbers from the integers. When passing to non-commutative rings such localisations do, in general, not exist. To overcome this problem one has to impose further conditions on the set of elements to be inverted (Ore localisation). Alternatively, one can change the approach and localise at sets of matrices over the given ring ([Co]) or, more generally, at sets of maps between finitely generated projective modules. This is the concept of universal localisation ([Sch]). Although universal localisations were successfully used in different branches of mathematics like algebraic K-theory ([N2],[NR]) and tilting theory ([AA],[AKL],[CX]), the concept itself is far from being understood.

Universal localisations always yield epimorphisms in the category of all (unital) rings. These ring epimorphisms are relevant to study certain abelian subcategories of our initial module category. In fact, it was proven in [GdP] that ring epimorphisms parametrise so-called *bireflective* subcategories, i.e., subcategories for which the inclusion functor admits both a left and a right adjoint. Furthermore, by imposing the condition $Tor_i^A(B,B) = 0$ for all $i > 0$ on a given ring epimorphism $A \to B$, which is then called *homological*, one also obtains a full embedding $D(B) \to D(A)$ of the corresponding derived categories ([GL]). Therefore, an explicit classification of the (homological) ring epimorphisms of a given ring is desirable. It would provide new insights into the representation theory of the ring.

In the hereditary case, homological ring epimorphisms are precisely given by universal localisations (see [KSt] and [Sch4]). In general, this correspondence is well-known not to hold. On the one hand, universal localisations do not always yield homological ring epimorphisms. A list of examples was constructed in [NRS]. There, it was shown that every finitely presented algebra occurs (up to Morita equivalence) as the universal localisation of an associated finite dimensional algebra having global dimension at most two. On the other hand, not all homological ring epimorphisms are localisations due to a (non-noetherian) example in [K2] (also compare [BS]). However, the question of which (homological) ring epimorphisms for a given ring are universal localisations seems widely open and is one motivation for this thesis.

In recent years localisations were also studied from a categorical perspective. A localisation of a given category is obtained by formally adding inverses to a given collection of morphisms. This process is widely used throughout mathematics. For example, it describes how to pass from a model category to its homotopy category (given as the localisation with respect to the class of weak equivalences) or, more specifically, from an abelian category $\mathcal{A}$ to its derived category (localise at the quasi-isomorphisms in the category of chain complexes over $\mathcal{A}$). In our setup, we are particularly interested in categorical localisations that appear as quotients of abelian or triangulated categories. More precisely, we want to study *useful* decompositions of our initial module category or of its derived category. This refers to the notion of recollement ([BBD]).

Many strong results have linked localisations to tilting theory, a fundamental tool in representation theory. The main target of representation theory is to study abstract structures in algebra by using matrices representing their elements and to classify these representations up to isomorphism. This goal can sometimes be achieved by explicitly computing representations of a given group or algebra. Another method is to compare categories of representations. Here, tilting theory comes into play. It provides a range of tools and techniques that allow to compare different categories of modules or complexes. Initially developed to compare finitely generated representations over finite dimensional algebras, more recently, the theory got extended to (possibly not finitely generated) tilting modules over arbitrary rings. These large tilting modules have, for example, shed new light on the longstanding finitistic dimension conjecture for Artin algebras ([AT]). Further applications of tilting theory have been found in Lie theory, algebraic geometry and topology. For an overview of the research area see [AHK].

A celebrated result by Brenner and Butler states that a finitely generated tilting module T over a finite dimensional algebra A yields equivalences between certain subcategories of A-*mod* (the category of finitely generated A-modules) and S-*mod*, where S is the endomorphism ring of T. Moreover, it was shown in [H] that the bounded derived categories of A-*mod* and S-*mod* are equivalent. A similar result does not hold for all tilting modules over any ring. It was shown in [B2] and [CX] that, in general, there is a recollement of derived module categories

$$\begin{tikzcd} D(C\text{-}Mod) \arrow[r] & D(S\text{-}Mod) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & D(A\text{-}Mod) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}$$

where the ring C is given as a universal localisation of S. This construction illustrates the importance of localisation techniques in tilting theory. In fact, recent results have shown that localisations provide powerful new tools in tilting theory – and not only in the context of triangulated categories. It turned out that all tilting modules over tame hereditary algebras or Dedekind domains can be classified by universal localisations ([AS],[AS2],[M]). A similar approach also applies to Prüfer domains and commutative noetherian rings ([AA],[APST]).

The general idea behind this classification goes back to a construction in [AS] (also see [GL]). There, it was shown that every injective homological ring epimorphism $A \to B$ with B of projective dimension at most one (when seen as an A-module) provides a tilting module, namely $B \oplus B/A$. Note that over a hereditary ring, this result applies to all injective universal localisations. However, over an arbitrary ring, we cannot expect universal localisations to be homological. Thus, we may ask for a class of generalised tilting modules which can be parametrised by universal localisations.

Recently, (support) τ-tilting modules over a finite dimensional algebra A were introduced in [AIR]. They generalise finitely generated tilting A-modules and complete this class from the point of view of mutation. Indeed, there is always a unique way of replacing indecomposable direct summands of support τ-tilting modules to obtain different ones. Moreover, it turns out that support τ-tilting modules parametrise many other interesting structures in the module category or in its bounded derived category such as torsion classes and 2-term silting complexes. However, the concept of τ-tilting relies on the existence of the Auslander-Reiten translation τ. In order to develop a similar theory over arbitrary rings, it is necessary to find an alternative description.

This problem was approached in [AMV]. The new class of silting modules introduced is defined over any ring and generalises tilting modules as well as support τ-tilting modules over a finite dimensional algebra. In particular, it is shown that silting modules correspond to 2-term silting complexes, to certain t-structures and to certain co-t-structures in the derived module category. In comparison to tilting, the setup of silting modules is better adapted to the intrinsic properties of localisations, as it will become clear throughout the thesis. In fact, over some rings, silting modules classify all universal localisations. Moreover, it turns out that most of the various constructions connecting tilting modules and localisations can be transferred to the silting context. In particular, to every (partial) silting module one can associate different types of localisations, namely ring epimorphisms and recollements.

In what follows, we provide a more detailed outline of the thesis and its main results. In the first chapter, we begin with discussing the underlying mathematical background. Already here, the focus is on drawing some first connections between the – a priori – different notions. The selection of contents in this chapter follows the idea of providing as much background material as necessary for reading the forthcoming chapters, but to refer to the literature for a more detailed historical background and related results. More specific assumptions and further notions, will only be introduced when needed later on.

In the second chapter, we study the interplay of universal localisations and ring epimorphisms. More precisely, we study ring epimorphisms $A \to B$ that turn B into a finitely presented A-module. All results in this chapter are joint work with Jorge Vitória (see [MV]). Motivated by the classification of universal localisations over hereditary rings in [KSt], we obtain the following theorem.

Theorem (Theorem 2.1.3) *Let $f : A \to B$ be a ring epimorphism such that B is a finitely presented A-module of projective dimension less or equal than one. Then f is homological if and only if it is a universal localisation.*

Moreover, the set of maps to localise at can be constructed explicitly by replacing the A-module map $A \to B$ by a quasi-isomorphic 2-term complex of finitely generated projective A-modules in the derived category. Note that this result cannot be generalised to B being of projective dimension two over A. Relevant examples will be provided.

Recent work uses universal localisations to construct interesting examples of recollements of derived module categories ([AKL], [CX], [CX2], [CX3]). In this setting, we prove the following.

Theorem (Theorem 2.2.1) *Let $f : A \to B$ be a homological ring epimorphism such that B is a finitely presented A-module of projective dimension less or equal than one. If*

$$Hom_A(coker(f), ker(f)) = 0$$

holds, then the derived restriction functor f_* induces a recollement of derived module categories

$$D(B) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(End_{D(A)}(K_f)),$$

where K_f is the cone of f in $D(A)$. Moreover, if B is a finitely presented projective A-module, then there is an isomorphism of rings $End_{D(A)}(K_f) \cong A/\tau_B(A)$, where $\tau_B(A)$ is the trace of B in A.

Recollements of derived module categories are particularly relevant to results obtained in [AKL2], [AKL3] and [LY], where a Jordan-Hölder-type theorem for derived module categories of some rings has been proved. Such a property cannot, however, hold for all rings and a counterexample can be constructed using universal localisations (see [CX]).

In the third chapter, we study universal localisations and their interactions with tilting modules in the context of finite dimensional algebras over a field $\mathbb{K}$ (see [M]). We suggest an approach, initially motivated by [Sch3], based on studying pairs of orthogonal subcategories in A-*mod*. This approach relies on the observation that for a finite dimensional algebra A every universal localisation A_Σ is given with respect to a set of finitely generated A-modules. In the hereditary case, our methods restrict to the consideration of *Ext*-orthogonal pairs, as studied in [KSt]. In the setting of hereditary algebras, we obtain the following result.

Theorem (Theorem 3.2.3, Proposition 3.2.8) *Let A be a finite dimensional and hereditary $\mathbb{K}$-algebra. There are bijections (related by restriction) between:*

(1) the set of equivalence classes of finitely generated support tilting A-modules and the set of epiclasses of finite dimensional universal localisations of A;

(2) the set of equivalence classes of finitely generated tilting A-modules and the set of epiclasses of finite dimensional and monomorphic universal localisations of A;

(3) the set of equivalence classes of finitely generated support tilting A/AeA-modules for an idempotent e in A and the set of epiclasses of finite dimensional universal localisations of A with $A_\Sigma \otimes_A Ae = 0$.

Moreover, the universal localisation associated to a tilting A-module T is given by localising at the set of non split-projective indecomposable direct summands of T in its torsion class.

As a consequence, all finitely generated tilting A-modules are (up to equivalence) of the form $A_\Sigma \oplus A_\Sigma/A$ for some finite dimensional and monomorphic universal localisation A_Σ of A. In the tame case, this completes the classification of tilting modules started in [AS2]. Note that not every infinitely generated tilting module over a tame hereditary algebra arises in this way ([AS2]).

Leaving the hereditary case, we will concentrate on universal localisations for Nakayama algebras. On the one hand, these algebras are sufficiently well-understood and they share particularly nice homological properties. On the other hand, from a representation theoretical point of view, Nakayama algebras are far away from hereditary algebras. They allow to approach universal localisations from a different perspective and may help to get a clearer picture in the general setting. We obtain a complete classification of the universal localisations by considering orthogonal collections of indecomposable A-modules.

Theorem (Theorem 3.3.8, Corollary 3.3.9) *Let A be a Nakayama algebra. There is a bijection between the universal localisations of A (up to epiclasses) and the sets $\{X_1,...,X_s\}$ of indecomposable A-modules (up to isomorphism) with $End_A(X_i) \cong \mathbb{K}$ for all i and $Hom_A(X_i,X_j) = 0$ for all $i \neq j$. Moreover, a ring epimorphism $A \to B$ is a universal localisation if and only if $Tor_1^A(B,B) = 0$.*

In particular, all homological ring epimorphisms are universal localisations. However, the projective dimension of a localisation A_Σ, when viewed as an A-module, can be infinite. Finally, we establish a link between universal localisations and support τ-tilting modules over Nakayama algebras. Note that the methods used in the proof rely significantly on the chosen setting and are unlikely to work in full generality.

Theorem (Theorem 3.4.6, Theorem 3.4.9) *Let A be a Nakayama algebra. There are bijections (related by restriction) between:*

(1) the set of equivalence classes of support τ-tilting A-modules and the set of epiclasses of universal localisations of A;

(2) the set of equivalence classes of τ-tilting A-modules and the set of epiclasses of universal localisations of A with $A_\Sigma \otimes_A Ae \neq 0$ for all idempotents $e \neq 0$ in A;

(3) the set of equivalence classes of support τ-tilting A/AeA-modules for an idempotent e in A and the set of epiclasses of universal localisations of A with $A_\Sigma \otimes_A Ae = 0$.

Moreover, the universal localisation associated to a τ-tilting module T is given by localising at the set of non split-projective indecomposable direct summands of T in its torsion class.

As a consequence, we can translate some of the combinatorics for universal localisations to the theory of τ-tilting modules over Nakayama algebras (also see [A]). Further connections between support τ-tilting modules and universal localisations will be discussed in the later chapters.

The fourth chapter discusses ring epimorphisms over certain self-injective algebras. The first main result deals with preprojective algebras of Dynkin type. These algebras have been successfully used in Lie theory and algebraic geometry (see, for example, [GLS] and [GLS2]). We show that preprojective algebras of Dynkin type only admit a finite number of ring epimorphisms $A \to B$ (up to equivalence) with $\mathrm{Tor}_1^A(B,B) = 0$ and where B is again finite dimensional. In this situation, it turns out that also B is a self-injective algebra that does not admit any loops (see Theorem 4.1.5). In particular, there are only finitely many universal localisations.

In the subsequent sections of the fourth chapter, we classify homological ring epimorphisms over some self-injective algebras. Since these ring epimorphisms help to study the structure of the corresponding derived module category, a classification is of great interest, as suggested in [CX3, §7.2, Question 4]. The following result collects some of the answers obtained in this direction.

Theorem (Theorem 4.2.6, Theorem 4.2.8, Theorem 4.3.3)

(1) Let A be a preprojective algebra of Dynkin type and $f : A \to B$ be a homological ring epimorphism that is neither zero nor an isomorphism and where B is again finite dimensional. Then A must be of type $\mathcal{A}_n (n \geq 2)$ and the algebra B is Morita-equivalent to $\mathbb{K}$. In fact, for each $n \geq 2$ there are precisely two such choices for f (up to equivalence).

(2) Let A be a connected and weakly symmetric algebra that fulfils the Tachikawa conjecture and let $f : A \to B$ be a non-zero homological ring epimorphism with B finite dimensional. Then f is an isomorphism.

(3) There is an explicit description of all homological ring epimorphisms over self-injective Nakayama algebras. In particular, most of the non-trivial examples are provided by universal localisations at certain sets of simple modules.

Recall that non-trivial homological ring epimorphisms always give rise to non-trivial recollements of the derived category of all A-modules (see Theorem 1.5.2 in Chapter 1). It turns out that for some of the above algebras (i.e. for those which are derived simple, see [AKLY]) the right hand side of the induced recollements cannot be equivalent to a derived module category. Explicit examples of derived simple and self-injective algebras with many different non-trivial homological ring epimorphisms are provided.

The fifth chapter contains joint work with Lidia Angeleri Hügel and Jorge Vitória (see [AMV]). We introduce the new concept of silting modules. These modules are intended to generalise tilting modules in a similar fashion as 2-term silting complexes generalise 2-term tilting complexes and also in the way support τ-tilting modules generalise finitely generated tilting modules over a finite dimensional algebra. Some related results were obtained in parallel work by Wei ([Wei2],[Wei3]).

This new class of modules shares some important features with tilting theory. In particular, the torsion class associated to a silting module provides left approximations, and partial silting modules admit an analogue of the Bongartz complement (see Theorem 5.1.15).

It turns out that silting modules are related to the class of quasitilting modules studied in [C] and [CDT]. As a main feature, these modules induce half of the equivalences occurring in Brenner-Butler's classical Tilting Theorem. This forces them to be finitely generated ([Tr]). In our work we drop this finiteness condition, and we show that large quasitilting modules can be used to classify certain torsion classes, including those generated by silting modules. We fix a ring A.

Theorem (Theorem 5.1.6) *The following are equivalent for a torsion class $\mathcal{T}$ in A-Mod.*

(1) For every A-module M there is a sequence

$$M \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$$

such that ϕ is a left $\mathcal{T}$-approximation and C is Ext-projective in $\mathcal{T}$.

(2) There is a finendo quasitilting A-module T such that $\mathcal{T} = Gen(T)$.

Notice that over a finite dimensional algebra A, finitely generated silting and finitely generated quasitilting modules coincide with support τ-tilting modules. By restricting the previous theorem to support τ-tilting modules, we obtain the known classification of functorially finite torsion classes in A-*mod* (compare [AIR, Theorem 2.7]).

Moreover, the proposed concept of silting modules allows us to parametrise certain structures in the derived category of A-modules, including 2-term silting complexes. Recall that silting complexes were first introduced by Keller and Vossieck ([KV]) to study t-structures in the bounded derived category of representations of Dynkin quivers. They generalise tilting complexes – and, thus, finitely generated tilting modules – in the sense that the associated t-structures yield hearts that are not necessarily derived equivalent to the initial algebra. The topic resurfaced recently, in particular through the work of Aihara and Iyama ([AI]), Keller and Nicolás ([KN]), Koenig and Yang ([KY]), and Mendoza, Sáenz, Santiago and Souto Salorio ([MSSS]).

We show that for an arbitrary ring there is a bijection between (not necessarily finitely generated) silting modules and (not necessarily compact) 2-term silting complexes. Moreover, every silting module gives rise to a t-structure which coincides both with the construction due to Happel, Reiten and Smalø in [HRS] and with the t-structure studied by Hoshino, Kato and Miyachi in [HKM]. We prove the following correspondences extending results in [AIR] and [AI].

Theorem (Theorem 5.2.11) *Let A be a ring. There are bijections between*

(1) equivalence classes of silting A-modules;

(2) equivalence classes of 2-term silting complexes in $D(A)$;

(3) 2-silting t-structures in $D(A)$;

(4) co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D(A)$ with $\mathcal{U}_{\leq 0}$ coproduct-closed and $D^{\leq -1} \subseteq \mathcal{U}_{\leq 0} \subseteq D^{\leq 0}$.

In fact, similar bijections hold for silting complexes of any finite length, thus, extending the correspondences established in [KN] and [KY] to the non-compact setting (see Theorem 5.2.6).

The sixth chapter contains ongoing joint work with Lidia Angeleri Hügel and Jorge Vitória. The target is to understand the interaction of silting modules and ring epimorphisms and to show that silting theory provides an appropriate context for studying universal localisations. Note that connections between silting modules and ring epimorphisms are to be expected by the results obtained in Chapter 3. We begin by associating to every partial silting module a bireflective subcategory and, thus, a ring epimorphism. The approach taken is motivated, on the one hand, by a known construction for partial tilting modules in [CTT]. On the other hand, in the setting of support τ-tilting modules, similar abelian categories were already considered in [J]. As a first main result, we obtain an explicit description of the ring epimorphism $A \to B$ associated to a partial silting module T_1. In fact, the ring B is isomorphic to $End_A(T)/I$, where $T := T_0 \oplus T_1$ denotes the *Bongartz completion* of T_1 to a silting module and I is the two-sided ideal generated by those endomorphisms factoring through an object in $Add(T_1)$ (see Theorem 6.1.4).

This result relates the representation theories of the initial ring A and the rings B and $End_A(T)$. Note that, if $Hom_A(T_1, T_0)$ is finitely generated, then the ideal I is generated by an idempotent e and one obtains a recollement of module categories of the form

$$\begin{tikzcd} B\text{-}Mod \arrow[r] & End_A(T)\text{-}Mod \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r] & eEnd_A(T)e\text{-}Mod. \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \end{tikzcd}$$

In the second part of the chapter, we restrict the setting to hereditary rings. Here, we prove that silting modules can be understood as support tilting modules, i.e., modules that are tilting over a quotient ring of A that is given by the trace ideal of a projective module. We then study (strongly) minimal silting modules T which are defined to admit certain minimal left $Add(T)$-approximations. It turns out that we can associate in a canonical way a homological ring epimorphism to every such silting module. We obtain the following classification result.

Theorem (Theorem 6.2.9) *Let A be a hereditary ring. Then there are bijections between:*

(1) Equivalence classes of strongly minimal silting A-modules;

(2) Epiclasses of homological ring epimorphisms of A.

Moreover, these bijections restrict to bijections between:

(1) Equivalence classes of strongly minimal tilting A-modules;

(2) Epiclasses of injective homological ring epimorphisms of A.

Recall that over a hereditary ring homological ring epimorphisms coincide with universal localisations. Moreover, they correspond bijectively to recollements of the derived module category (see [KSt] and [NS]). In case A is a finite dimensional hereditary algebra, a strongly minimal silting A-module is shown to be finitely generated if and only if the recollement given by the associated homological ring epimorphism is a recollement of derived modules categories (Proposition 6.2.13).

The seventh chapter contains ongoing joint work with Jan Šťovíček. The idea is to study universal localisations via torsion pairs which are naturally associated to a set Σ of maps between finitely generated projective modules. Similar torsion and torsionfree classes appeared before in the context of universal localisations (see, for example, [Sch3] and [AS, Chapter 4]). However, a systematic and general discussion of their properties and use is still missing. Furthermore, similar torsion pairs also appeared in the previous two chapters in the context of silting modules. The philosophy is to drop the presilting condition on the maps in Σ (i.e., for $\sigma \in \Sigma$, we do no longer suppose that $Hom_{D(A)}(\sigma, \sigma[1]) = 0$), but to assume that the projective modules involved are finitely generated. The first main result states that the torsion class

$$\mathcal{D}_\Sigma := \{X \in A\text{-}Mod \mid Hom_A(\sigma, X) \text{ is surjective } \forall \sigma \in \Sigma\}$$

is functorially finite in A-*Mod* (Theorem 7.2.7). As a consequence, we can construct reflections for every universal localisation explicitly (Corollary 7.2.10). The proof uses weak factorisation systems which can be thought of as a *weak* version of a model structure (see [Hov] and [Sto]).

In a second section, we use the language of exact categories of Grothendieck type (compare [SS] and [Sto]) to further study the relations between universal localisations and silting objects. More precisely, we obtain the following result.

Theorem (Theorem 7.3.3) *Let A be a ring. For every universal localisation A_Σ of A there is a map ∇ between projective A-modules with $Hom_{D(A)}(\nabla, \nabla[1]) = 0$ such that the A_Σ-modules in A-Mod are given by*

$$\{X \in A\text{-}Mod | Hom_A(\nabla, X) \text{ is an isomorphism}\}.$$

In particular, if ∇ is compact, then the universal localisations A_Σ and $A_{\{\nabla\}}$ coincide.

As an application for Artin algebras, we provide sufficient conditions for a universal localisation to arise from a presilting map. In this case, the localisation is of particular nice shape.

Theorem (Theorem 7.4.3) *Let A be an Artin algebra. Assume that $\mathcal{D}_\Sigma \cap A$-mod is functorially finite in A-mod. Then there is some map ∇ between finitely generated projective A-modules with $Hom_{D(A)}(\nabla, \nabla[1]) = 0$ such that the universal localisation A_Σ of A is given by localising at $\{\nabla\}$. Moreover, A_Σ is again an Artin algebra.*

We expect to use the above statement to prove that a universal localisation preserves the artinian property if and only if it is a localisation at a presilting map. Note that such a statement is true for hereditary finite dimensional algebras (see [KSt, Proposition 4.2]). Moreover, further evidence is provided by the bijections established in Chapter 3 for Nakayama algebras.

In summary, there are two main problems addressed in this thesis. First, the question of how to classify ring epimorphisms and universal localisations of a given ring. In other words, we are asking for a classification of certain abelian subcategories of a given module category. Solutions to this problem are mainly presented in the first half of the thesis. In Chapter 2, we provide sufficient conditions for a ring epimorphism to be a universal localisation. In Chapters 3 and 4, the setting is restricted to finite dimensional algebras and we obtain explicit classifications of relevant classes of ring epimorphisms and localisations.

The second main problem deals with the recently discovered connections between universal localisations and tilting modules. The aim is to describe the nature of these interactions. Motivated by the bijections between localisations and support τ-tilting modules established in Chapter 3, we first extend, simultaneously, the concept of tilting modules and support τ-tilting modules. This is done in the fifth chapter by developing a general theory of silting modules over an arbitrary ring. Silting modules turn out to provide an adequate setup to study universal localisations. In Chapter 6, it is shown that minimal silting modules classify all universal localisations over a hereditary ring. Finally, the general case of localisations over an arbitrary ring is treated in Chapter 7.

Chapter 1

Preliminaries

In this first chapter we fix some notation and introduce the relevant mathematical concepts used throughout. Proofs of the statements are only included in case they do not appear in the literature.

1.1 Notation

Throughout, A will be a (unitary) ring, A-Mod (respectively, A-mod) the category of (finitely generated) left A-modules, and A-$Proj$ (respectively, A-$proj$) its subcategory of (finitely generated) projective modules. Modules will always be left A-modules unless otherwise stated. In some contexts, A will be a finite dimensional algebra over an algebraically closed field $\mathbb{K}$ or an Artin algebra (i.e. an algebra over a commutative artinian ring R that is finitely generated as an R-module). Morphisms in A-$Proj$ will be interpreted, without change of notation, both as 2-term complexes concentrated in degrees -1 and 0 in the homotopy category $K(A\text{-}Proj)$, and as projective presentations of their cokernels. The unbounded derived (respectively, homotopy) category of A-Mod will be denoted by $D(A)$ (respectively, $K(A)$). If we restrict to bounded or right bounded complexes, we use the usual superscripts b and $-$, respectively. The term *subcategory* will always refer to a full subcategory closed under isomorphisms.

For a given A-module M, we denote by M° the subcategory of A-Mod consisting of the objects N such that $Hom_A(M,N)=0$, and by $M^{\perp_1}$ the subcategory of A-Mod consisting of the objects N such that $Ext_A^1(M,N)=0$. Moreover, we define $M^{\perp}$ to be $M^{\circ}\cap M^{\perp_1}$. Left orthogonal subcategories are analogously defined. Further, $Add(M)$ denotes the additive closure of M consisting of all modules isomorphic to a direct summand of a direct sum of copies of M, while $Gen(M)$ is the subcategory of M-generated modules (that is, all epimorphic images of modules in $Add(M)$), and $Pres(M)$ is the subcategory of M-presented modules. Dually, we define $Prod(M)$ to consist of all modules isomorphic to a direct summand of a product of copies of M and $Cogen(M)$ to be the subcategory containing all subobjects of modules in $Prod(M)$. In case A is an Artin algebra, we set $gen(T)$ to be $Gen(T) \cap A\text{-}mod$ and $add(T)$ to be $Add(T) \cap A\text{-}mod$.

For a given subcategory $\mathcal{C}$ of $A\text{-}Mod$, we say that an A-module P in $\mathcal{C}$ is **split-projective**, if all surjective morphisms $X \to P$ in $\mathcal{C}$ split, and **Ext-projective**, if $Ext^1_A(P,X) = 0$ for all X in $\mathcal{C}$.

In this thesis, we use the following convention. If f and g are morphisms in a category, then $g \circ f$ means g after f. However, when passing to the endomorphism ring of a given left A-module M, we define the multiplication in $End_A(M)$ to be read from the left. Hence, M carries the structure of a right $End_A(M)$-module and we can identify the ring A with its endomorphism ring $End_A({}_AA)$.

1.2 Approximations

For a subcategory $\mathcal{X}$ of $A\text{-}Mod$ and an object M in $A\text{-}Mod$ we call a morphism $f : M \to X$ with X in $\mathcal{X}$ a **left $\mathcal{X}$-approximation**, if every morphism $M \to X'$ with X' in $\mathcal{X}$ factors through f. If this factorisation is unique, f is said to be a $\mathcal{X}$-**reflection**. Moreover, we call f a **minimal left $\mathcal{X}$-approximation**, if the map f is left-minimal, i.e., for any endomorphism g of X, if $g \circ f = f$, then g is an isomorphism. Note that minimal left $\mathcal{X}$-approximations are unique up to isomorphism. In particular, every reflection is left-minimal. Dually, we can define **right approximations** and **coreflections**. Approximations will occur frequently throughout this thesis.

Examples of left approximations are provided by injective envelopes. Indeed, these approximations are left-minimal. The term pre-envelope will be used as a synonym for left approximation. Dually, projective covers are examples of right approximations with respect to $A\text{-}Proj$.

A subcategory $\mathcal{X}$ of $A\text{-}Mod$ will be called **covariantly** (respectively, **contravariantly**) **finite**, if every A-module admits a left (respectively, right) $\mathcal{X}$-approximation. Subcategories that are covariantly and contravariantly finite are called **functorially finite**.

1.3 Ring epimorphisms

Recall that a **ring epimorphism** is an epimorphism in the category of rings with unit. Two ring epimorphisms $f : A \to B$ and $g : A \to C$ are said to be **equivalent** if there is a ring isomorphism $h : B \to C$ such that $g = h \circ f$. We then say that B and C lie in the same **epiclass** of A. Clearly, all surjective ring homomorphisms provide examples of ring epimorphisms. A non-surjective example is given by the canonical embedding of $\mathbb{Z}$ into $\mathbb{Q}$.

Proposition 1.3.1. *[St, Proposition XI.1.2] For a ring homomorphism $f : A \to B$, the following statements are equivalent.*

(1) f is a ring epimorphism;

(2) The restriction functor $f_* : B\text{-}Mod \to A\text{-}Mod$ is fully faithful;

(3) $f \otimes_A B = B \otimes_A f : B \to B \otimes_A B$ is an isomorphism of B-B-bimodules;

(4) $B \otimes_A coker(f) = 0$.

Moreover, the functor $B \otimes_A -$ is left adjoint to f_*.

For a finite dimensional $\mathbb{K}$-algebra A, we call a ring epimorphism $A \to B$ **finite dimensional**, if B is finite dimensional over $\mathbb{K}$. Note that, in this case, restriction induces a fully faithful functor

$$f_* : B\text{-}mod \to A\text{-}mod.$$

Necessary and sufficient conditions for a ring epimorphism to be finite dimensional are given in [GdP, Proposition 2.2]. In some cases, all epiclasses of A are finite dimensional.

Lemma 1.3.2. *[GdP, Corollary 2.3] If A is a finite dimensional $\mathbb{K}$-algebra of finite representation type, then all ring epimorphisms $A \to B$ are finite dimensional. In particular, B is again a representation finite algebra.*

Epiclasses of a ring A can be classified by suitable subcategories of $A\text{-}Mod$. For a ring epimorphism $f : A \to B$ we denote by $\mathcal{X}_B$ the essential image of the associated restriction functor.

Theorem 1.3.3. *[GdP, Theorem 1.2][GL][I, Theorem 1.6.1] There is a bijection between:*

(1) ring epimorphisms $A \to B$ up to equivalence;

(2) bireflective subcategories $\mathcal{X}_B$ of $A\text{-}Mod$, i.e., full subcategories of $A\text{-}Mod$ closed under products, coproducts, kernels and cokernels.

If A is a finite dimensional $\mathbb{K}$-algebra, this bijection restricts to a bijection between:

(1) finite dimensional ring epimorphisms $A \to B$ up to equivalence;

(2) bireflective subcategories $\mathcal{X}_B$ of $A\text{-}mod$, i.e., full functorially finite subcategories of $A\text{-}mod$ closed under kernels and cokernels.

Given a ring epimorphism $f : A \to B$, consider the adjoint pair $(B \otimes_A -, f_*)$. For an A-module M, let $\psi_M : M \to B \otimes_A M$ be the unit of this adjunction at M. Clearly, we have that

$$\psi_M(m) = 1_B \otimes m,\ \forall m \in M.$$

Note that ψ_N for a B-module N is an isomorphism. In fact, ψ_M is the $\mathcal{X}_B$-reflection of M.

Lemma 1.3.4. *Let $f : A \to B$ be a ring epimorphism and M an A-module. For any B-module N and for any A-homomorphism $g : M \to N$, g factors uniquely through ψ_M.*

Proof. Since the map ψ_N is an isomorphism, we can define a homomorphism of A-modules

$$\tilde{g} := \psi_N^{-1} \circ (B \otimes_A g).$$

It is clear that $g = \tilde{g} \circ \psi_M$ and, by construction, $\tilde{g}$ is the unique map satisfying this property. □

Remark 1.3.5. *In particular, note that the ring epimorphism $f : A \to B$, regarded as a homomorphism of A-modules, is a $\mathcal{X}_B$-reflection. Moreover, if A is a finite dimensional $\mathbb{K}$-algebra, then f can be seen as the sum of the reflections of the indecomposable projective A-modules.*

Remark 1.3.6. *For a ring epimorphism $f : A \to B$ the functor $Hom_A(B, -)$ is right adjoint to the restriction functor f_*. Dually to the situation above, the $\mathcal{X}_B$-coreflection for an A-module M is given by the canonical map $\phi_M : Hom_A(B, M) \to M$, defined via evaluation at 1_B. In particular, M belongs to $\mathcal{X}_B$ if and only if ϕ_M is an isomorphism.*

The following two results will be frequently used throughout the thesis.

Proposition 1.3.7. *[Sch, Theorem 4.8] Let $A \to B$ be a ring epimorphism. Then the following are equivalent.*

(1) $Tor_1^A(B, B) = 0$;

(2) $Ext_A^1(M, N) \cong Ext_B^1(M, N)$ for all B-modules M and N.

Proposition 1.3.8. *Let $A \to \bar{A}$ be a surjective ring epimorphism with kernel I. The following holds.*

(1) The subcategory $\mathcal{X}_{\bar{A}}$ is closed under quotients and subobjects in A-Mod.

(2) I is an idempotent ideal if and only if $Tor_1^A(\bar{A}, \bar{A}) = 0$.

(3) Assume that I is idempotent and let X be in A-Mod. Then X lies in $\mathcal{X}_{\bar{A}}$ if and only if X belongs to I°. Moreover, if I is the trace ideal of a projective A-module P, then $I^\circ = P^\circ$.

Proof. For the first claim observe that all $\mathcal{X}_{\bar{A}}$-reflections are surjective and all $\mathcal{X}_{\bar{A}}$-coreflections are injective. Now let $X \to Y$ be an injection in $A\text{-}Mod$ with Y in $\mathcal{X}_{\bar{A}}$. By applying the functor $\bar{A} \otimes_A -$ to this injection, we obtain the following commutative diagram of A-modules

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Psi_X"'] & Y \arrow[d, "\Psi_Y"] \\ \bar{A} \otimes_A X \arrow[r] & \bar{A} \otimes_A Y \end{tikzcd}$$

The reflection Ψ_X is surjective and, by assumption, the map Ψ_Y is an isomorphism. Therefore, Ψ_X is also injective, since so is the composition $X \to \bar{A} \otimes_A Y$. It follows that X lies in $\mathcal{X}_{\bar{A}}$. Dually, we see that $\mathcal{X}_{\bar{A}}$ is closed under quotients, by applying the functor $Hom_A(\bar{A}, -)$ to a given surjection $X \to Y$ in $A\text{-}Mod$ with X in $\mathcal{X}_{\bar{A}}$.

The second statement follows from the fact that

$$Tor_1^A(\bar{A}, \bar{A}) = Tor_1^A(A/I, A/I) = I/I^2.$$

Finally, assume that $I = I^2$. By applying the functor $Hom_A(-, X)$ to the short exact sequence

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} \bar{A} \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow Hom_A(\bar{A}, X) \xrightarrow{\pi_*} Hom_A(A, X) \longrightarrow Hom_A(I, X) \longrightarrow Ext_A^1(\bar{A}, X) \longrightarrow 0$$

If $X \in \mathcal{X}_{\bar{A}}$, then π_* is an isomorphism and, since $Ext_A^1(\bar{A}, X) = 0$ by Proposition 1.3.7, it follows that $X \in I^\circ$. Conversely, if $Hom_A(I, X) = 0$, then π_* is an isomorphism turning X into an $\bar{A}$-module.

Moreover, if I is the trace ideal of a projective A-module P, there is some set J with a surjection $P^{(J)} \to I$. Consequently, $P^\circ \subseteq I^\circ$. For the other inclusion, take a map $f : P \to X$ in $A\text{-}Mod$. Then f factors through a surjection $A^{(J')} \to X$ for some set J', yielding the commutative diagram

$$\begin{tikzcd} P \arrow[rr, "g"] \arrow[dr, "f"'] & & A^{(J')} \arrow[dl] \\ & X & \end{tikzcd}$$

Since the image of every component map of g is contained in I, it follows that $f = 0$. Thus, $I^\circ = P^\circ$. $\square$

1.4 Homological ring epimorphisms

We are interested in ring epimorphisms with particularly nice homological properties. Following Geigle and Lenzing ([GL]), a ring epimorphism $f : A \to B$ is said to be **homological** if

$$Tor_i^A(B,B) = 0$$

for all $i > 0$. First examples of homological ring epimorphisms are provided by Ore localisations. In fact, such a localisation $A \to B$ turns B into a flat A-module. Further examples will be discussed in the context of universal localisations. For a ring epimorphism $f : A \to B$, we denote by K_f the object

$$A \xrightarrow{f} B$$

in the category of complexes of A-modules, where A lies in position -1. Note that, regarded as an object of $D(A)$, K_f is isomorphic to the cone of the map f, seen as a map of complexes concentrated in degree zero. The following well-known result is an analogue of Proposition 1.3.1 for homological ring epimorphisms.

Proposition 1.4.1. *The following are equivalent for a ring homomorphism $f : A \to B$.*

(1) f is a homological ring epimorphism;

(2) The derived restriction functor $f_* : D(B) \to D(A)$ is fully faithful;

(3) $B \otimes_A^{\mathbb{L}} f : B \to B \otimes_A^{\mathbb{L}} B$ is an isomorphism in $D(A)$;

(4) $B \otimes_A^{\mathbb{L}} K_f = 0$.

Moreover, the functor $B \otimes_A^{\mathbb{L}} -$ is left adjoint to f_*.

Proof. The fact that (1) is equivalent to (2) can be found in [GL, Theorem 4.4]. It is easy to see that (1) is equivalent to (3). Indeed, note that $H^0(B \otimes_A^{\mathbb{L}} f) = B \otimes_A f$ is an isomorphism if and only if f is a ring epimorphism. Also, for $i > 0$, $H^i(B \otimes_A^{\mathbb{L}} f) = Tor_i^A(B, f)$ is the zero map and it is an isomorphism if and only if $H^i(B \otimes_A^{\mathbb{L}} B) = Tor_i^A(B,B) = 0$.

Finally, we check that (3) is equivalent to (4). Consider the triangle in $D(A)$

$$A \xrightarrow{f} B \longrightarrow K_f \longrightarrow A[1]$$

and apply to it the triangle functor $B \otimes_A^{\mathbb{L}} -$. Clearly, $B \otimes_A^{\mathbb{L}} f$ is an isomorphism if and only if $B \otimes_A^{\mathbb{L}} K_f = 0$, thus finishing the proof. □

Homological ring epimorphisms of A play a role in understanding how to *decompose* the derived category $D(A)$ into other triangulated categories. This *decomposition* is formalised by the notion of recollement.

1.5 Recollements

The notion of recollement (in the context of triangulated categories) goes back to [BBD].

Definition 1.5.1. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be triangulated (respectively, abelian) categories. A **recollement** of $\mathcal{D}$ by $\mathcal{X}$ and $\mathcal{Y}$ is a diagram of six triangle (respectively, additive) functors such that*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{X} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, (j^*, j_*) are adjoint pairs;

(2) i_*, j_*, $j_!$ are full embeddings;

(3) $Im(i_*) = ker(j^*)$.

For every idempotent e of a finite dimensional $\mathbb{K}$-algebra A there is an induced recollement

$$\begin{tikzcd} A/AeA\text{-}Mod \arrow[r, "i_*"] & A\text{-}Mod \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & eAe\text{-}Mod \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where the functors involved are $i^* = A/AeA \otimes_A -$, $i^! = Hom_A(A/AeA, -)$, $j_! = Hom_{eAe}(A, -)$, $j_* = A \otimes_{eAe} -$, $j^* = Hom_A(eAe, -)$ and i_* equals the restriction functor associated to $f : A \to A/AeA$ (see [CPS]). In case f is homological, the recollement above can be derived yielding a recollement of $D(A)$ by $D(A/AeA)$ and $D(eAe)$. We also recall the following result from [NS], stating that homological ring epimorphisms, in general, give rise to recollements of triangulated categories.

Theorem 1.5.2. *[NS, §4] Let $f : A \to B$ be a homological ring epimorphism. Then the derived restriction functor f_* induces a recollement*

$$\begin{tikzcd} D(B) \arrow[r, "f_*"] & D(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & Tria(K_f) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $Tria(K_f)$ denotes the smallest triangulated subcategory of $D(A)$ containing K_f and closed under coproducts.

1.6 Universal localisations

The following theorem defines and shows existence of universal localisations.

Theorem 1.6.1. *[Sch, Theorem 4.1] Let A be a ring and Σ a set of maps between finitely generated projective A-modules. Then there is a ring A_Σ, unique up to isomorphism, and a ring homomorphism $f : A \to A_\Sigma$ such that*

(1) $A_\Sigma \otimes_A \sigma$ is an isomorphism of A-modules for all $\sigma \in \Sigma$;

(2) every ring homomorphism $g : A \to B$ such that $B \otimes_A \sigma$ is an isomorphism for all $\sigma \in \Sigma$ factors in a unique way through f, i.e., there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & B \\ & A_\Sigma. \arrow[ur, "\exists ! \tilde{g}"'] & \end{tikzcd}$$

We say that the ring A_Σ in the theorem is the **universal localisation** of A at Σ. It is well-known that the homomorphism $f : A \to A_\Sigma$ is a ring epimorphism with $Tor_1^A(A_\Sigma, A_\Sigma) = 0$. Equivalently, by Proposition 1.3.7, we have that $Ext_A^1(M,N) \cong Ext_{A_\Sigma}^1(M,N)$ for all A_Σ-modules M and N, showing that the category $\mathcal{X}_{A_\Sigma}$ is closed under extensions in A-Mod.

We can also define universal localisations with respect to a certain set of A-modules. Indeed, let $\mathcal{U}$ be a set of finitely presented A-modules of projective dimension less or equal than one. We denote by $A_\mathcal{U}$ the universal localisation of A at $\Sigma = \{\sigma_U | U \in \mathcal{U}\}$, where $\sigma_U : P \to Q$ is a projective resolution of U in A-mod. Note that $A_\mathcal{U}$ is well-defined by [Co] and we will call it the universal localisation of A at $\mathcal{U}$. We give two easy examples of universal localisations.

Example 1.6.2.

- *Let $A \to B$ be the Ore localisation at a multiplicative set $S \subset A$. Then B describes the universal localisation of A at the set $\{ \phi_s : A \xrightarrow{\cdot s} A \mid s \in S\}$. In fact,*

$$B \otimes \phi_s : B \to B$$

is an isomorphism given by right multiplication with an invertible element of B. Moreover, the Ore localisation B is, by definition, the universal ring inverting all the elements in S.

- *Let P be a finitely generated projective A-module. Then the universal localisation of A at $\{P\}$ is given by $A/\tau_P(A)$, where $\tau_P(A)$ denotes the trace ideal of P in A. To see this, first* *note that $A/\tau_P(A) \otimes_A P = 0$. Moreover, whenever there is a ring homomorphism $A \to B$ fulfilling $B \otimes_A P = 0$, then the A-module B belongs to P° and, thus, B carries a natural $A/\tau_P(A)$-module structure by Proposition 1.3.8. Consequently, there is a unique A-module map $A/\tau_P(A) \to B$ making the following diagram commute*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & A/\tau_P(A) \arrow[ur] & \end{tikzcd}$$

In fact, this is a commutative diagram of ring homomorphisms and, thus, $A/\tau_P(A)$ is the universal localisation of A at $\{P\}$.

In case the underlying ring is hereditary, it was shown in [KSt] that the universal localisations are precisely the homological ring epimorphisms. In general, however, such a parametrisation fails. In [K2], one finds a first explicit example of a homological ring epimorphism that is not a universal localisation. Moreover, it was shown in [BS, Theorem 7.8] that over a commutative semihereditary ring A (i.e. submodules of finitely generated projective A-modules are again projective) a ring epimorphism $A \to B$ is a universal localisation if and only if B is flat when viewed as an A-module. To provide examples of homological ring epimorphisms that are not universal localisations one can now pass to Prüfer domains (i.e. semihereditary integral domains). By [BS, Theorem 8.2 and the example afterwards], all homological ring epimorphisms over a Prüfer domain A are universal localisations if and only if the zero ideal is the unique idempotent prime ideal of A. An explicit example of a Prüfer domain with a non-trivial idempotent prime ideal is given by $\mathbb{Z}+X\mathbb{Q}[[X]]$. In fact, the non-maximal prime ideal $X\mathbb{Q}[[X]]$ is idempotent (see [B, Example B]).

Conversely, not all universal localisations yield homological ring epimorphisms (see [NRS]). Further examples for this observation will be provided in the forthcoming chapters of the thesis.

There is an explicit description of the modules over a given universal localisation.

Proposition 1.6.3. *Let $A \to A_\Sigma$ be a universal localisation of A and take a module X in A-Mod. Then the following are equivalent.*

(1) X belongs to $\mathcal{X}_{A_\Sigma}$;

(2) $Hom_A(\sigma, X)$ is an isomorphism for all $\sigma \in \Sigma$;

(3) $Hom_A(\sigma, A) \otimes_A X$ is an isomorphism for all $\sigma \in \Sigma$.

Proof. The first equivalence follows from [CX, Proposition 3.3]. For the second equivalence observe that, by [CX3, Lemma 2.4], there is a natural isomorphism of functors

$$Hom_A(\sigma,-) \cong Hom_A(\sigma,A) \otimes_A -.$$

□

Also note that the definition of universal localisation is left-right symmetric.

Lemma 1.6.4. *[CX, Lemma 3.2] Let Σ be a set of maps in A-$proj$. Now consider the induced set $\Sigma^{op} := \{Hom_A(\sigma,A) \mid \sigma \in \Sigma\}$ of maps between finitely generated projective right A-modules. Then the universal localisation $A_{\Sigma^{op}}$ of A is given by A_Σ.*

1.7 Tilting modules

We recall the definition of (not necessarily finitely generated) tilting modules.

Definition 1.7.1. *An A-module T is said to be* ***tilting*** *if $Gen(T) = T^{\perp_1}$, or equivalently, if T satisfies the following conditions (compare [CT, Proposition 1.3]):*

(T1) the projective dimension of T is less or equal than 1;

(T2) $Ext^1_A(T,T^{(I)}) = 0$ for any set I;

(T3) there is an exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where T_0 and T_1 lie in $Add(T)$ (and so ϕ is a left $Gen(T)$-approximation).

The subcategory $Gen(T)$ is then called a **tilting class**. It is a torsion class containing all the injective modules. Note that we can recover the additive closure of T from $Gen(T)$ by looking at the Ext-projective modules in the tilting class. We say that two tilting modules T and T' are **equivalent** whenever $Add(T) = Add(T')$. We will call a given tilting module **large**, if it is not equivalent to a finitely presented tilting module. Examples of (finitely presented) tilting modules can be found in [ASS] and [APR]. Moreover, large tilting modules are, for example, discussed in [AS], [AS2] and [CT]. In fact, we will see later that examples of (large) tilting modules can be constructed from ring epimorphisms and universal localisations (see Theorem 1.9.2).

The notion of partial tilting module is a weakening of the previous conditions.

Definition 1.7.2. *We say that an A-module T is* ***partial tilting*** *if*

(PT1) $T^{\perp_1}$ *is a torsion class;*

(PT2) T *lies in* $T^{\perp_1}$.

Condition (PT1) implies (T1) in the definition of a tilting module, and it is stronger than (T1) unless T is finitely presented (see [CT, Remark 1.5]). It was also shown in [CT] that every (not necessarily finitely presented) partial tilting module can be completed to a tilting module. This generalises a famous result by Bongartz initially stated for finitely generated tilting modules over a finite dimensional $\mathbb{K}$-algebra.

1.8 τ-tilting modules

Let A be a finite dimensional $\mathbb{K}$-algebra. We denote by τ the Auslander-Reiten translation in the category A-*mod* of finitely generated A-modules. We recall the following definitions from [AIR].

Definition 1.8.1. *A finitely generated A-module T is said to be*

- τ-***rigid*** *if* $Hom_A(T, \tau T) = 0$*;*
- τ-***tilting*** *if it is* τ*-rigid and the number of non-isomorphic indecomposable direct summands of T equals the number of isomorphism classes of simple A-modules;*
- ***support*** τ-***tilting*** *if there is an idempotent element e of A such that T is a* τ*-tilting* A/AeA*-module.*

Using the *Auslander-Reiten duality*, one checks that (finitely generated) tilting modules are τ-tilting and, conversely, that faithful τ-tilting modules are already tilting (see [AIR, Proposition 2.2]). In fact, if A is a hereditary algebra, then τ-tilting A-modules are tilting.

The introduction of τ-tilting modules in [AIR] was motivated by the idea of carrying out tilting theory from the perspective of **mutation**. In [AIR, Theorem 2.18], it was shown that every basic almost complete support τ-tilting module is a direct summand of precisely two basic support τ-tilting modules. This completion defines mutation between support τ-tilting modules and gives rise to a partial order which can be understood by comparing the associated torsion classes (see [AIR, Section 2.4]). More precisely, for two support τ-tilting modules T_1 and T_2 we have $T_1 \leq T_2$ if and only if $gen(T_1) \subseteq gen(T_2)$. Further examples and applications of τ-tilting theory are discussed in the forthcoming chapters of this thesis and can be found in [A], [AIR], [J] and [Mi].

1.9 Tilting modules arising from ring epimorphisms

Certain ring epimorphisms of A induce tilting modules. The next definition is taken from [AS].

Definition 1.9.1. *A tilting A-module T is said to* ***arise from a ring epimorphism****, if there is an injective ring epimorphism $f : A \to B$ such that $B \oplus B/A$ is a tilting A-module equivalent to T. We say that T* ***arises from universal localisation*** *if the epimorphism f is a universal localisation.*

In this definition B is uniquely determined up to epiclasses of A. In fact, if a tilting module T arises from a ring epimorphism $f : A \to B$, there is a canonical coresolution of A by $Add(T)$

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow B/A \longrightarrow 0.$$

The following theorem relates ring epimorphisms and tilting modules.

Theorem 1.9.2. *[AS, Theorem 3.5, Theorem 3.10]*

(1) Let $A \to B$ be an injective homological ring epimorphism where ${}_AB$ has projective dimension at most one. Then $B \oplus B/A$ is a tilting A-module and $\mathcal{X}_B$ equals $(B/A)^{\perp}$.

(2) Let T be a tilting A-module. Then T arises from a ring epimorphism if and only if there is a coresolution of A by $Add(T)$

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

such that $Hom_A(T_1, T_0) = 0$.

An example for this construction is provided by the large tilting $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ arising from the canonical embedding from $\mathbb{Z}$ into $\mathbb{Q}$.

Chapter 2

From ring epimorphisms to universal localisations

This chapter contains joint work with Jorge Vitória to appear in Forum Mathematicum (see [MV]). In the first section, we compare different classes of ring epimorphisms and localisation maps. More precisely, we discuss sufficient conditions for a ring epimorphism to be a universal localisation. As an application, in a second section, we consider recollements induced by some homological ring epimorphisms and investigate whether they yield recollements of derived module categories. In the last section of this chapter, we provide a list of examples illustrating our results. Throughout, we use the following terminology.

