Adiabatic theorems for general linear operators and well-posedness of linear evolution equations

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A mes parents

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And last, but certainly not least, I would like to express my deep gratitude to my parents for their loving care and support over all the years: thank you so much, this thesis is dedicated to you.

I hereby certify that this thesis has been composed by myself, and describes my own work, unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted, and all sources of information have been specifically acknowledged.

Summary

In this thesis, we are concerned with adiabatic theory for general – typically dissipative – linear operators and with the well-posedness of non-autonomous linear evolution equations. Well-posedness theory, at least to some extent, is a necessary preliminary to adiabatic theory.

In the well-posedness part of this thesis, we first consider the case of operators $A(t) : D(A(t)) \subset X \to X$ with time-independent domains $D(A(t)) = D$ in a Banach space X. We show that the quite involved regularity conditions of a well-posedness theorem by Yosida for contraction semigroup generators $A(t) : D \subset X \to X$ are equivalent to the simple condition that $t \mapsto A(t)$ be strongly continuously differentiable, which is known to be sufficient for well-posedness already by a well-known theorem of Kato. We also generalize another, less known, well-posedness theorem of Kato for skew self-adjoint operators $A(t)$ with time-independent domain D to quasicontraction group generators $A(t) : D \subset X \to X$ with time-independent domain D in a uniformly convex space X: for such operators well-posedness already follows if $t \mapsto A(t)$ is only continuous and of bounded variation. And finally, we construct simple examples with group generators $A(t) = A_0 + B(t)$ showing that the assumptions of the above theorems cannot be weakened too much or even dropped.

We then proceed to the case of operators $A(t) : D(A(t)) \subset X \to X$ with generally time-dependent domains $D(A(t))$ in a Banach space X. We prove the well-posedness of non-autonomous linear evolution equations for generators $A(t)$ whose pairwise commutators are complex scalars and, in addition, we establish an explicit representation formula for the evolution. We also prove well-posedness in the more general case where instead of the 1-fold commutators only the p-fold commutators of the operators $A(t)$ are complex scalars. All these results are furnished with rather mild stability and regularity assumptions: indeed, stability in the base space X and strong continuity conditions are sufficient. Applications include Segal field operators and Schrödinger operators for particles in external electric fields. Additionally, we improve a well-posedness result of Kato for group generators $A(t)$ with time-dependent domains by showing that the original norm continuity condition can be relaxed to strong continuity.

In the adiabatic theory part of this thesis, we establish adiabatic theorems with and without spectral gap condition for general operators $A(t) : D(A(t)) \subset X \to X$ with possibly time-dependent domains $D(A(t))$ in a Banach space X. We first prove adiabatic theorems with uniform and non-uniform spectral gap condition – including a slightly extended adiabatic theorem of higher order. In these theorems, the considered spectral subsets $\sigma(t)$ have only to be compact – in particular, they need not consist of eigenvalues, let alone semisimple eigenvalues. We then establish adiabatic theorems without

spectral gap condition for not necessarily (weakly) semisimple eigenvalues. In essence, it is only required there that the considered spectral subsets $\sigma(t) = \{\lambda(t)\}$ consist of eigenvalues $\lambda(t) \in \partial\sigma(A(t))$ and that there exist projections $P(t)$ reducing $A(t)$ such that $A(t)|_{P(t)D(A(t))} - \lambda(t)$ is nilpotent and $A(t)|_{(1-P(t))D(A(t))} - \lambda(t)$ is injective with dense range in $(1 - P(t))X$ for almost every t and such that a certain reduced resolvent estimate is satisfied. We show that spectral operators $A(t)$ that in a punctured neighborhood of $\lambda(t)$ are of scalar type provide a general class of examples for the adiabatic theorems without spectral gap. In all these theorems, the regularity conditions imposed on $t \mapsto A(t)$, $\sigma(t)$, $P(t)$ are fairly mild. With the help of numerous examples, we explore the strength of the presented adiabatic theorems.

We apply our adiabatic theorems for general dissipative operators with time-independent domains to generators of certain neutron transport semigroups describing the transport of neutrons in an infinite slab and to not necessarily dephasing generators of quantum dynamical semigroups describing the evolution of open quantum systems. Also, we apply our general adiabatic theorems for operators with time-dependent domains to obtain – in a very simple way – adiabatic theorems for skew self-adjoint operators $A(t)$ defined by symmetric sesquilinear forms $a(t)$.

And finally, we use the adiabatic theorem for skew self-adjoint operators without spectral gap condition, in a version for several eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$, to study adiabatic switching procedures: we extend the well-known Gell-Mann and Low theorem, which relates the eigenstates of a perturbed system to the ones of the unperturbed system, to the case of eigenstates belonging to non-isolated eigenvalues.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit Adiabatentheorie für allgemeine – typischerweise dissipative – lineare Abbildungen und mit der Wohlgestelltheit nichtautonomer linearer Evolutionsgleichungen. Wohlgestelltheitstheorie ist, zumindest zu einem gewissen Grad, eine notwendige Vorbereitung für die Adiabatentheorie.

Im Wohlgestelltheitsteil dieser Arbeit betrachten wir zunächst den Fall von linearen Abbildungen $A(t) : D(A(t)) \subset X \to X$ mit zeitunabhängigen domains $D(A(t)) = D$ in einem Banachraum X. Wir zeigen, dass die einigermaßen verwickelten Regularitätsbedingungen eines Wohlgestelltheitssatzes von Yosida für Kontraktionshalbgruppenerzeuger $A(t) : D \subset X \to X$ äquivalent sind zur einfachen Bedingung, dass $t \mapsto A(t)$ stark stetig differenzierbar ist, die bekanntermaßen hinreichend für die Wohlgestelltheit ist schon aufgrund eines wohlbekannten Satzes von Kato. Wir verallgemeinern außerdem einen weniger bekannten Wohlgestelltheitssatz von Kato für schiefselbstadjungierte Operatoren $A(t)$ mit zeitunabhängigem domain D auf Quasikontraktionshalbgruppenerzeuger $A(t) : D \subset X \to X$ in einem gleichmäßig konvexen Raum X: für solche Operatoren folgt Wohlgestelltheit schon, wenn $t \mapsto A(t)$ nur stetig und von beschränkter Variation ist. Und schließlich konstruieren wir einfache Beispiele mit Gruppenerzeugern $A(t) = A_0 + B(t)$, die zeigen, dass die Voraussetzungen der ogiben Sätze nicht allzu sehr abgeschwächt oder gar weggelassen werden können.

Wir gehen dann über zum Fall von linearen Abbildungen $A(t) : D(A(t)) \subset X \to X$ mit im allgemeinen zeitabhängigen domains $D(A(t))$ in einem Banachraum X. Wir zeigen die Wohlgestelltheit nichtautonomer Evolutionsgleichungen für Erzeuger $A(t)$, deren paarweise Kommutatoren komplexe Skalare sind und darüberhinaus beweisen wir eine explizite Darstellungsformel für die Zeitentwicklung. Wir zeigen Wohlgestelltheit auch in dem allgemeineren Fall, wo statt der 1-fachen Kommutatoren nur die p-fachen Kommutatoren der Operatoren $A(t)$ komplexe Skalare sind. All diese Sätze zeichnen sich durch ziemlich schwache Stabilitäts- und Regularitätsbedingungen aus: Stabilität im Ausgangsraum X und starke Stetigkeitsbedingungen genügen. Angewandt werden diese Sätze unter anderem auf Segalfeldoperatoren und Schrödingeroperatoren für Teilchen in einem äußeren elektrischen Feld. Außerdem verbessern wir einen Wohlgestelltheitssatz von Kato für Gruppenerzeuger $A(t)$ mit zeitabhängigen domains, indem wir zeigen, dass die ursprüngliche Norm-Stetigkeitsbedingung abgeschwächt werden kann zu starker Stetigkeit.

Im Adiabatenteil dieser Arbeit beweisen wir Adiabatensätze mit und ohne Spektrallückenbedingung für allgemeine lineare Abbildungen $A(t) : D(A(t)) \subset X \to X$ mit möglicherweise zeitabhängigen domains $D(A(t))$ in einem Banachraum X. Wir zeigen zunächst Adiabatensätze mit gleichmäßiger und nichtgleichmäßiger Spektrallückenbedin-

gung – einschließlich eines leicht verallgemeinerten Adiabatensatzes höherer Ordnung. In diesen Sätzen müssen die betrachteten spektralen Untermengen $\sigma(t)$ nur kompakt sein – insbesondere brauchen sie nicht aus Eigenwerten, geschweige denn halbeinfachen Eigenwerten, zu bestehen. Anschließend beweisen wir Adiabatensätze ohne Spektrallückenbedingung für nicht notwendig (schwach) halbeinfache Eigenwerte. Im wesentlichen wird dort nur verlangt, dass die betrachteten spektralen Untermengen $\sigma(t) = \{\lambda(t)\}$ aus Eigenwerten $\lambda(t) \in \partial\sigma(A(t))$ bestehen und dass $A(t)$ reduzierende Projektionen $P(t)$ existieren so, dass $A(t)|_{P(t)D(A(t))} - \lambda(t)$ nilpotent ist und $A(t)|_{(1-P(t))D(A(t))} - \lambda(t)$ injektiv ist mit dichtem Bild in $(1 - P(t))X$ für fast alle t und so, dass eine gewisse Abschätzung an die reduzierte Resolvente erfüllt ist. Wir zeigen, dass Spektraloperatoren, die in einer punktierten Umgebung von $\lambda(t)$ vom skalaren Typ sind, eine allgemeine Beispielklasse für die Adiabatensätze ohne Spektrallückenbedingung abgeben. In all diesen Sätzen sind die Regularitätsbedingungen an $t \mapsto A(t)$, $\sigma(t)$, $P(t)$ recht schwach. Anhand zahlreicher Beispiele loten wir die Stärke der vorgestellten Adiabatensätze aus.

Wir wenden unsere Adiabatensätze für allgemeine dissipative Operatoren mit zeitunabhängigen domains an auf Erzeuger gewisser Neutronentransporthalbgruppen, die den Neutronentransport in einer unendlich ausgedehnten Platte beschreiben, und auf nicht notwendig dephasierende Erzeuger quantendynamischer Halbgruppen, die die Dynamik offener Quantensysteme beschreiben. Außerdem wenden wir unsere allgemeinen Adiabatensätze für Operatoren mit zeitabhängigen domains an um – in sehr einfacher Weise – Adiabatensätze für schiefselbstadjungierte Operatoren zu erhalten, die über symmetrische Sesquilinearformen $a(t)$ definiert sind.

Schließlich benutzen wir den Adiabatensatz für schiefselbstadjungierte Operatoren ohne Spektrallückenbedingung, in einer Version für mehrere Eigenwerte $\lambda_1(t), \dots, \lambda_r(t)$, um adiabatische Anschaltvorgänge zu untersuchen: und zwar verallgemeinern wir den wohlbekannten Satz von Gell-Mann und Low, der die Eigenzustände eines gestörten Systems in Zusammenhang bringt mit denjenigen des ungestörten Systems, auf den Fall von Eigenzuständen, die zu nichtisolierten Eigenwerten gehören.

Contents

1 Introduction

In this thesis, we will be concerned with adiabatic theory for general – typically dissipative – linear operators and with the well-posedness of non-autonomous linear evolution equations. Well-posedness theory, at least to some extent, is a necessary preliminary to adiabatic theory.

1.1 Well-posedness theory

Well-posedness theory for non-autonomous linear evolution equations is concerned with evolution equations (initial value problems)

$$x' = A(t)x \ (t \in [s,1]) \quad \text{and} \quad x(s) = y \tag{1.1}$$

for densely defined linear operators $A(t) : D(A(t)) \subset X \to X$ $(t \in [0,1])$ in a Banach space X and initial values $y \in D(A(s))$ at initial times $s \in [0,1)$. Well-posedness of such evolution equations means something like unique (classical) solvability with continuous dependence of the initial data. When describing the time evolution of physical systems by means of (1.1), the well-posedness of (1.1) is of fundamental importance: for it guarantees that the uniquely existing solutions to (1.1) do not depend critically on the inaccuracies concomitant with the measurement of the initial state y and the initial time s.

In mathematically precise terms, *well-posedness* of (1.1) on the spaces $D(A(t))$ means the unique existence of a *solving evolution system for A on (the spaces) $D(A(t))$* or, for short, an *evolution system for A on $D(A(t))$*. Such an evolution system for A on $D(A(t))$ is defined to be a family U of bounded operators $U(t,s)$ in X for $(s,t) \in \Delta := \{(s,t) \in [0,1]^2 : s \leq t\}$ such that, for every $s \in [0,1)$ and $y \in D(A(s))$, the map

$$[s,1] \ni t \mapsto U(t,s)y \tag{1.2}$$

is a continuously differentiable solution to the initial value problem (1.1), and such that $U(t,s)U(s,r) = U(t,r)$ for all $(r,s),(s,t) \in \Delta$ and $\Delta \ni (s,t) \mapsto U(t,s)x$ is continuous for all $x \in X$.

A lot of work has been devoted to finding sufficient conditions for the well-posedness of evolution equations such as (1.1) and we will discuss some important milestones later on, after the necessary – relatively technical – terminology has been provided. In this introductory chapter we confine ourselves to recalling two of the very first general well-posedness theorems, which are both contained in Kato's seminal paper [62] from 1953. In the first theorem, general contraction semigroup generators $A(t) : D \subset X \to X$ with time-independent domain D in a Banach space X are considered and the well-posedness of (1.1) on D is established under the assumption that $t \mapsto A(t)y$ be continuously differentiable for every $y \in D$. In the second – less well-known – theorem, skew self-adjoint operators $A(t) : D \subset H \to H$, that is, operators of the form i times a self-adjoint operator, with time-independent domain D in a Hilbert space H are considered. It is shown that in this special case, the well-posedness of (1.1) on D already follows if $t \mapsto A(t) \in L(D, H)$ is only continuous and of bounded variation, where D is endowed with the graph norm of $A(0)$.

1.2 Adiabatic theory: setting and basic question

Adiabatic theory – in the form used and developed in this thesis – has its roots in quantum mechanics. It is concerned with slowly time-dependent systems described by evolution equations

$$x' = A(\varepsilon s)x \quad (s \in [0, 1/\varepsilon]) \quad \text{and} \quad x(0) = y \tag{1.3}$$

with linear operators $A(t) : D(A(t)) \subset X \to X$ for $t \in [0, 1]$ and some (small) slowness parameter $\varepsilon \in (0, \infty)$. Smaller and smaller values of ε mean that $A(\varepsilon s)$ depends more and more slowly on time s or, in other words, that the typical time where $A(\varepsilon\,.\,)$ varies appreciably gets larger and larger. What adiabatic theory is interested in is how certain distinguished solutions to (1.3) behave in the singular limit where the slowness parameter ε tends to 0. In this context, it is often convenient to rescale time and consider the rescaled equivalent of (1.3), namely

$$x' = \frac{1}{\varepsilon}A(t)x \quad (t \in [0, 1]) \quad \text{and} \quad x(0) = y. \tag{1.4}$$

As might be known to the reader, there are adiabatic theorems also in classical mechanics which, however, do not fit into the quantum-mechanically motivated – linear operator – framework just described. See [83], for instance. In this thesis, we will not enter the classical mechanics branch of adiabatic theory. We will also not go into the so-called space-adiabatic theory here and we refer to [132] which is the standard reference in this context. In contradistinction to space-adiabatic theory, adiabatic theory in the framework above is sometimes called time-adiabatic theory.

We now proceed to describe adiabatic theory in more specific terms and, in particular, formulate the setting and basic question of adiabatic theory in a mathematically precise manner. We do this first in the simpler and traditional case where the operators $A(t)$ are skew self-adjoint (typically, i times a Schrödinger operator) and then in the generally more complicated case where the $A(t)$ are general operators. In applications, the latter will typically be contraction semigroup generators or, in other words, densely defined dissipative operators having dense range (after translation).

1.2.1 Adiabatic theory for skew self-adjoint operators

Adiabatic theory for skew self-adjoint operators dates back to the early days of quantum mechanics. In rigorous form, it emerged in the paper [16], which sparked an extensive research activity first in physics and then – with some time lag – also in mathematics.

A typical application of adiabatic theory for skew self-adjoint operators is to switching procedures, where external perturbations (for example, an electric or magnetic field) are switched on infinitely slowly. Such switching procedures are desribed by operators

$$A(t) = A_0 + \kappa(t)V \tag{1.5}$$

with a smooth switching function $\kappa : [0,1] \to [0,1]$ satisfying $\kappa(0) = 0$ (perturbation completely switched off at time $t = 0$) and $\kappa(1) = 1$ (perturbation completely switched on at time $t = 1$). Considering the singular limit $\varepsilon \searrow 0$ in (1.3) corresponds to switching on the perturbation infinitely slowly. Another typical application of adiabatic theory is to (approximate) molecular dynamics, but we will not go into this subject here. See, for instance, [88] for the time-adiabatic approach going back to Born and Oppenheimer and [132] for the space-adiabatic approach to molecular dynamics.

In a nutshell, the setting and basic question of adiabatic theory for skew self-adjoint operators can be described as follows: one assumes that

- $A(t) : D(A(t)) \subset H \to H$ is skew self-adjoint in a Hilbert space H over $\mathbb{C}$ for every $t \in [0,1]$ and the initial value problems

$$x' = \frac{1}{\varepsilon} A(t)x \quad (t \in [t_0, 1]) \quad \text{and} \quad x(t_0) = y \tag{1.6}$$

  with initial values $y \in D(A(t_0))$ at initial times $t_0 \in [0,1)$ are well-posed on the spaces $D(A(t))$ for every value of the slowness parameter $\varepsilon \in (0,\infty)$,

- $\lambda(t)$ is an eigenvalue of $A(t)$ for every $t \in [0,1]$.

In this setting, one then wants to know the following: when – under which additional conditions on A and λ – does the evolution U_ε generated by (1.6) approximately follow the eigenspaces of $A(t)$ for $\lambda(t)$ as the slowness parameter ε tends to 0? With the help of the spectral measure $P^{A(t)}$ of $A(t)$ and the spectral projection

$$P(t) = P^{A(t)}(\{\lambda(t)\}) \tag{1.7}$$

of $A(t)$ onto $\{\lambda(t)\}$, this basic question of adiabatic theory can be formulated more precisely and concisely as follows: under which conditions is it true that

$$(1 - P(t))U_\varepsilon(t,0)P(0) \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{1.8}$$

with respect to a certain operator topology for all $t \in [0,1]$? According to the probabilistic interpretation of quantum mechanics, if $y = P(0)y$ is a (normed) eigenstate of $A(0)$ with corresponding eigenvalue $\lambda(0)$, then the quantity $\|(1 - P(t))U_\varepsilon(t,0)P(0)y\|^2$ is the probability for a transition from $P(0)H = P^{A(0)}(\{\lambda(0)\})H = \ker(A(0) - \lambda(0))$ to $(1 - P(t))H = P^{A(t)}(\sigma(A(t)) \setminus \{\lambda(t)\})H = \overline{\operatorname{ran}}(A(t) - \lambda(t))$ under the effect of the evolution

$U_\varepsilon(t,0)$. A bit more precisely, if $y = P(0)y$ is as before and $x_\varepsilon = U_\varepsilon(\,.\,,0)y$ is the solution to (1.6) with $t_0 = 0$, then

$$\|(1-P(t))U_\varepsilon(t,0)P(0)y\|^2 = P^{A(t)}_{x_\varepsilon(t),x_\varepsilon(t)}(\sigma(A(t)) \setminus \{\lambda(t)\}) \tag{1.9}$$

is the probability of not obtaining the value $\lambda(t)$ upon measuring $A(t)$ in the evolved state $x_\varepsilon(t)$. (In this probabilistic context, it is to be noted that $\|x_\varepsilon(t)\| = \|x_\varepsilon(0)\| = 1$ for all t by the skew symmetry of the operators $A(t)$.) So, strong convergence in (1.8) means precisely that the probability (1.9) of transitions vanishes in the limit $\varepsilon \searrow 0$.

Sometimes, it is desirable to know that suppression of transitions in the limit $\varepsilon \searrow 0$ occurs also for more general spectral subspaces, namely $P(t)H = P^{A(t)}(\sigma(t))H$ corresponding to a whole (compact) portion $\sigma(t)$ of the discrete or the essential spectrum of $A(t)$ or both. As above, suppression of transitions can be expressed by (1.8), where now

$$P(t) = P^{A(t)}(\sigma(t)). \tag{1.10}$$

1.2.2 Adiabatic theory for general operators

Adiabatic theory for general – as opposed to skew self-adjoint – operators, in a strict sense, originated in [98]. We point out, however, that as auxiliary objects, non-self-adjoint operators in adiabatic theory appear also in the so-called complex time method [55], [57], [58] going back to [76]. In recent years, the rather special results from [98] were considerably extended and developed further in the works [2], [60], and [12]. An important motivation – and source of applications – for these developments is the description of open quantum systems, whose evolution is governed by dissipative operators.

When generalizing the traditional adiabatic theory for skew self-adjoint operators to general (for instance, dissipative) operators a first question to be addressed is how one should replace the spectral projections (1.7) or (1.10) appearing in the very formulation (1.8) of the traditional theory. After all, these spectral projections are defined by means of the spectral measure of the pertinent skew self-adjoint operators and, for general operators, one does not have spectral measures. So, a first necessary preliminary in adiabatic theory for general operators, is to find natural substitutes for spectral projections and we shall call such substitutes *spectrally related projections* in the sequel.

With the help of spectrally related projections, the setting and basic question of general adiabatic theory can then be described as follows: one starts out from linear operators $A(t) : D(A(t)) \subset X \to X$ in a Banach space X over $\mathbb{C}$, compact subsets $\sigma(t)$ of the spectrum of $A(t)$ and projections $P(t)$ in X (for $t \in [0,1]$) such that

- $A(t)$ is densely defined closed for every $t \in [0,1]$ and the initial value problems

$$x' = \frac{1}{\varepsilon}A(t)x \quad (t \in [t_0,1]) \quad \text{and} \quad x(t_0) = y \tag{1.11}$$

  with initial values $y \in D(A(t_0))$ at initial times $t_0 \in [0,1)$ are well-posed on the spaces $D(A(t))$ for every value of the slowness parameter $\varepsilon \in (0,\infty)$,

- $P(t)$ is spectrally related with $A(t)$ and $\sigma(t)$ for every $t \in [0,1]$ except possibly for some few t.

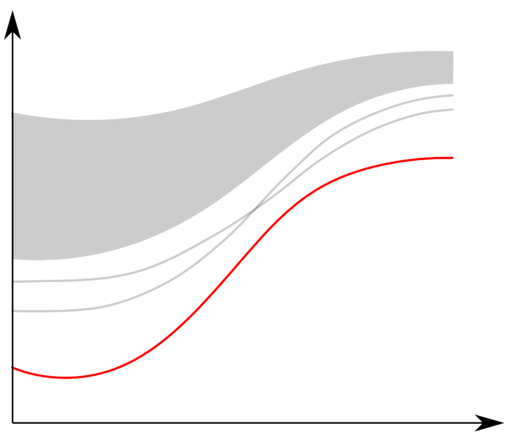
What one then wants to know is the following: when – under which additional conditions on A, σ and P – does the evolution U_ε generated by (1.11) approximately follow the spectral subspaces $P(t)X$ related to the spectral subsets $\sigma(t)$ of $A(t)$ as the slowness parameter ε tends to 0? In other – more precise and concise – terms: under which conditions is it true that

$$(1 - P(t))U_\varepsilon(t,0)P(0) \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{1.12}$$

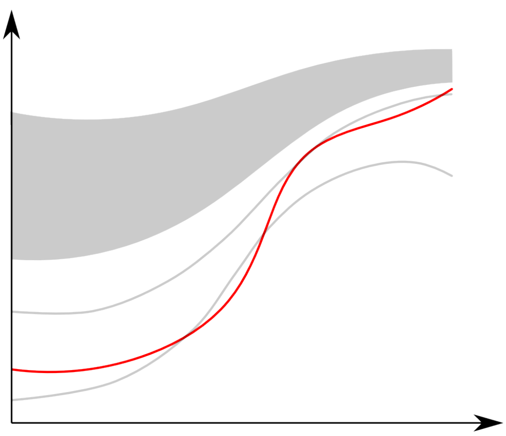
with respect to a certain operator topology for all $t \in [0,1]$? *Adiabatic theorems* are, by definition, theorems that give such conditions.

We will sometimes distinguish *quantitative* and *qualitative* adiabatic theorems depending on whether they yield information on the rate of convergence in (1.12) or not. If the rate of convergence in (1.12) can, for certain t, be shown to be of polynomial order ε^n in the slowness parameter, one speaks of *adiabatic theorems of higher order*. And if the rate of convergence in (1.12) can even be shown to be exponential, that is, of order $e^{-c/\varepsilon}$, one often speaks of *superadiabatic theorems*. An important distinction in adiabatic theory is between adiabatic theorems *with spectral gap condition* and adiabatic theorems *without spectral gap condition*, where one speaks of a *spectral gap* iff $\sigma(t)$ is isolated in the spectrum $\sigma(A(t))$ for every $t \in [0,1]$. It is also convenient to further divide adiabatic theorems with spectral gap condition into those *with uniform spectral gap condition* and those *with non-uniform spectral gap condition*, where a spectral gap is called uniform iff

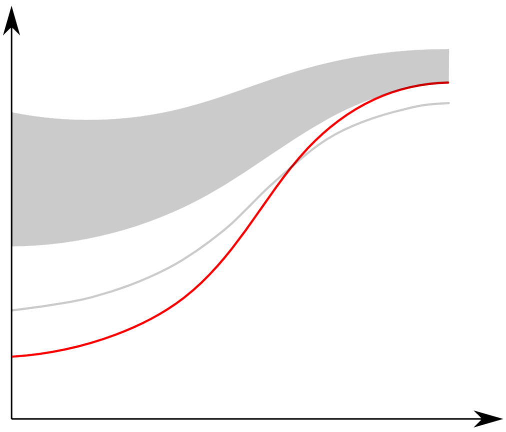
$$\inf_{t\in[0,1]} \operatorname{dist}\big(\sigma(t), \sigma(A(t)) \setminus \sigma(t)\big) > 0. \tag{1.13}$$



(a) A uniform spectral gap.



(b) A non-uniform spectral gap.



(c) Situation without spectral gap.

Schematic illustration of situations with spectral gap (uniform or non-uniform) and without spectral gap in the special case of skew self-adjoint operators $A(t)$ and $\sigma(t) = \{\lambda(t)\}$. In the figures above, the spectrum $\sigma(A(t))$ is plotted on the vertical axis $i\mathbb{R}$ against the horizontal t-axis and the red line represents the considered spectral values $\lambda(t)$.

In the context of spectral gaps, we will also use the convenient terminology of $\sigma(\,.\,)$ falling into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ at a point t_0 by which we mean that there exists a sequence (t_n) in $[0, 1]$ with $t_n \longrightarrow t_0$ as $n \to \infty$ such that

$$\operatorname{dist}\big(\sigma(t_n), \sigma(A(t_n)) \setminus \sigma(t_n)\big) \longrightarrow 0 \quad (n \to \infty) \tag{1.14}$$

With this terminology, the uniform spectral gap condition (1.13) can be equivalently reformulated by saying that there is no point t_0 at which $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$.

1.3 Some fundamental adiabatic theorems from the literature

We now recall those adiabatic theorems from the literature that are most relevant to the adiabatic theorems of this thesis. In doing so, we decidedly concentrate on mathematical aspects. Just as in the rest of this thesis, we will abbreviate $I := [0, 1]$ and if U is an evolution system, we will often write $U(t)$ instead of $U(t, 0)$ for brevity.

1.3.1 Case of skew self-adjoint operators

Adiabatic theory in rigorous form was born in 1928. In their paper [16], Born and Fock proved (1.8) for bounded skew self-adjoint operators $A(t)$ under the assumption that the spectrum of $A(t)$ be purely discrete for every t and that all eigenvalues of $A(t)$ have multiplicity 1 for every t except possibly for finitely many eigenvalue crossing points. In 1950, Kato [61] significantly relaxed the rather restrictive spectral assumptions from [16]: he assumed nothing about those parts of the spectrum away from the eigenvalue $\lambda(t)$

under consideration, but only assumed $\lambda(t)$ to be an isolated eigenvalue of finite multiplicity for every t. (As usual, an isolated eigenvalue is an eigenvalue that is an isolated point of the spectrum and not just of the point spectrum. See Section III.5 of [67], for instance.) In the case where the eigenvalues $\lambda(t)$ are uniformly isolated in the spectrum, he showed that

$$\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\| = O(\varepsilon) \quad (\varepsilon \searrow 0) \tag{1.15}$$

and in the case where the $\lambda(t)$ cross other eigenvalue curves finitely many times, he showed that

$$\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\| = o(1) \quad (\varepsilon \searrow 0). \tag{1.16}$$

Kato's proof of (1.15) proceeds in two steps. In the first step, he constructs an evolution W that *exactly* follows the eigenspaces $P(t)H = \ker(A(t) - \lambda(t))$ and their orthogonal complements $(1 - P(t))H = \overline{\operatorname{ran}}(A(t) - \lambda(t))$ in the precise sense that

$$W(t, t_0)P(t_0) = P(t)W(t, t_0) \tag{1.17}$$

for all t_0, t. An evolution system W satisfying (1.17) is called *adiabatic w.r.t.* P and Kato takes W to be the evolution system for $K = [P', P]$. In the second step, he then shows that this evolution W times a dynamical phase factor well approximates the true evolution U_ε on the subspace $P(0)H$, that is,

$$\sup_{t \in I} \left\| \big(V_\varepsilon(t) - U_\varepsilon(t)\big)P(0) \right\| = O(\varepsilon) \tag{1.18}$$

where $V_\varepsilon(t) = e^{1/\varepsilon \int_0^t \lambda(\tau)\, d\tau}\, W(t, 0)$. In view of (1.17) this implies (1.15).

Some years later, Lenard [81], Garrido [45], and Sancho [110] refined the convergence statement (1.15) in various directions, assuming that the (bounded) operators $A(t)$ depend smoothly on t. A typical corollary of their results is the following adiabatic theorem of higher order: if all the derivatives $A^{(k)}(t)$ at $t = t_0$ and $t = t_1$ vanish up to order $k = n$, then

$$(1 - P(t_1))U_\varepsilon(t_1, t_0)P(t_0) = O(\varepsilon^n) \tag{1.19}$$

instead of only $O(\varepsilon)$. In 1980, Nenciu [96] generalized Kato's adiabatic theorem for eigenvalues $\lambda(t)$ to general uniformly isolated compact subsets $\sigma(t)$ of the spectrum of bounded skew self-adjoint operators $A(t)$. In fact, he considered several such subsets $\sigma_1(t), \dots, \sigma_r(t)$. In 1987, Avron, Seiler, Yaffe [13] extended Nenciu's result, among other things, to unbounded skew self-adjoint operators $A(t) : D \subset H \to H$ with time-independent domain D. Similarly to Kato [61], they proceed in two steps: they construct an evolution V_ε that is adiabatic w.r.t. P in the sense of (1.17), where the $P(t)$ are the spectral projections $P^{A(t)}(\sigma(t))$ corresponding to the considered isolated compact spectral subsets $\sigma(t)$ of $A(t)$. Avron, Seiler, Yaffe take V_ε to be the evolution system for

$\frac{1}{\varepsilon}A + [P', P]$. And then they show that this adiabatic evolution well approximates the true evolution U_ε, namely

$$\sup_{t\in I} \|V_\varepsilon(t) - U_\varepsilon(t)\| = O(\varepsilon) \quad (\varepsilon \searrow 0) \tag{1.20}$$

on the entire space H (instead of only on $P(0)H$ as in [61]). In virtue of the adiabaticity of V_ε w.r.t. P, this then implies

$$\sup_{t\in I} \|(1-P(t))U_\varepsilon(t)P(0)\|, \quad \sup_{t\in I} \|P(t)U_\varepsilon(t)(1-P(0))\| = O(\varepsilon) \quad (\varepsilon \searrow 0). \tag{1.21}$$

Avron, Seiler, Yaffe also establish a higher order estimate of the type (1.19) and apply their results to the quantum Hall effect.

In 1993, Joye and Pfister [59] and Nenciu [99] considerably improved, under analyticity conditions, the higher order results from [81], [45], [110] and from [13] by pushing them to exponential order: for skew self-adjoint operators $A(t) : D \subset H \to H$ depending analytically on t and compact uniformly isolated spectral subsets $\sigma(t)$, they construct projections $P_\varepsilon(t)$ and evolutions V_ε, adiabatic w.r.t. P_ε, such that

$$\sup_{t\in I} \|P_\varepsilon(t) - P(t)\| = O(\varepsilon) \tag{1.22}$$

and such that V_ε approximates the evolution U_ε for $\frac{1}{\varepsilon}A$ exponentially well in ε:

$$\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(e^{-g/\varepsilon}) \quad (\varepsilon \searrow 0) \tag{1.23}$$

for some positive g. In particular, U_ε follows the subspaces $P_\varepsilon(t)X$ and $(1-P_\varepsilon(t))X$ to exponential order:

$$\sup_{t\in I} \|(1-P_\varepsilon(t))U_\varepsilon(t)P_\varepsilon(0)\|, \quad \sup_{t\in I} \|P_\varepsilon(t)U_\varepsilon(t)(1-P_\varepsilon(0))\| = O(e^{-g/\varepsilon}) \tag{1.24}$$

by (1.23) and the adiabaticity of V_ε w.r.t. P_ε. If $A'(t) = 0$ for t close to the initial and final time 0 and 1, respectively, then it further follows by the constructions from [59] and [99] that transitions across the spectral gap are exponentially suppressed:

$$(1-P(1))U_\varepsilon(1,0)P(0) = O(e^{-g/\varepsilon}). \tag{1.25}$$

Joye and Pfister's superadiabatic theorem can be applied in scattering situations to obtain exponential estimates on the transition probabilites across a general spectral gap as in [58] and, in more special situations, to obtain even explicit asymptotic formulas for the transition probabilities by reduction [59] to the Dykhne-type formulas from [57] for 2-level systems. In rough terms, Joye and Pfister's method of proof from [59] can be described as follows: starting from $A_{0\,\varepsilon}(t) := A(t)$ and $P_{0\,\varepsilon}(t) := P(t) = P^{A(t)}(\sigma(t))$ they iteratively construct a sequence $A_{1\,\varepsilon}(t), A_{2\,\varepsilon}(t), \dots$ of skew self-adjoint operators $A_{n\,\varepsilon}(t) := A(t) - \varepsilon[P'_{n-1\,\varepsilon}(t), P_{n-1\,\varepsilon}(t)]$ with pertinent spectral projections

$$P_{n-1\,\varepsilon}(t) := P^{A_{n-1\,\varepsilon}(t)}(\sigma(t)) = \frac{1}{2\pi i}\int_{\gamma_t} (z - A_{n-1\,\varepsilon}(t))^{-1}\, dz.$$

Also, they take $V_{n\,\varepsilon}$ for every $n \in \mathbb{N}$ to be the evolution system for $\frac{1}{\varepsilon}A_{n\,\varepsilon} + [P'_{n\,\varepsilon}, P_{n\,\varepsilon}]$, which is adiabatic w.r.t. $P_{n\,\varepsilon}$ by the same reason as in [13]. With the assumed analyticity condition, they can then show that

$$\sup_{t\in I} \|K_{n\,\varepsilon}(t) - K_{n-1\,\varepsilon}(t)\| \le c^n\, n!\, \varepsilon^n, \tag{1.26}$$

where $K_{k\,\varepsilon} := [P'_{k\,\varepsilon}, P_{k\,\varepsilon}]$. Since $K_{n\,\varepsilon} - K_{n-1\,\varepsilon}$ by construction is nothing but the difference of the generators $\frac{1}{\varepsilon}A_{n\,\varepsilon} + [P'_{n\,\varepsilon}, P_{n\,\varepsilon}]$ and $\frac{1}{\varepsilon}A$, it follows that the difference $V_{n\,\varepsilon} - U_\varepsilon$ of the pertaining unitary evolutions can be estimated by the same bound. Choosing then $n = n^*(\varepsilon)$ in an optimal way, namely $n^*(\varepsilon) \sim 1/\varepsilon$, and setting $P_\varepsilon := P_{n^*(\varepsilon)\,\varepsilon}$ as well as $V_\varepsilon := V_{n^*(\varepsilon)\,\varepsilon}$, the desired estimates (1.22) and (1.23) follow by Stirling's formula.

In 1998, Avron and Elgart [11] established the first (general) adiabatic theorem without spectral gap condition: they proved that if $A(t) : D \subset H \to H$ are skew self-adjoint operators depending sufficiently regularly on t and if $\lambda(t)$ are eigenvalues of $A(t)$ (isolated or not) with finite mulitplicity, then

$$\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0), \tag{1.27}$$

where U_ε and V_ε are the evolutions for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$ and $P(t) := P^{A(t)}(\{\lambda(t)\})$ for all t with $t \mapsto P(t)$ being assumed to be twice continuously differentiable. With the help of Kato's method from [61], they also treat the case where $\lambda(\,.\,)$ at finitely many points $t_1, \dots, t_m$ crosses other eigenvalue curves (under the assumption that $t \mapsto P^{A(t)}(\{\lambda(t)\})$ can be continued sufficiently regularly through the discontinuities $t_1, \dots, t_m$). Avron and Elgart base their proof of (1.27) on the following commutator equation method: they find a solution $B(t) = B_\delta(t)$ of the approximate commutator equation

$$[P'(t), P(t)] \supset B(t)A(t) - A(t)B(t) + C(t) \tag{1.28}$$

with an error $C(t) = C_\delta(t)$ whose size is controlled by the parameter δ. With this approximate commutator equation and partial integration, they rewrite the difference $V_\varepsilon(t) - U_\varepsilon(t)$ as

$$\begin{aligned} V_\varepsilon(t) - U_\varepsilon(t) &= \int_0^t U_\varepsilon(t,\tau)[P'(\tau), P(\tau)]V_\varepsilon(\tau)\, d\tau = \varepsilon\, U_\varepsilon(t,\tau)B_\delta(\tau)V_\varepsilon(\tau)\big|_{\tau=0}^{\tau=t} \\ &- \varepsilon \int_0^t U_\varepsilon(t,\tau)\big(B'_\delta(\tau) + B_\delta(\tau)[P'(\tau), P(\tau)]\big)V_\varepsilon(\tau)\, d\tau + \int_0^t U_\varepsilon(t,\tau)C_\delta(\tau)V_\varepsilon(\tau)\, d\tau \end{aligned} \tag{1.29}$$

and then show that the right-hand side of this equation can be made arbitrarily small as $\varepsilon \searrow 0$. Since the first two summands on the right-hand side of (1.29) are furnished with prefactors ε and since the evolutions U_ε and V_ε are unitary, these two summands are fine: namely, the explosion of the specifically chosen solutions $B_\delta(t)$ to (1.28) as δ tends to 0 can be compensated by choosing $\delta = \delta_\varepsilon$ such that it sufficiently slowly approaches 0. Since on the other hand, the errors $C_\delta(t)$ pertaining to the specific solutions $B_\delta(t)$ can be shown to tend to 0 as $\delta \searrow 0$ by means of the spectral theorem, the third summand on the right-hand side of (1.29) is fine as well.

In the same year and by a completely different method, Bornemann [17] obtained an adiabatic theorem for skew self-adjoint operators $A(t) = iA_{a(t)}$ defined by symmetric sesquilinear forms $a(t)$ with time-independent form domain and for discrete – in particular, isolated – eigenvalues $\lambda(t)$ of finite multiplicity. While the domains of the forms are assumed to be time-independent, the domains of the corresponding operators $A(t)$ may well depend on time in this result. Avron and Elgart's theorem, by contrast, is restricted to situations where the operators $A(t)$ have a common time-independent domain D. In particular, it does not allow applications to Schrödinger operators with general time-dependent Rollnik potentials $V(t)$ as discussed in [17]. Additionally, Bornemann's adiabatic theorem allows for infinitely many eigenvalue crossings: more precisely, the set of points where the considered eigenvalue curve $\lambda(\,.\,)$ crosses other eigenvalues is allowed to be a general null set.

In 2001, Teufel [131] gave a considerably simpler solution to (1.28) than the original one from [11], namely

$$B_\delta(t) = \big(\lambda(t) + \delta - A(t)\big)^{-1} P'(t)P(t) + P(t)P'(t)\big(\lambda(t) + \delta - A(t)\big)^{-1}. \tag{1.30}$$

Additionally, he observed that for the proof of [11] to work it is sufficient tho have $P(t) = P^{A(t)}(\{\lambda(t)\})$ only for almost every t (as long as $t \mapsto P(t)$ is still twice continuously differentiable); this allows for a more direct treatment of eigenvalue crossings without reference to the argument from [61].

1.3.2 Case of general operators

As has already been mentioned above, the first (rigorous) adiabatic theorem for not necessarily skew self-adjoint operators $A(t)$ was proven by Nenciu and Rasche [98] in 1992. In that paper, finite-dimensional (in fact, essentially 2-dimensional) spaces and uniformly isolated semisimple eigenvalues $\lambda(t)$ are considered. Semisimple eigenvalues of closed operators $A : D(A) \subset X \to X$ are defined as the poles of the resolvent $(\,.\, - A)^{-1}$ of order 1. In finite-dimensional spaces X these are precisely those eigenvalues with a trivial Jordan block (eigennilpotent) or, equivalently, those eigenvalues whose algebraic multiplicity equals the geometric multiplicity. Since the result from [98] is a rather special case of a theorem by Joye discussed below, we shall no further comment on it here.

In 2007, Abou Salem [2] considered a more general situation than the one from [98], namely, general contraction semigroup generators $A(t) : D \subset X \to X$ depending sufficiently regularly on t and uniformly isolated simple eigenvalues $\lambda(t)$ (where simplicity of eigenvalues means semisimplicity plus geometric multiplicity 1). In this situation, he proved that

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \quad (\varepsilon \searrow 0), \tag{1.31}$$

where U_ε and V_ε are the evolutions for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$ and where for every t

$$P(t) = \frac{1}{2\pi i} \int_{\gamma_t} (z - A(t))^{-1} \, dz \tag{1.32}$$

is the Riesz projection of $A(t)$ on $\lambda(t)$. Abou Salem applied this result in the context of quantum statistical mechanics to study the quasi-static evolution of non-equilibrium steady states. In essence, his proof of (1.31) rests upon solving the commutator equation

$$[P'(t), P(t)] \supset B(t)A(t) - A(t)B(t), \tag{1.33}$$

that is, (1.28) with vanishing error $C(t) = 0$, and his solution of (1.33) is already indicated in [11]. With (1.33) at hand, he can then perform a partial integration as in (1.29) which by virtue of $C(t) = 0$ yields the assertion because the evolutions U_ε and V_ε are bounded by the contraction semigroup assumption.

In the same year, Joye established a superadiabatic-type theorem for general operators $A(t) : D \subset X \to X$ that analytically depend on t and for uniformly isolated eigenvalues $\lambda(t)$ of finite algebraic multiplicity. In contrast to Abou Salem's result from [2], this theorem no longer requires the operators $A(t)$ to be contraction semigroup generators or the eigenvalues $\lambda(t)$ to be semisimple. Instead, it only assumes $A(t)(1-P(t))$ to generate a contraction semigroup and $\lambda(t)$ to lie in the closed left half-plane $\{\operatorname{Re} z \leq 0\}$, where $P(t)$ is the Riesz projection of $A(t)$ on $\lambda(t)$. Just like the result from [59], Joye's theorem then yields the existence of ε-dependent projections $P_\varepsilon(t)$ and evolutions V_ε, adiabatic w.r.t. P_ε, such that

$$\sup_{t \in I} \|P_\varepsilon(t) - P(t)\| = O(\varepsilon) \tag{1.34}$$

and such that V_ε approximates the evolution U_ε for $\frac{1}{\varepsilon}A$ exponentially well in ε:

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(e^{-g/\varepsilon}) \quad (\varepsilon \searrow 0) \tag{1.35}$$

for some positive g. In particular, U_ε follows the subspaces $P_\varepsilon(t)X$ and $(1 - P_\varepsilon(t))X$ to exponential order:

$$\sup_{t \in I} \|(1 - P_\varepsilon(t))U_\varepsilon(t)P_\varepsilon(0)\|, \quad \sup_{t \in I} \|P_\varepsilon(t)U_\varepsilon(t)(1 - P_\varepsilon(0))\| = O(e^{-g/\varepsilon}). \tag{1.36}$$

And provided that the evolution U_ε is bounded in ε, (1.34) and (1.36) further show that U_ε also follows the original spectral subspaces $P(t)X$ and $(1-P(t))X$ to first order in ε,

$$\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\|, \quad \sup_{t \in I} \|P(t)U_\varepsilon(t)(1 - P(0))\| = O(\varepsilon). \tag{1.37}$$

Since, in the situation of Abou Salem's theorem, the evolution is indeed bounded in ε (by the contraction semigroup assumption), that result – or, more precisely, the version of it with analyticity assumptions and slightly weakened conclusion (1.37) – is seen to be a special case of Joye's theorem above. In Joye's general situation, however, the evolution is generally unbounded in ε and Joye gave a simple finite-dimensional example where the transition probabilities on the left-hand side of (1.37) actually explode as $\varepsilon \searrow 0$. (It is because of this possible failure of adiabaticity that we referred to Joye's theorem only as a superadiabatic-type result above.) What is behind the possible unboundedness of U_ε

in the situation of [60], is that the eigenvalues are allowed to be non-semisimple *and* lie on the imaginary axis. Indeed, if $\lambda(t) \in i\mathbb{R}$ for all t in the situation of [60], then U_ε is bounded iff $\lambda(t)$ is semisimple for all t (as can be seen by an auxiliary result from [60]). Joye's proof essentially rests upon showing that U_ε, albeit generally unbounded in ε, grows only at most subexponentially in the sense that

$$\sup_{(s,t)\in\Delta} \|U_\varepsilon(t,s)\| \le c\, e^{c/\varepsilon^\beta} \quad (\varepsilon \in (0,\varepsilon^*]) \tag{1.38}$$

for some $\beta \in (0,1)$ strictly less than 1. Combining this subexponential growth with the exponential decay resulting from (1.26), which carries over mutatis mutandis from [59], one obtains the desired result (1.35).

In 2011, Avron, Fraas, Graf, Grech [12] and Schmid [112] independently of each other established the first adiabatic theorems for general operators in the case without spectral gap. In essence, their theorems coincide and the assumptions from [112] are essentially the following:

- $A(t) : D \subset X \to X$ for every $t \in I$ is a contraction semigroup generator such that $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$
- $\lambda(t)$ for $t \in I$ is an eigenvalue of $A(t)$ such that $\lambda(t) + \delta e^{i\vartheta_0} \in \rho(A(t))$ for every $\delta \in (0, \delta_0]$ and

$$\left\| \big(\lambda(t) + \delta e^{i\vartheta_0} - A(t)\big)^{-1} \right\| \le \frac{M_0}{\delta} \quad (\delta \in (0,\delta_0]), \tag{1.39}$$

- there exist projections $P(t)$ such that, for almost every $t \in I$,

$$P(t)X = \ker(A(t) - \lambda(t)) \quad \text{and} \quad (1 - P(t))X = \overline{\operatorname{ran}}(A(t) - \lambda(t)) \tag{1.40}$$

  and such that $t \mapsto P(t)$ is twice strongly continuously differentiable.

In [12] the special case $\lambda(t) = 0$ is considered, in which case the resolvent estimate (1.39) is, of course, automatically satisfied with $\vartheta_0 = 0$ (by the contraction semigroup assumption). It is shown in [112] that, under the above assumptions and the additional assumption that the $P(t)$ be of finite rank, one has

$$\sup_{t\in I} \|(1 - P(t))U_\varepsilon(t)P(0)\| \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{1.41}$$

and in [12] it is shown that, without an additional assumption on the rank of the $P(t)$,

$$\sup_{t\in I} \|(1 - P(t))U_\varepsilon(t)P(0)x\| \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{1.42}$$

for all $x \in X$. In essence, the proofs of this result – from [12] and [112] alike – are based on a suitable adaption of Kato's proof from [61] to the case without spectral gap and on showing that

$$\delta\big(\lambda(t) + \delta e^{i\vartheta_0} - A(t)\big)^{-1}(1 - P(t)) \longrightarrow 0 \quad (\delta \searrow 0)$$

in the strong operator topology for almost all t. Clearly, the above adiabatic theorem without spectral gap is a generalization of Avron and Elgart's result from [11] for skew self-adjoint operators $A(t)$ because the spectral projections $P(t) := P^{A(t)}(\{\lambda(t)\})$ appearing in that result obviously satisfy (1.40). Additionally, the above theorem is a generalization of Abou Salem's result from [2] – or, more precisely, the version of it without a bound on the rate of convergence: for if A is any closed operator and λ a semisimple eigenvalue of A, then the resolvent estimate

$$\left\| \left(\lambda + \delta - A\right)^{-1} \right\| \leq \frac{M_0}{\delta}$$

holds true for all complex $\delta \neq 0$ with $|\delta|$ small, and the Riesz projection P for A on λ satisfies

$$PX = \ker(A - \lambda) \quad \text{and} \quad (1 - P)X = \operatorname{ran}(A - \lambda) \tag{1.43}$$

and hence also

$$PX = \ker(A - \lambda) \quad \text{and} \quad (1 - P)X = \overline{\operatorname{ran}}(A - \lambda) \tag{1.44}$$

In fact, the semisimple eigenvalues of A are characterized as those spectral values λ for which there exists a projection P satisfying (1.43). And, by analogy, we call an eigenvalue λ *weakly semisimple* whenever there exists a projection P satisfying the weaker condition (1.44). We can thus qualify the adiabatic theorems from [12] and [112] as being concerned with weakly semisimple eigenvalues.

It is clear, however, that the eigenvalues of general contraction semigroup generators – as opposed to those of skew self-adjoint operators – will in many cases fail to be weakly semisimple. Consider, for example, block-diagonal operators

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} \lambda + N & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{with} \quad N := \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix} \tag{1.45}$$

in $X = \mathbb{C}^d \times \ell^2(\mathbb{N})$, where A_2 is an arbitrary contraction semigroup generator in $X_2 = \ell^2(\mathbb{N})$ and where $\lambda \in \{\operatorname{Re} z \leq 0\}$ stays sufficiently far away from the imaginary axis such that also $A_1 = \lambda + N$ is a contraction semigroup generator in $X_1 = \mathbb{C}^d$. It then follows that A generates a contraction semigroup in X and that

$$\ker(A - \lambda) \cap \overline{\operatorname{ran}}(A - \lambda) \supset \operatorname{span}\{(e_1, 0)\} \neq 0,$$
$$\ker(A - \lambda) + \overline{\operatorname{ran}}(A - \lambda) \subset \operatorname{span}\{(e_1, 0), \dots, (e_{d-1}, 0)\} + 0 \times \ell^2(\mathbb{N}) \neq X,$$

where the e_i denote the canonical unit vectors in $\mathbb{C}^d$. So, no projection P can exist with (1.44) or, in other words, the eigenvalue λ of A is not weakly semisimple. A different kind of example, with $\lambda = 0$, is given by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S_+ - 1 \end{pmatrix} \tag{1.46}$$

in $X = \mathbb{C}^d \times \ell^1(\mathbb{N})$, where S_+ denotes the right shift on $\ell^1(\mathbb{N})$ acting by $S_+(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. In this example, A is a contraction semigroup generator in X with eigenvalue $\lambda = 0$,

$$\ker(A - \lambda) \cap \overline{\operatorname{ran}}(A - \lambda) = 0 \quad \text{but} \quad \ker(A - \lambda) + \overline{\operatorname{ran}}(A - \lambda) \neq X \tag{1.47}$$

because $\lambda = 0$ belongs to the residual spectrum of $S_+ - 1$, that is, $\overline{\operatorname{ran}}(S_+ - 1) \neq \ell^1(\mathbb{N})$ (here it is decisive that we took S_+ to act in ℓ^1 rather than in ℓ^p with $p \neq 1$). So, the eigenvalue $\lambda = 0$ of A is not weakly semisimple.

A similar example from [12] (with $\ker A = 0$, however) indicates that the same type of difficulty (1.47) can arise for generators A of quantum dynamical semigroups, that is, a certain kind of contraction semigroups describing the dynamics of open quantum systems and defined on the trace class $X = S^1(\mathfrak{h})$ over a Hilbert space $\mathfrak{h}$. It is shown in [12] that the essential reason behind that difficulty is the non-reflexivity of $S^1(\mathfrak{h})$. In order to apply their adiabatic theorem without spectral gap condition to time-dependent generators of quantum dynamical semigroups with eigenvalue 0, Avron, Fraas, Graf and Grech therefore consider extensions $A(t)$ of such generators to the reflexive space $S^2(\mathfrak{h})$ of Hilbert–Schmidt operators.

1.4 Contributions of this thesis to well-posedness theory

In the well-posedness part (Chapter 2) of this thesis, we provide among other things the necessary dynamical preliminaries for our adiabatic theorems (Section 2.1), but above all we establish well-posedness theorems that are interesting in themselves and that go far beyond what is needed in adiabatic theory (Section 2.2 and Section 2.3). In fact, quite some of these results are furnished with too weak regularity conditions to be applicable in our adiabatic theorems.

1.4.1 Well-posedness for operators with time-independent domains

Section 2.3 contains our well-posedness results for operators $A(t)$ with time-independent domains $D(A(t)) = D$ and a more complete introduction, addressing also the central ideas behind these results, can be found in Section 2.2.1. In Section 2.2.2 we show that the regularity conditions of a well-posedness theorem by Yosida can be simplified quite considerably and we thereby clarify the relation of this theorem with other well-posedness theorems from the literature. In particular, the relation with the well-posedness result by Kato from [62] mentioned above (Section 1.1), which for contraction semigroup generators $A(t) : D \subset X \to X$ with time-independent domains establishes the well-posedness of (1.1) on D under the simple condition that

$$t \mapsto A(t)y \text{ is continuously differentiable for all } y \in D. \tag{1.48}$$

Yosida's well-posedness theorem can be found in his book [141] on functional analysis and it is reproduced in Reed and Simon's and Blank, Exner and Havlíček's books on mathematical physics, for instance. In large parts of mathematical physics (including adiabatic theory), Yosida's theorem is better known than the above-mentioned well-posedness theorem by Kato. Yet, the regularity conditions of Yosida's theorem are far more complicated and far less lucid than the simple strong continuous differentiability condition (1.48) from [62] and one might therefore think that, in return, Yosida's conditions should be more general than (1.48). We will see, however, that that they are not: we will show that Yosida's complicated regularity conditions are just equivalent to the simple continuous differentiability condition (1.48). In particular, this equivalence shows that the regularity condtions of quite some adiabatic theorems from the literature, for instance those from [13], [11], [131], [132], [1], [2] or [12], can be noticeably simplified. We also extend the above-mentioned equivalence of regularity conditions to the more general version of Yosida's well-posedness result for locally convex spaces from [140].

In Section 2.2.3 we slightly generalize the less known well-posedness theorem by Kato from [62] mentioned above (Section 1.1), which for skew self-adjoint operators $A(t) : D \subset H \to H$ with time-independent domains establishes the well-posedness of (1.1) on D under the condition that $t \mapsto A(t)$ is continuous and of bounded variation. We will show that for quasicontraction group generators $A(t)$ with time-independent domains in a uniformly convex space, this continuity and bounded variation condition still yields the well-posedness of (1.1) on D.

In Section 2.2.4 we show by simple examples that the assumptions of the previously discussed well-posedness theorems for operators with time-independent domains cannot be weakened too much or even dropped. Specifically, we show that in the well-posedness theorem for semigroup generators from [62], the strong continuous differentiability condition (1.48) cannot be weakened to Lipschitz continuity, and that in our well-posedness theorem for group generators in uniformly convex spaces, the continuity and bounded variation condition cannot be replaced by Hölder continuity of any degree $\alpha < 1$, and the uniform convexity condition cannot be dropped. In our examples, the operators $A(t)$ are of the simple form $A(t) = A_0 + B(t)$ with a contraction group generator A_0 and bounded perturbing operators $B(t)$. It seems that our examples are the first counterexamples to well-posedness involving group generators and, moreover, they are noticeably simpler than the previously known conterexamples from [105] and [41].

1.4.2 Well-posedness for operators with time-dependent domains

Section 2.3 contains our well-posedness results for operators $A(t)$ with generally time-dependent domains $D(A(t))$ and a more complete introduction, addressing also the central ideas behind these results, can be found in Section 2.3.1. In Section 2.3.2 we examine the special situation of semigroup generators $A(t)$ whose first (1-fold) or higher (p-fold) commutators at distinct times are complex scalars, in short:

$$[A(t_1), A(t_2)] = \mu(t_1, t_2) \in \mathbb{C} \tag{1.49}$$

or

$$\big[\dots \big[[A(t_1), A(t_2)], A(t_3)\big] \dots, A(t_{p+1})\big] = \mu(t_1, \dots, t_{p+1}) \in \mathbb{C} \tag{1.50}$$

in some sense to be made precise. In this special situation we prove well-posedness for (1.1) on suitable dense subspaces Y of X and, moreover, in the case (1.49) we prove the representation formula

$$U(t,s) = e^{\overline{\int_s^t A(\tau)\,d\tau}}\, e^{1/2 \int_s^t \int_s^\tau \mu(\tau,\sigma)\,d\sigma d\tau} \tag{1.51}$$

for the evolution generated by the operators $A(t)$. We thereby generalize a well-posedness result of Goldstein and of Nickel and Schnaubelt from [49], [101] dealing with the special case of (1.49) where $\mu \equiv 0$: in [49] contraction semigroup generators are considered, while in [101] contraction semigroup generators are replaced by general semigroup generators satisfying a certain stability condition and the formula (2.45) with $\mu \equiv 0$ is proved. Stability, in this context, is a generalization of contraction semigroup requirements.

Compared to the well-posedness theorems for general semigroup or group generators from [62], [65], [66] (Section 2.1.4) where no commutator conditions of the kind (1.49) or (1.50) are imposed, our well-posedness results for the special class of semigroup generators with (1.49) or (1.50) are furnished with fairly mild stability and regularity conditions: 1. It is sufficient – just as in the case of commuting operators from [49], [101] – to require stability of the family A only in X. In contrast to the well-posedness theorems from [65] or [66], for instance, it is not necessary to additionally require stability in a suitable invariant and suitably normed dense subspace Y of X contained in all the domains of the operators $A(t)$, which is generally difficult to verify unless the domains of the $A(t)$ are time-independent. 2. It is sufficient – similarly to the case of commuting operators from [49], [101] or to the elementary case of bounded operators – to require strong continuity conditions: indeed, it is sufficient if

$$t \mapsto A(t)y \quad \text{and} \quad (t_1, \dots, t_{k+1}) \mapsto \big[\dots,[[A(t_1),A(t_2)],A(t_3)]\dots,A(t_{k+1})\big]y$$

are continuous for $k \in \{1,\dots,p\}$ and y in a dense subspace Y of X contained in all the respective domains. In contrast to the well-posedness theorems from [65] or [66], this subspace Y need not be normed in any way whatsoever and $t \mapsto A(t)|_Y$ need not be norm continuous. And furthermore, it is not necessary to require an additional $W^{1,1}$-regularity condition on certain auxiliary operators $S(t) : Y \to X$ (as in the well-posedness theorems from [65], [66] for general semigroup generators $A(t)$) or an additional regularity condition on certain auxiliary norms $\|\,.\,\|_t^\pm$ on Y (as in the special well-posedness result from [65] for a certain kind of group generators). Such additional regularity conditions are necessary for well-posedness in general situations without commutator conditions of the kind (1.49) or (1.50), as is demonstrated by our counterexamples from Section 2.2.4, for instance.

In Section 2.3.3 we improve the special well-posedness result from [65] for group generators with time-dependent domains: in the spirit of [72] we show that strong (instead of norm) continuity is sufficient in this result – just like in our other well-posedness results for the case (1.49) or (1.50). And in a certain special case involving quasicontraction group generators with time-independent domains in a uniformly convex space, these other results can also be obtained by applying our improved well-posedness result for group generators. Incidentally, our result from Section 2.3.3 also generalizes the respective well-posedness theorem for group generators with time-independent domains from Section 2.2.3, while the result from [65] does not.

In Section 2.3.5 we finally give some applications of the abstract well-posedness theorems for generators with scalar 1-fold or p-fold commutators from Section 2.3.2, namely to Segal field operators $\Phi(f_t)$ as well as to the related operators $H_\omega + \Phi(f_t)$ describing a classical particle coupled to a time-dependent quantized field of bosons (Section 2.3.5.1) and finally to Schrödinger operators describing a quantum particle coupled to a time-dependent spatially constant electric field (Section 2.3.5.2).

1.5 Contributions of this thesis to adiabatic theory

In the adiabatic theory part (Chapter 3 to 6) of this thesis we extend and develop further the existing adiabatic theory – especially in the case without spectral gap. In a nutshell, our primary extensions can be described as follows. 1. We no longer require the considered spectral values $\lambda(t)$ to be (weakly) semisimple (which is motivated by the examples at the end of Section 1.3). 2. We no longer require the domains of the operators $A(t)$ to be time-independent (which is motivated, in parts, by the fact that the domains of operators $A(t)$ defined by sesquilinear forms $a(t)$ will in general be time-dependent). Additionally, we work with rather mild regularity conditions: in the case of time-independent domains, for instance, it will be sufficient to require a certain strong Sobolev regularity condition on $t \mapsto A(t)$ (which is satisfied if, for instance, $t \mapsto A(t)$ is continuously differentiable w.r.t. the strong or weak operator topology).

1.5.1 Spectrally related projections

In Chapter 3 we provide, among other things, the necessary spectral-theoretic preliminaries for general adiabatic theory, that is to say, we identify natural notions of spectrally related projections (generalized spectral projections). In the case with spectral gap this is canonical and one speaks of associated projections. In the case without spectral gap, however, this is not canonical and we will speak of weakly associated projections. In precise terms, associatedness and weak associatedness are defined as follows: let $A : D(A) \subset X \to X$ be a densely defined operator with $\rho(A) \neq \emptyset$ and let $\sigma \subset \sigma(A)$ be a compact subset. If σ is isolated in $\sigma(A)$, then a projection P is called *associated with A and σ* if and only if P commutes with A and $A|_{PD(A)}$ is bounded such that

$$\sigma(A|_{PD(A)}) = \sigma \quad \text{whereas} \quad \sigma(A|_{(1-P)D(A)}) = \sigma(A) \setminus \sigma.$$

Additionally, a spectral value $\lambda \in \sigma(A)$ is called *semisimple* iff it is a pole of $(\,.\, - A)^{-1}$ of order 1. If $\lambda \in \sigma(A)$ is a not necessarily isolated spectral value, then a projection P will be called *weakly associated (of order m) with A and λ* if and only if P commutes with A and $A|_{PD(A)}$ is bounded such that

$$A|_{PD(A)} - \lambda \text{ is nilpotent (of order } m) \text{ whereas } A|_{(1-P)D(A)} - \lambda \text{ is injective and has dense range in } (1-P)X$$

(where the order of nilpotence of a bounded operator N is the smallest positive integer m with $N^m = 0$). Additionally, λ will be called *weakly semisimple* iff there exists a projection P weakly associated with A and λ of order 1. (In view of (1.53), this definition coincides with the ad hoc definition (1.44) of weak semisimplicity used so far.)

In the case of an isolated spectral value $\lambda \in \sigma(A)$ the question arises of how the notions of associated and weakly associated projections are related. We will see that they conicide if λ is a pole of the resolvent and that they do not if λ is an essential singularity. In proving this, the following central properties of associatedness (which are completely well-known) and of weak associatedness (which seem to be new) will be used. In fact, they will be constantly used throughout this thesis.

- If σ is compact and isolated in $\sigma(A)$, then there exists a unique projection P associated with A and σ, and $P = (2\pi i)^{-1} \int_\gamma (z - A)^{-1}\, dz$, where γ is a cycle in $\rho(A)$ encircling σ but not $\sigma(A) \setminus \sigma$ (Riesz projection). If P is associated with A and $\sigma = \{\lambda\}$, and λ is a pole of $(\,.\, - A)^{-1}$ of order m, then

$$PX = \ker(A - \lambda)^k \quad \text{and} \quad (1 - P)X = \operatorname{ran}(A - \lambda)^k \quad (k \geq m). \tag{1.52}$$

- If λ is not isolated in $\sigma(A)$, then in general there exists no projection P weakly associated with A and λ, but if such a projection exists it is necessarily unique. If P is weakly associated with A and λ of order m and $\lambda \in \sigma(A)$ is an arbitrary spectral value, then

$$PX = \ker(A - \lambda)^k \quad \text{and} \quad (1 - P)X = \overline{\operatorname{ran}}(A - \lambda)^k \quad (k \geq m). \tag{1.53}$$

In the special case of skew self-adjoint operators A the notions of associatedness and weak associatedness reduce to the notion of spectral projections defined via the spectral measure of A: if λ is an arbitrary point of $\sigma(A)$, then a projection P is weakly associated with A and λ if and only if P is equal to the spectral projection $P^A(\{\lambda\})$. In particular, weakly associated projections always exist in the case of skew self-adjoint operators. In the case of general spectral operators (in the sense of [39]) we still have at least the following criterion for the existence of weakly associated projections:

- If A is a spectral operator (with spectral measure P^A) and
- if $\lambda \in \sigma(A)$ is such that for some bounded neighborhood σ of λ the bounded spectral operator $A|_{P^A(\sigma)X}$ is of finite type,

then the projection weakly associated with A and λ exists and is given by $P^A(\{\lambda\})$. In particular, this is true if A is spectral of scalar type.

1.5.2 Adiabatic theory for operators with time-independent domains

1.5.2.1 Case with spectral gap

In Section 4.1 we prove adiabatic theorems with spectral gap condition (uniform and non-uniform) for general operators $A(t) : D \subset X \to X$ with time-independent domain

$D(A(t)) = D$ and for compact spectral subsets $\sigma(t)$. We thereby generalize in a quite simple way the adiabatic theorem of Abou Salem from [2], which covers the case of singletons $\sigma(t) = \{\lambda(t)\}$ with uniformly isolated simple spectral values $\lambda(t)$. In simplified form, our our theorems (Theorem 4.1.1 and 4.1.2 combined) can be formulated as follows:

- If $A(t) : D \subset X \to X$ for every $t \in I$ generates a contraction semigroup and $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$,
- if $\sigma(t)$ for every $t \in I$ is a compact subset of $\sigma(A(t))$, $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ at only countably many points which, in turn, accumulate at only finitely many points, and $t \mapsto \sigma(t)$ is continuous,
- if $P(t)$ for every $t \in I \setminus N$ is associated with $A(t)$ and $\sigma(t)$ and $I \setminus N \ni t \mapsto P(t)$ extends to a twice strongly continuously differentiable map on the whole of I, where N denotes the set of those points where $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$, then

$$\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \quad \text{or} \quad \sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = o(1) \tag{1.54}$$

as $\varepsilon \searrow 0$, depending on whether $N = \emptyset$ (uniform spectral gap) or $N \neq \emptyset$ (non-uniform spectral gap). (In the above relation, U_ε and V_ε denote the evolution system for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$, respectively.)

In the case of uniform spectral gap (Section 4.1.1), an essential step in the proof of this theorem will be to solve the commutator equation (1.33) and we will do so in virtually the same way as in [2], namely we set

$$B(t) := \frac{1}{2\pi i} \int_{\gamma_t} (z - A(t))^{-1} P'(t) (z - A(t))^{-1} \, dz \tag{1.55}$$

with cycles γ_t in $\rho(A(t))$ encircling $\sigma(t)$ but not $\sigma(A(t)) \setminus \sigma(t)$. We then easily obtain, using the central properties of associatedness, the commutator equation $[P'(t), P(t)] \supset B(t)A(t) - A(t)B(t)$ for every t. With this commutator equation, in turn, and partial integration we can rewrite the difference $V_\varepsilon(t) - U_\varepsilon(t)$ as

$$\begin{aligned} V_\varepsilon(t) - U_\varepsilon(t) &= \int_0^t U_\varepsilon(t,\tau)[P'(\tau), P(\tau)] V_\varepsilon(\tau) \, d\tau \qquad (1.56) \\ &= \varepsilon\, U_\varepsilon(t,\tau) B(\tau) V_\varepsilon(\tau) \big|_{\tau=0}^{\tau=t} - \varepsilon \int_0^t U_\varepsilon(t,\tau) \big(B'(\tau) + B(\tau)[P'(\tau), P(\tau)] \big) V_\varepsilon(\tau) \, d\tau, \end{aligned}$$

from which the desired conclusion (1.54.a) follows by the boundedness of the evolutions U_ε and V_ε in ε. In the case with non-uniform spectral gap (Section 4.1.2), the desired conclusion (1.54.b) can be reduced by a standard argument from [61] to the case with uniform spectral gap.

In Section 4.1.3 we extend the above theorem to the case of several spectral subsets $\sigma_1(t), \dots, \sigma_r(t)$ and we do so by extending the commutator equation method just sketched. In this way, we also obtain a simple new proof for Nenciu's result from [96].

Section 4.1.3 also contains simple examples for the above theorem where the previously known adiabatic theorems from [2], [60], [12] cannot be applied. In particular, these examples show that even for singletons $\sigma(t) = \{\lambda(t)\}$ the spectral values $\lambda(t)$ may well be essential singularities of the resolvent of $A(t)$ and need not even be eigenvalues. Additionally, we show by example that the contraction semigroup condition on the operators $A(t)$ cannot be weakened too much, thereby complementing Joye's example from [60] by a different kind of counterexample.

In Section 4.1.4 we discuss two applied examples of the adiabatic theorem with spectral gap condition: one straightforward example for generators $A(t)$ of certain quantum dynamical semigroups with spectral value $\lambda(t) = 0$ and one non-straightforward example for generators $A(t)$ of certain neutron transport semigroups with $\lambda(t)$ being the rightmost spectral value of $A(t)$. In this latter example, $\lambda(t)$ can be shown to be a simple – and, in particular, uniformly isolated – eigenvalue by a Perron–Frobenius argument.

1.5.2.2 Case without spectral gap

In Section 4.2 we establish adiabatic theorems (qualitative and quantitative) without spectral gap condition for general operators $A(t) : D \subset X \to X$ with time-independent domain $D(A(t)) = D$ and for not necessarily weakly semisimple eigenvalues $\lambda(t)$. We thereby generalize the respective adiabatic theorems of Avron, Fraas, Graf, Grech from [12] and of Schmid from [112], which cover the case of weakly semisimple eigenvalues. In all our theorems, the eigenvalues $\lambda(t)$ are assumed to lie on the boundary of $\sigma(A(t))$ in such a way that $\lambda(t) + \delta e^{i\vartheta(t)} \in \rho(A(t))$ for all $\delta \in (0, \delta_0]$ with some t-independent δ_0.

Section 4.2.1 contains a qualitative adiabatic theorem which, in simplified form, can be formulated as follows:

- If $A(t) : D \subset X \to X$ for every $t \in I$ generates a contraction semigroup and $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$,
- if $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$ such that $\lambda(t) + \delta e^{i\vartheta(t)} \in \rho(A(t))$ for every $\delta \in (0, \delta_0]$ and $t \mapsto \lambda(t), e^{i\vartheta(t)}$ are continuously differentiable,
- if $P(t)$ is weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$,

$$\left\| \big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1}(1 - P(t)) \right\| \le \frac{M_0}{\delta} \quad (\delta \in (0, \delta_0]), \tag{1.57}$$

  $\operatorname{rk} P(0) < \infty$ and $t \mapsto P(t)$ is twice strongly continuously differentiable,

then $\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\| \longrightarrow 0$ as $\varepsilon \searrow 0$. If, in addition, X is reflexive, then

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0). \tag{1.58}$$

(In the above relations, U_ε and V_ε denote the evolution system for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$, respectively.)

Similarly to [11] and [131], a first essential step in our proof of this theorem will be to find a solution to the approximate commutator equation (1.28). In doing so, we will be guided by the case with spectral gap. In that case, an exact solution B of (1.28) with vanishing error $C = 0$ is given by (1.55) and, for singletons $\sigma(t) = \{\lambda(t)\}$ with poles $\lambda(t)$ of the resolvent $(. - A(t))^{-1}$ of order at most m_0, we can rewrite this exact solution as

$$B(t) = \sum_{k=0}^{m_0-1} \overline{R}(t)^{k+1} P'(t)P(t)(\lambda(t) - A(t))^k + \sum_{k=0}^{m_0-1} (\lambda(t) - A(t))^k P(t)P'(t)\overline{R}(t)^{k+1}$$

by means of Cauchy's theorem, where $\overline{R}(t) := (\lambda(t) - A(t)|_{\overline{P}(t)D(A(t))})^{-1}\overline{P}(t)$ and $\overline{P}(t) := 1 - P(t)$. In the case without spectral gap, the inverses $(\lambda(t) - A(t)|_{\overline{P}(t)D(A(t))})^{-1}$ do not exist as (bounded) operators on $\overline{P}(t)X$, but by the assumed spectral marginality of $\lambda(t)$ the slightly shifted inverses $\overline{R}_\delta(t) := (\lambda(t) + \delta e^{i\vartheta(t)} - A(t))^{-1}\overline{P}(t)$ do exist. We therefore set, suppressing t-dependence for convenience,

$$B_{\boldsymbol{\delta}} := \sum_{k=0}^{m_0-1} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big) P'P(\lambda - A)^k + \sum_{k=0}^{m_0-1} (\lambda - A)^k PP' \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big), \tag{1.59}$$

where $m_0 := \operatorname{rk} P(0) = \operatorname{rk} P(t)$ (the t-independence coming from the norm continuity of $t \mapsto P(t)$) and where $\boldsymbol{\delta} := (\delta_1, \dots, \delta_{m_0}) \in (0, \delta_0]^{m_0}$. In the special case of skew self-adjoint operators $A(t)$, this reduces to Teufel's solution (1.30) of (1.28) because the eigenvalues of skew self-adjoint operators are weakly semisimple and hence $P(\lambda - A)^k, (\lambda - A)^k P = 0$ for all $k \neq 0$ in that special case. With the above definition (1.59) we obtain, using $PX = \ker(A - \lambda)^{m_0}$ (central properties of weak associatedness!), the approximate commutator equation

$$[P'(t), P(t)] \supset B_{\boldsymbol{\delta}}(t)A(t) - A(t)B_{\boldsymbol{\delta}}(t) + C_{\boldsymbol{\delta}}(t)$$

for every $t \in I$, where the error $C_{\boldsymbol{\delta}} := C_{\boldsymbol{\delta}}^+ - C_{\boldsymbol{\delta}}^-$ is given by

$$C_{\boldsymbol{\delta}}^+ := \sum_{k=0}^{m_0-1} \delta_{k+1} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big) P'P(\lambda - A)^k, \quad C_{\boldsymbol{\delta}}^- := \sum_{k=0}^{m_0-1} (\lambda - A)^k PP' \delta_{k+1} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big).$$

With this approximate commutator equation, in turn, and partial integration we can then rewrite the difference $V_\varepsilon(t) - U_\varepsilon(t)$ as

$$\begin{aligned} V_\varepsilon(t) - U_\varepsilon(t) &= \int_0^t U_\varepsilon(t,\tau)[P'(\tau), P(\tau)]V_\varepsilon(\tau)\, d\tau = \varepsilon\, U_\varepsilon(t,\tau)B_{\boldsymbol{\delta}}(\tau)V_\varepsilon(\tau)\big|_{\tau=0}^{\tau=t} \\ &- \varepsilon \int_0^t U_\varepsilon(t,\tau)\big(B_{\boldsymbol{\delta}}'(\tau) + B_{\boldsymbol{\delta}}(\tau)[P'(\tau), P(\tau)]\big)V_\varepsilon(\tau)\, d\tau + \int_0^t U_\varepsilon(t,\tau)C_{\boldsymbol{\delta}}(\tau)V_\varepsilon(\tau)\, d\tau \end{aligned} \tag{1.60}$$

and it remains to show that the right-hand side of this equation can be made arbitrarily small as $\varepsilon \searrow 0$. In doing so, the assumed reduced resolvent estimate (1.57) will be important: it yields the estimates

$$\|B_{\boldsymbol{\delta}}(t)\| \le c\,(\delta_1 \cdots \delta_{m_0})^{-1}, \quad \left\|B'_{\boldsymbol{\delta}}(t)\right\| \le c\,(\delta_1 \cdots \delta_{m_0})^{-(m_0+1)},$$
$$\int_0^1 \left\|C^{\pm}_{\boldsymbol{\delta}}(\tau)\right\| \, d\tau \le c \sum_{k=0}^{m_0-1} (\delta_1 \cdots \delta_k)^{-1} \eta^{\pm}(\delta_{k+1}) \tag{1.61}$$

where $\eta^+(\delta) := \int_0^1 \delta \left\|\overline{R}_\delta(\tau) P'(\tau) P(\tau)\right\| \, d\tau$ and $\eta^-(\delta) := \int_0^1 \delta \left\|P(\tau)P'(\tau)\overline{R}_\delta(\tau)\right\| \, d\tau$ for $\delta \in (0, \delta_0]$. With

$$(1-P)X = \overline{\operatorname{ran}}(A-\lambda)^{m_0} \quad \text{and} \quad (1-P^*)X^* = \overline{\operatorname{ran}}(A^*-\lambda)^{m_0}$$

(central properties of weak associatedness plus: weak associatedness for semigroup generators in reflexive spaces carries over to the dual operators) and with the assumed resolvent estimate we will further show that

$$\delta\,\overline{R}_\delta(t), \quad \delta\,\overline{R}_\delta(t)^* \longrightarrow 0 \quad (\delta \searrow 0)$$

w.r.t. the strong operator topology for almost every $t \in I$. Since $\operatorname{rk} P(t)^* = \operatorname{rk} P(t) < \infty$, we even get $\eta^{\pm}(\delta) \longrightarrow 0$. Choosing then $\boldsymbol{\delta} = \boldsymbol{\delta}_\varepsilon = (\delta_{1\,\varepsilon}, \dots, \delta_{m_0\,\varepsilon})$ in a careful way, we will finally arrive at the desired conclusion (1.58). (In view of (1.61) it becomes clear that the simpler and a priori more natural choice $\delta_1 = \dots = \delta_{m_0}$ in (1.59) would not have worked out.)