Definition 2.0.3. *A ring epimorphism $f : A \to B$ is said to be*

- ***flat**, if f turns B into a flat left A-module;*
- ***finite**, if f turns B into a finitely generated projective left A-module;*
- ***1-finite**, if f turns B into a finitely presented left A-module of projective dimension less or equal than one.*

Clearly, every finite ring epimorphism is flat and 1-finite. Conversely, the following holds.

Proposition 2.0.4. *[Cu, Corollary 1.4] If A is a perfect ring, then a ring epimorphism $f : A \to B$ is flat if and only if it is finite.*

Remark 2.0.5. *For a perfect ring A, a ring epimorphism $A \to B$ is finite if and only if every finitely generated projective left B-module is finitely generated and projective as a left A-module. Equivalently, B is finitely generated as a left A-module and for all M in B-mod its projective cover in A-mod is also a left B-module.*

Consider the following sequence of A-modules given by a ring epimorphism $f : A \to B$

$$0 \longrightarrow ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow coker(f) \longrightarrow 0,$$

which we unfold into two short exact sequences, namely

$$0 \longrightarrow ker(f) \longrightarrow A \xrightarrow{\bar{f}} Im(f) \longrightarrow 0, \tag{2.0.1}$$

$$0 \longrightarrow Im(f) \longrightarrow B \longrightarrow coker(f) \longrightarrow 0. \tag{2.0.2}$$

The following easy observation is needed throughout this chapter.

Corollary 2.0.6. *Let $f : A \to B$ be a ring epimorphism. The following assertions hold.*

(1) $B \otimes_A Im(f) \cong B \otimes_A B \cong B$;

(2) $B \otimes_A ker(f) \cong Tor_1^A(B, Im(f))$;

(3) If $Tor_1^A(B,B) = 0$*, then one gets* $Tor_1^A(B, coker(f)) = 0$.

Proof. To prove (1), consider the commutative diagram given by the epi-mono factorisation of f

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & B \\ & Im(f) \arrow[ur] & \end{tikzcd}$$

and apply to it the functor $B \otimes_A -$. By Proposition 1.3.1, $B \otimes_A f : B \otimes_A A \to B \otimes_A B$ is an isomorphism and, therefore, the induced epimorphism $B \otimes_A \bar{f}$ is also a monomorphism.

The statements (2) and (3) follow from (1) by considering the long exact sequences given by applying the functor $B \otimes_A -$ to the sequences (2.0.1) and (2.0.2), respectively. □

2.1 A sufficient condition for universal localisation

In this section we provide sufficient conditions on a ring epimorphism for it to be a universal localisation. Recall that a **quasi-isomorphism** is a morphism of complexes inducing isomorphisms in the cohomologies.

Proposition 2.1.1. *Let $f : A \to B$ be a ring epimorphism. The following are equivalent.*

(1) There is a quasi-isomorphism from P_f, a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of projective left A-modules, to K_f, the cone of f in $D(A)$;

(2) B is a left A-module of projective dimension less or equal than one.

Moreover, if these conditions hold, B is finitely presented if and only if P_f can be chosen as a complex of finitely generated projective left A-modules.

Proof. (1) $\Rightarrow$ (2): Suppose we have a quasi-isomorphism as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & ker(g) \arrow[r, "k_1"] \arrow[d, "\cong"', "k"] & P_f^{-1} \arrow[r, "g"] \arrow[d, "\pi_2"] & P_f^0 \arrow[r, "c_1"] \arrow[d, "\pi_1"] & coker(g) \arrow[r] \arrow[d, "\cong"', "c"] & 0 \\
0 \arrow[r] & ker(f) \arrow[r, "k_2"] & A \arrow[r, "f"] & B \arrow[r, "c_2"] & coker(f) \arrow[r] & 0.
\end{tikzcd} \tag{2.1.1}$$

Define a complex as follows:

$$0 \longrightarrow P_f^{-1} \xrightarrow{p_1} A \oplus P_f^0 \xrightarrow{p_2} B \longrightarrow 0,$$

$$\begin{aligned} p_1 : P_f^{-1} &\longrightarrow A \oplus P_f^0 \\ x &\mapsto (\pi_2(x), g(x)) \end{aligned} \qquad \begin{aligned} p_2 : A \oplus P_f^0 &\longrightarrow B \\ (y,z) &\mapsto f(y) - \pi_1(z). \end{aligned}$$

A standard diagram chase in (2.1.1) shows that this is a short exact sequence. Hence, B has projective dimension less or equal than one.

(2) $\Rightarrow$ (1): Choose a projective resolution of B of shortest length

$$0 \longrightarrow P_1^B \xrightarrow{h} P_0^B \xrightarrow{\pi} B \longrightarrow 0$$

and consider a Cartan-Eilenberg resolution of K_f given by

$$\begin{tikzcd}
 & P_1^B \arrow[d, "h"] \\
A \arrow[r, "\hat{f}"] \arrow[d, dotted, "id"'] & P_0^B \arrow[d, dotted, "\pi"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

It is well-known (see [W, §5.7]) that there is a quasi-isomorphism from its total complex

$$A \oplus P_1^B \xrightarrow{\hat{f}+h} P_0^B$$

to K_f, thus finishing the proof. $\square$

Remark 2.1.2. *This proposition can be easily generalised to B of any finite projective dimension. Since our focus is on 1-finite ring epimorphisms, it is convenient to keep the statement and proof as above.*

The following theorem shows that certain homological ring epimorphisms can be characterised as universal localisations.

Theorem 2.1.3. *Let $f : A \to B$ be a 1-finite ring epimorphism. Then f is homological if and only if it is a universal localisation.*

Proof. Suppose that f is a universal localisation. Then $Tor_1^A(B,B) = 0$ and, since B is an A-module of projective dimension less or equal than one, f is homological.

Conversely, let P_f be a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of finitely generated projective left A-modules quasi-isomorphic to K_f, which exists by Proposition 2.1.1. Since f is homological, by Proposition 1.4.1, we have

$$0 = B \otimes_A^{\mathbb{L}} K_f \cong B \otimes_A^{\mathbb{L}} P_f = B \otimes_A P_f$$

in $D(A)$, showing that $B \otimes_A g$ is an isomorphism of left A-modules. Therefore, by Theorem 1.6.1, there is a commutative diagram of ring epimorphisms

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_g"'] & & B \\ & A_{\{g\}} \arrow[ur, "h"'] & \end{tikzcd}$$

showing that, in particular, the essential images of the corresponding restriction functors for right modules satisfy, by Proposition 1.3.1,

$$\mathcal{X}_B \subseteq \mathcal{X}_{A_{\{g\}}} \subseteq Mod\text{-}A.$$

In order to prove the reverse inclusion, we will see that $A_{\{g\}} \otimes_A f$ is an isomorphism of left (and right) A-modules. To do so, consider the short exact sequence

$$0 \longrightarrow ker(g) \longrightarrow P_f^{-1} \xrightarrow{\bar{g}} Im(g) \longrightarrow 0 \tag{2.1.2}$$

induced by the map g. Observe that a similar argument to the one in the proof of Corollary 2.0.6(1) shows that $A_{\{g\}} \otimes_A \bar{g}$ is an isomorphism. Using the commutative diagram (2.1.1) given by the quasi-isomorphism from P_f to K_f and applying the functor $A_{\{g\}} \otimes_A -$ to the short exact sequences (2.1.2) and (2.0.1) we get the following diagram of left A-modules

$$\begin{tikzcd}
A_{\{g\}} \otimes_A ker(g) \arrow[r, "0"] \arrow[d, "A_{\{g\}} \otimes_A k"'] & A_{\{g\}} \otimes_A P_f^{-1} \arrow[r, "\cong"] \arrow[d] & A_{\{g\}} \otimes_A Im(g) \arrow[r] \arrow[d] & 0 \\
A_{\{g\}} \otimes_A ker(f) \arrow[r, "A_{\{g\}} \otimes_A k_1"] & A_{\{g\}} \otimes_A A \arrow[r, "A_{\{g\}} \otimes_A \bar{f}"] & A_{\{g\}} \otimes_A Im(f) \arrow[r] & 0.
\end{tikzcd}$$

It shows that, since $A_{\{g\}} \otimes_A k$ is an isomorphism, $A_{\{g\}} \otimes_A k_1 = 0$ and, thus, $A_{\{g\}} \otimes_A \bar{f}$ is an isomorphism. Now, applying the functor $A_{\{g\}} \otimes_A -$ to the sequence (2.0.2), we get

$$Tor_1^A(A_{\{g\}}, coker(f)) \longrightarrow A_{\{g\}} \otimes_A Im(f) \longrightarrow A_{\{g\}} \otimes_A B \longrightarrow 0.$$

In order to compute $Tor_1^A(A_{\{g\}}, coker(f))$, consider a projective resolution of $coker(f)$ of the form

$$\cdots \longrightarrow P^{-2} \xrightarrow{d} P_f^{-1} \xrightarrow{g} P_f^0 \longrightarrow coker(f) \longrightarrow 0$$

and apply to it the functor $A_{\{g\}} \otimes_A -$. By definition, $A_{\{g\}} \otimes_A g$ is an isomorphism and, therefore, the first cohomology of the new complex is zero. This shows precisely that $Tor_1^A(A_{\{g\}}, coker(f)) = 0$ and, thus, using the epi-mono factorisation of f, we can conclude that

$$A_{\{g\}} \otimes_A f : A_{\{g\}} \otimes_A A \to A_{\{g\}} \otimes_A B$$

is an isomorphism of left A-modules. Moreover, it is also an isomorphism of right A-modules. Hence, $A_{\{g\}}$ has a natural right B-module structure, i.e, it lies in $\mathcal{X}_B$. Since $A_{\{g\}}$ is a generator of $\mathcal{X}_{A_{\{g\}}}$, this shows that $\mathcal{X}_{A_{\{g\}}} \subseteq \mathcal{X}_B$ and, thus, $\mathcal{X}_{A_{\{g\}}} = \mathcal{X}_B$. By Theorem 1.3.3, this means that $A_{\{g\}}$ and B lie in the same epiclass of A and, therefore, B is the universal localisation of A at $\{g\}$. $\square$

Remark 2.1.4. *Theorem 2.1.3 can also be derived from independent current work of Chen and Xi by observing that, under our assumptions, the generalised localisation in [CX3, Corollary 3.7] is a universal localisation.*

Remark 2.1.5. *Note that, for a homological 1-finite ring epimorphism $f : A \to B$, the above proof together with the proof of Proposition 2.1.1 explicitly constructs a map g in A-proj with $B = A_{\{g\}}$. Here, g depends only on the choice of a projective resolution of B of shortest length in A-mod.*

In particular, for finite ring epimorphisms we have the following result.

Corollary 2.1.6. *Let $f : A \to B$ be a finite ring epimorphism. Then B is the universal localisation of A at $\{f\}$, where f is seen as a morphism in A-proj.*

Note that the statement in Theorem 2.1.3 is no longer true, when we allow the A-module B to be of projective dimension greater than one.

Example 2.1.7. *Let A be the quotient of the path algebra over $\mathbb{K}$ of the quiver*

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

by the ideal generated by $\beta\alpha$. Note that A is of global dimension two. Consider the universal localisation of A at $\mathcal{U} := \{Ae_2\}$. By Example 1.6.2, $A_{\mathcal{U}}$ is given by the quotient ring A/Ae_2A. Moreover, $A_{\mathcal{U}}$, when viewed as an A-module, decomposes into the direct sum of the simple modules S_1 and S_3. Using that the A-module S_1 is of projective dimension two, it follows that $Tor_2^A(A_{\mathcal{U}}, A_{\mathcal{U}}) \neq 0$ and, hence, the ring epimorphism $A \to A_{\mathcal{U}}$ is not homological.

With further assumptions on the ring epimorphism f, the universal localisation in Theorem 2.1.3 takes a particularly nice form.

Corollary 2.1.8. *Let $f : A \to B$ be a homological 1-finite ring epimorphism. The following holds.*

(1) If f is injective, then $coker(f) = B/A$ is a finitely presented A-module of projective dimension less or equal than one and B is the universal localisation of A at $\{B/A\}$.

(2) If f is surjective, then $ker(f)$ is a finitely presented projective A-module and B is the universal localisation of A at $\{ker(f)\}$.

Moreover, if A is a finite dimensional $\mathbb{K}$-algebra and f is surjective then there is an idempotent e in A such that B and A/AeA lie in the same epiclass of A.

Proof. Let P_f be a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of finitely generated projective left A-modules quasi-isomorphic to K_f, which exists by Proposition 2.1.1.

(1) Since f is injective, g is injective and $coker(f) \cong coker(g)$ is a finitely presented A-module of projective dimension less or equal than one. By Theorem 2.1.3, it follows that B lies in the same epiclass of A as $A_{\{g\}} = A_{\{coker(f)\}}$.

(2) Since f is surjective, g is surjective and thus a split map. It follows that $ker(f) \cong ker(g)$ is a finitely presented projective A-module. Again, by Theorem 2.1.3, we get that B lies in the same epiclass of A as $A_{\{g\}}$, which is easily seen to be the universal localisation $A_{\{ker(f)\}}$.

Moreover, f surjective implies that $ker(f)$ is a two-sided idempotent ideal of A by Proposition 1.3.8. If A is a finite dimensional $\mathbb{K}$-algebra, this ideal is generated by an idempotent e in A. □

2.2 Recollements of derived module categories

We will now use homological 1-finite ring epimorphisms to construct recollements of derived module categories. For two A-modules M and N we denote by $\tau_M(N)$ the **trace** of M in N, i.e., the submodule of N given by the sum of the images of all A-homomorphisms from M to N.

Theorem 2.2.1. *Let $f : A \to B$ be a homological 1-finite ring epimorphism and assume that f fulfils $Hom_A(coker(f), ker(f)) = 0$. Then the derived restriction functor f_* induces a recollement of derived module categories*

$$D(B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(End_{D(A)}(K_f)).$$

Moreover, if f is finite, then there is an isomorphism of rings $End_{D(A)}(K_f) \cong A/\tau_B(A)$.

Proof. By Theorem 1.5.2, we have the following recollement of triangulated categories induced by the derived restriction functor f_*

$$D(B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} Tria(K_f).$$

Since f is 1-finite, by Proposition 2.1.1, K_f is quasi-isomorphic to P_f, a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of finitely generated projective left A-modules, and therefore it is compact in $D(A)$. We will prove that it is exceptional. Recall that (see, for example, [W, Corollary 10.4.7]), for all X in $D(A)$, $Hom_{D(A)}(K_f, X) \cong Hom_{K(A)}(P_f, X)$, where $K(A)$ denotes the homotopy category of complexes of left A-modules. Clearly, for all $i \geq 2$ and $i \leq -2$, we have

$$Hom_{D(A)}(K_f, K_f[i]) \cong Hom_{K(A)}(P_f, K_f[i]) = 0.$$

Since, by assumption, we know that

$$Hom_A(coker(g), ker(f)) \cong Hom_A(coker(f), ker(f)) = 0,$$

we also get

$$Hom_{D(A)}(K_f, K_f[-1]) \cong Hom_{K(A)}(P_f, K_f[-1]) = 0.$$

It remains to show that

$$Hom_{D(A)}(K_f, K_f[1]) \cong Hom_{K(A)}(P_f, K_f[1]) = 0.$$

Note that every element Φ in $Hom_{K(A)}(P_f, K_f[1])$ is uniquely determined by a morphism ϕ in $Hom_A(P_f^{-1}, B)$ which, by Lemma 1.3.4, factors through the $\mathcal{X}_B$-reflection $\psi_{P_f^{-1}}$. This shows that Φ factors through $B \otimes_A P_f$, which is zero in $D(A)$ (see argument in the proof of Theorem 2.1.3). Since $B \otimes_A P_f$ is a two term complex, it is also zero in $K(A)$. Thus, we have $\Phi = 0$ and

$$Hom_{D(A)}(K_f, K_f[i]) = 0, \ \forall i \neq 0.$$

We conclude that K_f is a compact exceptional object in $D(A)$. Therefore, by a result of Keller ([K, Theorem 8.5]), we get a recollement of derived module categories

$$D(B) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} D(A) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} D(End_{D(A)}(K_f)).$$

Suppose now that f is finite and $P_f = K_f$. We will describe $End_{D(A)}(K_f) \cong End_{K(A)}(K_f)$. Note that, for any element a in A, there is a unique morphism in $End_{K(A)}(K_f)$ defined by $k_A(1_A) = a$ and $k_B(1_B) = f(a)$ as in the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "k_A"'] & B \arrow[r] \arrow[d, "k_B"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

This assignment gives rise to a surjective ring homomorphism $\Omega : A \to End_{K(A)}(K_f)$, whose kernel can be described by homotopy. It turns out that an element a in A lies in the kernel of Ω if and only if it exists h in $Hom_A(B,A)$ with $h(1_B) = a$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dl, "h"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

commute. It remains to show that $ker(\Omega) = \tau_B(A)$. It is clear that $ker(\Omega) \subseteq \tau_B(A)$. Conversely, let a be an element in $\tau_B(A)$. Let h be a map in $Hom_A(B,A)$ such that $a = h(b)$ for some $b \in B$. We define a morphism $\tilde{h} \in Hom_A(B,B) \cong End_B(B)$ by mapping 1_B to b. Therefore, $h \circ \tilde{h}$ lies in $Hom_A(B,A)$ and it satisfies $h \circ \tilde{h}(1_B) = a$. Hence, a lies in $ker(\Omega)$, finishing the proof. $\square$

Following [Wi] and [AKL3, Definition 4.7], we say that a ring A is **derived simple** if it does not admit a non-trivial recollement of derived module categories.

Corollary 2.2.2. *If A admits a non-trivial homological 1-finite ring epimorphism $f : A \to B$ which is either injective or surjective, then A is not derived simple.*

Let $f : A \to B$ be a finite ring epimorphism. It is well-known that, as the trace of a projective A-module in A, $\tau_B(A)$ is a two-sided idempotent ideal. In particular, if A is a finite dimensional $\mathbb{K}$-algebra, then $\tau_B(A)$ is generated by an idempotent e, i.e., $\tau_B(A) = AeA$. More precisely, we have the following small lemma.

Lemma 2.2.3. *If A is a finite dimensional $\mathbb{K}$-algebra, B a finitely generated projective A-module and $I := \{e_1, ..., e_n\}$ a complete set of primitive orthogonal idempotents in A, then we have*

$$\tau_B(A) = \sum_{\substack{e_i \in I \\ Ae_i | B}} Ae_iA.$$

Following [CPS, §2.1], for a finite dimensional $\mathbb{K}$-algebra A, we call an idempotent ideal AeA of A **stratifying**, if the associated ring epimorphism $A \to A/AeA$ is homological.

2.3 Examples

In this section we will discuss recollements arising from Theorem 2.2.1 for three classes of homological 1-finite ring epimorphisms. Examples 2.3.1 and 2.3.2 consider the cases of injective and surjective ring epimorphisms, while Proposition 2.3.3 and Example 2.3.5 focus on finite ring epimorphisms which are neither injective nor surjective.

Example 2.3.1. *Let $f : A \to B$ be a 1-finite, homological and injective ring epimorphism. Then, by Corollary 2.1.8, B is the universal localisation of A at $\{B/A\}$ and, by Theorem 1.9.2, the finitely generated A-module $T := B \oplus B/A$ is tilting. Using Theorem 2.2.1, we get the following recollement of derived module categories*

$$\begin{tikzcd} D(B) \arrow[r] & D(A) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & D(End_A(B/A)). \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}$$

Note that B/A is isomorphic to K_f in $D(A)$. If B/A is an A-module of projective dimension one, this recollement is precisely the one induced by the universal localisation $A_{\{B/A\}}$ and by the tilting module T in [AKL, Theorem 4.8].

Indeed, take A to be the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$\begin{tikzcd} 1 \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & 2 \\ & 3 \arrow[ur, "\beta"'] & \end{tikzcd}$$

by the ideal generated by $\beta\alpha$. Consider the map $\gamma^* : P_2 \to P_1$ in A-$proj$ given by multiplication with γ. Using Remark 1.3.5, it is not difficult to see that $A \to A_{\{\gamma^*\}}$ is a 1-finite, homological and injective ring epimorphism and, thus, it yields the recollement

$$\begin{tikzcd} D(A_{\{\gamma^*\}}) \arrow[r] & D(A) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & D(End_{D(A)}(A_{\{\gamma^*\}}/A)). \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}$$

In fact, we can describe explicitly the outer terms of the recollement. On one hand, the universal localisation $A_{\{\gamma^*\}}$ is Morita equivalent to the $\mathbb{K}$-algebra C given by the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

by the ideal generated by $\beta\alpha$. On the other hand, since $A_{\{\gamma^*\}}/A$ is isomorphic to $coker(\gamma^*)^{\oplus 2}$ as a left A-module, we get that $End_{D(A)}(A_{\{\gamma^*\}}/A)$ is isomorphic to $\mathbb{K} \oplus \mathbb{K}$. Moreover, it follows from a case by case analysis, that this recollement is not induced by a stratifying ideal of A.

Example 2.3.2. *Let $f : A \to B$ be a 1-finite, homological and surjective ring epimorphism. Then, by Corollary 2.1.8, $ker(f)$ is a finitely generated projective A-module and $B \cong A/ker(f)$ is the universal localisation of A at $\{ker(f)\}$. Using Theorem 2.2.1, we get the following recollement of* *derived module categories*

$$D(A/ker(f)) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(End_A(ker(f))).$$

Note that we have $K_f \cong ker(f)[1]$ in $D(A)$.

Moreover, if A is a finite dimensional $\mathbb{K}$-algebra then, again by Corollary 2.1.8, B and A/AeA lie in the same epiclass of A, for some idempotent e in A. The above recollement is then the one induced by the stratifying ideal AeA of A, namely

$$D(A/AeA) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(eAe).$$

We now give sufficient conditions for universal localisations to yield finite ring epimorphisms. In what follows, an element $w \neq 0$ of an admissible ideal I of the path algebra of a quiver is called a **relation**, if it is a linear combination of paths with the same source and target such that for any non-trivial factorisation $w = uv$ neither u nor v lie in I. Note that I is generated by its relations.

Proposition 2.3.3. *Let $A = \mathbb{K}Q/I$ be a finite dimensional $\mathbb{K}$-algebra given by a connected quiver Q and an admissible ideal I in $\mathbb{K}Q$. Assume that there are vertices i and j and an arrow $\alpha : i \to j$ in Q such that:*

(1) α is the unique arrow in Q starting at vertex i;

(2) α is the unique arrow in Q ending at vertex j;

(3) there is no relation in I ending at vertex j.

Then the ring epimorphism $f : A \to A_{\{\alpha^*\}}$ is finite, where $\alpha^* : P_j \to P_i$ is the map in A-proj given by multiplication with α. Moreover, f_* induces a recollement of derived module categories

$$D(A_{\{\alpha^*\}}) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(\mathbb{K}).$$

Proof. By our combinatorial assumptions and Lemma 1.3.4, we have the following isomorphism of left A-modules for each indecomposable projective A-module P_k

$$A_{\{\alpha^*\}} \otimes_A P_k \cong \begin{cases} P_k, & k \neq j \\ P_i, & k = j. \end{cases}$$

Using Remark 1.3.5, we conclude that $f : A \to A_{\{\alpha^*\}}$ is a finite ring epimorphism and, when regarded as an A-module homomorphism,

$$f : \bigoplus_k P_k \longrightarrow \bigoplus_k (A_{\{\alpha^*\}} \otimes_A P_k)$$

is given by right multiplication with the square matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \dots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \dots & \\ & & & & & & 1 \end{pmatrix},$$

where α lies in position (j, j).

We now show that $Hom_A(coker(f), ker(f)) = 0$. Clearly, we have

$$coker(f) = coker(\alpha^*) = S_i,$$

$$ker(f) = ker(\alpha^*).$$

Note that f is injective if and only if there is no relation in I starting at vertex i. Now assume that $Hom_A(coker(f), ker(f)) = Hom_A(S_i, ker(\alpha^*)) \neq 0$. Consequently, there is a non-trivial element u in e_iAe_j such that αu is zero in A, a contradiction to condition (3) in the assumptions. Therefore, by Theorem 2.2.1, we get the following recollement of derived module categories

$$D(A_{\{\alpha^*\}}) \rightleftarrows D(A) \rightleftarrows D(A/\tau_{A_{\{\alpha^*\}}}(A)),$$

where, by Lemma 2.2.3, $\tau_{A_{\{\alpha^*\}}}(A)$ is isomorphic to AeA for $e := \sum_{k \neq j} e_k$. Hence, we have

$$A/\tau_{A_{\{\alpha^*\}}}(A) \cong A/AeA \cong \mathbb{K}.$$

□

Remark 2.3.4. *Note that similar conditions to the ones above are considered in [CK, Example 3.6.2], in the setting of expansions of abelian categories. Indeed, they prove that the inclusion functor $X_{A_{\{\alpha^*\}}} \hookrightarrow A$-mod is a right expansion. It is also a left expansion, if the map α^* is injective.*

We provide an application for the previous proposition.

Example 2.3.5. *Let $n \in \mathbb{N}_{>1}$ and A be the quotient of the path algebra over $\mathbb{K}$ of the quiver Q*

$$\begin{tikzcd} & 1 \arrow[r] & 2 \arrow[dr] & \\ n \arrow[ur] & & & 3 \arrow[dl] \\ & \cdots \arrow[ul] & 4 \arrow[l] & \end{tikzcd}$$

by an admissible ideal I which is not a power of the ideal generated by the arrows of Q. Consequently, there are vertices i and j and an arrow $\alpha : i \to j$ in Q such that there is no relation in I ending at vertex j. We can now apply Proposition 2.3.3, yielding the recollement

$$\begin{tikzcd} D(A_{\{\alpha^*\}}) \arrow[r] & D(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(\mathbb{K}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

In particular, A is not derived simple. This conclusion can also be obtained by observing that A admits a stratifying ideal AeA, for some idempotent e in A. Again by assumption, there are vertices r and s and an arrow $\beta : r \to s$ in Q such that there is no relation in I starting at vertex r. Hence, by multiplication with β we get an injective morphism $\beta^* : P_s \to P_r$ and coker$(\beta^*) = S_r$ is of projective dimension 1. Now consider the universal localisation of A at $\mathcal{U} := \{\bigoplus_{k \neq r} P_k\}$. By Example 1.6.2, $A_{\{\mathcal{U}\}}$ is given by A/AeA for $e := \sum_{k \neq r} e_k$. Since $\mathcal{X}_{A_{\{\mathcal{U}\}}} \subseteq A$-mod is equivalent to add$\{S_r\}$, the ring epimorphism $A \to A_{\{\mathcal{U}\}}$ is 1-finite and, hence, homological. We conclude that the idempotent ideal AeA is stratifying and it yields the following recollement of derived module categories

$$\begin{tikzcd} D(\mathbb{K}) \arrow[r] & D(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(eAe). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Note that in many cases the algebra eAe in the above recollement can be chosen to be Morita equivalent to $A_{\{\alpha^*\}}$. For example, let A be the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

by the ideal generated by $\beta\alpha\beta$. On one hand, the finite ring epimorphism $A \to A_{\{\alpha^*\}}$, where $A_{\{\alpha^*\}}$ is Morita equivalent to $\mathbb{K}[x]/x^2$, yields the recollement

$$\begin{tikzcd} D(\mathbb{K}[x]/x^2) \arrow[r] & D(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(\mathbb{K}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

On the other hand, the stratifying ideal Ae_2A induces the recollement

$$D(\mathbb{K}) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(A) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} D(e_2Ae_2),$$

where e_2Ae_2 and $\mathbb{K}[x]/x^2$ are isomorphic as rings.

Chapter 3

Universal localisations and tilting modules

We study universal localisations over finite dimensional algebras as well as their interplay with generalised tilting modules. The focus will be on finite dimensional universal localisations $A \to A_\Sigma$ for which A_Σ is again artinian. After introducing the relevant notions, in the first section, we discuss the specific nature of finite dimensional universal localisations in our setup and we provide a partial answer to the question of which ring epimorphisms are universal localisations. In Section 3.2, we prove that over a hereditary algebra finite dimensional universal localisations are in bijection with (finitely generated) support tilting modules and we discuss some consequences of the established correspondence. Section 3.3 is devoted to universal localisations over Nakayama algebras. Finally, in Section 3.4, we prove a correspondence between universal localisations and support τ-tilting modules over a Nakayama algebra. The results of this chapter are part of [M].

Throughout, A will denote a finite dimensional $\mathbb{K}$-algebra. By A-*ind* we denote the set containing one representative of each isomorphism class of finitely generated indecomposable A-modules. We say that a subcategory $\mathcal{C}$ of A-*mod*, which is closed under direct summands and (finite) direct sums, has a **finite generator**, if there is an A-module T in $\mathcal{C}$ such that for all X in $\mathcal{C}$ there is some $d \in \mathbb{N}$ and a surjection $T^d \to X$. We call a subcategory $\mathcal{C}$ in A-*mod*

- **wide**, if $\mathcal{C}$ is exact abelian and extension-closed;
- **f-wide**, if $\mathcal{C}$ is wide and has a finite generator;
- **bireflective**, if $\mathcal{C}$ is exact abelian and has a finite generator;
- **torsion**, if $\mathcal{C}$ is closed under quotients and extensions;
- **f-torsion**, if $\mathcal{C}$ is torsion and has a finite generator.

The set of all wide (respectively, f-wide, torsion or f-torsion) subcategories of A-*mod* is denoted accordingly. For a finitely generated A-module X we denote by

$$\cdots \longrightarrow P_2^X \xrightarrow{\sigma_1^X} P_1^X \xrightarrow{\sigma_0^X} P_0^X \xrightarrow{\pi^X} X \longrightarrow 0$$

the minimal projective resolution of X in A-*mod*. We are also interested in certain subcategories which are orthogonal to a subcategory $\mathcal{C}$ of A-*mod*, namely

$$^{\perp}\mathcal{C} := \{X \in A\text{-}mod \mid Hom_A(X,\mathcal{C}) = Ext^1_A(X,\mathcal{C}) = 0\};$$

$$\mathcal{C}^{\perp} := \{X \in A\text{-}mod \mid Hom_A(\mathcal{C},X) = Ext^1_A(\mathcal{C},X) = 0\};$$

$$\begin{aligned} {}^{*}\mathcal{C} &:= \{X \in A\text{-}mod \mid Hom_A(X,\mathcal{C}) = Ext^1_A(X,\mathcal{C}) = Hom_A(\sigma_1^X,\mathcal{C}) = 0\} \\ &= \{X \in A\text{-}mod \mid Hom_A(\sigma_0^X,\mathcal{C}) \text{ an isomorphism for all } C \in \mathcal{C}\}; \\ \mathcal{C}^{*} &:= \{X \in A\text{-}mod \mid Hom_A(\mathcal{C},X) = Ext^1_A(\mathcal{C},X) = Hom_A(\sigma_1^{C},X) = 0\} \\ &= \{X \in A\text{-}mod \mid Hom_A(\sigma_0^{C},X) \text{ an isomorphism for all } C \in \mathcal{C}\}. \end{aligned}$$

Note that for ${}^{*}\mathcal{C}$ and $\mathcal{C}^{*}$ to be well-defined, it is actually necessary to consider *minimal* projective resolutions of the corresponding A-modules. If A is hereditary, we have ${}^{\perp}\mathcal{C} = {}^{*}\mathcal{C}$ and $\mathcal{C}^{\perp} = \mathcal{C}^{*}$.

3.1 Universal localisations for finite dimensional algebras

In what follows, we will discuss some properties of universal localisations for finite dimensional $\mathbb{K}$-algebras. It turns out that we can define universal localisations with respect to a set of finitely generated A-modules. Take $\mathcal{U} \subseteq A$-*mod* and denote by $A_{\mathcal{U}}$ the universal localisation of A at the set $\{\sigma_0^X \mid X \in \mathcal{U}\}$. Note that $A_{\mathcal{U}}$ is well-defined, since the minimal projective resolutions are essentially unique. Conversely, if we start with a universal localisation A_Σ of A, we define $\mathcal{U}$ to be the set of cokernels of maps in Σ plus, additionally, the set of projective A-modules which are sent to zero by some map in Σ. It follows that A_Σ and $A_{\mathcal{U}}$ lie in the same epiclass of A, since an arbitrary map between finitely generated projective A-modules $f : P \to Q$ only differs from the minimal projective presentation of its cokernel by a trivial extension. In fact, there are finitely generated projective A-modules P' and Q' fitting into the following commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\cong"] & Q \arrow[r] \arrow[d, "\cong"] & coker(f) =: M \arrow[r] \arrow[d, "id_M"] & 0 \\
P_1^M \oplus Q' \oplus P' \arrow[r, "f'"] & P_0^M \oplus Q' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the map f' is given by the matrix

$$\begin{pmatrix} \sigma_0^M & 0 & 0 \\ 0 & id_{Q'} & 0 \end{pmatrix}.$$

Therefore, universal localisations of A can be defined with respect to a set of finitely generated A-modules. Throughout, we will not distinguish explicitly between localising with respect to a set of maps or a set of modules. However, the meaning of the given set Σ will become clear in the specific context. We call a universal localisation A_Σ of A

- **pure**, if $A_\Sigma \otimes_A Ae \neq 0$ for all idempotents $e \neq 0$ in A;
- ***e*-annihilating**, if $A_\Sigma \otimes_A Ae = 0$ for an idempotent e in A.

The set of all (respectively, all pure, e-annihilating or finite dimensional) universal localisations of A (up to epiclasses) will be denoted by $uniloc(A)$ (respectively, $uniloc^p(A)$, $uniloc_e(A)$ or $fd\text{-}uniloc(A)$). Note that all these sets are partially ordered by inclusion with respect to the essential image of the restriction functor $\mathcal{X}_{A_\Sigma}$. Some of the finite dimensional universal localisations of A are easy to compute. For example, the universal localisation at the projective A-module Ae for some idempotent e in A is given by the quotient ring A/AeA (see Example 1.6.2). In fact, all surjective universal localisations of A are of this form.

Lemma 3.1.1. *Let A be a finite dimensional $\mathbb{K}$-algebra and let $f : A \to B$ be a surjective ring epimorphism with $Tor_1^A(B,B) = 0$. Then there is an idempotent e in A such that B lies in the same epiclass of A as A/AeA.*

Proof. This follows from Proposition 1.3.8. $\square$

Next, we want to use certain pairs of orthogonal subcategories in $A\text{-}mod$ (defined above) to study finite dimensional universal localisations of A. Note that some of the following observations could also be stated for arbitrary universal localisations of A by considering suitable subcategories of $A\text{-}Mod$. Since, later on, we are mainly interested in finitely generated A-modules, we leave this possible generalisation to the reader. Thus, let A_Σ be a finite dimensional universal localisation of A. By Proposition 1.6.3, we know that $\mathcal{X}_{A_\Sigma}$ consists of those (finitely generated) A-modules for which $Hom_A(\sigma, X)$ is an isomorphism for all σ in Σ. It can also be described by Σ^*, if we understand Σ as a suitable set of finitely generated A-modules. Since $\mathcal{X}_{A_\Sigma}$ is closed under extensions in A-*mod*, we get an injective map

$$\omega : fd\text{-}uniloc(A) \longrightarrow f\text{-}wide(A)$$

by mapping A_Σ to $\mathcal{X}_{A_\Sigma} = \Sigma^*$. Now we can ask the following questions:

Question 3.1.2.

(1) How can we describe the image of ω *in* $f\text{-}wide(A)$*?*

(2) For which choices of A is the map ω *bijective?*

We ask for those finite dimensional ring epimorphisms $f : A \to B$ with $Tor_1^A(B, B) = 0$ that can be realised as universal localisations of A. Note that a very first answer is given by Lemma 3.1.1. The following proposition determines a candidate for the (partial) inverse of ω.

Proposition 3.1.3. *Let* $f : A \to B$ *be a finite dimensional ring epimorphism. The following holds.*

(1) ${}^*\mathcal{X}_B = \{X \in A\text{-}mod \mid B \otimes_A \sigma_0^X \text{ an isomorphism}\}$*, i.e.,* ${}^*\mathcal{X}_B$ *describes those finitely generated A-modules whose minimal projective presentation becomes invertible under* $B \otimes_A -$*.*

(2) ${}^*\mathcal{X}_B$ *is closed under finite direct sums, direct summands, extensions and cokernels of injective maps whose cokernel is of projective dimension less or equal to one.*

(3) $({}^*\mathcal{X}_B)^*$ *is a wide subcategory of A-mod with* $\mathcal{X}_B \subseteq ({}^*\mathcal{X}_B)^*$*. Moreover, if f is a universal localisation, then we get* $\mathcal{X}_B = ({}^*\mathcal{X}_B)^*$*.*

Proof. ad(1): Since the functor $B \otimes_A -$ is left adjoint to the restriction functor f_* and f_* induces a full embedding of the associated module categories, we have that $Hom_A(\sigma_0^X, Y)$ is an isomorphism for all Y in $\mathcal{X}_B$ if and only if $Hom_A(B \otimes_A \sigma_0^X, Y)$ is an isomorphism for all Y in $\mathcal{X}_B$. Consequently, if $B \otimes_A \sigma_0^X$ is an isomorphism, then $Hom_A(B \otimes_A \sigma_0^X, Y)$ and, therefore, $Hom_A(\sigma_0^X, Y)$ is an isomorphism for all Y in $\mathcal{X}_B$.

Conversely, let us assume that $Hom_A(\sigma_0^X, Y)$ and, thus, $Hom_A(B \otimes_A \sigma_0^X, Y)$ is an isomorphism for all Y in $\mathcal{X}_B$. It follows that $Hom_A(B \otimes_A X, Y) = 0$ for all Y in $\mathcal{X}_B$ and, hence, we get $B \otimes_A X = 0$. Consequently, the A-module homomorphism

$$B \otimes_A \sigma_0^X : B \otimes_A P_1^X \to B \otimes_A P_0^X$$

is surjective. In fact, it is split surjective, since $B \otimes_A P_1^X$ and $B \otimes_A P_0^X$ are projective B-modules. By assumption, we know that $Hom_A(B \otimes_A \sigma_0^X, B \otimes_A P_1^X)$ is an isomorphism and, therefore, the identity map on $B \otimes_A P_1^X$ must factor through $B \otimes_A \sigma_0^X$, turning $B \otimes_A \sigma_0^X$ into an isomorphism of A-modules. This finishes (1).

ad(2): Since the minimal projective resolution of a direct sum of finitely generated A-modules is given by the direct sum of the minimal projective resolutions of their direct summands, ${}^*\mathcal{X}_B$ is closed under (finite) direct sums and summands. On the other hand, by the Horseshoe Lemma, we know that for a short exact sequence of finitely generated A-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

with X and Z in ${}^*\mathcal{X}_B$, by taking the direct sum of the minimal projective presentations of X and Z, we get a (not necessarily minimal) projective presentation of Y that becomes invertible under the action of $B \otimes_A -$. Consequently, Y belongs to ${}^*\mathcal{X}_B$. Finally, if we assume that in the above sequence X and Y belong to ${}^*\mathcal{X}_B$, by applying the contravariant functor $Hom_A(-,V)$ for V in $\mathcal{X}_B$, we get that $Hom_A(Z,\mathcal{X}_B) = Ext_A^1(Z,\mathcal{X}_B) = 0$. If we further assume that Z is of projective dimension less or equal to one, we can conclude that Z lies in ${}^*\mathcal{X}_B$.

ad(3): We consider the universal localisation of A at ${}^*\mathcal{X}_B$. Then $({}^*\mathcal{X}_B)^*$ describes the finitely generated A-modules over this localisation. Therefore, $({}^*\mathcal{X}_B)^*$ is wide. The inclusion follows from a straightforward verification. Moreover, if f is a finite dimensional universal localisation, we get that $({}^*\mathcal{X}_B)^* = \mathcal{X}_B$ by (1). □

Let us add some remarks to this proposition. For a finite dimensional universal localisation A_Σ of A we call the modules in ${}^*\mathcal{X}_{A_\Sigma}$, according to [Sch3], A_Σ-**trivial**. Clearly, when seen as a set of modules, Σ is contained in ${}^*\mathcal{X}_{A_\Sigma}$ and the localisation A_Σ is given by $A_{{}^*\mathcal{X}_{A_\Sigma}}$. Consequently, a finite dimensional universal localisation of A is uniquely determined by its A_Σ-trivial modules. The partial order on $fd\text{-}uniloc(A)$, given by inclusion of the associated module categories, can be reformulated using these modules. More precisely, for A_{Σ_1} and A_{Σ_2} in $fd\text{-}uniloc(A)$ it follows that $A_{\Sigma_1} \leq A_{\Sigma_2}$ if and only if ${}^*\mathcal{X}_{A_{\Sigma_1}} \supseteq {}^*\mathcal{X}_{A_{\Sigma_2}}$. Besides, since ${}^*\mathcal{X}_{A_\Sigma}$ is closed under direct sums and summands, it is enough to focus on the indecomposable A_Σ-trivial modules. The further closure properties of ${}^*\mathcal{X}_{A_\Sigma}$ can be used to find a minimal subset among these indecomposable modules that still determines the localisation. But, in general, such a set will not be unique.

Concerning Proposition 3.1.3(3), one may consider $A_{{}^*\mathcal{X}_B}$ as the best approximation of B by a universal localisation of A, even though, a priori, it is not clear that $A_{{}^*\mathcal{X}_B}$ is again finite dimensional. In case it is finite dimensional (for example, if A is a representation finite algebra), then B is the universal localisation of A at ${}^*\mathcal{X}_B$ if and only if $\mathcal{X}_B = ({}^*\mathcal{X}_B)^*$. In many situations, this provides an explicit condition to decide whether a certain ring epimorphism is a universal localisation. Next, we will collect some answers to Question 3.1.2. The following statement can be deduced from [KSt, Theorem 6.1] using the language of Proposition 3.1.3.

Proposition 3.1.4. *Let A be a finite dimensional hereditary $\mathbb{K}$-algebra. Then we have a bijection*

$$\omega : fd\text{-}uniloc(A) \longrightarrow f\text{-}wide(A)$$

by mapping A_Σ to $\mathcal{X}_{A_\Sigma} = \Sigma^* = \Sigma^\perp$. The inverse is given by mapping $\mathcal{C}$ in $f\text{-}wide(A)$ to $A_{{}^*\mathcal{C}} = A_{{}^\perp\mathcal{C}}$.

In particular, ω is a bijection for every semisimple finite dimensional $\mathbb{K}$-algebra A. In this case, all universal localisations (up to epiclasses) are of the form A/AeA for e an idempotent in A (see Lemma 3.1.1).

Lemma 3.1.5. *Let A be a finite dimensional and local $\mathbb{K}$-algebra. Then the only finite dimensional ring epimorphisms $A \to B$ (up to epiclasses) with $Tor_1^A(B,B) = 0$ are the identity map on A and the zero map. In particular, for this choice of A the map ω induces a (trivial) bijection.*

Proof. Take a non-zero finite dimensional ring epimorphism $A \to B$ with $Tor_1^A(B,B) = 0$ and let X be an indecomposable A-module in $\mathcal{X}_B$. Since A is local, X is either simple or it admits, via a top-to-socle factorisation, a non-trivial endomorphism with kernel X' that again lies in $\mathcal{X}_B$. Since, in the second case, the length of the A-module X' is smaller than the length of X, by induction, we conclude that the unique simple A-module S belongs to $\mathcal{X}_B$. Thus, using that $\mathcal{X}_B$ is closed under extensions in A-*mod*, it actually contains all finitely generated A-modules and the ring epimorphism $A \to B$ is equivalent to the identity map on A. ☐

Remark 3.1.6. *Certain group algebras allow a classification of the finite dimensional universal localisations along these lines. For example, let A be the group algebra over $\mathbb{K}$ of a finite p-group for a prime p. Then the map*

$$\omega : fd\text{-}uniloc(A) \longrightarrow f\text{-}wide(A)$$

yields a bijection. In fact, if the characteristic of $\mathbb{K}$ equals the prime p, then A is local and we are in the case of Lemma 3.1.5. Otherwise, by Maschke's theorem, the algebra A is semisimple and the claim follows from Proposition 3.1.4.

In Section 3.3, we will obtain a further classification result for Nakayama algebras (see Corollary 3.3.9). Note that some partial answer to Question 3.1.2 can also be given by Theorem 2.1.3, here stated for finite dimensional algebras.

Theorem 3.1.7. *Let A be a finite dimensional $\mathbb{K}$-algebra and $f : A \to B$ be a finite dimensional and homological ring epimorphism such that the projective dimension of ${}_AB$ is at most 1. Then f is a universal localisation. In particular, $\mathcal{X}_B$ belongs to the image of ω and fulfils the condition $\mathcal{X}_B = ({}^*\mathcal{X}_B)^*$.*

For the sake of completeness, we finish the section with two examples of universal localisations of a finite dimensional algebra which are infinite dimensional over the ground field. Note that this phenomena occurs rather frequently, keeping in mind [NRS]. There it was shown that (up to Morita equivalence) every finitely presented algebra appears as the universal localisation of a finite dimensional algebra.

Example 3.1.8. *[NRS, Section 1] Let B be the first Weyl algebra, i.e., B is given as the quotient of $\mathbb{K}< x,y >$ by the two-sided ideal generated by $xy - yx - 1$. In particular, B is infinite dimensional over $\mathbb{K}$. Now consider the bound path algebra A over $\mathbb{K}$ given by the quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & 2 \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & 3 \arrow[r, shift left, "\gamma_1"] \arrow[r, shift right, "\gamma_2"'] & 4 \end{tikzcd}$$

and the two-sided ideal generated by $\gamma_2\beta_1\alpha_1 - \gamma_1\beta_1\alpha_2$ and $\gamma_2\beta_2\alpha_1 - \gamma_1\beta_2\alpha_2 - \gamma_1\beta_1\alpha_1$. Then the universal localisation of A one obtains by inverting the arrows α_1, β_1 and γ_1 is given by the matrix algebra $M_4(B)$. Note that all non-trivial modules over the localisation are infinitely generated over A. Consequently, the example tells us that to check if a universal localisation of a finite dimensional algebra A is finite dimensional, it is not sufficient to see that the finitely generated A-modules over the localisation admit a finite generator.

Remark 3.1.9. *If A is a finite dimensional and hereditary $\mathbb{K}$-algebra, it follows from [KSt, Proposition 4.2] that a universal localisation A_Σ of A is finite dimensional if and only if there is a finitely generated A-module X with $Ext_A^1(X,X) = 0$ such that A_Σ and $A_{\{X\}}$ lie in the same epiclass of A.*

In general, such an A-module X will not exist for a given universal localisation.

Example 3.1.10. *Consider the Kronecker algebra $A = \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K}^2 & \mathbb{K} \end{pmatrix}$ and a quasi-simple regular A-module S. In particular, we have $Ext_A^1(S,S) \neq 0$. It is well-known (see, for example, [Sch2]) that the universal localisation of A at $\{S\}$ is given by the matrix algebra $M_2(\mathbb{K}[x])$, which is clearly infinite dimensional over $\mathbb{K}$. Note that the A-module structure of $M_2(\mathbb{K}[x])$, induced by the ring epimorphism, depends on the choice of S.*

3.2 Tilting modules and universal localisations for hereditary algebras

We begin this section with a small lemma, stated in [Sch4] without a proof.

Lemma 3.2.1. *Let A be a finite dimensional and hereditary $\mathbb{K}$-algebra. Then a universal localisation $A \to A_\Sigma$ is monomorphic if and only if it is pure.*

Proof. First, observe that as a map of A-modules we can write the ring homomorphism $f : A \to A_\Sigma$ in the following form:

$$f : A \to A_\Sigma \otimes_A A$$

$$a \mapsto f(a) \otimes 1_A = 1_{A_\Sigma} \otimes a$$

Now assume that f is monomorphic and suppose there is some idempotent $e \neq 0$ in A with $A_\Sigma \otimes_A Ae = 0$. It follows that $f(Ae) = 1_{A_\Sigma} \otimes Ae = 0$ and, therefore, $Ae \subseteq ker(f)$, a contradiction. Conversely, assume that the localisation A_Σ is pure and suppose that $ker(f) \neq 0$. Take some $x \neq 0$ in $ker(f)$ and consider the left ideal I of A generated by x. Clearly, $I \subseteq ker(f)$. Since A is hereditary, I is a projective left A-module of the form Ae for some idempotent $e \neq 0$ in A. Now it follows that $0 = f(Ae) = 1_{A_\Sigma} \otimes Ae$ and, thus, we get $A_\Sigma \otimes_A Ae = 0$, again yielding a contradiction. $\square$

Note that monomorphic universal localisations $A \to A_\Sigma$ are always pure. But the converse will fail in general (compare Example 3.3.5 and Example 3.4.7).

We need the following notion of support tilting module. A finitely generated A-module T is called **support tilting**, if T is a tilting module over the $\mathbb{K}$-algebra A/AeA for some idempotent e in A. Clearly, all tilting modules are support tilting. The set of isomorphism classes of finitely generated basic tilting (respectively, support tilting) A-modules will be denoted by $tilt(A)$ (respectively, $s\text{-}tilt(A)$). Note that there is a natural way of associating a torsion class to a support tilting module T by considering $gen(T)$. We say that two support tilting A-modules T and T' are **equivalent**, if $gen(T) = gen(T')$. If A is a hereditary $\mathbb{K}$-algebra, we get the following correspondences:

Theorem 3.2.2. *[IT, §2] Let A be a finite dimensional hereditary $\mathbb{K}$-algebra. There are bijections*

$$s\text{-}tilt(A) \to f\text{-}tors(A) \to f\text{-}wide(A)$$

given by mapping a (basic) support tilting module T to $gen(T)$ and a finitely generated torsion class $\mathcal{T}$ to

$$\alpha(\mathcal{T}) := \{X \in \mathcal{T} \mid \forall (g : Y \to X) \in \mathcal{T}, ker(g) \in \mathcal{T}\}.$$

The inverse is given by assigning to a finitely generated wide subcategory $\mathcal{C}$ the torsion class $gen(\mathcal{C})$ and to a finitely generated torsion class $\mathcal{T}$ the (basic) support tilting module T, given by the sum of the indecomposable Ext-projectives in $\mathcal{T}$. Furthermore, the split-projective modules in the torsion class coincide with the projective modules in the wide subcategory.

For an arbitrary finite dimensional $\mathbb{K}$-algebra A these bijections, in general, will fail. In order to get a similar classification of the finitely generated torsion classes, we have to consider τ-tilting modules (see Theorem 3.4.3). In the hereditary case, the following theorem establishes a bijection between support tilting A-modules and finite dimensional universal localisations of A.

Theorem 3.2.3. *Let A be a finite dimensional and hereditary $\mathbb{K}$-algebra.*

(1) There is a bijection

$$\Psi_A : s\text{-}tilt(A) \longrightarrow fd\text{-}uniloc(A)$$

by mapping a support tilting A-module T to $A_{\Sigma_T} := A_{\perp(\alpha(gen(T)))}$. The inverse maps a universal localisation A_Σ to T_Σ, the sum of the indecomposable Ext-projectives in $gen(\Sigma^\perp)$.