Similarly to the case with spectral gap, we will extend the above theorem to the case of several eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$. We will achieve this by solving a suitable extended approximate commutator equation and in that undertaking the case with spectral gap will again be an indispensable guideline. It seems that the thus obtained extension of the adiabatic theorem without spectral gap condition is new even in the special case of skew self-adjoint operators.

We will also identify relatively simple and convenient criteria for the assumptions – in particular, the reduced resolvent estimate (1.57) – of the above adiabatic theorem to be satisfied: we will see that these assumptions are satisfied if $A(t)$ for every t is a spectral operator (with spectral measure $P^{A(t)}$) generating a contraction semigroup and $\lambda(t)$ is an eigenvalue of $A(t)$ such that, apart from regularity conditions,

- $A(t)|_{P^{A(t)}(\sigma(t))D}$ is spectral of scalar type for some punctured neighborhood

  $$\sigma(t) := \sigma(A(t)) \cap \overline{U}_{r_0}(\lambda(t)) \setminus \{\lambda(t)\}$$

  of $\lambda(t)$ in $\sigma(A(t))$ ($r_0 \in (0,\infty) \cup \{\infty\}$) and $\operatorname{rk} P^{A(t)}(\{\lambda(t)\}) < \infty$ for a. e. t,
- the open sector $\{\lambda(t) + \delta e^{i\vartheta} : \delta \in (0, \delta_0), \vartheta \in (\vartheta(t) - \vartheta_0, \vartheta(t) + \vartheta_0)\}$ of radius $\delta_0 \in (0, \infty)$ and angle $2\vartheta_0 \in (0, \pi)$ is contained in $\rho(A(t))$.

Section 4.2.2 contains some quantitative refinements of the qualitative adiabatic theorem above. In particular, it contains a quantitative adiabatic theorem for scalar type spectral operators $A(t)$ whose spectral measures $P^{A(t)}$ are Hölder continuous in t around $\lambda(t)$ in some sense. We thereby generalize a result for skew self-adjoint operators from [11] and [131] and, in particular, slightly improve the respective quantitative bound on the rate of convergence in (1.58).

Section 4.2.3 gives simple (classes of) examples for the adiabatic theorem above where the previously known adiabatic theorems from [12] or [112] cannot be applied: one example where the operators $A(t)$ are spectral and one example where they are not. In the context of finding examples going beyond [12] or [112], it is important to notice the following fact: if the eigenvalues $\lambda(t)$ from the adiabatic theorem above are purely imaginary for all t, these eigenvalues are automatically weakly semisimple (by virtue of the contraction semigroup assumption on $A(t)$). So, in the special case of purely imaginary eigenvalues $\lambda(t)$, the adiabatic theorem above essentially (save for regularity subtleties) reduces to the adiabatic theorem from [12] or [112]. We also show by example that the regularity condition on $t \mapsto P(t)$ cannot be weakened to strong continuity (while it can be weakened to continuous differentiability, as we will see).

In Section 4.2.4 we apply the adiabatic theorem without spectral gap condition to generators $A(t)$ of quantum dynamical semigroups in $X = S^p(\mathfrak{h})$ (Schatten-p class over a Hilbert space $\mathfrak{h}$) with eigenvalue $\lambda(t) = 0$, that is,

$$A(t)\rho = Z_0(t)(\rho) + \sum_{j \in J} B_j(t)\rho B_j(t)^* - 1/2\{B_j(t)^* B_j(t), \rho\} \qquad (\rho \in D(Z_0(t))) \quad (1.62)$$

where the operators $Z_0(t)$ are the generators of the group in $S^p(\mathfrak{h})$ defined by $e^{Z_0(t)\tau}(\rho) := e^{-iH(t)\tau}\rho\, e^{iH(t)\tau}$ with self-adjoint operators $H(t)$ on $\mathfrak{h}$ and where the operators $B_j(t)$ are bounded operators on $\mathfrak{h}$ satisfying

$$\sum_{j \in J} B_j(t)^* B_j(t) = \sum_{j \in J} B_j(t) B_j(t)^* < \infty \quad (1.63)$$

for every $t \in I$ (J an arbitrary index set). We thereby extend the respective application from [12] where

- the generators $A(t)$ are assumed to be dephasing, and
- the self-adjoint operators $H(t)$ are assumed to be bounded (so that the $A(t)$ are bounded as well).

Since dephasingness means precisely that each of the operators $B_j(t)$ belongs to the double commutant $\{H(t)\}''$ of $H(t)$ which for separable $\mathfrak{h}$, in turn, is given by

$$\{H(t)\}'' = \{f(H(t)) : f \text{ bounded measurable function } \sigma(H(t)) \to \mathbb{C}\},$$

the equality condition in (1.63) is clearly a weaker requirement than dephasingness. In fact, it is noticeably weaker as will become clear already by very simple examples. Just as in [12], we consider the operators $A(t)$ from (1.62) not in the natural space $S^1(\mathfrak{h})$ but in $X = S^p(\mathfrak{h})$ for $p \in (1, \infty)$ because, in the natural space $S^1(\mathfrak{h})$, projections weakly associated with $A(t)$ and $\lambda(t) = 0$ will quite often fail to exist. (In fact, such projections fail to exist in a significantly wider range of examples than indicated in [12], as we shall see.) Condition (1.63) is what turns $A(t)$ into a well-defined operator in $S^p(\mathfrak{h})$ in the first place, with natural properties for arbitrary $p \in (1, \infty)$.

1.5.3 Adiabatic theory for operators with time-dependent domains

In Chapter 5 we establish our adiabatic theorems for operators $A(t) : D(A(t)) \subset X \to X$ with generally time-dependent domains. Section 5.1 is devoted to fully general operators $A(t)$ while Section 5.2 is devoted to the particularly interesting special case of skew self-adjoint operators $A(t) = iA_{a(t)}$ defined by symmetric sesquilinear forms $a(t)$.

In Section 5.1.1 and Section 5.1.2 we extend the adiabatic theorems with and without spectral gap condition from the previous chapter to the case of time-dependent domains. In this extension process, the necessary changes in the assumptions are essentially confined to regularity conditions: most importantly, the regularity condition on $t \mapsto A(t)$ from the adiabatic theorems for operators with time-independent domains will be replaced by

- strong continuous differentiability conditions on the resolvents $t \mapsto (z - A(t))^{-1}$ for suitable points z,
- and the condition that the evolution U_ε for $\frac{1}{\varepsilon}A$ exist on the spaces $D(A(t))$ for every ε and be bounded in ε.

(In the case of time-independent domains $D(A(t)) = D$, these two conditions are implicit in the strong continuous differentiability assumption on $t \mapsto A(t)$ and the contraction semigroup assumption on $A(t)$ made in the simplified theorems displayed above.) What makes our extensions work is, basically, the following two simple facts:

- An evolution system $U = (U(t, s))$ for operators $A(t)$ on the spaces $D(A(t))$ is right differentiable w.r.t. the second variable s in some appropriate sense, namely: for every $s_0 \in [0, t)$ and every $y \in D(A(s_0))$, the mapping $[0, t] \ni s \mapsto U(t, s)y$ is right differentiable at s_0 with right derivative $-U(t, s_0)A(s_0)y$.
- A continuous, right differentiable map $f : [0, t] \to X$ with continuous right derivative $\partial_+ f$ is already continuously differentiable on $[0, t)$ and therefore the fundamental theorem of calculus is available.

With these two facts, the proofs from the case of time-independent domains can easily be carried over. In fact, the only major changes are in the justification of the partial integration step in (1.56) and (1.60). Almost all other steps – in particular, the resolution of the (approximate) commutator equations – are pointwise in t and can therefore be taken over without change to case of time-dependent domains.

Section 5.1.3 is devoted to an extension of the adiabatic theorem of higher order from [59] or [99] to the case of general – not necessarily skew self-adjoint – operators with generally time-dependent domains. Albeit a bit technical, this extension is not difficult – the essential ingredients for it to work being the same as in Section 5.1.1 and Section 5.1.2.

Section 5.2 gives applications of the general theory from Section 5.1 to the special case of skew self-adjoint operators $A(t) = iA_{a(t)}$ defined by symmetric sesquilinear forms $a(t)$ with time-independent form domain. Schrödinger operators with time-dependent Rollnik potentials are typical examples for such operators. Specifically, we shall consider the situation of two Hilbert spaces H^+ and H, where H^+ is continuously and densely embedded in H, where $a(t)$ is a symmetric sesquilinear form on H^+ such that, for some $m \in (0, \infty)$,

$$\langle \,.\,,\,..\,\rangle_t^+ := a(t)(\,.\,,\,..\,) + m\,\langle \,.\,,\,..\,\rangle$$

is a scalar product on H^+ and $\|\,.\,\|_t^+$ is equivalent to $\|\,.\,\|^+$ for every $t \in I$, and where $t \mapsto a(t)(x, y)$ is twice continuously differentiable for all $x, y \in H^+$. Applying the adiabatic theorems for general operators from the preceding section, we obtain among other things the following adiabatic theorem without spectral gap condition. It is a generalization of the adiabatic theorem of Bornemann from [17] where, recall, the considered eigenvalues are required to belong to the discrete spectrum (and hence are isolated).

- If $A(t) = iA_{a(t)} : D(A(t)) \subset H \to H$ for every $t \in I$ is the skew self-adjoint operator associated with $a(t)$ and $a(t)$ is as above,
- if $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$ such that $t \mapsto \lambda(t)$ continuous,
- if $P(t)$ is weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$, $\operatorname{rk} P(0) < \infty$ and $t \mapsto P(t)$ is continously differentiable, then

$$\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\|\,, \quad \sup_{t \in I} \|P(t)U_\varepsilon(t)(1 - P(0))\| \longrightarrow 0$$

as $\varepsilon \searrow 0$. Above, U_ε denotes the evolution system for $\frac{1}{\varepsilon}A$ on the spaces $D(A(t))$ whose existence is guaranteed by a well-posedness theorem of Kisyński from [71]. Apart from the fact that the theorem above is more general than the result from [17], our method of proof is also considerably simpler than the – completely different – method employed in [17].

1.5.4 Adiabatic switching

In Chapter 6 we use the adiabatic theorem for skew self-adjoint operators without spectral gap condition, in the version for several eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$ (Section 4.2.1), to study adiabatic switching procedures similar to (1.5) for not necessarily isolated eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$. We extend the well-known Gell-Mann and Low formula, which relates the eigenstates of a perturbed system to the ones of the unperturbed system, to the case of eigenstates belonging to non-isolated eigenvalues. And thereby, we generalize a recent result by Brouder, Panati, Stoltz from [20] where the case of isolated eigenvalues is treated.

1.6 Structure and organization of this thesis

Some remarks on the interdependence of the various chapters and sections as well as on the relation with already published or pre-published works are in order. Chapter 2 can be read independently of all other chapters. Chapters 3 to 6 presuppose the dynamical preliminaries from Section 2.1 and the spectral-theoretic preliminaries from Section 3.1 and, above all, from Section 3.2. We point out that the other well-posedness sections (Section 2.2 and 2.3) are not necessary for the adiabatic theory chapters. Also, at first reading, one may well confine oneself to Section 2.1.2 where the constantly used notions of well-posedness and evolution systems are defined. Sections 2.1.1 and 2.1.3 may be ignored at first reading because the less common notions of $W_*^{m,1}$-regularity or $(M,0)$-stability explained in these sections can be replaced at any occurrence by the simpler notions of m times strong continuous differentiability or contraction semigroups, respectively. Section 2.1.4 recalls some fundamental well-posedness results from the literature and thereby provides the necessary background information for Sections 2.2 and 2.3. Section 3.3 properly defines spectral gaps (uniform and non-uniform) and the continuity of set-valued functions $t \mapsto \sigma(t)$, while Section 3.4 properly introduces the basic concept of adiabatic evolutions and discusses circumstances under which one has an adiabatic theorem already on trivial grounds. In Section 3.5 we explain the standard structure that is behind all our non-applied examples from the adiabatic theory part of this thesis. And finally, Section 3.6 provides the necessary preliminaries on generators – especially dephasing generators – of quantum dynamical semigroups needed for the applied examples from Section 4.1.4 and Section 4.2.4.

Chapter 2 combines the papers [114], [116], and [117]. Section 2.2.2 is based on the paper [114], but in the present thesis the case of locally convex spaces is treated in detail. Section 2.2.3 and Section 2.2.4 are improvements of the paper [116]. In the well-posedness theorem of that paper, $t \mapsto A(t)$ was assumed to be Lipschitz or, more generally, to be absolutely continuous in a certain sense. In the corresponding theorem of this thesis, $t \mapsto A(t)$ is only assumed to be continuous and of bounded variation. Section 2.3 is essentially identical with the paper [117]. All that has been changed is that the well-posedness theorem for group generators has slightly been generalized: it now contains the respective theorem for operators with time-independent domains from Section 2.2.3 as a special case, while its counterpart from [117] did not.

Chapters 3 to 6 are an expansion of the paper [113]. Some central parts of that paper are summarized in [115]. In detail, the changes made to [113] are the following. In the spectral-theoretic preliminaries, we now provide more details (Section 3.1 and Section 3.2), in particular, as far as spectral operators are concerned. We also provide here a quite general class of examples, not present in [113], for the adiabatic theorem without spectral gap condition, namely in terms of spectral operators $A(t)$ that are of scalar type in a punctured neighorhood of the considered eigenvalue. While in [113], the adiabatic theorems for operators with time-independent domains feature a strong $W^{1,\infty}$-regularity condition on $t \mapsto A(t)$, the respective theorems of this thesis only require a strong $W^{1,1}$-regularity condition, which goes back to the classic well-posedness theorem of Kato [66]. Correspondingly, the preliminaries on the regularity of operator-valued maps (Section 2.1.1) had to be adapted and the proofs of the adiabatic theorems also had to be slightly modified. Additionally, we also adapted the examples accordingly. And finally, we now discuss applied examples of the adiabatic theorems for general operators (Section 4.1.4 and Section 4.2.4). Such applications were not present in [113]. Chapter 6 on adiabatic switching is new as well.

1.7 Some global conventions on notation

We finally record some global notational conventions. In all sections except Section 2.2, the symbol X will denote a Banach space over $\mathbb{C}$ unless explicitly stated otherwise, and the same goes for the symbols Y and Z except in Section 2.1.2 to Section 2.3.5. Similarly, H will stand for a Hilbert space over $\mathbb{C}$ except in the context of quantum dynamical semigroups where it denotes a self-adjoint operator on a Hilbert space $\mathfrak{h}$. Sometimes, SOT and WOT will be used to denote the strong or weak operator topology of the Banach space $L(X,Y)$ of bounded linear operators from X to Y or – unless explicitly stated otherwise – of the Banach space $L(X) = L(X,X)$, where X and Y will be clear from the context in most cases. We will abbreviate

$$I := [0,1] \quad \text{and} \quad \Delta := \{(s,t) \in I^2 : s \leq t\},$$

and for evolution systems U defined on Δ we will write $U(t) := U(t,0)$ for all $t \in I$, while U_ε for $\varepsilon \in (0,\infty)$ will always denote the evolution system for $\frac{1}{\varepsilon}A$ on $D(A(t))$ provided it exists. As far as notational conventions on general spectral theory are concerned (in particular, concerning the not completely universal notion of continuous and residual spectrum), we follow the standard textbooks [39], [129] or [141]. As far as semigroup theory is concerned, we take [41] and [104] as standard references. When speaking of a semigroup, we will always mean a strongly continuous semigroup unless explicitly stated otherwise. And finally, the notation employed in the examples will be explained in Section 3.5.

2 Well-posedness theorems for non-autonomous linear evolution equations

2.1 Some preliminaries on regularity and well-posedness

2.1.1 Sobolev regularity of operator-valued functions and one-sided differentiability

We begin by briefly recalling those facts on vector-valued Sobolev spaces that will be needed in the sequel (see [7] and [10] for more detailed expositions). We follow the notational conventions of [10]. In particular, *(μ-)measurability* of a Y-valued map on a complete measure space $(X_0, \mathcal{A}, \mu)$ will not only mean that this map is $\mathcal{A}$-measurable but also that it is μ-almost separably-valued, whereas the notion of *(μ-)strong measurability* will be reserved for operator-valued maps that are pointwise μ-measurable. Suppose J is a non-trivial bounded interval and $p \in [1, \infty) \cup \{\infty\}$. Then $W^{1,p}(J, X)$ is defined to consist of those (equivalence classes of) p-integrable functions $f : J \to X$ for which there is a p-integrable function $g : J \to X$ (called a weak derivative of f) such that

$$\int_J f(t)\varphi'(t)\, dt = -\int_J g(t)\varphi(t)\, dt$$

for all $\varphi \in C_c^\infty(J^\circ, \mathbb{C})$. As usual, *$p$-integrability* of a function $f : J \to X$ (with $p \in [1, \infty) \cup \{\infty\}$) means that f is measurable and $\|f\|_p := (\int_J \|f(\tau)\|^p \, d\tau)^{1/p} < \infty$ if $p \in [1, \infty)$ or $\|f\|_p := \text{ess-sup}_{t \in J} \|f(t)\| < \infty$ if $p = \infty$. If f is in $W^{1,p}(J, X)$ and g_1, g_2 are two weak derivatives of f, then $g_1 = g_2$ almost everywhere, so that up to almost everywhere equality there is only one weak derivative of f which is denoted by ∂f. It is well-known that $W^{1,p}(J, X)$ is a Banach space w.r.t. the norm $\|\,.\,\|_{1,p}$ whith $\|f\|_{1,p} := \|f\|_p + \|\partial f\|_p$ for $f \in W^{1,p}(J, X)$. It is also well-known that the space $W^{1,p}(J, X)$ for $p \in [1, \infty) \cup \{\infty\}$ can be characterized by means of indefinite integrals: $W^{1,p}(J, X)$ consists of those (equivalence classes of) p-integrable functions $f : J \to X$ for which there is a p-integrable function g such that for some (and hence every) $t_0 \in J$

$$f(t) = f(t_0) + \int_{t_0}^t g(\tau)\, d\tau \text{ for all } t \in J,$$

or, equivalently (by Lebesgue's differentiation theorem), $W^{1,p}(J, X)$ consists of (equivalence classes of) p-integrable functions $f : J \to X$ which are differentiable almost everywhere and whose (pointwise) derivative f' is p-integrable such that for some (and hence every) $t_0 \in J$

$$f(t) = f(t_0) + \int_{t_0}^{t} f'(\tau)\, d\tau \ \text{ for all } t \in J.$$

Additionally, the pointwise derivative f' of an $f \in W^{1,p}(J, X)$ equals the weak derivative ∂f almost everywhere. It follows from this characterization that, in case X is reflexive (or more generally, satisfies the Radon–Nikodým property), $W^{1,1}(J, X)$ respectively $W^{1,\infty}(J, X)$ consists exactly of the (equivalence classes of) absolutely continuous or Lipschitz continuous functions, respectively (where one inclusion is completely trivial and independent of the Radon–Nikodým property, of course). We refer to [34] for a measure-theoretic definition of the Radon–Nikodým property (Section III.1) and for a host of characterizations of that property (Section IV.3 and VII.6) along with many examples and counterexamples (Section VII.7).

We now move on to define – inspired by the introduction of Kato's work [68] – the notion of $W_*^{m,p}$-*regularity* for $p \in [1, \infty) \cup \{\infty\}$, which shall be used in all our adiabatic theorems with time-independent domains. An operator-valued function $J \ni t \mapsto A(t) \in L(X, Y)$ on a compact interval J is said to belong to $W_*^{0,p}(J, L(X, Y)) = L_*^p(J, L(X, Y))$ if and only if $t \mapsto A(t)$ is strongly measurable and $t \mapsto \|A(t)\|$ has a p-integrable majorant, that is, a p-integrable function

$$\alpha : J \to [0, \infty) \cup \{\infty\} \quad \text{such that} \quad \|A(t)\| \le \alpha(t) \quad (t \in J).$$

(Sometimes, for instance in [66], the property of having a p-integrable majorant is called *upper p-integrability*.) And $t \mapsto A(t)$ is said to belong to $W_*^{1,p}(J, L(X, Y))$ if and only if there is a $B \in L_*^p(J, L(X, Y))$ (called a $W_*^{1,p}$-*derivative* of A) such that for some (and hence every) $t_0 \in J$

$$A(t)x = A(t_0)x + \int_{t_0}^{t} B(\tau)x\, d\tau \ \text{ for all } t \in J \text{ and } x \in X. \tag{2.1}$$

$W_*^{m,p}(J, L(X, Y))$ for arbitrary $m \in \mathbb{N}$ is defined recursively, of course.

We point out that the $W_*^{m,p}$-spaces (unlike the $W^{m,p}$-spaces), by definition, consist of functions (of operators) rather than equivalence classes of such functions. It is obvious from the characterization of $W^{1,p}(J, Y)$ by way of indefinite integrals that, if $t \mapsto A(t)$ is in $W_*^{1,p}(J, L(X, Y))$, then $t \mapsto A(t)x$ is (the continuous representative of an element) in $W^{1,p}(J, Y)$. It is also obvious that

$$W_*^{1,\infty}(J, L(X, Y)) \subset W_*^{1,p}(J, L(X, Y)) \subset W_*^{1,1}(J, L(X, Y)) \tag{2.2}$$

and that $W_*^{1,1}$- and $W_*^{1,\infty}$-regularity imply absolute continuity or Lipschitz continuity w.r.t. the norm topology, respectively. In fact, if Y has the Radon–Nikodým property, then $W_*^{1,1}$- and $W_*^{1,\infty}$-regularity can be thought of as being not much more than absolute or Lipschitz continuity (in view of the above remarks in conjunction with the Radon–Nikodým property), but for general spaces Y this is certainly not true: for example, $t \mapsto A(t)$ with

$$A(t)g := f(t)g \quad (g \in C(I, \mathbb{C})) \qquad (f(t) := (t - \,.\,)\chi_{[0,t]}(\,.\,) \in C(I, \mathbb{C}))$$

is Lipschitz continuous from I to $L(X,Y)$ ($X = Y := C(I, \mathbb{C})$), but not $W_*^{1,\infty}$-regular because $t \mapsto A(t)g$ is non-differentiable at every $t \in (0,1)$ for $g := 1$ (Example 1.2.8 of [10]). See also Example 2.2.12 for a similar counterexample. A simple and important criterion for $W_*^{1,\infty}$-regularity is furnished by the following proposition.

Proposition 2.1.1. *Suppose $J \ni t \mapsto A(t) \in L(X,Y)$ is continuously differentiable w.r.t. the strong or weak operator topology, where J is a compact interval. Then $t \mapsto A(t)$ is in $W_*^{1,\infty}(J, L(X,Y))$.*

Proof. It is well-known that a weakly continuous map $J \to Y$ is almost separably valued, whence $t \mapsto A'(t)x$ is measurable (by Pettis' characterization of measurability). With the help of the Hahn–Banach theorem the conclusion readily follows. ■

It follows from Lebesgue's differentiation theorem that $W_*^{1,p}$-derivatives are essentially unique, more precisely: if $t \mapsto A(t)$ is in $W_*^{1,p}(J, L(X,Y))$ for a $p \in [1,\infty) \cup \{\infty\}$ and B_1, B_2 are two $W_*^{1,p}$-derivatives of A, then one has for every $x \in X$ that $B_1(t)x = B_2(t)x$ for almost every $t \in J$. It should be emphasized that this last condition does *not* imply that $B_1(t) = B_2(t)$ for almost every $t \in J$. (Indeed, take $J := [0,1]$, $X := \ell^2(J)$ and define

$$A(t) := 0 \text{ as well as } B_1(t)x := \langle e_t, x \rangle \, e_t \text{ and } B_2(t)x := 0$$

for $t \in J$ and $x \in X$, where $e_t(s) := \delta_{s\,t}$. Then, for every $x \in X$, $B_1(t)x$ is different from 0 for at most countably many $t \in J$, and it follows that B_1 and B_2 both are $W_*^{1,\infty}$-derivatives of A, but $B_1(t) \neq B_2(t)$ for every $t \in J$.)

In the presented adiabatic theorems for time-independent domains (Section 4.1 and 4.2), we will make much use of the following lemma stating that $W_*^{1,p}$-regularity carries over to products and inverses. It is used implicitly in [36] for $p = 1$ and noted explicitly in the introduction of [68] for $p = \infty$ and for separable spaces. We prove this lemma here since it is not proved in [36] and [68] and, more importantly, since it is not a priori clear – (almost) separability being crucial for measurability – that the separability assumption of [68] is actually not necessary. An analogue of this lemma for SOT- and WOT-continuous differentiability is well-known (and easily proved with the help of the theorem of Banach–Steinhaus).

Lemma 2.1.2. *Suppose that $J = [a,b]$ is compact and $p \in [1,\infty) \cup \{\infty\}$.*

(i) If $t \mapsto A(t)$ is in $W_*^{1,p}(J, L(X,Y))$ and $t \mapsto B(t)$ is in $W_*^{1,p}(J, L(Y,Z))$, then $t \mapsto B(t)A(t)$ is in $W_*^{1,p}(J, L(X,Z))$ and $t \mapsto B'(t)A(t) + B(t)A'(t)$ is a $W_*^{1,p}$-derivative of BA for every $W_*^{1,p}$-derivative A', B' of A or B, respectively.

(ii) If $t \mapsto A(t)$ is in $W^{1,p}_*(J, L(X,Y))$ and $A(t)$ is bijective onto Y for every $t \in J$, then $t \mapsto A(t)^{-1}$ is in $W^{1,p}_*(J, L(Y,X))$ and $t \mapsto -A(t)^{-1}A'(t)A(t)^{-1}$ is a $W^{1,p}_*$-derivative of A^{-1} for every $W^{1,p}_*$-derivative A' of A.

Proof. We first prove the lemma in the special case where $p = 1$, from which the general case will easily follow. In essence, our proof rests upon the following two facts: 1. If $t \mapsto A(t) \in L(X,Y)$ and $t \mapsto B(t) \in L(Y,Z)$ are both strongly measurable, then their product $t \mapsto B(t)A(t) \in L(X,Z)$ is strongly measurable as well (Lemma A 4 of [66]). 2. If $f : J \mapsto X$ is absolutely continuous and differentiable almost everywhere, then f' is integrable and $f(t) = f(a) + \int_a^t f'(\tau)\, d\tau$ for every $t \in J$ (Proposition 1.2.3 of [10]). (Alternatively, the proof could also be based on a mollification argument. See Lemma 2.5 of [113], for instance.)

(i) We fix arbitrary $W^{1,1}_*$-derivatives A', B' of A and B and prove that $t \mapsto C(t) := B'(t)A(t) + B(t)A'(t)$ is in $L^1_*(J, L(X,Z))$ and that

$$B(t)A(t)x = B(a)A(a)x + \int_a^t B'(\tau)A(\tau)x + B(\tau)A'(\tau)x\, d\tau$$

for every $t \in J$ and $x \in X$. In order to see that $t \mapsto C(t)$ is strongly measurable, invoke Lemma A 4 of [66] and to see that $t \mapsto \|C(t)\|$ has an integrable majorant, notice that

$$\|C(t)\| \le c(\alpha(t) + \beta(t)) \quad (t \in J),$$

where $c := \max\{\sup_{t\in J} \|A(t)\|, \sup_{t\in J} \|B(t)\|\}$ is finite by the continuity of $t \mapsto A(t), B(t)$ and where α, β denote integrable majorants of A' and B'. So, $t \mapsto C(t)$ is indeed in $L^1_*(J, L(X,Z))$. Also, it is obvious from the absolute continuity of $t \mapsto A(t), B(t)$ that $t \mapsto B(t)A(t)x$ for every $x \in X$ is an absolutely continuous map $J \to Z$, and it therefore remains to show that $t \mapsto B(t)A(t)x$ for every $x \in X$ is differentiable almost everywhere with (almost-everywhere) derivative $t \mapsto C(t)x$ (Proposition 1.2.3 of [10]). So, let $x \in X$. Choose a countable dense subset $\{t_k : k \in \mathbb{N}\}$ of J and define

$$N_x := N_{A(\,.\,)x} \cup \bigcup_{k\in\mathbb{N}} N_{B(\,.\,)A(t_k)x} \cup N_\beta$$

where $N_{A(\,.\,)x}$ and $N_{B(\,.\,)y}$ denote the sets of those $t \in J$ for which $(A(t+h)x - A(t)x)/h$ and $(B(t+h)y - B(t)y)/h$ do not converge to $A'(t)x$ and $B'(t)y$ as $h \to 0$, respectively, and where

$$N_\beta := \Big\{ t \in J : \frac{1}{h}\int_t^{t+h} \beta(\tau)\, d\tau \nrightarrow \beta(t) \ (h \to 0) \Big\}.$$

Since A', B' are $W^{1,1}_*$-derivatives of A and B and since β is integrable, N_x is a null set by Lebesgue's differentiation theorem, and furthermore we have

$$\frac{B(t+h)A(t+h)x - B(t)A(t)x}{h} \longrightarrow C(t)x \quad (h \to 0)$$

for every $t \in J \setminus N_x$. Indeed, let be an arbitrary point of $J \setminus N_x$. We then see first that

$$B(t+h)\frac{A(t+h)x - A(t)x}{h} \longrightarrow B(t)A'(t)x \tag{2.3}$$

because $\tau \mapsto B(\tau)$ is continuous and $t \notin N_{A(\,.\,)x}$, and second that

$$\Big(\frac{B(t+h) - B(t)}{h} - B'(t)\Big)A(t)x \longrightarrow 0 \quad (h \to 0) \tag{2.4}$$

because $\tau \mapsto A(\tau)$ is continuous and $t \notin \bigcup_{k\in\mathbb{N}} N_{B(\,.\,)A(t_k)x} \cup N_\beta$. In order to obtain this second convergence, notice that for every $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that

$$\|A(t_k) - A(t)\|_{X,Y} < \varepsilon / c_t,$$
$$c_t := \sup_{h\neq 0 \text{ with } t+h\in J} \Big|\frac{1}{h}\int_t^{t+h} \beta(\tau)\,d\tau\Big| \,\|x\|_X + \big\|B'(t)\big\|_{Y,Z}\,\|x\|_X < \infty,$$

so that the left-hand side of (2.4), which can be expressed as

$$\Big(\frac{B(t+h) - B(t)}{h} - B'(t)\Big)A(t_k)x + \frac{1}{h}\int_t^{t+h} B'(\tau)\big(A(t) - A(t_k)\big)x\,d\tau \\ - B'(t)\big(A(t) - A(t_k)\big)x,$$

has norm less than 3ε for $|h| \neq 0$ small enough.

(ii) We fix an arbitrary $W^{1,1}_*$-derivative A' of A and prove that the map $t \mapsto B(t) := -A(t)^{-1}A'(t)A(t)^{-1}$ is in $L^1_*(J, L(Y,X))$ and that

$$A(t)^{-1}y = A(a)^{-1}y - \int_a^t A(\tau)^{-1}A'(\tau)A(\tau)^{-1}y\,d\tau$$

for every $t \in J$ and $y \in Y$. In order to see that $t \mapsto B(t)$ is strongly measurable, invoke Lemma A 4 of [66] and to see that $t \mapsto \|B(t)\|$ has an integrable majorant, notice that

$$\|B(t)\| \leq c^2\alpha(t) \quad (t \in J),$$

where $c := \sup_{t\in J}\big\|A(t)^{-1}\big\|$ is finite by the continuity of $t \mapsto A(t)$ and where α denotes an integrable majorant of A'. So, $t \mapsto B(t)$ is indeed in $L^1_*(J, L(Y,X))$. Also, it is obvious from the absolute continuity of $t \mapsto A(t)$ that $t \mapsto A(t)^{-1}y$ for every $y \in Y$ is an absolutely continuous map $J \to X$, and it therefore remains to show that $t \mapsto A(t)^{-1}y$ for every $y \in Y$ is differentiable almost everywhere with (almost-everywhere) derivative $t \mapsto B(t)y$ (Proposition 1.2.3 of [10]). So, let $y \in Y$. Choose a countable dense subset $\{t_k : k \in \mathbb{N}\}$ of J and define

$$N_y := \bigcup_{k\in\mathbb{N}} N_{A(\,.\,)A(t_k)^{-1}y} \cup N_\alpha$$

where $N_{A(\,.\,)x}$ and N_α are defined as in (i) above. Since A' is a $W_*^{1,1}$-derivative of A and since α is integrable, N_y is a null set by Lebesgue's differentiation theorem, and furthermore we have

$$\frac{A(t+h)^{-1}y - A(t)^{-1}y}{h} - B(t)y = -A(t+h)^{-1}\Big(\frac{A(t+h)-A(t)}{h} - A'(t)\Big)A(t)^{-1}y$$
$$-\big(A(t+h)^{-1} - A(t)^{-1}\big)A'(t)A(t)^{-1}y \longrightarrow 0 \quad (h \to 0)$$

for every $t \in J \setminus N_y$. Indeed, $\tau \mapsto A(\tau)^{-1}$ is continuous and

$$\Big(\frac{A(t+h)-A(t)}{h} - A'(t)\Big)A(t)^{-1}y \longrightarrow 0 \quad (h \to 0) \tag{2.5}$$

for every $t \in J \setminus N_y$ by the same arguments as for (2.4).

With the special case $p = 1$ at hand, we can now also prove the lemma for general $p \in [1,\infty) \cup \{\infty\}$. Indeed, for $p \in [1,\infty) \cup \{\infty\}$, every $W_*^{1,p}$-regular map on the compact interval J is, in particular, $W_*^{1,1}$-regular and every $W_*^{1,p}$-derivative of such a map is also a $W_*^{1,1}$-derivative. So, in the situation of (i) or (ii),

$$t \mapsto B'(t)A(t) + B(t)A'(t) \quad \text{ or } \quad t \mapsto -A(t)^{-1}A'(t)A(t)^{-1} \tag{2.6}$$

is a $W_*^{1,1}$-derivative of BA or A^{-1}, respectively. Since $t \mapsto A'(t), B'(t)$ or $t \mapsto A'(t)$ have p-integrable majorants in the situation of (i) or (ii), the same is true of the maps in (2.6) and so these maps are even $W_*^{1,p}$-derivatives of BA and A^{-1} respectively, as desired. ■

We shall need the following simple product rule very often: it will always be used for estimating the difference of two evolution systems and for establishing adiabaticity of evolution systems. And furthermore, it will take the role of Lemma 2.1.2 in the adiabatic theorems for time-dependent domains (Section 5.1).

Lemma 2.1.3. *Suppose $C(t)$ is a bounded linear map in X for every $t \in J = [a,b]$, let $t_0 \in [a,b)$, and let Y_{t_0} be a dense subspace of X. Suppose that $t \mapsto C(t)y$ is right differentiable at t_0 for all $y \in Y_{t_0}$ and that the map $f : J \to X$ is right differentiable at t_0 and $f(t_0) \in Y_{t_0}$. Suppose finally that $\sup_{t \in J_{t_0}} \|C(t)\| < \infty$ for a neighbourhood J_{t_0} of t_0. Then $t \mapsto C(t)f(t)$ is right differentiable at t_0 with right derivative*

$$\partial_+(C(\,.\,)f(\,.\,))(t_0) = \partial_+C(t_0)f(t_0) + C(t_0)\partial_+f(t_0).$$

Proof. We have

$$\frac{C(t_0+h)f(t_0+h) - C(t_0)f(t_0)}{h}$$
$$= C(t_0+h)\frac{f(t_0+h)-f(t_0)}{h} - \frac{C(t_0+h)f(t_0) - C(t_0)f(t_0)}{h}$$

for positive and sufficiently small h. Since $\sup_{t \in J_{t_0}} \|C(t)\| < \infty$ and Y_{t_0} is dense, we easily get that $C(t_0+h) \longrightarrow C(t_0)$ as $h \searrow 0$ w.r.t. the strong operator topology, and the desired conclusion follows. ■

We shall also need the following lemma on the relation between right differentiability and the class $W^{1,\infty}$, which is a generalized version of Corollary 2.1.2 of [104]. It will be used very often – especially in Section 5.1 – in conjunction with the lemma above: Lemma 2.1.3 will yield right differentiability of a given product and Lemma 2.1.4 will then yield an integral representation for this product.

Lemma 2.1.4. *Suppose $f : J \to X$ is a continuous, right differentiable map on a compact interval $J = [a, b]$ such that the right derivative $\partial_+ f : [a, b) \to X$ is bounded. Then f is in $W^{1,\infty}(J, X)$ and*

$$f(t) = f(t_0) + \int_{t_0}^{t} \partial_+ f(\tau)\, d\tau$$

for all $t_0, t \in J$. In particular, if $\partial_+ f$ is even continuous and continuously extendable to the right endpoint b, then f is continuously differentiable.

Proof. Since $\partial_+ f$ is measurable (as the pointwise limit of a sequence of difference quotients) and $\partial_+ f$ is bounded, we have only to show that

$$\int_{(a,b)} f(t)\varphi'(t)\, dt = -\int_{(a,b)} \partial_+ f(t)\varphi(t)\, dt \quad \text{for all } \varphi \in C_c^\infty((a, b), \mathbb{C})$$

in order to get $f \in W^{1,\infty}(J, X)$ (from which, in turn, the asserted integral representation follows by the continuity of f). So, let $\varphi \in C_c^\infty((a, b), \mathbb{C})$ and denote by $\tilde{\varphi}$ and $\tilde{f}$ the zero extension of φ and f to the whole real line. Then

$$\begin{aligned}\int_{(a,b)} f(t)\varphi'(t)\, dt &= \lim_{h \searrow 0} \int_{\mathbb{R}} \tilde{f}(t) \frac{\tilde{\varphi}(t-h) - \tilde{\varphi}(t)}{-h}\, dt \\ &= -\lim_{h \searrow 0} \int_{\mathbb{R}} \frac{\tilde{f}(t+h) - \tilde{f}(t)}{h} \tilde{\varphi}(t)\, dt = \int_{(a,b)} \partial_+ f(t)\varphi(t)\, dt,\end{aligned}$$

since $\operatorname{supp} \varphi \subset [a + \delta, b - \delta]$ for some $\delta > 0$ and since

$$\left\| \frac{f(t+h) - f(t)}{h} \varphi(t) \right\| \le \sup_{\tau \in (a,b)} \|\partial_+ f(\tau)\| \, \|\varphi\|_\infty < \infty$$

for all $t \in [a+\delta, b-\delta]$ and $h \in (0, \delta)$ (which mean value estimate can be derived from the continuity and right differentiability of f in a similar way as Lemma III.1.36 of [67]). ■

2.1.2 Well-posedness and evolution systems

We recall here from [41] the fundamental concepts of well-posedness and (solving) evolution systems for non-autonomous linear evolution equations

$$x' = A(t)x \ (t \in [s, b]) \quad \text{and} \quad x(s) = y \tag{2.7}$$

for densely defined linear operators $A(t) : D(A(t)) \subset X \to X$ $(t \in [a, b])$ and initial values $y \in Y_s \subset D(A(s))$ at initial times $s \in [a, b)$. Well-posedness of such evolution equations means, of course, something like unique (classical) solvability with continuous dependence of the initial data. Instead of giving a direct definition (especially of continuous dependence on the data) (Definition VI.9.1 of [41]), we give here a more convenient equivalent definition using so-called evolution systems (Definition VI.9.2 of [41]).

If $A(t) : D(A(t)) \subset X \to X$ is a densely defined linear operator and Y_t is a dense subspace of $D(A(t))$ for every t in a compact interval $J = [a, b]$, then the initial value problems (2.7) for A are called *well-posed on (the spaces)* Y_t if and only if there exists a *solving evolution system for* A *on (the spaces)* Y_t or, for short, an *evolution system for* A *on* Y_t. Such an evolution system for A on Y_t is, by definition, a family U of bounded operators $U(t, s)$ in X for $(s, t) \in \Delta_J := \{(s, t) \in J^2 : s \le t\}$ such that

(i) for every $s \in [a, b)$ and $y \in Y_s$, the map $[s, b] \ni t \mapsto U(t, s)y$ is a continuously differentiable solution to the initial value problem (2.7) satisfying $U(t, s)y \in Y_t$ for all $t \in [s, b]$ (where a solution to (2.7) is a differentiable map $x : [s, b] \to X$ such that $x(t) \in D(A(t))$ and $x'(t) = A(t)x(t)$ for all $t \in [s, b]$ and $x(s) = y$),

(ii) $U(t, s)U(s, r) = U(t, r)$ for all $(r, s), (s, t) \in \Delta_J$ and $\Delta_J \ni (s, t) \mapsto U(t, s)x$ is continuous for all $x \in X$.

A family U of bounded operators $U(t, s)$ for $(s, t) \in \Delta_J$ that satisfies the chain and continuity property (ii) but not necessarily the solution property (i) above is called an *evolution system* as such. In our adiabatic theorems below, we will always be in the situation $Y_t = D(A(t))$ for $t \in J$.

If, for a given family A of densely defined operators $A(t) : D(A(t)) \subset X \to X$ and dense subspaces Y_t of $D(A(t))$, there exists any solving evolution system, then it is already unique. In order to see this we need the following simple lemma, which will always be used when the difference of two evolution systems has to be dealt with.

Lemma 2.1.5. *Suppose $A(t) : D(A(t)) \subset X \to X$ is a densely defined linear operator and Y_t is a dense subspace of $D(A(t))$ for every $t \in J$. Suppose further that U is an evolution system for A on Y_t. Then, for every $s_0 \in [a, t)$ and every $x_0 \in Y_{s_0}$, the map $[a, t] \ni s \mapsto U(t, s)x_0$ is right differentiable at s_0 with right derivative $-U(t, s_0)A(s_0)x_0$. In particular, if $Y_t = D(A(t)) = D$ for all $t \in J$ and $s \mapsto A(s)x$ is continuous for all $x \in D$, then $[a, t] \ni s \mapsto U(t, s)x$ is continuously differentiable for all $x \in D$.*

Proof. Since $U(t, s)U(s, r) = U(t, r)$ for $(r, s), (s, t) \in \Delta_J$ and since $\Delta_J \ni (s, t) \mapsto U(t, s)$ is strongly continuous, we obtain for every $s_0 \in [a, t)$ and $x_0 \in D(A(s_0))$ that

$$\begin{aligned}\frac{U(t, s_0 + h)x_0 - U(t, s_0)x_0}{h} &= -U(t, s_0 + h)\frac{U(s_0 + h, s_0)x_0 - x_0}{h} \\ &\longrightarrow -U(t, s_0)A(s_0)x_0\end{aligned}$$

as $h \searrow 0$. If $Y_t = D(A(t)) = D$ for all $t \in J$ and $s \mapsto A(s)x$ is continuous for all $x \in D$, then the asserted continuous differentiability of $[a, t] \ni s \mapsto U(t, s)x$ for $x \in D$ follows by Lemma 2.1.4 or Corollary 2.1.2 of [104]. ■

Corollary 2.1.6. *Suppose $A(t) : D(A(t)) \subset X \to X$ is a densely defined linear operator and Y_t is a dense subspace of $D(A(t))$ for every $t \in J$. If U and V are two evolution systems for A on Y_t, then $U = V$.*

Proof. If U and V are two evolution systems for A on the spaces Y_t, then for every $(s,t) \in \Delta_J$ with $s < t$ and $y \in Y_s$ the map $[s,t] \ni \tau \mapsto U(t,\tau)V(\tau,s)y$ is continuous and right differentiable with vanishing right derivative by virtue of Lemma 2.1.5 and Lemma 2.1.3. With the help of Lemma 2.1.4 it then follows that

$$V(t,s)y - U(t,s)y = U(t,\tau)V(\tau,s)y\big|_{\tau=s}^{\tau=t} = 0,$$

which by the density of Y_s in X implies $U(\,.\,,s) = V(\,.\,,s)$. Since s was arbitrary in $[a,b)$ we obtain $U = V$, as desired. ■

In the special – autonomous – situation where $A(t) = A_0$ $(t \in J)$ for some densely defined operator A_0 in X, it is easy to see that the initial value problems (2.7) for A are well-posed on $D(A_0)$ if and only if A_0 is a semigroup generator in X. In this case, the (uniquely existing!) evolution system U for A on $D(A_0)$ is given by the semigroup:

$$U(t,s) = e^{A_0(t-s)} \quad ((s,t) \in \Delta_J).$$

In general non-autonomous situations, however, a characterization of well-posedness – particularly, a characterization as simple as in the autonomous case – still seems out of reach [93]. See, for instance, Example VI.9.4 of [41].

2.1.3 Stable families of operators and admissible subspaces

We also briefly recall from [65] or [104] the concepts of stable families of operators, of parts of operators in a subspace, and of admissible subspaces. A family A of linear operators $A(t) : D(A(t)) \subset X \to X$ (where $t \in J$) is called (M,ω)-*stable* (for some $M \in [1,\infty)$ and $\omega \in \mathbb{R}$) if and only if $A(t)$ generates a strongly continuous semigroup on X for every $t \in J$ and

$$\left\| e^{A(t_n)s_n} \cdots e^{A(t_1)s_1} \right\| \le M e^{\omega(s_1 + \cdots + s_n)} \tag{2.8}$$

for all $s_1, \dots, s_n \in [0,\infty)$ and all $t_1, \dots, t_n \in J$ satisfying $t_1 \le \cdots \le t_n$ with arbitrary $n \in \mathbb{N}$. Alternatively, (M,ω)-stability could be defined with the help of the resolvents of the $A(t)$ (Proposition 3.3 of [65]) or certain monotonic families of norms $\|\,.\,\|_t$ (Proposition 1.3 of [102]): for instance, A is (M,ω)-stable if and only if $(\omega,\infty) \subset \rho(A(t))$ for all $t \in J$ and

$$\left\| (\lambda - A(t_n))^{-1} \cdots (\lambda - A(t_1))^{-1} \right\| \le \frac{M}{(\lambda - \omega)^n} \tag{2.9}$$

for all $\lambda \in (\omega,\infty)$ and all $t_1, \dots, t_n \in J$ satisfying $t_1 \le \cdots \le t_n$ with arbitrary $n \in \mathbb{N}$.

Clearly, a family A of linear operators in X is $(1,0)$-stable if and only if each member $A(t)$ of the family generates a contraction semigroup on X. It is simple to produce examples – relevant to adiabatic theory – of $(M,0)$-stable families that fail to be $(1,0)$-stable (Example 4.1.3).

When it comes to estimating perturbed evolution systems in Section 4.1 and 4.2, the following important criterion for stability will always – and tacitly – be used: if A is an (M,ω)-stable family of linear operators $A(t) : D(A(t)) \subset X \to X$ for $t \in J$, $B(t)$ is a bounded operator in X for $t \in J$ and $b := \sup_{t \in J} \|B(t)\|$ is finite, then $A + B$ is $(M, \omega + Mb)$-stable (Proposition 3.5 of [65]). See also Proposition 3.4 and Proposition 4.4 of [65] for further criteria for stability that play an important role in the classic well-posedness theorems of Kato for group and semigroup generators, respectively (Section 2.1.4.2). In our examples to adiabatic theory the following lemma will be important.

Lemma 2.1.7. *Suppose A_0 is an (M_0,ω_0)-stable family of operators $A_0(t) : D(A_0(t)) \subset X \to X$ for $t \in J$ and $R(t) : X \to X$ for every $t \in J$ is a bijective bounded operator such that $t \mapsto R(t)$ is in $W^{1,\infty}_*(J, L(X))$. Then the family A with $A(t) := R(t)^{-1}A_0(t)R(t)$ is (M,ω)-stable for some $M \in [1,\infty)$ and $\omega = \omega_0$.*

Proof. We may assume that $\omega_0 = 0$, since $(\tilde{M},\tilde{\omega})$-stability of a family $\tilde{A}$ is equivalent to the $(\tilde{M},0)$-stability of $\tilde{A} - \tilde{\omega}$. Set $\|x\|_t := d\, e^{-M_0 ct} \, \|R(t)x\|_{0\,t}$ for $x \in X$ and $t \in J$, where

$$c := \operatorname*{ess\text{-}sup}_{t \in J} \left\| R'(t)R(t)^{-1} \right\| \quad \text{and} \quad d := \sup_{t \in J} e^{M_0 ct} \left\| R(t)^{-1} \right\|$$

and the $\| \,.\, \|_{0\,t}$ are norms on X associated with A_0 according to Proposition 1.3 of [102]. It then easily follows – in a similar way as in the proof of Theorem 4.2 of [71] – that the norms $\| \,.\, \|_t$ satisfy the conditions (a), (b), (c) of Proposition 1.3 in [102] for the family A with a certain $M \in [1,\infty)$ and therefore A is $(M,0)$-stable, as desired. ■

If A is an arbitrary operator in X and Y is an arbitrary subspace of X, then the operator $\tilde{A}$ defined by

$$D(\tilde{A}) := \big\{ y \in D(A) \cap Y : Ay \in Y \big\} \quad \text{and} \quad \tilde{A}y := Ay \quad (y \in D(\tilde{A}))$$

is called the *part of A in Y* or, for short, the *Y-part of A*. If A is a semigroup generator on X, then a subspace Y of $X = (X, \| \,.\, \|)$ endowed with a norm $\| \,.\, \|_*$ is called *A-admissible* if and only if

(i) $(Y, \| \,.\, \|_*)$ is a Banach space densely and continuously embedded in $(X, \| \,.\, \|)$,

(ii) $e^{As} Y \subset Y$ for all $s \in [0,\infty)$ and the restriction $e^{A\,\cdot}|_Y$ is a strongly continuous semigroup in $(Y, \| \,.\, \|_*)$.

In this case, the semigroup $e^{A\,\cdot}|_Y$ is generated by the part $\tilde{A}$ of A in Y (Proposition 2.3 of [65]). See also Proposition 2.4 of [65] for a useful criterion for a densely and continuously embedded subspace to be A-admissible.

2.1.4 Some fundamental well-posedness results from the literature

With the preliminaries from the previous subsections at hand, we can now discuss some fundamental well-posedness results for (2.7) from the literature, in particular, the classic well-posedness theorems of Kato for semigroup and group generators, respectively. We confine ourselves to the so-called *hyperbolic case*, that is, the case of general (semi)group generators $A(t)$, and we refer to [125], [104], [4] for the so-called *parabolic case*, that is, the case of generators $A(t)$ of holomorphic semigroups.

2.1.4.1 Case of time-independent domains

We begin with the case of operators $A(t)$ with time-independent domains. In this case, the following condition will be important and extensively used.

Condition 2.1.8. *$A(t) : D \subset X \to X$ for every $t \in I$ is a densely defined closed linear map such that A is (M, ω)-stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$ and such that $t \mapsto A(t)$ is in $W_*^{1,1}(I, L(Y, X))$, where Y is the space D endowed with the graph norm of $A(0)$.*

As was noted in Proposition 2.1.1 above, the $W_*^{1,1}$-regularity requirement of Condition 2.1.8 is satisfied if, for instance, $t \mapsto A(t)$ is strongly or weakly continuously differentiable. It follows from a classic theorem of Kato (Theorem 1 of [66]), which we reproduce below for convenience, that Condition 2.1.8 guarantees well-posedness.

Theorem 2.1.9 (Kato)**.** *Suppose $A(t) : D \subset X \to X$ for every $t \in I$ is a linear map such that Condition 2.1.8 is satisfied. Then there is a unique evolution system U for A on D and, moreover, $\Delta \ni (s, t) \mapsto U(t, s)|_Y \in L(Y)$ is strongly continuous and the following estimate holds true:*

$$\|U(t, s)\| \leq M e^{\omega(t-s)} \text{ for all } (s, t) \in \Delta.$$

Proof. Setting $S(t) := A(t) - (\omega + 1)$ for $t \in I$ and $Y := D$ endowed with the graph norm of $A(0)$, one easily verifies the assumptions of Theorem 2.1.11 below. ■

See also [68] (Section 1) for a weaker $W_*^{1,\infty}$-version of the theorem above. Important precursors of the above theorem are to be found in Kato's and Kisyński's papers [62] (Theorem 4) and [71] (Theorem 3.0): in these theorems, well-posedness is established under the condition that $t \mapsto A(t)$ is strongly or weakly continuously differentiable, respectively. Another – and perhaps the first – noteworthy precursor is to be found in Phillips' paper [105] (Theorem 6.2) where $A(t)$ is assumed to be of the form $A(t) = A_0 + B(t)$ with a contraction semigroup generator A_0 and bounded operators $B(t)$ such that $t \mapsto B(t)$ is strongly continuously differentiable.

Condition 2.1.8 does not only guarantee well-posedness, but it is also essentially everything we have to require of A in the adiabatic theorems of Section 4.1 and 4.2 for time-independent domains: indeed, we have only to add the requirement that $\omega = 0$ to Condition 2.1.8 to arrive at the assumptions on A of these theorems. In most adiabatic theorems in the literature – for example those of [13], [11], [131], [132], [1], [2] or [12] – by contrast, the assumptions on A rest upon Yosida's theorem (Theorem XIV.4.1 of [141]): in these theorems it is required of A that each $A(t)$ generate a contraction semigroup on X and that an appropriate translate $A - z_0$ of A satisfy the rather involved hypotheses of Yosida's theorem (or – for example in the case of [13] or [12] – more convenient strengthenings thereof). It is shown in Section 2.2.2 that this is the case if and only if $A(t) - z_0$, for every $t \in I$, is a boundedly invertible generator of a contraction semigroup on X and

$$t \mapsto A(t)x \text{ is continuously differentiable for all } x \in D.$$

In particular, it follows that the regularity conditions on A of the adiabatic theorems of the present thesis are more general than the respective assumptions of the previously known adiabatic theorems – and, of course, they are also striclty more general (which is demonstrated by the examples of Section 4.1 and 4.2).

2.1.4.2 Case of time-dependent domains

We now turn to the case of operators $A(t)$ with time-dependent domains. In this case, a first simple – but nonetheless useful – well-posedness result deals with operators $A(t)$ that arise through similarity transformations from operators $A_0(t)$ with time-independent domains: $A(t) = R(t)^{-1}A_0(t)R(t)$. See [63] (Theorem 2) or [71] (Theorem 4.2).

Corollary 2.1.10. *Suppose A_0 is a family of linear maps $A_0(t) : D \subset X \to X$ that satisfies Condition 2.1.8 and let $A(t) := R(t)^{-1}A_0(t)R(t)$ for $t \in I$, where $t \mapsto R(t)$ is in $W_*^{2,1}(I, L(X))$ and $R(t)$ is bijective onto X for every $t \in I$. Then there is a unique evolution system U for A on $D(A(t))$.*

Proof. Since $t \mapsto A_0(t) + R'(t)R(t)^{-1}$ is in $W_*^{1,1}(I, L(Y, X))$ by Lemma 2.1.2 and since $A_0 + R'R^{-1}$ is $(M, \omega + Mb)$-stable with

$$b := \sup_{t \in I} \left\| R'(t)R(t)^{-1} \right\| < \infty,$$

it follows from Theorem 2.1.9 that there is a unique evolution system $\tilde{U}_0$ for $A_0 + R'R^{-1}$ on D. Set $U(t, s) := R(t)^{-1}\tilde{U}_0(t, s)R(s)$ for $(s, t) \in \Delta$. Then U is an evolution system for A on $D(A(t))$, as is easily verified. ■

In [71] Kisyński uses a (weakened) version of the above corollary (Theorem 4.2 of [71]) to establish well-posedness of (2.7) on the spaces $D(A(t))$ for skew self-adjoint operators $A(t)$ defined by semibounded symmetric sesquilinear forms $a(t)$ with time-independent (form) domain (Theorem 8.1 of [71]). In our adiabatic theorems of Section 5.2, this result will be relevant.

A very important and classic well-posedness result for general semigroup generators with time-dependent domains is given by the following theorem of Kato (Theorem 1 of [66]). It is – to the present day – paradigmatic in the well-posedness theory of (abstract) hyperbolic evolution equations. See [69], [127] or the introduction of the papers [101],

[94], for instance. (See also [24] whose abstract results are, in fact, special cases of the theorem below, as is explained in the third remark of Section 2.3.4 below.) A simplified proof was given by Dorroh in [36].

Theorem 2.1.11 (Kato). *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous group on X such that A is (M, ω)-stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$. Suppose further that Y for every $t \in I$ is an $A(t)$-admissible subspace of X contained in $\cap_{\tau \in I} D(A(\tau))$ and that $A(t)|_Y$ is a bounded operator from Y to X such that*

$$t \mapsto A(t)|_Y$$

is norm continuous. And finally, suppose for each $t \in I$ there exists an isomorphism $S(t)$ from Y onto X and a bounded operator $B(t)$ in X such that

$$S(t)A(t)S(t)^{-1} = A(t) + B(t) \tag{2.10}$$

and such that $t \mapsto S(t)$ is in $W_*^{1,1}(I, L(Y, X))$ and $t \mapsto B(t)$ is in $W_*^{0,1}(I, L(X))$. Then there exists a unique evolution system U for A on Y (and, moreover, $\Delta \ni (s,t) \mapsto U(t,s)|_Y \in L(Y)$ is strongly continuous).

If in the above theorem one requires $t \mapsto B(t)$ to be even strongly continous (as in Theorem 6.1 of [65]), then the regularity condition on $t \mapsto A(t)|_Y$ can be improved from norm to strong continuity. In essence, this improvement is due to Kobayasi [72]. (See the remark after Theorem 2.3.5 below.)

Aside from the above well-posedness theorem for general semigroup generators, there is another important well-posedness theorem of Kato (Theorem 5.2 together with Remark 5.3 of [65]) which is tailored to a certain kind of group – instead of mere semigroup – generators. We will show that also in this result for group generators, the regularity condition on $t \mapsto A(t)|_Y$ can be improved from norm to strong continuity (Theorem 2.3.5).

Theorem 2.1.12 (Kato). *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous group on X such that $A^+ := A(\,.\,)$ and $A^- := -A(1 - \,.\,)$ are (M, ω)-stable for some $M \in [1, \infty)$ and $\omega \in \mathbb{R}$. Suppose further that Y for every $t \in I$ is an $A^\pm(t)$-admissible subspace of X contained in $\cap_{\tau \in I} D(A(\tau))$ and that $A(t)|_Y$ is a bounded operator from Y to X such that*

$$t \mapsto A(t)|_Y$$

is norm continuous. And finally, suppose there exists for each $t \in I$ a norm $\|\,.\,\|_t^\pm$ on Y equivalent to the original norm of Y such that $Y_t^\pm := (Y, \|\,.\,\|_t^\pm)$ is uniformly convex and

$$\|y\|_t^\pm \le e^{c^\pm |t-s|} \|y\|_s^\pm \qquad (y \in Y \text{ and } s, t \in I) \tag{2.11}$$

for some constant $c^\pm \in (0, \infty)$ and such that the Y-part $\tilde{A}^\pm(t)$ of $A^\pm(t)$ generates a quasicontraction semigroup in $Y_t^\pm$, more precisely

$$\left\| e^{\tilde{A}^\pm(t)\tau} y \right\|_t^\pm \le e^{\omega_0 \tau} \|y\|_t^\pm \qquad (y \in Y, \tau \in [0, \infty) \text{ and } t \in I) \tag{2.12}$$

for some t-independent growth exponent $\omega_0 \in \mathbb{R}$. Then there exists a unique evolution system U for A on Y (and, moreover, $\Delta \ni (s,t) \mapsto U(t,s)|_Y \in L(Y)$ is strongly continuous).

In rough terms, the assumptions of the above well-posedness theorem for general semigroup generators and of the above well-posedness theorem for group generators can be, and are often, classified (and memorized) as stability conditions plus regularity conditions. Indeed, in both of the above theorems, the family A is explicitly required to be stable in the base space X and implicitly – through the condition (2.10) or the conditions (2.11) and (2.12), respectively – required to be stable also in a suitably normed subspace Y of all the domains (Proposition 3.4 and Proposition 4.4 of [65], respectively). And then, $t \mapsto A(t)|_Y$ is required, in both theorems, to be continuous and the operators $S(t)$ or the norms $\|\,.\,\|_t^{\pm}$ are required to depend sufficiently regularly on t (namely $W_*^{1,1}$-regularly or as in (2.11)), respectively. In the proofs of the above theorems, the various assumptions play, roughly speaking, the following roles:

- the stability conditions in X and Y together with the regularity condition on $t \mapsto A(t)|_Y$ are used to show that certain standard approximants $U_n(t,s)$ are strongly convergent to an evolution system U on X and that $t \mapsto U(t,s)y$ is right differentiable at s with right derivative $A(s)y$ for $y \in Y$ (Theorem 4.1 of [65]),
- the other conditions are then used to show that $U(t,s)y$ belongs to Y for all $(s,t) \in \Delta$ and $y \in Y$ and that $t \mapsto U(t,s)y$ is continuous in the norm of Y for all $y \in Y$.

Combining these two principal steps, one then concludes that U is a solving evolution system for A on Y, as desired. Compare Theorem 5.4.3 of [104].

2.1.4.3 Series expansion and estimates for perturbed evolutions

In the adiabatic theorems with spectral gap condition of Section 5.1 (especially in the adiabatic theorem of higher order) the following well-expected perturbation result will be needed. It gives a perturbation series expansion for a perturbed evolution system if only this perturbed evolution exists. (See the classical example of Phillips (Example 6.4 of [105]) showing that the existence of the perturbed evolution really has to be required.)

Proposition 2.1.13. *Suppose that $A(t) : D(A(t)) \subset X \to X$ is a densely defined linear map for every $t \in I$ and that $t \mapsto B(t) \in L(X)$ is WOT-continuous. Suppose further that there is an evolution system U for A on $D(A(t))$ and an evolution system V for $A + B$ on $D(A(t))$. Then*

(i) $V(t,s) = \sum_{n=0}^{\infty} V_n(t,s)$, *where* $V_0(t,s) := U(t,s)$ *and*

$$V_{n+1}(t,s)x := \int_s^t U(t,\tau)B(\tau)V_n(\tau,s)x \, d\tau \text{ for } x \in X \text{ and } n \in \mathbb{N} \cup \{0\}.$$

(ii) If there are $M \in [1,\infty)$, $\omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for $(s,t) \in \Delta$, then

$$\|V(t,s)\| \leq Me^{(\omega+Mb)(t-s)}$$

for all $(s,t) \in \Delta$, where $b := \sup_{t \in I} \|B(t)\|$. And if, for every $(s,t) \in \Delta$, $U(t,s)$ is unitary and $B(t)$ is skew symmetric, then $V(t,s)$ is unitary as well.

Proof. Since weakly continuous maps on compact intervals are integrable (see the proof of Proposition 2.1.1), it easily follows that the integrals defining the V_n really exist and that

$$\tilde{V}(t,s) := \sum_{n=0}^{\infty} V_n(t,s)$$

exists uniformly in $(s,t) \in \Delta$. Also, it is easy to see – applying Lemma 2.1.3 and Lemma 2.1.4 to $[s,t] \ni \tau \mapsto U(t,\tau)V(\tau,s)x$ with $x \in D(A(s))$ – that V satisfies the same integral equation as $\tilde{V}$ from which assertion (i) follows. Assertion (ii) is a simple consequence of the series expansion in (i). ■

2.2 Well-posedness for operators with time-independent domains

2.2.1 Introduction

In this section, we are concerned with the well-posedness of non-autonomous linear evolution equations

$$x' = A(t)x \ (t \in [s,1]) \quad \text{and} \quad x(s) = y \tag{2.13}$$

for densely defined linear operators $A(t) : D(A(t)) \subset X \to X$ $(t \in [0,1])$ with time-independent domains $D(A(t)) = D$ and initial values $y \in D = D(A(s))$ at initial times $s \in [0,1)$.

In Section 2.2.2 we show that the regularity conditions of a well-posedness theorem by Yosida from [141], [140] can be simplified quite considerably and we thereby clarify the relation of this theorem with other well-posedness theorems from the literature, in particular, with an early well-posedness result by Kato from [62]. In this latter theorem, well-posedness of (2.13) is established for contraction semigroup generators $A(t)$ with time-independent domains $D(A(t)) = D$ in a normed space under the condition that $t \mapsto A(t)$ be strongly continuously differentiable, that is,

$$t \mapsto A(t)y \text{ is continuously differentiable for all } y \in D. \tag{2.14}$$

(In fact, in [62] this regularity condition is stated in a somewhat implicit way. In the form above it explicitly appears in [63] and then in all the standard textbooks on evolution equations such as [75], [125], [104], [50], [41], for instance.) Some years later, Yosida proved a similar well-posedness theorem, namely for contraction semigroup generators $A(t)$ with time-independent domains in a normed space or, more generally, in a locally convex space. It can be found in Yosida's book [141] on functional analysis (case of normed spaces) and in Yosida's article [140] (case of locally convex spaces) and it is reproduced, in the normed space case, in Reed and Simon's and Blank, Exner and Havlíček's books on mathematical physics, for instance. In large parts of mathematical physics, Yosida's theorem is therefore better known than the above-mentioned well-posedness theorem by Kato. In particular, this is true for adiabatic theory. Yet, the regularity conditions of Yosida's theorem are far more complicated and far less lucid than the simple strong continuous differentiability condition (2.14) from [62] and one might therefore think that, in return, Yosida's conditions should be more general than (2.14). We will see, however, that that they are not: we will show that Yosida's complicated regularity conditions are just equivalent to the simple continuous differentiability condition (2.14) – both in the case of normed spaces (Section 2.2.2.2) and, with a small proviso, also in the case of locally convex spaces (Section 2.2.2.3). In essence, these equivalences – or rather, the non-straightforward implication of these equivalences – are based upon the following observation: Yosida's assumptions require the uniform convergence of certain discrete one-sided difference quotients of the map $t \mapsto A(t)y$ for $y \in D$ and this requirement, by a suitable mean value theorem for disretely one-sided differentiable maps (Section 2.2.2.1), implies the continuous differentiability of $t \mapsto A(t)y$. In spite of the simplicity of this observation, the equivalence of Yosida's regularity conditions with (2.14) cannot be found in the literature. In particular, it is not recorded in the standard textbooks [75], [125], [104], [50], [41] or the review articles [69], [118].

In Section 2.2.3 we slightly generalize a less known well-posedness theorem by Kato from [62]. In this theorem, well-posedness of (2.13) is established for skew self-adjoint operators $A(t)$ with time-independent domains under the condition that $t \mapsto A(t)$ is continuous and of bounded variation; this is a weaker condition than the regularity conditions from the well-posedness theorems for general semigroup generators in general spaces. We show that for quasicontraction group generators $A(t)$ with time-independent domains in a uniformly convex space, this continuity and bounded variation condition still yields the well-posedness of (2.13). In essence, we follow the arguments from [62], but the proof given below is shorter and more direct than the one in [62]. In particular, it is considerably simpler than the known well-posedness proofs for semigroup generators in general spaces. In spite of its relevance to non-autonomous Schrödinger equations, the well-posedness theorem for skew self-adjoint operators from [62] has – quite astonishingly – not made it to the textbook literature, at least not to the standard books [104], [125], [50], [75], [141], [107], [15]. In some cases, for example in Theorem X.71 of [107], this leads to unnecessarily strong assumptions. In this context, also see [138] and [135].

In Section 2.2.4 we show by example that the assumptions of the previously discussed well-posedness theorems for operators with time-independent domains cannot be weakened too much or even dropped. Specifically, we show that

- in the well-posedness theorem for semigroup generators in general spaces (Section 2.1.4.1 and Section 2.2.2), the $W_*^{1,1}$-regularity condition cannot be replaced by Lipschitz continuity

- in the well-posedness theorem for group generators in uniformly convex spaces (Section 2.2.3), the continuity and bounded variation condition cannot be replaced by Hölder continuity of any degree $\alpha < 1$, and the uniform convexity condition cannot be dropped.

In this context, it should be recalled from [124], [120], [104] that in the case of generators $A(t)$ of holomorphic semigroups, Hölder continuity does suffice for well-posedness. In our examples the operators $A(t)$ will be of the simple form $A(t) = A_0 + B(t)$ with a contraction group generator A_0 and bounded perturbing operators $B(t)$, and in one example, A_0 and $B(t)$ will even be skew self-adjoint. It seems that our examples are the first counterexamples to well-posedness involving group generators and, moreover, they are noticeably simpler than the previously known conterexamples from [105] and [41].

2.2.2 Well-posedness for semigroup generators: simplification of a theorem by Yosida

2.2.2.1 Some preparations

We begin by recalling some basic facts about locally convex spaces that are needed in the sequel. We follow the terminology of [111] so that, in particular, a locally convex space will always be assumed to be Hausdorff. In detail, a *locally convex space* is a Hausdorff topological vector space $X = (X, \mathcal{T})$ over $\mathbb{C}$ with a topology $\mathcal{T} = \mathcal{T}_P$ generated by a family P of seminorms on X, that is, a subset U of X is open iff for every $x \in U$ there exists an $\varepsilon > 0$ and a finite subset F of P such that

$$U_{F,\varepsilon} + x \subset U \qquad (U_{F,\varepsilon} := \{y \in X : p(y) < \varepsilon \text{ for all } p \in F\}).$$

So, a net (x_i) in X converges to a point $x \in X$ if and only if $p(x_i - x) \longrightarrow 0$ for all $p \in P$. In particular, every $p \in P$ is a continuous map from X to $[0, \infty)$. Standard examples of locally convex spaces are

- normed spaces $(X, \|\,.\,\|)$ (here, $P = \{p_0\}$ with $p_0 := \|\,.\,\|$),
- spaces $L(X, Y)$ of bounded operators between normed spaces X and Y endowed with the strong or the weak operator topology (that is, $P = \{p_x : x \in X\}$ with $p_x(A) := \|Ax\|$ or $P = \{p_{x^*,x} : x^* \in X^*, x \in X\}$ with $p_{x^*,x}(A) := |\langle x^*, Ax\rangle|$, respectively).

A sequence (x_n) in a locally convex space $X = (X, \mathcal{T}_P)$ with topology generated by P is called *Cauchy sequence* iff for every $p \in P$ and every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $p(x_n - x_m) < \varepsilon$ for all $m, n \geq n_0$; X is called *sequentially complete* iff every Cauchy sequence in X has a limit in X. A subset M of $X = (X, \mathcal{T}_P)$ is called *bounded* iff $p(M)$ is bounded in $[0, \infty)$ for all $p \in P$; a function f with values in X is called *bounded* iff $\operatorname{ran} f$ is bounded.

We now turn to functions f on an interval J with values in a locally convex space $X = (X, \mathcal{T}_P)$ and we spell out, for convenience, basic properties of such functions $f : J \to X$.

- f is uniformly continuous if and only if for every $p \in P$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that $p(f(t') - f(t)) < \varepsilon$ whenever $|t' - t| < \delta$,

- f is differentiable at t_0 if and only if there exists a – then unique – element $x \in X$ (denoted $f'(t_0)$) such that $p\big((f(t) - f(t_0))/(t - t_0) - x\big) \longrightarrow 0$ as $t \to t_0$,
- f on a compact intverval J is Riemann integrable if and only if there exists a – then unique – element $x \in X$ (denoted $\int_a^b f(\tau)\,d\tau$) such that for every $p \in P$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every partition $\{t_0, \dots, t_m\}$ of J with mesh less than δ and every choice of intermediate points $\tau_i \in [t_{i-1}, t_i]$ one has

$$p\Big(\sum_{i=1}^{m} f(\tau_i)(t_i - t_{i-1}) - x\Big) < \varepsilon.$$

As in the special case of normed spaces, one easily verifies that a continuous function f on a compact interval J with values in a sequentially complete locally convex space X is automatically bounded, uniformly continuous, and Riemann integrable.

With the terminology provided above, we can now prove a mean value theorem for discretely left differentiable maps, which will be the crucial ingredient in our simplification of Yosida's conditions for well-posedness. It is a variant of the well-known, elementary fact that a continuous and left differentiable map with vanishing left derivative is constant (Lemma III.1.36 of [67] or Corollary 2.1.2 of [104]).

Lemma 2.2.1. *(i) Suppose X is a sequentially complete locally convex space with topology generated by a family P of seminorms on X and $f : I \to X$ is a continuous map such that the limit $g(t) := \lim_{k\to\infty} k\left(f(t) - f(t - \frac{1}{k})\right)$ exists uniformly in $t \in (0,1]$, that is, the limit exists for every $t \in (0,1]$ and*

$$\sup_{t\in[\frac{1}{k},1]} p\Big(k\left(f(t) - f(t - \frac{1}{k})\right) - g(t)\Big) \longrightarrow 0 \quad (k \to \infty)$$

for every $p \in P$. Then, for every $t \in I$ and $p \in P$,

$$p(f(t) - f(0)) \leq \sup_{\tau\in(0,1)} p(g(\tau))\,(t - 0).$$

(ii) Suppose, in addition, that the limit $g(0) := \lim_{t\searrow 0} g(t)$ exists. Then f is continuously differentiable on I and $f' = g$.

Proof. (i) We show by a simple telescoping sum argument that for every rational point $t = \frac{r}{s} \in (0,1) \cap \mathbb{Q}$ and every $p \in P$ and $\varepsilon > 0$ the estimate

$$p\Big(f\Big(\frac{r}{s}\Big) - f(0)\Big) \leq (M_p + \varepsilon)\,\frac{r}{s} \qquad \big(M_p := \sup_{\tau\in(0,1)} p(g(\tau))\big) \tag{2.15}$$

holds true, from which the asserted estimate immediately follows by the density of $(0,1)\cap\mathbb{Q}$ in I and the continuity of f. So let $\frac{r}{s} \in (0,1)\cap\mathbb{Q}$ with $r, s \in \mathbb{N}$ and let $\varepsilon > 0$. We then write the difference $f(\frac{r}{s}) - f(0)$ as a telescoping sum

$$f\Big(\frac{r}{s}\Big) - f(0) = f\Big(\frac{nr}{ns}\Big) - f(0) = \sum_{i=1}^{nr} f\Big(\frac{i}{ns}\Big) - f\Big(\frac{i}{ns} - \frac{1}{ns}\Big) \quad (n \in \mathbb{N}) \tag{2.16}$$

and, using the uniform convergence assumption, choose $n \in \mathbb{N}$ so large that

$$\sup_{t \in [\frac{1}{ns}, 1)} p\Big(f(t) - f\Big(t - \frac{1}{ns}\Big)\Big) \leq (M_p + \varepsilon)\, \frac{1}{ns}. \tag{2.17}$$

Combining (2.16) and (2.17) we immediately obtain (2.15) and are done.

(ii) Since, by the continuity of f and the uniform convergence assumption, g is continuous on $(0, 1]$ and since the limit $\lim_{t \searrow 0} g(t)$ exists, g is continuous on I. We can therefore define

$$h(t) := f(t) - \int_0^t g(\tau)\, d\tau$$

for $t \in I$. It is straightforward to check that h is continuous and that

$$k\left(h(t) - h(t - \frac{1}{k})\right) = k\left(f(t) - f(t - \frac{1}{k})\right) - g(t) + g(t) - k \int_{t - \frac{1}{k}}^{t} g(\tau)\, d\tau \longrightarrow 0$$

uniformly in $t \in (0, 1]$ as $k \to \infty$. So, it follows by part (i) that h is constant (remember that locally convex spaces are assumed to be Hausdorff) and from this, in turn, the assertion readily follows. ■

We finally turn to linear operators in a locally convex space $X = (X, \mathcal{T}_P)$. A linear operator $A : X \to X$ is called *bounded* iff for every $q \in P$ there exists an $M \in [0, \infty)$ and a finite subset F of P such that

$$q(Ax) \leq M \max\{p(x) : p \in F\} \qquad (x \in X);$$

a linear operator $A : D(A) \subset X \to X$ is called *boundedly invertible* iff it is bijective from $D(A)$ onto X and its inverse A^{-1} is bounded from X to X. We call a family A of densely defined linear operators $A(t) : D(A(t)) \subset X \to X$ to be $\big((M_p)_{p \in P}, \omega\big)$*-stable* (where $M_p \in [1, \infty)$ for every $p \in P$ and $\omega \in \mathbb{R}$) if and only if $\lambda - A(t)$ is boundedly invertible for every $\lambda \in (\omega, \infty)$ and every $t \in J$, and

$$p\Big((\lambda - A(t_n))^{-1} \cdots (\lambda - A(t_1))^{-1}\, x\Big) \leq \frac{M_p}{(\lambda - \omega)^n}\, p(x) \tag{2.18}$$

for all $\lambda \in (\omega, \infty)$ and all $t_1, \dots, t_n \in J$ satisfying $t_1 \leq \cdots \leq t_n$ (with arbitrary $n \in \mathbb{N}$) and all $x \in X$. It is clear by (2.9) that this is a generalization of the notion of (M, ω)-stable families of operators in normed spaces and it follows by Yosida's characterization of generators of so-called equicontinuous semigroups from [141] that every member of an $\big((M_p)_{p \in P}, \omega\big)$-stable family A of operators is the generator of an equicontinuous semigroup. We finally recall that – just as in the case of normed spaces – a linear operator $A : X \to X$ is continuous if and only if it is bounded.

2.2.2.2 Case of normed spaces

In this subsection, we simplify the rather complicated regularity conditions of Yosida's well-posedness theorem from [141] (Theorem XIV.4.1): we show that they are equivalent to the strong continuous differentiability of $t \mapsto A(t)$. In detail, Yosida's sufficient conditions for well-posedness from [141] read as follows. (It should be noted that the (uniform) continuity condition in (i) and the uniform convergence condition in (ii) below already imply the continuity of $(0,1] \ni t \mapsto C(t)x$ and that, therefore, Condition 2.2.2 (iii) is implicit in Conditions 2.2.2 (i) and (ii).)