(2) Ψ_A restricts to a bijection between

$$tilt(A) \longrightarrow fd\text{-}uniloc^p(A).$$

Moreover, regarding the inverse, T_Σ is equivalent to $A_\Sigma \oplus A_\Sigma/A$.

(3) Ψ_A restricts to a bijection between

$$s\text{-}tilt(A/AeA) \longrightarrow fd\text{-}uniloc_e(A)$$

for $e = e^2$ in A. In particular, if T is equivalent to ${}_A(A/AeA)$, it is mapped to $A_{\Sigma_T} = A/AeA$.

Proof. ad(1): Follows from Theorem 3.2.2 and Proposition 3.1.4.

ad(2): First, take a basic tilting A-module T and let P be an indecomposable projective A-module. We want to show that $Hom_A(P, \alpha(gen(T))) \neq 0$. Since T is tilting, we have a short exact sequence of the form

$$0 \longrightarrow P \xrightarrow{f'} T_0 \longrightarrow T_1 \longrightarrow 0$$

with T_0 and T_1 in $add(T)$. Now suppose that $T_0 \notin \alpha(gen(T))$. Since, by [IT, Proposition 2.15], we know that $\alpha(gen(T))$ is given by

$$\{X \in gen(T) \mid \forall (g : Y \twoheadrightarrow X) \in gen(T), Y \text{ split-projective} : ker(g) \in gen(T)\},$$

there is a split-projective module Z in $gen(T)$ (in fact, Z belongs to $add(T)$) and a surjection $g : Z \twoheadrightarrow T_0$ such that $ker(g) \notin gen(T)$. Since P is projective, we can lift the map f' to get an injective map $h : P \to Z$ with $f' = g \circ h$. But the split-projective modules in $gen(T)$ must also belong to $\alpha(gen(T))$ (see Theorem 3.2.2) and we get that $Hom_A(P, \alpha(gen(T))) \neq 0$. Therefore, P does not lie in ${}^{\perp}(\alpha(gen(T))) = {}^{\perp}\mathcal{X}_{A_{\Sigma_T}}$. It follows that A_{Σ_T} is pure.

Conversely, let A_Σ be a pure and finite dimensional universal localisation of A. By Lemma 3.2.1, $f : A \to A_\Sigma$ is monomorphic and we get the following short exact sequence of A-modules

$$0 \longrightarrow A \xrightarrow{f} A_\Sigma \longrightarrow A_\Sigma/A \longrightarrow 0 .$$

By Theorem 1.9.2, $T'_\Sigma := A_\Sigma \oplus A_\Sigma/A$ is a tilting A-module. Therefore, it suffices to show that $gen(T_\Sigma) = gen(T'_\Sigma)$. This follows from Theorem 3.2.2 and the construction of Ψ_A in (1), since

$$gen(T_\Sigma) \overset{Thm.3.2.2}{=} gen(\alpha(gen(T_\Sigma))) \overset{(1)}{=} gen(\mathcal{X}_{A_\Sigma}) = gen(A_\Sigma) = gen(T'_\Sigma).$$

ad(3): For a given idempotent e in A, a support tilting A-module T belongs to $s\text{-}tilt(A/AeA)$ if and only if T carries the natural structure of an A/AeA-module (i.e., $T \in \mathcal{X}_{A/AeA}$) or, equivalently, T is annihilated by e (i.e., $Hom_A(Ae, T) = 0$). Now first assume that A_Σ is a finite dimensional and e-annihilating universal localisation of A. It follows that all the modules in $\mathcal{X}_{A_\Sigma}$ are annihilated by e and so is the corresponding support tilting A-module, when constructed as in Theorem 3.2.2. Conversely, if a support tilting A-module T is annihilated by the idempotent e, so are the modules in $gen(T)$ and $\alpha(gen(T))$. Hence, Ae lies in ${}^{\perp}(\alpha(gen(T))) = {}^{\perp}\mathcal{X}_{A_{\Sigma_T}}$ and, thus, $A_{\Sigma_T} \otimes_A Ae = 0$.

Finally, if T is equivalent to ${}_A(A/AeA)$, then $gen(T)$ is already abelian and we get

$$\mathcal{X}_{A_{\Sigma_T}} = \alpha(gen(T)) = gen(T) = gen(A/AeA) = \mathcal{X}_{A/AeA}.$$

Therefore, A_{Σ_T} and A/AeA lie in the same epiclass of A. $\square$

Corollary 3.2.4. *Let A be a finite dimensional and hereditary $\mathbb{K}$-algebra. For an idempotent e in A there is a commutative diagram of bijections*

$$\begin{tikzcd} fd\text{-}uniloc_e(A) \arrow[rr, "\Phi_e"] & & fd\text{-}uniloc(A/AeA) \\ & s\text{-}tilt(A/AeA) \arrow[ul, "\Psi_A"] \arrow[ur, "\Psi_{A/AeA}"'] & \end{tikzcd}$$

where Φ_e maps a universal localisation A_Σ of A to the universal localisation of A/AeA at the set

$$^{\perp}\mathcal{X}_{A_\Sigma} \cap \mathcal{X}_{A/AeA}$$

of finitely generated A/AeA-modules. The inverse is given by mapping a universal localisation $(A/AeA)_{\Sigma'}$ of A/AeA to the universal localisation of A at the set $\Sigma' \cup \{Ae\}$ of A-modules.

Proof. On the one hand, by Theorem 3.2.3(3), we can identify the (basic) support tilting A/AeA-modules via Ψ_A with the finite dimensional and e-annihilating universal localisations of A. On the other hand, by applying Theorem 3.2.3(1) to the finite dimensional and hereditary $\mathbb{K}$-algebra A/AeA, the map $\Psi_{A/AeA}$ describes a bijection between the (basic) support tilting A/AeA-modules and the finite dimensional universal localisations of A/AeA. The map Φ_e is now defined as the composition $\Psi_{A/AeA} \circ \Psi_A^{-1}$. The precise assignments follow from the construction. □

Remark 3.2.5. *Note that the correspondence Φ_e in Corollary 3.2.4 can also be established directly without first passing through s-tilt(A/AeA). In fact, for every finite dimensional $\mathbb{K}$-algebra A and an idempotent e in A there is a bijective correspondence between the universal localisations of A/AeA and the e-annihilating universal localisations of A.*[1] *In what follows, however, we want to keep in mind the support tilting modules associated to our specific localisations together with the commutative diagram provided in Corollary 3.2.4.*

Remark 3.2.6. *The inverse of the map Ψ_A in Theorem 3.2.3(1) can also be expressed as follows: Take a finite dimensional universal localisation A_Σ of A. Either A_Σ is already pure and, thus, T_Σ is equivalent to $A_\Sigma \oplus A_\Sigma/A$ or it exists an idempotent e in A such that A_Σ is e-annihilating and the universal localisation $\Phi_e(A_\Sigma)$ of A/AeA is pure. Consequently, by Corollary 3.2.4 and Theorem 3.2.3(2), T_Σ is equivalent to the support tilting A-module $\Phi_e(A_\Sigma) \oplus \Phi_e(A_\Sigma)/(A/AeA)$, where $\Phi_e(A_\Sigma)$, regarded as an A-module, is isomorphic to $_AA_\Sigma$.*

Corollary 3.2.7. *For a finite dimensional and hereditary $\mathbb{K}$-algebra A every tilting A-module (up to equivalence) arises from a universal localisation. In particular, for every tilting A-module T there is a short exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with T_1, T_2 in $add(T)$ and $Hom_A(T_2, T_1) = 0$.*

Proof. Follows from Theorem 3.2.3(2) and Theorem 1.9.2. The short exact sequence

$$0 \to A \to A_{\Sigma_T} \to A_{\Sigma_T}/A \to 0$$

fulfils the wanted properties. □

[1] I would like to thank the anonymous referee of [M] for pointing this out to me.

We will further the comparison between a support tilting A-module and its associated universal localisation. The following proposition tells us how to read off A_{Σ_T} from the A-module T.

Proposition 3.2.8. *Let A be a finite dimensional and hereditary $\mathbb{K}$-algebra and T be a basic support tilting A-module which is tilting over the algebra A/AeA for an idempotent e in A. Then A_{Σ_T} is given by localising at the set of all non split-projective indecomposable direct summands of T in $gen(T)$ and the A-module Ae.*

Proof. By Corollary 3.2.4 and Remark 3.2.6, T is equivalent to the support tilting A-module

$$\Phi_e(A_{\Sigma_T}) \oplus \Phi_e(A_{\Sigma_T})/(A/AeA),$$

induced by the finite dimensional and pure, thus, monomorphic universal localisation $\Phi_e(A_{\Sigma_T})$ of A/AeA. Note that if T is a tilting A-module, then e is zero and Φ_e equals the identity on $fd\text{-}uniloc(A)$. By Corollary 2.1.8, $\Phi_e(A_{\Sigma_T})$ is given by localising with respect to the finitely generated A/AeA-module

$$\Phi_e(A_{\Sigma_T})/(A/AeA)$$

and, thus, by Corollary 3.2.4, A_{Σ_T} is given by localising at the set $\{\Phi_e(A_{\Sigma_T})/(A/AeA), Ae\}$ of A-modules. Consequently, it remains to show that $\Phi_e(A_{\Sigma_T})/(A/AeA)$, viewed as an A-module, describes precisely the non split-projective indecomposable direct summands of T in $gen(T)$. In other words, we have to check that an indecomposable A-module in $add(T)$ is not split-projective in $gen(T)$ if and only if it belongs to $add(\Phi_e(A_{\Sigma_T})/(A/AeA))$. One implication follows from the fact that the split-projective A-modules in $gen(T)$ are precisely given by $add(A_{\Sigma_T}) = add(\Phi_e(A_{\Sigma_T}))$ (see Theorem 3.2.2). For the other implication, we observe that there are no A-homomorphisms from $\Phi_e(A_{\Sigma_T})/(A/AeA)$ to the module $\Phi_e(A_{\Sigma_T})$ (see Corollary 3.2.7) saying that the A-module $\Phi_e(A_{\Sigma_T})/(A/AeA)$ has no indecomposable direct summand which is split-projective in $gen(T)$. □

In particular, if T is a (basic) tilting A-module, then A_{Σ_T} is given by localising at the set of the non split-projective indecomposable direct summands of T in $gen(T)$.

Example 3.2.9. *Let A be a finite dimensional basic and hereditary $\mathbb{K}$-algebra with a sink in the underlying quiver. Let S be a simple and projective A-module which is not injective. We write ${}_AA$ as a direct sum $P \oplus S$ of projective A-modules. By τ we denote the usual Auslander-Reiten translation. Then the A-module*

$$T := \tau^{-1}S \oplus P$$

is tilting, following [APR]. T is usually called an APR-tilting module. Using Proposition 3.2.8, we conclude that the associated universal localisation A_{Σ_T} of A is given by $A_{\{\tau^{-1}S\}}$.

In the last part of this section we will discuss how the notion of mutation for support tilting modules or, more precisely, the induced partial order (given by inclusion of the associated torsion classes) translates to the set of universal localisations. Again, by A we denote a finite dimensional and hereditary $\mathbb{K}$-algebra. It is not hard to see that the partial order on $s\text{-}tilt(A)$ is finer than the natural partial order on $fd\text{-}uniloc(A)$. In fact, if A_{Σ_1} and A_{Σ_2} are finite dimensional universal localisations of A with $A_{\Sigma_1} \leq A_{\Sigma_2}$, then we have $\mathcal{X}_{A_{\Sigma_1}} \subseteq \mathcal{X}_{A_{\Sigma_2}}$ and, therefore, $gen(\mathcal{X}_{A_{\Sigma_1}}) \subseteq gen(\mathcal{X}_{A_{\Sigma_2}})$, showing that for the associated support tilting modules T_{Σ_1} and T_{Σ_2} it follows $T_{\Sigma_1} \leq T_{\Sigma_2}$. However, the following easy example illustrates that the converse, in general, does not hold true.

Example 3.2.10. *Consider the path algebra $A := \mathbb{K}(1 \to 2)$ and the two support tilting A-modules $T_1 := P_1 \oplus S_1$ and $T_2 := S_1$, which are clearly mutations of each other. We have $T_2 \leq T_1$. But the associated universal localisations $A_{\Sigma_{T_1}}$ and $A_{\Sigma_{T_2}}$ are not related. In fact, $A_{\Sigma_{T_1}}$ is the universal localisation of A at $\{S_1\}$ and $A_{\Sigma_{T_2}}$ is the localisation at $\{P_2\}$. We can also compare the Hasse quivers for the different partial orders on $s\text{-}tilt(A)$ and $fd\text{-}uniloc(A)$. The first one is given by*

$$\begin{tikzcd}
 & P_1 \oplus S_1 \arrow[r] & S_1 \arrow[dr] & \\
P_1 \oplus P_2 \arrow[ur] \arrow[dr] & & & 0 \\
 & P_2 \arrow[urr] & &
\end{tikzcd}$$

In the Hasse quiver for the natural partial order on $fd\text{-}uniloc(A)$ the universal localisations of A are indicated by the corresponding indecomposable A_Σ-trivial modules (see Section 3.1).

$$\begin{tikzcd}
 & \{S_1\} \arrow[dr] & \\
\{0\} \arrow[ur] \arrow[r] \arrow[dr] & \{P_1\} \arrow[r] & \{P_1, P_2, S_1\} \\
 & \{P_2\} \arrow[ur] &
\end{tikzcd}$$

3.3 Nakayama algebras and universal localisations

In this section, we classify the universal localisations of a Nakayama algebra A by certain subcategories of A-*mod*. More precisely, we show that the map ω discussed in Section 3.1 is bijective (see Question 3.1.2). Note that for Nakayama algebras all universal localisations are finite dimensional and all (relevant) subcategories of A-*mod* have a finite generator, since A is representation finite (see Proposition 3.3.1). Throughout, we denote by $l(X)$ the **Loewy length** of a finitely generated A-module X and by $rad(X)$ its **radical**. We first recall the definition of a Nakayama algebra. A finite dimensional $\mathbb{K}$-algebra A is called **Nakayama** if every indecomposable projective A-module and every indecomposable injective A-module is uniserial. The following well-known result helps to understand the representation theory of A.

Proposition 3.3.1. *[ASS, Theorem V.3.5] Let A be a Nakayama algebra and M an indecomposable A-module. Then it exists an indecomposable projective A-module P and a positive integer t with* $1 \leq t \leq l(P)$ *such that* $M \cong P/rad^t(P)$. *In particular, A is representation finite and every indecomposable A-module is uniserial.*

We want to realise Nakayama algebras as bound path algebras. Consider for $n \in \mathbb{N}$ the quivers

$$\Delta_n := \; 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

$$\tilde{\Delta}_n := \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n. \arrow[llll, bend left] \end{tikzcd}$$

It is a well-known fact that a basic and connected $\mathbb{K}$-algebra A is a Nakayama algebra if and only if A is isomorphic to a quotient $\mathbb{K}Q_A/I$, where Q_A is a quiver of the form Δ_n or $\tilde{\Delta}_n$ and I is an admissible ideal of $\mathbb{K}Q_A$. Moreover, A is a self-injective Nakayama algebra not isomorphic to the ground field if and only if $Q_A = \tilde{\Delta}_n$ and the admissible ideal I is a power of the arrow ideal of $\mathbb{K}Q_A$ (see [ASS, Chapter V.3]).

Later on, the following Nakayama algebras will play an important role

$$A_n^h := \mathbb{K}\Delta_n/R^h \qquad \text{and} \qquad \tilde{A}_n^h := \mathbb{K}\tilde{\Delta}_n/R^h,$$

where $h \in \mathbb{N}_{>1}$ and R denotes the arrow ideal of the associated path algebra. The following lemma will be useful throughout.

Lemma 3.3.2. *[D, Lemma 2.2.2] Let A be a Nakayama algebra and X_1 and X_2 be indecomposable A-modules. If*

$$0 \longrightarrow X_1 \longrightarrow Y \longrightarrow X_2 \longrightarrow 0$$

is a non-split short exact sequence of A-modules, then Y has at most two indecomposable direct summands Y_1 and Y_2 and the short exact sequence is of the form

$$\begin{tikzcd} & & Y_1 \arrow[dr] & & & \\ 0 \arrow[r] & X_1 \arrow[ur, hook] \arrow[dr] & & X_2 \arrow[r] & 0. \\ & & Y_2 \arrow[ur, hook] & & & \end{tikzcd}$$

Next, we want to understand universal localisations of Nakayama algebras.

Lemma 3.3.3. *Let A be a Nakayama algebra and A_Σ be a universal localisation of A. Then also A_Σ is a Nakayama algebra.*

Proof. First of all, by Proposition 3.3.1 and Lemma 1.3.2, A_Σ is a finite dimensional and representation finite $\mathbb{K}$-algebra. Now let X be an indecomposable projective or injective A_Σ-module. Since, via restriction, ${}_AX$ is an indecomposable (not necessarily projective or injective) A-module, it is uniserial by Proposition 3.3.1. Consequently, X is uniserial as an A_Σ-module. $\square$

Remark 3.3.4. *In general, a universal localisation A_Σ of a basic and connected Nakayama algebra A is neither basic nor connected.*

Example 3.3.5. *Consider the Nakayama algebra $A := A_3^2$ and the universal localisation of A at $\Sigma := \{S_2\}$, which one obtains by inverting the arrow $2 \to 3$ in the quiver Δ_3. The A-module ${}_AA_\Sigma$ is five-dimensional of the form $P_2^{\oplus 2} \oplus S_1$ and the algebra A_Σ is Morita-equivalent to $\mathbb{K} \times \mathbb{K}$. In particular, A_Σ is neither basic nor connected.*

In order to classify universal localisations for Nakayama algebras we use some of the methods developed in Section 3.1. By Proposition 3.1.3, we know that a universal localisation A_Σ of an algebra A is determined by the indecomposable A-modules in ${}^*\mathcal{X}_{A_\Sigma}$. In the Nakayama case we can be more precise. We will consider a minimal and explicitly given set of indecomposable A_Σ-trivial modules which determines the localisation. Let n be the number of non-isomorphic simple A-modules, $i \in \{1,...,n\}$ and P_i be the corresponding indecomposable projective A-module. Then we define X_i^Σ to be $P_i/rad^{t_i}(P_i)$ for $t_i \geq 0$ minimal such that $P_i/rad^{t_i}(P_i)$ lies in ${}^*\mathcal{X}_{A_\Sigma}$, whenever such t_i exists. By convention, we define $P_i/rad^0(P_i)$ to be P_i. The set of all the indecomposable X_i^Σ is denoted by W_Σ.

Lemma 3.3.6. *Let A be a Nakayama algebra and A_Σ be a universal localisation of A. Then A_Σ is uniquely determined by the set W_Σ, i.e., A_Σ and A_{W_Σ} lie in the same epiclass of A.*

Proof. We have to show that ${}^*\mathcal{X}_{A_\Sigma}$ equals ${}^*\mathcal{X}_{A_{W_\Sigma}}$. However, one of the inclusions is immediate. Thus, let X be an indecomposable A-module in ${}^*\mathcal{X}_{A_\Sigma}$ and take the minimal projective presentation $P_1^X \to P_0^X$ of X in A-mod. We can assume that X is not projective (otherwise it would already belong to W_Σ). Then either P_0^X belongs to W_Σ or there is $j \in \{1,...,n\}$ such that X surjects onto $X_j^\Sigma = P_j/rad^{t_j}(P_j)$ for some $t_j \geq 1$. In the first case, since X is in ${}^*\mathcal{X}_{A_\Sigma}$, also P_1^X has to be in W_Σ. Consequently, by definition, X belongs to ${}^*\mathcal{X}_{A_{W_\Sigma}}$. In the second case, we get a short exact sequence

$$0 \longrightarrow ker(\pi) \longrightarrow X \xrightarrow{\pi} X_j^\Sigma \longrightarrow 0$$

and it follows, by comparing minimal projective presentations, that also $ker(\pi)$ lies in ${}^*\mathcal{X}_{A_\Sigma}$. By Proposition 3.1.3(2), ${}^*\mathcal{X}_{A_{W_\Sigma}}$ is closed under extensions and, therefore, X lies in ${}^*\mathcal{X}_{A_{W_\Sigma}}$ if and only if $ker(\pi)$ belongs to ${}^*\mathcal{X}_{A_{W_\Sigma}}$. Now we can repeat the whole argument with $ker(\pi)$ instead of X and, since the length of $ker(\pi)$ is smaller than the length of X, we are done after finitely many steps. □

Note that a non-projective A-module X_i^Σ in W_Σ represents the "shortest" non-trivial morphism, from an indecomposable projective A-module P_j to the module P_i, which becomes invertible after tensoring with A_Σ. By "shortest" we mean a minimal number of factorisations through other indecomposable A-modules. The classification of the universal localisations of A will work via the notion of orthogonal collections. We call a set of A-modules $\{X_1,...,X_s\}$ an **orthogonal collection**, if $End_A(X_i) \cong \mathbb{K}$ for all i and $Hom_A(X_i,X_j) = 0$ for all $i \neq j$. Since A is a Nakayama algebra, we clearly have that $s \leq n$ and that $s = n$ already implies that all X_i are simple A-modules. The following proposition can be deduced from [D].

Proposition 3.3.7. *[D, Proposition 2.2.8, Theorem 2.2.10, §2.6] Let A be a Nakayama algebra. There is a bijection between the wide subcategories and the isomorphism classes of orthogonal collections in A-mod by mapping a wide subcategory $\mathcal{C}$ to the set of $\mathcal{C}$-simple A-modules.*

Now we are able to state the main result of this section.

Theorem 3.3.8. *Let A be a Nakayama algebra. There is a bijection between the universal localisations of A (up to epiclasses) and the isomorphism classes of orthogonal collections in A-mod.*

Proof. Without loss of generality, we may assume A to be connected. Let A_Σ be a universal localisation of A. By Lemma 3.3.6, A_Σ is uniquely determined by the set W_Σ. The general idea of the proof is to deform W_Σ uniquely into an orthogonal collection of A-modules and to show that every orthogonal collection of A-modules occurs in this way. In a first step, we list and prove five important properties of W_Σ:

(1) For all non-projective X in W_Σ, we have $l(X) \leq n-1$.

(2) Composing minimal projective presentations of modules in W_Σ never gives an endomorphism.

(3) For all non-projective X in W_Σ, the minimal projective presentation σ_0^X of X does not factor through any projective A-module in W_Σ.

(4) The minimal projective presentations of two different non-projective modules in W_Σ cannot have the same domain.

(5) The minimal projective presentation of a non-projective A-module in W_Σ factors properly through the projective cover P_0^X of a non-projective A-module X in W_Σ if and only if it factors through P_1^X.

ad(1): Suppose that $l(X) \geq n$. Consequently, the minimal projective presentation of X

$$\sigma_0^X : P_1^X \to P_0^X$$

factors through a non-trivial endomorphism α of P_1^X (respectively, through a non-trivial endomorphism of P_0^X). Since X belongs to ${}^*\mathcal{X}_{A_\Sigma}$, the morphism σ_0^X becomes invertible after tensoring with A_Σ and, thus, also does α. Note that $A_\Sigma \otimes_A \alpha$ is not zero by assumption. This leads to a contradiction, since A is a finite dimensional algebra and, therefore, all non-trivial endomorphisms of indecomposable projective A-modules must be nilpotent. ad(2): Follows from the arguments used in the proof of (1). ad(3): Suppose the map σ_0^X factors through some P_j in W_Σ. Since P_j is getting annihilated by tensoring with A_Σ while the map σ_0^X becomes invertible at the same time, it follows that also the projective cover P_0^X of X gets annihilated and, thus, P_0^X must belong to ${}^*\mathcal{X}_{A_\Sigma}$. Therefore, by the definition of W_Σ, X is projective, contradicting our assumption. ad(4): Suppose that the negation of (4) holds for two non-projective A-modules X_1 and X_2 in W_Σ. Hence, without loss of generality, we can assume that the minimal projective presentation of X_1

$$\sigma_0^{X_1} : P_1^{X_1} \to P_0^{X_1}$$

factors non-trivially through $P_0^{X_2}$, the projective cover of X_2. Since X_1 and X_2 belong to ${}^*\mathcal{X}_{A_\Sigma}$, also the induced map from $P_0^{X_2}$ to $P_0^{X_1}$ becomes invertible under the action of $A_\Sigma \otimes_A -$, contradicting the minimality of X_1 in the definition of W_Σ. ad(5): Let X_1 and X_2 be two non-projective A-modules in W_Σ such that the minimal projective presentation of X_1 factors properly through $P_0^{X_2}$, the projective cover of X_2. Now suppose that condition (5) is not fulfilled. Thus, we get the following commutative diagram of A-modules

$$\begin{tikzcd}
 & P_1^{X_1} \arrow[rr, "\sigma_0^{X_1}"] \arrow[dr, "f_2"] & & P_0^{X_1} \\
P_1^{X_2} \arrow[ur, "f_1"] \arrow[rr, "\sigma_0^{X_2}"] & & P_0^{X_2} \arrow[ur, "f_3"'] &
\end{tikzcd}$$

Since X_1 and X_2 belong to ${}^*\mathcal{X}_{A_\Sigma}$, also the cokernels of the f_i lie in ${}^*\mathcal{X}_{A_\Sigma}$, again contradicting the minimality of X_1 and X_2 in the definition of W_Σ. Since the argument is symmetric, the reverse implication follows.

In explicit terms, the above properties guarantee that two minimal projective presentations represented by non-projective A-modules in W_Σ, when seen as arcs on a line or on a circle, respectively, are either completely separated, consecutive or they cover each other properly. The projective A-modules in W_Σ can be seen as uncovered and unattached points in this picture. Moreover, conditions (1) and (2) put restrictions on the length of these arcs as well as on the length of their possible chains. It follows that every set $\mathcal{X} := \{X_1, ..., X_s\}$ of indecomposable A-modules (up to isomorphism), fulfilling the above properties, such that every X_i is a quotient of a different indecomposable projective A-module, equals the set $W_\mathcal{X}$, induced by the universal localisation $A_\mathcal{X}$.

Next, we will modify W_Σ to get another set $\tilde{W}_\Sigma$ of indecomposable A-modules. In fact, whenever there is a maximal subset $\{X_{i_j}\} \subseteq W_\Sigma$ such that the minimal projective presentations of the pairwise different X_{i_j} form a non-trivial chain of the form

$$\sigma_0^{X_{i_1}} \circ ... \circ \sigma_0^{X_{i_l}} =: \sigma^*$$

for $j \in \{1, ..., l\}$, we replace X_{i_1} by $\tilde{X}_{i_1} := coker(\sigma^*)$ and X_{i_j}, for $j \neq 1$, by $\tilde{X}_{i_j} := P_0^{X_{i_j}}$, the projective cover of X_{i_j} in A-mod. In other words, we replace a maximal chain of consecutive morphisms, each represented by a non-projective A-module in W_Σ, by a long composition and we add indecomposable projective A-modules in-between, which no longer belong to ${}^*\mathcal{X}_{A_\Sigma}$. Note that all non-projective $\tilde{X}_i$ in $\tilde{W}_\Sigma$ still belong to ${}^*\mathcal{X}_{A_\Sigma}$. Also, we can get back W_Σ from $\tilde{W}_\Sigma$ by reversing the above process: for a non-projective $\tilde{X}_i$ in $\tilde{W}_\Sigma$, we consider its minimal projective presentation and all projective modules $\tilde{X}_{i_j}$ in $\tilde{W}_\Sigma$ for $j \in \{1,...,l\}$ such that this presentation factors through $\tilde{X}_{i_j}$. Say we have the commutative diagram

$$\begin{tikzcd} P_1^{\tilde{X}_i} \arrow[r, "\sigma^{\tilde{X}_i}"] \arrow[d, "f_l"] & P_0^{\tilde{X}_i} \\ \tilde{X}_{i_l} \arrow[r, "f_{l-1}"] & \cdots \arrow[r, "f_1"] & \tilde{X}_{i_1} \arrow[u, "f_0"'] \end{tikzcd}$$

Now we replace $\tilde{X}_i$ by the cokernel of f_0 and $\tilde{X}_{i_j}$ by the cokernel of f_j for $j \in \{1,...,l\}$. If we start with $\tilde{X}_i$ being of minimal length and continue afterwards by only considering the projective modules in $\tilde{W}_\Sigma$ that have not appeared before, we get back W_Σ from $\tilde{W}_\Sigma$. We conclude that the universal localisation A_Σ is uniquely determined by the set $\tilde{W}_\Sigma$. Moreover, $\tilde{W}_\Sigma$ fulfils the following properties, induced by W_Σ:

(1') For all non-projective X in $\tilde{W}_\Sigma$, we have $l(X) \leq n-1$.

(2') Minimal projective presentations of modules in $\tilde{W}_\Sigma$ can never be composed.

(3') The minimal projective presentations of two different non-projective modules in $\tilde{W}_\Sigma$ cannot have the same domain.

(4') The minimal projective presentation of a non-projective A-module in $\tilde{W}_\Sigma$ factors properly through the projective cover P_0^X of a non-projective A-module X in $\tilde{W}_\Sigma$ if and only if it factors through P_1^X.

In explicit terms, the minimal projective presentations represented by the A-modules in $\tilde{W}_\Sigma$ are either completely separated or they cover each other properly, when seen as arcs and loops on a line or on a circle. By $\mathbf{W}$ we denote the set of all isomorphism classes of sets $\{X_1,...,X_s\}$ of indecomposable A-modules with $s \leq n$, fulfilling the properties (1'), (2'), (3') and (4'), where every X_i is a quotient of a different indecomposable projective A-module. We get a bijection

$$uniloc(A) \to \mathbf{W}$$

by mapping a universal localisation A_Σ to $\tilde{W}_\Sigma$. We already stated injectivity. Surjectivity follows from reversing the idea on how to pass from W_Σ to $\tilde{W}_\Sigma$ and previous observations.

It remains to prove the bijective correspondence between $\mathbf{W}$ and the isomorphism classes of all orthogonal collections in A-*mod*. We consider a bijection Φ on A-*ind* given by mapping an indecomposable projective A-module P to its simple top and an indecomposable non-projective A-module $P/rad^t(P)$ to $P/rad^{t+1}(P)$ for $1 \leq t < l(P)$. We claim that Φ induces a bijection between

W and the isomorphism classes of all orthogonal collections in A-*mod* by mapping $\{X_1,...,X_s\}$ in **W** to $\{\Phi(X_1),...,\Phi(X_s)\}$. Let us first check that the assignment yields a well-defined map. Clearly, the empty set in **W** corresponds to the trivial orthogonal collection. Moreover, using property (1'), we know that the length of the $\Phi(X_i)$ is bounded by n such that $End_A(\Phi(X_i))$ is isomorphic to $\mathbb{K}$. Now let Q_A be the underlying quiver of A. Since A is connected, so is Q_A. We number the vertices of Q_A from 1 to n. If X_i is not projective, then the z-th entry of the dimension vector of $\Phi(X_i)$ is given as follows:

$$(dim\Phi(X_i))_z = \begin{cases} 1, & \text{if } \sigma_0^{X_i} : P_1^{X_i} \to P_0^{X_i} \text{ factors through } P_z \\ 0, & \text{else} \end{cases}$$

Recall that $\Phi(X_i)$ is simple in case X_i is projective. Now consider $Hom_A(\Phi(X_i),\Phi(X_j))$ for $i \neq j$. By construction, there are no surjective maps from $\Phi(X_i)$ to $\Phi(X_j)$. Keeping in mind the shape of the dimension vector, by property (2') and (3'), there also cannot be any injective morphisms. Orthogonality finally follows using property (4'). Consequently, Φ induces a well-defined map from **W** to the set of all isomorphism classes of orthogonal collections in A-*mod*. Moreover, this map is injective, since Φ is a bijection on A-*ind*. It remains to prove surjectivity. Take an arbitrary orthogonal collection $\mathcal{X} := \{X_1,...,X_s\}$ in A-*mod*. Clearly, every X_i is a quotient of a different indecomposable projective A-module and we have $s \leq n$. Now we apply the obvious inverse Φ^{-1} of Φ to get the set

$$\Phi^{-1}(\mathcal{X}) := \{\Phi^{-1}(X_1),...,\Phi^{-1}(X_s)\}$$

of indecomposable A-modules. We have to show that $\Phi^{-1}(\mathcal{X})$ belongs to **W**. Since $End_A(X_i) \cong \mathbb{K}$, we know that $\Phi^{-1}(\mathcal{X})$ fulfils property (1'). The properties (2'), (3') and (4') follow from the orthogonality of the X_i. This finishes the proof. □

We have the following immediate corollaries.

Corollary 3.3.9. *Let A be a Nakayama algebra. There is a bijection*

$$\omega : uniloc(A) \longrightarrow wide(A)$$

by mapping a universal localisation A_Σ to $\mathcal{X}_{A_\Sigma} = \Sigma^*$. The inverse is given by mapping a wide subcategory $\mathcal{C}$ of A-mod to $A_{*\mathcal{C}}$. Thus, a ring epimorphism $A \to B$ is a universal localisation if and only if $Tor_1^A(B,B) = 0$.

Proof. Combining Proposition 3.3.7 and Theorem 3.3.8, we get a bijective correspondence between the epiclasses of the universal localisations of A and the wide subcategories of A-*mod*.

However, it follows from the construction that this bijection is not given by ω. But since both of the sets are finite and since, by Section 3.1, ω already defines an injective map, we are done. $\square$

Corollary 3.3.10. *Consider the Nakayama algebras A_n^h and $\tilde{A}_n^h$ for $h > 1$. The following holds.*

(1) There is a bijective correspondence between the epiclasses of the universal localisations of A_n^h and the possible configurations of non-crossing arcs with length at most the minimum of $\{n-1, h-1\}$ on a line with n linearly ordered points. By convention, a loop has length zero.

(2) There is a bijective correspondence between the epiclasses of the universal localisations of $\tilde{A}_n^h$ and the possible configurations of non-crossing arcs with length at most the minimum of $\{n-1, h-1\}$ on a circle with n linearly ordered points. Again, a loop is considered to have length zero.

Proof. On the one hand, part two of the corollary can be deduced from [D, Corollary 2.6.12] combined with Theorem 3.3.8. On the other hand, the whole statement follows from a careful analysis of the proof of Theorem 3.3.8. More precisely, the set $\mathbf{W}$ in the proof corresponds naturally to the wanted set of configurations of non-crossing arcs. Indeed, for a fixed set of indecomposable A-modules $\mathcal{X}$ in $\mathbf{W}$ we draw an arc from j to i (respecting the given orientation on the points), whenever the cokernel of the map $P_i \to P_j$ belongs to $\mathcal{X}$. Moreover, a projective A-module P_k in $\mathcal{X}$ gives rise to a loop at the point k. $\square$

The previous discussion allows us to count the universal localisations of A_n^h and $\tilde{A}_n^h$. In [D, Section 2.6], this was done with respect to the orthogonal collections in the module category. Note that for $\tilde{A}_n^h$ with $h \geq n$ its number is given by $\binom{2n}{n}$, independent of the choice of h.

By Corollary 3.3.9, we already know that all homological ring epimorphisms of A are universal localisations. But the converse is far from being true, as the following example illustrates.

Example 3.3.11. *Consider the Nakayama algebra $A := A_n^h$ for $n \geq 3$ and $2 \leq h < n$. Take x and $r < h$ in $\mathbb{N}_0$ with $n = xh + r$. By [HS, Proposition 1.4], the global dimension of A is given by*

$$gldim(A) = pd(S_1) = \begin{cases} 2x-1, & \text{if } r = 0 \\ 2x, & \text{if } r = 1 \\ 2x+1, & \text{else} \end{cases}$$

Now let P be the projective A-module appearing last in the minimal projective resolution of S_1. Note that P is indecomposable and we have $P \neq P_1, P_2$. By $\mathcal{C}_{S_1}$ we denote the wide and semisimple *subcategory $add(P \oplus S_1)$ of A-mod. By construction, we clearly have $Ext^i_A(S_1, P) \neq 0$ for some $i > 1$. It follows that the universal localisation $A_{*\mathcal{C}_{S_1}}$ is not homological, since for all $i \geq 1$*

$$Ext^i_{A_{*\mathcal{C}_{S_1}}}(S_1, P) = 0.$$

In the minimal case, for $n = 3$ and $h = 2$, the universal localisation $A_{*\mathcal{C}_{S_1}}$, here given by A/Ae_2A, is the unique universal localisation of A not yielding a homological ring epimorphism. In general, we get plenty of those by localising at certain subsets of ${}^*\mathcal{C}_{S_1}$. For example, if $n > 3$ the ring epimorphisms $A \to A/Ae_2A$ and $A \to A/Ae_3A$ and $A \to A/A(e_2 + e_3)A$ are never homological.

3.4 τ-tilting modules and universal localisations for Nakayama algebras

In this section, we will prove a similar result to Theorem 3.2.3 for Nakayama algebras, now using τ-tilting modules. Let A be a Nakayama algebra. The first step will be to compare the torsion classes and the wide subcategories in A-*mod*. By [IT, Proposition 2.12], we know that for any $\mathcal{T}$ in $tors(A)$ the subcategory

$$\alpha(\mathcal{T}) := \{X \in \mathcal{T} \mid \forall (g : Y \to X) \in \mathcal{T}, ker(g) \in \mathcal{T}\}$$

forms an exact abelian and extension-closed, thus wide, subcategory of A-*mod*. Note that, in contrast to the hereditary case (see Theorem 3.2.2), the split-projective A-modules in $\mathcal{T}$ do not necessarily belong to $\alpha(\mathcal{T})$. We want to show that α yields a bijection between

$$tors(A) \longrightarrow wide(A).$$

We have to construct an inverse to α. Let $\mathcal{C}$ be a wide subcategory of A-*mod*. Note that, again in contrast to the hereditary case, $gen(\mathcal{C})$ is not, in general, closed under extensions. Thus, we set

$$\beta(\mathcal{C}) := add\{X \in A\text{-}ind \mid X \text{ is an extension of modules in } gen(\mathcal{C})\}.$$

Using Lemma 3.3.2, it follows that $\beta(\mathcal{C})$ describes precisely the subcategory of A-*mod* containing *all* modules that are extensions of modules in $gen(\mathcal{C})$. The next lemma justifies the definition.

Lemma 3.4.1. *Take a wide subcategory $\mathcal{C}$ and a torsion class $\mathcal{T}$ in A-mod. The following holds.*

(1) $\beta(\mathcal{C})$ is the smallest torsion class in A-mod containing $\mathcal{C}$.

(2) $\alpha(\mathcal{T}) = \alpha(\mathcal{T})_{ind} := \{X \in \mathcal{T} \mid \forall (g : Y \to X) \in \mathcal{T}$ and Y indecomposable, $ker(g) \in \mathcal{T}\}$.

(3) Every split-projective A-module in $\mathcal{T}$ admits a non-zero quotient in $\alpha(\mathcal{T})$.

Proof. ad(1): We will first show that $\beta(\mathcal{C})$ is closed under quotients. Thus, take X in $\beta(\mathcal{C})$ and a surjection $f : X \twoheadrightarrow X'$ in $A\text{-}mod$. We have to show that X' belongs to $\beta(\mathcal{C})$. We can assume that X and X' are indecomposable. Consider the short exact sequence

$$0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} Z \longrightarrow 0$$

where Y and Z are indecomposable A-modules in $gen(\mathcal{C})$. If f factors through π, then X' belongs to $gen(\mathcal{C}) \subseteq \beta(\mathcal{C})$, since $gen(\mathcal{C})$ is closed under quotients. Otherwise, since every indecomposable A-module is uniserial, π factors through f and we can consider the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & ker(f\circ i) \arrow[d, "i'"] & & & \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d] & X \arrow[r, "\pi"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & coker(i') = ker(\pi') \arrow[r] & X' \arrow[r, "\pi'"] & Z \arrow[r] & 0
\end{tikzcd}$$

Since $gen(\mathcal{C})$ is closed under quotients, $ker(\pi')$ belongs to $gen(\mathcal{C})$ and X' can be written as an extension of modules in $gen(\mathcal{C})$. Hence, X' lies in $\beta(\mathcal{C})$.

Next, we want to see that $\beta(\mathcal{C})$ is closed under extensions. We start with a general statement about the structure of a module in $\beta(\mathcal{C})$. This observation will be crucial in the actual proof afterwards. Let X be an indecomposable A-module in $\beta(\mathcal{C})$ together with a short exact sequence

$$0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} Z \longrightarrow 0$$

where Y and Z are again indecomposable A-modules in $gen(\mathcal{C})$. Let C_Y and C_Z be indecomposable A-modules in $\mathcal{C}$ surjecting onto Y and Z, respectively. Since every indecomposable A-module is uniserial, either the A-module X belongs to $gen(\mathcal{C})$ or the map π factors through C_Z such that Z belongs to $\mathcal{C}$, as the cokernel of the induced map from C_Y to C_Z. We call this property $(*)$.

Now let X_1 and X_2 be two indecomposable A-modules in $\beta(\mathcal{C})$. By Lemma 3.3.2, a non-trivial extension of these two modules is of the form

$$\begin{tikzcd}
 & & V \arrow[dr] & & \\
0 \arrow[r] & X_1 \arrow[ur, hook, "j"] \arrow[dr] & & X_2 \arrow[r] & 0. \\
 & & W \arrow[ur, hook, "k"] & &
\end{tikzcd}$$

Since $\beta(\mathcal{C})$ is closed under quotients, W and $coker(j) = coker(k)$ belong to $\beta(\mathcal{C})$. It suffices to show that V is in $\beta(\mathcal{C})$. We consider two cases with respect to the following short exact sequence

$$0 \longrightarrow X_1 \xrightarrow{j} V \longrightarrow coker(j) \longrightarrow 0.$$

<u>Case1</u>: Assume that $coker(j)$ lies in $gen(\mathcal{C})$. If also X_1 belongs to $gen(\mathcal{C})$, we are done by the definition of $\beta(\mathcal{C})$. Otherwise, by the property $(*)$, we get a short exact sequence of the form

$$0 \longrightarrow Y \longrightarrow X_1 \longrightarrow Z \longrightarrow 0$$

with Y indecomposable in $gen(\mathcal{C})$ and Z indecomposable in $\mathcal{C}$, yielding the induced sequence

$$0 \longrightarrow Y \xrightarrow{j'} V \longrightarrow coker(j') \longrightarrow 0.$$

If now $coker(j')$ belongs to $gen(\mathcal{C})$, we are done by the definition of $\beta(\mathcal{C})$. Otherwise, using that $coker(j)$ lies in $gen(\mathcal{C})$, there is an indecomposable A-module V' in $\mathcal{C}$, fitting into the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "j"] \arrow[d] & V \arrow[r] \arrow[d] & coker(j) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r] & coker(j') \arrow[r] \arrow[d] & coker(j) \arrow[r] & 0 \\
 & & V' \arrow[ur] \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Consequently, $coker(j)$ equals the cokernel of the induced map from Z to V' and, thus, it belongs to $\mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, also $coker(j')$ lies in $\mathcal{C}$, leading to a contradiction.

<u>Case2</u>: Assume that $coker(j) \notin gen(\mathcal{C})$. By the property $(*)$, there is a short exact sequence

$$0 \longrightarrow Y_j \longrightarrow coker(j) \longrightarrow C_j \longrightarrow 0$$

with Y_j indecomposable in $gen(\mathcal{C})$ and C_j indecomposable in $\mathcal{C}$, yielding the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & ker(s) \arrow[r] \arrow[d, "p"] & V \arrow[r, "s"] \arrow[d] & C_j \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_j \arrow[r] & coker(j) \arrow[r] & C_j \arrow[r] & 0
\end{tikzcd}$$

with surjective vertical morphisms. Now either $ker(s)$ belongs to $gen(\mathcal{C})$ and, thus, V lies in $\beta(\mathcal{C})$, as wanted, or, the map p must factor through some indecomposable A-module C_{Y_j} in $\mathcal{C}$, since Y_j belongs to $gen(\mathcal{C})$. In the second case, we get a short exact sequence of the form

$$0 \longrightarrow C_{Y_j} \longrightarrow E \longrightarrow C_j \longrightarrow 0$$

where E is an indecomposable A-module in $\mathcal{C}$ surjecting onto $coker(j)$. This contradicts our assumption that $coker(j)$ does not lie in $gen(\mathcal{C})$. Consequently, $\beta(\mathcal{C})$ forms a torsion class in A-*mod*. Moreover, by construction, $\beta(\mathcal{C})$ is the smallest torsion class in A-*mod* containing $\mathcal{C}$.

ad(2): Clearly, we have $\alpha(\mathcal{T}) \subseteq \alpha(\mathcal{T})_{ind}$. Conversely, take X in $\alpha(\mathcal{T})_{ind}$, Y in $\mathcal{T}$ and a map $g: Y \to X$. We have to show that $ker(g)$ belongs to $\mathcal{T}$. To begin with, we can assume X to be indecomposable, since $\alpha(\mathcal{T})_{ind}$ is closed under direct summands. In particular, the image of g is an indecomposable A-module. Moreover, without loss of generality, we can assume that also the kernel of g is indecomposable and that g is not a split map. It follows, by Lemma 3.3.2, that $Y = Y_1 \oplus Y_2$ for Y_1 and Y_2 indecomposable and that g is induced by (π, i), like in the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
 & & Y_1 \arrow[dr, two heads, "\pi"] & \\
 & ker(g) \arrow[ur, hook] \arrow[dr, two heads] & & Im(g) \\
ker(\pi) \arrow[ur, hook, "i'"] & & Y_2 = coker(i') \arrow[ur, hook, "i"] &
\end{tikzcd}$$

Since X lies in $\alpha(\mathcal{T})_{ind}$, it follows that $ker(\pi)$ belongs to $\mathcal{T}$ and, thus, $ker(g)$ can be written as an extension of modules in $\mathcal{T}$. Therefore, $ker(g)$ belongs to $\mathcal{T}$.

ad(3): It is enough to show the statement for an indecomposable split-projective A-module T in $\mathcal{T}$. If T belongs to $\alpha(\mathcal{T})$, we are done. Now assume that $T \notin \alpha(\mathcal{T})$ and that $gen(T) \cap \alpha(\mathcal{T}) = \{0\}$. By S we denote $T/rad(T)$, the simple top of T. Since S does not lie in $\alpha(\mathcal{T})$, there must be a quotient T' of T and a surjection $\pi : T' \to S$ with $ker(\pi) \notin \mathcal{T}$. But $ker(\pi)$ is a quotient of $rad(T)$ and, therefore, $rad(T)$ cannot lie in $\mathcal{T}$. Next, we consider $T/rad^2(T)$. Since $T/rad^2(T)$ does not belong to $\alpha(\mathcal{T})$, there is a map $f : T'' \to T/rad^2(T)$ in $\mathcal{T}$ with $ker(f)$ not lying in $\mathcal{T}$. Using part (2) of this lemma we can choose T'' to be indecomposable. Since $rad(T)$ does not belong to $\mathcal{T}$, the module T'' must be a quotient of T or a quotient of $rad(T)$, saying that $ker(f)$ is a quotient of $rad^2(T)$. Consequently, also $rad^2(T)$ cannot lie in $\mathcal{T}$. By repeating this argument, after finitely many steps, we conclude that all submodules of T in A-mod do not belong to $\mathcal{T}$. Hence, maps to T in $\mathcal{T}$ are trivial such that T lies in $\alpha(\mathcal{T})$, a contradiction. $\square$

The following proposition establishes the wanted bijection.

Proposition 3.4.2. *Let A be a Nakayama algebra. There is a bijection between*

$$tors(A) \longrightarrow wide(A)$$

by mapping a torsion class $\mathcal{T}$ to $\alpha(\mathcal{T})$. The inverse of α is given by β.

Proof. We first show that for $\mathcal{C}$ in $wide(A)$, we have $\alpha(\beta(\mathcal{C})) = \mathcal{C}$.

ad" $\supseteq$ ": Take C in $\mathcal{C}$ indecomposable, X in $\beta(\mathcal{C})$ and a map $f : X \to C$. We have to check that $ker(f)$ lies in $\beta(\mathcal{C})$. Using Lemma 3.4.1(2), we can assume that X is indecomposable. First of all, if X belongs to $gen(\mathcal{C})$, we are done, since there is an indecomposable A-module C_X in $\mathcal{C}$ surjecting onto X such that the kernel of the induced map from C_X to C forces the kernel of f to lie in $gen(\mathcal{C}) \subseteq \beta(\mathcal{C})$. Otherwise, by the property $(*)$ in the proof of Lemma 3.4.1(1), we have a short exact sequence of the form

$$0 \longrightarrow Y \longrightarrow X \xrightarrow{\pi} Z \longrightarrow 0$$

with Y indecomposable in $gen(\mathcal{C})$ and Z indecomposable in $\mathcal{C}$. First assume that π factors through

$Im(f)$. Consequently, we get the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
 & C_Y \arrow[r, "g"] \arrow[d, two heads] & X \arrow[r, "f"] \arrow[d, two heads] & C \\
ker(f \circ g) \arrow[ur, hook] \arrow[d, two heads] & Y \arrow[ur, hook] & Im(f) \arrow[ur, hook] \arrow[d, two heads] & \\
ker(f) \arrow[ur, hook] & & Z &
\end{tikzcd}$$

It follows that $ker(f)$ belongs to $gen(\mathcal{C}) \subseteq \beta(\mathcal{C})$. Otherwise, we get the commutative diagram

$$\begin{tikzcd}
 & & X \arrow[d, two heads, "\pi"] \\
 & ker(f) \arrow[ur, hook] \arrow[d, two heads] & Z \arrow[d, two heads, "\pi'"] \\
Y \arrow[ur, hook] & ker(\pi') \arrow[ur, hook] & Im(f)
\end{tikzcd}$$

Since the kernel of π' belongs to $\mathcal{C}$, as the kernel of the induced map from Z to C, it follows that $ker(f)$ is an extension of modules in $gen(\mathcal{C})$ and, thus, it lies in $\beta(\mathcal{C})$, completing the argument.

ad" $\subseteq$ ": Take X in $\alpha(\beta(\mathcal{C}))$ indecomposable. We have to show that it belongs to $\mathcal{C}$. We first assume that X lies in $gen(\mathcal{C})$, getting the following short exact sequence

$$0 \longrightarrow ker(\pi) \longrightarrow C \xrightarrow{\pi} X \longrightarrow 0$$

where C in $\mathcal{C}$ is indecomposable and $ker(\pi)$ belongs to $\beta(\mathcal{C})$, by assumption. Thus, by the definition of $\beta(\mathcal{C})$, there is an indecomposable A-module C_π in $\mathcal{C}$ mapping non-trivially to $ker(\pi)$ and, hence, yielding an induced map $g : C_\pi \to C$. Then π factors through the cokernel of g, which again belongs to $\mathcal{C}$. By repeating the argument with $coker(g)$ instead of C, we get, after finitely many steps, that X lies in $\mathcal{C}$.