Condition 2.2.2. *$A(t) : D \subset X \to X$ for every $t \in I$ is a boundedly invertible generator of a contraction semigroup on X such that the following holds true:*

(i) $\{(s', t') \in I^2 : s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s} C(t,s)x$ is bounded and uniformly continuous for all $x \in X$, where $C(t,s) := A(t)A(s)^{-1} - 1$

(ii) $C(t)x := \lim_{k\to\infty} k\, C(t, t - \frac{1}{k})x$ exists uniformly in $t \in (0,1]$ for all $x \in X$

(iii) $(0,1] \ni t \mapsto C(t)x$ is continuous for all $x \in X$.

While the theorem of Yosida is formulated in terms of Condition 2.2.2 above, Yosida's proof of this theorem only requires the following modified – and a priori weaker – Condition 2.2.3 (as can be explicitly seen, for instance, from the exposition in [112]). It is obtained from Yosida's original Condition 2.2.2 by omitting the uniform continuity requirement and by adding the requirement that $(0,1] \ni t \mapsto C(t)x$ be continuously extendable to the left endpoint 0.

Condition 2.2.3. *$A(t) : D \subset X \to X$ for every $t \in I$ is a boundedly invertible generator of a contraction semigroup on X such that the following holds true:*

(i) $\{(s', t') \in I^2 : s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s} C(t,s)x$ is bounded for all $x \in X$, where $C(t,s) := A(t)A(s)^{-1} - 1$

(ii) $C(t)x := \lim_{k\to\infty} k\, C(t, t - \frac{1}{k})x$ exists uniformly in $t \in (0,1]$ for all $x \in X$

(iii) $(0,1] \ni t \mapsto C(t)x$ is continuous and continuously extendable to the left endpoint 0 for all $x \in X$.

Consider finally the following simple conditions which – apart from the inessential bounded invertibility requirement – coincide with Kato's sufficient conditions for well-posedness from [62] (Theorem 4).

Condition 2.2.4. *$A(t) : D \subset X \to X$ for every $t \in I$ is a boundedly invertible generator of a contraction semigroup on X such that $t \mapsto A(t)x$ is continuously differentiable for all $x \in D$.*

With the help of Lemma 2.2.1 we can now prove the equivalence of the above sufficient conditions for the well-posedness of (2.13).

Theorem 2.2.5. *Condition 2.2.2, Condition 2.2.3, and Condition 2.2.4 are equivalent to each other.*

Proof. Suppose that Condition 2.2.2 is satisfied. It is easy to see, using the uniform continuity in Condition 2.2.2 (i) and the uniform convergence in Condition 2.2.2 (ii), that $(0,1] \ni t \mapsto C(t)x$ is uniformly continuous, whence Condition 2.2.3 is satisfied.

Suppose now that Condition 2.2.3 is satisfied and let $x \in D$. We show that the map $t \mapsto f(t) = A(t)x$ satisfies the hypotheses of Lemma 2.2.1. As a first step one concludes from the the boundedness in Condition 2.2.3 (i) that f is continuous (hence uniformly continuous). Indeed, for every $t \in [0,1]$, one has

$$f(t+h) - f(t) = \Big(A(t+h)A(t)^{-1} - 1\Big)A(t)x = h \cdot \frac{1}{h}\,C(t+h,t)A(t)x \longrightarrow 0 \qquad (2.19)$$

as $h \to 0$. As a second step one deduces from the uniform convergence in Condition 2.2.3 (ii), the uniform continuity of f just proved, and the boundedness in Condition 2.2.3 (i) that

$$k\Big(f(t) - f\big(t - \frac{1}{k}\big)\Big) = k\,C\big(t, t - \frac{1}{k}\big)f\big(t - \frac{1}{k}\big) \longrightarrow C(t)f(t) \quad (k \to \infty) \qquad (2.20)$$

for every $t \in (0,1]$ and that this convergence is uniform in $t \in (0,1]$. Indeed, by Condition 2.2.3 (i) and the uniform boundedness principle, there exists a finite constant c such that

$$\Big\|k\,C(t, t - \frac{1}{k})x\Big\| \leq c\,\|x\| \quad \text{and hence also} \quad \|C(t)x\| \leq c\,\|x\| \qquad (2.21)$$

for all $x \in X$ and all $t \in (0,1]$. In order to see that the convergence

$$k\,C\big(t, t - \frac{1}{k}\big)\Big(f\big(t - \frac{1}{k}\big) - f(t)\Big) \longrightarrow 0 \quad (k \to \infty) \qquad (2.22)$$

is uniform in $t \in (0,1]$, use the uniform continuity of f and (2.21). In order to see that the convergence

$$\Big(k\,C\big(t, t - \frac{1}{k}\big) - C(t)\Big)f(t) \longrightarrow 0 \quad (k \to \infty) \qquad (2.23)$$

is uniform, use the uniform continuity of f again to see that for every $\varepsilon > 0$ there exist finitely many points $t_1, \dots, t_m$ of I such that $\min\{\|f(t) - f(t_i)\| : i \in \{1, \dots, m\}\} < \varepsilon/3c$ for every $t \in I$, and then use Condition 2.2.3 (ii) to see that there exists a $k_\varepsilon \in \mathbb{N}$ such that

$$\sup_{t \in [\frac{1}{k}, 1]} \Big\|\Big(k\,C\big(t, t - \frac{1}{k}\big) - C(t)\Big)f(t_i)\Big\| < \varepsilon/3$$

for $k \geq k_\varepsilon$ and all $i \in \{1, \dots, m\}$. Combining (2.22) and (2.23) one then obtains the uniform convergence in (2.20). As a third step one finally observes that the limit map

$(0,1] \ni t \mapsto C(t)f(t)$ in (2.20) is continuously extendable to the left endpoint 0 by Condition 2.2.3 (iii). We have thus verified all hypotheses of Lemma 2.2.1, and this lemma yields the continuous differentiability of $t \mapsto A(t)x$, that is, Condition 2.2.4.

Suppose finally that Condition 2.2.4 is satisfied. Then $s \mapsto A(s)A(0)^{-1}$ is strongly continuously differentiable and hence

$$s \mapsto \big(A(s)A(0)^{-1}\big)^{-1} = A(0)A(s)^{-1}$$

is norm continuous. Since $A'(\tau)A(s)^{-1}x = A'(\tau)A(0)^{-1}\,A(0)A(s)^{-1}x$ for $(s,\tau) \in I^2$ and $x \in X$, we see that $I^2 \ni (\tau, s) \mapsto A'(\tau)A(s)^{-1}x$ is continuous and hence

$$\{s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\,C(t,s)x = \frac{1}{t-s}\int_s^t A'(\tau)A(s)^{-1}x\,d\tau, \tag{2.24}$$

extends to a continuous map on the whole of I^2. And from this, in turn, Condition 2.2.2 readily follows. ■

2.2.2.3 Case of locally convex spaces

In this subsection we extend the simplification of Yosida's regularity conditions to the case of locally convex spaces: we show that the regularity conditions of Yosida's well-posedness theorem from [140] are equivalent to the strong continuous differentiability of $t \mapsto A(t)$ and a certain boundedness condition (namely, (2.26) which, in the special case of normed spaces, is implicit in the strong continuous differentiability condition). In detail, Yosida's sufficient conditions for well-posedness from [140] read as follows. (It should be noted that (2.25) in (i) and the (uniform) convergence condition in (ii) below already imply that $p(C(t)x) \leq c_p\,p(x)$ for all $x \in X$ and $t \in (0,1]$ and hence that Condition 2.2.6 (iii) is implicit in Conditions 2.2.6 (i) and (ii).)

Condition 2.2.6. *$A(t) : D \subset X \to X$, for every $t \in I$, is a boundedly invertible densely defined linear map in a sequentially complete locally convex space X over $\mathbb{C}$. A is $\big((M_p)_{p\in P}, 0\big)$-stable for a family P of seminorms on X generating the topology of X, and the following holds true:*

(i) $\{(s',t') \in I^2 : s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\,C(t,s)x$ is bounded and uniformly continuous for all $x \in X$, and for every $p \in P$ there is a constant $c_p \in [0,\infty)$ such that

$$p\Big(\frac{1}{t-s}\,C(t,s)x\Big) \leq c_p\,p(x) \tag{2.25}$$

for all $(s,t) \in \{s' \neq t'\}$ and all $x \in X$, where $C(t,s) := A(t)A(s)^{-1} - 1$

(ii) $C(t)x := \lim_{k\to\infty} k\,C(t, t-\frac{1}{k})x$ exists uniformly in $t \in (0,1]$ for all $x \in X$

(iii) $C(t)$ is a bounded linear map for every $t \in (0,1]$.

Consider now the following simpler conditions.

Condition 2.2.7. *$A(t) : D \subset X \to X$, for every $t \in I$, is a boundedly invertible densely defined linear map in a sequentially complete locally convex space X over $\mathbb{C}$. A is $\big((M_p)_{p\in P}, 0\big)$-stable for a family P of seminorms on X generating the topology of X, $t \mapsto A(t)x$ is continuously differentiable for all $x \in D$, and for every $p \in P$ there is a constant $c_p \in [0,\infty)$ such that*

$$p\Big(\frac{1}{t-s}\, C(t,s)x\Big) \le c_p\, p(x) \tag{2.26}$$

for all $(s,t) \in \{s' \neq t'\}$ and all $x \in X$ (which last estimate is satisfied if, for instance, $p(A'(t)A(s)^{-1}x) \le c_p\, p(x)$ for all $(s,t) \in I^2$ and all $x \in X$).

With the help of Lemma 2.2.1 we can now prove the equivalence of the above conditions.

Theorem 2.2.8. *Condition 2.2.6 and Condition 2.2.7 are equivalent to each other.*

Proof. Suppose that Condition 2.2.6 is satisfied and let $x \in D$. We show in much the same way as in the previous subsection that the map $t \mapsto f(t) = A(t)x$ satisfies the hypotheses of Lemma 2.2.1. In order to see that f is continuous and that

$$k\Big(f(t) - f\big(t - \frac{1}{k}\big)\Big) = k\, C\big(t, t-\frac{1}{k}\big) f\big(t - \frac{1}{k}\big) \longrightarrow C(t)f(t) \quad (k \to \infty) \tag{2.27}$$

uniformly in $t \in (0,1]$, one argues in the same way as in the proof of Theorem 2.2.5 – the only thing to notice here is that the estimates (2.21) carry over to the present case of sequentially complete locally convex spaces (although the principle of uniform boundedness used in the derivation of (2.21) would be available in the present setting only if the spaces were additionally barreled):

$$p\Big(k\, C(t, t-\frac{1}{k})x\Big) \le c_p\, p(x) \quad \text{and hence also} \quad p(C(t)x) \le c_p\, p(x) \tag{2.28}$$

for all $x \in X$ and $t \in (0,1]$ by virtue of the estimate (2.25) required in Condition 2.2.6. In order to see that the limit map $(0,1] \ni t \mapsto C(t)f(t)$ in (2.27) is continuously extendable to the left endpoint 0, one uses the uniform continuity in Condition 2.2.6 (i) and the uniform convergence in Condition 2.2.6 (ii) to see that $(0,1] \ni t \mapsto C(t)y$ is uniformly continuous and, hence, that $C(0)y := \lim_{h \searrow 0} C(h)y$ exists for every $y \in X$. It then follows by (2.28) that

$$p(C(t)x) \le c_p\, p(x) \tag{2.29}$$

for all $x \in X$ and all $t \in [0,1]$ (including $t = 0$), which in turn yields the continuity of $[0,1] \ni t \mapsto C(t)f(t)$, as desired. We thus see that all hypotheses of Lemma 2.2.1 are satisfied, and this lemma yields the continuous differentiability of $t \mapsto A(t)x$, that is, Condition 2.2.7.

Suppose now that Condition 2.2.7 is satisfied. We have only to show that $I^2 \ni (\tau, s) \mapsto A'(\tau)A(s)^{-1}x$ is continuous because then

$$\{s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\, C(t,s)x = \frac{1}{t-s} \int_s^t A'(\tau)A(s)^{-1}x \, d\tau, \tag{2.30}$$

extends to a continuous map on the whole of I^2, from which Condition 2.2.6 readily follows as in the previous subsection. In virtue of (2.26) we have

$$p(A(t)A(s)^{-1}x) \le |t-s|c_p\, p(x) + p(x) \le c_p'\, p(x) \tag{2.31}$$

for all $(s,t) \in I^2$ and $x \in X$. Also, $\tau \mapsto A'(\tau)A(0)^{-1}y$ is continuous for $y \in X$ with

$$p(A'(\tau)A(0)^{-1}y) = \lim_{h\to 0} p\Big(\frac{1}{h}C(\tau+h,\tau)A(\tau)A(0)^{-1}y\Big) \le c_p\, p(A(\tau)A(0)^{-1}y) \le c_p c_p'\, p(y)$$

for every $\tau \in I$; and $s \mapsto A(0)A(s)^{-1}x$ is continuous for every $x \in X$ because

$$\begin{aligned} p\big(A(0)A(s+h)^{-1}x - A(0)A(s)^{-1}x\big) &= p\big(A(0)A(s+h)^{-1}\big(A(s)-A(s+h)\big)A(s)^{-1}x\big) \\ &\le c_p'\, p\big(\big(A(s)-A(s+h)\big)A(s)^{-1}x\big) \longrightarrow 0 \end{aligned}$$

as $h \to 0$. The continuity of $I^2 \ni (\tau,s) \mapsto A'(\tau)A(s)^{-1}x$ is now obvious and we are done. ■

In the special case of (complete) normed spaces, the estimate (2.26) is implicit in the strong continuous differentiability of $t \mapsto A(t)$ by (2.24) and the uniform boundedness principle. In the above case of (sequentially complete) locally convex spaces, however, this is no longer true and so the condition (2.26) cannot be dropped. Indeed, even if one additionally assumes that the space be barreled (so that the uniform boundedness principle is available) and that $s \mapsto A(0)A(s)^{-1}x$ be bounded for every $x \in X$, the strong continuous differentiability only yields that for every $p \in P$ there exists a continuous seminorm q such that

$$p\Big(\frac{1}{t-s}\,C(t,s)x\Big) \le q(x) \tag{2.32}$$

for all $(s,t) \in \{s' \neq t'\}$ and all $x \in X$. (Use similar arguments as in the last implication of the proof of Theorem 2.2.5 and exploit the uniform boundedness principle.) It should be noted that (2.32) is not sufficient for Yosida's proof from [140]: for this proof to work it is essential to have on the right-hand side of the estimate the same seminorm as on the left (as in (2.26)) and not just any continuous seminorm.

2.2.3 Well-posedness for group generators in uniformly convex spaces

2.2.3.1 Some preparations

We begin by recalling some definitions and some basic facts about quasicontraction groups, uniformly convex spaces, and functions of bounded variation. A *quasicontraction group in* X is a strongly continuous group $e^{A\,\cdot}$ in X such that for some $\omega_0 \in \mathbb{R}$

$$\big\|e^{A\tau}\big\| \le e^{\omega_0|\tau|} \qquad (\tau \in \mathbb{R}).$$

A normed space $X = (X, \|\,.\,\|)$ is called *uniformly convex* if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|(x+y)/2\| > 1-\delta$ for two normed vectors $x, y \in X$ implies $\|x-y\| < \varepsilon$. Simple examples of uniformly convex spaces are given by

- inner product spaces,
- the sequence spaces $\ell^p(J)$ for $p \in (1,\infty)$, where J is an arbitrary index set, or more generally, the function spaces $L^p(X_0,\mu,E)$ for $p \in (1,\infty)$, where $(X_0,\mathcal{A},\mu)$ is an arbitrary measure space and E a uniformly convex Banach space,
- the Schatten-p classes $S^p(\mathfrak{h})$ for $p \in (1,\infty)$, where $\mathfrak{h}$ is an arbitrary Hilbert space.

It should be noted that uniform convexity is *not* invariant under transition to an equivalent norm, as can be seen by considering $\mathbb{C}^2$ endowed with the ℓ^2-norm (uniformly convex) and $\mathbb{C}^2$ endowed with the ℓ^1- or the ℓ^∞-norm (not uniformly convex), respectively. It is well-known that uniformly convex Banach spaces are reflexive (Milman's theorem). It is also well-known and easy to see that uniformly convex spaces X have the following, so-called *Kadec–Klee property*: if $x \in X$ and (x_n) is a sequence in X such that $x_n \longrightarrow x$ weakly and $\|x_n\| \longrightarrow \|x\|$, then $\|x_n - x\| \longrightarrow 0$. An example of a Kadec–Klee space that is not uniformly convex is given by the Schatten-1 class $S^1(\mathfrak{h})$ (trace class), where $\mathfrak{h}$ is an arbitrary infinite-dimensional Hilbert space. See [9] or [119], for instance.

And finally, a function f from $I = [0,1]$ to a normed space Z is said to be of *bounded variation* if and only if its variation $V_f(I)$ is finite, where the variation $V_f(J)$ over closed subintervals J of I is defined as

$$V_f(J) := \sup\left\{V_{f,\pi} : \pi \text{ a partition of } J\right\} \quad \text{and} \quad V_{f,\{t_0,\dots,t_m\}} := \sum_{i=1}^{m} \|f(t_i) - f(t_{i-1})\| \,.$$

In the theorem below, the following facts about functions of bounded variation will be crucial; their proof, however, is elementary and therefore omitted.

Lemma 2.2.9. *Suppose $f : I \to Z$ is of bounded variation and Z a Banach space. Then*

(i) the limits $f(a+) := \lim_{t \searrow a} f(t)$ and $f(b-) := \lim_{t \nearrow b} f(t)$ exist for every $b \in (0,1]$ and $a \in [0,1)$,

(ii) $V_f([a,c]) = V_f([a,b]) + V_f([b,c])$ for all $a,b,c \in I$ with $a \le b \le c$,

(iii) $V_f([a,y]) \longrightarrow 0$ as $y \searrow a$ and $V_f([x,b]) \longrightarrow 0$ as $x \nearrow b$ for every $a \in [0,1)$ for which $f(a+) = f(a)$ and every $b \in (0,1]$ for which $f(b-) = f(b)$, respectively.

2.2.3.2 Slight generalization of a theorem by Kato

With these preparations at hand, we can now prove the announced generalization of the well-posedness theorem by Kato for skew self-adjoint operators with time-independent domains from [62] (Theorem 3). In doing so, we basically follow the arguments from [62] (Section 3.1 to 3.11), but the proof below is shorter and more direct – in particular, because in contrast to [62] we do not gradually strengthen the assumptions to get gradually closer and closer to well-posedness (and achieve it only in the last step). Also, the proof given below is simpler than the (known) proofs of well-posedness for the case of semigroup generators in general spaces, for instance, those from [36], [104], [125], [141],

[107]. Alternatively, the theorem below could also be concluded from Theorem 2.3.5 whose proof, however, is much more technical – and hence much less instructive – than the one given here. See the second remark after the theorem.

Theorem 2.2.10. *Suppose $A(t) : D \subset X \to X$ for every $t \in I$ is a quasicontraction group generator with time-independent domain D in a uniformly convex Banach space X such that*

$$\left\|e^{A(t)\tau}\right\| \le e^{\omega_0|\tau|} \qquad (\tau \in \mathbb{R}) \tag{2.33}$$

with a t-independent constant $\omega_0 \in \mathbb{R}$. Suppose further that $t \mapsto A(t) \in L(Y, X)$ is continuous and of bounded variation, where Y is the space D endowed with the graph norm of $A(0)$. Then there exists a unique evolution system U for A on Y.

Proof. We construct the sought evolution U by approximation with the standard approximants U_n from hyperbolic evolution equations theory (recall that uniqueness of U will then be automatic by Corollary 2.1.6). So, we choose partitions π_n of I with $\operatorname{mesh}(\pi_n) \longrightarrow 0$ as $n \to \infty$ and, for any such partition, we evolve piecewise according to the values of $t \mapsto A(t)$ at the finitely many partition points of π_n, that is, we set

$$U_n(t, s) := e^{A(r_n(t))(t-s)} \tag{2.34}$$

for $(s, t) \in \Delta$ with s, t lying in the same partition subinterval of π_n and

$$U_n(t, s) := e^{A(r_n(t))(t-r_n(t))} e^{A(r_n^-(t))(r_n(t)-r_n^-(t))} \cdots e^{A(r_n(s))(r_n^+(s)-s)} \tag{2.35}$$

for $(s, t) \in \Delta$ with s, t lying in different partition subintervals of π_n. We also set $U_n(s, t) := U_n(t, s)^{-1}$ for $(s, t) \in \Delta$. In the equations above, $r_n(u)$ for $u \in I$ denotes the largest partition point of π_n less than or equal to u and $r_n^-(u)$, $r_n^+(u)$ is the neighboring partition point below or above $r_n(u)$, respectively.

We now show, in six steps, that the operators $U_n(t, s)$ are strongly convergent to an invertible evolution system U for A on Y uniformly in $(s, t) \in I^2$. An invertible evolution system U for A on Y is a family of bounded operators $U(t, s)$ in X for $(s, t) \in I^2$ such that

(i) $[s, 1] \ni t \mapsto U(t, s)y$ for $y \in Y$ and $s \in [0, 1)$ is a continuously differentiable solution of (2.42) with values in Y,

(ii) $U(t, s)U(s, r) = U(t, r)$ for all $(r, s), (s, t) \in I^2$ and $I^2 \ni (s, t) \mapsto U(t, s)$ is strongly continuous.

If U satisfies only (ii), we just speak of an invertible evolution system. In our proof, we will make extensive use of the following t-dependent norms $\|\,.\,\|_t$ on Y defined by

$$\|y\|_t := \|\underline{A}(t)y\| \qquad (y \in Y \text{ and } t \in I),$$

where $\underline{A}(t) := A(t) - (\omega_0 + 1)$. Since $\|\,.\,\|_t$ is equivalent to the graph norm of $\underline{A}(t)$, the space $Y_t := (Y, \|\,.\,\|_t)$ is a Banach space and, just like X, it is uniformly convex (and in particular reflexive) as can be easily verified using the definition of uniform convexity. (In the special case where X is a Hilbert space and $A(t)$ is skew self-adjoint, it follows that $\|y\|_t^2 = \|A(t)y\|^2 + \|y\|^2$ and hence Y_t is a Hilbert space, too.)

As a first step, we observe that there exists a constant C such that for all $s, t \in I$ and all $y \in Y$

$$\|y\|_t \leq e^{CV_A(J(s,t))} \|y\|_s \qquad (J(s,t) := [\min\{s,t\}, \max\{s,t\}]) \tag{2.36}$$

and that, in particular, the norms $\|\,.\,\|_t$ are equivalent to the norm of Y uniformly in $t \in I$. Indeed, by the continuity of $t \mapsto \underline{A}(t) \in L(Y, X)$, the map $t \mapsto \underline{A}(t)^{-1} \in L(X, Y)$ is continuous and hence $C := \sup_{s \in I} \|\underline{A}(s)^{-1}\|_{X,Y} < \infty$. In view of $\|y\|_t \leq \|\underline{A}(t)\underline{A}(s)^{-1}\| \|y\|_s$, the desired relation (2.36) follows from

$$\begin{aligned}\|\underline{A}(t)\underline{A}(s)^{-1}\| &= \|1 + (\underline{A}(t) - \underline{A}(s))\underline{A}(s)^{-1}\| \leq 1 + \|\underline{A}(t) - \underline{A}(s)\|_{Y,X} \|\underline{A}(s)^{-1}\|_{X,Y} \\ &\leq 1 + CV_A(J(s,t)) \leq e^{CV_A(J(s,t))}.\end{aligned}$$

And from (2.36) in turn it is clear that the norms $\|\,.\,\|_t$ are equivalent to $\|\,.\,\|_0$ uniformly in $t \in I$ and hence also to the graph norm of $A(0)$ which, recall, was defined to be the norm of Y.

As a second step, we show that for all $t > s$, $n \in \mathbb{N}$, and $y \in Y$,

$$\|U_n(t,s)y\|_t \leq e^{CV_A([s,t]) + 2CV_A([r_n(s),s])} \, e^{\omega_0(t-s)} \|y\|_s, \tag{2.37}$$

$$\|U_n(s,t)y\|_s \leq e^{CV_A([s,t]) + 2CV_A([r_n(s),s])} \, e^{\omega_0(t-s)} \|y\|_t, \tag{2.38}$$

and that, in particular, $\sup_{(s,t) \in I^2} \|U_n(t,s)\|_{Y,Y} \leq M < \infty$ for all $n \in \mathbb{N}$ and some finite number M. With the help of (2.36) we pass from $\|\,.\,\|_t$ to $\|\,.\,\|_{r_n(t)}$, then from $\|\,.\,\|_{r_n(t)}$ to $\|\,.\,\|_{r_n^-(t)}$ and so on, where in each step we use that $e^{A(t)\tau}$ is a quasicontraction in Y_t satisfying (2.33) for any $t \in I$. In this way we arrive at

$$\|U_n(t,s)y\|_t \leq e^{CV_A([r_n(s),t])} \, e^{\omega_0(t-s)} \|y\|_{r_n(s)}$$

and, using (2.36) once again together with the additiviy

$$V_A([r_n(s), t]) = V_A([r_n(s), s]) + V_A([s,t])$$

of the variation (Lemma 2.2.9 (ii)), we obtain (2.37); (2.38) is proved analogously. Since the norms $\|\,.\,\|_t$ are equivalent to the norm of Y uniformly in $t \in I$ by the first step, the asserted uniform boundedness of $\|U_n(t,s)\|_{Y,Y}$ now follows from (2.37) and (2.38).

As a third step, we show that for all $x \in X$ the limit $U(t,s)x := \lim_{n \to \infty} U_n(t,s)x$ exists uniformly in $(s,t) \in I^2$ and that it defines an invertible evolution system U. Indeed, for every $y \in Y$ the map $\tau \mapsto U_m(t,\tau)U_n(\tau,s)y$ is piecewise continuously differentiable with possible jumps in the derivative at the partition points from $\pi_m \cup \pi_n$. It follows that

$$\begin{aligned}U_n(t,s)y - U_m(t,s)y &= U_m(t,\tau)U_n(\tau,s)y\Big|_{\tau=s}^{\tau=t} \\ &= \int_s^t U_m(t,\tau)\big(A(r_n(\tau)) - A(r_m(\tau))\big)U_n(\tau,s)y \, d\tau\end{aligned}$$

and by the second step and the continuity of $\tau \mapsto A(\tau) \in L(Y, X)$ we conclude that

$$\sup_{(s,t)\in I^2} \|U_n(t,s)y - U_m(t,s)y\| \leq \int_0^1 e^{\omega_0|t-\tau|}\|A(r_n(\tau)) - A(r_m(\tau))\|_{Y,X}\, M\|y\|_Y \, d\tau \longrightarrow 0$$

as $m, n \to \infty$. In other words, $(U_n(t,s)y)$ for every $y \in Y$ is a Cauchy sequence in X uniformly in $(s,t) \in I^2$. So, the uniform existence of $U(t,s)x := \lim_{n\to\infty} U_n(t,s)x$ for every $x \in X$ follows by the density of Y in X and the uniform boundedness $\|U_n(t,s)\| \leq e^{\omega_0|t-s|}$ while the invertible evolution system properties for U are inherited from the approximants U_n due to the uniform convergence.

As a fourth step, we show that for all $y \in Y$ and $s, t \in I$ one has $U(t,s)y \in Y$ and

$$\|U(t,s)y\|_t \leq e^{CV_A([s,t])}\, e^{\omega_0|t-s|}\|y\|_s. \tag{2.39}$$

Indeed, for every $y \in Y$, the sequence $(U_n(t,s)y)$ is bounded in Y_t by the second and the first step, and $U_n(t,s)y \longrightarrow U(t,s)y$ in X by the third step. Since Y_t is reflexive (as a uniformly convex Banach space), it follows that $U(t,s)y \in Y$ and that $U_n(t,s)y \longrightarrow U(t,s)y$ weakly in Y_t. So,

$$\|U(t,s)y\|_t \leq \liminf_{n\to\infty} \|U_n(t,s)y\|_t \leq e^{CV_A([s,t])}\, e^{\omega_0|t-s|}\|y\|_s$$

by (2.37) and (2.38) and by the continuity of $\tau \mapsto A(\tau) \in L(Y, X)$ and the continuity properties of the variation (Lemma 2.2.9 (iii)).

As a fifth step, we show that for all $y \in Y$ the map $I \ni t \mapsto U(t,s)y$ is differentiable in the norm of X with derivative $t \mapsto A(t)U(t,s)y$. Since $U(t,s)Y \subset Y$ and $U(t+h,s) = U(t+h,t)U(t,s)$ by the fourth step, it suffices to prove the assertion for $s = t$. We have for every $y \in Y$

$$\begin{aligned} U(t+h,t)y - e^{A(t)h}y &= \lim_{n\to\infty} e^{A(t)(t+h-\tau)}U_n(\tau,t)y\Big|_{\tau=t}^{\tau=t+h} \\ &= \lim_{n\to\infty} \int_t^{t+h} e^{A(t)(t+h-\tau)}\big(A(r_n(\tau)) - A(t)\big)U_n(\tau,s)y\, d\tau. \end{aligned}$$

It therefore follows that

$$\begin{aligned} &\Big\|\frac{1}{h}(U(t+h,t)y - e^{A(t)h}y)\Big\| \\ &\qquad \leq \lim_{n\to\infty} \frac{1}{|h|}\left|\int_t^{t+h} e^{\omega_0|t+h-\tau|}\|A(r_n(\tau)) - A(t)\|_{Y,X}\, d\tau\right| M\|y\|_Y \\ &\qquad = \frac{1}{|h|}\left|\int_t^{t+h} e^{\omega_0|t+h-\tau|}\|A(\tau) - A(t)\|_{Y,X}\, d\tau\right| M\|y\|_Y \longrightarrow 0 \quad (h \to 0) \end{aligned}$$

by the second step and the continuity of $\tau \mapsto A(\tau) \in L(Y, X)$. Since $(e^{A(t)h}y - y)/h \longrightarrow A(t)y$ as $h \to 0$, the assertion of the fifth step now follows.

As a sixth and last step, we show that for all $y \in Y$ the map $t \mapsto A(t)U(t,s)y$ is continuous in the norm of X. Since $t \mapsto A(t) \in L(Y, X)$ is continuous, it suffices to show that $t \mapsto U(t,s)y$ is continuous in the norm of Y. And to see this, in turn, it suffices to show that $\lim_{h\to 0} U(t+h,t)y = y$ in the norm of Y or, equivalently, in the norm of Y_t. We already know that $U(t+h,t)y \longrightarrow y$ in X as $h \to 0$ by the third step and that $h \mapsto U(t+h,t)y$ is bounded in Y_t by (2.39) and (2.36). It follows that $U(t+h,t)y \longrightarrow y$ weakly in Y_t as $h \to 0$ by a similar argument as in the fourth step. Consequently,

$$\begin{aligned}\|y\|_t &\le \liminf_{h\to 0} \|U(t+h,t)y\|_t \le \limsup_{h\to 0} \|U(t+h,t)y\|_t \\ &\le \limsup_{h\to 0} e^{CV_A(J(t,t+h))} \|U(t+h,t)y\|_{t+h} \le \limsup_{h\to 0} e^{2CV_A(J(t,t+h))+\omega_0|h|} \|y\|_t = \|y\|_t.\end{aligned}$$

Since now Y_t is uniformly convex, the weak convergence $U(t+h,t)y \longrightarrow y$ in Y_t and the convergence $\|U(t+h,t)y\|_t \longrightarrow \|y\|_t$ of the norms just shown implies the desired convergence $U(t+h,t)y \longrightarrow y$ in the norm of Y_t as $h \to 0$. ■

Some remarks are in order which, in particular, clarify the relation of the theorem above with similar results.

1. It is easy to see that the regularity condition of the above theorem is weaker – and strictly weaker – than the $W^{1,1}_*$-regularity condition from Theorem 2.1.9. Indeed, if $t \mapsto A(t) \in L(Y,X)$ is $W^{1,1}_*$-regular, then it is absolutely continuous by (2.1) and, in particular, continuous and of bounded variation; on the other hand, there are simple examples of skew self-adjoint operators $A(t)$ with time-independent domains such that $t \mapsto A(t) \in L(Y,X)$ is continuous and of bounded variation but not $W^{1,1}_*$-regular: choose, for instance,

$$A(t) := A_0 + \kappa(t)B,$$

where A_0 is skew self-adjoint and $B \neq 0$ is bounded skew symmetric and where $\kappa : I \to \mathbb{R}$ is the Cantor singular function (see, for instance, [23]).

2. It is also not difficult to see that the above theorem is a special case of Theorem 2.3.5 below. Indeed, one has only to choose the t-dependent norms $\|\,.\,\|_t^{\pm}$ and the functions $c^{\pm} : \mathcal{I} := \{\text{closed subintervals of } I\} \to [0,\infty)$ appearing in that theorem in the following way and then to recall (2.36) as well as Lemma 2.2.9 (ii) and (iii):

$$\|y\|_t^{\pm} := \left\|\underline{A}^{\pm}(t)y\right\| \quad (y \in Y) \quad \text{and} \quad c^{\pm}(J) := C^{\pm} V_{A^{\pm}}(J) \quad (J \in \mathcal{I}),$$

where $\underline{A}^{\pm}(t) := A^{\pm}(t) - (\omega_0 + 1)$ with $A^{+}(t) := A(t)$ and $A^{-}(t) := -A(1-t)$ and where $C^{\pm} := \sup_{s\in I} \left\|\underline{A}^{\pm}(s)^{-1}\right\|$.

3. When applying the above theorem, it is sufficient to verify that $t \mapsto A(t)$ is continuous w.r.t. the weak operator topology and of bounded variation, because the bounded variation implies the existence of the limits $A(t+) := \lim_{s\searrow t} A(s)$ and $A(t-) := \lim_{s\nearrow t} A(s)$ (in the norm topology) for all t by Lemma 2.2.9 (i) and the weak continuity in turn implies that these limits have to coincide for each t, which yields the continuity of $t \mapsto A(t)$. See Condition C_3 of [62].

2.2.4 Counterexamples to well-posedness

We now investigate the optimality of the assumptions of the well-posedness theorems for operators with time-independent domains discussed so far. We will show by example that the assumptions of these theorems cannot be drastically weakened – not even in the case of group generators $A(t)$ that are as close to bounded operators as could be in the sense that $A(t) = A_0 + B(t)$ for an unbounded contraction group generator A_0 and some bounded operators $B(t)$. (In the elementary case of bounded operators, by contrast, the assumptions can of course be weakened drastically, namely to strong continuity of $t \mapsto A(t)$. See [104] (Theorem 5.1.1) or [107] (Theorem X.69), for instance.) Similar examples can be found in [105] (Example 6.4) and in [41] (Example VI.9.21), but the examples below seem to be the first counterexamples for group instead of semigroup generators and, moreover, they are noticeably simpler than those from [105] and [41] because no perturbation series has to be computed to get an explicit expression for the putative evolution.

Lemma 2.2.11. *Suppose $A(t) = A_0 + B(t)$ for a group generator A_0 and bounded operators $B(t)$ in a Banach space X such that $t \mapsto B(t)$ is strongly continuous. Suppose further that there exists a – then unique – evolution system U for A on $D(A_0)$. Then*

$$U(t,s) = e^{A_0 t}\tilde{U}(t,s)e^{-A_0 s} \quad ((s,t) \in \Delta),$$

where $\tilde{U}$ denotes the (trivially existing) evolution system for $\tilde{B}$ on X and where $\tilde{B}(t) := e^{-A_0 t}B(t)e^{A_0 t}$ for $t \in I$. In particular, if $B(t) = e^{A_0 t}Be^{-A_0 t}$ for a fixed bounded operator B, then U is simply given by $U(t,s) = e^{A_0 t}e^{B(t-s)}e^{-A_0 s}$ for all $(s,t) \in \Delta$.

Proof. We have to show $\tilde{V} = \tilde{U}$, where $\tilde{V}(t,s) := e^{-A_0 t}U(t,s)e^{A_0 s}$ for $(s,t) \in \Delta$. And this is simple: since U is an evolution system for A on $D(A_0)$, one has that for every $y \in D(A_0)$ the vectors $U(\tau,s)e^{A_0 s}y$ lie in $D(A_0)$ and hence $\tau \mapsto \tilde{V}(\tau,s)y$ is differentiable with derivative $\tau \mapsto \tilde{B}(\tau)\tilde{V}(\tau,s)y$. So, $\tau \mapsto \tilde{U}(t,\tau)\tilde{V}(\tau,s)y$ is continuous and right differentiable with right derivative 0 (Lemma 2.1.5) and therefore

$$\tilde{U}(t,s)y - \tilde{V}(t,s) = \tilde{U}(t,\tau)\tilde{V}(\tau,s)y\big|_{\tau=s}^{\tau=t} = 0$$

(Lemma 2.1.4) for all $y \in D(A_0)$ and, by density, also for all $y \in X$, as desired. ■

In our first example, we show that the $W^{1,1}_*$-regularity condition from Theorem 2.1.9 (and, a fortiori, the strong continuous differentiability condition from Theorem 4 of [62]) cannot be replaced by Lipschitz continuity, and that the uniform convexity condition from Theorem 2.2.10 cannot be dropped.

Example 2.2.12. Choose A_0 to be the generator of the left translation group in $X := C_0(\mathbb{R}) = \{g \in C(\mathbb{R}) : g(t) \longrightarrow 0 \ (|t| \to \infty)\}$, that is,

$$e^{A_0 t}g = g(\,.\,+t) \quad (g \in X \text{ and } t \in \mathbb{R}),$$

and choose $B := M_f$ to be multiplication with the function f defined by $f(\xi) := 0$ for $\xi \le 0$, $f(\xi) := \xi$ for $\xi \in [0,1]$, and $f(\xi) := 1$ for $\xi \ge 1$. Set

$$A(t) := A_0 + B(t) \quad \text{with} \quad B(t) := e^{A_0 t} B e^{-A_0 t}$$

for $t \in I$. Then $A(t)$ is the generator of a quasicontraction group with time-independent domain $D(A_0)$ in X such that

$$\left\| e^{A(t)\tau} \right\| \le e^{\|B(t)\| |\tau|} \le e^{\|B\| |\tau|} = e^{\|f\|_\infty |\tau|} \qquad (\tau \in \mathbb{R})$$

and, moreover, $t \mapsto A(t)$ is Lipschitz continuous because $B(t)$ is multiplication with the function $f(\,.\,+t)$ and f is Lipschitz. An evolution system for A on $D(A_0)$ does not exist, however. Indeed, if such an evolution system U existed, it would satisfy $e^{-A_0 t}U(t,0) = e^{Bt}$ for all $t \in I$ (Lemma 2.2.11) and hence we would obtain

$$e^{Bt} D(A_0) \subset D(A_0) \quad (t \in I) \tag{2.40}$$

because $U(t,0)D(A_0) \subset D(A_0)$. Yet, e^{Bt} acts by multiplication with the non-differentiable function e^{ft} and $D(A_0) = C_0^1(\mathbb{R}) = \{g \in C^1(\mathbb{R}) : g, g' \in X\}$, and therefore (2.40) is not satisfied for $t \ne 0$. So, an evolution system U for A on $D(A_0)$ cannot exist. (Alternatively, we could obtain a contradiction also by proving that $t \mapsto U(t,0)g = e^{A_0 t} e^{Bt} g = e^{f(\,.\,+t)t} g(\,.\,+t)$ is not differentiable, for instance, for $g \in D(A_0)$ with $g|_{[-1,2]} = 1$.) ◀

In our second example, we show that the continuity and bounded variation condition from Theorem 2.2.10 cannot be replaced by Hölder continuity of any degree $\alpha < 1$, even though it can be replaced by Lipschitz continuity, of course. In this example, the operators $A(t)$ are skew self-adjoint. It is remarkable that, contrastingly, for generators $A(t)$ of holomorphic semigroups Hölder continuity does suffice for well-posedness. See [124], [120] or [104] (Theorem 5.6.1), for instance.