Now we assume that $X \notin gen(\mathcal{C})$. Since X lies in $\beta(\mathcal{C})$, we can use the property $(*)$ to get

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \\
 & C_Y \arrow[u, two heads] \arrow[ur, "\psi"'] & & &
\end{tikzcd} \tag{3.4.1}$$

with C_Y and Z indecomposable in $\mathcal{C}$ and Y indecomposable in $gen(\mathcal{C})$. Since X lies in $\alpha(\beta(\mathcal{C}))$, we get that $ker(\psi)$ belongs to $\beta(\mathcal{C})$. If $ker(\psi)$ lies even in $gen(\mathcal{C})$, then the A-module Y has to be in $\mathcal{C}$, as the cokernel of a map between indecomposable A-modules in $\mathcal{C}$. Thus, also X lies in $\mathcal{C}$, as an extension of modules in $\mathcal{C}$, contradicting our assumption. Otherwise, if we assume that $ker(\psi)$ does not belong to $gen(\mathcal{C})$, we can again use the property $(*)$ to get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y_\psi \arrow[r] & ker(\psi) \arrow[r] & C_\psi \arrow[r] & 0 \\ & C_{Y_\psi} \arrow[u, two heads] \arrow[ur, "\psi'"'] & & & \end{tikzcd}$$

with C_ψ and C_{Y_ψ} indecomposable in $\mathcal{C}$. By composition, we now get a new map $\phi : C_{Y_\psi} \to C_Y$ such that the morphism ψ factors through $coker(\phi)$ in $\mathcal{C}$. Therefore, we can replace C_Y by $coker(\phi)$ in the diagram (3.4.1)

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \\ & coker(\phi) \arrow[u, two heads] \arrow[ur, "\tilde{\psi}"'] & & & \end{tikzcd}$$

and repeat the whole argument. After finitely many steps, we conclude that Y and, thus, also X lies in $\mathcal{C}$, again yielding a contradiction.

Next, we have to verify that for $\mathcal{T}$ in $tors(A)$, we have $\beta(\alpha(\mathcal{T})) = \mathcal{T}$.

ad" $\supseteq$ ": It suffices to show that all indecomposable split-projective modules in $\mathcal{T}$ belong to $\beta(\alpha(\mathcal{T}))$. Let T in $\mathcal{T}$ be indecomposable and split-projective. If T belongs to $\alpha(\mathcal{T})$, we are done. Now assume that $T \notin \alpha(\mathcal{T})$. By Lemma 3.4.1(3), there is an indecomposable A-module X in $\alpha(\mathcal{T})$ yielding the sequence

$$0 \longrightarrow ker(\pi) \longrightarrow T \xrightarrow{\pi} X \longrightarrow 0$$

with $ker(\pi)$ in $\mathcal{T}$. If $ker(\pi)$ also belongs to $gen(\alpha(\mathcal{T}))$, we get that T lies in $\beta(\alpha(\mathcal{T}))$, by definition. Otherwise, if $ker(\pi)$ is not in $gen(\alpha(\mathcal{T}))$, we can deduce from Lemma 3.4.1(3) that there must be an indecomposable A-module X' in $\alpha(\mathcal{T})$ yielding the short exact sequence

$$0 \longrightarrow ker(\pi') \longrightarrow ker(\pi) \xrightarrow{\pi'} X' \longrightarrow 0$$

where $ker(\pi')$ lies in $\mathcal{T}$. If now $ker(\pi')$ also belongs to $gen(\alpha(\mathcal{T}))$, we get that $ker(\pi)$ is in $\beta(\alpha(\mathcal{T}))$ and, hence, T lies in the torsion class $\beta(\alpha(\mathcal{T}))$. Otherwise, we can repeat the previous argument, until, after finitely many steps, the corresponding kernel must belong to $gen(\alpha(\mathcal{T}))$.

ad" $\subseteq$ ": The inclusion holds, since $\alpha(\mathcal{T}) \subseteq \mathcal{T}$ and $\beta(\alpha(\mathcal{T}))$ is, by construction, the smallest torsion class in A-*mod* containing $\alpha(\mathcal{T})$, see Lemma 3.4.1(1). $\square$

Torsion classes in A-*mod* can be classified by support τ-tilting A-modules (see Definition 1.8.1). The set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting) A-modules will be denoted by $\tau\text{-}tilt(A)$ (respectively, $s\tau\text{-}tilt(A)$). Note that every support τ-tilting module T gives rise to a torsion class $gen(T)$. We say that two support τ-tilting A-modules T and T' are **equivalent**, if $gen(T) = gen(T')$. We get the following correspondence between support τ-tilting modules and finitely generated torsion classes.

Theorem 3.4.3. *[AIR, Theorem 2.7] Let A be a finite dimensional $\mathbb{K}$-algebra. There is a bijection between*

$$s\tau\text{-}tilt(A) \longrightarrow f\text{-}tors(A)$$

given by mapping a (basic) support τ-tilting module T to the torsion class $gen(T)$. Conversely, we assign to a finitely generated torsion class $\mathcal{T}$ the sum of the indecomposable Ext-projectives in $\mathcal{T}$.

Applying this result to our context, we get the following corollary.

Corollary 3.4.4. *Let A be a Nakayama algebra. There are bijections between the following sets:*

(1) isomorphism classes of basic support τ-tilting A-modules;

(2) torsion classes in A-mod;

(3) wide subcategories in A-mod;

(4) isomorphism classes of orthogonal collections in A-mod;

(5) epiclasses of universal localisations of A.

Proof. The bijection between (1) and (2) follows from Theorem 3.4.3. The correspondences between (3), (4) and (5) are given by Proposition 3.3.7 and Theorem 3.3.8. Finally, Proposition 3.4.2 finishes the proof. □

Remark 3.4.5. *The presented list of bijections can be extended taking into account further results in [AIR] and [BY]. For example, there are correspondences between support τ-tilting modules and certain silting or cluster tilting objects, (co-)t-structures and g-matrices for a given finite dimensional algebra. Nevertheless, in order to keep notation low, we focus on the presented objects in the corollary above. We refer to the literature and Chapter 5 for further directions.*

In what follows, we explore the correspondence between the support τ-tilting modules and the universal localisations of A.

Theorem 3.4.6. *Let A be a Nakayama algebra.*

(1) There is a bijection

$$\Psi_A : s\tau\text{-}tilt(A) \longrightarrow uniloc(A)$$

by mapping a support τ-tilting A-module T to $A_{A_{\Sigma_T}} := A_{*(\alpha(gen(T)))}$. The inverse maps a universal localisation A_Σ to T_Σ, the sum of the indecomposable Ext-projectives in $\beta(\Sigma^*)$.

(2) Ψ_A restricts to a bijection between

$$\tau\text{-}tilt(A) \longrightarrow uniloc^p(A).$$

(3) Ψ_A restricts to a bijection between

$$s\tau\text{-}tilt(A/AeA) \longrightarrow uniloc_e(A)$$

for $e = e^2$ in A. In particular, if T is equivalent to ${}_A(A/AeA)$, it is mapped to $A_{\Sigma_T} = A/AeA$.

Proof. ad(1): Follows from Theorem 3.4.3, Proposition 3.4.2 and Corollary 3.3.9.

ad(2): Let T be a basic τ-tilting A-module. By [AIR, Proposition 2.2], T is sincere such that $Hom_A(P,T) \neq 0$ for all indecomposable projective A-modules P. We have to show that

$$Hom_A(P, \alpha(gen(T))) \neq 0$$

for all P. Now let T' be an indecomposable direct summand of T, P be an indecomposable projective A-module and $f : P \to T'$ be a non-trivial morphism. If T' is in $\alpha(gen(T))$, there is nothing to show. We distinguish cases with respect to the cokernel of f. If $coker(f)$ lies in $gen(\alpha(gen(T)))$, we are done, keeping in mind that every indecomposable A-module is uniserial and P is projective. In fact, in the extremal case when $coker(f)$ already belongs to $\alpha(gen(T))$, we know that $Im(f)$ lies in $gen(T)$ and, by Lemma 3.4.1(3), we get a surjection from P to an indecomposable module in $\alpha(gen(T))$. As a consequence, again by Lemma 3.4.1(3), it remains to consider the case when there is an indecomposable A-module X in $\alpha(gen(T))$ yielding the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
P \arrow[rr, "f"] \arrow[dr, "\tilde{f}"] \arrow[dd] & & T' \arrow[d, "\pi"] \\
& ker(\tilde{\pi} \circ \pi) \arrow[ur, hook] & coker(f) \arrow[d, "\tilde{\pi}"] \\
Im(f) \arrow[ur, hook] & & X
\end{tikzcd}$$

Since X belongs to $\alpha(gen(T))$, we know that $ker(\tilde{\pi} \circ \pi)$ lies in $gen(T)$. Certainly, if $ker(\tilde{\pi} \circ \pi)$ belongs to $gen(\alpha(gen(T)))$, we are done, using that P is projective. Otherwise, we can repeat the whole argument with $\tilde{f}$ instead of f. Since the length of the indecomposable A-module $ker(\tilde{\pi} \circ \pi)$ is smaller than the length of T', after finitely many steps, we get that $Hom_A(P, \alpha(gen(T))) \neq 0$. Consequently, P does not lie in ${}^*(\alpha(gen(T)))$ and we have $A_{\Sigma_T} \otimes_A P \neq 0$. It follows that the localisation A_{Σ_T} is pure.

Conversely, let A_Σ be a pure universal localisation of A and let P be an indecomposable projective A-module. Now consider the non-trivial A-module map

$$\phi_P : P \to A_\Sigma \otimes_A P.$$

Since $A_\Sigma \otimes_A P$ lies in $\mathcal{X}_{A_\Sigma}$, there is a basic split-projective module T_P in $\beta(\mathcal{X}_{A_\Sigma})$ surjecting onto $A_\Sigma \otimes_A P$. Since P is projective, ϕ_P factors through T_P and we get a non-trivial map from P to T_P. Using the fact that $\beta(\mathcal{X}_{A_\Sigma})$ is closed under extensions, we conclude that all split-projective modules in $\beta(\mathcal{X}_{A_\Sigma})$ are Ext-projective such that T_P becomes a direct summand of T_Σ. Therefore, we get $Hom_A(P, T_\Sigma) \neq 0$ for all indecomposable projective A-modules P, telling that T_Σ is sincere and, thus, by [AIR, Proposition 2.2], τ-tilting.

ad(3): From (2) we deduce that for a support τ-tilting A-module T and a finitely generated projective A-module $P = Ae$ we have $Hom_A(Ae, T) = 0$ if and only if $A_{\Sigma_T} \otimes_A Ae = 0$. Thus, T belongs to $s\tau\text{-}tilt(A/AeA)$ if and only if A_{Σ_T} is e-annihilating. For the last part of the statement see the proof of Theorem 3.2.3(3). □

Note that, in general, tilting A-modules do not arise from universal localisations.

Example 3.4.7. *Consider the Nakayama algebra $A := A_3^2$ and the tilting A-module*

$$T := P_2 \oplus P_1 \oplus S_2.$$

The associated universal localisation A_{Σ_T} of A is given by localising at the A-module S_2 (see Example 3.3.5) and, therefore, the map $f : A \to A_{\Sigma_T}$ is not monomorphic. Moreover, since the identity map on A is the only monomorphic universal localisation of A (up to epiclasses), T cannot arise from universal localisation.

In the hereditary setting of Section 3.2, we compared a basic support tilting A-module T directly with its associated universal localisation A_{Σ_T} (see Proposition 3.2.8). The following proposition initiates a similar comparison in the given Nakayama context.

Proposition 3.4.8. *Let A be a Nakayama algebra and T be a basic support τ-tilting A-module.*

(1) If T' is an indecomposable direct summand of T, then the following are equivalent.

(i) T' is not split-projective in $gen(T)$;

(ii) T' belongs to ${}^{*}\mathcal{X}_{A_{\Sigma_T}}$.

(2) If X is an indecomposable A-module in ${}^{*}\mathcal{X}_{A_{\Sigma_T}}$, then the following are equivalent.

(i) $X \in add(T)$;

(ii) $X \in gen(\mathcal{X}_{A_{\Sigma_T}})$;

(iii) $X \in gen(T) = \beta(\mathcal{X}_{A_{\Sigma_T}})$.

Proof. ad(1): $(i) \Leftarrow (ii)$: If T' lies in ${}^{*}\mathcal{X}_{A_{\Sigma_T}}$, we have $Hom_A(T', \mathcal{X}_{A_{\Sigma_T}}) = 0$ and, therefore, by Lemma 3.4.1(3), T' cannot be split-projective in $gen(T)$.
$(i) \Rightarrow (ii)$: Now assume that T' is not split-projective in $gen(T)$. We have to show that T' belongs to ${}^{*}\mathcal{X}_{A_{\Sigma_T}}$. Equivalently, we will show that

(I) $Ext^1_A(T', \mathcal{X}_{A_{\Sigma_T}}) = 0$;

(II) $Hom_A(T', \mathcal{X}_{A_{\Sigma_T}}) = 0$;

(III) $Hom_A(\sigma_1^{T'}, \mathcal{X}_{A_{\Sigma_T}}) = 0$.

(I) follows, in particular, from the fact that T' is Ext-projective in $\beta(\mathcal{X}_{A_{\Sigma_T}})$. To prove (II) we suppose that there is an indecomposable A-module X in $\mathcal{X}_{A_{\Sigma_T}}$ with a non-trivial map $f : T' \to X$.

By assumption, we get the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
 & \tilde{T} \arrow[d] \arrow[dr, "\tilde{f}"] & \\
ker(\tilde{f}) \arrow[ur, hook] \arrow[d] & T' \arrow[r, "f"] \arrow[d] & X \\
ker(f) \arrow[ur, hook] & Im(f) \arrow[ur, hook] &
\end{tikzcd}$$

where $\tilde{T}$ is split-projective in $\beta(\mathcal{X}_{A_{\Sigma_T}}) = gen(T)$. Since X lies in $\mathcal{X}_{A_{\Sigma_T}}$, we know that $ker(\tilde{f})$ is in $\beta(\mathcal{X}_{A_{\Sigma_T}})$. Consequently, we have $Ext^1_A(T', \beta(\mathcal{X}_{A_{\Sigma_T}})) \neq 0$, a contradiction, since T' is Ext-projective in $\beta(\mathcal{X}_{A_{\Sigma_T}})$. For (III) we have to show that every map $g : P_1^{T'} \to X$ for X in $\mathcal{X}_{A_{\Sigma_T}}$ factors through $ker(\pi^{T'})$, where

$$P_1^{T'} \xrightarrow{\sigma_0^{T'}} P_0^{T'} \xrightarrow{\pi^{T'}} T' \longrightarrow 0$$

describes the minimal projective presentation of T' in A-mod. Let us suppose that we have a non-trivial map $g : P_1^{T'} \to X$ not factoring through $ker(\pi^{T'})$ and with X indecomposable in $\mathcal{X}_{A_{\Sigma_T}}$. Then we get the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
P_1^{T'} \arrow[r, "g"] \arrow[d] & X \arrow[r, "g'"] \arrow[d] & P_0^{T'} \arrow[d] \\
Im(g) \arrow[ur, hook] \arrow[d] & M \arrow[ur, hook] \arrow[d] & T' \arrow[d, "\pi'"] \\
ker(\pi^{T'}) \arrow[ur, hook] & ker(\pi') \arrow[ur, hook] & coker(g')
\end{tikzcd}$$

where M lies in $\beta(\mathcal{X}_{A_{\Sigma_T}})$, since X is in $\mathcal{X}_{A_{\Sigma_T}}$. Thus, we get $Ext^1_A(T', \beta(\mathcal{X}_{A_{\Sigma_T}})) \neq 0$, a contradiction.

ad(2): $(i) \Rightarrow (ii)$: Since X belongs to ${}^*\mathcal{X}_{A_{\Sigma_T}}$, we know that X cannot surject onto any object in $\mathcal{X}_{A_{\Sigma_T}}$. Using that $X \in add(T)$, we know that X lies in $\beta(\mathcal{X}_{A_{\Sigma_T}})$ and therefore, by Lemma 3.4.1(3), we get $X \in gen(\mathcal{X}_{A_{\Sigma_T}})$. Clearly, we have $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$: We have to show that X is Ext-projective in $\beta(\mathcal{X}_{A_{\Sigma_T}})$. Suppose there is an indecomposable A-module M in $\beta(\mathcal{X}_{A_{\Sigma_T}})$ with $Ext^1_A(X, M) \neq 0$. Using Lemma 3.3.2 and the minimal projective presentation of X, we get the commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
 & & P_0^X \arrow[d] \arrow[dd, bend left, "\pi^X"] \\
P_1^X \arrow[d] & L \arrow[ur, hook] \arrow[d] & Y_1 \arrow[d, "\pi"] \\
ker(\pi^X) \arrow[ur, hook] \arrow[d] & M \arrow[ur, hook] \arrow[d] & X \\
ker(\pi) \arrow[ur, hook] & Y_2 \arrow[ur, hook] &
\end{tikzcd}$$

(Note that Y_2 possibly can be zero. In this situation, we have $ker(\pi^X) = L$ and $ker(\pi) = M$.) Now since M belongs to $\beta(\mathcal{X}_{A_{\Sigma_T}})$, by Lemma 3.4.1(3), there is an indecomposable A-module X_T in $\mathcal{X}_{A_{\Sigma_T}}$ such that the projective cover P_0^M of M in A-*mod* surjects onto X_T. Since X lies in ${}^*\mathcal{X}_{A_{\Sigma_T}}$, we have $Hom_A(\sigma_1^X, \mathcal{X}_{A_{\Sigma_T}}) = 0$ and, therefore, L has to surject onto X_T. Moreover, we know that

$$Ext_A^1(X, \mathcal{X}_{A_{\Sigma_T}}) = 0$$

such that Y_2 (or M in case Y_2 equals zero) must surject onto X_T via some map g with $ker(g)$ in $\beta(\mathcal{X}_{A_{\Sigma_T}})$. If $Y_2 \neq 0$ and $Y_2 \neq X_T$, by using again Lemma 3.4.1(3), we can repeat the previous argument with $ker(g)$ instead of M to conclude that $ker(g)$ has to surject onto an indecomposable A-module in $\mathcal{X}_{A_{\Sigma_T}}$ such that, after finitely many steps, we get that $ker(\pi)$ belongs to $\beta(\mathcal{X}_{A_{\Sigma_T}})$. Note that this conclusion is immediate, if $Y_2 = 0$ or $Y_2 = X_T$. But now $ker(\pi)$ must surject onto some indecomposable A-module X_T' in $\mathcal{X}_{A_{\Sigma_T}}$ with $Ext_A^1(X, X_T') \neq 0$, a contradiction, since $Ext_A^1(X, \mathcal{X}_{A_{\Sigma_T}})$ must be zero, by assumption. $\square$

The following theorem allows us to read off completely the associated universal localisation A_{Σ_T} from a basic support τ-tilting A-module T.

Theorem 3.4.9. *Let A be a Nakayama algebra and T be a basic support τ-tilting A-module which is τ-tilting over the algebra A/AeA for an idempotent e in A. Then A_{Σ_T} is given by localising at the set Σ_T', containing the A-module Ae and all non split-projective indecomposable direct summands of T in $gen(T)$.*

Proof. One way of proving the above statement is to show that ${}^*\mathcal{X}_{A_{\Sigma_T}}$ equals ${}^*\mathcal{X}_{A_{\Sigma_T'}}$. By Theorem

3.4.6(3) and Proposition 3.4.8(1), we know that

$$^*\mathcal{X}_{A_{\Sigma'_T}} \subseteq {}^*\mathcal{X}_{A_{\Sigma_T}}.$$

Now take X indecomposable in $^*\mathcal{X}_{A_{\Sigma_T}}$. If X belongs to $gen(T) = \beta(\mathcal{X}_{A_{\Sigma_T}})$, by Proposition 3.4.8(2), it already lies in $add(T)$ and, thus, using Proposition 3.4.8(1), we get that X lies in Σ'_T. Moreover, if we consider the minimal projective resolution of X in A-mod and assume that $Hom_A(P_0^X, \mathcal{X}_{A_{\Sigma_T}}) = 0$, using Theorem 3.4.6(3), we get that P_0^X is a direct summand of Ae and already belongs to Σ'_T. Since X lies in $^*\mathcal{X}_{A_{\Sigma_T}}$, we also get that $Hom_A(P_1^X, \mathcal{X}_{A_{\Sigma_T}}) = 0$. Again by Theorem 3.4.6(3), this implies that P_1^X belongs to Σ'_T and, therefore, X belongs to $^*\mathcal{X}_{A_{\Sigma'_T}}$. Consequently, we can assume that X does not belong to $gen(\mathcal{X}_{A_{\Sigma_T}}) \subseteq gen(T)$ and that there is an indecomposable A-module X_T in $\mathcal{X}_{A_{\Sigma_T}}$ together with a non-trivial map $g : P_0^X \to X_T$ yielding the following commutative diagram of indecomposable A-modules

$$\begin{tikzcd}
P_1^{X_1} = P_1^X \arrow[d, two heads] & P_1^{X_2} = P_0^X \arrow[d, two heads] & P_0^{X_2} = P_0^{X_1} \arrow[d, two heads] \\
ker(\pi^X) \arrow[ur, hook] \arrow[d, two heads] & ker(\pi^{X_2}) \arrow[ur, hook] \arrow[d, two heads] & X_T \arrow[d, two heads] \\
ker(\pi^{X_1}) \arrow[ur, hook] & Im(g) \arrow[ur, hook] \arrow[d, two heads] & X_1 \arrow[d, two heads] \\
 & X \arrow[ur, hook, "i"] & X_2 = coker(i)
\end{tikzcd} \tag{3.4.2}$$

Note that, by assumption on X, $Hom_A(X, X_T)$ must be zero. Now we will show that the A-module X_T can be chosen in a way such that X_1 and X_2 belong to $^*\mathcal{X}_{A_{\Sigma_T}}$. Equivalently, we will show that for $i = 1, 2$

(I) $Hom_A(X_i, \mathcal{X}_{A_{\Sigma_T}}) = 0$;

(II) $Ext_A^1(X_i, \mathcal{X}_{A_{\Sigma_T}}) = 0$;

(III) $Hom_A(\sigma_1^{X_i}, \mathcal{X}_{A_{\Sigma_T}}) = 0$.

(I): Since X is in $^*\mathcal{X}_{A_{\Sigma_T}}$, we have $Hom_A(X, \mathcal{X}_{A_{\Sigma_T}}) = 0$. Thus, every map from X_1 to an indecomposable A-module in $\mathcal{X}_{A_{\Sigma_T}}$ must factor through X_2. Suppose we have a non-trivial map $f : X_2 \to X'_T$ with X'_T indecomposable in $\mathcal{X}_{A_{\Sigma_T}}$. We can lift f to a map $\tilde{f} : X_T \to X'_T$ such that the inclusion $Im(g) \to X_T$ factors through $ker(\tilde{f})$. Since $ker(\tilde{f})$ lies in $\mathcal{X}_{A_{\Sigma_T}}$, we can replace X_T by $ker(\tilde{f})$ and repeat the whole argument with different X_1 and X_2. Since X does not belong to $gen(\mathcal{X}_{A_{\Sigma_T}})$, we know that $Im(g)$ does not lie in $\mathcal{X}_{A_{\Sigma_T}}$ and, thus, after finitely many steps (I) is fulfilled. For (II), we first apply the functor $Hom_A(-,X'_T)$ for X'_T in $\mathcal{X}_{A_{\Sigma_T}}$ to the short exact sequence

$$0 \longrightarrow X \xrightarrow{i} X_1 \longrightarrow X_2 \longrightarrow 0$$

to see that it suffices to show that $Ext^1_A(X_2,\mathcal{X}_{A_{\Sigma_T}}) = 0$. Now suppose that there is an indecomposable X'_T in $\mathcal{X}_{A_{\Sigma_T}}$ with $Ext^1_A(X_2,X'_T) \neq 0$. Since we have $Hom_A(X,\mathcal{X}_{A_{\Sigma_T}}) = 0$, by using Lemma 3.3.2, we conclude that there is a non-trivial morphism from $ker(\pi^{X_2})$ to X_1 factoring through X'_T. Since X does not belong to $gen(\mathcal{X}_{A_{\Sigma_T}})$, the A-module X'_T cannot surject onto X. Thus, we can replace X_T by X'_T and repeat the whole argument with different X_1 and X_2, until (I) and (II) are fulfilled. Regarding (III), we will first show that $Hom_A(\sigma_1^{X_2},\mathcal{X}_{A_{\Sigma_T}}) = 0$. Suppose there is an indecomposable A-module X'_T in $\mathcal{X}_{A_{\Sigma_T}}$ and a map $f : P_1^{X_2} \to X'_T$ not factoring through $ker(\pi^{X_2})$. Consequently, the minimal projective presentation of X_2 factors through X'_T

$$\begin{tikzcd}
P_1^{X_2} \arrow[rr] \arrow[dr] & & P_0^{X_2} \\
& X'_T \arrow[ur, "f'"'] &
\end{tikzcd}$$

and via f' we get the induced morphism

$$h : X'_T \to X_T.$$

Moreover, since $ker(\pi^{X_2}) \subseteq Im(f')$, it follows that X_2 surjects onto the cokernel of h, which belongs to $\mathcal{X}_{A_{\Sigma_T}}$. This yields a contradiction. Using that $Hom_A(\sigma_1^{X_2},\mathcal{X}_{A_{\Sigma_T}}) = 0$ and the fact that $Hom_A(\sigma_1^{X},\mathcal{X}_{A_{\Sigma_T}}) = 0$ as well as $Ext^1_A(X,\mathcal{X}_{A_{\Sigma_T}}) = 0$, since X lies in ${}^*\mathcal{X}_{A_{\Sigma_T}}$, we can also conclude that $Hom_A(\sigma_1^{X_1},\mathcal{X}_{A_{\Sigma_T}}) = 0$.

Altogether, we have seen that we can choose the A-module X_T in $\mathcal{X}_{A_{\Sigma_T}}$ in a way such that X_1 and X_2 belong to ${}^*\mathcal{X}_{A_{\Sigma_T}}$. Since, by construction, X_1 and X_2 also belong to $gen(\mathcal{X}_{A_{\Sigma_T}})$, they belong to Σ'_T, by Proposition 3.4.8. Consequently, keeping in mind diagram (3.4.2), we conclude that X lies in ${}^*\mathcal{X}_{A_{\Sigma'_T}}$. This finishes the proof. $\square$

In particular, if T is a τ-tilting A-module, then A_{Σ_T} is just given by localising at the set of indecomposable non split-projective A-modules in $add(T)$. Let us finish with an example illustrating the previous result.

Example 3.4.10. *Let A be the algebra $\tilde{A}_3^3$. By Theorem 3.4.6 and Corollary 3.3.10, we know that*

$$|s\tau\text{-}tilt(A)| = |uniloc(A)| = \binom{6}{3} = 20.$$

If we restrict ourselves to proper τ-tilting A-modules, by Theorem 3.4.6(2) and the classification of the universal localisations of A, we get that

$$|\tau\text{-}tilt(A)| = |uniloc^p(A)| = 10.$$

The following table lists these τ-tilting modules and universal localisations (indicated by Σ'_T).

τ-**tilt**$(\mathbf{A})$	**uniloc**$^{\mathbf{p}}(\mathbf{A})$
$T := A = P_1 \oplus P_2 \oplus P_3$	$\Sigma'_T = \{0\}$
$T := P_1 \oplus P_3 \oplus S_1$	$\Sigma'_T = \{S_1\}$
$T := P_1 \oplus P_2 \oplus S_2$	$\Sigma'_T = \{S_2\}$
$T := P_2 \oplus P_3 \oplus S_3$	$\Sigma'_T = \{S_3\}$
$T := P_1 \oplus P_1/rad^2(P_1) \oplus S_1$	$\Sigma'_T = \{S_1, P_1/rad^2(P_1)\}$
$T := P_2 \oplus P_2/rad^2(P_2) \oplus S_2$	$\Sigma'_T = \{S_2, P_2/rad^2(P_2)\}$
$T := P_3 \oplus P_3/rad^2(P_3) \oplus S_3$	$\Sigma'_T = \{S_3, P_3/rad^2(P_3)\}$
$T := P_1 \oplus P_1/rad^2(P_1) \oplus S_2$	$\Sigma'_T = \{P_1/rad^2(P_1)\}$
$T := P_2 \oplus P_2/rad^2(P_2) \oplus S_3$	$\Sigma'_T = \{P_2/rad^2(P_2)\}$
$T := P_3 \oplus P_3/rad^2(P_3) \oplus S_1$	$\Sigma'_T = \{P_3/rad^2(P_3)\}$

Chapter 4

Ring epimorphisms for some self-injective algebras

In this chapter, we study finite dimensional ring epimorphisms for self-injective algebras. In a first section, we prove that over a preprojective algebra of Dynkin type, there is only a finite number of finite dimensional universal localisations. Moreover, these localisations turn out to be self-injective algebras again. In a second section, we classify the homological ring epimorphisms among these localisations and provide a large class of algebras that do not admit finite dimensional homological ring epimorphisms. In the last section of this chapter, we give a complete and explicit classification of the homological ring epimorphisms over a self-injective Nakayama algebra. This classification was already published in [M].

Let us start by recalling some basic definitions and results. Throughout, A will denote a finite dimensional and self-injective $\mathbb{K}$-algebra. Let $A\text{-}\underline{mod}$ be the **stable module category** defined to be the quotient of A-*mod* by the ideal generated by all the A-homomorphisms that factor through a projective module. By π_A we denote the canonical quotient functor $A\text{-}mod \to A\text{-}\underline{mod}$. It is well-known that $A\text{-}\underline{mod}$ carries the structure of a triangulated category with shift functor Ω^{-1} (compare [HJ]). Here Ω denotes the syzygy functor on the (stable) module category (see [SY, Chapter IV,8]). Also recall that the Auslander-Reiten translate τ, the Nakayama functor ν and the syzygy functor Ω induce autoequivalences of $A\text{-}\underline{mod}$ related by the following natural isomorphisms of functors (see [SY, Chapter IV., Theorem 8.5])

$$\tau \cong \Omega^2\nu \cong \nu\Omega^2$$

$$\tau^{-1} \cong \Omega^{-2}\nu^{-1} \cong \nu^{-1}\Omega^{-2}.$$

4.1 Ring epimorphisms for preprojective algebras

In this section we are interested in preprojective algebras of Dynkin type and their finite dimensional ring epimorphisms. We recall the definition of the preprojective algebra. First, fix a Dynkin quiver $Q = (Q_0, Q_1)$ of type $\mathcal{A}_n (n \geq 2)$, $\mathcal{D}_n (n \geq 4)$, $\mathcal{E}_6, \mathcal{E}_7$ or $\mathcal{E}_8$ and denote by $\overline{Q}$ its double quiver obtained by adding an arrow $\alpha^* : j \to i$ for all arrows $\alpha : i \to j$ in Q_1. The **preprojective algebra** $A = A_Q$ associated to Q is given by the quotient of the path algebra $\mathbb{K}\overline{Q}$ by the ideal I generated by

$$\sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha).$$

These algebras were first introduced and studied in [GP]. Since the quiver Q is Dynkin, the algebra $A = A_Q$ is a finite dimensional and self-injective $\mathbb{K}$-algebra (see [SY, Chapter IV, Theorem 14.1]). Moreover, the stable module category A-$\underline{mod}$ has Calabi-Yau dimension 2, i.e., there is a natural isomorphism of functors $\nu \cong \Omega^{-3}$ (compare, for example, [ES]). In particular, it follows that $\tau \cong \Omega^{-1}$ (respectively, $\tau^{-1} \cong \Omega$) as functors from A-$\underline{mod}$ to A-$\underline{mod}$.

To the quiver Q we associate the quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ for $|Q_0| = n$ that maps $x \in \mathbb{Z}^n$ to

$$q(x) := \sum_{i \in Q_0} x_i^2 - \sum_{\substack{\alpha \in Q_1 \\ \alpha: i \to j}} x_i x_j.$$

Note that, since Q is Dynkin, q is positive-definite. We call the elements in $\{x \in \mathbb{Z}^n | q(x) = 1\}$ the **positive real roots** of Q. By Gabriel's Theorem they are in bijection with the isomorphism classes of indecomposable $\mathbb{K}Q$-modules. In particular, their number is finite (see [ASS, Chapter VII]).

To the quiver $\overline{Q}$ we associate the symmetric bilinear form $(-,-) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by

$$(x,y) := \sum_{i \in Q_0} 2x_i y_i - \sum_{\substack{\alpha \in \overline{Q}_1 \\ \alpha: i \to j}} x_i y_j$$

for $x, y \in \mathbb{Z}^n$. By $\underline{dim}(M)$ we denote the dimension vector of a finite dimensional A-module M. The following lemma can be found in [CB, Lemma 1].

Lemma 4.1.1. *Let A be a preprojective algebra of Dynkin type and let M and N be finite dimensional A-modules. Then the following holds.*

$$dim Ext_A^1(M,N) = dim Hom_A(M,N) + dim Hom_A(N,M) - (\underline{dim}(M), \underline{dim}(N)).$$

This lemma can be used to prove the following proposition.

Proposition 4.1.2. *Let A be a preprojective algebra of Dynkin type and let M be an indecomposable finite dimensional A-module. Then the following are equivalent.*

(1) $End_A(M)$ is isomorphic to $\mathbb{K}$;

(2) $\underline{dim}(M)$ is a positive real root of the Dynkin quiver Q and $Ext_A^1(M,M) = 0$.

Proof. We denote by x the dimension vector of M. Then (x,x) is given by

$$2\sum_{i\in Q_0} x_i^2 - 2\sum_{\substack{\alpha\in Q_1\\ \alpha:i\to j}} x_i x_j = 2q(x) > 0.$$

Now assume that (1) holds. Using Lemma 4.1.1, it follows that

$$dim\, Ext_A^1(M,M) = 2 - 2q(x) \leq 0$$

and, thus, $Ext_A^1(M,M)$ is zero and $q(x)$ equals 1. In particular, x is a positive real root of Q.

Conversely, assume that (2) holds. Again by Lemma 4.1.1, we get that

$$2 \cdot dim\, End_A(M) = 2q(x).$$

Now the claim follows, since $x = \underline{dim}(M)$ is a positive real root of Q. □

As a consequence of this proposition we obtain the following result mentioned in [CKW].

Corollary 4.1.3. *Every preprojective algebra $A = A_Q$ of Dynkin type is Schur-representation-finite, i.e., for all $x \in \mathbb{Z}^n$ with $|Q_0| = n$ there are only finitely many isomorphism classes of A-modules with trivial endomorphism ring and dimension vector x.*

Proof. Fix x in $\mathbb{Z}^n$. We consider the module variety $mod(A,x)$ whose points are given by the A-modules of dimension vector x (for a general introduction we refer to [CB1] and the references therein). Note that the general linear group of suitable size acts on $mod(A,x)$ by conjugation such that the orbits of this action correspond to the isomorphism classes of the x-dimensional A-modules. Now suppose that $mod(A,x)$ contains infinitely many non-isomorphic A-modules M with trivial endomorphism ring. By Proposition 4.1.2, we have $Ext_A^1(M,M) = 0$ for all such M. Using [G, Corollary 1.2], the orbit O_M of such M is open in $mod(A,x)$ such that its closure $\overline{O}_M$ yields an irreducible component of $mod(A,x)$. Note that $\overline{O}_M$ is given by the union of O_M and further orbits of smaller dimension (see [CB1]). Therefore, any A-module M' in $\overline{O}_M$ with trivial endomorphism ring must already be isomorphic to M. This yields a contradiction, since $mod(A,x)$ decomposes only into finitely many irreducible components. □

Recall that for a finite dimensional ring epimorphism $f : A \to B$ we denote by $\mathcal{X}_B$ the essential image of the restriction functor f_* in $A\text{-}mod$. For a given preprojective algebra A of Dynkin type we want to understand the finite dimensional ring epimorphisms $A \to B$ with $Tor_1^A(B,B) = 0$. We need the following small lemma.

Lemma 4.1.4. *Let A be a finite dimensional $\mathbb{K}$-algebra and $A \to B$ be a finite dimensional ring epimorphism with $Tor_1^A(B,B) = 0$. Then $\mathcal{X}_B$ is uniquely determined by its simple modules.*

Proof. Since $Tor_1^A(B,B) = 0$, the subcategory $\mathcal{X}_B$ is closed under extensions in $A\text{-}mod$. Now the claim follows by observing that every module in $\mathcal{X}_B$ has a filtration by simple B-modules. □

Now we are able to state the main result of this section.

Theorem 4.1.5. *Let A be a preprojective algebra of Dynkin type. Then the following holds.*

(1) Up to epiclasses, there is only a finite number of finite dimensional ring epimorphisms of the form $A \to B$ with $Tor_1^A(B,B) = 0$.

(2) If $A \to B$ is a finite dimensional ring epimorphism with $Tor_1^A(B,B) = 0$, then B is again a self-injective $\mathbb{K}$-algebra without loops.

Proof. We begin with proving statement (1). Let $A \to B$ be a finite dimensional ring epimorphism with $Tor_1^A(B,B) = 0$ and let S be a simple B-module. In particular, we have

$$\mathbb{K} \cong End_B(S) \cong End_A(S).$$

Therefore, by Proposition 4.1.2, $\underline{dim}(S)$ is a positive real root of the quiver Q. Recall that, since Q is Dynkin, there are only finitely many such roots. Moreover, by Corollary 4.1.3, there can be only finitely many isomorphism classes of A-modules with trivial endomorphism ring and dimension vector $\underline{dim}(S)$. Consequently, there is only a finite number of A-modules (up to isomorphism) to choose S from. Now the claim follows from Lemma 4.1.4.

Regarding part (2), we first show that the $\mathbb{K}$-algebra B is again self-injective. Let X be in $B\text{-}mod$. Since $Tor_1^A(B,B) = 0$, it follows that

$$0 = Ext_B^1(B,X) \cong Ext_A^1(B,X).$$

Since, by Lemma 4.1.1, we have $dim\, Ext_A^1(X,B) = dim\, Ext_A^1(B,X)$, also $Ext_A^1(X,B) \cong Ext_B^1(X,B)$ must be zero. Hence, B is an injective B-module and the $\mathbb{K}$-algebra B is self-injective. Finally, we have to check that $Ext_B^1(S,S)$ is zero for all simple B-modules S. This follows, since $End_A(S)$ is isomorphic to $\mathbb{K}$ and, hence, by Proposition 4.1.2, $Ext_A^1(S,S) = 0$ and, thus, also $Ext_B^1(S,S) = 0$. □

In particular, there is only a finite number of finite dimensional universal localisations for every preprojective algebra of Dynkin type. In fact, I conjecture that for these algebras every finite dimensional ring epimorphism $A \to B$ with $Tor_1^A(B,B) = 0$ is a universal localisation. Moreover, I conjecture that there is a bijection between the finite dimensional universal localisations of A and the support τ-tilting A-modules. Note that the support τ-tilting modules over a preprojective algebra of Dynkin type were classified in [Mi]. Indeed, their number is finite and given by the order of the Weyl group corresponding to the underlying Dynkin quiver. The above conjectures are known to be true in case A is of type $\mathcal{A}_2$ and, thus, a Nakayama algebra (see Theorem 3.3.8 and Theorem 3.4.6). We provide a further example.

Example 4.1.6. *Let A be the preprojective $\mathbb{K}$-algebra of type $\mathcal{A}_3$. Note that A is representation finite and its Auslander-Reiten quiver is given by*

$$\begin{tikzcd}
P_1 \arrow[dr] & & & & & & P_3 \\
& M_{12} \arrow[dr] & & S_3 \arrow[dr] & & M_{21} \arrow[ur] \arrow[dr] & \\
S_2 \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[r] \arrow[dr] & P_2 \arrow[r] & W \arrow[ur] \arrow[dr] & & S_2 \\
& M_{32} \arrow[ur] & & S_1 \arrow[ur] & & M_{23} \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & & & & & P_1
\end{tikzcd}$$

The following table presents a complete list of all ring epimorphisms $A \to B$ with $Tor_1^A(B,B) = 0$ (described by $\mathcal{X}_B$), all universal localisations of A (described by a possible set Σ of modules to localise at) and all support τ-tilting A-modules. Note that the set Σ associated to a support τ-tilting A-module T can be chosen to consist of the indecomposable projective A-modules P with $Hom_A(P,T) = 0$ and those direct summands of T that are not split-projective in $gen(T)$ (compare Proposition 3.2.8 in the hereditary case and Theorem 3.4.9 for Nakayama algebras).

ring epimorphisms	*universal localisations*	*support τ-tilting modules*
$\{0\}$	$\Sigma = \{P_1, P_2, P_3\}$	$T = 0$
A-*mod*	$\Sigma = \emptyset$	$T = A$
$add\{S_1\}$	$\Sigma = \{P_2, P_3\}$	$T = S_1$
$add\{S_2\}$	$\Sigma = \{P_1, P_3\}$	$T = S_2$
$add\{S_3\}$	$\Sigma = \{P_1, P_2\}$	$T = S_3$
$add\{P_1\}$	$\Sigma = \{M_{12}, S_1\}$	$T = P_1 \oplus M_{12} \oplus S_1$
$add\{P_3\}$	$\Sigma = \{M_{32}, S_3\}$	$T = P_3 \oplus M_{32} \oplus S_3$
$add\{M_{12}\}$	$\Sigma = \{S_1, P_3\}$	$T = M_{12} \oplus S_1$
$add\{M_{32}\}$	$\Sigma = \{S_3, P_1\}$	$T = M_{32} \oplus S_3$
$add\{M_{21}\}$	$\Sigma = \{S_2, P_3\}$	$T = M_{21} \oplus S_2$
$add\{M_{23}\}$	$\Sigma = \{S_2, P_1\}$	$T = M_{23} \oplus S_2$
$add\{M\}$	$\Sigma = \{S_1, S_3\}$	$T = M \oplus S_1 \oplus S_3$
$add\{W\}$	$\Sigma = \{M_{21}, M_{23}\}$	$T = W \oplus M_{21} \oplus M_{23}$
$add\{P_1, S_2\}$	$\Sigma = \{M_{12}\}$	$T = P_1 \oplus S_2 \oplus M_{12}$
$add\{P_3, S_2\}$	$\Sigma = \{M_{32}\}$	$T = P_3 \oplus S_2 \oplus M_{32}$
$add\{S_1, S_3\}$	$\Sigma = \{P_2\}$	$T = S_1 \oplus S_3$
$add\{M_{12}, M_{32}\}$	$\Sigma = \{M\}$	$T = P_1 \oplus P_3 \oplus M$
$add\{M_{21}, M_{23}\}$	$\Sigma = \{S_2\}$	$T = M_{21} \oplus M_{23} \oplus S_2$
$add\{S_1, S_2, M_{12}, M_{21}\}$	$\Sigma = \{P_3\}$	$T = M_{12} \oplus M_{21}$
$add\{S_2, S_3, M_{32}, M_{23}\}$	$\Sigma = \{P_1\}$	$T = M_{32} \oplus M_{23}$
$add\{P_1, S_3, M_{12}, M\}$	$\Sigma = \{S_1\}$	$T = P_1 \oplus M \oplus S_1$
$add\{P_1, S_1, M_{23}, W\}$	$\Sigma = \{M_{21}\}$	$T = P_1 \oplus W \oplus M_{21}$
$add\{P_3, S_3, M_{21}, W\}$	$\Sigma = \{M_{23}\}$	$T = P_3 \oplus W \oplus M_{23}$
$add\{P_3, S_1, M_{32}, M\}$	$\Sigma = \{S_3\}$	$T = P_3 \oplus M \oplus S_3$

4.2 Homological ring epimorphisms & Tachikawa's conjecture

In this section, we will study the relationship between finite dimensional homological ring epimorphisms of self-injective algebras and the Tachikawa conjecture (see [T, Section 8]).

Conjecture 4.2.1 (Tachikawa)**.** *Let A be a finite dimensional self-injective $\mathbb{K}$-algebra and M be in A-mod. If $Ext^i_A(M,M) = 0$ for all $i > 0$, then M is projective.*

There are several affirmative answers to the conjecture, namely, it is known to hold for

- group algebras of finite groups ([Schu, Chapter 3]);
- self-injective algebras of finite representation type ([Schu, Chapter 3]);
- symmetric algebras with radical cube zero ([Ho, Theorem 3.1]);
- local self-injective algebras with radical cube zero ([Ho, Theorem 3.4]).

Recall that a finite dimensional $\mathbb{K}$-algebra A is called **periodic**, if A is a periodic A-A-bimodule with respect to the syzygy Ω. In this case, A is a self-injective algebra and every finite dimensional A-module without projective direct summands is periodic (compare [SY, Chapter IV,11]).

Lemma 4.2.2. *Let A be a finite dimensional and self-injective $\mathbb{K}$-algebra.*

(1) If A is periodic, then it fulfils the Tachikawa conjecture.

(2) Let B be a finite dimensional and self-injective $\mathbb{K}$-algebra that is stable equivalent to A. Then B fulfils the Tachikawa conjecture if and only if so does A.

Proof. We first show (1). Let M be a non-projective module in A-*mod* and take $d > 0$ such that $\Omega^d(M) = M$. It suffices to check that $Ext_A^d(M,M) \neq 0$. But this follows by [SY, Chapter IV, Theorem 9.4] using the $\mathbb{K}$-linear isomorphism

$$0 \neq \underline{Hom}_A(M,M) = \underline{Hom}_A(\Omega^d(M),M) \cong Ext_A^d(M,M).$$

We now prove (2). Denote by $\psi : B\text{-}\underline{mod} \to A\text{-}\underline{mod}$ the triangle equivalence between the two stable module categories. Since statement (2) is symmetric, we only prove one implication. Assume that the algebra A fulfils the Tachikawa conjecture. Let N be an indecomposable non-projective B-module. By assumption, we know that there is some $d > 0$ such that

$$Ext_A^d(\psi(N),\psi(N)) \neq 0.$$

Using the same $\mathbb{K}$-linear isomorphism as in (1) it follows that

$$0 \neq \underline{Hom}_A(\Omega_A^d(\psi(N)),\psi(N)) = \underline{Hom}_A(\Omega_B^d(N),N) \cong Ext_B^d(N,N).$$

Consequently, also B fulfils the Tachikawa conjecture. $\square$

Next, we relate the Tachikawa conjecture to the study of homological ring epimorphisms.

Proposition 4.2.3. *Let A be a self-injective $\mathbb{K}$-algebra that fulfils the Tachikawa conjecture and let $f : A \to B$ be a finite dimensional ring epimorphism. Then the following are equivalent.*

(1) f is homological;

(2) B is a projective A-module;

(3) $\mathcal{X}_B$ is closed under syzygies in A-mod (i.e., $X \in \mathcal{X}_B$ implies $\Omega_A^z(X) \in \mathcal{X}_B$ for all $z \in \mathbb{Z}$);

(4) f_* induces a fully faithful triangle functor f_Δ making the following diagram commute

$$\begin{tikzcd} B\text{-}mod \arrow[r, "f_*"] \arrow[d, "\pi_B"'] & A\text{-}mod \arrow[d, "\pi_A"] \\ B\text{-}\underline{mod} \arrow[r, "f_\Delta"] & A\text{-}\underline{mod} \end{tikzcd}$$

In particular, $\pi_A(\mathcal{X}_B)$ is a triangulated subcategory of A-$\underline{mod}$.

Moreover, if these conditions are fulfilled, then f is the universal localisation of A at the A-module map $A \to B$ and the $\mathbb{K}$-algebra B is again self-injective.

Proof. First assume that (1) holds. Since f is homological, we know that

$$Ext_A^n(B,B) \cong Ext_B^n(B,B) = 0$$

for all $n > 0$. Since, by assumption, the A-module ${}_AB$ is finite dimensional and the Tachikawa conjecture holds for A, ${}_AB$ must be projective. Since (2) clearly implies (1), we get (1)⇔(2). Next, note that if B is a projective A-module, it is also an injective A-module and, equivalently, the category $\mathcal{X}_B$ is closed under taking projective covers and injective envelopes in A-*mod*. Moreover, since $\mathcal{X}_B$ is closed for kernels, cokernels and extensions in A-*mod*, (2) is equivalent to (3). Further, observe that the diagram of categories in (4) can only commute, if ${}_AB$ is projective. Hence, (4) implies (2). Now assume that (1)-(3) hold. We have to show (4). We define the functor f_Δ to be f_* on objects and to map a morphism $\overline{g} \in B$-$\underline{mod}$ to the corresponding coset $\overline{g} \in A$-$\underline{mod}$. First of all, f_Δ is a well-defined functor, since, by (2), a morphism between A-modules M and N in $\mathcal{X}_B$ that factors through a projective B-module also factors through a projective A-module. Moreover, f_Δ is full, since so is f_*. Faithfulness follows from the following observation: let $g : M \to N$ be a morphism in $\mathcal{X}_B$ that factors through a projective A-module P

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr] & & N \\ & P. \arrow[ur, "h"'] & \end{tikzcd}$$

Now the map h must factor through the $\mathcal{X}_B$-reflection $P \to B \otimes_A P$ where $B \otimes_A P$ is a projective B-module. Consequently, the map g is also zero in B-$\underline{mod}$. Finally, f_Δ is a triangle functor, since, by (2) and (3), f_* induces a natural isomorphism $\Omega_B \cong \Omega_A$.

Moreover, using that ${}_AB$ is a projective A-module, by Corollary 2.1.6, the ring epimorphism f is the universal localisation of A at the A-module map $A \to B$. Since the $\mathbb{K}$-algebra A is self-injective, ${}_AB$ is also an injective A-module. Using that $\mathcal{X}_B$ is a full subcategory of A-*mod*, it follows that ${}_BB$ is an injective B-module. Thus, the $\mathbb{K}$-algebra B is self-injective. $\square$

In general, it seems that self-injective $\mathbb{K}$-algebras do not admit many homological ring epimorphisms. The next lemma points in this direction. Recall that a two-sided idempotent ideal I of A is called **stratifying**, if the induced ring epimorphism $A \to A/I$ is homological.

Lemma 4.2.4. *Let A be a connected self-injective $\mathbb{K}$-algebra fulfilling the Tachikawa conjecture and let $f : A \to B$ be a non-zero finite dimensional homological ring epimorphism. Then the following are equivalent.*

(1) the map f is injective;

(2) the map f is surjective.

In particular, it follows that 0 *and A are the only stratifying ideals of A.*

Proof. First, assume that the map f is injective. Since the algebra A is self-injective and, by Proposition 4.2.3, ${}_AB$ is a projective A-module, it is also an injective A-module such that ${}_AA$ becomes a direct summand of ${}_AB$. Consequently, f_* yields an equivalence between A-*mod* and B-*mod* showing that f lies in the same epiclass as the identity map $A \to A$. In particular, f is an isomorphism.