Example 2.2.13. Choose A_0 and B as in the previous example with the sole exception that now $X := L^2(\mathbb{R})$ and $f := iw$ where $w : \mathbb{R} \to \mathbb{R}$ is the Weierstraß function

$$w(\xi) := \sum_{n=1}^{\infty} 2^{-n} \cos(2^n \xi) \quad (\xi \in \mathbb{R}),$$

which is Hölder continuous of degree α for every $\alpha < 1$ but nowhere differentiable (and in particular not Lipschitz). See, for instance, Theorem II.4.9 of [143]. Set

$$A(t) := A_0 + B(t) \quad \text{with} \quad B(t) := e^{A_0 t} B e^{-A_0 t}$$

for $t \in I$. Then $A(t)$ is skew self-adjoint with time-independent domain $D(A_0)$ in the Hilbert space X and, moreover, $t \mapsto A(t)$ is Hölder continuous of degree α for every $\alpha < 1$ because $B(t)$ is multiplication with the function $f(\,.\,+t)$ and f is Hölder continuous of any degree $\alpha < 1$. An evolution system for A on $D(A_0)$ does not exist because otherwise we would obtain

$$e^{Bt} D(A_0) \subset D(A_0) \quad (t \in I) \tag{2.41}$$

by the same arguments as for (2.40). Yet, e^{Bt} acts by multiplication with the nowhere differentiable function e^{ft} and $D(A_0) = W^{1,2}(\mathbb{R}) \subset \{g \in X : g \text{ differentiable a. e.}\}$, and therefore (2.41) is not satisfied for $t \neq 0$. So, an evolution system U for A on $D(A_0)$ cannot exist. ◀

2.3 Well-posedness for operators with time-dependent domains

2.3.1 Introduction

In this section, we are concerned with non-autonomous linear evolution equations

$$x' = A(t)x \ (t \in [s, 1]) \quad \text{and} \quad x(s) = y \tag{2.42}$$

for densely defined linear operators $A(t) : D(A(t)) \subset X \to X$ $(t \in [0, 1])$ and initial values $y \in Y \subset D(A(s))$ at initial times $s \in [0, 1)$. Well-posedness of such evolution equations has been studied by many authors in a large variety of situations. See, for instance, [104], [118], [92], [94] for an overview.

In Section 2.3.2 we examine the special situation of semigroup generators $A(t)$ whose first (1-fold) or higher (p-fold) commutators at distinct times are complex scalars, in short:

$$[A(t_1), A(t_2)] = \mu(t_1, t_2) \in \mathbb{C} \tag{2.43}$$

or

$$\big[\dots \big[[A(t_1), A(t_2)], A(t_3)\big] \dots, A(t_{p+1})\big] = \mu(t_1, \dots, t_{p+1}) \in \mathbb{C} \tag{2.44}$$

in some sense to be made precise (see the commutation relations (2.46), (2.56) and (2.59), (2.63)). In this special situation we prove well-posedness for (2.42) on suitable dense subspaces Y of X and, moreover, in the case (2.43) we prove the representation formula

$$U(t, s) = e^{\overline{\int_s^t A(\tau)\, d\tau}} \, e^{1/2 \int_s^t \int_s^\tau \mu(\tau,\sigma)\, d\sigma d\tau} \tag{2.45}$$

for the evolution generated by the operators $A(t)$. We thereby generalize a well-posedness result of Goldstein and of Nickel and Schnaubelt from [49], [101] dealing with the special case of (2.43) where $\mu \equiv 0$: in [49] contraction semigroup generators are considered, while in [101] contraction semigroup generators are replaced by general semigroup generators and the formula (2.45) with $\mu \equiv 0$ is proved.

What one gains by restricting oneself to the special class of semigroup generators with (2.43) or (2.44) – instead of considering general semigroup generators as in [62], [65], [66], for instance – is that well-posedness can be established under fairly weak stability and regularity conditions: 1. It is sufficient – just as in the case of commuting operators from [49], [101] – to require stability of the family A only in X. In contrast to the well-posedness theorems from [65] or [66], for instance, it is not necessary to additionally require stability in a suitable invariant and suitably normed dense subspace

Y of X contained in all the domains of the operators $A(t)$, which is generally difficult to verify unless the domains of the $A(t)$ are time-independent. 2. It is sufficient – similarly to the case of commuting operators from [49], [101] or to the elementary case of bounded operators – to require strong continuity conditions: indeed, it is sufficient if

$$t \mapsto A(t)y \quad \text{and} \quad (t_1, \dots, t_{k+1}) \mapsto \big[\dots,[[A(t_1),A(t_2)],A(t_3)]\dots,A(t_{k+1})\big]y$$

are continuous for $k \in \{1, \dots, p\}$ and y in a dense subspace Y of X contained in all the respective domains. In contrast to the well-posedness theorems from [65] or [66], this subspace Y need not be normed in any way whatsoever and $t \mapsto A(t)|_Y$ need not be norm continuous. And furthermore, it is not necessary to require an additional $W^{1,1}$-regularity condition on certain auxiliary operators $S(t) : Y \to X$ (as in the well-posedness theorems from [65], [66] for general semigroup generators $A(t)$) or an additional regularity condition on certain auxiliary norms $\|\,.\,\|_t^{\pm}$ on Y (as in the special well-posedness result from [65] for a certain kind of group generators). Such additional regularity conditions are necessary for well-posedness in general situations without commutator conditions of the kind (2.43) or (2.44) – even if the domains of the $A(t)$ are time-independent (see the examples in [105], [41] or [116], for instance).

As is well-known from [86], [42], [134], in the case of bounded operators $A(t)$ one has representation formulas of Campbell–Baker–Hausdorff and Zassenhaus type for the evolution, which in the case (2.43) reduce to our representation formula (2.45). It should be noticed, however, that for bounded operators condition (2.43) can be satisfied only if $\mu \equiv 0$, so that (2.45) is independent of [86], [42], [134] (for non-zero μ). In view of the representation formulas from [134] it is desirable to prove representation formulas analogous to (2.45) also in the case (2.44), but this is left to future research.

All proofs in connection with the special situations (2.43) or (2.44) are, in essence, based upon the observation that in these situations the operators $A(r)$ can be commuted – up to controllable errors – through the exponential factors of the standard approximants $U_n(t,s)$ from [49], [65], [66], [101] for the sought evolution, which are of the form

$$U_n(t,s) = e^{A(r_m)\tau_m} \cdots e^{A(r_1)\tau_1}$$

with partition points $r_1, \dots, r_m$ of the interval $[s,t]$. See (2.50) and (2.61) respectively.

In Section 2.3.3 we improve the special well-posedness result from [65] for group generators with time-dependent domains: in the spirit of [72] we show that strong (instead of norm) continuity is sufficient in this result – just like in our other well-posedness results for the case (2.43) or (2.44). And in a certain special case involving quasicontraction group generators with time-independent domains in a uniformly convex space, these other results can also be obtained by applying the improved well-posedness result for group generators.

Section 2.3.4 discusses, among other things, the relation of our well-posedness results from Section 2.3.2.1 and 2.3.2.2 to the results from [65], [66], [72] and to the result from Section 2.3.3. In Section 2.3.5 we give some applications of the abstract well-posedness theorems for generators with scalar 1-fold or p-fold commutators from Section 2.3.2, namely to Segal field operators $\Phi(f_t)$ as well as to the related operators $H_\omega + \Phi(f_t)$ describing a classical particle coupled to a time-dependent quantized field of bosons (Section 2.3.5.1) and finally to Schrödinger operators describing a quantum particle coupled to a time-dependent spatially constant electric field (Section 2.3.5.2).

2.3.2 Well-posedness for semigroup generators whose commutators are complex scalars

We will use the notion of well-posedness and evolution systems from [41] which is recalled in Section 2.1.2. We will always be in the situation where $Y_t = Y$ for $t \in I$. At some places we will also use the notion of evolution systems from [101], which is slightly weaker than the one above in that it does not require that $[s,1] \ni t \mapsto U(t,s)y$ have values in Y for every $y \in Y$ and $s \in [0,1)$ (while all other conditions from the definition in Section 2.1.2 with $Y_t = Y$ are taken over). We will then speak of *evolution systems in the wide sense for A on Y* and, in case there exists exactly one such evolution system in the wide sense, we will speak of *well-posedness in the wide sense on Y*. Commutators of possibly unbounded operators are taken in the operator-theoretic sense,

$$D([A,B]) := D(AB - BA) = D(AB) \cap D(BA),$$

except in some formal heuristic computations (whose formal character will always be pointed out). We will finally also need the standard notions of (M,ω)-stability, of the part of an operator A in a subspace Y, and of A-admissible subspaces from [65] or [104], which are recalled in Section 2.1.3.

2.3.2.1 Scalar 1-fold commutators

In this subsection we prove well-posedness for (2.42) in the case (2.43) where the 1-fold commutators of the operators $A(t)$ are complex scalars. We have to make precise the merely formal commutation relation (2.43), of course, and we begin with a well-posedness result where (2.43) is replaced by the formally equivalent commutation relation (2.46) for the semigroups $e^{A(t)\,\cdot}$ with the generators $A(s)$. In addition to well-posedness this theorem also yields a representation formula for the evolution. It is a generalization of a well-posedness result of Goldstein [49] (Theorem 1.1) and – after the slight modifications discussed in (2.80) and (2.81) below – of Nickel and Schnaubelt [101] (Theorem 2.3 and Proposition 2.5).

Theorem 2.3.1. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M,ω)-stable for some $M \in [1,\infty)$ and $\omega \in \mathbb{R}$ and such that for some complex numbers $\mu(s,t) \in \mathbb{C}$*

$$A(s)e^{A(t)\tau} \supset e^{A(t)\tau}\big(A(s) + \mu(s,t)\tau\big) \tag{2.46}$$

for all $s,t \in I$ and $\tau \in [0,\infty)$. Suppose further that the maximal continuity subspace

$$Y^\circ := \{y \in \cap_{\tau \in I} D(A(\tau)) : t \mapsto A(t)y \text{ is continuous}\} \tag{2.47}$$

is dense in X and that $(s,t) \mapsto \mu(s,t)$ is continuous. Then there exists a unique evolution system U for A on Y° and it is given by

$$U(t,s) = e^{\overline{(\int_s^t A(\tau)\,d\tau)^\circ}} e^{1/2 \int_s^t \int_s^\tau \mu(\tau,\sigma)\,d\sigma\,d\tau} \quad ((s,t) \in \Delta),$$

where $\left(\int_s^t A(\tau)\,d\tau\right)^\circ$ is the (closable) operator defined by $y \mapsto \int_s^t A(\tau)y\,d\tau$ on Y°.

Proof. (i) We first show, in three steps, the existence of an evolution system U for A on Y°, which is then necessarily unique by Corollary 2.1.6. In order to do so we approximate the sought evolution U by the standard approximants U_n from hyperbolic evolution equations theory, that is, we choose partitions

$$\pi_n = \{r_{n\,i} : i \in \{0, \dots, m_n\}\}$$

of I with $\mathrm{mesh}(\pi_n) \longrightarrow 0$ as $n \to \infty$ and, for any such partition, we evolve piecewise according to the values of $t \mapsto A(t)$ at the finitely many partition points of π_n. So,

$$U_n(t,s) := e^{A(r_n(t))(t-s)} \tag{2.48}$$

for $(s,t) \in \Delta$ with s, t lying in the same partition subinterval of π_n and

$$U_n(t,s) := e^{A(r_n(t))(t-r_n(t))} e^{A(r_n^-(t))(r_n(t)-r_n^-(t))} \cdots e^{A(r_n(s))(r_n^+(s)-s)} \tag{2.49}$$

for $(s,t) \in \Delta$ with s, t lying in different partition subintervals of π_n. In the equations above, $r_n(u)$ for $u \in I$ denotes the largest partition point of π_n less than or equal to u and $r_n^-(u)$, $r_n^+(u)$ is the neighboring partition point below or above $r_n(u)$, respectively.

We then obtain, by repeatedly applying the assumed commutation relation (2.46), the following important commutation relation which allows us to take $A(r)$ from the left of $U_n(t,s)$ to the right and which is central to the entire proof:

$$A(r)U_n(t,s)y = U_n(t,s)\Big(A(r) + \int_s^t \mu(r, r_n(\sigma))\,d\sigma\Big)y \tag{2.50}$$

for all $y \in D(A(r))$. As a first step, we observe that

$$U_n(t,s)U_n(s,r) = U_n(t,r) \quad \text{and} \quad \|U_n(t,s)\| \le Me^{\omega(t-s)} \tag{2.51}$$

for all $(s,t), (r,s) \in \Delta$ and that $\Delta \ni (s,t) \mapsto U_n(t,s)$ is strongly continuous.

As a second step, we show that $(U_n(t,s)x)$ for every $x \in X$ is a Cauchy sequence in X uniformly in $(s,t) \in \Delta$. Since $\cap_{r' \in I} D(A(r'))$ is invariant under the semigroups $e^{A(r)\cdot}$ for all $r \in I$ by (2.46), it follows that $[s,t] \ni \tau \mapsto U_m(t,\tau)U_n(\tau,s)y$ for every $y \in \cap_{r' \in I} D(A(r'))$ is piecewise continuously differentiable (with the partition points of $\pi_m \cup \pi_n$ as exceptional points) and therefore

$$\begin{aligned}
U_n(t,s)y - U_m(t,s)y &= U_m(t,\tau)U_n(\tau,s)y\Big|_{\tau=s}^{\tau=t} \\
&= \int_s^t U_m(t,\tau)\big(A(r_n(\tau)) - A(r_m(\tau))\big)U_n(\tau,s)y\,d\tau = \int_s^t U_m(t,\tau)U_n(\tau,s) \\
&\qquad \Big(A(r_n(\tau)) - A(r_m(\tau)) + \int_s^\tau \mu(r_n(\tau), r_n(\sigma)) - \mu(r_m(\tau), r_n(\sigma))\,d\sigma\Big)y\,d\tau
\end{aligned}$$

for every $y \in \cap_{r' \in I} D(A(r'))$ where, for the last equation, (2.50) has been used. So,

$$\sup_{(s,t)\in\Delta} \|U_n(t,s)y - U_m(t,s)y\| \le M^2 e^{w(b-a)} \biggl(\int_a^b \|A(r_n(\tau))y - A(r_m(\tau))y\| \, d\tau$$
$$+ \int_a^b \int_a^b \bigl|\mu(r_n(\tau), r_n(\sigma)) - \mu(r_m(\tau), r_n(\sigma))\bigr| \, \|y\| \, d\sigma \, d\tau \biggr) \longrightarrow 0 \quad (m, n \to \infty)$$

for every $y \in Y^\circ$ by the uniform continuity of $\tau \mapsto A(\tau)y$ and $(\tau, \sigma) \mapsto \mu(\tau, \sigma)$. And by (2.51) this uniform Cauchy property extends to all $y \in X$. Consequently,

$$U(t,s)x := \lim_{n\to\infty} U_n(t,s)x$$

for every $x \in X$ exists uniformly in $(s,t) \in \Delta$ and hence the properties observed in the first step carry over from U_n to U.

As a third step, we show that $t \mapsto U(t,s)y$ for every $y \in Y^\circ$ is a continuously differentiable solution to (2.42) with values in Y°. Since $\tau \mapsto U_n(\tau,s)y$ for $y \in \cap_{r' \in I} D(A(r'))$ is piecewise continuously differentiable with piecewise derivative

$$[s,t] \setminus \pi_n \ni \tau \mapsto A(r_n(\tau))U_n(\tau,s)y = U_n(\tau,s)\Bigl(A(r_n(\tau)) + \int_s^\tau \mu(r_n(\tau), r_n(\sigma)) \, d\sigma\Bigr) y$$

by virtue of (2.50), we have

$$U_n(t,s)y = y + \int_s^t U_n(\tau,s)\Bigl(A(r_n(\tau)) + \int_s^\tau \mu(r_n(\tau), r_n(\sigma)) \, d\sigma\Bigr) y \, d\tau$$

and therefore

$$U(t,s)y = y + \int_s^t U(\tau,s)\Bigl(A(\tau) + \int_s^\tau \mu(\tau, \sigma) \, d\sigma\Bigr) y \, d\tau$$

for all $y \in Y^\circ$. So, $t \mapsto U(t,s)y$ is continuously differentiable for every $y \in Y^\circ$ with derivative

$$t \mapsto U(t,s)\Bigl(A(t) + \int_s^t \mu(t,\sigma) \, d\sigma\Bigr) y = \lim_{n\to\infty} A(t)U_n(t,s)y = A(t)U(t,s)y,$$

where the last two equations hold by (2.50) and the closedness of $A(t)$. Also, since for all $y \in Y^\circ$ and $r \in I$

$$A(r)U_n(t,s)y \longrightarrow U(t,s)\Bigl(A(r) + \int_s^t \mu(r,\sigma) \, d\sigma\Bigr) y \quad (n \to \infty),$$

we see by the closedness of the operators $A(r)$ that $U(t,s)y \in Y^\circ$ for $y \in Y^\circ$. So, in summary, we have shown that U is an evolution system for A on Y°.

(ii) We now show, in three steps, that $\bigl(\int_s^t A(\tau) \, d\tau\bigr)^\circ$ for every fixed $(s,t) \in \Delta$ is closable and that its closure generates a strongly continuous semigroup in X with

$$e^{\overline{(\int_s^t A(\tau)\, d\tau)^\circ}} = U(t,s) e^{-1/2 \int_s^t \int_s^\tau \mu(\tau,\sigma) \, d\sigma \, d\tau}.$$

As a first step, we show a discrete version of the above representation formula: more precisely, we show that $B_n := \int_s^t A(r_n(\tau))\,d\tau$ is closable and that B_n generates a strongly continuous semigroup with the following decomposition of Zassenhaus type:

$$e^{\overline{B_n}r} = U_n^r(t,s)e^{-1/2(\int_s^t \int_s^\tau \mu(r_n(\tau),r_n(\sigma))\,d\sigma\,d\tau)r^2} \quad (r \in [0,\infty)), \tag{2.52}$$

where the operators $U_n^r(t,s)$ are defined in the same way as the operators $U_n(t,s)$ above with the only difference that now the generators $A(u)$ are all multiplied by the number r. Indeed, by the assumed commutation relations, we obtain the following commutation relations for semigroups,

$$e^{A_i\sigma}e^{A_j\tau} = e^{A_j\tau}e^{A_i\sigma}e^{\mu_{ij}\sigma\tau} \quad (\sigma,\tau \in [0,\infty)), \tag{2.53}$$

where $A_k := A(t_k)h_k$ and $\mu_{kl} := \mu(t_k,t_l)h_kh_l$ for arbitrary $t_k,t_l \in I$ and $h_k,h_l \in [0,\infty)$. (In fact, if $y \in D(A_i)$, then

$$e^{A_j\tau}e^{A_i\sigma}e^{\mu_{ij}\sigma\tau}y - e^{A_i\sigma}e^{A_j\tau}y = e^{A_i(\sigma-r)}e^{A_j\tau}e^{A_ir}e^{\mu_{ij}r\tau}y\big|_{r=0}^{r=\sigma}$$

and $[0,\sigma] \ni r \mapsto e^{A_i(\sigma-r)}e^{A_j\tau}e^{A_ir}e^{\mu_{ij}r\tau}y$ is differentiable with derivative 0.) With the help of (2.53) one verifies that

$$[0,\infty) \ni r \mapsto e^{A_mr}\cdots e^{A_1r}e^{-1/2\sum_{i\le j}\mu_{ji}r^2} \tag{2.54}$$

is a strongly continuous semigroup in X. As this semigroup, by the assumed commutation relation, leaves the subspace $D(A_1)\cap\cdots\cap D(A_m)$ invariant, its generator contains the operator $A_1+\cdots+A_m$, which is therefore closable with closure equal to the generator. Since B_n is of the form $A_1+\cdots+A_m$ and since the right-hand side of (2.52) is of the form (2.54) (because $\mu_{ii}=0$ by virtue of (2.53)), the assertion of the first step follows.

As a second step, we observe that the limit $T(r)x := \lim_{n\to\infty} e^{\overline{B_n}r}x$ exists locally uniformly in $r \in [0,\infty)$ for every $x \in X$ and that T is a strongly continuous semigroup in X. Indeed, with the same arguments as in (i), it follows that $(U_n^r(t,s)x)$ is convergent locally uniformly in r for every $x \in X$ with limit denoted by $U^r(t,s)x$ and therefore the strongly continuous semigroups $e^{\overline{B_n}\,\cdot}$ by (2.52) are strongly convergent locally uniformly in r, so that

$$T(r)x := \lim_{n\to\infty} e^{\overline{B_n}r}x = U^r(t,s)e^{-1/2(\int_s^t\int_s^\tau \mu(\tau,\sigma)\,d\sigma\,d\tau)r^2}x \quad (x \in X) \tag{2.55}$$

defines a strongly continuous semigroup T on X.

As a third step, we show that the generator A_T of this semigroup is given by $\overline{B^\circ}$ where $B^\circ := \big(\int_s^t A(\tau)\,d\tau\big)^\circ$, from which the desired representation formula for U then follows by (2.55) (because $U^1(t,s) = U(t,s)$). Indeed, for all $y \in Y^\circ$,

$$\frac{T(h)y-y}{h} = \lim_{n\to\infty}\frac{e^{\overline{B_n}h}y-y}{h} = \lim_{n\to\infty}\frac{1}{h}\int_0^h e^{\overline{B_n}r}\,\overline{B_n}y\,dr = \frac{1}{h}\int_0^h T(r)B^\circ y\,dr$$
$$\longrightarrow B^\circ y \quad (h \searrow 0)$$

by the dominated convergence theorem. So, B° is closable with $\overline{B^\circ} \subset A_T$. We now want to show that $D(\overline{B^\circ})$ is a core for A_T by verifying the invariance $T(r)D(\overline{B^\circ}) \subset D(\overline{B^\circ})$ for all $r \in [0, \infty)$. If $y \in Y^\circ$, then

$$B_m e^{\overline{B_n} r} y = e^{\overline{B_n} r}\big(B_m + \nu_{m,n} r\big) y \quad \text{with} \quad \nu_{m,n} := \int_s^t \int_s^t \mu(r_m(\tau), r_n(\sigma))\, d\sigma\, d\tau$$

by the product decomposition of $e^{\overline{B_n} r}$ from (2.52) and by the central commutation relation (2.50). So,

$$B^\circ e^{\overline{B_n} r} y = e^{\overline{B_n} r}\big(B^\circ + \lim_{m\to\infty} \nu_{m,n} r\big) y$$

for all $y \in Y^\circ$, from which it further follows that

$$T(r)y \in D(\overline{B^\circ}) \quad \text{and} \quad \overline{B^\circ}\, T(r) y = T(r)\big(B^\circ + \lim_{n\to\infty}\lim_{m\to\infty} \nu_{m,n} r\big) y = T(r) B^\circ y$$

for all $y \in Y^\circ$. In the last equation, we used that $\mu(\tau,\sigma) = -\mu(\sigma,\tau)$ for all $\sigma, \tau \in I$ which can be seen from (2.53). It follows that $\overline{B^\circ} T(r) \supset T(r)\overline{B^\circ}$ and, in particular, $T(r)D(\overline{B^\circ}) \subset D(\overline{B^\circ})$ for all $r \in [0,\infty)$. So, $D(\overline{B^\circ})$ is a core for A_T and hence $A_T = \overline{B^\circ}$, as desired. ■

We also note the following variant of the above theorem where the form (2.56) of the imposed commutation relation is closer to (2.43). In return, one has to require relatively strong invariance conditions.

Corollary 2.3.2. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M,ω)-stable for some $M \in [1,\infty)$ and $\omega \in \mathbb{R}$. Suppose further that Y is an $A(t)$-admissible subspace of X for every $t \in I$ such that*

$$Y \subset \bigcap_{\tau \in I} D(A(\tau)) \qquad \text{and} \qquad A(t)Y \subset \bigcap_{\tau\in I} D(A(\tau)),$$

$A(t)|_Y$ is a bounded operator from Y to X, and

$$[A(s), A(t)]\big|_{D(\tilde{A}(t))} \subset \mu(s,t) \in \mathbb{C} \tag{2.56}$$

for all $s, t \in I$, where $\tilde{A}(t)$ is the part of $A(t)$ in Y. Suppose finally that $(s,t) \mapsto \mu(s,t)$ and $t \mapsto A(t)y$ are continuous for all $y \in Y$. Then the conclusions of the above theorem hold true.

Proof. We verify the assumptions of the previous theorem and, to that purpose, we establish the commutation relations

$$e^{A_1\sigma} e^{A_2\tau} = e^{A_2\tau} e^{A_1\sigma} e^{\mu_{12}\tau\sigma} \quad (\sigma, \tau \in [0,\infty)), \tag{2.57}$$

where $A_k := A(t_k)$ and $\mu_{kl} := \mu(t_k, t_l)$ for arbitrary $t_1, t_2 \in I$. In order to see (2.57), one shows that

$$A_1 e^{A_2\tau} y = e^{A_2\tau}\big(A_1 + \mu_{12}\tau\big) y \tag{2.58}$$

for $y \in Y$ by differentiating $[0,\tau] \ni r \mapsto e^{A_2(\tau-r)} A_1 e^{A_2 r} y$ for vectors y in the domain of the part $\tilde{A}_2$ of A_2 in Y which by the A_2-admissibility of Y is the generator of the strongly continuous semigroup $t \mapsto e^{A_2 t}|_Y$ in Y (Proposition 2.3 of [65]). (In addition to the A_2-admissibility, the boundedness of $A_1|_Y$ from Y to X and the invariance condition $A_1 Y \subset D(A_2)$ come into play here.) Along the same lines as (2.53), the relation (2.57) then follows. And since (2.57) is equivalent to the commutation relations (2.46) and since $Y^\circ \supset Y$ is dense in X, the assumptions of Theorem 2.3.1 are satisfied, as desired. ∎

2.3.2.2 Scalar p-fold commutators

In this subsection we prove well-posedness for (2.42) in the case (2.44) where the p-fold commutators of the operators $A(t)$ are complex scalars for some $p \in \mathbb{N}$. We have to make precise the merely formal commutation relation (2.44), of course, and we begin with a well-posedness result where (2.44) is replaced by the formally equivalent commutation relations (2.59) for the semigroups $e^{A(t)\cdot}$ with the generators $A(s_1) = C^{(0)}(s_1)$ and certain operators $C^{(k)}(s_1, \dots, s_{k+1})$ which are formally given as the k-fold commutator of the operators $A(s_1), \dots, A(s_{k+1})$.

Theorem 2.3.3. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M,ω)-stable for some $M \in [1,\infty)$ and $\omega \in \mathbb{R}$ and such that for some closed operators $C^{(k)}(s_1, \dots, s_{k+1})$, where $k \in \{0, \dots, p-1\}$ and $C^{(0)}(s) := A(s)$, and for some complex numbers $\mu(t_1, \dots, t_{p+1}) \in \mathbb{C}$*

$$\begin{aligned} C^{(k)}(\underline{s}) e^{A(t)\tau} \supset e^{A(t)\tau}\big(C^{(k)}(\underline{s}) + C^{(k+1)}(\underline{s}, t)\tau + \dots + C^{(p-1)}(\underline{s}, t, \dots, t)\frac{\tau^{p-1-k}}{(p-1-k)!} + \\ + \mu(\underline{s}, t, \dots, t)\frac{\tau^{p-k}}{(p-k)!}\big) \qquad (\underline{s} := (s_1, \dots, s_{k+1})) \end{aligned} \tag{2.59}$$

for all $k \in \{0, \dots, p-1\}$ and $s_i, t \in I$ and $\tau \in [0,\infty)$. Suppose further that the maximal continuity subspace

$$Y^\circ := \bigcap_{k=0}^{p-1} \{y \in D_k : (t_1, \dots, t_{k+1}) \mapsto C^{(k)}(t_1, \dots, t_{k+1}) y \text{ is continuous}\} \tag{2.60}$$

$$D_k := \cap_{\tau_1, \dots, \tau_{k+1} \in I} D(C^{(k)}(\tau_1, \dots, \tau_{k+1}))$$

is dense in X and that $(t_1, \dots, t_{p+1}) \mapsto \mu(t_1, \dots, t_{p+1})$ is continuous. Then there exists a unique evolution system U for A on Y°.

Proof. We define U_n as in (2.48) and (2.49) and, for $u \in I$, we define $i_{n\,u}$ to be the index $i \in \{0, \dots, m_n\}$ with $u \in [r_{n\,i}, r_{n\,i+1})$. We then obtain, by the assumed commutation relations (2.59), the following important commutation relation which allows us to take the operators $A(r)$ from the left of $U_n(t,s)$ to the right:

$$A(r)U_n(t,s)y = U_n(t,s)\big(A(r) + S_n^{(1)}(t,s,r) + \dots + S_n^{(p)}(t,s,r)\big)y \tag{2.61}$$

$$S_n^{(l)}(t,s,r) := \int_s^t \int_s^{t_n(\tau_1)} \cdots \int_s^{t_n(\tau_{l-1})} C^{(l)}(r, r_n(\tau_1), \dots, r_n(\tau_l))/\alpha_{i_{n\,\tau_1},\dots,i_{n\,\tau_l}} \, d\tau_l \dots d\tau_2 \, d\tau_1$$

for all $y \in Y^\circ$ and $r \in I$, $(s,t) \in \Delta$ and $l \in \{1, \dots, p\}$, where $C^{(p)} := \mu$ and $t_n(\tau) := \min\{r_n^+(\tau), t\}$ for $\tau \in I$ and where $\alpha_{j_1,\dots,j_l}$ for an l-tupel $(j_1, \dots, j_l)$ of natural numbers denotes the number of permutations σ leaving the l-tupel invariant, that is,

$$(j_{\sigma(1)}, \dots, j_{\sigma(l)}) = (j_1, \dots, j_l).$$

(In verifying (2.61), it is best to write $A(r) = A_r = C_r^{(0)}$ and $U_n(t,s) = e^{A_m h_m} \cdots e^{A_1 h_1}$ with $A_j = A(s_j)$ and to prove by induction over $m \in \mathbb{N}$, with the help of the assumed commutation relations, that

$$A_r e^{A_m h_m} \dots e^{A_1 h_1} y = e^{A_m h_m} \dots e^{A_1 h_1} \big(A_r + S^{(1)} + \dots + S^{(p)}\big)y$$

$$S^{(l)} := \sum_{1 \le j_l \le \dots \le j_1 \le m} C^{(l)}_{r;j_1,\dots,j_l}/\alpha_{j_1,\dots,j_l} \, h_{j_1} \cdots h_{j_l} \quad \text{with} \quad C^{(l)}_{r;j_1,\dots,j_l} := C^{(l)}(r, s_{j_1}, \dots, s_{j_l}).$$

It is easy to see that the sums $S^{(l)}$ are nothing but the integrals $S_n^{(l)}(t,s,r)$ in (2.61) and therefore (2.61) follows.) With the help of the commutation relation (2.61), the continuity of the maps $(t_1, \dots, t_{k+1}) \mapsto C^{(k)}(t_1, \dots, t_{k+1})y$ for $y \in Y^\circ$, the fact that $\alpha_{i_{n\,\tau_1},\dots,i_{n\,\tau_k}} \longrightarrow k!$ as $n \to \infty$ for every $(\tau_1, \dots, \tau_k) \in I^k$ with $\tau_1 > \dots > \tau_k$, and the closedness of the operators $A(r)$, we see in the same way as in the proof of Theorem 2.3.1 that

- $(U_n(t,s)x)$ is a Cauchy sequence in X uniformly in $(s,t) \in \Delta$ for every $x \in X$ with limit denoted by $U(t,s)x$,
- $U(t,s)U(s,r) = U(t,r)$ for every $(r,s), (s,t) \in \Delta$ and $(s,t) \mapsto U(t,s)$ is strongly continuous,
- $[s,1] \ni t \mapsto U(t,s)y$ for every $y \in Y^\circ$ is a continuously differentiable solution to (2.42).

Consequently, U is at least an evolution system for A on Y° in the wide sense, and it remains to show that $[s,1] \ni t \mapsto U(t,s)y$ has values in Y° for every $y \in Y^\circ$. In order to do so one establishes, using the same arguments as for (2.61), the commutation relation

$$C^{(k)}(\underline{r})U_n(t,s)y = U_n(t,s)\big(C^{(k)}(\underline{r}) + S_n^{(k+1)}(t,s,\underline{r}) + \dots + S_n^{(p)}(t,s,\underline{r})\big)y \tag{2.62}$$

for all $y \in Y^\circ$ and $\underline{r} \in I^{k+1}$, $(s,t) \in \Delta$ and $k \in \{0, \dots, p-1\}$, where $S_n^{(k+l)}(t,s,\underline{r})$ is defined as the integral of

$$(\tau_1, \dots, \tau_l) \mapsto C^{(k+l)}(\underline{r}, r_n(\tau_1), \dots, r_n(\tau_l))/\alpha_{i_{n\,\tau_1},\dots,i_{n\,\tau_l}}$$

over the same domain of integration as in the definition of $S_n^{(l)}(t,s,r)$ in (2.61). Since the operators $C^{(k)}(\underline{r})$ are closed for $\underline{r} \in I^{k+1}$ and $k \in \{0,\dots,p-1\}$ by assumption, it follows from (2.62) that for every $y \in Y^\circ$ and $k \in \{0,\dots,p-1\}$ one has:

$$U(t,s)y \in D(C^{(k)}(\underline{r})) \text{ for every } \underline{r} \in I^{k+1} \text{ and } \underline{r} \mapsto C^{(k)}(\underline{r})U(t,s)y \text{ is continuous}$$

or, in other words, that $U(t,s)y \in Y^\circ$ for every $y \in Y^\circ$, as desired. ■

We also note the following variant of the above theorem where the form (2.63) of the imposed commutation relation is closer to (2.44). In return, one has to require relatively strong invariance conditions.

Proposition 2.3.4. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous semigroup on X such that A is (M,ω)-stable for some $M \in [1,\infty)$ and $\omega \in \mathbb{R}$ and recursively define $C^{(0)}(t) := A(t)$ as well as $C^{(k)}(t_1,\dots,t_{k+1}) := [C^{(k-1)}(t_1,\dots,t_k), A(t_{k+1})]$ for $k \in \mathbb{N}$. Suppose further that Y is an $A(t)$-admissible subspace of X for every $t \in I$, and $p \in \mathbb{N}$ a natural number such that for all $t_i \in I$*

$$Y \subset \bigcap_{\tau_1,\dots,\tau_p \in I} D(C^{(p-1)}(\tau_1,\dots,\tau_p)) \qquad \textit{and} \qquad C^{(p-1)}(t_1,\dots,t_p)Y \subset \bigcap_{\tau\in I} D(C^{(0)}(\tau)),$$

$C^{(k)}(t_1,\dots,t_{k+1})|_Y$ is a bounded operator from Y to X for all $k \in \{0,\dots,p-1\}$, and

$$C^{(p)}(t_1,\dots,t_{p+1})\big|_{D(\tilde{A}(t_{p+1}))} \subset \mu(t_1,\dots,t_{p+1}) \in \mathbb{C}, \tag{2.63}$$

where $\tilde{A}(t)$ is the part of $A(t)$ in Y. Suppose finally that $(t_1,\dots,t_{p+1}) \mapsto \mu(t_1,\dots,t_{p+1})$ and $(t_1,\dots,t_{k+1}) \mapsto C^{(k)}(t_1,\dots,t_{k+1})y$ are continuous for all $y \in Y$ and $k \in \{0,\dots,p-1\}$. Then there exists a unique evolution system U in the wide sense for A on Y.

Proof. We recall that, by our convention from the beginning of Section 2.3.2, the commutators $C^{(k)}(t_1,\dots,t_{k+1})$ are to be understood in the operator-theoretic sense, and we can therefore conclude that

$$Y \subset \bigcap_{\tau_1,\dots,\tau_{k+1} \in I} D(C^{(k)}(\tau_1,\dots,\tau_{k+1})) \qquad \text{and} \qquad C^{(k)}(t_1,\dots,t_{k+1})Y \subset \bigcap_{\tau\in I} D(C^{(0)}(\tau)),$$

for all $k \in \{0,\dots,p-1\}$ by successively proceeding from $p-1$ to 0. With this in mind, one verifies the commutation relations

$$\begin{aligned} C^{(k)}(\underline{s})e^{A(t)\tau}y = e^{A(t)\tau}\big(C^{(k)}(\underline{s}) + C^{(k+1)}(\underline{s},t)\tau + \dots + C^{(p-1)}(\underline{s},t,\dots,t)\frac{\tau^{p-1-k}}{(p-1-k)!} + \\ + \mu(\underline{s},t,\dots,t)\frac{\tau^{p-k}}{(p-k)!}\big)y \qquad (\underline{s} := (s_1,\dots,s_{k+1})) \end{aligned} \tag{2.64}$$

for all $y \in Y$ and $k \in \{0,\dots,p-1\}$ by proceeding from $p-1$ to 0 and by using, at each successive step, the same arguments as for (2.58). And from (2.64), in turn, one obtains the existence of an evolution system U in the wide sense for A on Y in exactly the same way as in the proof of Theorem 2.3.3. (It is not to be expected, however, that U is even an evolution system for A on Y in the strict sense. See the sixth remark in Section 2.3.4.) In order to obtain uniqueness, one has only to observe that for any evolution system V in the wide sense for A on Y,

$$U_n(t,s)y - V(t,s)y = V(t,\tau)U_n(\tau,s)y\big|_{\tau=s}^{\tau=t} = \int_s^t V(t,\tau)\big(A(r_n(\tau)) - A(\tau)\big)U_n(\tau,s)y\,d\tau$$

converges to 0 for every $y \in Y$ and $(s,t) \in \Delta$ by (2.61). ■

2.3.3 Well-posedness for group generators

After having proved well-posedness results for semigroup generators with (2.43) or (2.44), we now improve, inspired by [72], the special well-posedness result from [65] (Theorem 5.2 in conjunction with Remark 5.3) for a certain kind of group (instead of semigroup) generators $A(t)$ and certain uniformly convex subspaces Y of the domains $D(A(t))$: we show that this result is still valid if $t \mapsto A(t)|_Y$ is assumed to be only strongly continuous (instead of norm continuous as in [65]). In [72] the same is done for the general well-posedness theorem from [65] (Theorem 6.1). We point out that although several arguments from [72] can be used here as well, it is by no means obvious that the improvement made in [72] can be carried over to the special well-posedness result of [65]. In particular, the possibility of such an improvement is not mentioned in the literature – at least, not in [72], [136], [137], [68], [69], [126], [127]. In addition to the improvement from norm to strong continuity (which is the main point here), we also slightly generalize the compatibility condition for certain t-dependent norms $\|\,.\,\|_t^\pm$ from [65]. Instead of requiring

$$\|y\|_t^\pm \le e^{c^\pm|t-s|}\,\|y\|_s^\pm \tag{2.65}$$

for some constants $c^\pm$ and all $y \in Y$ and $s,t \in I$, we only require the compatibility condition (2.67) below for some functions

$$c^\pm : \mathcal{I} \to [0,\infty) \qquad (\mathcal{I} := \{\text{closed subintervals of } I\})$$

that are additive and continuous on $\mathcal{I}$ in the sense that

$$c^\pm([a,c]) = c^\pm([a,b]) + c^\pm([b,c]) \quad (a \le b \le c) \quad \text{and} \quad c^\pm(J) \longrightarrow 0 \quad (\lambda(J) \to 0). \tag{2.66}$$

In this way, the well-posedness theorem for quasicontraction group generators with time-independent domains in a uniformly convex space proved in Section 2.2.3 above becomes a special case of the theorem below. See the second remark after Theorem 2.2.10.