Conversely, assume that the map f is surjective. Thus, there is some two-sided ideal I of A with $B \cong A/I$. Since $Tor_1^A(B,B) = 0$, the ideal I is idempotent by Proposition 1.3.8 and, hence, of the form AeA for an idempotent e in A. Since, by Proposition 4.2.3, the A-module $B \cong A/AeA$ is projective, we get a decomposition of A, as a module, into $AeA \oplus A/AeA$. Since the algebra A is self-injective and $Hom_A(AeA, A/AeA) = 0$, it follows that also $Hom_A(A/AeA, AeA) = 0$. Consequently, A splits as an algebra into the direct sum of $End_A(AeA)$ and B. By assumption, this implies that e is zero and, thus, f is an isomorphism. $\square$

Note that the lack of stratifying ideals for an algebra A or, more generally, the absence of finite dimensional homological ring epimorphisms is related to the notion of derived simplicity (compare, for example, Corollary 2.2.2). It is well-known that every non-trivial stratifying ideal of A induces a non-trivial recollement of derived module categories of the form

$$D(A/AeA) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(eAe).$$

In particular, an algebra A with a non-trivial stratifying ideal is not derived simple. More generally, most examples of recollements of derived module categories with middle term $D(A)$ do arise from finite dimensional homological ring epimorphisms starting in A (compare [AKLY]). Therefore, a classification of all these ring epimorphisms would be desirable (see [CX3, Question 4]).

Next, we give a complete classification of the finite dimensional homological ring epimorphisms for preprojective $\mathbb{K}$-algebras of Dynkin type. We start by providing an example.

Example 4.2.5. *Let A be a preprojective $\mathbb{K}$-algebra of type $\mathcal{A}_n (n \geq 2)$. By P we denote the indecomposable projective A-module associated to vertex 1 or to vertex n. It follows from the construction of A that $End_A(P) \cong \mathbb{K}$. Consequently, $add(P)$ yields a bireflective subcategory of A-mod and the associated ring epimorphism $A \to B$ is homological. In fact, by looking at the corresponding representation of P, this ring epimorphism can be described explicitly. If we assume $\mathcal{A}_n$ to be linearly oriented, then B is either given by the universal localisation of A at the set $\{P_j \xrightarrow{\alpha} P_i \mid \alpha \in Q_1\}$, in case $P = P_1$, or by the universal localisation of A at the set $\{P_i \xrightarrow{\alpha^*} P_j \mid \alpha^* \in \bar{Q}_1\}$, otherwise.*

Now we are able to state the main result of this section.

Theorem 4.2.6. *Let A be a preprojective $\mathbb{K}$-algebra of Dynkin type and $f : A \to B$ be a finite dimensional homological ring epimorphism that is neither zero nor an isomorphism. Then A must be of type $\mathcal{A}_n (n \geq 2)$ and the algebra B is Morita-equivalent to $\mathbb{K}$. In fact, for each $n \geq 2$ there are precisely two such choices for f (up to epiclasses).*

Proof. First of all, recall that A is a periodic algebra (see [SY, Chapter IV, Theorem 14.1]) and, hence, it fulfils the Tachikawa conjecture, by Lemma 4.2.2. Now let $f : A \to B$ be a finite dimensional homological ring epimorphism that is neither zero nor an isomorphism. Suppose that the essential image of the restriction functor $\mathcal{X}_B$ contains a non-projective indecomposable A-module X. Consequently, by Proposition 4.2.3, $\mathcal{X}_B$ also contains $\Omega^s(X)$ for all integers s and, thus, using that A-$\underline{mod}$ is 2-Calabi-Yau, also $\nu(X)$, where ν denotes the Nakayama functor. In particular, the projective A-covers P^X of X and $P^{\nu(X)} = \nu(P^X)$ of $\nu(X)$ belong to $\mathcal{X}_B$. Now let P be an indecomposable direct summand of P^X. Consider the following sequence of A-modules

$$0 \longrightarrow rad(P) \longrightarrow P \longrightarrow top(P) = soc(\nu(P)) \longrightarrow \nu(P).$$

Since P and $\nu(P)$ belong to $\mathcal{X}_B$, so does $rad(P)$, as the kernel of the induced map from P to $\nu(P)$, and $top(P)$, as the image of this map. Now we can repeat the whole argument with any indecomposable direct summand X' of $rad(P)$. Note that $rad(P)$ cannot have any projective direct summand, since the algebra A is self-injective. Say $P = P_i$ for $i \in Q_0$. It follows that also the projective A-cover

$$P^{rad(P_i)} = \bigoplus_{\substack{\alpha \in \overline{Q}_1 \\ \alpha : i \to j}} P_j$$

of $rad(P_i)$ as well as its Nakayama shift $\nu(P^{rad(P_i)})$ lie in $\mathcal{X}_B$. Keeping in mind the shape of the underlying quiver, after finitely many steps, we conclude that $\mathcal{X}_B$ contains the free A-module of rank one. Thus, f must be an isomorphism contradicting our assumption. It follows that $\mathcal{X}_B$ can only contain projective A-modules. Indeed, since $\mathcal{X}_B$ is closed under kernels and cokernels in A-mod and $Hom_A(P,Q) \neq 0$ for all indecomposable projective A-modules P and Q, $\mathcal{X}_B$ must be of the form $add(P)$ for P projective with $End_A(P) \cong \mathbb{K}$. But such projectives do only exist in case A is of type $\mathcal{A}_n$ (otherwise there are non-trivial paths starting and ending at any given vertex of the underlying quiver). In fact, for every $n \geq 2$ the indecomposable projective A-modules with trivial endomorphism ring are the ones corresponding to the outer vertices of $\mathcal{A}_n$. Clearly, following Example 4.2.5, the bireflective subcategories $add(P)$ for those P give rise to finite dimensional homological ring epimorphisms. This completes the proof. $\square$

We finish the section by establishing a similar classification for weakly symmetric algebras. By $\mathfrak{D}(-)$ we denote the $\mathbb{K}$-dual $Hom_{\mathbb{K}}(-,\mathbb{K})$. Recall the following definition.

Definition 4.2.7. *A given finite dimensional $\mathbb{K}$-algebra A is called*

- ***symmetric**, if A is isomorphic to $\mathfrak{D}(A)$ as A-A-bimodules;*
- ***weakly symmetric**, if $top(P) \cong soc(P)$ for all indecomposable projective A-modules P.*

Note that weakly symmetric algebras are self-injective. In case A is symmetric, the Nakayama functor $\nu = \mathfrak{D}(A) \otimes_A -$ is naturally isomorphic to the identity functor on A-mod.

Theorem 4.2.8. *Let A be a connected and weakly symmetric $\mathbb{K}$-algebra fulfilling the Tachikawa conjecture and let $f : A \to B$ be a non-zero finite dimensional homological ring epimorphism. Then f is an isomorphism.*

Proof. The idea of the proof is motivated by the arguments used in the proof of Theorem 4.2.6. Without loss of generality, we can assume that A is basic and, thus, given as a bound path algebra $\mathbb{K}Q/I$. Since the ring epimorphism f is non-zero, by Proposition 4.2.3, there is an indecomposable projective A-module P that lies in $\mathcal{X}_B$. Since A is weakly symmetric, via a top-to-socle factorisation, we conclude that also $rad(P)$ belongs to $\mathcal{X}_B$. Say P is given by P_i for $i \in Q_0$. Again by Proposition 4.2.3, the projective A-cover of $rad(P_i)$ lies in $\mathcal{X}_B$ and, thus, so do all the P_j for $i \to j$ being an arrow in Q_1. Now we repeat the argument with all such P_j. Since A is connected and self-injective (i.e. every arrow in Q_1 is part of an oriented cycle), after finitely many steps, we conclude that $\mathcal{X}_B$ contains the free module of rank one. Thus, f is an isomorphism. □

In particular, this result applies to blocks of group algebras that were shown to be derived simple in [LY]. In the next section, we will see an example of a derived simple algebra that admits many non-trivial homological ring epimorphisms.

4.3 Homological ring epimorphisms for Nakayama algebras

In this section, we want to discuss and classify the homological ring epimorphisms for self-injective Nakayama algebras. Note that, since these algebras are representation finite, they fulfil the Tachikawa conjecture and, moreover, all ring epimorphisms will be finite dimensional. Using Proposition 4.2.3, for a self-injective Nakayama algebra A, a ring epimorphism $f : A \to B$ is homological if and only if f turns B into a projective A-module. We are now looking for a more explicit description. For the rest of this section fix A to be $\tilde{A}_n^h$ for $n, h \geq 2$ (see Section 3.3) and let M be a non-projective and indecomposable A-module. Then M is of infinite projective dimension and periodic with respect to the syzygy-functor Ω. The following lemma describes this periodicity and the corresponding *Ext*-groups.

Lemma 4.3.1. *Let A and M be as above. Denote by s the Loewy length of M. The following holds.*

(1) If $s \neq \frac{1}{2}h$, we have $\Omega^z(M) = M$ if and only if $z = 2x$ for $x \in \mathbb{N}_{\geq 1}$ with $xh \equiv 0$ modulo n. If these equivalent conditions hold, we get $Ext_A^z(M, M) \cong \mathbb{K}$.

(2) If $s = \frac{1}{2}h$, we have $\Omega^z(M) = M$ if and only if $z \in \mathbb{N}_{\geq 1}$ with $zs \equiv 0$ modulo n. If these equivalent conditions hold, we get $Ext_A^z(M, M) \cong \mathbb{K}$.

Proof. Note that, since M is not projective, we have $s < h$. Assume that M is a quotient of the indecomposable projective A-module P_i for $i \in \{1, ..., n\}$. The minimal projective resolution of M

is of the following form

$$\mathcal{P}_M : \ldots \longrightarrow P_{\overline{i+2h}} \longrightarrow P_{\overline{i+h+s}} \longrightarrow P_{\overline{i+h}} \longrightarrow P_{\overline{i+s}} \longrightarrow P_i \longrightarrow M$$

where the indices are to be read modulo n. Now if $s \neq \frac{1}{2}h$, then the length of the indecomposable A-module $\Omega^z(M)$, for $z \in \mathbb{N}_{\geq 1}$ odd, is $h-s \neq s$ such that $\Omega^z(M) \neq M$. On the other hand, the length of $\Omega^z(M)$ always equals s, if z is even. Hence, keeping in mind that every indecomposable A-module is uniserial, we get

$$\Omega^z(M) = M \Longleftrightarrow z \in 2\mathbb{N}_{\geq 1} \wedge \frac{1}{2}zh \equiv 0 \bmod n.$$

If $s = \frac{1}{2}h$, then the length of the module $\Omega^z(M)$ for $z \in \mathbb{N}_{\geq 1}$ always equals s and, thus,

$$\Omega^z(M) = M \Longleftrightarrow zs \equiv 0 \bmod n.$$

Finally, $\Omega^z(M) = M$ implies $Ext_A^z(M,M) \neq 0$ and, in fact, by looking at $Hom_A(\mathcal{P}_M, M)$, it follows

$$Ext_A^z(M,M) \cong \mathbb{K}.$$

□

Now let $f : A \to B$ be a homological ring epimorphism. We call f **semisimple**, if B is a semisimple $\mathbb{K}$-algebra. Since, by Proposition 4.2.3, B is a projective A-module, the semisimple homological ring epimorphisms of A are classified by the orthogonal collections of projective A-modules. From now on, we will assume that f is not semisimple. Let M be an indecomposable non-projective A-module in $\mathcal{X}_B$. Again by Proposition 4.2.3, the minimal projective A-resolution $\mathcal{P}_M$ of M is contained in $\mathcal{X}_B$ and coincides with the minimal projective resolution of M as a B-module. Let $\mathcal{C}_M$ be the smallest additive subcategory of A-*mod* containing M, all indecomposable projective A-modules appearing in $\mathcal{P}_M$ and the objects $\Omega^r(M)$ for $r > 0$. Then $\mathcal{C}_M$ is the smallest (not necessarily abelian) *higher extension-closed* subcategory of A-*mod* containing M and we have $\mathcal{C}_M \subseteq \mathcal{X}_B$. Using Lemma 4.3.1, we have the following immediate consequences.

- If $h = 2$, then all non-isomorphic homological ring epimorphisms $A \to B$ are semisimple. Note that for $h = 2$ all non-projective indecomposable A-modules are simple and, by Lemma 4.3.1(2), we get that $\mathcal{C}_S$ already equals A-*mod* for any simple A-module S.

- If $h = n-1$ or if n is a prime number with $h < n$, then a similar analysis of Lemma 4.3.1 yields that all non-isomorphic homological ring epimorphisms $A \to B$ are semisimple.

Example 4.3.2. *In case $h < n$, the first example of a non-isomorphic and non-semisimple homological ring epimorphism $f : A \to B$ occurs for $\tilde{A}_6^3$. There are precisely three such choices given by the universal localisations at $\Sigma = \{S_2, S_5\}$, $\Sigma = \{S_3, S_6\}$ or $\Sigma = \{S_1, S_4\}$. In all these cases, the $\mathbb{K}$-algebra A_Σ is Morita-equivalent to $\tilde{A}_4^2$ and $\mathcal{X}_{A_\Sigma}$ is given by $\mathcal{C}_{S_1}$, $\mathcal{C}_{S_2}$ or $\mathcal{C}_{S_3}$, respectively. The framed modules in the following picture of the AR-quiver describe the indecomposables in $\mathcal{C}_{S_1}$.*

$$\begin{tikzcd}
 & P_6 \arrow[d] & P_5 \arrow[d] & P_4 \arrow[d] & P_3 \arrow[d] & P_2 \arrow[d] & P_1 \arrow[d] & \\
\cdots & M_6 \arrow[d] \arrow[ur] & M_5 \arrow[d] \arrow[ur] & M_4 \arrow[d] \arrow[ur] & M_3 \arrow[d] \arrow[ur] & M_2 \arrow[d] \arrow[ur] & M_1 \arrow[d] & \cdots \\
 & S_6 \arrow[ur] & S_5 \arrow[ur] & S_4 \arrow[ur] & S_3 \arrow[ur] & S_2 \arrow[ur] & S_1 &
\end{tikzcd}$$

Next, we classify all homological ring epimorphisms for connected self-injective Nakayama algebras by using the classification of the universal localisations from Theorem 3.3.8.

Theorem 4.3.3. *Let A be a self-injective Nakayama algebra of the form $\tilde{A}_n^h$ for $n, h \geq 2$.*

(1) A admits a non-zero semisimple homological ring epimorphism if and only if $h \leq n$. These ring epimorphisms are classified by the non-empty orthogonal collections of indecomposable projective A-modules.

(2) A admits a non-isomorphic and non-semisimple homological ring epimorphism if and only if $gcd(n,h) = d > 1$ and $h > 2$. These ring epimorphisms are classified by the orthogonal collections of simple modules $\{S_{i_1}, ..., S_{i_k}\}$ with $i_j \in \{1, ..., d\}$ pairwise distinct for

$$k \in \begin{cases} \{1, ..., d-1\}, & \text{if } h \neq d \\ \{1, ..., d-2\}, & \text{if } h = d. \end{cases}$$

Proof. ad(1): This follows from Proposition 4.2.3 and the fact that indecomposable projective A-modules have trivial endomorphism algebras if and only if $h \leq n$.

ad(2): Let $f : A \to B$ be a non-isomorphic and non-semisimple homological ring epimorphism. By previous arguments, we already know that h must be greater than 2. Combining Proposition 4.2.3 and Lemma 3.3.3, we know that B is a self-injective Nakayama algebra and that ${}_AB$ is a projective A-module. Consequently, since A is connected and B is not semisimple, also B is connected and, thus, (up to Morita-equivalence) of the form $\tilde{A}_{\tilde{n}}^{\tilde{h}}$ for $2 \leq \tilde{n} \leq n$ and $2 \leq \tilde{h} \leq h$. By Corollary 3.3.9, we can write f as a universal localisation $f : A \to A_{\Sigma_B}$. We will consider the set W_{Σ_B}, which determines the localisation (see Lemma 3.3.6). We claim that W_{Σ_B} only contains simple A-modules.

First of all, since B is of infinite global dimension, W_{Σ_B} cannot contain any projective A-modules. Now suppose there is an indecomposable A-module M in W_{Σ_B} with $1 < l(M) = s < min\{n,h\}$ (recall that a non-projective module in W_{Σ_B} has length at most $n-1$). Without loss of generality, we can choose M to be of minimal length among the non-simple A-modules in W_{Σ_B}. Let P_i for $i \in \{1,...,n\}$ be the projective cover of M in A-*mod* and we choose $j \in \{1,...,n\}$ such that the corresponding simple A-module fulfils $Ext^1_A(S_i,S_j) \neq 0$. Consequently, the A-module $P_j/rad^{s-1}(P_j)$, the radical of M, belongs to $\mathcal{X}_{A_{\Sigma_B}}$ (see Lemma 3.3.6 and the defining properties for W_{Σ_B} discussed in the proof of Theorem 3.3.8). But the projective A-module P_j does not carry an A_{Σ_B}-module structure, since $Hom_A(\sigma_0^M, P_j)$ is not an isomorphism. This yields a contradiction, keeping in mind that the A-module ${}_AB$ must be projective. Consequently, W_{Σ_B} contains only simple A-modules or, equivalently, A_{Σ_B} is given by inverting certain arrows in the underlying quiver $\tilde{\Delta}_n$.

Now the fact that A_{Σ_B} is Morita-equivalent to an algebra $\tilde{A}^{\tilde{h}}_{\tilde{n}}$ for $2 \leq \tilde{n} \leq n$ and $2 \leq \tilde{h} \leq h$ induces some periodicity of length $2 \leq d \leq min\{h,n\}$ on the simple modules in W_{Σ_B}, where d divides h and n. More precisely, W_{Σ_B} is determined by a subset of the form $\{S_{i_1},...,S_{i_k}\}$ for $i_j \in \{1,...,d\}$ pairwise distinct and

$$k \in \begin{cases} \{1,...,d-1\}, & \text{if } h \neq d \\ \{1,...,d-2\}, & \text{if } h = d \end{cases}$$

such that a simple A-module S_m belongs to W_{Σ_B} if and only if there is some $j \in \{1,...,k\}$ with $m \equiv i_j$ modulo d. Note that we can choose d to be $gcd(n,h)$. In particular, we get $gcd(n,h) > 1$. Conversely, if $d > 1$ is the greatest common divisor of $h > 2$ and n, then every universal localisation at a set of simple A-modules $\mathcal{S}$, admitting a periodicity like above with respect to d, yields a non-isomorphic and non-semisimple homological ring epimorphism $A \to A_{\mathcal{S}}$. □

Note that this result allows us to count the homological ring epimorphisms for self-injective Nakayama algebras (up to epiclasses). For example, take A to be the algebra $\tilde{A}^h_n$ for $n = h \geq 2$. Then there are precisely

$$\sum_{i=0}^{n-1} \binom{n}{i}$$

non-zero homological ring epimorphisms out of a total number of $\binom{2n}{n}$ universal localisations. The algebras $\tilde{A}^h_n$ for $h > n$ and $gcd(n,h) = 1$ do not admit proper homological ring epimorphisms.

In some cases it is known that connected self-injective Nakayama algebras are derived simple. As a first example, this was shown in [Wi] for the algebra $\tilde{A}^2_2$. Nowadays, it is known to the experts that also the algebras $\tilde{A}^2_n$ and $\tilde{A}^n_2$ for $n \geq 2$ are derived simple. This can, indeed, be checked by classifying the indecomposable and exceptional compact objects in $D(A)$. A general overview on derived simplicity can be found in [AKLY]. We provide some further examples of derived simple self-injective Nakayama algebras that, by the previous theorem, admit many non-trivial homological ring epimorphisms.

Example 4.3.4. *Consider the algebra $A = \tilde{A}_n^h$ with $h = xn$ for $n, h \geq 2$ and $x \geq 1$. We will show that A is derived simple. Suppose we have a recollement of derived module categories of the form*

$$\begin{tikzcd} D(B) \arrow[r, "i_*"] & D(A) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r] & D(C). \arrow[l, shift right=3ex, "j_!"'] \arrow[l, shift left=3ex] \end{tikzcd}$$

By [LY, Proposition 4.1], the object $i_*(B)$ is compact and exceptional in $D(A)$. The same holds true for $j_!(C)$ and we have that $Hom_{D(A)}(j_!(C), i_*(B)[i]) = 0$ for all $i \in \mathbb{Z}$.

Now consider an indecomposable and compact object $\mathcal{P}$ in $D(A)$ given as

$$0 \longrightarrow P^{-m} \xrightarrow{d^{m-1}} \cdots \longrightarrow P^{-1} \xrightarrow{d^0} P^0 \longrightarrow 0$$

with $P^i \neq 0$ and $m \in \mathbb{N}$. Since A is uniserial, all the P^i are indecomposable projective A-modules. Further, assume that $d^0 \neq 0$. Using that $h = xn$, it follows that $ker(d^0) = coker(d^0)$. Consequently, we can construct the following morphism Ψ of chain complexes

$$\begin{tikzcd} \cdots \arrow[r] & P^{-2} \arrow[r, "d^{-1}"] \arrow[d, "0"] & P^{-1} \arrow[r, "d^0"] \arrow[d, "0"] & P^0 \arrow[r] \arrow[d, "\psi"] & 0 \arrow[r] \arrow[d, "0"] & 0 \\ \cdots \arrow[r] & P^{-3} \arrow[r, "d^{-2}"] & P^{-2} \arrow[r, "d^{-1}"] & P^{-1} \arrow[r, "d^0"] & P^0 \arrow[r] & 0 \end{tikzcd}$$

where the image of the map ψ is given by $ker(d^0) = coker(d^0)$. Next, suppose that Ψ is homotopic to zero. Therefore, it exists a map $\phi : P^0 \to P^{-2}$ such that $\phi \circ d^0 = 0$ and $d^{-1} \circ \phi = \psi$

$$\begin{tikzcd} P^{-1} \arrow[r, "d^0"] \arrow[d, "0"'] & P^0 \arrow[d, "\psi"] \arrow[dl, "\phi"'] \\ P^{-2} \arrow[r, "d^{-1}"] & P^{-1}. \end{tikzcd}$$

Now, by construction of ψ, there is also a map η such that $\eta \circ \psi = \phi$ and, thus, $\phi = \eta \circ d^{-1} \circ \phi$. It follows that the endomorphism $\eta \circ d^{-1}$ of P^{-2} must be an isomorphism, forcing d^{-1} to be injective. Since P^{-2} and P^{-1} are indecomposable and A is self-injective, d^{-1} is an isomorphism turning d^0 into the zero map. This contradicts our assumption. Consequently, the morphism Ψ is non-zero in

$D(A)$ and, thus, the complex $\mathcal{P}$ is not exceptional.

Therefore, the only indecomposable compact and exceptional objects in $D(A)$ are of the form

$$\cdots \longrightarrow 0 \longrightarrow P \longrightarrow 0 \longrightarrow \cdots$$

where P denotes an indecomposable projective A-module. In particular, the complexes $j_!(C)$ and $i_*(B)$ are concentrated in one degree and can be identified with projective A-modules. Finally, since $Hom_A(P,Q) \neq 0$ for all indecomposable projective A-modules P and Q and, moreover, $Hom_{D(A)}(j_!(C), i_*(B)[i]) = 0$ for all $i \in \mathbb{Z}$, it follows that either $j_!(C) = A$ (and $i_*(B) = 0$) or $i_*(B) = A$ (and $j_!(C) = 0$). Thus, the above recollement is trivial and A is derived simple.

Chapter 5

Silting modules

This chapter contains joint work with Lidia Angeleri Hügel and Jorge Vitória (see [AMV]). We introduce the new concept of silting modules. These modules generalise tilting modules over an arbitrary ring, as well as support τ-tilting modules over a finite dimensional algebra.

One motivation to develop such a theory is based on the observation that the class of tilting modules seems too *small* to parametrise all universal localisations or homological ring epimorphisms of a given ring. In fact, the various connections between tilting modules and localisations that were recently discovered seem to be part of a bigger picture – as suggested by the correspondences between support τ-tilting modules and universal localisations discussed in Chapter 3. Before going back to ring epimorphisms in Chapter 6, in this chapter, we set the foundation for a *general* τ*-tilting theory* over an arbitrary ring. The new modules we obtain, will be called silting.

We start this chapter with a small reminder on tilting and τ-tilting modules. Moreover, since support τ-tilting modules are known to parametrise certain structures in the derived module category – that we intend to generalise – we also recall some of the relevant notions needed later on. Afterwards, in the first section, we introduce the concept of silting modules. One way to approach silting modules is via quasitilting modules, originally defined in [CDT]. We begin by discussing a large version of quasitilting modules and we show that these modules can be used to classify certain torsion classes (see Theorem 5.1.6). Subsequently, we define silting and partial silting modules proving that every partial silting module admits an analogue of the Bongartz complement (see Theorem 5.1.15). In the last section, we study the interplay of silting complexes and silting modules. First, we show bijections between (not necessarily compact) silting complexes of finite length and certain t-structures and co-t-structures in the derived category (see Theorem 5.2.6). Finally, we see how these bijections restrict to a correspondence between 2-term silting complexes and silting modules (see Theorem 5.2.11).

Recall from Chapter 1 (see Definition 1.7.1 and Definition 1.7.2) that a module T is said to be **tilting**, if $Gen(T) = T^{\perp_1}$ and it is called **partial tilting** if

(PT1) $T^{\perp_1}$ is a torsion class;

(PT2) T lies in $T^{\perp_1}$.

Furthermore, once (PT1) is satisfied, (PT2) is equivalent to $Gen(T)$ lying in $T^{\perp_1}$. In fact, also $Gen(T)$ is then a torsion class which we prove by following the arguments in [C, Proposition 4.4].

Lemma 5.0.5. *If a module T satisfies $Gen(T) \subseteq T^{\perp_1}$, then $(Gen(T), T^{\circ})$ is a torsion pair.*

Proof. We verify that $Gen(T) = {}^{\circ}(T^{\circ})$. Clearly, we have $Gen(T) \subseteq {}^{\circ}(T^{\circ})$. For M in ${}^{\circ}(T^{\circ})$, consider the short exact sequence

$$0 \to \tau_T(M) \to M \to M/\tau_T(M) \to 0$$

given by the trace $\tau_T(M)$ of T in M. Applying the functor $Hom_A(T, -)$ to the sequence and using the fact that $Ext^1_A(T, \tau_T(M)) = 0$, we see that $M/\tau_T(M)$ lies in T°. Thus, we have $M \cong \tau_T(M)$. $\square$

Silting modules are also intended to generalise support τ-tilting modules over a finite dimensional $\mathbb{K}$-algebra (see Definition 1.8.1). Passing to possibly infinitely generated modules over arbitrary rings, we need a description of these modules that does not rely on the Auslander-Reiten translation τ. Such a description is given in the next result extending work of [AIR] and [ASm].

Theorem 5.0.6. *Let A be a finite dimensional $\mathbb{K}$-algebra and let T be a finitely generated A-module with minimal projective presentation σ.*

(1) An A-module M satisfies $Hom_A(M, \tau T) = 0$ if and only if the morphism of abelian groups $Hom_A(\sigma, M)$ is surjective.

(2) T is τ-rigid if and only if $Gen(T) \subseteq T^{\perp_1}$.

(3) T is support τ-tilting if and only if $Gen(T)$ consists of the modules M such that $Hom_A(\widetilde{\sigma}, M)$ is surjective, where $\widetilde{\sigma}$ is the projective presentation of T obtained as the direct sum of σ with the complex $(Ae \to 0)$ for a suitable idempotent element e of A.

Proof. Statement (1) follows by similar arguments to the ones used in [AIR, Proposition 2.4] and statement (2) is a direct generalisation of [ASm, Proposition 5.8], using a more general version of the Auslander-Reiten formula (see, for example, [Kr]).

By [AIR, Corollary 2.13], T is support τ-tilting if and only if $gen(T)$ consists precisely of the finitely generated A-modules M such that $Hom_A(\widetilde{\sigma}, M)$ is surjective. Consider the torsion pair $(gen(T), T^\circ \cap A\text{-}mod)$ in A-mod. Note that the subcategory of A-mod formed by the A-modules M such that $Hom_A(\widetilde{\sigma}, M)$ is surjective forms a torsion class in A-mod, whose associated torsion-free class contains $T^\circ \cap A\text{-}mod$. Moreover, by (2) and Lemma 5.0.5, we also have that $(Gen(T), T^\circ)$ is a torsion pair in A-mod. Our claim now follows from the fact that there is a unique torsion pair $(\mathcal{T}, \mathcal{F})$ in A-mod with $gen(T) \subseteq \mathcal{T}$ and $T^\circ \cap A\text{-}mod \subseteq \mathcal{F}$, given by the direct limit closure of $(gen(T), T^\circ \cap A\text{-}mod)$ in A-mod (compare Proposition 7.4.2 in the last chapter of this thesis). $\square$

Support τ-tilting modules turn out to be in bijection with certain (2-term) complexes, called silting, and they are closely related with certain t-structures and co-t-structures. Let us recall some definitions. First of all, for an object X in $D(A)$, we say that $\{X[i] : i \in \mathbb{Z}\}$ **generates** $D(A)$, if whenever a complex Y in $D(A)$ satisfies $Hom_{D(A)}(X[i], Y) = 0$ for all $i \in \mathbb{Z}$, then $Y = 0$.

Definition 5.0.7. *A bounded complex of finitely generated projective A-modules σ is **silting** if*

(1) $Hom_{D(A)}(\sigma, \sigma[i]) = 0$ for all $i > 0$;

(2) the set $\{\sigma[i] : i \in \mathbb{Z}\}$ generates $D(A)$.

*A silting complex σ is said to be **2-silting** if σ is a 2-term complex of projective A-modules.*

Remark 5.0.8. *The notion of silting complex has appeared in different references with different generation requirements. In order to remove any ambiguity we remark that all these generation properties are equivalent. Given a ring A and an object X in $D(A)$, the set $\{X[i] : i \in \mathbb{Z}\}$ generates $D(A)$ if and only if $D(A)$ is the smallest triangulated subcategory of $D(A)$ which contains X and is closed under coproducts. In fact, the if-part is clear, and the converse implication follows from [AJS, Proposition 4.5] and [NS, Lemma 2.2(1)].*

If X is compact in $D(A)$, then the two equivalent conditions above are furthermore equivalent to say that $K^b(A\text{-}proj)$ is the smallest triangulated subcategory of $D(A)$ containing X and closed under direct summands (i.e., the smallest thick subcategory containing X). Indeed, under this assumption $\{X[i] : i \in \mathbb{Z}\}$ is clearly a generating set for $D(A)$. The converse holds as argued in [AI, Proposition 4.2], using arguments of Ravenel ([R]) and Neeman ([N]).

Throughout, we will use the following notation. For a subcategory $\mathcal{X}$ of a triangulated category D we denote by $\mathcal{X}^{\perp_{>0}}$ the subcategory consisting of the objects Y in D such that $Hom_D(X,Y[i])=0$ for all $i>0$ and all $X\in\mathcal{X}$. Similarly, one defines $\mathcal{X}^{\perp_{<0}}$ and $\mathcal{X}^{\perp_0}$. If the subcategory consists of a single object X, we write just $X^{\perp_{>0}}$, $X^{\perp_{<0}}$, and $X^{\perp_0}$. The notation for left orthogonal subcategories is defined analogously.

Definition 5.0.9. *[BBD, Bo, P] Let D be a triangulated category. A* ***t-structure*** *(respectively, a* ***co-t-structure****) in D is a pair of subcategories $(\mathcal{V}^{\leq 0},\mathcal{V}^{\geq 0})$ (respectively, $(\mathcal{U}_{\geq 0},\mathcal{U}_{\leq 0})$) such that*

(1) $Hom_D(\mathcal{V}^{\leq 0},\mathcal{V}^{\geq 0}[-1])=0$ (respectively, $Hom_D(\mathcal{U}_{\geq 0},\mathcal{U}_{\leq 0}[1])=0$);

(2) $\mathcal{V}^{\leq 0}[1]\subseteq\mathcal{V}^{\leq 0}$ (respectively, $\mathcal{U}_{\geq 0}[-1]\subseteq\mathcal{U}_{\geq 0}$);

(3) For every X in D, there is a triangle

$$Y\longrightarrow X\longrightarrow W\longrightarrow Y[1]$$

such that $Y\in\mathcal{V}^{\leq 0}$ and $W\in\mathcal{V}^{\geq 0}[-1]$ (respectively, Y lies in $\mathcal{U}_{\geq 0}$ and W lies in $\mathcal{U}_{\leq 0}[1]$).

We use the notations $\mathcal{V}^{\leq n}:=\mathcal{V}^{\leq 0}[-n]$, $\mathcal{V}^{\geq n}:=\mathcal{V}^{\geq 0}[-n]$, $\mathcal{U}_{\geq n}:=\mathcal{U}_{\geq 0}[-n]$ and $\mathcal{U}_{\leq n}:=\mathcal{V}_{\leq 0}[-n]$. A t-structure (respectively, co-t-structure) is furthermore said to be ***bounded*** *if*

$$\bigcup_{n\in\mathbb{Z}}\mathcal{V}^{\leq n}=D=\bigcup_{n\in\mathbb{Z}}\mathcal{V}^{\geq n}\quad(\textit{respectively,}\quad\bigcup_{n\in\mathbb{Z}}\mathcal{U}_{\geq n}=D=\bigcup_{n\in\mathbb{Z}}\mathcal{U}_{\leq n}).$$

For a t-structure $(\mathcal{V}^{\leq 0},\mathcal{V}^{\geq 0})$, the intersection $\mathcal{V}^{\leq 0}\cap\mathcal{V}^{\geq 0}$ is called the **heart** and $\mathcal{V}^{\leq 0}$ is called the **aisle**. Note that the aisle completely determines the t-structure since $\mathcal{V}^{\geq 0}=(\mathcal{V}^{\leq 0})^{\perp_0}[1]$. Furthermore, for a t-structure, the triangles in axiom (3) are functorial, giving rise to **truncation functors** (see [BBD]).

Example 5.0.10. *(1) The pair $(D^{\leq 0},D^{\geq 0})$ in $D(A)$, where $D^{\leq 0}$ (respectively, $D^{\geq 0}$) is the subcategory of complexes with cohomologies lying in non-positive (respectively, non-negative) degrees, is a t-structure, called the* ***standard t-structure****. We denote its associated truncation functors by $\tau^{\leq n}$ and $\tau^{\geq n}$, for all $n\in\mathbb{Z}$.*

(2) [Bo, P] Consider the triangulated subcategory $K_p(A)$ of $K(A)$ of homotopically projective complexes. Recall that an object X in $K(A)$ is ***homotopically projective****, if $Hom_{K(A)}(X,Y)=0$ for all acyclic complexes Y. For example, right bounded complexes of projective A-modules are homotopically projective. In fact, homotopically projective complexes play the role of right bounded*

*complexes of projective A-modules in $K^-(A)$. The canonical functor from $K(A)$ to $D(A)$ is known to induce a triangle equivalence between $K_p(A)$ and $D(A)$ (see, for example, [K3]). We use this fact throughout without further mention. The pair $(K_{\geq 0}, K_{\leq 0})$ in $K_p(A)$, where $K^{\geq 0}$ (respectively, $K_{\leq 0}$) is the subcategory of complexes whose negative (respectively, positive) components are zero, is a co-t-structure, called the **standard co-t-structure**. The triangles in axiom (3) can be obtained (non-functorially) using the so-called **stupid truncations**, where zero replaces the components of the complex which are outside the required bound.*

(3) [HRS, Theorem 2.1] A torsion pair $(\mathcal{T}, \mathcal{F})$ in A-Mod induces a t-structure $(D^{\leq 0}_{\mathcal{T}}, D^{\geq 0}_{\mathcal{F}})$ in $D(A)$ given by

$$D^{\leq 0}_{\mathcal{T}} := \{X \in D(A) : H^0(X) \in \mathcal{T}, H^i(X) = 0, \forall i > 0\}$$

$$D^{\geq 0}_{\mathcal{F}} := \{X \in D(A) : H^{-1}(X) \in \mathcal{F}, H^i(X) = 0, \forall i < -1\}$$

*which is called the **HRS-tilt with respect to** $(\mathcal{T}, \mathcal{F})$.*

*(4) [AJS2, Proposition 3.2] For every object X in $D(A)$ there is a t-structure $(aisle(X), X^{\perp_{<0}})$, called **the t-structure generated by** X, where $aisle(X)$ is the smallest coproduct-closed suspended subcategory of $D(A)$ containing X. Recall that an additive subcategory of $D(A)$ is called **suspended**, if it is closed under extensions and positive shifts.*

The following theorems, which we will generalise to a larger context, relate some of the concepts introduced above. For details on the notion of a silting t-structure we refer to [AI, Definition 4.9] and Definition 5.2.4. For related results, see also [IJY] and [MSSS].

Theorem 5.0.11. *[KY, Theorem 6.1][KN] Let A be a finite dimensional $\mathbb{K}$-algebra. There are bijections between*

(1) isomorphism classes of basic silting complexes in $K^b(A\text{-}proj)$;

(2) bounded t-structures in $D^b(A\text{-}mod)$ whose heart is equivalent to Γ-mod for a $\mathbb{K}$-algebra Γ;

(3) bounded co-t-structures in $K^b(A\text{-}proj)$.

Theorem 5.0.12. *[AIR, Theorem 3.2],[AI, Theorem 4.10] Let A be a finite dimensional $\mathbb{K}$-algebra. There are bijections between*

(1) isomorphism classes of basic support τ-tilting A-modules;

(2) isomorphism classes of basic 2-silting complexes in $K^b(A\text{-}proj)$;

(3) 2-silting t-structures $(\mathcal{U}^{\leq 0}, \mathcal{U}^{\geq 0})$ in $D(A)$.

5.1 Introduction to silting modules

We want to introduce a class of modules that generalises tilting modules over arbitrary rings, and at the same time, coincides with support τ-tilting modules when restricting to finitely generated modules over a finite dimensional algebra. One of the main common features of tilting and support τ-tilting modules is their connection to torsion classes that provide left (and right) approximations. Therefore, we start by discussing the existence of such approximations. Afterwards, we define silting modules and study further properties and examples.

5.1.1 Approximations and quasitilting modules

A crucial feature of tilting theory is that tilting classes provide special pre-envelopes. Recall that, given a subcategory $\mathcal{T}$ of A-Mod, a **special $\mathcal{T}$-pre-envelope** of an A-module M is a short exact sequence

$$0 \longrightarrow M \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$$

such that B lies in $\mathcal{T}$ and $Ext^1_A(C, \mathcal{T}) = 0$ (and so ϕ is a left $\mathcal{T}$-approximation of M).

Theorem 5.1.1. *[ATT, Theorem 2.1] A torsion class $\mathcal{T}$ in A-Mod is a tilting torsion class if and only if every A-module admits a special $\mathcal{T}$-pre-envelope.*

Also support τ-tilting modules induce approximation sequences, but the map ϕ is not injective in general. So, we now turn to torsion classes providing left approximations with Ext-projective cokernel. The classification of such torsion classes will lead us to the notion of a quasitilting module, and it will allow to recover a result from [AIR] relating support τ-tilting modules with functorially finite torsion classes (see Remark 5.1.17).

First, we recall the notion of a $*$-module ([C]). Such modules arise in the literature as capturing *half* of the categorical equivalences of the Brenner-Butler theorem in tilting theory. In fact $*$-modules are precisely those A-modules T such that the functor $Hom_A(T, -)$ induces an equivalence between $Gen(_AT)$ and $Cogen(_B\mathsf{D}(T))$, where $B = End_A(T)$ and $\mathsf{D}(T)$ is the dual of T with respect to an injective cogenerator of A-Mod. This forces them to be finitely generated ([Tr]). For our purpose we have to drop this finiteness condition and work with the following "large version" of the notion of a $*$-module.

Definition 5.1.2. *An A-module T is a **$*$-module** if $Gen(T) = Pres(T)$, and $Hom_A(T, -)$ is exact for short exact sequences in $Gen(T)$.*

Quasitilting modules were introduced in [CDT] as the (self-small) $*$-modules T for which $Gen(T)$ is a torsion class. In fact, there are many equivalent ways of defining such modules (see [CDT, Proposition 2.1]). For a subcategory $\mathcal{C}$ of A-Mod, we denote by $\overline{\mathcal{C}}$ the subcategory formed by the submodules of all modules in $\mathcal{C}$.

Lemma and Definition 5.1.3. *The following statements are equivalent for an A-module T.*

(1) T is a $*$-module and $Gen(T)$ is a torsion class;

(2) $Pres(T) = Gen(T)$ and T is Ext-projective in $Gen(T)$;

(3) $Gen(T) = \overline{Gen(T)} \cap T^{\perp_1}$.

We say that T is ***quasitilting*** *if it satisfies any of the equivalent conditions above.*

Proof. (1)⇒(2): We only have to show that $Ext^1_A(T, Gen(T)) = 0$. Consider a short exact sequence

$$0 \longrightarrow M \longrightarrow N \xrightarrow{g} T \longrightarrow 0$$

with M in $Gen(T)$. Since $Gen(T)$ is a torsion class, N lies in $Gen(T)$ and, by assumption, the sequence remains exact when applying the functor $Hom_A(T, -)$. But then it is split exact as 1_T factors through g.

(2)⇒(1): It is clear that if T is Ext-projective in $Gen(T)$ then $Hom_A(T, -)$ is exact for short exact sequences in $Gen(T)$. By Lemma 5.0.5, $Gen(T)$ is a torsion class and, thus, we have (1).

(2)⇒(3): It is clear that $Gen(T) \subseteq \overline{Gen(T)} \cap T^{\perp_1}$. For the reverse inclusion, let N lie in $\overline{Gen(T)} \cap T^{\perp_1}$ and let M be an object in $Gen(T)$ such that there is a monomorphism $f : N \to M$. Clearly, $C := coker(f)$ lies in $Gen(T)$ and, thus, in $Pres(T)$. So there is a surjection $g : T' \to C$ with T' in $Add(T)$ such that $K := ker(g)$ lies in $Gen(T)$. Since $Ext^1_A(T', N) = 0$ by assumption, we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "a"] & T' \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "f"] & M \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

Now, the Snake Lemma shows that $coker(a) = coker(b)$ and, thus, $coker(a)$ lies in $Gen(T)$. Since $Gen(T)$ is extension-closed by Lemma 5.0.5, we conclude that N lies in $Gen(T)$.

(3)⇒(2) We only need to show that $Gen(T) \subseteq Pres(T)$. Let M lie in $Gen(T)$ and consider the universal map $u : T^{(I)} \to M$, where $I = Hom_A(T,M)$. Clearly u is surjective, and since

$$Ext^1_A(T, T^{(I)}) = 0,$$

it follows that $ker(u)$ is in $T^{\perp_1}$. By assumption, $ker(u)$ lies in $Gen(T)$, so M is in $Pres(T)$. □

We will require our modules to be **finendo**, i.e. finitely generated over their endomorphism ring, as this characterises the modules T for which $Gen(T)$ provides left approximations ([ATT, Proposition 1.2]). Note that this is further equivalent to $Gen(T)$ being closed for direct products ([CM, Lemma on p.408]). Recall that a module is called **faithful**, if its annihilator is zero. The following lemma extends results in [DH, Corollary 2] and [C2, Corollary 6], relating $*$-modules to tilting, to the context of arbitrary rings.

Lemma 5.1.4. *An A-module T is a finendo $*$-module if and only if it is a tilting $A/Ann(T)$-module.*

Proof. Set $\bar{A} = A/Ann(T)$. Clearly, T lies in $\mathcal{X}_{\bar{A}}$, since $Hom_A(\bar{A},T) \cong T$, by evaluation. Consequently, using Proposition 1.3.8, it follows that

$$Gen(_AT) = Gen(_{\bar{A}}T).$$

Therefore, T is a finendo $*$-module over A if and only if T is a finendo $*$-module over $\bar{A}$. So, without loss of generality, it is enough to show that T is a faithful finendo $*$-module over A if and only if T is a tilting A-module.

The if-part is clear. For the only-if-part, consider a faithful finendo $*$-module T. As in [C2, Theorem 3], we see that all injective A-modules are contained in $Gen(T)$, and $Gen(T) \subseteq T^{\perp_1}$. We repeat the arguments for the reader's convenience. Since T is faithful there is a monomorphism $\phi : A \to T^\alpha$ for some set α, where T^α lies in $Gen(T)$ as T is finendo. Now every surjection $A^{(I)} \to E$ to an injective module E extends to a surjection $(T^\alpha)^{(I)} \to E$, showing the first claim. Further, given M in $Gen(T)$, the functor $Hom_A(T,-)$ is exact on the short exact sequence in $Gen(T)$ induced by an injective envelope $M \to E(M)$ and, since $Ext^1_A(T,E(M)) = 0$, we get $Ext^1_A(T,M) = 0$.

By Lemma 5.0.5, we have that $Gen(T)$ is a torsion class. Thus, by Lemma and Definition 5.1.3, T is a quasitilting module and

$$Gen(T) = \overline{Gen(T)} \cap T^{\perp_1}.$$

But $\overline{Gen(T)} = A\text{-}Mod$, since every injective module is in $Gen(T)$. Hence, $Gen(T) = T^{\perp_1}$. □

Let us turn to the existence of approximations.

Proposition 5.1.5. *The following are equivalent for an A-module T.*

(1) T is a finendo quasitilting module.

(2) T is Ext-projective in $Gen(T)$ and there is an exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0,$$

with T_0 and T_1 in $Add(T)$ and ϕ a left $Gen(T)$-approximation.

Proof. (1)⇒(2): T is by definition Ext-projective in $Gen(T)$ and, moreover, T is a tilting $\bar{A}$-module by Lemma 5.1.4. Then there is a short exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\bar{\phi}} T_0 \longrightarrow T_1 \longrightarrow 0$$

with T_0 and T_1 in $Add(T)$ and $\bar{\phi}$ is a left $Gen(T)$-approximation in $\bar{A}$-Mod. The composition with the canonical projection $\pi : A \to \bar{A}$ yields the desired left $Gen(T)$-approximation $\phi = \bar{\phi} \circ \pi : A \to T_0$ in A-Mod.

(2)⇒(1): By Lemma 5.1.4, it is enough to show that T is an $\bar{A}$-tilting module. First, we see that $Gen(T)$ is contained $ker(Ext^1_{\bar{A}}(T,-))$. Indeed, every short exact sequence

$$0 \to M \to N \to T \to 0$$

in $\bar{A}$-Mod with M in $Gen(T)$, splits in A-Mod as T is Ext-projective, and, thus, it splits in $\bar{A}$-Mod. Now, we show that $Ann(T) = ker(\phi)$. In fact, $Ann(T) \subseteq ker(\phi)$ as T_0 lies in $Gen(T)$ and

$$Ann(T) = Ann(Gen(T)).$$

Conversely, we use that $Ann(T)$ is the intersection of the kernels of all maps in $Hom_A(A,T)$. Since every map $f : A \to T$ factors through ϕ, we infer $ker(\phi) \subseteq ker(f)$.

Therefore, ϕ factors as $\phi = \bar{\phi} \circ \pi$ through the canonical projection $\pi : A \to \bar{A}$. From the sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\bar{\phi}} T_0 \longrightarrow T_1 \longrightarrow 0$$

we deduce that every X in $ker(Ext^1_{\bar{A}}(T,-))$, since it is generated by $\bar{A}$ and satisfies $Ext^1_{\bar{A}}(T_1,X) = 0$, it is also generated by T_0, and, thus, by T. Hence, $Gen(T) = ker(Ext^1_{\bar{A}}(T,-))$. □

We now classify the torsion classes that yield left approximations with Ext-projective cokernel.

Theorem 5.1.6. *The following are equivalent for a torsion class $\mathcal{T}$ in A-Mod.*

(1) For every A-module M there is a sequence

$$M \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$$

such that ϕ is a left $\mathcal{T}$-approximation and C is Ext-projective in $\mathcal{T}$.

(2) There is a finendo quasitilting A-module T such that $\mathcal{T} = Gen(T)$.

Proof. (1)⇒(2): Choose $M = A$ with an approximation sequence

$$A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$$

and set $T = B \oplus C$. Clearly, we have $Gen(T) \subseteq \mathcal{T}$. Conversely, if X is a module in $\mathcal{T}$, any surjection $f : A^{(I)} \to X$ factors through the $\mathcal{T}$-approximation $\phi^{(I)}$ via a surjection $B^{(I)} \to X$, showing that X lies in $Gen(T)$. Thus, we have that $Gen(T) = \mathcal{T}$. By Proposition 5.1.5, it remains to show that T is Ext-projective in $Gen(T)$. In fact, by assumption, we have to verify this only for B. As in the proof of Proposition 5.1.5 we obtain a short exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\bar{\phi}} B \longrightarrow C \longrightarrow 0$$

over $\bar{A} = A/Ann(T)$, and we see that $Gen(T)$ is contained $ker(Ext^1_{\bar{A}}(C,-))$. Using the projectivity of ${}_{\bar{A}}\bar{A}$, we infer that $Gen(T)$ is also contained $ker(Ext^1_{\bar{A}}(B,-))$. Consider now a short exact sequence $0 \to M \to N \to B \to 0$ in A-Mod with M in $Gen(T)$. Since $Gen(T)$ is a torsion class, also N belongs to $Gen(T)$ and the sequence actually lies in $\bar{A}$-Mod. Then it splits in $\bar{A}$-Mod, and, thus, it also splits in A-Mod. So B is Ext-projective in $Gen(T)$.

(2)⇒(1): As in [ATT, Proposition 1.2], we use the approximation sequence for A in Proposition 5.1.5 to construct approximation sequences for all A-modules M, where the cokernels turn out to lie in $Add(T)$ and, thus, are Ext-projective modules in $\mathcal{T}$. □

The next lemma tells how to recover a quasitilting module from its associated torsion class.

Lemma 5.1.7. *If T is quasitilting, then $Add(T)$ is the class of Ext-projective modules in $Gen(T)$.*

Proof. If T is Ext-projective in $Gen(T)$, then so is every module in $Add(T)$. Conversely, given an Ext-projective module M in $Gen(T) = Pres(T)$, there is a surjection $f : T' \to M$, for some T' in $Add(T)$, with $ker(f)$ in $Gen(T)$. The Ext-projectivity of M implies that the short exact sequence induced by f splits and, thus, M lies in $Add(T)$. □

Consequently, two quasitilting modules have the same additive closure if and only if they generate the same torsion class. We will, thus, say that two quasitilting modules T_1 and T_2 are **equivalent** if $Add(T_1) = Add(T_2)$. Theorem 5.1.6 can now be rephrased as follows.

Corollary 5.1.8. *There is a bijection between the equivalence classes of finendo quasitilting A-modules and torsion classes $\mathcal{T}$ in A-Mod such that every A-module has a left $\mathcal{T}$-approximation with Ext-projective cokernel.*

5.1.2 Silting modules

In this subsection we study (partial) silting modules, the main objects under consideration in this chapter. These modules will be defined in a way suggested by Theorem 5.0.6. For a morphism σ in A-$Proj$, we consider the class of A-modules

$$\mathcal{D}_\sigma := \{X \in A\text{-}Mod | Hom_A(\sigma, X) \text{ is surjective}\}.$$

We collect some useful properties of $\mathcal{D}_\sigma$.

Lemma 5.1.9. *Let σ be a map in A-$Proj$ with cokernel T.*

(1) $\mathcal{D}_\sigma$ is closed under quotients, extensions, and direct products.

(2) The class $\mathcal{D}_\sigma$ is contained in $T^{\perp_1}$.

(3) An A-module X belongs to $\mathcal{D}_\sigma$ if and only if for some (respectively, all) projective presentation(s) ω of X the condition $Hom_{D(A)}(\sigma, \omega[1]) = 0$ is satisfied.

Proof. (1) By construction, $\mathcal{D}_\sigma$ is closed for products. Now set $\sigma : P_{-1} \to P_0$ and consider a short exact sequence in A-Mod

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

We get the following induced commutative diagram of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & Hom_A(P_0, X) \arrow[r] \arrow[d] & Hom_A(P_0, Y) \arrow[r] \arrow[d] & Hom_A(P_0, Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom_A(P_{-1}, X) \arrow[r] & Hom_A(P_{-1}, Y) \arrow[r] & Hom_A(P_{-1}, Z) \arrow[r] & 0
\end{tikzcd}$$

Now the missing closure conditions follow by the Snake Lemma.

(2) Set $\sigma : P_{-1} \to P_0$ and write $\sigma = i \circ \pi$ with $\pi : P_{-1} \to Im(\sigma)$ and $i : Im(\sigma) \to P_0$. By applying the functor $Hom_A(-,N)$, with N in $\mathcal{D}_\sigma$, to the short exact sequence induced by the monomorphism $i : Im(\sigma) \to P_0$ we get the exact sequence

$$Hom_A(P_0,N) \xrightarrow{i_*} Hom_A(Im(\sigma),N) \longrightarrow Ext^1_A(T,N) \longrightarrow 0.$$

We show that i_* is surjective. Consider a test map $f : Im(\sigma) \to N$. Since N belongs to $\mathcal{D}_\sigma$, there is a map $g : P_0 \to N$ such that $f \circ \pi = g \circ i \circ \pi$. Hence, since π is an epimorphism, we get $f = g \circ i$.

(3) This is a consequence of [AIR, Lemma 3.4]. □

Definition 5.1.10. *We say that an A-module T is*

- ***partial silting*** *if there is a projective presentation σ of T such that*
  - *(S1) $\mathcal{D}_\sigma$ is a torsion class.*
  - *(S2) T lies in $\mathcal{D}_\sigma$.*
- ***silting*** *if there is a projective presentation σ of T such that $Gen(T) = \mathcal{D}_\sigma$.*

We will then say that T is (partial) silting ***with respect to*** *σ.*

Remark 5.1.11.

(1) If T is partial silting, then $Gen(T) \subseteq \mathcal{D}_\sigma \subseteq T^{\perp_1}$ by Lemma 5.1.9(2), and $(Gen(T), T^\circ)$ is a torsion pair by Lemma 5.0.5. The same arguments show that every silting module is partial silting.

(2) Since $\mathcal{D}_\sigma$ is always closed for quotients and extensions, condition (S1) is equivalent to require that $\mathcal{D}_\sigma$ is closed for coproducts. This is always true when σ is a map in A-$proj$ and, thus, a compact object in $D(A)$. So, in this case, T is partial silting if and only if

$$Hom_{D(A)}(\sigma, \sigma[1]) = 0.$$

The latter property hints on the choice of the name silting for our modules, which will indeed be justified by the relation with (2-term) silting complexes (to be explored in Section 5.2).

Notice, however, that in general $\mathcal{D}_\sigma$ can contain T, and even all direct sums of copies of T, without being a torsion class. For example, the generic module G over the Kronecker algebra (see Example 3.1.10) satisfies the condition $Gen(G) \subseteq G_1^\perp$. Taking a monomorphic presentation σ of G, we obtain the class $\mathcal{D}_\sigma = G^{\perp_1}$. But $\mathcal{D}_\sigma$ is not a torsion class (and G is not partial tilting according to Definition 1.7.2), because it is not closed under direct sums. Indeed, every adic module $S_{-\infty}$ belongs to $G^{\perp_1}$, while ${S_{-\infty}}^{(\omega)}$ does not. This follows from [O, Proposition 1 and Remark on p.265] *stating that a torsion-free regular module belongs to $G^{\perp_1}$ if and only if it is pure-injective. For details on infinite dimensional modules over hereditary algebras we refer to [Ri] and [RR].*

(3) Note that the definitions in 5.1.10 depend on the choice of σ: not all projective presentations of a silting or partial silting module will fulfil conditions (S1) and (S2). Further, T can be partial silting with respect to different projective presentations giving rise to different associated torsion classes. However, there is a unique torsion class, $Gen(T)$, turning T into a silting module.

There is an evident parallel between the conditions (S1) and (S2) in Definition 5.1.10 and the axioms (PT1) and (PT2) defining partial tilting modules and, thus, also with (T1) and (T2) in the definition of a tilting module (compare Chapter 1). We will later obtain an analogue of (T3) in Theorem 5.1.14. Moreover, the definition of silting clearly resembles the condition $Gen(T) = T^{\perp_1}$ defining tilting. Let us make this comparison more precise. An A-module T is said to be **sincere** if $Hom_A(P,T) \neq 0$ for all non-zero projective A-modules P.

Proposition 5.1.12. *The following statements hold.*

(1) An A-module T is (partial) tilting if and only if T is a (partial) silting module with respect to a monomorphic projective presentation.

(2) A module T of projective dimension at most one is tilting if and only if it is sincere silting.

Proof. (1) If T is a partial tilting module, there is a monomorphic projective presentation σ of T, and $\mathcal{D}_\sigma = T^{\perp_1}$. Since $Ext^1_A(T,T) = 0$, T lies in $\mathcal{D}_\sigma$, so that T is partial silting with respect to σ. If, furthermore, T is tilting, then $Gen(T) = T^{\perp_1} = \mathcal{D}_\sigma$, thus, showing that T is silting. The converse implication is shown similarly.

(2) If T is tilting, then it is a faithful module and, therefore, sincere. Conversely, assume that T is a sincere silting module with respect to a projective presentation $\sigma : P_{-1} \to P_0$. Since T has projective dimension at most one, $Im(\sigma)$ is a projective A-module and $ker(\sigma)$ is a direct summand of P_{-1}. But then, as T lies in $\mathcal{D}_\sigma$ and every morphism $P_{-1} \to T$ factors through σ, we have

$$Hom_A(ker(\sigma), T) = 0.$$

Since $ker(\sigma)$ is projective and T is sincere, it follows that $ker(\sigma) = 0$ and T is tilting by (1). $\square$

Notice that even if a module has projective dimension one, it can happen that monomorphic presentations are not the ones to consider for verifying the silting condition. So not all silting modules of projective dimension 1 are tilting, as illustrated in the examples at the end of this section. The next proposition relates silting modules to quasitilting modules.

Proposition 5.1.13. *(1) All silting modules are finendo quasitilting.*

(2) A module is tilting if and only if it is faithful silting (and if and only if it is faithful finendo quasitilting).

Proof. (1): Let T be silting with respect to a projective presentation $\sigma : P_{-1} \to P_0$. Then we know from Lemma 5.1.9(1) that $Gen(T) = \mathcal{D}_\sigma$ is closed under direct products, which means that T is finendo. Further, T is Ext-projective in $Gen(T)$ by Remark 5.1.11(1). It remains to show that $Gen(T) \subseteq Pres(T)$. Let M lie in $Gen(T)$, let I be $Hom_A(T,M)$, and consider the universal map $u : T^{(I)} \to M$ (which is then surjective). It suffices to show that $K := ker(u)$ lies in $\mathcal{D}_\sigma = Gen(T)$. Pick $f : P_{-1} \to K$. Since $T^{(I)} \in \mathcal{D}_\sigma$, we have the following commutative diagram

$$\begin{tikzcd} & P_{-1} \arrow[r, "\sigma"] \arrow[d, "f"] & P_0 \arrow[r, "\pi"] \arrow[d, "g"] & T \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & K \arrow[r, "k"] & T^{(I)} \arrow[r, "u"] & M \arrow[r] & 0. \end{tikzcd}$$

By the universality of u, there is a map $\tilde{h} : T \to T^{(I)}$ such that $u \circ \tilde{h} = h$. Now consider the map $\psi := g - \tilde{h} \circ \pi$. Since $u \circ \psi = 0$, there is a map $\tilde{g} : P_0 \to K$ such that $k \circ \tilde{g} = \psi$. Consequently, we get $k \circ \tilde{g} \circ \sigma = g \circ \sigma = k \circ f$. Using the injectivity of k, it follows that $\tilde{g} \circ \sigma = f$ and, thus, K belongs to $\mathcal{D}_\sigma$, as wanted.

Statement (2) is an immediate consequence of Lemma 5.1.4. □

In particular, it follows that we can recover the additive closure of a silting module from its associated torsion class (see Lemma 5.1.7). We will say that two silting modules T and T' are **equivalent** if $Add(T) = Add(T')$. The next result characterises silting modules in terms of a condition (S3) which is the silting counterpart of condition (T3) in Definition 1.7.1.

Proposition 5.1.14. *The following are equivalent for a module T with projective presentation σ.*

(1) T is a silting module with respect to σ.

(2) T is a partial silting module with respect to σ and

(S3) there is an exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0,$$

with T_0 and T_1 in $Add(T)$ and ϕ a left $\mathcal{D}_\sigma$-approximation.

Proof. (1)⇒(2): Follows from Proposition 5.1.13(1) and Proposition 5.1.5 using $\mathcal{D}_\sigma = Gen(T)$.

(2)⇒(1): Since T is partial silting with respect to σ, it is clear that $Gen(T) \subseteq \mathcal{D}_\sigma$. If M lies in $\mathcal{D}_\sigma$, any surjection $f : A^{(I)} \to M$ factors through the $\mathcal{D}_\sigma$-approximation $\phi^{(I)}$ via a surjection $g : T_0^{(I)} \to M$. Thus, M lies in $Gen(T)$. □

A well-known result of Bongartz – later proved in full generality in [CT] – states that every partial tilting module can be completed to a tilting module. The following theorem now generalises it to our setting.

Theorem 5.1.15. *Every partial silting A-module T with respect to a projective presentation σ is a direct summand of a silting A-module $\bar{T} = T \oplus M$ with the same associated torsion class, that is, $Gen(\bar{T}) = \mathcal{D}_\sigma$.*

Proof. Let T be a partial silting A-module and let $\sigma : P_{-1} \to P_0$ be a projective presentation of T. In order to find a complement for T, we begin by constructing an approximation sequence for A in $\mathcal{D}_\sigma$. Consider the universal map $\psi : {P_{-1}}^{(I)} \to A$ with $I = Hom_A(P_{-1}, A)$. We get the following pushout diagram

$$\begin{tikzcd} {P_{-1}}^{(I)} \arrow[r, "\sigma^{(I)}"] \arrow[d, "\psi"] & {P_0}^{(I)} \arrow[r] \arrow[d, "\psi_1"] & T^{(I)} \arrow[r] \arrow[d, equal] & 0 \\ A \arrow[r, "\phi"] & M \arrow[r, "\pi"] & T^{(I)} \arrow[r] & 0. \end{tikzcd} \tag{5.1.1}$$

If M lies in $\mathcal{D}_\sigma$, then it follows from the universal property of the pushout that ϕ is a left $\mathcal{D}_\sigma$-approximation. We will, therefore, show that any map $g : P_{-1} \to M$ factors through σ. Since $T^{(I)}$ lies in $\mathcal{D}_\sigma$, the composition $\pi \circ g$ must factor through σ via some map $g_1 : P_0 \to T^{(I)}$, yielding the following commutative diagram

$$\begin{tikzcd} P_{-1} \arrow[r, "\sigma"] \arrow[d, "g"] & P_0 \arrow[d, "g_1"] \\ M \arrow[r, "\pi"] & T^{(I)} \end{tikzcd}$$

Moreover, since P_0 is projective, there is a map $g_2 : P_0 \to M$ such that $g_1 = \pi \circ g_2$. It follows that $g_2 \circ \sigma - g$ factors through ϕ. Now, by the construction of ψ and the commutativity of diagram (5.1.1), there are component maps $\psi' : P_{-1} \to X$ and $\psi_1' : P_0 \to M$ fulfilling $g_2 \circ \sigma - g = f \circ \psi' = \psi_1' \circ \sigma$. Consequently, the map g factors through σ, proving that M lies in $\mathcal{D}_\sigma$.

We will now prove that $\bar{T} := T \oplus M$ is a silting A-module. Since the left square of diagram (5.1.1) is a pushout diagram, it yields a projective presentation of M

$${P_{-1}}^{(I)} \xrightarrow{(g\ \sigma^{(I)})} A \oplus {P_0}^{(I)} \xrightarrow{\binom{-\phi}{f}} M \longrightarrow 0.$$

This gives us a projective presentation of $\bar{T}$ by considering the direct sum $\gamma := \sigma \oplus (g\ \sigma^{(I)})$. Set $\delta := (g\ \sigma^{(I)})$. We show that $\mathcal{D}_\gamma = \mathcal{D}_\sigma$. First, note that $\mathcal{D}_\gamma = \mathcal{D}_\sigma \cap \mathcal{D}_\delta$. Therefore, it suffices to check that $\mathcal{D}_\sigma \subseteq \mathcal{D}_\delta$. Take $X \in \mathcal{D}_\sigma$. We will prove that $Hom_A(\delta, X)$ is surjective. Let $h : P_1^{(I)} \to X$ be a test map and $i : P_1 \to P_1^{(I)}$ and $\pi : P_0^{(I)} \to P_0$ be canonical component maps for the coproduct. For every component of the coproduct we get the following commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r, "\sigma"] \arrow[d, "i"] & P_0 \\
P_1^{(I)} \arrow[r, "\delta"] \arrow[dr, "h"] & A \oplus P_0^{(I)} \arrow[u, "\pi"] \\
& X
\end{tikzcd}$$

Since X lies in $\mathcal{D}_\sigma$, there is a map $\psi : P_0 \to X$ such that $\psi \circ \sigma = h \circ i$. Thus, we get $\psi \circ \pi \circ \delta \circ i = h \circ i$. By the universal property of the coproduct, it follows that $\psi \circ \pi \circ \delta = h$, showing that $Hom_A(\delta, X)$ is surjective. Consequently, $\bar{T}$ is a partial silting module, since it is in $\mathcal{D}_\gamma = \mathcal{D}_\sigma$, and it is even silting by Proposition 5.1.14. $\square$

We have seen in Proposition 5.1.12 that (partial) tilting modules are examples of (partial) silting modules. We now discuss non-tilting examples of silting modules. An important class of examples is given by τ-rigid and support τ-tilting modules over a finite dimensional $\mathbb{K}$-algebra.

Proposition 5.1.16. *Let A be a finite dimensional $\mathbb{K}$-algebra and let T be in A-mod. Then the following hold.*

(1) T is partial silting if and only if it is τ-rigid.

(2) T is silting if and only if it is support τ-tilting.

(3) T is (finendo) quasitilting if and only if it is support τ-tilting.

Proof. (1): This follows from Theorem 5.0.6(1) and (2) and Remark 5.1.11(1) and (2).

(2) If T is silting, then by (1) it is τ-rigid, and it satisfies condition (S3) in Theorem 5.1.14, where the $Gen(T)$-approximation sequence $A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$ can be taken in A-*mod*. Now the claim follows by [J, Proposition 2.14]. The converse follows from Theorem 5.0.6(3).

(3) First, note that this statement was independently proven in [Wei2]. We give an alternative proof. Recall that finitely generated A-modules are always finendo. By (2), the statement can be rephrased by saying that T is quasitilting if and only if it is silting. Now the if-part is just Proposition 5.1.13(1). We show that for T in A-*mod* also the converse holds true. If T is quasitilting,

then by Proposition 5.1.5, it satisfies condition (S3) in Theorem 5.1.14, and it is Ext-projective in $Gen(T)$. By Theorem 5.0.6(2) the latter means that T is τ-rigid. We conclude from (1) that T is a partial silting module satisfying (S3), or equivalently, a silting module. $\square$

Remark 5.1.17.

(1) Corollary 5.1.8 can now be viewed as an analogue of [AIR, Theorem 2.7] stating that over a finite dimensional algebra A, there is a bijection between isomorphism classes of basic support τ-tilting modules and functorially finite torsion classes $\mathcal{T}$ in A-mod. Indeed, left $\mathcal{T}$-approximations in A-mod can be chosen to be minimal, and then the cokernel is always Ext-projective by a well-known lemma due to Wakamatsu (see, for example, [X, Lemma 2.1.2]).

(2) A further consequence of Proposition 5.1.16 is that for any support τ-tilting module T over a finite dimensional $\mathbb{K}$-algebra A, the functor $Hom_A(T,-)$ induces half of the categorical equivalences of the Brenner-Butler theorem in tilting theory (compare [J, Proposition 3.5] and [HKM, Theorem 4.4]). For more details on such equivalences see [CDT].

We finish this section with two explicit examples of silting modules.

Example 5.1.18. *Let A be the $\mathbb{K}$-algebra given as the quotient of the path algebra of the quiver*

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

by the ideal generated by $\beta\alpha$ (see Example 2.1.7 and Example 3.3.5). By T we denote the τ-tilting A-module $P_3 \oplus P_1 \oplus S_1$. Note that T has projective dimension 2. Its minimal projective presentation

$$P_2 \xrightarrow{\sigma} P_3 \oplus P_1 \oplus P_1 \longrightarrow T$$

is given by $\sigma = (0\,0\,\alpha^*)$, where the map $\alpha^* : P_2 \to P_1$ is induced by multiplication with α. Consequently, an A-module X belongs to $\mathcal{D}_\sigma$ if and only if $Hom_A(\alpha^*, X)$ is surjective if and only if X lies in $Add(T) = Gen(T)$. Hence, T is a silting A-module with respect to σ.

The following is an example (taken from [CDT, Example 5.3]) of a finitely generated silting module which is neither tilting nor finitely presented.

Example 5.1.19. *Let Q be a quiver with two vertices, 1 and 2, and countably many arrows from 1 to 2. Let P_i be the indecomposable projective $\mathbb{K}Q$-module $\mathbb{K}Qe_i$ for $i = 1,2$. We show that $T := P_1/soc(P_1)$ is a silting module (of projective dimension one) which is not tilting. Indeed, as observed in [CDT], the class $Gen(T)$ consists precisely of the semisimple injective $\mathbb{K}Q$-modules* *and, thus, we have $Gen(T) = (P_2)^\circ \subsetneq T^{\perp_1}$. In particular, T is not a tilting module. Of course, T is not finitely presented. It admits the following projective presentation*

$$0 \longrightarrow P_2^{(\mathbb{N})} \xrightarrow{\sigma} P_1 \longrightarrow T \longrightarrow 0,$$

with $\mathcal{D}_\sigma = T^{\perp_1}$. Let γ be the projective presentation of T obtained as the direct sum of σ with the trivial map $P_2 \to 0$. Then we have that

$$\mathcal{D}_\gamma = T^{\perp_1} \cap {P_2}^\circ = {P_2}^\circ = Gen(T),$$

thus, proving that T is a silting module.

Note that it remains open whether all finendo quasitilting modules are silting.

5.2 Silting complexes

In this section, we discuss (large) silting complexes and how they relate to t-structures, co-t-structures and silting modules. Motivated by Theorem 5.0.11, we first investigate the bijections between silting complexes and certain t-structures and co-t-structures. Then we show that mapping a 2-term silting complex to its cohomology defines a bijection between (equivalence classes of) 2-term silting complexes and (equivalence classes of) silting modules. In particular, this justifies our choice of name for the class of modules under study.

5.2.1 Silting complexes, t-structures and co-t-structures

We begin by extending the notion of silting and presilting complexes in order to include complexes of large projective modules. We adopt a definition due to Wei [Wei, Definition 3.1], who called such complexes semi-tilting.

Definition 5.2.1. *A bounded complex of projective A-modules σ is said to be **presilting** if*

(1) $Hom_{D(A)}(\sigma, \sigma^{(I)}[i]) = 0$, for all sets I and $i > 0$.

*It is furthermore **silting** if it also satisfies*

(2) the smallest triangulated subcategory of $D(A)$ containing $Add(\sigma)$ is $K^b(A\text{-}Proj)$.

*We call σ n-**presilting**, respectively n-**silting**, if it is an n-term complex of projective A-modules. Hereby, and throughout this section, an n-**term** complex of projective modules means a complex concentrated between degrees $-n+1$ and 0.*

For a presilting complex σ, we investigate the subcategory $aisle(\sigma)$ from Example 5.0.10(4), and the subcategory $\sigma^{\perp_{>0}}$. They play an important role in determining whether σ is silting or not (see [HKM, Theorem 1.3] and [AI, Corollary 4.7]).

Proposition 5.2.2. *The following statements are equivalent for a n-term complex $\sigma \in K^b(A\text{-}Proj)$.*

(1) The complex σ is (n-)silting.

(2) σ is presilting, $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed for coproducts in $D(A)$, and the set $\{\sigma[i] : i \in \mathbb{Z}\}$ generates $D(A)$.

(3) $aisle(\sigma) = \sigma^{\perp_{>0}}$.

(4) σ is presilting and $\sigma^{\perp_{>0}}$ lies in $D^{\leq 0}$.

Proof. (1)⇒(2): It follows from [Wei, Proposition 4.2] that $\sigma^{\perp_{>0}}$ is closed for coproducts in $D(A)$ and, thus, so is $\sigma^{\perp_{>0}} \cap D^{\leq 0}$. By definition, the smallest triangulated subcategory of $D(A)$ containing $Add(\sigma)$ contains A. Then the smallest triangulated subcategory of $D(A)$ closed under coproducts and containing σ is $D(A)$. It then follows from Remark 5.0.8 that $D(A)$ is generated by $\{\sigma[i], i \in \mathbb{Z}\}$.

(2)⇒(3): The arguments are similar to those in the proof of [AI, Corollary 4.7]. The subcategory $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is suspended and, by assumption, closed for coproducts in $D(A)$. It follows that $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ contains $aisle(\sigma)$ (see Example 5.0.10(4)). For any X in $\sigma^{\perp_{>0}}$, there is a triangle associated with the t-structure $(aisle(\sigma), \sigma^{\perp_{<0}})$

$$Y \to X \to Z \to Y[1],$$

with Y in $aisle(\sigma)$ and Z in $\sigma^{\perp_{\leq 0}}$. Since Y then also lies in $\sigma^{\perp_{>0}}$, and X lies in $\sigma^{\perp_{>0}}$ by assumption, we conclude that Z lies in $\sigma^{\perp_{>0}}$. But then Z lies in $\sigma^{\perp_{\leq 0}} \cap \sigma^{\perp_{>0}}$, and so $Z = 0$ by (2). It follows that $\sigma^{\perp_{>0}} = aisle(\sigma)$.

(3)⇒(4): Since σ lies in $D^{\leq 0}$, then so does $aisle(\sigma) = \sigma^{\perp_{>0}}$.

(4)⇒(1): This follows from [Wei, Proposition 3.12]. □

Remark 5.2.3. *It follows from the equivalence (1)⇔(2) in Proposition 5.2.2 that Definition 5.0.7 and Definition 5.2.1 agree on complexes $\sigma \in K^b(A\text{-}proj)$. Note that σ being compact implies that $\sigma^{\perp_{>0}}$ is closed for coproducts.*

We will now study the (co-)t-structures arising from silting complexes in some more detail.

Definition 5.2.4.

*(1) A t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ (respectively, a co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$) in $D(A)$ is said to be **intermediate** if there are integers $a, b \in \mathbb{Z}$, $a \leq b$, such that $D^{\leq a} \subseteq \mathcal{V}^{\leq 0} \subseteq D^{\leq b}$ (respectively, $D^{\leq a} \subseteq \mathcal{U}_{\leq 0} \subseteq D^{\leq b}$).*

*(2) A t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is said to be **silting** if it is intermediate and there is a silting complex σ in $D(A)$ such that $\mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = Add(\sigma)$. It is furthermore said to be **n-silting** if $D^{\leq -n+1} \subseteq \mathcal{V}^{\leq 0} \subseteq D^{\leq 0}$.*

Lemma 5.2.5. *A t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is n-silting with $\mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = Add(\sigma)$ if and only if σ is an n-silting complex and $\mathcal{V}^{\leq 0} = \sigma^{\perp_{>0}}$.*

Proof. Suppose that $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is an n-silting t-structure with $\mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = Add(\sigma)$. It is clear that σ is a silting complex (since it has the same additive closure as a silting complex). From Proposition 5.2.2 we have that $\sigma^{\perp_{>0}} = aisle(\sigma)$ and, hence, $\sigma^{\perp_{>0}}$ is contained in $\mathcal{V}^{\leq 0}$ as so is σ. It remains to see that $\mathcal{V}^{\leq 0} \subseteq \sigma^{\perp_{>0}}$. By the orthogonality relations of t-structures, it is enough to prove that $\sigma^{\perp_{<0}}$ is contained in $\mathcal{V}^{\geq 0}$. Let X lie in $\sigma^{\perp_{<0}}$ and consider the canonical triangle associated with the t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$

$$Y \to X \to Z \to Y[1],$$

where Y lies in $\mathcal{V}^{\leq -1}$ and Z lies in $\mathcal{V}^{\geq 0}$. By assumption, we have that $Hom_{D(A)}(\sigma, X[i]) = 0$ for all $i < 0$ and, since σ lies in $\mathcal{V}^{\leq 0}$, we also have that $Hom_{D(A)}(\sigma, Z[i]) = 0$ for all $i < 0$. Thus, we have that $Hom_{D(A)}(\sigma, Y[i]) = 0$ for all $i < 0$. On the other hand, since σ lies in ${}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = {}^{\perp_0}(\mathcal{V}^{\leq -1})$ we see that $Hom_{D(A)}(\sigma, Y[i]) = 0$ for all $i \geq 0$. Recalling from Proposition 5.2.2 that $\{\sigma[i] : i \in \mathbb{Z}\}$ is a set of generators for $D(A)$, we conclude that $Y = 0$. Thus, $X \cong Z$ and X lies in $\mathcal{V}^{\geq 0}$ as wanted.

It remains to show that σ is an n-term complex. Let σ be a complex of projective A-modules of the form $(P_i, d_i)_{i \in \mathbb{Z}}$ with $P_i = 0$ for all $i > 0$. Since the t-structure is n-silting, $D^{\leq -n+1}$ lies in $\mathcal{V}^{\leq 0}$ and σ lies in ${}^{\perp_0}(\mathcal{V}^{\leq 0}[1])$, so $Hom_{D(A)}(\sigma, D^{\leq -n}) = 0$. Consider now the canonical co-t-structure $(K_{\geq 0}, K_{\leq 0})$ in $K_p(A)$ from Example 5.0.10(2), and take a triangle given by stupid truncations

$$Y \longrightarrow \sigma \xrightarrow{u} Z \longrightarrow Y[1],$$

with Y in $K_{\geq -n+1} \cap K_{\leq 0}$ and Z in $K_{\leq -n}$. Since Z lies in $D^{\leq -n}$, the map u is zero and, thus, σ lies in $K_{\geq -n+1} \cap K_{\leq 0}$ because it is a summand of Y (in fact, we even have $Y \cong \sigma$).

Conversely, let σ be an n-silting complex. Then it follows that $D^{\leq -n+1} \subseteq \sigma^{\perp_{>0}}$, and from Proposition 5.2.2 we have that $\sigma^{\perp_{>0}} \subseteq D^{\leq 0}$. Moreover, we have $Add(\sigma) \subseteq \sigma^{\perp_{>0}} \cap {}^{\perp_0}(\sigma^{\perp_{>0}}[1])$. We show the reverse inclusion. Let X lie in $\sigma^{\perp_{>0}} \cap {}^{\perp_0}(\sigma^{\perp_{>0}}[1])$ and let I be the set $Hom_{D(A)}(\sigma, X)$. The canonical universal map $u : \sigma^{(I)} \to X$ gives rise to a triangle

$$K \longrightarrow \sigma^{(I)} \xrightarrow{u} X \xrightarrow{v} K[1].$$

Applying the functor $Hom_{D(A)}(\sigma, -)$ to the triangle, since $Hom_{D(A)}(\sigma, \sigma^{(I)}[1]) = 0$ and further $Hom_{D(A)}(\sigma, u)$ is surjective, we deduce that $Hom_{D(A)}(\sigma, K[1]) = 0$. For $i > 0$, since

$$Hom_{D(A)}(\sigma, \sigma^{(I)}[i+1]) = 0 = Hom_{D(A)}(\sigma, X[i])$$

we also conclude that $Hom_{D(A)}(\sigma, K[i+1]) = 0$. Thus, K lies in $\sigma^{\perp_{>0}}$. Since X lies in ${}^{\perp_0}(\sigma^{\perp_{>0}}[1])$, we infer that $v \in Hom_{D(A)}(X, K[1])$ is zero. Therefore, u splits and X lies in $Add(\sigma)$ as wanted. Thus, $(\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$ is an n-silting t-structure. □

It follows from the lemma that two silting complexes σ and γ in $D(A)$ satisfy $Add(\sigma) = Add(\gamma)$ if and only if $\sigma^{\perp_{>0}} = \gamma^{\perp_{>0}}$. Therefore, we can define, unambiguously, a notion of equivalence of silting complexes: two silting complexes σ and γ are said to be **equivalent** if $Add(\sigma) = Add(\gamma)$.

The following theorem generalises the correspondence of (compact) silting complexes with t-structures and co-t-structures in [AI] and [KY]. It has been partly treated in [Wei, Theorem 5.3].

Theorem 5.2.6. *There are bijections between*

(1) equivalence classes of silting complexes in $D(A)$;

(2) silting t-structures in $D(A)$;

(3) intermediate co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D(A)$ with $\mathcal{U}_{\leq 0}$ closed for coproducts in $D(A)$;

(4) triples $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of subcategories of $D(A)$ such that $(\mathcal{A}, \mathcal{B})$ is a co-t-structure and $(\mathcal{B}, \mathcal{C})$ is an intermediate t-structure.

Proof. Consider the following assignments.

Bijection	Assignment
$(1) \to (2)$	$\Psi:\ \sigma \mapsto\ (\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$
$(2) \to (1)$	$\Theta:\ (\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0}) \mapsto\ \sigma$ with $Add(\sigma) = \mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1])$
$(1) \to (3)$	$\Phi:\ \sigma \mapsto\ ({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}})$
$(1) \to (4)$	$\Omega:\ \sigma \mapsto\ ({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$

We have seen above that the assignments Ψ, Φ and Ω do not depend on the representative of the equivalence class of the silting complex σ. Note also that these assignments commute with the shift functor [1], which is an auto-equivalence of the derived category. To show that Ψ, Φ and Θ are bijections, we will assume without loss of generality that silting complexes are concentrated in degrees less or equal than 0 or that $\sigma^{\perp_{>0}}$ is contained in $D^{\leq 0}$. It follows immediately from Lemma 5.2.5 that the assignments Ψ and Θ are inverse to each other. It is also clear that if Φ and Ψ are bijections, then so is Ω.

We prove that Φ is a bijection in two steps. The first step provides a bijection between (1) and certain co-t-structures in $D^-(A)$, and the second step will relate them to the co-t-structures in (2).

Step 1: In [Wei, Theorem 5.3] it is shown that assigning to a silting complex σ the subcategory $\sigma^{\perp_{>0}}$ yields a bijection between equivalence classes of silting complexes and subcategories $\mathcal{U}$ of $D^-(A)$ satisfying four properties. We leave to the reader to check that two of those properties (being specially covariantly finite and coresolving, as defined in [Wei]) correspond exactly to the statement that $({}^{\perp_0}(\mathcal{U}[1]),\mathcal{U})$ is a co-t-structure in $D^-(A)$. Notice that here the left orthogonal is computed in $D^-(A)$. A third property states that $\mathcal{U}$ is closed for coproducts.

We turn to the fourth property. It asserts that every object X in $D^-(A)$ admits a finite coresolution by $\mathcal{U}$, i.e. there are a positive integer m, a collection of objects $(U_i)_{0\leq i\leq m}$ in $\mathcal{U}$, and a finite sequence of triangles as follows

$$\begin{gathered}X \to U_0 \to C_0 \to X[1]\\ C_0 \to U_1 \to C_1 \to C_0[1]\\ \cdots\\ C_{m-2} \to U_{m-1} \to U_m \to C_m[1].\end{gathered}$$

We now prove that this property can be rephrased by saying that the co-t-structure $({}^{\perp_0}(\mathcal{U}[1]),\mathcal{U})$ in $D^-(A)$ is intermediate. In fact, the classes $\mathcal{U}$ occurring in [Wei, Theorem 5.3] satisfy this condition by [Wei, Lemma 4.1]. Conversely, given a co-t-structure $(\mathcal{U}_{\geq 0},\mathcal{U}_{\leq 0})$ in $D^-(A)$ such that $D^{\leq -n}\subseteq \mathcal{U}_{\leq 0}\subseteq D^{\leq 0}$ for some n, we take a complex X in $D^-(A)$, say with $H^i(X)=0$ for all $i>k$, and construct a sequence of triangles as above. Let us first reduce this analysis to the case where X lies in $D^{\leq 0}$. Indeed, by the axioms of co-t-structure we have a triangle

$$X \to U_0 \to C_0 \to X[1]$$

such that U_0 lies in $\mathcal{U}_{\leq 0}$ and C_0 lies in $\mathcal{U}_{\geq 0}$. Using that $\mathcal{U}_{\leq 0}\subseteq D^{\leq 0}$, we see that $H^i(C_0)=0$ for all $i > k-1$. So we can find a finite sequence of triangles yielding an object C_{k-1} in $D^{\leq 0}$. Hence, without loss of generality, we may assume to start with X in $D^{\leq 0}$. We build a sequence of triangles

$$\begin{gathered} X \to U_0 \to C_0 \to X[1] \\ C_0 \to U_1 \to C_1 \to C_0[1] \\ \dots \\ C_{n-1} \to U_n \to C_n \to C_{n-1}[1] \end{gathered}$$

where U_i lies in $\mathcal{U}_{\leq 0}$ and C_i lies in $\mathcal{U}_{\geq 0}$ for all $0 \leq i \leq n$. Here n is the natural number above with $D^{\leq -n} \subseteq \mathcal{U}_{\leq 0}$. We claim that $Hom_{D(A)}(C_n, C_{n-1}[1]) = 0$. This will show that the last triangle splits, so C_{n-1} will belong to $\mathcal{U}_{\leq 0}$ as wished. To prove this claim, we apply the functor $Hom_{D(A)}(C_n, -)$ to all the triangles. From the orthogonality properties of the co-t-structure we infer that

$$Hom_{D(A)}(C_n, C_{n-1}[1]) \cong Hom_{D(A)}(C_n, C_{n-i}[i])$$

for all $1 \leq i \leq n$. From the first triangle we get an isomorphism

$$Hom_{D(A)}(C_n, C_0[n]) \cong Hom_{D(A)}(C_n, X[n+1]).$$

But C_n lies in $\mathcal{U}_{\geq 0} = {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$, and $X[n+1]$ lies in $D^{\leq -n-1} = D^{\leq -n}[1] \subseteq \mathcal{U}_{\leq 0}[1]$. So we conclude that $Hom_{D(A)}(C_n, X[n+1]) = 0$, which proves our claim.

Step 2. We have shown that Φ defines a bijection between equivalence classes of silting complexes in $D(A)$ and intermediate co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D^-(A)$ such that $\mathcal{U}_{\leq 0}$ is closed for coproducts in $D(A)$. It remains to prove that such co-t-structures in $D^-(A)$ and the corresponding co-t-structures in $D(A)$ are in bijection. To this end, we prove that for such a co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D^-(A)$, the pair $({}^{\perp_0}(\mathcal{U}_{\leq 0}[1]), \mathcal{U}_{\leq 0})$ in $D(A)$ (now with the orthogonal computed in $D(A)$) is an intermediate co-t-structure in $D(A)$. Then we immediately obtain an injective assignment with an obvious inverse given by the intersection with $D^-(A)$, completing our proof.

We only have to verify axiom (3) in the definition (5.0.9) of a co-t-structure for the pair $({}^{\perp_0}(\mathcal{U}_{\leq 0}[1]), \mathcal{U}_{\leq 0})$ in $D(A)$. We use the equivalence between $D(A)$ and $K_p(A)$ and consider the standard co-t-structure $(K_{\geq 0}, K_{\leq 0})$ in $K_p(A)$ from Example 5.0.10(2). For any X in $K_p(A)$, using stupid truncation, there is a triangle

$$Y \longrightarrow X \xrightarrow{\psi} Z \longrightarrow Y[1]$$

where Y in $K_{\geq 1}$ and Z in $K_{\leq 0}$. Now, Z lies in $D^{-}(A)$ and, thus, there is a triangle

$$C[-1] \longrightarrow Z \xrightarrow{\theta} U \longrightarrow C$$

with U in $\mathcal{U}_{\leq 0}$ and C in $\mathcal{U}_{\geq 0} \subset {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$. Using the octahedral axiom, there is a triangle

$$Y[1] \longrightarrow Cone(\theta \circ \psi) \longrightarrow C \longrightarrow Y[2].$$

Since $Y[1]$ lies in $K_{\geq 0}$ and homotopically projective resolutions of complexes in $\mathcal{U}_{\leq 0}[1]$ lie in $K_{\leq -1}$, we have that $Y[1]$ lies in ${}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$. Since C also lies in ${}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$, so does $Cone(\theta \circ \psi)$, thus, yielding a co-t-structure triangle

$$Cone(\theta \circ \psi)[-1] \longrightarrow X \xrightarrow{\theta\psi} U \longrightarrow Cone(\theta \circ \psi)$$

with $Cone(\theta \circ \psi)$ in ${}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$ and U in $\mathcal{U}_{\leq 0}$, as wanted. □

Remark 5.2.7.

- *In [MSSS, Corollary 5.9], a bijection between silting subcategories and co-t-structures is established. This can be used to show that the items (1)-(4) above are in bijection with bounded co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $K^b(A\text{-}Proj)$ such that $\mathcal{U}_{\geq 0} \cap \mathcal{U}_{\leq 0} = Add(\sigma)$ for some object σ in $K^b(A\text{-}Proj)$.*

- *We clarify in some more detail how the result above generalises Theorem 5.0.11 for compact silting complexes over a finite dimensional $\mathbb{K}$-algebra A. In [KY], it is shown that if σ is compact, then $(\sigma^{\perp_{>0}} \cap D^b(A\text{-}mod), \sigma^{\perp_{<0}} \cap D^b(A\text{-}mod))$ is a t-structure in $D^b(A\text{-}mod)$. Adopting the notation in the proof of Theorem 5.2.6, this corresponds to the restriction of the t-structure $\Psi(\sigma)$ in $D(A)$ to $D^b(A)$. Indeed, it can be checked that such a restriction must be bounded (because $\Psi(\sigma)$ is intermediate) and that the heart is a module category (the zero cohomology of σ with respect to the t-structure is a projective generator of the heart and it is small because σ is compact). Moreover, the co-t-structure associated to σ under Theorem 5.0.11 can be checked (using the description provided in [MSSS]) to coincide with the restriction of $\Phi(\sigma)$ to $K^b(A\text{-}proj)$. Again, this restriction will be bounded because the co-t-structure is intermediate.*

5.2.2 2-term silting complexes and silting modules

In this subsection, we turn back to silting modules and their relation to silting complexes. The following lemma establishes a useful connection between $\sigma^{\perp_{>0}}$ and the torsion class $\mathcal{D}_\sigma$ introduced in Subsection 5.1.2, for any 2-term complex σ in $K^b(A\text{-}Proj)$.

Lemma 5.2.8. *The following hold for a 2-term complex σ in $K^b(A\text{-}Proj)$ with $T = H^0(\sigma)$.*

(1) An object X in $D^{\leq 0}$ belongs to $\sigma^{\perp_{>0}}$ if and only if $H^0(X)$ lies in $\mathcal{D}_\sigma$. Moreover, we have $\mathcal{D}_\sigma = \sigma^{\perp_{>0}} \cap A\text{-}Mod$.

(2) An object X in $D^{\geq 0}$ belongs to $\sigma^{\perp_{\leq 0}}$ if and only if $H^0(X)$ lies in T°. Moreover, we have $T^\circ = \sigma^{\perp_{\leq 0}} \cap A\text{-}Mod$.

(3) The module T is partial silting with respect to σ if and only if the complex σ is presilting and $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed for coproducts in $D(A)$.

Proof. We set $\sigma : P_{-1} \to P_0$.

(1) Let $X = (X_j, d_j)_{j\in\mathbb{Z}}$ be a complex in $D^{\leq 0}$ (assume without loss of generality that $X_j = 0$ for all $j > 0$). Suppose that X lies in $\sigma^{\perp_{>0}}$. Any map from $h : P_{-1} \to H^0(X)$ lifts to a map $f : P_{-1} \to X_0$ via the projection map $\pi : X_0 \to H^0(X)$, since P_{-1} is projective. Now, f induces a map in $Hom_{K(A)}(\sigma, X[1])$ which we assume to be zero. Thus, there are maps $s_0 : P_0 \to X_0$ and $s_{-1} : P_{-1} \to X_{-1}$ such that $f = s_0 \circ \sigma + d_{-1} \circ s_{-1}$. Since $h = \pi \circ f$, we easily see that $h = \pi \circ s_0 \circ \sigma$ and, thus, $H^0(X)$ lies in $\mathcal{D}_\sigma$.

Conversely, suppose that $H^0(X)$ lies in $\mathcal{D}_\sigma$. Then, for a morphism in $Hom_{K(A)}(\sigma, X[1])$ defined by a map $f : P_{-1} \to X_0$, there is $h : P_0 \to H^0(X)$ such that $\pi \circ f = h \circ \sigma$. Since P_0 is projective, there is $s_0 : P_0 \to X_0$ such that $\pi \circ s_0 = h$. It is then easy to observe that there is $s_{-1} : P_{-1} \to X_{-1}$ such that $f - s_0 \circ \sigma = d_{-1} \circ s_{-1}$, showing that f is null-homotopic.

(2) Let X be an object in $\sigma^{\perp_{\leq 0}} \cap D^{\geq 0}$. Since X lies in $D^{\geq 0}$, we have a (standard) t-structure triangle of the form

$$(\tau^{\geq 1}X)[-1] \to H^0(X) \to X \to \tau^{\geq 1}X.$$

Since $Hom_{D(A)}(\sigma, (\tau^{\geq 1}X)[-1]) = 0 = Hom_{D(A)}(\sigma, X)$, we get that $Hom_{D(A)}(\sigma, H^0(X)) = 0$ and, thus, $H^0(X)$ lies in T°. Similarly, one proves the converse.

(3) First we claim that $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed for coproducts if and only if $\mathcal{D}_\sigma$ is closed for coproducts, i.e., condition (S1) in the definition of partial silting module holds for T. Indeed,

consider the canonical triangle

$$\tau^{\leq -1}\bigoplus_{i\in I} X_i \longrightarrow \bigoplus_{i\in I} X_i \longrightarrow H^0(\bigoplus_{i\in I} X_i) \longrightarrow (\tau^{\leq -1}\bigoplus_{i\in I} X_i)[1],$$

for any family of objects $(X_i)_{i\in I}$ in $\sigma^{\perp_{>0}}\cap D^{\leq 0}$. Since $D^{\leq -1}$ is contained in $\sigma^{\perp_{>0}}\cap D^{\leq 0}$ and H^0 commutes with coproducts, our claim follows from (1). Condition (S2) is equivalent to σ lying in $\sigma^{\perp_{>0}}$, by Lemma 5.1.9(3). □

The following theorem is a non-compact version of [HKM, Theorem 2.10], in the sense that it extends the statements from compact silting complexes to silting complexes in $K^b(A\text{-}Proj)$. Note that the torsion pair $(\mathcal{D}_\sigma, T^\circ)$ coincides with the torsion pair $(\mathcal{X}(\sigma), \mathcal{Y}(\sigma))$ in [HKM].

Theorem 5.2.9. *Let σ be 2-term complex in $K^b(A\text{-}Proj)$ and $T = H^0(\sigma)$. The following statements are equivalent.*

(1) σ is a 2-silting complex;

(2) σ is a presilting complex, and $\{\sigma[i] : i \in \mathbb{Z}\}$ is a set of generators in $D(A)$;

(3) T is a silting module with respect to σ;

(4) $(\mathcal{D}_\sigma, T^\circ)$ is a torsion pair in $A\text{-}Mod$.

Moreover, if the conditions above are satisfied, we have

$$\sigma^{\perp_{>0}} = D^{\leq 0}_{\mathcal{D}_\sigma} = \{X \in D(A) : H^0(X) \in \mathcal{D}_\sigma,\ H^i(X) = 0\ \forall i > 0\}.$$

Proof. (1)⇒(2): This follows from Proposition 5.2.2.

(2)⇒ (1): By Proposition 5.2.2, we have to show that $\sigma^{\perp_{>0}}$ is contained in $D^{\leq 0}$. Let X be in $\sigma^{\perp_{>0}}$ and consider its triangle decomposition with respect to the canonical t-structure in $D(A)$

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \to (\tau^{\leq 0}X)[1].$$

It is clear that $Hom_{D(A)}(\sigma[i], \tau^{\geq 1}X) = 0$ for $i \geq 0$. Moreover, applying $Hom_{D(A)}(\sigma[i], -)$, with $i < 0$ to the triangle we get by assumption that $Hom_{D(A)}(\sigma, X[-i]) = 0$, and also that

$$Hom_{D(A)}(\sigma, (\tau^{\leq 0}X)[-i+1]) = 0$$

since $(\tau^{\leq 0}X)[-i+1]$ lies in $D^{\leq -2}$ for all $i<0$. Therefore, we have that $Hom_{D(A)}(\sigma[i],\tau^{\geq 1}X)=0$ for all $i\in\mathbb{Z}$, and $\tau^{\geq 1}X=0$ as $\{\sigma[i]:i\in\mathbb{Z}\}$ is a set of generators for $D(A)$.

(1)⇒ (3): Combining Proposition 5.2.2 with Lemma 5.2.8, we see that T is partial silting, and so $Gen(T)\subseteq\mathcal{D}_\sigma$. Let now M be a module in $\mathcal{D}_\sigma$ and take the universal map $u:\sigma^{(I)}\to M$, where $I=Hom_{D(A)}(\sigma,M)$. We will show that $H^0(u):T^{(I)}\to M$ is a surjection. For this purpose, we consider the triangle

$$\sigma^{(I)}\xrightarrow{u}M\to C\to\sigma^{(I)}[1]$$

and prove that $H^0(C)=0$. We use the generating property of the set $\{\sigma[i]:i\in\mathbb{Z}\}$, see (2) above. Note that the long exact sequence of cohomologies for the triangle above shows that there is a surjection $M\to H^0(C)$. Hence, the module $H^0(C)$ lies in the torsion class $\mathcal{D}_\sigma$, and, by Lemma 5.2.8, it lies also in $\sigma^{\perp_{>0}}$, that is, $Hom_{D(A)}(\sigma,H^0(C)[1])=0$. Since σ is a two-term complex, it remains to see that $Hom_{D(A)}(\sigma,H^0(C))=0$. As C lies in $D^{\leq 0}$, we have the following canonical triangle given by the standard t-structure in $D(A)$

$$\tau^{\leq -1}C\to C\to H^0(C)\to\tau^{\leq -1}C[1].$$

On one hand, since σ is presilting, it follows from the definition of C that $Hom_{D(A)}(\sigma,C)=0$. On the other hand, since σ is a 2-term complex we also get that $Hom_{D(A)}(\sigma,\tau^{\leq -1}C[1])=0$. Therefore, we have $Hom_{D(A)}(\sigma,H^0(C))=0$, as wanted.

(3)⇒ (4): This follows immediately from Remark 5.1.11(1) as $\mathcal{D}_\sigma=Gen(T)$.

(4)⇒(1): Suppose that $(\mathcal{D}_\sigma,T^\circ)$ is a torsion pair. Then clearly T is partial silting with respect to σ, which implies by Lemma 5.2.8 that σ is presilting and $\sigma^{\perp_{>0}}\cap D^{\leq 0}$ is closed for coproducts in $D(A)$. By Proposition 5.2.2, it remains to show that $\{\sigma[i]:i\in\mathbb{Z}\}$ generates $D(A)$. Let X be an object of $D(A)$ such that $Hom_{D(A)}(\sigma,X[i])=0$ for all $i\in\mathbb{Z}$. Since σ is concentrated in degrees -1 and 0, this is equivalent to $Hom_{D(A)}(\sigma,\tau^{\leq 0}(X[i]))=0$ for all $i\in\mathbb{Z}$. Then $Hom_{D(A)}(\sigma,H^i(X))=0$, and ,thus, $H^i(X)$ lies in T° for all $i\in\mathbb{Z}$. Consider the triangle

$$H^0(X[i-1])[-1]\to\tau^{\leq -1}(X[i-1])\to\tau^{\leq 0}(X[i-1])\to H^0(X[i-1])$$

and apply to it the functor $Hom_{D(A)}(\sigma,-)$. Since

$$Hom_{D(A)}(\sigma,\tau^{\leq 0}(X[i-1]))=0=Hom_{D(A)}(\sigma,H^0(X[i-1])[-1]),$$

we conclude that

$$0 = Hom_{D(A)}(\sigma, \tau^{\leq -1}(X[i-1])) = Hom_{D(A)}(\sigma, \tau^{\leq 0}(X[i])[1]),$$

thus, showing that $\tau^{\leq 0}(X[i])$ belongs to $\sigma^{\perp_{>0}}$ for all $i \in \mathbb{Z}$. By Lemma 5.2.8, it follows that $H^i(X) = H^0(\tau^{\leq 0}X[i])$ lies in $\mathcal{D}_\sigma$ for all $i \in \mathbb{Z}$. Since the pair $(\mathcal{D}_\sigma, T^\circ)$ is a torsion pair, we conclude that $H^i(X) = 0$ for all $i \in \mathbb{Z}$, as wanted.