Theorem 2.3.5. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is the generator of a strongly continuous group on X such that $A^+ := A(\,.\,)$ and $A^- := -A(1-\,.\,)$ are (M,ω)-stable for some $M \in [1,\infty)$ and $\omega \in \mathbb{R}$. Suppose further that Y for every $t \in I$*

is an $A^{\pm}(t)$-admissible subspace of X contained in $\cap_{\tau\in I}D(A(\tau))$ and that $A(t)|_Y$ is a bounded operator from Y to X such that

$$t \mapsto A(t)|_Y$$

is strongly continuous. And finally, suppose there exist functions $c^{\pm}: \mathcal{I} \to [0,\infty)$ satisfying (2.66) and for each $t \in I$ there exists a norm $\|\,.\,\|_t^{\pm}$ on Y equivalent to the original norm of Y such that $Y_t^{\pm} := (Y, \|\,.\,\|_t^{\pm})$ is uniformly convex and

$$\|y\|_t^{\pm} \le e^{c^{\pm}(J(s,t))} \|y\|_s^{\pm} \qquad (J(s,t) := [\min\{s,t\}, \max\{s,t\}]) \tag{2.67}$$

for all $y \in Y$ and $s, t \in I$, and such that the Y-part $\tilde{A}^{\pm}(t)$ of $A^{\pm}(t)$ generates a quasi-contraction semigroup in $Y_t^{\pm}$, more precisely

$$\left\| e^{\tilde{A}^{\pm}(t)\tau} y \right\|_t^{\pm} \le e^{\omega_0 \tau} \|y\|_t^{\pm} \quad (\tau \in [0,\infty), y \in Y, t \in I) \tag{2.68}$$

for some t-independent growth exponent $\omega_0 \in \mathbb{R}$. Then there exists a unique evolution system U for A on Y (and, moreover, $\Delta \ni (s,t) \mapsto U(t,s)|_Y \in L(Y)$ is strongly continuous).

Proof. We adopt from [72] the shorthand notation $U^{\pm}(t,s,\pi)$ for products of the semigroups $e^{A^{\pm}(t)\,\cdot}$ associated with finite or infinite partitions π in I. Without further specification, convergence or continuity in X, Y will always mean convergence or continuity in the norm of X, Y.

As a first step we show that for each $y \in Y$ and $s \in [0,1)$ there exists a sequence $(\pi_n^{\pm}) = (\pi_{y,s,n}^{\pm})$ of partitions of I such that $(U^{\pm}(t,s,\pi_{y,s,n}^{\pm})y)$ is a Cauchy sequence in X for $t \in [s,1]$. What we have to show here is that for every sequence $\pi = (t_k)$, strictly monotonically increasing in I, and arbitrary $t_k' \in [t_k, t_{k+1})$, the following assertions are satisfied (Lemma 1 of [72]):

(i) $(U^{\pm}(t_k', t_0, \pi)x)$ is a Cauchy sequence in X for every $x \in X$ whose limit will be denoted by $U^{\pm}(t_\infty, t_0, \pi)x$ where $t_\infty := \lim_{k\to\infty} t_k'$,

(ii) $(U^{\pm}(t_k', t_0, \pi)y)$ is a Cauchy sequence in Y for every $y \in Y$.

With the help of Lemma 2 and 3 of [72], whose proofs carry over without change to the present situation, the existence of sequences $(\pi_{y,s,n}^{\pm})$ of partitions with the claimed properties then follows. Assertion (i) is simple and is proven in the same way as in [72], while assertion (ii) has to be proven in a completely different way because the proof of [72] essentially rests on the existence of certain isomorphisms $S(t)$ from Y onto X which are not available here. We show, using ideas from [65] (Section 5), that

$$U^{\pm}(t_\infty, t_0, \pi)y \in Y \quad \text{and} \quad U^{\pm}(t_k', t_0, \pi)y \longrightarrow U^{\pm}(t_\infty, t_0, \pi)y \quad \text{weakly in } Y \tag{2.69}$$

for every $y \in Y$ and that

$$\limsup_{k\to\infty} \left\|U^{\pm}(t_k', t_0, \pi)y\right\|_{\bar{t}_\infty}^{\mp} \le \left\|U^{\pm}(t_\infty, t_0, \pi)y\right\|_{\bar{t}_\infty}^{\mp} \quad (\bar{t}_\infty := 1 - t_\infty) \tag{2.70}$$

for $y \in Y$, which two things by the uniform convexity of $Y_{\bar{t}_\infty}$ imply the convergence

$$U^{\pm}(t'_k, t_0, \pi)y \longrightarrow U^{\pm}(t_\infty, t_0, \pi)y \text{ in } Y$$

and in particular assertion (ii). In order to see (2.69) notice first that $\tilde{A}^{\pm}$ is $(\tilde{M}, \tilde{\omega})$-stable for some $\tilde{M} \in [1, \infty)$ and $\tilde{\omega} = \omega_0$ by (2.67) and (2.68) (argue as in Proposition 3.4 of [65] using (2.66.a)), so that the sequence $(U^{\pm}(t'_k, t_0, \pi)y)$ is bounded in the norm of Y (recall that

$$e^{A^{\pm}(t)\tau}\big|_Y = e^{\tilde{A}^{\pm}(t)\tau}$$

by Proposition 2.3 of [65]). Since Y is reflexive (Milman's theorem), every subsequence of $(U^{\pm}(t'_k, t_0, \pi)y)$ has in turn a weakly convergent subsequence in Y whose weak limit must be equal to $U^{\pm}(t_\infty, t_0, \pi)y$ by assertion (i), and therefore (2.69) follows. In order to see (2.70) notice first that $(U^{\pm}(t'_k, t_n, \pi)x)_{n\in\mathbb{N}}$ is a Cauchy sequence in X for every $x \in X$ and $k \in \mathbb{N}$, where

$$U^{\pm}(t'_k, \tau, \pi) := U^{\pm}(\tau, t'_k, \pi)^{-1} = e^{-A^{\pm}(t_k)(t_{k+1}-t'_k)} \cdots e^{-A^{\pm}(r_\pi(\tau))(\tau - r_\pi(\tau))}$$

for $\tau \in (t'_k, t_\infty)$ and where $r_\pi(\tau)$ denotes the largest point of π less than or equal to τ. Indeed, for every $x \in Y$,

$$U^{\pm}(t'_k, t_m, \pi)x - U^{\pm}(t'_k, t_n, \pi)x = -\int_{t_m}^{t_n} U^{\pm}(t'_k, \tau, \pi)A^{\pm}(r_\pi(\tau))x \, d\tau \longrightarrow 0 \quad (m, n \to \infty)$$

in X and by the (M, ω)-stability of $A^{\mp}$, this convergence extends to all $x \in X$. We denote the limit by $U^{\pm}(t'_k, t_\infty, \pi)x$ and note for later use that

$$U^{\pm}(t'_k, t_\infty, \pi)y \in Y \quad \text{and} \quad U^{\pm}(t'_k, t_n, \pi)y \longrightarrow U^{\pm}(t'_k, t_\infty, \pi)y \quad \text{weakly in } Y \tag{2.71}$$

by the same arguments as those for (2.69). Since $U^{\pm}(t'_k, t_0, \pi) = U^{\pm}(t'_k, t_n, \pi)U^{\pm}(t_n, t_0, \pi)$ for all $n \in \mathbb{N}$, it follows that

$$U^{\pm}(t'_k, t_0, \pi) = U^{\pm}(t'_k, t_\infty, \pi)U^{\pm}(t_\infty, t_0, \pi). \tag{2.72}$$

Also, since

$$U^{\pm}(t'_k, t_n, \pi) = e^{A^{\mp}(\bar{t}_k)(t_{k+1}-t'_k)} \cdots e^{A^{\mp}(\bar{t}_{n-1})(t_n - t_{n-1})} \quad (\bar{t}_i := 1 - t_i)$$

for $n \geq k+1$, it follows by successively passing from $\|\,.\,\|^{\mp}_{\bar{t}_\infty}$ to $\|\,.\,\|^{\mp}_{\bar{t}_k}$ to ... to $\|\,.\,\|^{\mp}_{\bar{t}_{n-1}}$ and back to $\|\,.\,\|^{\mp}_{\bar{t}_\infty}$ with the help of (2.67), by using (2.68) at each successive step, and by using (2.66.a) that

$$\left\|U^{\pm}(t'_k, t_n, \pi)z\right\|^{\mp}_{\bar{t}_\infty} \leq e^{2c^{\mp}([\bar{t}_\infty, \bar{t}_k])}\, e^{\omega_0(t_n - t'_k)}\, \|z\|^{\mp}_{\bar{t}_\infty}$$

for every $z \in Y$, and therefore

$$\left\|U^{\pm}(t'_k, t_\infty, \pi)z\right\|^{\mp}_{\bar{t}_\infty} \leq e^{2c^{\mp}([\bar{t}_\infty, \bar{t}_k])}\, e^{\omega_0(t_\infty - t'_k)}\, \|z\|^{\mp}_{\bar{t}_\infty} \tag{2.73}$$

for $z \in Y$ by virtue of (2.71). Combining now (2.72), (2.73) and (2.66.b) we obtain (2.70), which concludes our first step.

As a second step we observe that $U_0^{\pm}(t,s)y := \lim_{n\to\infty} U^{\pm}(t,s,\pi^{\pm}_{y,s,n})y$ for $y \in Y$ and $(s,t) \in \Delta$ defines a linear operator from Y to X extendable to a bounded operator $U^{\pm}(t,s)$ in X, and that $U^{\pm}$ is an evolution system in X such that $t \mapsto U^{\pm}(t,s)y$ for every $y \in Y$ is right differentiable (in the norm of X) at s with right derivative $A^{\pm}(s)y$. All this follows in the same way as in [72] (Lemma 4 and 5). In particular, it follows from the right differentiability and evolution system properties just mentioned that $[0,t] \ni s \mapsto U^{\pm}(t,s)y$ is continuously differentiable (from both sides) for every $y \in Y$ with derivative $s \mapsto -U^{\pm}(t,s)A^{\pm}(s)y$ by Corollary 2.1.2 of [104].

As a third step we show that $U^{\pm}(t,s)$ leaves the subspace Y invariant for every $(s,t) \in \Delta$ and that $[s,1] \ni t \mapsto U^{\pm}(t,s)y$ is right continuous in Y for every $y \in Y$. In order to see that $U^{\pm}(t,s)y$ lies in Y for $y \in Y$, notice that the sequence $(U^{\pm}(t,s,\pi^{\pm}_{y,s,n})y)$ is bounded in the norm of Y, whence by the same argument as for (2.69)

$$U^{\pm}(t,s)y \in Y \quad \text{and} \quad U^{\pm}(t,s,\pi^{\pm}_{y,s,n})y \longrightarrow U^{\pm}(t,s)y \quad \text{weakly in } Y. \tag{2.74}$$

In order to see that $[s,1] \ni t \mapsto U^{\pm}(t,s)y$ is right continuous in Y for every $y \in Y$, we have only to show, by the invariance property just established, that $U^{\pm}(t+h,t)y \longrightarrow y$ in Y as $h \searrow 0$ for every $t \in [0,1)$. And for this in turn it is sufficient to show, by the uniform convexity of Y_t, that

$$U^{\pm}(t+h,t)y \longrightarrow y \quad \text{weakly in } Y \text{ as } h \searrow 0 \tag{2.75}$$

and

$$\limsup_{h\searrow 0} \left\| U^{\pm}(t+h,t)y \right\|_t^{\pm} \le \|y\|_t^{\pm} \tag{2.76}$$

Since this can be achieved in a way similar to the proof of (2.69) and (2.70), we may omit the details.

We can now show that $t \mapsto U^{+}(t,s)y$ is continuous in Y for every $y \in Y$ and then conclude the proof. Indeed, $\tau \mapsto U^{\mp}(1-s,1-\tau)z$ is differentiable for $z \in Y$ with derivative $\tau \mapsto U^{\mp}(1-s,1-\tau)A^{\mp}(1-\tau)z$ by the last remark of our second step and $\tau \mapsto U^{\pm}(\tau,s)y$ is right differentiable for $y \in Y$ with right derivative $\tau \mapsto A^{\pm}(\tau)U^{\pm}(\tau,s)y$ because for every $\tau \in [s,1)$ the vector $z := U^{\pm}(\tau,s)y$ lies in Y and

$$\frac{1}{h}\big(U^{\pm}(\tau+h,s)y - U^{\pm}(\tau,s)y\big) = \frac{1}{h}\big(U^{\pm}(\tau+h,\tau)z - z\big) \longrightarrow A^{\pm}(\tau)z \quad (h \searrow 0)$$

by our second and third step. So, the map $[s,t] \ni \tau \mapsto U^{\mp}(1-s,1-\tau)U^{\pm}(\tau,s)y$ is right differentiable for every $y \in Y$ with right derivative 0. Corollary 2.1.2 of [104] therefore yields

$$U^{\mp}(1-s,1-t)U^{\pm}(t,s)y - y = U^{\mp}(1-s,1-\tau)U^{\pm}(\tau,s)y\big|_{\tau=s}^{\tau=t} = 0$$

for every $y \in Y$ and hence

$$U^{\mp}(1-s,1-t)U^{\pm}(t,s) = 1 = U^{\mp}(1-\bar{t},1-\bar{s})U^{\pm}(\bar{s},\bar{t}) = U^{\mp}(t,s)U^{\pm}(1-s,1-t)$$

for all $(s,t) \in \Delta$. It follows that

$$U^+(t-h,s)y = U^+(t,t-h)^{-1}U^+(t,s)y = U^-(1-t+h,1-t)U^+(t,s)y \longrightarrow U^+(t,s)y$$

in Y as $h \searrow 0$ by our third step, whence $t \mapsto U^+(t,s)y$ right *and* left continuous and hence continuous in Y. Combining this with the previous steps, we see with the help of Corollary 2.1.2 of [104] that $t \mapsto U^+(t,s)y$ is continuously differentiable in X for every $y \in Y$ with derivative $t \mapsto A^+(t)U^+(t,s)y$ and therefore $U := U^+$ is an evolution system for $A = A^+$ on Y, as desired. ■

Incidentally, it is also possible to improve (a version of) the well-posedness theorem from [66] (Theorem 1) in the spirit of [72]: in this theorem strong continuity of $t \mapsto A(t)|_Y$ is sufficient as well, provided that A is (M,ω)-stable (instead of only quasistable) and that $t \mapsto \|B(t)\|$ is bounded (instead of only upper integrable). (We make this proviso in order to make sure that the boundedness condition (2.1) of [72] is still satisfied for arbitrary partitions π and that (2.2) of [72] is satisfied with the modified right hand side

$$C\,\|x\| \int_{t_i}^{t_k'} \alpha(\tau)\,d\tau,$$

where α is a suitable integrable function. All other arguments from [72] carry over without formal change, a bit more care being necessary in the justification of assertion (c) of [72] because of the weaker regularity of $t \mapsto S(t)$ – see [36].)

2.3.4 Some remarks on the relation with the literature

We close this section about abstract well-posedness results with some remarks concerning, in particular, the relation of the results from Section 2.3.2.1 and 2.3.2.2 with the results from [65], [66], [72], [101] and with the result from Section 2.3.3.

1. Compared to the well-posedness theorems from [65], [66], [72] where no commutator conditions of the kind (2.43) or (2.44) are imposed, the well-posedness theorems from Section 2.3.2.1 and 2.3.2.2 are furnished with rather mild stability and regularity conditions: Concerning stability, we had only to require in the theorems from Section 2.3.2.1 and 2.3.2.2 that the family A be (M,ω)-stable in X (or that the slightly weaker stability condition (2.80) be satisfied). In the well-posedness theorems from [65], [66], [72], by contrast, it has to be required in addition that there exist an $A(t)$-admissible subspace Y of X contained in all the domains of the $A(t)$ such that the induced family $\tilde{A}$ consisting of the Y-parts $\tilde{A}(t)$ of the $A(t)$ is $(\tilde{M},\tilde{\omega})$-stable in Y. Such a subspace Y is generally difficult to find – unless the domains of the $A(t)$ are time-independent. (In this latter case, one can choose $Y := D(A(0)) = D(A(t))$ endowed with the graph norm of $A(0)$, provided only that $t \mapsto A(t)$ is of bounded variation – just apply Proposition 4.4 of [65] with $S(t) := A(t) - (\omega + 1)$.) Concerning regularity, we had only to require strong continuity conditions in the theorems from Section 2.3.2.1 and 2.3.2.2: namely, we had to require that

$$t \mapsto C^{(0)}(t)y = A(t)y \quad \text{and} \quad (t_1,\dots,t_{k+1}) \mapsto C^{(k)}(t_1,\dots,t_{k+1})y$$

be continuous for $k \in \{1, \dots, p\}$ and y in a dense subspace Y of X contained in all the respective domains or, equivalently, that the maximal continuity subspaces (2.47) or (2.60) be dense in X and that μ be continuous. In general situations without commutator conditions of the kind (2.43) or (2.44), by contrast, strong continuity conditions are not sufficient for well-posedness – not even if the domains of the $A(t)$ are time-independent. (See the respective counterexamples in [105] (Example 6.4), [41] (Example VI.9.21), [116] (Example 1 and 2).) Accordingly, in the general well-posedness results from [65] (Theorem 6.1), [72], and [66] (Theorem 1) for general semigroup generators $A(t)$, there is a strong $W^{1,1}$-regularity condition on certain auxiliary operators $S(t)$ defined on an $A(t)$-admissible subspace Y of X contained in all the domains $D(A(t))$, which boils down to a strong $W^{1,1}$-regularity condition on $t \mapsto A(t)$ in the case of time-independent domains $D(A(t)) = Y$ (Remark 6.2 of [65]); and in the special well-posedness result (Theorem 5.2 and Remark 5.3) from [65] for group generators $A(t)$, there still is a norm continuity condition on $t \mapsto A(t)|_Y$ and a regularity condition on certain auxiliary norms $\|\,.\,\|_t^{\pm}$ on Y, which boils down to a Lipschitz continuity condition on $t \mapsto A(t)$ in the case of time-independent domains $D(A(t)) = Y$ (Theorem 2.1 of [116]).

2. In a certain special case involving group generators $A(t)$ with time-independent domains, the well-posedness assertion of the theorems from Section 2.3.2.1 and 2.3.2.2 can alternatively also be inferred from the well-posedness theorem from Section 2.3.3. In fact, if in addition to the assumptions of Theorem 2.3.3 the following three conditions are satisfied, then the well-posedness assertion of this theorem (but no representation formula, of course) also follows from Theorem 2.3.5:

- $A(t)$ for every $t \in I$ is a quasicontraction group generator with time-independent domain $D(A(t)) = Y$ in the uniformly convex space X such that

$$\left\| e^{\pm A(t)\tau} \right\| \le e^{\omega \tau} \quad (\tau \in [0, \infty)) \tag{2.77}$$

  for some t-independent growth exponent $\omega \in \mathbb{R}$,

- $C^{(k)}(t_1, \dots, t_{k+1})$ is a bounded operator on X for every $(t_1, \dots, t_{k+1}) \in I^{k+1}$ and

$$\sup_{(t_1,\dots,t_{k+1}) \in I^{k+1}} \left\| C^{(k)}(t_1, \dots, t_{k+1}) \right\| < \infty \tag{2.78}$$

  for every $k \in \{1, \dots, p-1\}$ (an empty condition for $p = 1$!),

- $t \mapsto A(t)y$ is continuous for every $y \in Y$.

Indeed, under these conditions the norms $\|\,.\,\|_t^{\pm}$ appearing in Theorem 2.3.5 can be chosen to be $\|\,.\,\|_* := \|(A(0) - \omega - 1)\,.\,\|$ for every $t \in I$ (t-independent!): with this norm, Y becomes a uniformly convex subspace admissible for the group generators $\pm A(t)$ and

$$\left\| e^{\pm A(t)\tau} y \right\|_* \le e^{\omega_0 \tau} \|y\|_* \quad (y \in Y \text{ and } \tau \in [0, \infty)) \tag{2.79}$$

for a suitable $\omega_0 \in \mathbb{R}$, and finally $Y^\circ = Y$. (In order to see (2.79) and the $\pm A(t)$-admissibility of Y one checks that (2.59) holds true for $\tau \in (-\infty, 0)$ as well, so that in particular

$$A(0)e^{\pm A(t)\tau}y = e^{\pm A(t)\tau}\big(A(0) + C^{(1)}(0,t)(\pm\tau) + \dots + C^{(p-1)}(0,t,\dots,t)(\pm\tau)^{p-1}/(p-1)! \\ + \mu(0,t,\dots,t)(\pm\tau)^p/p!\big)y$$

for all $y \in Y$ and $\tau \in [0,\infty)$. With the help of (2.77) and (2.78) the desired $\pm A(t)$-admissibility and the quasicontraction group property (2.79) then readily follow.)

3. In the well-posedness theorems from [56] and [94] weaker notions of well-posedness are used than here [95], which in return allows for weaker regularity assumptions than those of [66] and [72] (but the stability conditions are the same). In the second product representation theorem from [102] (Proposition 4.9) which also asserts well-posedness, there seems to be missing, in the hyperbolic case, an additional stability and regularity assumption of the kind of condition (ii") from [65]. At least, it is not clear [93] how the asserted well-posedness should be established and how the range condition from Chernoff's theorem (invoked in [102]) should be verified without such an additional assumption. (In this respect, see in particular Theorem 4.19 of [100] and the remarks preceding it, which state that $\mathcal{Y}$ is a core for $\mathcal{G}$ only under the additional condition (ii") from [65].) As far as [24] is concerned, it should be remarked that the abstract well-posedness theorem of this paper is actually a corollary of the well-posedness theorem of [66]. (Indeed, if for every $y \in Y$ the map $t \mapsto S(t)y$ is continuous on the whole of I and differentiable at all except countably many points of I with an exceptional set N not depending on y and if $\sup_{t\in I\setminus N} \|S'(t)y\| < \infty$, then $t \mapsto S(t)y$ is already absolutely continuous (Theorem 6.3.11 of [23]) and

$$S(t)y = S(0)y + \int_0^t S'(\tau)y\,d\tau$$

(Proposition 1.2.3 of [10]) for every $y \in Y$, so that the strong $W^{1,1}$-regularity condition for $t \mapsto S(t)$ from [66] is satisfied.)

4. It is clear from the proofs of Theorem 2.3.1 and Theorem 2.3.3 that the well-posedness statements remain valid if the (M,ω)-stability condition of these theorems is replaced by the condition from [101] that there exist a sequence (π_n) of partitions of I such that $\operatorname{mesh}(\pi_n) \longrightarrow 0$ and

$$\left\| e^{A(r_n(t))(t-r_n(t))} \dots e^{A(r_n(s))(r_n^+(s)-s)} \right\| \le Me^{\omega(t-s)} \quad ((s,t)\in\Delta). \tag{2.80}$$

In [101] this stability condition is shown to be strictly weaker than (M,ω)-stability. Also, it is clear from the proof of Theorem 2.3.1 that the representation formula for the evolution is still valid if (2.80) is sharpened to

$$\left\| e^{A(r_n(t))r(t-r_n(t))} \dots e^{A(r_n(s))r(r_n^+(s)-s)} \right\| \le Me^{\omega r(t-s)} \quad ((s,t)\in\Delta, r\in[0,\infty)). \tag{2.81}$$

In particular, the method of proof of Theorem 2.3.1 yields an alternative and more elementary proof (without reference to the Trotter–Kato theorem) of Proposition 2.5 from [101] (or, rather, of a slightly corrected version of it: in order for the proof of [101] to work one has to choose as the domain of $\int_s^t A(\tau)\,d\tau$ the maximal continuity subspace Y° of A as defined in (2.47), instead of the quite arbitrary subspace denoted by Y in [101] because such a subspace, in contrast to Y°, is not left invariant by the operators $(\overline{B}_n - \lambda)^{-1}$ in general).

5. In the situation of Theorem 2.3.1, one might think that it should be possible to (more efficiently) obtain the well-posedness of the initial value problems (2.42) on Y° by first defining a candidate U for the sought evolution system through the representation formula

$$U(t,s) := e^{\overline{(\int_s^t A(\tau)\,d\tau)^\circ}} e^{1/2 \int_s^t \int_s^\tau \mu(\tau,\sigma)\,d\sigma\,d\tau},$$

and by then verifying that this candidate is indeed an evolution system for A on Y°. In order to prove that the closure of $(\int_s^t A(\tau)\,d\tau)^\circ$ exists and is a semigroup generator, one might want to employ the theorem of Trotter and Kato as in [101] – instead of exploiting the locally uniform convergence of the sequences $(U_n^r(t,s)x)$ as we did. And in order to verify the evolution system properties for U, one might want to make rigorous the following formal differentiation rule for exponential operators (appearing in [134], for instance):

$$\frac{e^{B(t+h)} - e^{B(t)}}{h} = \frac{e^{B(t+h)\tau} e^{B(t)(1-\tau)}}{h}\Big|_{\tau=0}^{\tau=1} = \int_0^1 e^{B(t+h)\tau}\,\frac{B(t+h)-B(t)}{h}\,e^{B(t)(1-\tau)}\,d\tau$$
$$\longrightarrow \int_0^1 e^{B(t+h)\tau} B'(t)\, e^{B(t)(1-\tau)}\,d\tau \quad (h \to 0) \tag{2.82}$$

with $B(t) := \overline{(\int_s^t A(\tau)\,d\tau)^\circ}$. Yet, this is possible only if $\operatorname{Re}\mu(\tau,\sigma) \geq 0$ for all $\sigma \leq \tau$ because only then can the right hand side of (2.52) be dominated by a bound $M' e^{\omega' r}$ for all $r \in [0,\infty)$ uniformly in $n \in \mathbb{N}$ (a first crucial assumption of the Trotter–Kato theorem). And moreover, the verification of the density of $\operatorname{ran}((\int_s^t A(\tau)\,d\tau)^\circ - \lambda)$ in X for $\lambda > \omega'$ (a second crucial assumption of the Trotter–Kato theorem) and the verifications of the evolution system properties for U with the help of (2.82) are more involved than the arguments in our approach.

6. In Proposition 2.3.4 we obtained well-posedness on the given subspace Y only in the wide sense, that is, the existence of a unique evolution system U for A on Y in the wide sense. We could not prove the invariance of Y under the operators $U(t,s)$, however (while in the special case $p=1$ we could prove such an invariance for a different subspace, namely (2.47), in Corollary 2.3.2). And, in fact, we do not expect it to be true in general: at least, it is not possible to obtain this invariance – as in Theorem 2.3.3 – by a closedness argument from (2.62) (which equation is still true in the situation of Proposition 2.3.4 for vectors $y \in Y$) because in general, under the assumptions of Proposition 2.3.4, none of the operators $C^{(k)}(t_1,\dots,t_{k+1})|_Y$ will be a closed operator in X. Choose, for instance,

$$A(t) := A_0 + B(t) = A_0 + b(t)B_0 \quad \text{in } X := L^1(I),$$

where $A_0 f := \partial_x f$ for $f \in D(A_0) = \{f \in W^{1,1}(I) : f(1) = 0\}$ and $(B_0 f)(x) := x^p f(x)$ for $f \in X$ and where $t \mapsto b(t) \in \mathbb{C}$ is continuous, and then choose

$$Y := D(A_0^2) \text{ if } p = 1 \quad \text{and} \quad Y := D(A_0^p) \text{ if } p \in \mathbb{N} \setminus \{1\}$$

endowed with the norm of $W^{2,1}(I)$ or $W^{p,1}(I)$, respectively. It is then easy to see that, indeed, all the assumptions of Proposition 2.3.4 are satisfied, but $C^{(k)}(t_1, \dots, t_{k+1})|_Y$ is non-closed for every $k \in \{0, \dots, p\}$ and every $(t_1, \dots, t_{k+1}) \in I^{k+1}$. (We point out that in this specific example one nevertheless does have the invariance of Y under the operators $U(t,s)$ for $s \le t$, but this seems to essentially depend on the specific structure of the example: since $A(t)$ is A_0 plus a bounded perturbation $B(t)$, the evolution system U for A on Y in the wide sense satisfies

$$U(t,s)f = e^{A_0(t-s)} f + \int_s^t e^{A_0(t-\tau)} B(\tau) U(\tau, s) f \, d\tau \quad (f \in X)$$

and hence is given by the respective perturbation series expansion; and since $e^{A_0 \cdot}|_Y$ is a strongly continuous semigroup in Y and $B(t)|_Y$ is a bounded operator in Y, this perturbation series leaves Y invariant.)

2.3.5 Some applications of the well-posedness theorems for operators with scalar commutators

We now discuss some applications of the abstract results from Section 2.3.2. In all of them the operators $A(t)$ will be skew self-adjoint in a Hilbert space X.

2.3.5.1 Segal field operators

In this subsection we apply the well-posedness result of Section 2.3.2.1 to Segal field operators $\Phi(f_t)$ in $\mathcal{F}_+(\mathfrak{h})$, the symmetric Fock space over a complex Hilbert space $\mathfrak{h}$. Segal field operators $\Phi(f)$ are defined for $f \in \mathfrak{h}$ as the closure of $2^{-1/2}(a(f) + a^*(f))$, where $a(f)$ and $a^*(f)$ are the usual annihilation and creation operators in $\mathcal{F}_+(\mathfrak{h})$ corresponding to f. It is well-known that the operators $\Phi(f)$ are self-adjoint and, as a consequence of the canonical commutation relations for creation and annihilation operators, they satisfy the commutation relations

$$[\Phi(f), \Phi(g)] = i \operatorname{Im} \langle f, g \rangle \quad (f, g \in \mathfrak{h}) \tag{2.83}$$

on a suitable dense subspace of $\mathcal{F}_+(\mathfrak{h})$. See [18] (Section 5.2.1), [107] (Section X.7) or [35] (Section 5.4) for these and other standard facts about such operators and basic concepts from quantum field theory. In view of (2.83) we expect to obtain well-posedness for the operators $A(t) = i\Phi(f_t)$ by means of Theorem 2.3.1. Indeed, we have:

Corollary 2.3.6. *Set $A(t) = i\Phi(f_t)$ in $X := \mathcal{F}_+(\mathfrak{h})$ and suppose that $t \mapsto f_t \in \mathfrak{h}$ is continuous. Then there exists a unique evolution system U for A on the maximal continuity subspace Y° for A and it is given by*

$$U(t,s) = e^{\overline{(\int_s^t i\Phi(f_\tau)\, d\tau)^\circ}} e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im}\langle f_\tau, f_\sigma \rangle \, d\sigma \, d\tau} = W\Big(\int_s^t f_\tau \, d\tau \Big) e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im}\langle f_\tau, f_\sigma \rangle \, d\sigma \, d\tau}$$

where $W(h) := e^{i\Phi(h)}$ *denotes the Weyl operator for* $h \in \mathfrak{h}$.

Proof. We have already remarked that the operators $A(t)$ are skew self-adjoint and hence (semi)group generators. We also see, by the Weyl form

$$\Phi(f)e^{i\Phi(g)} = e^{i\Phi(g)}\big(\Phi(f) - \operatorname{Im}\langle f, g\rangle\big) \tag{2.84}$$

of the canonical commutation relations (Proposition 5.2.4 (1) in [18]), that the generators $A(s)$ can be commuted through the groups $e^{A(t)\cdot}$ in the way required in (2.46) with $\mu(s,t) = -i\operatorname{Im}\langle f_s, f_t\rangle$. It remains to show that the maximal continuity subspace Y° for A is a dense subspace of X. In order to do so, one uses that for every $f \in \mathfrak{h}$ one has: $D(N^{1/2}) \subset D(\Phi(f))$ and

$$\|\Phi(f)\psi\| = \left\|2^{-1/2}(a(f) + a^*(f))\psi\right\| \leq 2^{1/2}\,\|f\|\,\left\|(N+1)^{1/2}\psi\right\| \tag{2.85}$$

for every $\psi \in D(N^{1/2})$ (Lemma 5.3 of [35]), where N is the number operator in $\mathcal{F}_+(\mathfrak{h})$. Since $t \mapsto f_t$ is continuous by assumption, the estimate (2.85) shows that the maximal continuity subspace Y° for A contains the dense subspace $D(N^{1/2})$ of X and is therefore dense itself. So, the desired well-posedness statement and the first of the asserted representation formulas for U follow from Theorem 2.3.1. In order to see the second representation formula for U, repeatedly apply the identity

$$W(f)W(g) = W(f+g)e^{-i/2\operatorname{Im}\langle f,g\rangle} \tag{2.86}$$

(Proposition 5.2.4 (2) of [18]) to the approximants U_n for U from the proof of Theorem 2.3.1 and use the strong continuity of $\mathfrak{h} \ni h \mapsto W(h)$ (Proposition 5.2.4 (4) of [18]). Alternatively, the well-posedness statement and the first representation formula could also be concluded from Corollary 2.3.2 with $Y := D(N)$ endowed with the graph norm of N. Indeed, Y with this norm is an $A(t)$-admissible subspace of X because

$$Ne^{i\Phi(f)} = e^{i\Phi(f)}\big(N + \Phi(if) + \|f\|^2/2\big)$$

for all $f \in \mathfrak{h}$ (Proposition 2.2 of [87]), $Y \subset \cap_{\tau\in I} D(A(\tau))$ and $A(t)Y \subset D(N^{1/2}) \subset \cap_{\tau\in I} D(A(\tau))$ by the definition of creation and annihilation operators, $A(t)|_Y$ is a bounded operator from Y to X by (2.85), and finally $[A(s), A(t)]|_{D(N)} \subset -i\operatorname{Im}\langle f_s, f_t\rangle$ (Proposition 5.2.3 (3) of [18]). ■

It is possible to give at least two alternative proofs of variants of the above result and we briefly comment on these alternative approaches (which, however, are not necessary for understanding Corollary 2.3.7 below). A first alternative approach is based upon the fifth remark from Section 2.3.4, which is applicable here because $\operatorname{Re}\mu(\tau,\sigma) = 0$ for all $\sigma, \tau \in I$. It yields the following version of Corollary 2.3.6: if $t \mapsto f_t \in \mathfrak{h}$ is continuous, then there exists a unique evolution system U for A on the maximal continuity subspace Y° for A and U is given by the first representation formula of the corollary. A second alternative – and more pedestrian – approach is based upon a well-known exponential series expansion for Weyl operators, namely (2.88) below, and yields the following version of Corollary 2.3.6 for $\mathfrak{h} = L^2(\mathbb{R}^3)$: if both

$$t \mapsto f_t \in \mathfrak{h} = L^2(\mathbb{R}^3) \quad \text{and} \quad t \mapsto f_t/\sqrt{\omega} \in \mathfrak{h} = L^2(\mathbb{R}^3)$$

are continuous for a measurable function $\omega : \mathbb{R}^3 \to \mathbb{R}$ with $\omega(k) > 0$ for almost all $k \in \mathbb{R}^3$, then there exists a unique evolution system U in the wide sense for A on $Y = D(H_\omega^{1/2})$ (where H_ω is the second quantization of ω defined in (2.89) and (2.90) below) and U is given by the second representation formula from the corollary above. In order to see this by pedestrian arguments, one defines a candidate for the sought evolution U in the wide sense through

$$U(t,s) := W\Big(\int_s^t f_\tau \, d\tau\Big) \, e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im}\langle f_\tau, f_\sigma\rangle \, d\sigma \, d\tau} \tag{2.87}$$

and exploits the exponential series expansion

$$W(g)\psi = e^{i\Phi(g)}\psi = \sum_{n=0}^{\infty} \frac{i^n}{n!} \, \Phi(g)^n \psi \qquad (g \in \mathfrak{h}) \tag{2.88}$$

for Weyl operators $W(g)$ on vectors ψ in the finite particle subspace $\mathcal{F}_+^0(\mathfrak{h}) := \{\psi \in \mathcal{F}_+(\mathfrak{h}) : \psi^{(n)} = 0 \text{ for all but finitely many } n\}$. (See the proof of Theorem X.41 of [107].) With this expansion, one can show by term-wise differentiation and repeated application of the commutation relation $[\Phi(f), \Phi(g)]|_{\mathcal{F}_+^0(\mathfrak{h})} \subset i \operatorname{Im}\langle f, g\rangle$ (Theorem X.41 (c) of [107]) that the mapping $t \mapsto U(t,s)\psi$ is differentiable for $\psi \in \mathcal{F}_+^0(\mathfrak{h})$ with the desired derivative $t \mapsto i\Phi(f_t)U(t,s)\psi$. Since $\mathcal{F}_+^0(\mathfrak{h})$ is a core for $\Phi(f_t)|_Y$ uniformly in $t \in I$ by virtue of (2.91) below (recall, $Y = D(H_\omega^{1/2})$) and since the operators $\Phi(f_t)$ can be commuted through $U(t,s)$ up to scalar errors by virtue of (2.84), there exists for every $\psi \in Y$ a sequence (ψ_n) in $\mathcal{F}_+^0(\mathfrak{h})$ such that $\psi_n \longrightarrow \psi$ and

$$i\Phi(f_t)U(t,s)\psi_n \longrightarrow U(t,s)\Big(i\Phi(f_t)\psi - i\int_s^t \operatorname{Im}\langle f_t, f_\tau\rangle \, d\tau\psi\Big) = i\Phi(f_t)U(t,s)\psi$$

uniformly in $t \in I$ as $n \to \infty$. It follows that $t \mapsto U(t,s)\psi = \lim_{n\to\infty} U(t,s)\psi_n$ is continuously differentiable even for $\psi \in Y$ with the desired derivative. So, U defined by (2.87) is indeed an evolution system in the wide sense for A on $Y = D(H_\omega^{1/2})$, and it is also unique by virtue of [62] (Theorem 1).