Let us now assume that the equivalent conditions (1)-(4) hold. In particular, $(\mathcal{D}_\sigma, T^\circ)$ is a torsion pair in $A\text{-}Mod$ and so Example 5.0.10(3) gives us a t-structure $(D^{\leq 0}_{\mathcal{D}_\sigma}, D^{\geq 0}_{T^\circ})$. We want to prove that $\sigma^{\perp_{>0}} = D_{\mathcal{D}_\sigma}$. Proposition 5.2.2 shows that $aisle(\sigma) = \sigma^{\perp_{>0}} \subseteq D^{\leq 0}$ and, thus, by Example 5.0.10(4) and Lemma 5.2.8(1),

$$\sigma^{\perp_{>0}} = aisle(\sigma) \subseteq \{X \in D(A) : H^0(X) \in \mathcal{D}_\sigma, H^i(X) = 0,\ \forall i > 0\} = D^{\leq 0}_{\mathcal{D}_\sigma}.$$

We will show that $aisle(\sigma)^{\perp_0} \subseteq D^{\geq 1}_{T^\circ}$, thus, proving that the inclusion above is in fact an equality. Let X be an object in $aisle(\sigma)^{\perp_0} = \sigma^{\perp_{\leq 0}}$. It is clear that $Hom_{D(A)}(\sigma, (\tau^{\leq -1}X)[i]) = 0$ for all $i > 0$. Consider now the triangle

$$(\tau^{\geq 0}X)[i-1] \to (\tau^{\leq -1}X)[i] \to X[i] \to (\tau^{\geq 0}X)[i].$$

Since σ lies in $D^{\leq 0}$, we have that $Hom_{D(A)}(\sigma, (\tau^{\geq 0}X)[i-1]) = 0$ for all $i \leq 0$ and also, by the assumption on X, $Hom_{D(A)}(\sigma, X[i]) = 0$ for all $i \leq 0$. This shows that $Hom_{D(A)}(\sigma, (\tau^{\leq -1}X)[i]) = 0$ for all $i \leq 0$. Since $\{\sigma[i] : i \in \mathbb{Z}\}$ is a set of generators for $D(A)$, we conclude that $\tau^{\leq -1}X = 0$. By Lemma 5.2.8(2), we get that $H^0(X)$ lies in T°. $\square$

Remark 5.2.10. *(1) Theorem 5.2.9 shows that the t-structure generated by a silting complex σ equals both the t-structure $(\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$ studied by Hoshino-Kato-Miyachi in [HKM] and the t-structure associated to the torsion pair $(\mathcal{D}_\sigma, T^\circ)$ in the sense of Happel-Reiten-Smalø [HRS].*

(2) We know from Theorem 5.2.9 that the cohomology $H^0(\sigma)$ is a silting module for any 2-silting complex σ. Further, σ and γ are equivalent 2-silting complexes if and only if the silting modules $T = H^0(\sigma)$ and $T' = H^0(\gamma)$ are equivalent. Indeed, recall that the silting modules T and T' are equivalent if $Add(T) = Add(T')$, which in turn means that $Gen(T) = Gen(T')$. So the only-if-part follows from the fact that H^0 commutes with coproducts. Conversely, if T and T' are equivalent, then they generate the same torsion pair, and, therefore, the associated Happel-Reiten-Smalø t-structures coincide, which means that $\sigma^{\perp_{>0}} = \gamma^{\perp_{>0}}$, by Theorem 5.2.9.

We finish by specialising Theorem 5.2.6 to 2-term complexes.

Theorem 5.2.11. *There are bijections between*

(1) equivalence classes of 2-silting complexes;

(2) equivalence classes of silting A-modules;

(3) 2-silting t-structures in $D(A)$;

(4) co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D(A)$ with $D^{\leq -1} \subseteq \mathcal{U}_{\leq 0} \subseteq D^{\leq 0}$ and $\mathcal{U}_{\leq 0}$ closed for coproducts in $D(A)$.

Proof. Consider the following assignments.

Bijection	Assignment
$(1) \to (2)$	$H^0 : \sigma \mapsto H^0(\sigma)$
$(1) \to (3)$	$\Psi : \sigma \mapsto (\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$
$(1) \to (4)$	$\Phi : \sigma \mapsto ({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}})$

Remark 5.2.10(2) above shows that H^0 is well-defined and injective. The surjectivity follows from Theorem 5.2.9, where it is shown that if T is a silting module with respect to a projective presentation σ, then σ is a 2-silting complex. Moreover, it follows from Lemma 5.2.5 that the map Ψ from Theorem 5.2.6 induces a bijection between equivalence classes of 2-silting complexes and 2-silting t-structures. Finally the co-t-structure $({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}})$ in $D(A)$ associated to a 2-silting complex σ clearly satisfies $D^{\leq -1} \subseteq \sigma^{\perp_{>0}} \subseteq D^{\leq 0}$. The map Φ from Theorem 5.2.6, therefore, restricts to the stated bijection. $\square$

Chapter 6

Silting modules and ring epimorphisms

This chapter contains ongoing joint work with Lidia Angeleri Hügel and Jorge Vitória. The aim is to relate silting modules to ring epimorphisms. More precisely, in the first section, we study ring epimorphisms that arise from partial silting modules. These ring epimorphisms will be described explicitly in ring theoretical terms by using a completion of the partial silting module to a silting module. In a second part, we restrict the setting to certain minimal silting modules over hereditary rings. These modules are shown to parametrise the homological ring epimorphisms and, thus, the universal localisations of the given ring (see Theorem 6.2.9). This result can be understood as a significant generalisation of Theorem 3.2.3 in Chapter 3.

6.1 Ring epimorphisms arising from partial silting modules

We start by generalising some ideas from [CTT] on partial tilting modules. First, fix a ring A and a partial silting A-module T_1 with associated torsion class $\mathcal{D}_\sigma$ given by a projective presentation σ of T_1. By T we denote the silting module obtained from the completion of T_1 in the sense of Theorem 5.1.15. We have the following two torsion pairs associated to T_1 (note that $Gen(T_1)$ is a torsion class by Lemma 5.0.5):

- $(\mathcal{D},\mathcal{R}) := (\mathcal{D}_\sigma = Gen(T), T^\circ)$
- $(\mathcal{T},\mathcal{F}) := ({}^\circ(T_1^\circ) = Gen(T_1), T_1^\circ)$

We are interested in the full subcategory $\mathcal{Y} := \mathcal{D}\cap\mathcal{F}$ of A-Mod. Note that, by definition,

$$\mathcal{F} = \{X \in A\text{-}Mod \,|\, Hom_A(\sigma, X) \text{ is injective}\}$$

and, therefore,

$$\mathcal{Y} = \{X \in A\text{-}Mod | Hom_A(\sigma, X) \text{ is bijective}\}.$$

In case, σ lies in A-$proj$, $\mathcal{Y}$ describes precisely the essential image of the restriction functor of the universal localisation of A at $\{\sigma\}$ (see Proposition 1.6.3). We will show that $\mathcal{Y}$ is always a bireflective subcategory of A-Mod and, therefore, we can associate a ring epimorphism $A \to B$ such that $\mathcal{Y} = \mathcal{X}_B$. The following arguments mimic the approach taken in [CTT, Proposition 1.4].

Lemma 6.1.1. *Consider a module $M \in \mathcal{Y}$ together with a short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then we have the following equivalent conditions

$$L \in \mathcal{Y} \Leftrightarrow L \in \mathcal{D} \Leftrightarrow N \in \mathcal{F} \Leftrightarrow N \in \mathcal{Y}.$$

Proof. Since M belongs to $\mathcal{Y} = \mathcal{D} \cap \mathcal{F}$, we know that L lies in $\mathcal{F}$ and N lies in $\mathcal{D}$. This proves the two outer equivalences. For the remaining equivalence consider the following commutative diagram induced by $\sigma : P \to Q$

$$\begin{tikzcd}
0 \arrow[r] & Hom_A(Q,L) \arrow[r] \arrow[d, "\sigma_*^L"] & Homa_A(Q,M) \arrow[r] \arrow[d, "\cong"] & Hom_A(Q,N) \arrow[r] \arrow[d, "\sigma_*^N"] & 0 \\
0 \arrow[r] & Hom_A(P,L) \arrow[r] & Homa_A(P,M) \arrow[r] & Hom_A(P,N) \arrow[r] & 0.
\end{tikzcd}$$

By the Snake Lemma, σ_*^L is surjective if and only if σ_*^N is injective. This finishes the proof. □

Proposition 6.1.2. *$\mathcal{Y}$ is a bireflective and extension closed subcategory of A-Mod.*

Proof. We have to show that $\mathcal{Y} = \{X \in A\text{-}Mod | Hom_A(\sigma, X) \text{ is bijective}\}$ is closed under products, coproducts, kernels, cokernels and extensions. Clearly, $\mathcal{Y}$ is closed under extensions, since so are $\mathcal{D}$ and $\mathcal{F}$. Secondly, $\mathcal{Y}$ is closed under products, since $Hom_A(\sigma, \prod X_i) = \prod Hom_A(\sigma, X_i)$ for X_i in A-Mod and products are exact. Moreover, $\mathcal{Y}$ is closed under coproducts, since so are $\mathcal{D}$ and $\mathcal{F}$. Finally, take a map $\omega : M \to N$ in $\mathcal{Y}$. Clearly, $Im(\omega)$ belongs to $\mathcal{Y}$, since it is a quotient of M (thus, in $\mathcal{D}$) and a submodule of N (thus, in $\mathcal{F}$). Now the claim follows by Lemma 6.1.1. □

For a module X, we construct its $\mathcal{Y}$-reflection explicitly. Take the $\mathcal{D}$-approximation sequence

$$X \xrightarrow{\phi_X} M_X \longrightarrow T_1^{(I)} \longrightarrow 0$$

constructed as in Theorem 5.1.15. Recall that $T = T_1 \oplus M_A$. Now consider the composition ψ_X

$$X \xrightarrow{\phi_X} M_X \longrightarrow M_X/\tau_{T_1}(M_X) =: \bar{M}_X$$

where τ_{T_1} denotes the module trace of T_1 that defines the torsion-radical with respect to $\mathcal{T}$.

Proposition 6.1.3. *The $\mathcal{Y}$-reflection for an A-module X is given by ψ_X.*

Proof. First of all, by construction, $\bar{M}_X$ belongs to $\mathcal{Y}$. Moreover, ψ_X – as a composition of a left $\mathcal{D}$-approximation and a left $\mathcal{F}$-approximation – becomes a left $\mathcal{Y}$-approximation. Thus, it remains to show that ψ_X induces an isomorphism (not only a surjective map), for all Y in $\mathcal{Y}$, between

$$Hom_A(\bar{M}_X, Y) \xrightarrow{\psi_{X*}} Hom_A(X,Y).$$

The kernel of ψ_{X*} is given by $Hom_A(coker(\psi_X), Y)$. Since Y lies in $\mathcal{F}$, it suffices to show that $coker(\psi_X)$ belongs to $\mathcal{T} = Gen(T_1)$. The following commutative diagram with surjective vertical maps finishes the proof

$$\begin{tikzcd}
X \arrow[r, "\phi_X"] \arrow[d, equal] & M_X \arrow[r] \arrow[d] & T_1^{(I)} \arrow[r] \arrow[d] & 0 \\
X \arrow[r, "\psi_X"] & \bar{M}_X \arrow[r] & coker(\psi_X) \arrow[r] & 0.
\end{tikzcd}$$

□

Following the previous discussion, to every partial silting module we can associate a bireflective and extension-closed subcategory $\mathcal{Y}$ and, thus, a ring epimorphism $A \to B$ with $Tor_1^A(B,B) = 0$.

Theorem 6.1.4. *Let T_1 be a partial silting A-module with associated ring epimorphism $A \to B$ and let $T = T_1 \oplus M_A$ be the completion of T_1 to a silting A-module in the above sense. Then there is an isomorphism of rings between B and $End_A(T)/I$, where I is the two-sided ideal generated by the endomorphisms factoring through an object in $Add(T_1)$. Moreover, if $Hom_A(T_1, M_A)$ admits a finite set of generators, then I is generated by the idempotent in $End_A(T)$ corresponding to T_1.*

Proof. Observe that, by Proposition 6.1.3, the ring B is isomorphic to $End_A(\bar{M}_A)$. In a first step, we will define a ring homomorphism $g : End_A(M_A) \to End_A(\bar{M}_A)$. Consider the short exact sequence

$$0 \longrightarrow \tau_{T_1}(M_A) \xrightarrow{i} M_A \xrightarrow{\pi} \bar{M}_A \longrightarrow 0.$$

Note that the trace of T_1 in M_A will be preserved by any endomorphism of M_A. Consequently, for all γ in $End_A(M_A)$ there is a unique endomorphism $\bar{\gamma}$ in $End_A(\bar{M}_A)$ such that $\bar{\gamma} \circ \pi = \pi \circ \gamma$. We define $g(\gamma)$ to be $\bar{\gamma}$. This assignment turns g into a ring homomorphism.

We show that g is surjective. Take any δ in $End_A(\bar{M}_A)$. By applying the functor $Hom_A(M_A,-)$ to the short exact sequence above we get the exact sequence

$$End_A(M_A) \xrightarrow{\pi_*} Hom_A(M_A,\bar{M}_A) \longrightarrow Ext_A^1(M_A,\tau_{T_1}(M_A)).$$

Since $M_A \in Add(T)$ and $\tau_{T_1}(M_A) \in \mathcal{D}$, by Lemma 5.1.7, we know that $Ext_A^1(M_A,\tau_{T_1}(M_A)) = 0$. Hence, there is a morphism γ in $End_A(M_A)$ such that $\delta \circ \pi = \pi \circ \gamma$ showing that g is surjective.

Next, we claim that a morphism γ in $End_A(M_A)$ belongs to the kernel of g if and only if it factors through a module in $Add(T_1)$. First, suppose that γ factors through $Add(T_1)$. Consequently, the image of γ lies in the trace of T_1 and, thus, $\pi \circ \gamma = 0$. Therefore, we also have $\bar{\gamma} \circ \pi = 0$ showing that $\bar{\gamma} = 0$, since π is surjective. Conversely, if γ lies in the kernel of g, we get $\pi \circ \gamma = 0$, meaning that the image of γ lies in the trace of T_1. Hence, there is a map $\omega : M_A \to \tau_{T_1}(M_A)$ making the following diagram commute

$$\begin{tikzcd}
 & & M_A \arrow[dl, "\omega"'] \arrow[d, "\gamma"] & & \\
0 \arrow[r] & \tau_{T_1}(M_A) \arrow[r, "i"] & M_A \arrow[r, "\pi"] & \bar{M}_A \arrow[r] & 0.
\end{tikzcd}$$

Now choose the set I to be $Hom_A(T_1,\tau_{T_1}(M_A))$ and consider the short exact sequence

$$0 \longrightarrow ker(\mu) \longrightarrow T_1^{(I)} \xrightarrow{\mu} \tau_{T_1}(M_A) \longrightarrow 0.$$

We obtain the following commutative diagram induced by the presentation $\sigma : P \to Q$ of T_1

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Hom_A(T_1,ker(\mu)) \arrow[r] \arrow[d] & Hom_A(T_1,T_1^{(I)}) \arrow[r, "\mu_*"] \arrow[d] & Hom_A(T_1,\tau_{T_1}(M_A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom_A(Q,ker(\mu)) \arrow[r] \arrow[d, "\eta"] & Hom_A(Q,T_1^{(I)}) \arrow[r] \arrow[d] & Hom_A(Q,\tau_{T_1}(M_A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom_A(P,ker(\mu)) \arrow[r] & Hom_A(P,T_1^{(I)}) \arrow[r] \arrow[d] & Hom_A(P,\tau_{T_1}(M_A)) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where the morphism μ_* is surjective. By the Snake Lemma, it follows that η is surjective and, thus, $ker(\mu)$ lies in $\mathcal{D}$. Consequently, since M_A is in $Add(T)$, we get $Ext^1_A(M_A, ker(\mu)) = 0$, using Lemma 5.1.7. Hence, there is a map $\nu : M_A \to T_1^{(I)}$ such that $\omega = \mu \circ \nu$. We conclude that $\gamma = i \circ \omega = i \circ \mu \circ \nu$ factors through $T_1^{(I)}$, proving the claim. All together, we get an isomorphism of rings

$$End_A(M_A)/J \xrightarrow{\cong} End_A(\bar{M}_A) \cong B$$

where J is the ideal of $End_A(M_A)$ generated by the endomorphisms factoring through $Add(T_1)$.

In a second step, we will now establish the isomorphism to $End_A(T)/I$. The endomorphism ring of T can be written as a 2×2-matrix as follows

$$End_A(T) = \begin{pmatrix} End_A(M_A) & Hom_A(M_A, T_1) \\ Hom_A(T_1, M_A) & End_A(T_1) \end{pmatrix}.$$

We construct a ring homomorphism $h : End_A(M_A) \to End_A(T)/I$ that maps $\gamma \in End_A(M_A)$ to

$$\left[\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} \right].$$

It is not hard to check that h is, indeed, a ring homomorphism. Furthermore, h is clearly surjective, since any element in $End_A(T)$ can be decomposed as follows

$$\left[\begin{pmatrix} \gamma & \beta \\ \delta & \varepsilon \end{pmatrix} \right] = \left[\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix} \right]$$

where the last three classes of this decomposition are zero, since their representatives lie in J. It remains to check that $ker(h) = J$. On the one hand, if γ in $End_A(M_A)$ factors through some module in $Add(T_1)$, then, clearly, so does $h(\gamma)$. On the other hand, if γ lies in the kernel of h, then

$$\tilde{\gamma} := \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$$

is an endomorphism of T factoring through $Add(T_1)$. Note that we can write $\tilde{\gamma}$ as the composition

$$T \xrightarrow{\oplus} M_A \xrightarrow{\gamma} M_A \xrightarrow{\oplus} T$$

where $\oplus$ denotes the canonical split injection or split surjection, respectively. Furthermore, we can write γ as the composition

$$M_A \xrightarrow{\oplus} T \xrightarrow{\oplus} M_A \xrightarrow{\gamma} M_A \xrightarrow{\oplus} T \xrightarrow{\oplus} M_A.$$

Hence, γ factors through a module in $Add(T_1)$ and, thus, it is in J. We get an isomorphism of rings

$$End_A(M_A)/J \xrightarrow{\cong} End_A(T)/I.$$

Combining this isomorphism with the previous one induced by g, it follows that $End_A(T)/I \cong B$.

Moreover, assume that $Hom_A(T_1, M_A)$ admits a finite set of generators. Consequently, the ideal I is given by those endomorphisms of T that factor through an A-module in $add(T_1)$. Now let e_{T_1} be the idempotent in $End_A(T)$ corresponding to T_1, i.e., e_{T_1} is given by the composition

$$T \xrightarrow{\oplus} T_1 \xrightarrow{\oplus} T.$$

Clearly, e_{T_1} lies in I and, hence, so does the two-sided ideal $< e_{T_1} >$ generated by it. Conversely, take an endomorphism ω of T that factors through $add(T_1)$ such that there is some $d \geq 0$ and a commutative diagram

$$\begin{tikzcd} T \arrow[rr, "\omega"] \arrow[dr, "a"'] & & T \\ & T_1^d \arrow[ur, "b"'] & \end{tikzcd}$$

We can write ω as a finite sum $\sum(b_i \circ a_i)$, where a_i and b_i are the corresponding component maps of a and b for $1 \leq i \leq d$. Note that we can write $b_i \circ a_i$ also as the following chain of compositions

$$T \xrightarrow{a_i} T_1 \xrightarrow{\oplus} T \xrightarrow{e_{T_1}} T \xrightarrow{\oplus} T_1 \xrightarrow{b_1} T.$$

It follows that $b_i \circ a_i$ lies in $< e_{T_1} >$ and, hence, so does ω. We conclude that $I =< e_{T_1} >$. $\square$

Note that, if the ideal I is idempotent, we get the following recollement of abelian categories relating the representation theories of A, B and $End_A(T)$ (compare, for example, [PV])

$$\mathcal{X}_B \cong B\text{-}Mod \begin{smallmatrix}\longleftarrow\\ \longrightarrow \\ \longleftarrow\end{smallmatrix} End_A(T)\text{-}Mod \begin{smallmatrix}\longleftarrow\\ \longrightarrow \\ \longleftarrow\end{smallmatrix} \mathcal{H}.$$

It would be interesting to find sufficient criteria to make sure that the ring epimorphisms $A \to B$ and $\mathrm{End}_A(T) \to End_A(T)/I$ are homological. In fact, this would allow us to compare the derived module categories $D(A), D(B)$ and $D(End_A(T))$. However, in general, the vanishing of all Tor-groups is not to be expected, as the following two examples illustrate.

Example 6.1.5. *Let A be the preprojective $\mathbb{K}$-algebra of type $\mathcal{A}_3$ (see Example 4.1.6). Consider the partial silting A-module $T_1 := S_1 \oplus S_3$ with respect to its minimal projective presentation σ. By construction, the corresponding ring epimorphism $A \to B$ is given by the universal localisation of A at $\Sigma = \{\sigma\}$. Using the table in Example 4.1.6, it follows that A_Σ is Morita-equivalent to the field $\mathbb{K}$ and, when viewed as an A-module, A_Σ has infinite projective dimension. Consequently, by Proposition 4.2.3, the ring epimorphism $A \to B$ is not homological.*

Even the ring epimorphisms associated to partial tilting modules are generally not homological.

Example 6.1.6. *Consider the finite dimensional $\mathbb{K}$-algebra A given as the quotient of the path algebra for the quiver*

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & 4 \\
& 3 \arrow[ur, "\delta"'] \arrow[d, "\varepsilon"] & \\
& 5 &
\end{tikzcd}$$

by the ideal generated by $\beta\alpha - \delta\gamma$ and $\varepsilon\gamma$. The Auslander-Reiten quiver of A is given by

$$\begin{tikzcd}
& P_2 \arrow[dr] & & I_5 \arrow[dr] & & & & \\
P_4 \arrow[ur] \arrow[dr] & & M_1 \arrow[ur] \arrow[dr] & & S_3 \arrow[dr] & & I_2 \arrow[dr] & \\
& P_3 \arrow[ur] \arrow[dr] & & M_3 \arrow[ur] \arrow[r] \arrow[dr] & P_1 \arrow[r] & M_4 \arrow[ur] \arrow[dr] & & I_1 \\
P_5 \arrow[ur] & & M_2 \arrow[ur] & & S_2 \arrow[ur] & & I_3 \arrow[ur] &
\end{tikzcd}$$

Consider the partial tilting (and, thus, also partial silting) A-module M_1. The associated bireflective subcategory $\mathcal{Y}$ is described by $M_1^{\perp} = Add(P_2 \oplus P_3 \oplus P_5 \oplus M_2 \oplus I_1)$. It follows that the corresponding ring B is hereditary. But, by considering the minimal projective A-resolution of I_1, we conclude that $Ext_A^2(I_1, P_5) \neq 0$. Therefore, the ring epimorphism $A \to B$ cannot be homological.

6.2 Minimal silting modules over hereditary rings

In this section we study silting modules over hereditary rings that turn out to play the role of generalised support tilting modules. Afterwards, we define (strongly) minimal silting modules. This definition will allow us to associate a unique ring epimorphism to every such silting module. We have the following useful lemma.

Lemma 6.2.1. *Let A be a hereditary ring and $\mathcal{T}$ be a subcategory of A-Mod with $\mathcal{T} = Add(\mathcal{T})$. If $\phi : A \to T_0$ is a left $\mathcal{T}$-approximation with $Ext^1_A(T_0,T_0) = 0$, then $J := ker(\phi)$ is a two-sided idempotent ideal.*

Proof. Note that J is given by $Ann(\mathcal{T})$ (compare the proof of Proposition 5.1.5) and, therefore, it is a two-sided ideal (and projective both as left and as right A-module). Since A is hereditary, ${}^{\perp_1}(T_0)$ is closed for subobjects. Thus, it follows that $Ext^1_A(A/J,T_0) = 0$. By applying the functor $Hom_A(-,T_0)$ to the short exact sequence induced by the inclusion of J in A and using the fact that any map from A to T_0 factors through ϕ (and thus through the quotient A/J), we conclude that $Hom_A(J,T_0) = 0$. Since J is a projective A-module and ϕ is a $\mathcal{T}$-approximation with $\mathcal{T} \subseteq Gen(T_0)$, we get $Hom_A(J,\mathcal{T}) = 0$.

Consider now the monomorphism $\bar{\phi} : A/J \to T_0$ induced by ϕ. Applying the functor $J \otimes_A -$ to the short exact sequence induced by $\bar{\phi}$, since $Tor^A_1(J,-) = 0$, we see that there is a monomorphism $J \otimes_A A/J \to J \otimes_A T_0$. Now, let $f : A^{(I)} \to T_0$ be an epimorphism, for some set I. Then it follows that there is a surjection $J \otimes_A f : J^{(I)} \to J \otimes_A T_0$. Since J is projective, $J \otimes_A T_0$ lies in $Add(T_0) \subseteq \mathcal{T}$ and, therefore, $J \otimes f = 0$, which implies that $J \otimes_A T_0 = 0$. This shows that $J \otimes_A A/J = J/J^2 = 0$ and, thus, J is idempotent. □

The following proposition is a consequence of the above lemma.

Proposition 6.2.2. *Let A be a hereditary ring.*

(1) An A-module T is silting if and only if T is tilting over $A/Ann(T)$ and the ideal $Ann(T)$ is idempotent. In other words, silting A-modules are support tilting.

(2) Let $f : A \to B$ be a homological ring epimorphism. Then the kernel of f is an idempotent ideal. In particular, the ring epimorphism $A \to A/ker(f)$ is homological.

Proof. We first prove statement (1). Assume that T is silting. Thus, by Lemma 5.1.4, T is tilting over the quotient ring $\bar{A} := A/Ann(T)$. Moreover, by Proposition 5.1.14, there is a left $Gen(T)$-approximation $\phi : A \to T_0$ with T_0 in $Add(T)$ and $ker(\phi) = Ann(T)$. Since T is silting, T_0 has no self-extensions and, thus, by Lemma 6.2.1, $Ann(T)$ is idempotent. Conversely, suppose that T is a tilting $\bar{A}$-module with $Ann(T)$ idempotent. Consider the projective A-presentation σ of T given as the direct sum of a monomorphic presentation of T with the trivial map $Ann(T) \to 0$. Since $Ann(T)$ is idempotent, it follows from Proposition 1.3.7 and Proposition 1.3.8 that

$$\mathcal{D}_\sigma = T^{\perp_1} \cap Ann(T)^\circ = ker(Ext^1_{\bar{A}}(T,-)) = Gen(T).$$

Consequently, T is a silting A-module.

Statement (2) follows from Lemma 6.2.1 by observing that f, when seen as a map of A-modules, is the $\mathcal{X}_B$-reflection of A and we have $Ext^1_A(B,B) = 0$, by assumption. Moreover, the ring epimorphism $A \to A/ker(f)$ is homological by Proposition 1.3.8. □

Note that a similar statement does not hold without the hereditary assumption.

Example 6.2.3. *Let T be a τ-tilting module over a finite dimensional $\mathbb{K}$-algebra A that is not tilting. Since T is sincere, $Ann(T) \neq 0$ cannot contain any idempotent $e \neq 0$ in A and, thus, it is not an idempotent ideal. Similarly, we can construct a homological ring epimorphism over A whose kernel is not idempotent.*

Definition 6.2.4. *Let A be a hereditary ring and T be a silting A-module. Then T is called*

- ***minimal**, if ${}_AA$ admits a minimal left $Add(T)$-approximation;*
- ***strongly minimal**, if all free A-modules admit minimal left $Add(T)$-approximations.*

We have the following equivalent way of defining (strongly) minimal silting modules.

Lemma 6.2.5. *Let A be a hereditary ring. Then the following are equivalent.*

(1) T is a (strongly) minimal silting A-module;

(2) ${}_AA$ (every free A-module) admits a minimal left $Gen(T)$-approximation.

Proof. Statement (2) follows from (1), using that free A-modules are, in particular, projective, which yields the needed factorisation property. Now assume that (2) holds and let $\phi : A^{(I)} \to M$ be a minimal left $Gen(T)$-approximation for some set I. Since T is a silting A-module, we also have a left $Add(T)$-approximation $f : A^{(I)} \to T_0$, which is also a $Gen(T)$-approximation. Consequently, we get the following commutative diagram of A-modules

$$\begin{tikzcd} A^{(I)} \arrow[rr, "\phi"] \arrow[dr, "f"'] & & M \arrow[dl, shift left, "g_2"] \\ & T_0 \arrow[ur, shift left, "g_1"] & \end{tikzcd}$$

Using that the map ϕ is left-minimal, it follows that M is a direct summand of T_0 and, thus, lies in $Add(T)$. Therefore, (1) holds. □

The following remark collects some useful facts about minimal left approximations.

Remark 6.2.6. *(1) Suppose a module M admits a minimal left $\mathcal{Y}$-approximation for an additive subcategory $\mathcal{Y} \subseteq A\text{-}Mod$. Then a left $\mathcal{Y}$-approximation $g : M \to Y$, with Y in $\mathcal{Y}$, is minimal if and only if $Im(g)$ is not contained in any proper summand of Y ([X, Corollary 1.2.3]);*

(2) If T is a strongly minimal silting module over a hereditary ring A, then the minimal left $Add(T)$-approximation of $A^{(I)}$ is given by taking the coproduct indexed by the set I of the minimal left $Add(T)$-approximation $A \to T_0$. This follows from [X, Theorem 1.4.6].

Clearly, the definition of (strongly) minimal silting modules also applies to tilting modules. Note that already in the setting of tilting, we obtain many non-trivial examples.

Example 6.2.7. *Let A be a hereditary ring.*

- *Let T be an endofinite silting A-module (i.e. T has finite length over its endomorphism ring). Consequently, by [KS, Theorem 4.1], $Add(T)$ is closed for products and, thus, by [KS, Theorem 3.1], every A-module admits a minimal left $Add(T)$-approximation. In particular, finitely generated silting modules over hereditary Artin algebras are strongly minimal.*

- *Let A be noetherian and consider the minimal injective coresolution*

$$0 \longrightarrow A \longrightarrow E_1 \longrightarrow E_2 \longrightarrow 0.$$

It follows that $T := E_1 \oplus E_2$ is a tilting A-module where $Gen(T)$ is given by the class of injective A-modules. Since injective envelopes are left-minimal, by Lemma 6.2.5, T is a strongly minimal tilting module.

- *Let T be a tilting A-module that arises from a ring epimorphism (see Definition 1.9.1), say $T = B \oplus B/A$ for a monomorphic homological ring epimorphism $A \to B$. Then T is strongly minimal. In fact, we have the following $Add(T)$-approximation sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow B/A \longrightarrow 0.$$

Since ϕ is a reflection map, it is clearly left-minimal. Moreover, arbitrary coproducts of ϕ will again yield reflections.

It remains open if all minimal silting modules are strongly minimal. In fact, the notion of strong minimality is only needed for a technical argument (see Proposition 6.2.8) necessary to prove the main result of this section. Minimal silting modules are motivated by the following construction.

Let A be a hereditary ring and T be a minimal silting module with associated torsion class $\mathcal{D}_\sigma = Gen(T)$. Consider the minimal $Add(T)$-approximation sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0.$$

Following Proposition 6.2.2, T is a tilting module over the quotient ring $\bar{A} := A/Ann(T)$ with $Ann(T)$ being idempotent. We get the induced minimal $Add(T)$-approximation sequence in $\bar{A}$-Mod

$$0 \longrightarrow \bar{A} \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0.$$

Since the ideal $Ann(T)$ is idempotent and, thus, $\mathcal{X}_{\bar{A}}$ is closed under extensions in A-Mod, we get

$$Gen(T) = T_1^{\perp_1} \cap \mathcal{X}_{\bar{A}} = T_1^{\perp_1} \cap Ann(T)^\circ.$$

In particular, T_1 is a partial silting A-module with respect to the projective presentation σ_1, given as the direct sum of a monomorphic presentation of T_1 with the trivial map $Ann(T) \to 0$. In fact, we have $\mathcal{D}_{\sigma_1} = Gen(T) = \mathcal{D}_\sigma$. Following Proposition 6.1.2, we consider the bireflective subcategory

$$\mathcal{Y} = T_1^{\perp} \cap \mathcal{X}_{\bar{A}}$$

associated to T_1. The corresponding ring epimorphism will be denoted by $A \to B_T$. Since the approximation ϕ was chosen minimal and, hence, the module T_1 is uniquely determined, we obtain a well-defined map from (equivalence classes of) minimal silting modules to (epiclasses of) ring epimorphisms by mapping T to the ring epimorphism $A \to B_T$. Before stating the main result of this section, we need the following technical proposition motivated by Theorem 3.2.2.

Proposition 6.2.8. *Let A be a hereditary ring and T be a strongly minimal silting A-module with associated ring epimorphism $A \to B_T$. Then $\mathcal{X}_{B_T}$ coincide with*

$$\alpha(Gen(T)) = \{X \in Gen(T) \mid \forall (g : Y \to X) \in Gen(T), ker(g) \in Gen(T)\}.$$

Moreover, if $A \to T_0$ is the minimal left $Add(T)$-approximation, then we have $\mathcal{X}_{B_T}$-$Proj = Add(T_0)$.

Proof. Since T is a minimal silting module, we have the minimal approximation sequence

$$A \xrightarrow{\phi} T_0 \xrightarrow{\pi} T_1 \longrightarrow 0 \tag{6.2.1}$$

with T_0 and T_1 in $Add(T)$. We begin by discussing some properties of the category $\alpha(Gen(T))$. By [IT, Proposition 2.12 and Proposition 2.15], $\alpha(Gen(T))$ is an abelian and extension-closed subcategory of A-Mod that coincides with

$$\{X \in Gen(T) \mid \forall (g : Y \twoheadrightarrow X) \in Gen(T), ker(g) \in Gen(T)\}.$$

In other words, it is enough to consider kernels of surjective maps in $Gen(T)$. We first show that $Add(T_0)$ describes precisely the projective objects in $\alpha(Gen(T))$. Consider $T_0^{(I)}$ for some set I and a surjection $g : Y \twoheadrightarrow T_0^{(I)}$ in $Gen(T)$. Since $A^{(I)}$ is projective we get the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] & T_0^{(I)} \arrow[r] & 0 \\ & A^{(I)} \arrow[ul, "h"] \arrow[u, "\phi^{(I)}"'] & \end{tikzcd}$$

Since $\phi^{(I)}$ is a left $Gen(T)$-approximation, there is a map g' with $h = g' \circ \phi^{(I)}$. Thus, one gets $g \circ g' \circ \phi^{(I)} = \phi^{(I)}$. But T is strongly minimal and, hence, the map $\phi^{(I)}$ is left-minimal by Remark 6.2.6. Consequently, the map g splits showing that the modules in $Add(T_0)$ are projective in $\alpha(Gen(T))$. Conversely, for all objects X in $Gen(T)$, there is a surjection $T_0^{(I)} \twoheadrightarrow X$ for some set I, since $Gen(T) = Gen(T_0)$. Using that $Add(T_0) \subseteq \alpha(Gen(T))$, we conclude that all projective objects in $\alpha(Gen(T))$ belong to $Add(T_0)$.

Next, we verify that $\alpha(Gen(T)) = \mathcal{X}_{B_T}$. Recall that $\mathcal{X}_{B_T} = T_1^{\perp} \cap \mathcal{X}_{\bar{A}}$ for $\bar{A} = A/Ann(T)$. Since $Ann(T)$ is idempotent and, hence, $\mathcal{X}_{\bar{A}}$ is closed under extensions in A-Mod, it follows that

$$\mathcal{X}_{B_T} = \{X \in \mathcal{X}_{\bar{A}} \mid Hom_{\bar{A}}(T_1, X) = 0 = Ext^1_{\bar{A}}(T_1, X)\}.$$

We first prove that $\mathcal{X}_{B_T} \subseteq \alpha(Gen(T))$. Take X in $\mathcal{X}_{B_T}$. Since $Ext^1_{\bar{A}}(T_1, X) = 0$ and T is a tilting module over $\bar{A}$, the module X lies in $Gen(T)$. Now consider a surjection $g : Y \twoheadrightarrow X$ with Y in $Gen(T)$. Note that $ker(g)$ belongs to $\mathcal{X}_{\bar{A}}$, since so do X and Y. By applying the functor $Hom_{\bar{A}}(T_1, -)$ to the induced short exact sequence we obtain the exact sequence

$$Hom_{\bar{A}}(T_1, Y) \longrightarrow Hom_{\bar{A}}(T_1, X) \longrightarrow Ext^1_{\bar{A}}(T_1, ker(g)) \longrightarrow Ext^1_{\bar{A}}(T_1, Y).$$

Since, by assumption, $Hom_{\bar{A}}(T_1, X) = 0$ and Y lies in $Gen(T)$ (showing that $Ext^1_{\bar{A}}(T_1, Y) = 0$), it follows that $Ext^1_{\bar{A}}(T_1, ker(g)) = 0$. This proves that $ker(g)$ lies in $Gen(T)$ and, thus, $X \in \alpha(Gen(T))$.

To show the other inclusion, we turn back to $Add(T_0)$. We have already checked that $Add(T_0)$ describes precisely the split-projectives modules in $Gen(T)$, respectively, the projective objects in $\alpha(Gen(T))$. Next, we see that $Add(T_0)$ is closed for subobjects in $Gen(T)$. Take Z in $Gen(T)$ with an inclusion map $Z \to T_0^{(I)}$ for some set I. Note that, by definition, Z belongs to $\alpha(Gen(T))$. Now the claim follows by observing that the subcategory $\alpha(Gen(T))$ is closed under extensions in A-Mod and, therefore, it is also a hereditary category. Consequently, Z lies in $Add(T_0)$, since it is a subobject of a projective.

Now we prove the missing inclusion $\alpha(Gen(T)) \subseteq \mathcal{X}_{B_T}$. Take a module X in $\alpha(Gen(T))$. Since X lies in $Gen(T)$, we have $Ext^1_{\bar{A}}(T_1, X) = 0$. Suppose that there is a non-trivial map $h : T_1 \to X$ and take a surjection $g : T_0^{(I)} \twoheadrightarrow X$ for some set I. Since X lies in $\alpha(Gen(T))$, the kernel of g belongs to $Gen(T)$ and, therefore, $Ext^1_A(T_1, ker(g)) = 0$. Consequently, the map h lifts via a map h' yielding the following commutative diagram

$$\begin{tikzcd}
T_0^{(I)} \arrow[r, "g"] & X \arrow[r] & 0 \\
& T_1 \arrow[u, "h"'] \arrow[ul, "h'"] &
\end{tikzcd}$$

Since $Add(T_0)$ is closed for subobjects in $Gen(T)$, we get that $Im(h') \in Add(T_0)$. Consequently, $Im(h')$ is split-projective in $Gen(T)$ and, hence, a direct summand of T_1. Let $p : T_1 \to Im(h')$ be the canonical projection. Then also the composition $p \circ \pi : T_0 \to Im(h')$ is a split surjection (see Diagram 6.2.1) and the restriction $\pi\restriction_{Im(h')}$ is an isomorphism. Consequently, $Im(\phi)$ is contained in a proper summand of T_0, contradicting the minimality of ϕ (compare Remark 6.2.6). $\square$

Now we are able to state the main result of this section.

Theorem 6.2.9. *Let A be a hereditary ring. Then there are bijections between:*

(1) Equivalence classes of strongly minimal silting modules;

(2) Epiclasses of homological ring epimorphisms of A.

Moreover, these bijections restrict to bijections between:

(1) Equivalence classes of strongly minimal tilting modules;

(2) Epiclasses of injective homological ring epimorphisms of A.

Proof. We will show that the following assignment yields the desired bijection:

$$\Psi : T \mapsto (f : A \to B_T).$$

First of all, we check that the assignment is well-defined. By construction, the ring epimorphism f is uniquely determined by T. Moreover, by Proposition 6.1.2, the subcategory $\mathcal{X}_{B_T}$ is closed under extensions in A-Mod and, therefore, $Tor_1^A(B_T, B_T) = 0$. Since A is hereditary, all higher Tor-groups vanish showing that the ring epimorphism f is homological.

The injectivity of the map Ψ follows from Proposition 6.2.8. In fact, if T and T' are two strongly minimal silting modules with $\Psi(T) = \Psi(T')$, then the previous proposition tells us that $Add(T_0) = Add(T_0')$ and, thus, T and T' are equivalent.

Next, we prove surjectivity of Ψ. Let $f : A \to B$ be a homological ring epimorphism. By Proposition 6.2.2, we get a commutative diagram of homological ring epimorphisms

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & A/ker(f) \arrow[ur, "f'"] & \end{tikzcd}$$

where f' is injective and the quotient ring $\bar{A} := A/ker(f)$ is again hereditary. Consequently, by Theorem 1.9.2, $T := B \oplus B/\bar{A}$ is a tilting module over $\bar{A}$ and it is strongly minimal by Example 6.2.7. Using Proposition 6.2.2, T becomes a strongly minimal silting A-module with respect to a projective A-presentation σ of T that is given as the direct sum of a monomorphic presentation of T with the trivial map $ker(f) \to 0$. It remains to check that $\Psi(T)$ lies in the same epiclass as the ring epimorphism $f : A \to B$. Since the minimal left $Add(T)$-approximation is given by the A-module map $f : A \to B$, by construction, it suffices to show that

$$(B/\bar{A})^{\perp} \cap \mathcal{X}_{\bar{A}} = \mathcal{X}_B.$$

But this follows again from Theorem 1.9.2 and, thus, completing the argument.

As a consequence of the previous arguments, the inverse Ψ^{-1} of Ψ assigns to a homological ring epimorphism $f : A \to B$ the strongly minimal silting A-module $B \oplus coker(f)$. Therefore, in case f is injective, the module $\Psi^{-1}(f)$ is actually a tilting module by Theorem 1.9.2. It remains to check the restriction of the map Ψ. Let T be a strongly minimal tilting A-module with a monomorphic minimal $Add(T)$-approximation $\phi : A \to T_0$. Using Proposition 6.2.8, it follows that $Hom_A(T_0/A, T_0) = 0$ and, therefore, by Theorem 1.9.2, T arises from an injective homological ring epimorphism. In fact, this ring epimorphism has to be $A \to B_T$. This finishes the proof. $\square$

Remark 6.2.10.

(1) If we restrict Theorem 6.2.9 to support τ-tilting modules over a hereditary finite dimensional $\mathbb{K}$-algebra (i.e. to finitely generated support tilting modules), we obtain back the correspondences established in Theorem 3.2.3.

(2) The construction of the map Ψ in Theorem 6.2.9 can be generalised to the setting of non-hereditary rings. To do so, we first have to define minimal silting modules over arbitrary rings and then use the construction discussed in the first section of this chapter. However, the ring epimorphisms we obtain in this way will usually not be homological (see Example 6.1.5 and Example 6.1.6). Moreover, we do not know whether the map Ψ remains injective.

We have the following immediate corollary of Theorem 6.2.9.

Corollary 6.2.11. *Let A be a hereditary ring. Then a tilting A-module T arises from a ring epimorphism if and only if T is strongly minimal.*

We present an example of a tilting module that is not strongly minimal.

Example 6.2.12. *Let A be the Kronecker algebra (see Example 3.1.10) and consider the Lukas tilting module T (see [L] and [L2]). Recall that $Gen(T)$ is given by the A-modules without any indecomposable preprojective summands. By [AS2], T does not arise from a ring epimorphism. In fact, since $Add(T') = Add(T)$ for all non-zero direct summands T' of T (see [L, Theorem 6.1] and [L2, Theorem 3.1]), we conclude that for all $Add(T)$-approximation sequences*

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

the bireflective subcategory $T_1^{\perp}$ only contains the zero-module.

Theorem 6.2.9 has some further applications. It is well-known that over a hereditary ring A (epiclasses of) homological ring epimorphisms correspond bijectively to (equivalence classes of) recollements of $D(A)$ (see [NS] and [KSt, Theorem 8.1]). In fact, this bijection maps a homological ring epimorphism $A \to B$ to the recollement

$$\begin{tikzcd} D(B) \arrow[r, "f_*"] \arrow[r, leftarrow, shift left=2ex] \arrow[r, leftarrow, shift right=2ex] & D(A) \arrow[r] \arrow[r, leftarrow, shift left=2ex] \arrow[r, leftarrow, shift right=2ex] & Tria(K_f) \end{tikzcd} ,$$

where K_f denotes the cone of f in $D(A)$ (see Theorem 1.5.2).

Proposition 6.2.13. *Let A be a hereditary finite dimensional $\mathbb{K}$-algebra and T be a strongly minimal silting module with associated ring epimorphism $f : A \to B_T$. The following are equivalent.*

(1) T is finitely generated;

(2) B_T is a finite dimensional $\mathbb{K}$-algebra;

(3) $Tria(K_f)$ is equivalent to a derived module category.

Proof. If T is finitely generated, then so is T_0, given by the minimal left $Add(T)$-approximation $A \to T_0$. Consequently, by Theorem 6.2.9, B_T (which is isomorphic to T_0 when seen as an A-module) is a finite dimensional $\mathbb{K}$-algebra. Conversely, if B_T is finite dimensional, then the module $T = B_T \oplus coker(f)$ is finitely generated. It follows that (1) is equivalent to (2).

Now assume that (2) holds. Then K_f, given by the complex $A \to B_T$, is compact in $D(A)$. Now, by [K, Theorem 8.5], it suffices to check that K_f is also exceptional. Since A is hereditary, K_f decomposes into the direct sum $ker(f)[1] \oplus coker(f)$, where $ker(f)$ is a projective A-module given by $Ann(T)$. We denote, as usual, $A/Ann(T)$ by $\bar{A}$. We have to show that

(i) $Hom_A(ker(f), coker(f)) = 0$;

(ii) $Hom_A(coker(f), ker(f)) = 0$;

(iii) $Ext^1_A(coker(f), coker(f)) = 0$.

Statement (i) follows by observing that $coker(f)$ belongs to $Gen(T) \subseteq \mathcal{X}_{\bar{A}} = ker(f)^{\circ}$. For (ii) observe that $coker(f)$ cannot have a projective direct summand, since the map f is left-minimal. Therefore, using that A is hereditary, $Hom_A(coker(f), P)$ must be zero for all projective A-modules P. Finally, (iii) holds, since $coker(f)$ lies in $Add(T)$.

It remains to prove that (3) implies (2). But if $Tria(K_f)$ is equivalent to a derived module category, it follows from [AKLY, Lemma 2.10 (b)] that the algebra B_T has to be finite dimensional. $\square$

Chapter 7

Universal localisations via torsion pairs

This chapter contains ongoing joint work with Jan Šťovíček. We study universal localisations of a given ring via torsion pairs. More precisely, to every set Σ of maps between finitely generated projective modules, we associate two torsion pairs which, in special cases, already appeared in the context of silting modules in Chapters 5 and 6. But now we are interested in the universal localisation $A \to A_\Sigma$. After defining the relevant objects, in the second section, we use the previously defined torsion pairs to construct explicitly the reflections of the corresponding localisation. It turns out that, different to the setting of silting modules, the underlying approximation theory is more involved in the sense that it uses transfinite compositions. In the third section, we show that universal localisations are, indeed, intrinsically connected to silting objects. More precisely, every universal localisation can be described by a two-term complex of projective modules which has no positive self-extensions. Finally, in the last section of the chapter, this observation is applied to Artin algebras.

7.1 Setup

For a set Σ of maps between finitely generated projective A-modules we consider the universal localisation $f : A \to A_\Sigma$. The essential image of the associated restriction functor f_* is denoted by $\mathcal{X}_\Sigma = \mathcal{X}_{A_\Sigma}$. To the set Σ we associate two torsion pairs in $A\text{-}Mod$. One is given by $(\mathcal{D}_\Sigma, \mathcal{R}_\Sigma)$ with

$$\mathcal{D}_\Sigma := \{X \in A\text{-}Mod \mid Hom_A(\sigma, X) \text{ is surjective} \, \forall \sigma \in \Sigma\}.$$

The modules in $\mathcal{D}_\Sigma$ (respectively, $\mathcal{R}_\Sigma$) are called **Σ-divisible** (respectively, **Σ-reduced**). Following the arguments in Lemma 5.1.9, we see that $\mathcal{D}_\Sigma$ is closed under quotients and extensions in $A\text{-}Mod$.

Moreover, since the projective modules involved in the maps in Σ are finitely generated, $\mathcal{D}_\Sigma$ is also closed under (arbitrary) direct sums and, thus, it is a torsion class. The second torsion pair $(\mathcal{T}_\Sigma, \mathcal{F}_\Sigma)$ is the one generated by all the cokernels of maps in Σ. Hence,

$$\mathcal{F}_\Sigma := \{X \in A\text{-}Mod | Hom_A(\sigma, X) \text{ is injective} \, \forall \sigma \in \Sigma\}.$$

The modules in $\mathcal{F}_\Sigma$ (respectively, $\mathcal{T}_\Sigma$) are called Σ-**torsion-free** (respectively, Σ-**torsion**). Note that, by Proposition 1.6.3, it follows that

$$\mathcal{X}_\Sigma = \mathcal{D}_\Sigma \cap \mathcal{F}_\Sigma.$$

Similar torsion and torsionfree classes appeared before in the context of universal localisations (see, for example, [Sch3] and [AS, Chapter 4]). However, a systematic and general discussion of their properties and use is still missing. The following easy example shows that the above torsion pairs depend on the set Σ and not only on the universal localisation A_Σ of A.

Example 7.1.1. *Let P be a finitely generated projective A-module and consider the two sets $\Sigma = \{0 \to P\}$ and $\Sigma' = \{P \to 0\}$. Clearly, we have $A_\Sigma = A'_\Sigma$. The torsion class $\mathcal{D}_\Sigma$ associated to Σ equals A-Mod. But $\mathcal{D}_{\Sigma'}$, on the other hand, is given by $\mathcal{X}_\Sigma \subsetneq A$-$Mod$. Similarly, we have $\mathcal{F}_\Sigma \neq \mathcal{F}_{\Sigma'}$.*

However, to keep notation simple, throughout, we only write $\mathcal{D}, \mathcal{R}, \mathcal{T}$ and $\mathcal{F}$ whenever the set Σ is fixed beforehand. We have the following lemma on how to pass some information from Σ to the class of Σ-divisible modules.

Lemma 7.1.2. *Let Σ be a set of maps in A-$proj$. The following are equivalent.*

(1) The torsion class $\mathcal{D}$ contains all injective A-modules;

(2) The set Σ consists of monomorphisms.

Proof. We first assume that (1) holds. Let C be an injective cogenerator of A-Mod lying in $\mathcal{D}$. Note that an A-module X belongs to $\mathcal{D}$ if and only if for all $\sigma : P \to Q$ in Σ every given map $P \to X$ factors through σ. Take $\sigma : P \to Q$ in Σ and a monomorphism $\phi : P \to C^I$ for some set I. Since C^I is an injective A-module, it belongs to $\mathcal{D}$. Consequently, the map ϕ must factor through σ, forcing also the map σ to be a monomorphism. Hence, we get (2). The converse follows directly from the definition of $\mathcal{D}$. $\square$

If the equivalent conditions in Lemma 7.1.2 are fulfilled, the universal localisation of A at Σ turns out to be easier to describe (see, for example, the approach taken in [Sch3]). In this situation,

$\mathcal{D}$ is given by the class

$$(Coker(\Sigma))^{\perp_1} := \{X \in A\text{-}Mod | Ext^1_A(coker(\sigma), X) = 0\ \forall \sigma \in \Sigma\}$$

and, thus, $\mathcal{D}$ is a tilting class with an associated complete cotorsion pair $({}^{\perp_1}\mathcal{D}, \mathcal{D})$ (see [AC], [ST]). In Section 7.2, this information will be useful to better understand the $\mathcal{X}_\Sigma$-reflections in $A\text{-}Mod$. In general, not every universal localisation of A is given by localising at a set of injective morphisms between finitely generated projective A-modules. However, the next proposition verifies that we can factor every universal localisation into a surjective ring homomorphism and another universal localisation at a set of injective maps. So, the situation gets simplified. Recall that for a given set Σ in $A\text{-}proj$ the universal localisation $f : A \to A_\Sigma$ coincides with the universal localisation of A at the set $\Sigma^* := \{Hom_A(\sigma, A) | \sigma \in \Sigma\}$ of maps between finitely generated projective right A-modules (see Lemma 1.6.4). For a fixed set Σ, we denote by A_{TF} the ring we obtain by factoring out the $\mathcal{T}$-torsion part of A. The associated surjective ring homomorphism is given by $\pi : A \to A_{TF}$. Note that, by construction, there is a unique A-module map $g : A_{TF} \to A_\Sigma$ turning the following diagram into a commutative diagram of ring epimorphisms

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\pi"'] & & A_\Sigma \\ & A_{TF} \arrow[ur, "g"'] & \end{tikzcd}$$

Proposition 7.1.3. *The morphism $g : A_{TF} \to A_\Sigma$ describes the universal localisation of A_{TF} at $\Sigma_{TF} := \{A_{TF} \otimes_A \sigma | \sigma \in \Sigma\}$ and g is given by localising at the set of injective maps in $(\Sigma_{TF})^*$.*

Proof. First of all, A_Σ is Σ_{TF}-invertible, since, by assumption,

$$A_\Sigma \otimes_{A_{TF}} (A_{TF} \otimes_A \sigma) \cong A_\Sigma \otimes_A \sigma$$

is an isomorphism for all σ in Σ. We have to check the universal property for g. Let $\psi : A_{TF} \to S$ be a Σ_{TF}-invertible ring homomorphism. Consequently,

$$S \otimes_A \sigma \cong S \otimes_{A_{TF}} (A_{TF} \otimes_A \sigma)$$

is an isomorphism for all σ in Σ, yielding, by the universal property of A_Σ, a unique ring homomorphism $h : A_\Sigma \to S$ such that $\psi \circ \pi = h \circ f$. Hence, we get that $\psi \circ \pi = h \circ g \circ \pi$. Using the surjectivity of π, we obtain the wanted factorisation $\psi = h \circ g$. It follows that g is the universal localisation of

A_{TF} at Σ_{TF}. Moreover, by the construction of A_{TF}, we know that $Hom_A(\sigma, A_{TF})$ is injective and, thus, using adjunction, also

$$Hom_{A_{TF}}(A_{TF} \otimes_A \sigma, A_{TF})$$

must be injective. This finishes the proof. $\square$

Remark 7.1.4. *The ring epimorphism π is, in general, not a universal localisation of A. In fact, $Tor_1^A(A_{TF}, A_{TF})$ does not always vanish. For example, if A is a finite dimensional $\mathbb{K}$-algebra and $A \to A_\Sigma$ is a pure universal localisation of A (compare Chapter 3), then $\mathcal{X}_{A_\Sigma}$ is not contained in $\mathcal{X}_{A/AeA}$ for any non-zero idempotent e in A. Also, the map g is generally not injective.*

Remark 7.1.5. *We can iterate the process described above. To do so, we first need to define – similar to the situation for left A-modules – the torsion pairs $(\mathcal{D}^{op}, \mathcal{R}^{op})$ and $(\mathcal{T}^{op}, \mathcal{F}^{op})$ in Mod-A with respect to a given set Σ^{op} of maps between finitely generated projective right A-modules. Now we can reduce A transfinitely by factoring out, step by step, the $\mathcal{T}$-torsion and the $\mathcal{T}^{op}$-torsion part. As a direct limit, we obtain a ring A_{TF} that is torsionfree from both sides, meaning, with respect to $\mathcal{F}$ and $\mathcal{F}^{op}$. Note that the reduction process is finite if the ring A is noetherian. Again, we get the following commutative diagram of ring epimorphisms*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\pi"'] & & A_\Sigma \\ & A_{TF} \arrow[ur, "g"'] & \end{tikzcd}$$

where g is the universal localisation of A_{TF} at the induced set Σ_{TF}. But now all the maps in Σ_{TF} and in Σ_{TF}^* are injective. In other words, all the cokernels of the maps in Σ_{TF} are bound, i.e.,

$$Hom_{A_{TF}}(coker(\sigma), A_{TF}) = 0$$

for all σ in Σ. In general, this condition is only necessary for the localisation g to be injective (see [AS, Remark 4.4] and [Sch3]). Over a hereditary ring, however, it will force the kernel of g to vanish (compare [Sch4]).

7.2 Divisible approximations and reflections

Since universal localisations yield ring epimorphisms, $\mathcal{X}_\Sigma$ admits reflections in *A-Mod*. The aim of this section is to construct these reflections explicitly in a two step process using the torsion pairs defined above. By doing so, we generalise some ideas used in Theorem 5.1.15 and Proposition 6.1.3 to construct approximations and reflections. First, we have to introduce some terminology discussed in [Sto]. We begin with an orthogonality relation on morphisms in a given category $\mathcal{C}$.

Definition 7.2.1. *For two morphisms $f : V \to W$ and $g : X \to Y$ in $\mathcal{C}$ we write $f \Box g$, if for all commutative squares in $\mathcal{C}$ of the form*

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there is a diagonal dotted morphism $W \to X$ making the two triangles in the diagram commute. Note that we do not require uniqueness of the dotted arrow.

We provide some easy examples to illustrate this definition.

Example 7.2.2. *Let $\mathcal{C}$ be any category.*

- *Take an object $V \in \mathcal{C}$ with the identity morphism $1_V : V \to V$ and let $g : X \to Y$ be any morphism in $\mathcal{C}$. Consider a commutative square in $\mathcal{C}$ of the form*

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "1_V"'] & X \arrow[d, "g"] \\ V \arrow[r] & Y. \end{tikzcd}$$

  It follows that we can choose h to be the required dotted morphism to complete this diagram. Hence, $1_V \Box g$. Dually, one checks that $g \Box 1_V$ for all morphisms g in $\mathcal{C}$.

- *Suppose the category $\mathcal{C}$ is abelian. Take $V \in \mathcal{C}$ together with the zero morphism $0_V : V \to 0$ and let $g : X \to Y$ be any morphism in $\mathcal{C}$. Then it easily follows that $0_V \Box g$ if and only if $Hom_{\mathcal{C}}(V, ker(g))$ only contains the zero morphism.*

- *Suppose the category $\mathcal{C}$ is abelian. Let $f : V \to W$ be an epimorphism and $g : X \to Y$ be a monomorphism in $\mathcal{C}$. Consider a commutative square in $\mathcal{C}$ of the form*

$$\begin{tikzcd} V \arrow[r, "h_1"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h_2"] & Y. \end{tikzcd}$$

By i we denote the kernel morphism $ker(f) \to V$. Since g is a monomorphism and the above square commutes, the composition $h_1 \circ i$ must be zero and, by the universal property of the cokernel, we get a unique morphism $s : W \to X$ such that $s \circ f = h_1$. Consequently, again using the commutativity of the square, it follows that $g \circ s \circ f = g \circ h_1 = h_2 \circ f$. Since f is an epimorphism, we actually get $g \circ s = h_2$. Consequently, $f \square g$ holds.

The main concept needed later on is the following.

Definition 7.2.3. *Let $(\mathcal{L}, \mathcal{U})$ be a pair of classes of morphisms in $\mathcal{C}$. We say that $(\mathcal{L}, \mathcal{U})$ is a* ***weak factorisation system****, if*

(FS1) $\mathcal{L}$ and $\mathcal{U}$ are closed under direct summands, i.e., whenever there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "h"] & V \arrow[r] \arrow[d, "f"] & X \arrow[d, "h"] \\ Y \arrow[r] & W \arrow[r] & Y \end{tikzcd}$$

such that f lies in $\mathcal{L}$ (respectively, in $\mathcal{U}$) and the horizontal arrows compose to the identity, then also h belongs to $\mathcal{L}$ (respectively, to $\mathcal{U}$).

(FS2) We have $f \square g$ for all f in $\mathcal{L}$ and g in $\mathcal{U}$.

(FS3) For all morphisms $h : X \to Y$ in $\mathcal{C}$ there is a factorisation $h = g \circ f$ with f in $\mathcal{L}$ and g in $\mathcal{U}$.

This concept is motivated by the notion of a model category (see [Hov]). As a first example, we can take an abelian category $\mathcal{C}$ and we define $\mathcal{L}$ to be the class of all epimorphisms in $\mathcal{C}$ and $\mathcal{U}$ to be the class of all monomorphisms in $\mathcal{C}$ (compare Example 7.2.2). However, we will see that this example is not typical, in the sense that there are weak factorisation systems $(\mathcal{L}, \mathcal{U})$, for which all maps in $\mathcal{L}$ are monomorphisms and all maps in $\mathcal{U}$ are epimorphisms (see Proposition 7.2.8).

Note that we do not require uniqueness of the factorisation in (FS3). Part (FS2) of the above definition can be used to compare different possible factorisations. Moreover, it is not hard to check that the classes $\mathcal{L}$ and $\mathcal{U}$ determine each other (compare [Sto, Lemma 4.5]). In fact, we have

$$\mathcal{L} = \{f \mid f \square g \text{ for all } g \in \mathcal{U}\} \qquad \text{and} \qquad \mathcal{U} = \{g \mid f \square g \text{ for all } f \in \mathcal{L}\}.$$

Moreover, the class $\mathcal{L}$ is closed under pushouts and transfinite compositions (see [Sto, Lemma 4.6]). Let us recall the definition of a transfinite composition.

Definition 7.2.4. *For an ordinal number λ, we call a direct system $(X_\alpha, f_{\alpha\beta})_{\alpha<\beta<\lambda}$ indexed by λ in a category $\mathcal{C}$ a λ-**sequence**, if for every limit ordinal $\mu < \lambda$, the object X_μ together with the morphisms $f_{\alpha\mu}$ for $\alpha < \mu$ is a direct limit of the direct subsystem $(X_\alpha, f_{\alpha\beta})_{\alpha<\beta<\mu}$.*

*The **composition** of the λ-sequence is given by the colimit morphism*

$$X_0 \longrightarrow \varinjlim_{\alpha<\lambda} X_\alpha.$$

*If I is a class of morphisms in $\mathcal{C}$, we define a **transfinite composition of morphisms from** I to be the composition of a λ-sequence $(X_\alpha, f_{\alpha\beta})_{\alpha<\beta<\lambda}$ where $f_{\alpha,\alpha+1}$ belongs to I for all $\alpha+1 < \lambda$.*

*Finally, for a proper set I of morphisms in $\mathcal{C}$, we define a **relative** I-**cell complex** to be a transfinite composition of pushouts of morphisms from I.*

If we assume λ to be the smallest infinite ordinal ω, then a λ-sequence will be an infinite sequence in the usual sense. Moreover, to get a better intuition about transfinite compositions, we can assume $\mathcal{C}$ to be a module category and we consider a λ-sequence built up in the following way: take X_0 to be 0 and the module X_α to be contained in the module $X_{\alpha+1}$ with $X_{\alpha+1}/X_\alpha$ being in a fixed subcategory $\mathcal{S}$ of $\mathcal{C}$ for all $\alpha+1 < \lambda$. Then the composition of this λ-sequence yields an $\mathcal{S}$-filtered module. For example, if we assume $\mathcal{S}$ to be the subcategory of all simple modules, this module will be semiartinian (compare [GT, Chapter 3.1]). In Subsection 7.3, we will use similar λ-sequences to define filtrations – also called transfinite extensions – in exact categories.

The following result, which is a special version of Quillen's small object argument, is crucial in our context. As a reference compare [Sto, Theorem 4.8 and Corollary 4.9]. The setting is now restricted to module categories.

Theorem 7.2.5. *Let I be a set of morphisms in A-Mod and set*

$$\mathcal{U} := \{g \mid f \square g \text{ for all } f \in I\},$$

$$\mathcal{L} := \{f \mid f \square g \text{ for all } g \in \mathcal{U}\}.$$

Then $(\mathcal{L}, \mathcal{U})$ is a weak factorisation system in A-Mod and $\mathcal{L}$ consists precisely of direct summands of relative I-cell complexes.

We need the following small lemma.

Lemma 7.2.6. *Let Σ be a set of morphisms between finitely generated projective A-modules and take X in A-Mod such that the map $0 \to X$ is a relative Σ-cell complex. Then X belongs to the torsion class $\mathcal{T}$.*

Proof. We can write the map $0 \to X$ as a transfinite composition of pushouts of maps from Σ

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_\alpha \longrightarrow X_{\alpha+1} \longrightarrow \cdots$$

We proceed by transfinite induction on α. Suppose that X_α already belongs to $\mathcal{T}$. To construct $X_{\alpha+1}$, we consider the following pushout diagram with σ in Σ

$$\begin{tikzcd}
P \arrow[r, "\sigma"] \arrow[d] & Q \arrow[r] \arrow[d] & coker(\sigma) \arrow[r] \arrow[d, equal] & 0 \\
X_\alpha \arrow[r] & X_{\alpha+1} \arrow[r] & coker(\sigma) \arrow[r] & 0
\end{tikzcd}$$

Since, by construction, $coker(\sigma)$ lies in $\mathcal{T}$ and $\mathcal{T}$ is a torsion class, also $X_{\alpha+1}$ belongs to $\mathcal{T}$. Now the claim follows by observing that $\mathcal{T}$ is closed under direct limits. $\square$

Now we are able to state the main theorem of this section.

Theorem 7.2.7. *For every set of maps Σ in A-$proj$ the class $\mathcal{D}$ of all Σ-divisible A-modules is covariantly finite. Moreover, we can choose the left $\mathcal{D}$-approximations to be (direct summands of) relative Σ-cell complexes with cokernels in $\mathcal{D} \cap \mathcal{T}$.*

Proof. Take X in A-Mod and consider the weak factorisation system $(\mathcal{L}, \mathcal{U})$ generated by Σ in the sense of Theorem 7.2.5. We get the following factorisation of the trivial map $X \to 0$

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"'] & & 0 \\
& D \arrow[ur, "g"'] &
\end{tikzcd}$$

with $f \in \mathcal{L}$ and $g \in \mathcal{U}$. We claim that f is a left $\mathcal{D}$-approximation. Note that, by construction, an A-module Y belongs to $\mathcal{D}$ if and only if the map $Y \to 0$ lies in $\mathcal{U}$. Indeed, we know that Y lies in $\mathcal{D}$ if and only if for all $\sigma : P \to Q$ in Σ and every map $P \to Y$ there exists a map $Q \to Y$ making the following diagram commute

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\sigma"'] & Y \\
Q. \arrow[ur] &
\end{tikzcd}$$

Equivalently, we have $\sigma \square (Y \to 0)$ for all σ in Σ showing that $Y \to 0$ belongs to $\mathcal{U}$. Consequently, since the morphism g lies in $\mathcal{U}$, the A-module D belongs to $\mathcal{D}$. Now take a test map $\phi : X \to D'$

for D' in $\mathcal{D}$. It follows that $D' \to 0$ lies in $\mathcal{U}$. Thus, by (FS2), we get $f\square(D' \to 0)$ yielding a factorisation of the form

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & D' \\ D. \arrow[ur] & \end{tikzcd}$$

Therefore, f is a left $\mathcal{D}$-approximation. It remains to show that $coker(f)$ belongs to $\mathcal{T}$. First, observe that the map $0 \to coker(f)$ belongs to $\mathcal{L}$, since it can be obtained from the following pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ 0 \arrow[r] & coker(f) \end{tikzcd}$$

Consequently, the map $0 \to coker(f)$ is a direct summand of a relative Σ-cell complex (see Theorem 7.2.5) and, thus, the claim follows from Lemma 7.2.6. $\square$

Note that we can replace the weak factorisation system $(\mathcal{L},\mathcal{U})$ in the proof of Theorem 7.2.7 by the weak factorisation system $(\mathcal{L}',\mathcal{U}')$ induced by the set $\Sigma' := \Sigma \cup \{0 \to A\}$ with $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{L} \subseteq \mathcal{L}'$ in order to construct $\mathcal{D}$-approximations. Indeed, the torsion class $\mathcal{D}_{\Sigma'}$ coincides with $\mathcal{D}_\Sigma = \mathcal{D}$. But, a priori, these $\mathcal{D}$-approximations may differ from the ones induced by $(\mathcal{L},\mathcal{U})$. They will be direct summands of relative Σ'-cell complexes with cokernels in $\mathcal{T}_{\Sigma'} \supseteq \mathcal{T} = \mathcal{T}_\Sigma$. On first sight, this seems to be a superfluous step. But the next proposition tells that changing the system helps to understand the classes of maps being involved.

Proposition 7.2.8. *Let $(\mathcal{L}',\mathcal{U}')$ be the weak factorisation system induced by the set Σ'. The following holds.*

(1) A map g in A-Mod belongs to $\mathcal{U}'$ if and only if g is an epimorphism and $ker(g)$ lies in $\mathcal{D}$.

(2) If $\mathcal{D}$ contains all injective A-modules, then a map f belongs to $\mathcal{L}'$ if and only if f is a monomorphism and $coker(f)$ is in ${}^{\perp_1}\mathcal{D}$.

Proof. ad(1): First, suppose that the map $g : X \to Y$ belongs to $\mathcal{U}'$. Let $\pi : A^{(I)} \to Y$ be an epimorphism for some set I. Since, by construction, the map $0 \to A$ belongs to $\mathcal{L}'$, so does the map $0 \to A^{(I)}$. Consequently, there is a map $g_1 : A^{(I)} \to X$ making the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ A^{(I)} \arrow[r, "\pi"] \arrow[ur, "g_1"] & Y. \end{tikzcd}$$

Thus, also the map g must be an epimorphism. Moreover, since $\mathcal{U}'$ is closed under pullbacks (dually to the fact that the left hand side of a weak factorisation system is always closed under pushouts), also the map $ker(g) \to 0$ lies in $\mathcal{U}'$. Consequently, $ker(g)$ belongs to $\mathcal{D}$ (compare with the proof of Theorem 7.2.7).

Conversely, let $g : X \to Y$ be an epimorphism in A-Mod such that $ker(g)$ is in $\mathcal{D}$. By $\sigma' : P \to Q$ we denote a test map from Σ'. We have to show that $\sigma' \Box g$ holds. Consider the commutative square

$$\begin{tikzcd} P \arrow[r, "h_1"] \arrow[d, "\sigma'"'] & X \arrow[d, "g"] \\ Q \arrow[r, "h_2"] & Y. \end{tikzcd} \tag{7.2.1}$$

We have to construct a map $Q \to X$ making the new diagram commutative. Since g is an epimorphism and Q is projective, there is a map $g_1 : Q \to X$ such that $h_2 = g \circ g_1$. Hence, the map $h_1 - g_1 \circ \sigma'$ must factor through the kernel of g yielding the commutative diagram

$$\begin{tikzcd} & ker(g) \arrow[d, "i"] \\ P \arrow[ur, "s"] \arrow[r, "h_1 - g_1 \circ \sigma'"'] & X. \end{tikzcd}$$

Moreover, since $ker(g)$ belongs to $\mathcal{D}$, there exists a map $t : Q \to ker(g)$ such that $s = t \circ \sigma'$. Consequently, we get $h_1 - g_1 \circ \sigma' = i \circ t \circ \sigma'$ or, equivalently, $h_1 = (i \circ t + g_1) \circ \sigma'$. We claim that the map $i \circ t + g_1$ yields the wanted factorisation in diagram (7.2.1). It remains to check that $h_2 = g \circ (i \circ t + g_1)$. But this follows from the fact that $g \circ i \circ t$ must be zero, by construction.

ad(2): If $\mathcal{D}$ contains all injective A-modules, by Lemma 7.1.2, we know that Σ consists of monomorphisms. Thus, the torsion class $\mathcal{D}$ is given by $(Coker(\Sigma))^{\perp_1}$ and it is tilting (see Section 7.1). Let $(\mathcal{A}, \mathcal{D})$ be the associated complete cotorsion pair. By [Sto, Theorem 5.13], we can associate to $(\mathcal{A}, \mathcal{D})$ the weak factorisation system $(Mon(\mathcal{A}), Epi(\mathcal{D}))$, where $Mon(\mathcal{A})$ describes the class of all monomorphisms in A-Mod with cokernels in $\mathcal{A}$ and $Epi(\mathcal{D})$ is given by the class of all epimorphisms in A-Mod with kernels in $\mathcal{D}$. Now the claim follows by (1), keeping in mind that the two classes of morphisms in a weak factorisation system determine each other. $\Box$

We have the following immediate corollary.

Corollary 7.2.9. *Let Σ be a set of injective maps in A-$proj$. There are monomorphic left $\mathcal{D}$-approximations $f : X \to D$ for every A-module X such that $coker(f)$ lies in ${}^{\perp_1}\mathcal{D} \cap \mathcal{D} \cap \mathcal{T}$.*

Now we can use Theorem 7.2.7 to construct the $\mathcal{X}_\Sigma$-reflections for a given universal localisation A_Σ of A. The approach is motivated by Proposition 6.1.3. For an A-module X we first take a left $\mathcal{D}$-approximation $f : X \to D$, given by the weak factorisation system $(\mathcal{L}, \mathcal{U})$ associated to Σ, and factor out the $\mathcal{T}$-torsion part of D. Let ψ_X be the map defined by the composition

$$X \xrightarrow{f} D \longrightarrow D/\tau_{\mathcal{T}}(D).$$

Corollary 7.2.10. *The $\mathcal{X}_\Sigma$-reflection of the A-module X is given by ψ_X.*

Proof. Clearly, the module $D/\tau_{\mathcal{T}}(D)$ lies in $\mathcal{X}_\Sigma = \mathcal{D} \cap \mathcal{F}$. Moreover, the map ψ_X – as a composition of a left $\mathcal{D}$-approximation and a left $\mathcal{F}$-approximation – is a left $\mathcal{X}_\Sigma$-approximation. It remains to check that ψ_X induces an isomorphism (not only a surjective map) for all X' in $\mathcal{X}_\Sigma$ between

$$Hom_A(D/\tau_{\mathcal{T}}(D), X') \xrightarrow{(\psi_X)_*} Hom_A(X, X').$$

Note that the kernel of $(\psi_X)_*$ is given by $Hom_A(coker(\psi_X), X')$. Moreover, the cokernel of ψ_X is a quotient of the cokernel of f and, thus, it belongs to $\mathcal{T}$, by Theorem 7.2.7. On the other hand, we know that X' lies in $\mathcal{F}$ and, hence, $Hom_A(coker(\psi_X), X')$ must be zero. $\square$

In some cases the process of constructing these $\mathcal{X}_\Sigma$-reflections can be made more explicit.

Example 7.2.11. *Let σ be an injective map in A-proj with $Hom_{D(A)}(\sigma, \sigma[1]) = 0$. In other words, we assume that the cokernel T_1 of σ is a finitely presented partial tilting A-module. We set $\Sigma = \{\sigma\}$. The associated torsion pairs are given by*

$$(\mathcal{D}, \mathcal{R}) = (Gen(T) = T_1^{\perp_1}, T^\circ)$$

$$(\mathcal{T}, \mathcal{F}) = (Gen(T_1), T_1^\circ),$$

where T denotes the Bongartz completion of T_1. It is well-known that for every A-module X there is a special left $\mathcal{D}$-approximation $f : X \to D_X$ – the Bongartz pre-envelope of X – where the map f is injective and its cokernel is given by a direct sum of copies of T_1 (see [CTT]). By construction, f is a relative Σ-cell complex that is given by an only one step pushout construction starting from a direct sum of copies of σ. Note that the tilting A-module T is defined to be $T_1 \oplus D_A$. To get the $\mathcal{X}_\Sigma$-reflection of X with respect to the universal localisation A_Σ of A it now suffices to take D_X and factor out the trace of T_1 in D_X. In particular, it follows from the construction that the ring A_Σ is finitely generated when seen as an A-module.

We can weaken the assumptions of the previous example in the following way.

Example 7.2.12. *Let σ be a not necessarily injective map in A-$proj$ with $Hom_{D(A)}(\sigma, \sigma[1]) = 0$. The cokernel of σ is then a partial silting A-module (compare Remark 5.1.11(2)). Again, we set $\Sigma = \{\sigma\}$. The associated torsion pairs are given by*

$$(\mathcal{D}, \mathcal{R}) = (Gen(T), T^{\circ})$$

$$(\mathcal{T}, \mathcal{F}) = (Gen(T_1), T_1^{\circ}),$$

where T denotes the completion of T_1 to a silting A-module (compare setup in Chapter 6). Similarly to the tilting situation, we can construct $\mathcal{D}$-approximations in A-Mod (not necessarily injective) that are relative Σ-cell complexes – also obtained by a one step pushout construction – and whose cokernels are given by a direct sum of copies of T_1 (see the proof of Theorem 5.1.15 in Chapter 5). Again, we get the $\mathcal{X}_\Sigma$-reflections for the universal localisation A_Σ by factoring out the module trace of T_1 (see Proposition 6.1.3) such that, in particular, the ring A_Σ will be finitely generated when seen as an A-module.

7.3 The morphism category and big generating maps

Recall that different sets of maps between finitely generated projective A-modules can yield the same universal localisation of A. In this section, we ask for a canonical candidate among these different sets. Note that one can always consider the saturated set of all maps between finitely generated projective A-modules which become invertible under the localisation. But, in general, this saturation is hard to deal with (compare, for example, [Sch3]). Instead, we will ask for a particularly small set of generating maps. To start with, we consider the following exact categories

$$\mathcal{L} := \{f : P \to Q | f \in A\text{-}proj\}$$

$$\mathcal{BL} := \{f : P \to Q | f \in A\text{-}Proj\}.$$

The exact structure is induced from the abelian category $\mathcal{C}^b(A)$ – the category of bounded cochain complexes of A-modules.

Lemma 7.3.1. *The exact categories $\mathcal{L}$ and $\mathcal{BL}$ are hereditary with enough projectives and injectives. Moreover, for all objects Z_1 and Z_2 in $\mathcal{BL}$, we have*

$$Ext^1_{\mathcal{BL}}(Z_1, Z_2) \cong Hom_{D(A)}(Z_1, Z_2[1]).$$

Proof. We prove the claim for $\mathcal{BL}$ (the statement for $\mathcal{L}$ follows analogously). Note that for P in A-$Proj$ the objects $id : P \to P$ and $0 \to P$ are projective in $\mathcal{BL}$. Dually, the objects $id : P \to P$ and $P \to 0$ are injective. Take some Z_1 in $\mathcal{BL}$ given by the map $f : P_1 \to Q_1$. A projective resolution

$$0 \longrightarrow P^1(Z_1) \xrightarrow{\phi} P^0(Z_1) \longrightarrow Z_1 \longrightarrow 0$$

of Z_1 in $\mathcal{BL}$ is given by the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "id"] \arrow[d, "\oplus"] & P_1 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\binom{-f}{id}"] & P_0 \oplus P_1 \arrow[r, "(id\ f)"] & P_0 \arrow[r] & 0
\end{tikzcd}$$

Note that an injective coresolution of Z_1 in $\mathcal{BL}$ can be constructed dually. Now we take another object Z_2 in $\mathcal{BL}$ given by the map $g : P_2 \to Q_2$. It follows that the assignment

$$\psi : Hom_{D(A)}(Z_1, Z_2[1]) \longrightarrow Hom_{C^b(A)}(P^1(Z_1), Z_2)/Im(\phi^*) = Ext^1_{\mathcal{BL}}(Z_1, Z_2)$$

that maps $h \in Hom_{D(A)}(Z_1, Z_2[1])$ given by the diagram

$$\begin{tikzcd}
 & P_1 \arrow[r] \arrow[d, "h"] & Q_1 \\
P_2 \arrow[r] & Q_2 &
\end{tikzcd}$$

to $\psi(h)$, the coset of the cochain map

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_1 \arrow[d, "h"] \\
P_2 \arrow[r] & Q_2,
\end{tikzcd}$$

yields an isomorphism, where ϕ^* is the map $Hom_{C^b(A)}(P^0(Z_1), Z_2) \to Hom_{C^b(A)}(P^1(Z_1), Z_2)$ given by precomposing with ϕ. Indeed, h is homotopic to zero if and only if $\psi(h)$ factors through ϕ. $\square$

It turns out that the exact category $\mathcal{BL}$ is of Grothendieck type, i.e., it fulfils similar structural properties like Grothendieck categories in the abelian setting (see [SS] and [Sto, Chapter 3], also for a more general discussion of exact categories and the related concepts we need). Now let Σ be a subset of objects of $\mathcal{L}$ and define

$$\mathcal{E}_\Sigma := \text{ the closure of } \Sigma \text{ under finite extensions and direct summands in } \mathcal{L};$$

$$\mathcal{BE}_\Sigma := \text{ the closure of } \Sigma \text{ under transfinite extensions and direct summands in } \mathcal{BL}.$$

Recall that a **transfinite extension** of objects from a class I (or an I-**filtration**) in an exact category $\mathcal{C}$ is given by a λ-sequence – for some ordinal λ – of the form

$$0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\alpha \longrightarrow X_{\alpha+1} \longrightarrow \cdots$$

such that for every $\alpha + 1 < \lambda$ the map $X_\alpha \to X_{\alpha+1}$ is an admissible monomorphism whose cokernel lies in I. The closure of I in $\mathcal{C}$ under transfinite extensions contains then precisely the objects X for which $0 \to X$ is the composition of such a λ-sequence (see Definition 7.2.4).

By construction, $\mathcal{E}_\Sigma$ and $\mathcal{BE}_\Sigma$ are again exact categories with the induced structure coming from $\mathcal{L}$ and $\mathcal{BL}$, respectively. Since, by construction and [Sto, Proposition 3.19], the class $\mathcal{BE}_\Sigma$ is deconstructible (it consists precisely of the direct summands of the Σ-filtered objects in $\mathcal{BL}$), we get, by [Sto, Theorem 3.16], that $\mathcal{BE}_\Sigma$ is an exact category of Grothendieck type. In particular, $\mathcal{BE}_\Sigma$ is closed under taking coproducts. Furthermore, we define

$$\Sigma^{\perp_1} := \{X \in \mathcal{BL} \mid Ext^1_{\mathcal{BL}}(\sigma, X) = 0 \text{ for all } \sigma \in \Sigma\}.$$

Using [SS, Theorem 2.13(4)], for all X in $\mathcal{BL}$, there is an exact $\Sigma^{\perp_1}$-approximation sequence

$$0 \longrightarrow X \longrightarrow \nabla_X \longrightarrow C_X \longrightarrow 0,$$

where ∇_X belongs to $\Sigma^{\perp_1}$ and C_X is Σ-filtered. Note that, if X belongs to $\mathcal{BE}_\Sigma$, then so does ∇_X. We set $\nabla := \bigoplus_{\sigma \in \Sigma} \nabla_\sigma$ with $\nabla \in \mathcal{BE}_\Sigma$. The following lemma verifies that $\mathcal{BE}_\Sigma$ has enough injective objects which, in fact, can be described explicitly. By $\Sigma^{\perp_1}_{\mathcal{E}}$ we denote $\Sigma^{\perp_1} \cap \mathcal{BE}_\Sigma$.

Lemma 7.3.2. *We have $\mathcal{BE}_\Sigma$-$Inj = \Sigma^{\perp_1}_{\mathcal{E}}$. Moreover, ∇ is an injective cogenerator for $\mathcal{BE}_\Sigma$ with*

$$\Sigma^{\perp_1}_{\mathcal{E}} = Add(\nabla).$$

Proof. Clearly, every injective object in $\mathcal{BE}_\Sigma$ belongs to $\Sigma^{\perp_1}_{\mathcal{E}}$. Conversely, take some object Y in $\Sigma^{\perp_1}_{\mathcal{E}}$. We have to show that $Ext^1_{\mathcal{BL}}(X,Y) = 0$ for all X in $\mathcal{BE}_\Sigma$. Without loss of generality, we can assume that X is Σ-filtered. Thus, we can write X as a direct limit

$$X = \varinjlim_{\alpha<\lambda} X_\alpha,$$

where $(X_\alpha)_{\alpha<\lambda}$ is a transfinite extension of objects from Σ. But now the claim follows from our assumption on Y, using [SS, Proposition 2.12].

For the second statement, first observe that $\Sigma_{\mathcal{E}}^{\perp_1}$ is closed under taking coproducts, since the maps in Σ are small in $D(A)$. Here we make use of Lemma 7.3.1. Consequently, we get

$$Add(\nabla) \subseteq \Sigma_{\mathcal{E}}^{\perp_1}.$$

It remains to prove that ∇ cogenerates $\mathcal{BE}_\Sigma$, i.e., we have to construct an admissible monomorphism $X \to \nabla_X$ for all X in $\mathcal{BE}_\Sigma$ where ∇_X belongs to $Add(\nabla)$. Here we are looking at coproducts of ∇, since, in general, exact categories of Grothendieck type are not closed under taking products (see [Sto, Remark 5.10]). Again, without loss of generality, we can assume that the object X is Σ-filtered of the form

$$X = \varinjlim_{\alpha<\lambda} X_\alpha,$$

where $(X_\alpha)_{\alpha<\lambda}$ is a transfinite extension of objects from Σ. First, suppose the claim holds for some X_α. Then we can consider the following commutative diagram given by the Horseshoe Lemma (see, for example, [Bü])

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_\alpha \arrow[r] \arrow[d] & X_{\alpha+1} \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nabla_{X_\alpha} \arrow[r] & \nabla_{X_\alpha} \oplus \nabla_C \arrow[r] & \nabla_C \arrow[r] & 0
\end{tikzcd}$$

where C belongs to Σ and $\nabla_C, \nabla_{X_\alpha}$ are in $Add(\nabla) \subseteq \mathcal{BE}_\Sigma\text{-}Inj$. Thus, we obtain an admissible monomorphism $X_{\alpha+1} \to \nabla_{X_\alpha} \oplus \nabla_C$, showing that also $X_{\alpha+1}$ is cogenerated by ∇. Finally, we have to verify that the property of being cogenerated by ∇ is preserved under taking (special) direct limits in $\mathcal{BE}_\Sigma$. Let μ be some ordinal and $(X_\alpha)_{\alpha<\mu}$ be a transfinite extension of objects from Σ where all the X_α are cogenerated by ∇. By assumption on the X_α, there are admissible monomorphisms $X_\alpha \to \nabla_{X_\alpha}$ with ∇_{X_α} in $Add(\nabla)$ yielding a direct system $(\nabla_{X_\alpha})_{\alpha<\mu}$ where the induced maps $\nabla_{X_\alpha} \to \nabla_{X_{\alpha+1}}$ for $\alpha+1<\mu$ are split monomorphisms. Consequently, the direct limit ∇^* of this induced system is given by a coproduct that belongs to $Add(\nabla)$. Now the claim follows by observing that the limit morphism

$$X \to \nabla^*$$

can be written as a transfinite composition of admissible monomorphism in $\mathcal{BE}_\Sigma$ (compare the proof of Lemma 1.4 in [SS]). Thus, it is again an admissible monomorphism, since $\mathcal{BE}_\Sigma$ is of Grothendieck type. $\square$

Now we are able to state the main result of this section.

Theorem 7.3.3. *The full subcategory $\mathcal{X}_\Sigma$ of A-Mod is given by*

$$\{X \in A\text{-}Mod | Hom_A(\nabla, X) \text{ is an isomorphism}\}.$$

In particular, if ∇ belongs to $\mathcal{E}_\Sigma$, then the universal localisations A_Σ and $A_{\{\nabla\}}$ coincide.

Proof. First, take X in $\mathcal{X}_\Sigma$, i.e., we know that $Hom_A(\sigma, X)$ is an isomorphism for all σ in Σ. We have to show that also $Hom_A(\nabla, X)$ is an isomorphism. We may assume that ∇ is Σ-filtered, so

$$\nabla = \varinjlim_{\alpha<\lambda} X_\alpha,$$

where $(X_\alpha)_{\alpha<\lambda}$ is a transfinite extension of objects from Σ. Using the fact that

$$Hom_A(\varinjlim X_\alpha, X) \cong \varprojlim Hom_A(X_\alpha, X)$$

and the fact that inverse limits preserve isomorphisms, it suffices to check that $Hom_A(X_\alpha, X)$ is an isomorphism for all $\alpha < \lambda$. But this follows directly by a transfinite induction on α.

Conversely, assume that $Hom_A(\nabla, X)$ and, thus, also $Hom_A(\nabla^{(I)}, X) \cong \prod_{i\in I} Hom_A(\nabla, X)$ is an isomorphism for every set I. Let σ be in Σ. We have to show that $Hom_A(\sigma, X)$ is an isomorphism. By Lemma 7.3.2, there is some set I yielding the exact $Add(\nabla)$-approximation sequence in $\mathcal{BE}_\Sigma$

$$0 \longrightarrow \sigma \longrightarrow \nabla_\sigma \longrightarrow \nabla^{(I)}.$$

Note that the involved degreewise exact sequences of A-modules are split. By applying the contravariant functor $Hom_A(-, X)$, we obtain the following exact sequence in $\mathcal{C}^b(Ab)$

$$Hom_A(\nabla^{(I)}, X) \longrightarrow Hom_A(\nabla_\sigma, X) \longrightarrow Hom_A(\sigma, X) \longrightarrow 0.$$

Now the claim follows by the Five-Lemma. $\square$

Remark 7.3.4. *Note that Theorem 7.3.3 remains true, if we replace ∇ by an arbitrary injective cogenerator of $\mathcal{BE}_\Sigma$. Moreover, the arguments used in the proof of Theorem 7.3.3 actually show that for all X in $\mathcal{X}_\Sigma$ and ω in $\mathcal{BE}_\Sigma$ the map $Hom_A(\omega, X)$ is an isomorphism.*

Moreover, it should be pointed out that, in general, the exact category $\mathcal{E}_\Sigma$ will not admit an injective cogenerator and, thus, ∇ is not always an object of $\mathcal{E}_\Sigma$. But in case it is, the situation becomes significantly nicer. First of all, instead of localising at the given set Σ, it is enough to consider just the single morphism ∇. Furthermore, since ∇ is injective in $\mathcal{E}_\Sigma$, by Lemma 7.3.1, we know that $Hom_{D(A)}(\nabla, \nabla[1]) = 0$. Consequently, we are in the setting of Example 7.2.12 that allows us to construct explicitly the reflections of the localisation and that assures that A_Σ is finitely generated when seen as an A-module. We further explore this situation in the next section.

7.4 An application for Artin algebras

We start this section with a general lemma that relates objects in $\Sigma^{\perp_1}$ to divisible A-modules, i.e., objects of the torsion class $\mathcal{D}$. We first fix a set Σ of maps in A-$proj$.

Lemma 7.4.1. *Let f be in $\mathcal{BL}$. Then f lies in $\Sigma^{\perp_1}$ if and only if $coker(f)$ is in $\mathcal{D}$.*

Proof. The idea of the proof is based on [AIR, Lemma 3.4]. Note that we do not require A to be a finite dimensional algebra over a field. The arguments needed are of pure homological nature. For the sake of completeness we sketch the idea. First, assume that f belongs to $\Sigma^{\perp_1}$. By Lemma 7.3.1, we have $Hom_{D(A)}(\sigma, f[1]) = 0$ for all σ in Σ. Now take a map $\sigma : P \to Q$ in Σ and let $g : P \to coker(f)$ be a morphism of A-modules. We have to show that g factors through σ. But, since P is projective, g lifts to a map in $Hom_{D(A)}(\sigma, f[1])$ that must be zero, by assumption. This yields the wanted factorisation.

Conversely, assume that the cokernel of f belongs to $\mathcal{D}$. Let $\sigma : P \to Q$ be a map in Σ. Then we know that any map $h : P \to coker(f)$ of A-modules factors through σ. This information can now be used to construct homotopy for every map in $Hom_{D(A)}(\sigma, f[1])$. Hence, by Lemma 7.3.1, it follows that f lies in $\Sigma^{\perp_1}$. $\square$

From now on, let A be an Artin algebra. Recall that the category of all finitely generated left A-modules will be denoted by A-mod. By τ (respectively, τ^{-1}) we denote the usual Auslander-Reiten translate (respectively, its inverse). We define the torsion class $\mathcal{D}_0 \subseteq A$-$mod$ to be

$$\mathcal{D}_0 := \mathcal{D} \cap A\text{-}mod.$$

Note that $\mathcal{D}_0$ still depends on the chosen set Σ in A-$proj$. Moreover, in general, $\mathcal{D}_0$ will not be covariantly finite in A-mod (like $\mathcal{D}$ is in A-Mod, by Theorem 7.2.7). The torsionfree class associated to $\mathcal{D}_0$ is given by $\mathcal{R}_0 = \mathcal{R} \cap A\text{-}mod$. Under the following assumption we can recover the torsion pair $(\mathcal{D}, \mathcal{R})$ from $(\mathcal{D}_0, \mathcal{R}_0)$.

Proposition 7.4.2. *If $\mathcal{D}_0$ (respectively, $\mathcal{R}_0$) is functorially finite in A-mod, then the torsion pair $(\mathcal{D}, \mathcal{R})$ is given by the limit closure $(\varinjlim \mathcal{D}_0, \varinjlim \mathcal{R}_0)$ in A-Mod.*

Proof. Note that, by the main theorem in [Sm], $\mathcal{D}_0$ is functorially finite in A-mod if and only if so is $\mathcal{R}_0$. Moreover, if these conditions are fulfilled, there is a finitely generated A-module C cogenerating $\mathcal{R}_0$ with $Hom_A(\tau^{-1}C, C) = 0$ (compare the main theorem and Lemma 0.1 in [Sm]). In fact, C is a cotilting $\bar{A}$-module, where $\bar{A}$ denotes the quotient of A we obtain by factoring out the $\mathcal{D}_0$-torsion part. We shall use this information to show that there is only one way of extending $(\mathcal{D}_0, \mathcal{R}_0)$ to a torsion pair in A-Mod, namely by considering $(\varinjlim \mathcal{D}_0, \varinjlim \mathcal{R}_0)$. First, note that, by [CB2, Lemma 4.4], $(\varinjlim \mathcal{D}_0, \varinjlim \mathcal{R}_0)$ is a torsion pair in A-Mod with the smallest torsion class among the possible extensions of $(\mathcal{D}_0, \mathcal{R}_0)$. On the other hand, every torsion-free class containing $\mathcal{R}_0$ also has to contain $Cogen(C)$. In a first step, we will show that $Cogen(C)$ already is a torsionfree class in A-Mod.

Clearly, $Cogen(C)$ is closed under products and subobjects in A-Mod. Moreover, by assumption on C, $Hom_A(\tau^{-1}C, X)$ vanishes for all X in $Cogen(C)$. Consequently, by the Auslander-Reiten-formula (see [Kr]), it follows that $Ext^1_A(X, C) = 0$ for all X in $Cogen(C)$. Now consider the exact sequence in A-Mod

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{\pi} Z \longrightarrow 0$$

with X and Z in $Cogen(C)$. Let i_X (respectively, i_Z) be an inclusion of X (respectively, Z) into some product of copies of C. Since $Ext^1_A(Z, C)$ vanishes and, hence, also $Ext^1_A(Z, C^I)$ must be zero for all sets I, there is a map h_1 such that $i_X = h_1 \circ i$. By h_2 we denote the composition $i_Z \circ \pi$. It follows that $h := (h_1, h_2)$ yields an injective map from Y into some product of copies of C. Hence, $Cogen(C)$ is also closed under extensions and, thus, a torsionfree class in A-Mod.

It remains to prove that $Cogen(C) = \varinjlim \mathcal{R}_0$ and the non-trivial inclusion is $\varinjlim \mathcal{R}_0 \subseteq Cogen(C)$. Let $X = \varinjlim X_i$ be an object in $\varinjlim \mathcal{R}_0$ with all the X_i in $\mathcal{R}_0$. Since C is a cogenerator of $\mathcal{R}_0$ in A-mod, there are inclusions $X_i \to C^{n_i}$ with $n_i \in \mathbb{N}$ for all i. Using that direct limits are exact in A-Mod, we get an injective morphism $X \to \varinjlim C^{n_i}$. Now it suffices to check that $\varinjlim C^{n_i}$ lies in $Prod(C)$. But this follows by observing that C is a finite length module with $Prod(C) = Add(C)$ and $Add(C)$ is closed under direct limits (see [KS]). $\square$

Now we are able to state the main result of this section.

Theorem 7.4.3. *Let Σ be a set of maps in A-proj. Assume that $\mathcal{D}_0$ (respectively, $\mathcal{R}_0$) is functorially finite in A-mod. Then there is some map ∇ in A-proj with $Hom_{D(A)}(\nabla, \nabla[1]) = 0$ such that the universal localisation A_Σ of A is given by localising at $\{\nabla\}$. Moreover, also the torsion class $\mathcal{D}_{\{\nabla\}} \cap$A-mod is functorially finite in A-mod and A_Σ is again an Artin algebra.*

Proof. First, recall that, by [Sm], $\mathcal{D}_0$ is functorially finite in *A-mod* if and only if so is $\mathcal{R}_0$. From Section 7.3 we know that for every σ in Σ there is a short exact sequence in $\mathcal{BE}_\Sigma$ of the form

$$0 \longrightarrow \sigma \longrightarrow \nabla_\sigma \longrightarrow C_\sigma \longrightarrow 0 \tag{7.4.1}$$

with ∇_σ in $\Sigma^{\perp_1}$. Morally, this approximation sequence translates into a $\mathcal{D}$-approximation sequence when passing to cokernels. We consider the induced morphism

$$f : coker(\sigma) \to coker(\nabla_\sigma).$$

By Lemma 7.4.1, $coker(\nabla_\sigma)$ belongs to $\mathcal{D}$. Additionally, since the short exact sequence in diagram (7.4.1) is a $\Sigma^{\perp_1}$-approximation sequence, f is a left $\mathcal{D}$-approximation. To see this, recall that, by Lemma 7.4.1, every projective presentation of a divisible A-module belongs to $\Sigma^{\perp_1}$. By assumption, there is also a minimal left $\mathcal{D}_0$-approximation

$$f' : coker(\sigma) \to D_\sigma$$

in *A-mod* that, by Proposition 7.4.2, actually turns out to be a left $\mathcal{D}$-approximation. Consequently, we can decompose $coker(\nabla_\sigma)$ into the direct sum $D_\sigma \oplus V$ for some V in *A-Mod* such that f is given by the morphism

$$(f'\ 0) : coker(\sigma) \to D_\sigma \oplus V.$$

This information can now be translated back to $\mathcal{BE}_\Sigma$. More precisely, we can choose projective presentations ∇_{D_σ} of D_σ (not necessarily minimal) and ∇_V of V with $\nabla_\sigma = \nabla_{D_\sigma} \oplus \nabla_V$ such that the short exact approximation sequence

$$0 \longrightarrow \sigma \longrightarrow \nabla_{D_\sigma} \longrightarrow Coker \longrightarrow 0$$

belongs to $\mathcal{E}_\Sigma$. Recall that, by Lemma 7.3.2, $\nabla := \bigoplus_{\sigma\in\Sigma} \nabla_{D_\sigma}$ is an injective cogenerator of $\mathcal{BE}_\Sigma$. Next, we check that ∇ can be chosen in $\mathcal{E}_\Sigma$. Since ∇ is injective in $\mathcal{BE}_\Sigma$, by Lemma 7.3.1,

$$Hom_{D(A)}(\nabla, \nabla[1]) = 0.$$

It follows, by [AI, Corollary 2.28], that there is a bound on the number of non-isomorphic indecomposable direct summands of ∇. Thus, ∇ can be built from a finite sum of objects in $\mathcal{E}_\Sigma$. Finally, Theorem 7.3.3 shows that A_Σ coincides with the universal localisation of A at $\{\nabla\}$.

Moreover, we can construct left $\mathcal{D}_{\{\nabla\}}$-approximations by a one step pushout construction starting from ∇ (see Theorem 5.1.15). In fact, for X in A-*mod*, $\nabla : P \to Q$ and a universal map $s : P^n \to X$ with $n \in \mathbb{N}$ we consider the pushout diagram

$$\begin{array}{ccc} P^n & \xrightarrow{\nabla^n} & Q^n \\ {\scriptstyle s}\downarrow & & \downarrow \\ X & \xrightarrow{t} & D. \end{array}$$

By the same arguments used in Theorem 5.1.15, it follows that t defines a left $\mathcal{D}_{\{\nabla\}}$-approximation. By construction, D is a finitely generated A-module and, thus, $\mathcal{D}_{\{\nabla\}} \cap A$-*mod* is functorially finite. In particular, keeping in mind Corollary 7.2.10, $A_\Sigma = A_{\{\nabla\}}$ is again an Artin algebra. □

We have the following immediate corollary.

Corollary 7.4.4. *Let A be a representation finite Artin algebra. Then every universal localisation of A is of the form $A_{\{\nabla\}}$ for ∇ in A-$proj$ with $Hom_{D(A)}(\nabla, \nabla[1]) = 0$.*

Note that we cannot expect to easily generalise the above theorem to the context of arbitrary rings. The presented proof relies heavily on Proposition 7.4.2, the Krull-Schmidt property needed in [AI, Corollary 2.28] and the existence of projective covers.

Remark 7.4.5.

(1) Note that, in general, the torsion class $\mathcal{D}_0$ in Theorem 7.4.3 is not functorially finite. In fact, every universal localisation $A \to A_\Sigma$ for which A_Σ is no longer an Artin algebra provides such an example.

(2) Theorem 7.4.3 can be made more explicit using the results obtained for Nakayama algebras in Chapter 3. By Theorem 3.4.6, universal localisations over Nakayama algebras correspond bijectively to support τ-tilting modules (and, thus, to 2-term silting complexes). Moreover, by Theorem 3.4.9, every universal localisation is given by localising at a certain (explicitly described) direct summand ∇ of the associated 2-term silting complex. In particular, we have $Hom_{D(A)}(\nabla, \nabla[1]) = 0$.

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