With the help of the above well-posedness result for Segal field operators we will now establish the well-posedness of the initial value problems for operators $H_\omega + \Phi(f_t)$ in $\mathcal{F}_+(\mathfrak{h})$ with $\mathfrak{h} := L^2(\mathbb{R}^3)$. Such operators are sometimes called van Hove Hamiltonians and they describe a classical particle coupled to a time-dependent quantized field of bosons: H_ω describes the energy of the field while $\Phi(f_t)$ describes the interaction of the particle with the field. (See, for instance, [33] or [70].) The operator H_ω is the second quantization of the dispersion relation $\omega : \mathbb{R}^3 \to \mathbb{R}$, a measurable function with $\omega(k) > 0$

for almost every $k \in \mathbb{R}^3$, that is, H_ω is the operator on $\mathcal{F}_+(\mathfrak{h}) = \bigoplus_{n\in\mathbb{N}\cup\{0\}} \mathfrak{h}_+^{(n)}$ defined by

$$(H_\omega\psi)^{(n)} := H_\omega^{(n)}\psi^{(n)} \quad \text{for} \quad \psi \in D(H_\omega) := \{\psi \in \mathcal{F}_+(\mathfrak{h}) : (H_\omega^{(n)}\psi^{(n)}) \in \mathcal{F}_+(\mathfrak{h})\}, \quad (2.89)$$

where the operators $H_\omega^{(n)}$ act by multiplication as follows:

$$H_\omega^{(0)}\psi^{(0)} := 0 \quad \text{and} \quad (H_\omega^{(n)}\psi^{(n)})(k_1, \dots, k_n) := \sum_{i=1}^{n} \omega(k_i)\psi^{(n)}(k_1, \dots, k_n) \qquad (2.90)$$

for $\psi^{(0)} \in \mathfrak{h}_+^{(0)} = \mathbb{C}$ and $\psi^{(n)} \in \mathfrak{h}_+^{(n)} = L^2_+(\mathbb{R}^{3n}) := \{\varphi \in L^2(\mathbb{R}^{3n}) : \varphi(k_{\sigma(1)}, \dots, k_{\sigma(n)}) = \varphi(k_1, \dots, k_n)$ for all permutations $\sigma\}$ (Example 1 in Section X.7 of [107]). It is well-known and easy to see that H_ω is a positive self-adjoint operator and that for all f with $f \in \mathfrak{h}$ and $f/\sqrt{\omega} \in \mathfrak{h}$ one has: $D(H_\omega^{1/2}) \subset D(\Phi(f))$ and

$$\|\Phi(f)\psi\| \leq 2^{1/2}\big(\|f\|^2 + \|f/\sqrt{\omega}\|^2 \big)^{1/2} \|(H_\omega + 1)^{1/2}\psi\| \qquad (2.91)$$

for all $\psi \in D(H_\omega^{1/2})$. (See, for instance, (13.70) of [121] or (20.33) and (20.34) of [54].) With the help of (2.91) it easily follows that $\Phi(f)$ is infinitesimally bounded w.r.t. H_ω and hence that $H_\omega + \Phi(f)$ is self-adjoint on $D(H_\omega)$ provided $f \in \mathfrak{h}$ and $f/\sqrt{\omega} \in \mathfrak{h}$.

Corollary 2.3.7. *Set $A(t) = -i(H_\omega + \Phi(f_t))$ in $X := \mathcal{F}_+(\mathfrak{h})$, where $\mathfrak{h} := L^2(\mathbb{R}^3)$ and $\omega : \mathbb{R}^3 \to \mathbb{R}$ is measurable with $\omega(k) > 0$ for almost all $k \in \mathbb{R}^3$, and suppose that $t \mapsto f_t/\sqrt{\omega} \in \mathfrak{h}$ is continuous and $t \mapsto f_t \in \mathfrak{h}$ is absolutely continuous. Then there exists a unique evolution system U for A on $D(H_\omega)$ and it is given by* (2.92) *and* (2.94) *below.*

Proof. It follows from the remarks above that the operators $A(t)$ are skew self-adjoint with time-independent domain $D(H_\omega)$ because $f_t, f_t/\sqrt{\omega} \in \mathfrak{h}$ by assumption. Since at least formally $[H_\omega, i\Phi(g)] = \Phi(i\omega g)$ by virtue of Lemma 2.5 (ii) of [32], the p-fold commutators (2.44) will not collapse to a complex scalar in general. We can therefore not hope to apply the results from Section 2.3.2.1 and 2.3.2.2 directly. We can, however, reduce the desired assertion to Corollary 2.3.6 by switching to the interaction picture, that is, we define a candidate for the sought evolution system U as the interaction picture evolution,

$$U(t,s) := e^{-iH_\omega t}\tilde{U}(t,s)e^{iH_\omega s}, \qquad (2.92)$$

where $\tilde{U}$ denotes the evolution system for $\tilde{A}$ with $\tilde{A}(t) := -ie^{iH_\omega t}\Phi(f_t)e^{-iH_\omega t}$. It has to be shown, of course, that this evolution exists on an appropriate dense subspace, and this can be done by way of Corollary 2.3.6. Indeed,

$$e^{iH_\omega t}\Phi(f)e^{-iH_\omega t} = \Phi(e^{i\omega t}f) \quad (f \in \mathfrak{h}, t \in \mathbb{R}) \qquad (2.93)$$

by Theorem X.41 (e) of [107], that is, the operator $\tilde{A}(t)$ is (i times) a Segal field operator,

$$\tilde{A}(t) = -ie^{iH_\omega t}\Phi(f_t)e^{-iH_\omega t} = i\Phi(\tilde{f}_t) \quad \text{with} \quad \tilde{f}_t := -e^{i\omega t}f_t$$

and $t \mapsto \tilde{f}_t$ is obviously continuous. So, by Corollary 2.3.6, the evolution system $\tilde{U}$ exists on the maximal continuity subspace $\tilde{Y}^\circ$ for $\tilde{A}$ and is given by

$$\tilde{U}(t,s) = W(g_{t,s}) e^{-i/2 \int_s^t \int_s^\tau \operatorname{Im}\langle \tilde{f}_\tau, \tilde{f}_\sigma \rangle \, d\sigma \, d\tau} \quad \text{with} \quad g_{t,s} := \int_s^t \tilde{f}_\tau \, d\tau. \tag{2.94}$$

We now show that U, given by (2.92) and (2.94), is an evolution system for A on $D(H_\omega)$. In order to see that $t \mapsto U(t,s)\psi$ is differentiable for all $\psi \in D(H_\omega)$ with the desired derivative

$$t \mapsto -i(H_\omega + \Phi(f_t))U(t,s)\psi, \tag{2.95}$$

we have to show in view of (2.92) that

$$D(H_\omega) \subset \tilde{Y}^\circ \quad \text{and} \quad \tilde{U}(t,s)D(H_\omega) \subset D(H_\omega) \quad ((s,t) \in \Delta). \tag{2.96}$$

And in order to show that (2.95) is continuous, we would like to move the unbounded operators H_ω and $\Phi(f_t)$ through the constituents $e^{-iH_\omega t}$ and $W(g_{t,s})$ of $U(t,s)$ in a suitable way. We first show the inclusion (2.96.a) and the continuity of $t \mapsto \Phi(f_t)U(t,s)\psi$ for all $\psi \in D(H_\omega)$. It easily follows from (2.91) and the assumed continuity of $t \mapsto f_t, f_t/\sqrt{\omega} \in \mathfrak{h}$ that

$$D(H_\omega) \subset D(H_\omega^{1/2}) \subset \tilde{Y}^\circ \tag{2.97}$$

and hence that (2.96.a) holds true. It also follows from (2.93) and (2.84) that

$$\begin{aligned} \Phi(f_t)e^{-iH_\omega t} &= e^{-iH_\omega t}\Phi(e^{i\omega t} f_t), \\ \Phi(e^{i\omega t} f_t)W(g_{t,s}) &= W(g_{t,s})\big(\Phi(e^{i\omega t} f_t) - \operatorname{Im}\langle e^{i\omega t} f_t, g_{t,s}\rangle\big). \end{aligned}$$

So, $t \mapsto \Phi(f_t)e^{-iH_\omega t}W(g_{t,s})\psi$ and hence $t \mapsto \Phi(f_t)U(t,s)\psi$ is continuous for $\psi \in D(H_\omega)$ because $t \mapsto \Phi(e^{i\omega t} f_t)\psi$ is continuous for $\psi \in D(H_\omega)$ by (2.91) and because $t \mapsto W(g_{t,s})$ is strongly continuous by Proposition 5.2.4 (4) of [18]. We now show the inclusion (2.96.b) and the continuity of $t \mapsto H_\omega U(t,s)\psi$ for all $\psi \in D(H_\omega)$ by showing that $W(g_{t,s})D(H_\omega) \subset D(H_\omega)$ and that H_ω can be moved through $W(g_{t,s})$ in a suitable way. It is here that the assumed absolute continuity of $t \mapsto f_t$ will come into play. Since

$$W(g)D(H_\omega) = D(H_\omega) \quad \text{and} \quad H_\omega W(g) = W(g)\big(H_\omega + \Phi(i\omega g) + \langle g, \omega g\rangle/2\big) \tag{2.98}$$

for every $g \in D(\omega) = \{h \in \mathfrak{h} : \omega h \in \mathfrak{h}\}$ (Lemma 2.5 (ii) of [32]), we are led to showing that

$$g_{t,s} \in D(\omega) \quad \text{and} \quad t \mapsto \omega g_{t,s} \in \mathfrak{h} \text{ is continuous.} \tag{2.99}$$

In order to do so, notice that the map $\tau \mapsto f_\tau$, being absolutely continuous with values in the reflexive space $\mathfrak{h}$, is differentiable almost everywhere (Corollary 1.2.7 of [10]) and that $\tau \mapsto e^{i\omega\tau}(i\omega+1)^{-1}$ is strongly continuously differentiable. We can therefore (Proposition 1.2.3 of [10]) perform the following integration by parts:

$$\begin{aligned}-g_{t,s} &= \int_s^t e^{i\omega\tau}(i\omega+1)^{-1} f_\tau \, d\tau + \int_s^t e^{i\omega\tau} i\omega \, (i\omega+1)^{-1} f_\tau \, d\tau \\ &= (i\omega+1)^{-1}\Big(\int_s^t e^{i\omega\tau} f_\tau \, d\tau + e^{i\omega\tau} f_\tau \Big|_{\tau=s}^{\tau=t} - \int_s^t e^{i\omega\tau} f_\tau' \, d\tau \Big).\end{aligned}$$

So, (2.99) follows. With the help of (2.99) we now obtain from (2.98) the following conclusions: first, that $W(g_{t,s})D(H_\omega) = D(H_\omega)$ for all $(s,t) \in \Delta$ and hence that (2.96.b) holds true and second, that $t \mapsto H_\omega W(g_{t,s})\psi$ and hence $t \mapsto H_\omega U(t,s)\psi$ is continuous for $\psi \in D(H_\omega)$ because $t \mapsto \Phi(i\omega g_{t,s})\psi$ is continuous for $\psi \in D(H_\omega)$ by (2.91) and (2.99) and because $t \mapsto W(g_{t,s})$ is strongly continuous by Proposition 5.2.4 (4) of [18]. ■

If one suitably strengthens or modifies the assumptions of Corollary 2.3.7, one can conclude the well-posedness statement of that corollary (but not the representation (2.92) and (2.94) for the evolution, of course) by means of various general well-posedness theorems. Indeed, if for instance one adds the assumption that $t \mapsto f_t/\sqrt{\omega} \in \mathfrak{h}$ be absolutely continuous as well, then the well-posedness statement of Corollary 2.3.7 can also be concluded from [66] (Theorem 1) because, under the thus strengthened assumptions, the strong $W^{1,1}$-regularity condition on $t \mapsto A(t)$ required in [66] can be verified by means of (2.91). Similarly, if one replaces the absolute continuity condition on $t \mapsto f_t$ by the assumption that both $t \mapsto f_t$ and $t \mapsto f_t/\sqrt{\omega}$ be of bounded variation and continuous, then the well-posedness statement of Corollary 2.3.7 can be concluded from [62] (Theorem 3). It is not difficult to find functions ω and f_t as in the above corollary such that $t \mapsto f_t/\sqrt{\omega}$ is not of bounded variation (so that Corollary 2.3.7 does not follow from [62]). Choose, for instance, $f_0 \in \mathfrak{h}$ with $f_0(k) = 1$ for $|k| \le 1$, and $\alpha \in [3/2, 3)$ and then set

$$\omega(k) := |k|^\alpha \quad \text{and} \quad f_t(k) := e^{i\omega(k)^{-1/2}t} f_0(k) \quad (k \in \mathbb{R}^3).$$

2.3.5.2 Schrödinger operators for external electric fields

In this subsection we apply the well-posedness result of Section 2.3.2.2 to Schrödinger operators $-\Delta + b(t)\cdot x$ in $L^2(\mathbb{R}^d)$. Such operators describe a quantum particle in a time-dependent spatially constant electric field $b(t) \in \mathbb{R}^d$ and they are shown to be essentially self-adjoint below. Setting $A(t) = \overline{i\Delta - ib(t)\cdot x}$, we obtain by formal computation

$$[A(t_1), A(t_2)] = 2\sum_{\kappa=1}^{d} (b_\kappa(t_2) - b_\kappa(t_1))\partial_\kappa, \quad \big[[A(t_1), A(t_2)], A(t_3)\big] = \mu(t_1,t_2,t_3) \quad (2.100)$$

with $\mu(t_1,t_2,t_3) := -2i\sum_{\kappa=1}^d (b_\kappa(t_2) - b_\kappa(t_1))b_\kappa(t_3) \in \mathbb{C}$. In view of (2.100) we expect to obtain well-posedness for the operators $A(t)$ by means of Theorem 2.3.3 with $p = 2$. Indeed, we have (see also the remarks below):

Corollary 2.3.8. *Set $A(t) = \overline{A_0 + B(t)}$ in $X := L^2(\mathbb{R}^d)$ (existence of the closure is shown below), where $A_0 := i\Delta$ with $D(A_0) = W^{2,2}(\mathbb{R}^d)$ and where $B(t)$ is multiplication by $-ib(t) \cdot x$, and suppose $t \mapsto b(t) \in \mathbb{R}^d$ is continuous. Then there exists a unique evolution system U for A on the maximal continuity subspace Y° for $A = C^{(0)}$ and $C^{(1)}$ defined in* (2.104). *Additionally, U is given by* (2.105) *and* (2.106) *below.*

Proof. (i) We first show that $A_0 + B(t_0)$ for every $t_0 \in I$ is essentially skew self-adjoint and that the unitary group generated by $A := \overline{A_0 + B(t_0)}$ is given by

$$e^{At} = e^{A_0 t} e^{Bt} e^{-\partial_1 b_1 t^2} \cdots e^{-\partial_d b_d t^2} e^{2ib^2 t^3/3} \quad (t \in \mathbb{R}), \tag{2.101}$$

where $B := B(t_0)$ and $b = (b_1, \dots, b_d) := b(t_0) \in \mathbb{R}^d$. We do so by showing that the right hand side of (2.101), which we abbreviate as $T(t)$, defines a strongly continuous unitary group in X with

$$A_0 + B \subset A_T \quad \text{and} \quad T(t)D(A_0 + B) \subset D(A_0 + B) \quad (t \in \mathbb{R}),$$

where A_T stands for the generator of T. (In order to understand why $e^{A\,\cdot}$ should decompose as in (2.101), plug the following formal commutators

$$[B, A_0] = -2\sum_{\kappa=1}^{d} b_\kappa \partial_\kappa, \quad [[B, A_0], B] = 2ib^2, \quad [[B, A_0], A_0] = 0$$

into the Zassenhaus formula [86], [122], [21] for bounded operators.) With the help of the explicit formulas for the groups $e^{A_0\,\cdot}$ (free Schrödinger group), $e^{B\,\cdot}$ (multiplication group), $e^{\partial_\kappa\,\cdot}$ (translation group) we find the following commutation relations,

$$\begin{gathered} e^{A_0 t} e^{\partial_\kappa s} = e^{\partial_\kappa s} e^{A_0 t}, \quad e^{Bt} e^{\partial_\kappa s} = e^{\partial_\kappa s} e^{Bt} e^{i b_\kappa t s}, \\ e^{A_0 t} e^{Bs} = e^{Bs} e^{A_0 t} e^{2\partial_1 b_1 ts} \cdots e^{2\partial_d b_d ts} e^{-ib^2 ts^2} \qquad (s, t \in \mathbb{R}). \end{gathered} \tag{2.102}$$

It follows from (2.102) that T is indeed a strongly continuous unitary group and that

$$\begin{gathered} e^{\partial_\kappa s} D(A_0) \subset D(A_0), \quad e^{\partial_\kappa s} D(B) \subset D(B), \quad e^{Bs} D(A_0) \subset D(A_0), \\ e^{A_0 t} D(A_0 + B) \subset D(B) \qquad (s, t \in \mathbb{R}), \end{gathered}$$

so that $T(t)D(A_0 + B) \subset D(A_0 + B)$ for all $t \in \mathbb{R}$ and $A_0 + B \subset A_T$. Consequently, $A_0 + B$ is essentially skew self-adjoint and $A = \overline{A_0 + B}$ is equal to A_T. After these preparations we can now verify the assumptions of Theorem 2.3.3 for $p = 2$. Indeed, using the commutation relations (2.102) we find that

$$e^{C_{12}\sigma} e^{A_3 \tau} = e^{A_3 \tau} e^{C_{12}\sigma} e^{\mu_{123}\tau\sigma}, \quad e^{A_1 \sigma} e^{A_2 \tau} = e^{A_2 \tau} e^{A_1 \sigma} e^{C_{12}\tau\sigma} e^{\mu_{122}\tau^2\sigma/2} e^{\mu_{121}\tau\sigma^2/2} \tag{2.103}$$

for all $\sigma, \tau \in \mathbb{R}$, where $A_j := A(t_j) = C^{(0)}(t_j)$, $b_j := b(t_j)$, $\mu_{jkl} := -2i\sum_{\kappa=1}^{d}(b_{k\,\kappa} - b_{j\,\kappa}) b_{l\,\kappa}$, and

$$C_{jk} = C^{(1)}(t_j, t_k) \text{ is the closure of } 2\sum_{\kappa=1}^{d}(b_{k\,\kappa} - b_{j\,\kappa})\partial_\kappa, \tag{2.104}$$

that is, C_{jk} generates the translation group $t \mapsto e^{2(b_{k\,1}-b_{j\,1})\partial_1 t} \dots e^{2(b_{k\,d}-b_{j\,d})\partial_d t}$. And from (2.103), in turn, the commutation relations imposed in Theorem 2.3.3 follow by differentiation at $\sigma = 0$. Since, moreover, the maximal continuity subspace for $A = C^{(0)}$ and $C^{(1)}$ contains the dense subspace of Schwartz functions on $\mathbb{R}^d$, the existence of a unique evolution system U for A on Y° follows by Theorem 2.3.3.

(ii) We now show the following representation formula for U:

$$U(t,s) = W(t)\tilde{U}(t,s)W(s)^{-1} = e^{\overline{(\int_0^t B(\tau)\,d\tau)^\circ}}\, e^{\overline{\int_s^t \tilde{A}(\tau)\,d\tau}}\, e^{-\overline{(\int_0^s B(\tau)\,d\tau)^\circ}}, \tag{2.105}$$

where $\tilde{U}$ is the evolution system for $\tilde{A}$ on $D := W^{2,2}(\mathbb{R}^d)$ with $\tilde{A}(t) := -i(-i\nabla - c(t))^2$ and $c(t) := \int_0^t b(\tau)\,d\tau$ and where the gauge transformation W is the evolution system for B on Z°, the maximal continuity subspace for B. Clearly, since $B(\tau) = -ib(\tau)\cdot x$ and $\tilde{A}(\tau) = -i\mathcal{F}^{-1}(\xi - c(\tau))^2\mathcal{F}$,

$$e^{\overline{(\int_0^t B(\tau)\,d\tau)^\circ}} = e^{-i\int_0^t b(\tau)\cdot x\,d\tau} \quad \text{and} \quad e^{\overline{\int_s^t \tilde{A}(\tau)\,d\tau}} = \mathcal{F}^{-1}e^{-i\int_s^t (\xi - c(\tau))^2\,d\tau}\mathcal{F} \tag{2.106}$$

(which last expression could be cast into a more explicit integral form similar to the explicit integral representation of the free Schrödinger group). It should be noticed that, due to the pairwise commutativity of the opertors $\tilde{A}(t)$ and of the operators $B(t)$, the existence of the evolution systems $\tilde{U}$ and W, and the second equality in (2.105) already follow by [49] and [101]. In order to see the first equality in (2.105), one shows by similar arguments as those of part (i) above that the subspace $Y_0^\circ := D \cap Z^\circ$ of Y° is invariant under $W(s)^{-1}$, $\tilde{U}(t,s)$, $W(t)$ and that

$$A_0 W(t)f = W(t)\tilde{A}(t)f$$

$$B(r)\tilde{U}(t,s)f = \tilde{U}(t,s)\Big(B(r)f - 2\sum_{\kappa=1}^d b_\kappa(r)\,(t-s)\,\partial_\kappa f + 2i\sum_{\kappa=1}^d b_\kappa(r)\int_s^t c_\kappa(\tau)\,d\tau\,f\Big)$$

for $f \in Y_0^\circ$. (Show commutation relations for $e^{\tilde{A}(r_1)\sigma}$ and $e^{B(r_2)\tau}$ analogous to (2.102) to obtain commutation relations for $B(r_2)$ with $e^{\tilde{A}(r_1)\sigma}$ and then use the standard product approximants for the evolution systems W and $\tilde{U}$.) It then follows that U_0 defined by $U_0(t,s) := W(t)\tilde{U}(t,s)W(s)^{-1}$ is an evolution system for A on Y_0°, which by the standard uniqueness argument for evolution systems must coincide with U. ■

We see from part (ii) of the above proof that the existence of an evolution system U_0 for A on the subspace Y_0°, after a suitable gauge transformation, already follows by [49], [101] – but in order to obtain well-posedness on Y°, the results from [49], [101] do not suffice, because the subspace Y_0° is strictly contained in Y° in general. (Indeed, if for instance $b(t) \equiv 1 \in \mathbb{R}^d$ with $d = 1$, then the function ψ with $\psi(\xi) := e^{i\xi^3/3}/\xi$ for $\xi \in [1,\infty)$ and $\psi(\xi) := 0$ for $\xi \in (-\infty, 1)$ does not belong to the range of $C - i := i\partial_\xi + \xi^2 - i$. Consequently, $-\partial_x^2 + x - i = \mathcal{F}^{-1}(C-i)\mathcal{F}$ is not surjective so that

$$Y_0^\circ = D(A_0 + B) = D(-\partial_x^2 + x) \subsetneq D(\overline{-\partial_x^2 + x}) = D(A) = Y^\circ$$

by the standard criterion for self-adjointness.) We finally remark that the results of [138] do not apply to the situation of this section.

3 Spectral-theoretic and other preliminaries for general adiabatic theory

3.1 Spectral operators: basic facts

3.1.1 Spectral measures, spectral integrals, spectral operators

We recall here some basic facts about spectral operators that will be needed in the sequel and we begin with the definition of spectral measures. A *spectral measure P on* $(\mathbb{C}, \mathcal{B}_{\mathbb{C}}, X)$ is a map from $\mathcal{B}_{\mathbb{C}}$ to the set of bounded projections on X such that

(i) $P(\emptyset) = 0$ and $P(\mathbb{C}) = 1$,

(ii) $P(E \cap F) = P(E)P(F)$ for all $E, F \in \mathcal{B}_{\mathbb{C}}$,

(iii) $P(\cup_{n=1}^{\infty} E_n)x = \sum_{n=1}^{\infty} P(E_n)x$ for all $x \in X$ and all pairwise disjoint sets $E_n \in \mathcal{B}_{\mathbb{C}}$.

If, in addition, $X = H$ is Hilbert space and $P(E)$ is an orthogonal projection for every $E \in \mathcal{B}_{\mathbb{C}}$, then P is a spectral measure in the sense of spectral theory for normal operators and we sometimes call P an *orthogonal spectral measure on* $(\mathbb{C}, \mathcal{B}_{\mathbb{C}}, X)$. Sometimes, we will also use the notation $P_E := P(E)$ for a spectral measure P.

Clearly, every spectral measure P induces $\mathbb{C}$-valued measures $P_{x^*,x}$ through

$$P_{x^*,x}(E) := \langle x^*, P(E)x \rangle \quad (x^* \in X^*, x \in X)$$

and it is often convenient to reduce statements about spectral measures to $\mathbb{C}$-valued measures in this way. (We point out that spectral measures in the sense above are precisely the *countably addititve* spectral measures in the terminology of [39] (Definition XV.2.1). Since we only work with countably additive measures here, this slight deviation in terminology should not cause any confusion.)

We will also need *spectral integrals*

$$\int f \, dP = \int f(z) \, dP(z)$$

of $\mathcal{B}_{\mathbb{C}}$-measurable functions $f : \mathbb{C} \to \mathbb{C}$ with respect to arbitrary spectral measures P, which are defined in essentially the same way as in the case of orthogonal spectral measures: for simple measurable functions f one defines

$$\int f \, dP := \sum_{k=1}^{m} \alpha_k P(E_k) \qquad (f = \sum_{k=1}^{m} \alpha_k \chi_{E_k}). \tag{3.1}$$

Since $M := \sup_{E\in\mathcal{B}_{\mathbb{C}}} \|P(E)\|$ is finite by Corollary XV.2.4 of [39] (which basically uses only that the $\mathbb{C}$-valued measures $P_{x^*,x}$ are majorized by their total variation measures $|P_{x*,x}|$ in conjunction with the Banach–Steinhaus theorem), it can be shown by using the Jordan–Hahn decomposition of the the $\mathbb{C}$-valued measures $P_{x^*,x}$ that

$$\left\| \int f \, dP \right\| \le 4M \, \|f\|_\infty$$

for all simple measurable functions f. (In the case of orthogonal spectral measures the right-hand side of this estimate can be improved to $\|f\|_\infty$, of course.) With the help of this estimate and the fact that every bounded measurable function f can be uniformly approximated by simple measurable functions, one can then extend (3.1) to bounded measurable functions f, namely:

$$\int f \, dP := \lim_{n\to\infty} \int f_n \, dP, \tag{3.2}$$

where (f_n) is an arbitrary sequence of simple measurable functions converging uniformly to f. See Chapter X.1 of [39]. And finally, (3.2) is extended to possibly unbounded measurable functions f in the following way:

$$\begin{aligned} D\Big(\int f \, dP\Big) &:= \Big\{x \in X : \Big(\int f \, \chi_{\{|f|\le n\}} \, dPx\Big) \text{ converges in } X\Big\}, \\ \int f \, dPx &:= \lim_{n\to\infty} \int f \, \chi_{\{|f|\le n\}} \, dPx \end{aligned} \tag{3.3}$$

for $x \in D(\int f \, dP)$. See Chapter XVIII.1 of [39]. It can be shown that spectral integrals with respect to arbitrary spectral measures have the following properties (Theorem XVIII.1.11 of [39]), familiar from the case of orthogonal spectral measures.

Proposition 3.1.1. *Suppose P is a spectral measure on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}}, X)$ and $f, g : \mathbb{C} \to \mathbb{C}$ are measurable functions.*

(i) $\int f \, dP$ is a densely defined closed operator. It is bounded if and only if f is P-essentially bounded (which means that there is $N \in \mathcal{B}_{\mathbb{C}}$ with $P(N) = 0$ such that $f\chi_{\mathbb{C}\setminus N}$ is bounded), and in that case

$$\left\| \int f \, dP \right\| \le 4M \inf \big\{ \sup |f|\chi_{\mathbb{C}\setminus N} : N \in \mathcal{B}_{\mathbb{C}} \text{ with } P(N) = 0\big\}.$$

(ii) $\int f \, dP + \int g \, dP \subset \int f + g \, dP$ and $(\int f \, dP)(\int g \, dP) \subset \int fg \, dP$ with domains

$$\begin{aligned} D\Big(\int f \, dP + \int g \, dP\Big) &= D\Big(\int f + g \, dP\Big) \cap D\Big(\int g \, dP\Big), \\ D\Big(\int f \, dP \int g \, dP\Big) &= D\Big(\int fg \, dP\Big) \cap D\Big(\int g \, dP\Big). \end{aligned}$$

(iii) $\int f\,dP$ *is injective if and only if* $P_{\{f=0\}} = 0$. *In that case* $(\int f\,dP)^{-1} = \int \frac{1}{f}\,dP$.

(iv) If $x \in D(\int f\,dP)$ *and* $x^* \in X^*$, *then* f *is* $P_{x^*,x}$*-integrable and*

$$\left\langle x^*, \int f\,dPx \right\rangle = \int f\,dP_{x^*,x}.$$

We now recall the definition of spectral operators. A densely defined closed operator $A : D(A) \subset X \to X$ is called *spectral operator* if and only if there exists a spectral measure P on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}}, X)$ such that

$$P(E)A \subset AP(E) \quad \text{and} \quad \sigma(A|_{P(E)D(A)}) \subset \overline{E}$$

for every $E \in \mathcal{B}_{\mathbb{C}}$ and such that $P(E)D(A) = P(E)X$ for every bounded $E \in \mathcal{B}_{\mathbb{C}}$. Such a spectral measure P is called a *spectral measure for* A or a *resolution of the identity for* A. It can be shown (Corollary XV.3.8 and Theorem XVIII.1.5 of [39]) that for a given spectral operator A there exists only one spectral measure (called the *spectral measure of* A and often denoted by P^A).

Simple examples of spectral operators are furnished by the class of normal operators on a Hilbert space and the class of arbitrary operators on a finite-dimensional space: in the first case the spectral measure is, of course, given by the orthogonal spectral measure from the spectral theorem for normal operators, and in the second case the spectral measure is given by

$$P(E) = \sum_{\lambda \in E} P_\lambda = \sum_{\lambda \in E \cap \sigma(A)} P_\lambda \qquad (E \in \mathcal{B}_{\mathbb{C}})$$
$$P_\lambda := \frac{1}{2\pi i} \int_{\gamma_\lambda} (z - A)^{-1}\,dz \quad (\lambda \in \mathbb{C}),$$

where $\gamma_\lambda = \partial U_r(\lambda)$ and $r = r_\lambda$ is chosen so small that $\overline{U}_r(\lambda) \cap \sigma(A) \subset \{\lambda\}$. In particular, it follows that all multiplication operators $A = M_f$ on $X = L^2(X_0, \mu)$ (with $(X_0, \mathcal{A}, \mu)$ an arbitrary measure space) are spectral operators. See Chapter XV.11 and XV.12 and Chapter XIX and XX of [39] for more interesting – differential operator – examples of spectral operators.

In order to get some intuition for spectral operators we note three immediate consequences of the definition above.

1. If A is a spectral operator and $E \in \mathcal{B}_{\mathbb{C}}$, then the restriction $A|_{P^A(E)D(A)}$ is a spectral operator as well with spectral measure given by

$$P^{A|_{P^A(E)D(A)}}(F) = P^A(F)|_{P^A(E)X} = P^A(F \cap E)|_{P^A(E)X} \quad (F \in \mathcal{B}_{\mathbb{C}}). \tag{3.4}$$

In particular, if the set E is bounded, then the operator $A|_{P^A(E)D(A)} = A|_{P^A(E)X}$ is bounded.

2. If A is a spectral operator, then $P^A(\sigma(A)) = 1$ and $P^A(E) = 0$ for every $E \in \mathcal{B}_{\mathbb{C}}$ with $E \subset \mathbb{C} \setminus \sigma(A)$. In particular, if $\sigma(A)$ is bounded, then the operator $A = AP^A(\sigma(A))$

is bounded as well. (In order to see that indeed the spectral measure of A vanishes outside $\sigma(A)$, approximate $\mathbb{C} \setminus \sigma(A)$ by the bounded closed sets

$$E_n := \left\{ z \in \overline{U}_n(0) : \operatorname{dist}(z, \sigma(A)) \geq 1/n \right\} \in \mathcal{B}_{\mathbb{C}}.$$

It follows that $A|_{P^A(E_n)D(A)} = A|_{P^A(E_n)X}$ is a bounded operator with

$$\sigma(A|_{P^A(E_n)D(A)}) \subset E_n \cap \sigma(A) = \emptyset \qquad (n \in \mathbb{N})$$

and therefore $P^A(E_n)X$ must be 0 (because a bounded operator can have empty spectrum only on the trivial space 0) or, in other words, $P^A(E_n) = 0$ for all $n \in \mathbb{N}$. Since $E_n \nearrow \mathbb{C} \setminus \sigma(A)$, it further follows that

$$(1 - P^A(\sigma(A)))x = P^A(\mathbb{C} \setminus \sigma(A))x = \lim_{n\to\infty} P^A(E_n)x = 0$$

for all $x \in X$. Consequently, $P^A(E) = P^A(E \cap \mathbb{C} \setminus \sigma(A)) = P^A(E)P^A(\mathbb{C} \setminus \sigma(A)) = 0$ for every subset $E \in \mathcal{B}_{\mathbb{C}}$ of $\mathbb{C} \setminus \sigma(A)$ and, in particular, for $E = \mathbb{C} \setminus \sigma(A)$.)

3. If A is a spectral operator and $E \in \mathcal{B}_{\mathbb{C}}$ is an isolated subset of $\sigma(A)$, then

$$\sigma(A|_{P^A(E)D(A)}) = E \quad \text{and} \quad \sigma(A|_{(1-P^A(E))D(A)}) = \sigma(A) \setminus E. \tag{3.5}$$

If, in addition, $E \neq \emptyset$ is bounded, the above equalities imply an explicit expression for the projection $P^A(E)$, namely

$$P^A(E) = \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} \, dz,$$

where γ is a cycle in $\rho(A)$ with indices $\mathrm{n}(\gamma, E) = 1$ and $\mathrm{n}(\gamma, \sigma(A) \setminus E) = 0$. See Theorem 3.2.1 below. (In order to see (3.5), one can argue as follows. Since E is isolated in $\sigma(A)$, both E and $\sigma(A) \setminus E$ are closed sets and therefore

$$\sigma(A|_{P^A(E)D(A)}) \subset E \quad \text{and} \quad \sigma(A|_{(1-P^A(E))D(A)}) \subset \sigma(A) \setminus E$$

because $1 - P^A(E) = P^A(\sigma(A) \setminus E)$ by the preceding remark. Since, moreover,

$$\sigma(A|_{P^A(E)D(A)}) \cup \sigma(A|_{(1-P^A(E))D(A)}) = \sigma(A)$$

by the commutativity of the projection $P^A(E)$ with A, the inclusions above cannot be strict, as desired.)

3.1.2 Special classes of spectral operators: scalar type and finite type

We will also need certain special classes of spectral operators, namely the spectral operators of scalar type and of finite type, respectively. An operator $A : D(A) \subset X \to X$ is called

(i) *spectral operator of scalar type* if and only if $A = \int z \, dP(z)$ for some spectral measure P on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}}, X)$,

(ii) *spectral operator of finite type* if and only if $A = S + N$ for some bounded spectral operator S of scalar type and some nilpotent operator N with $SN = NS$.

Simple examples of spectral operators of scalar type are, of course, the normal operators on a Hilbert space. In fact, every spectral operator A of scalar type on a Hilbert space $X = H$ is essentially (up to similarity transformation) a normal operator. (In order to see this, notice that there exists a bijective bounded operator T such that $P_0(E) := T^{-1}P^A(E)T$ is self-adjoint for every $E \in \mathcal{B}_\mathbb{C}$ (Theorem 1 of [133]). So, P_0 is an orthogonal spectral measure on $(\mathbb{C}, \mathcal{B}_\mathbb{C}, H)$ and

$$A = \int z \, dP^A(z) = T A_0 T^{-1},$$

where $A_0 := \int z \, dP_0(z)$ is normal.) In [47] one finds more specific examples: it is shown there that the generic one-dimensional periodic Schrödinger operator is spectral of scalar type (Remark 8.7). Simple examples of spectral operators of finite type are the operators on finite-dimensional spaces (Jordan normal form theorem!).

It can be shown that spectral operators of scalar or finite type really are spectral operators: for every spectral measure P on $(\mathbb{C}, \mathcal{B}_\mathbb{C}, X)$, the operator $\int z \, dP(z)$ is spectral with spectral measure P (Lemma XVIII.2.13 of [39]); and for every bounded spectral operator S of scalar type and every nilpotent operator N with $SN = NS$, the operator $S + N$ is bounded spectral with spectral measure P^S. In fact, one has the following characterization of bounded spectral operators (Theorem XV.4.5 of [39]).

Theorem 3.1.2. *An operator A on X is a bounded spectral operator if and only if $A = S + N$ for some bounded spectral operator S of scalar type and some quasinilpotent operator N with $SN = NS$. Additionally, S and N with the above properties are uniquely determined by A, namely $S = \int z \, dP^A(z)$ and $N = A - S$.*

It is natural to ask whether an analogous characterization holds true for unbounded spectral operators: for instance, one could conjecture that an operator $A : D(A) \subset X \to X$ is a spectral operator if and only if $A = S + N$ for some spectral operator S of scalar type and some quasinilpotent operator N with $SN \supset NS$. In fact, the "if" implication is true by the theorem below (a special case of Theorem XVIII.2.28 of [39]), while the "only if" implication is false at least for the canonical candidate $S = \int z \, dP^A(z)$ (by the remarks preceding Theorem XVIII.2.28 of [39]).

Theorem 3.1.3. *If $A = S + N$ for a spectral operator S of scalar type and some quasinilpotent operator N with $SN \supset NS$, then A is a spectral operator.*

Proof. Since N commutes with S by assumption, N also commutes with the spectral measure P^S of S: $NP^S(E) = P^S(E)N$ for all $E \in \mathcal{B}_\mathbb{C}$ (Corollary XVIII.1.4 of [39]). So the desired conclusion follows from Theorem XVIII.2.28 of [39]. ■

3.1.3 Spectral theory of spectral operators

We will finally also need some facts from the spectral theory of bounded spectral operators, most importantly, the facts from the following proposition (Theorem XV.8.2, Theorem XV.8.3 and Theorem XV.8.6 of [39]).

Proposition 3.1.4. *Suppose A is a bounded spectral operator on X (with spectral measure P^A) and $\lambda \in \sigma(A)$.*

(i) If $\lambda \in \sigma_p(A)$, then $P^A(\{\lambda\}) \neq 0$.

(ii) If $P^A(\{\lambda\}) = 0$, then $\lambda \in \sigma_c(A)$.

If, in particular, A is of finite type, then $\sigma_r(A) = \emptyset$ and for every $\lambda \in \sigma(A)$ one has: $\lambda \in \sigma_p(A)$ iff $P^A(\{\lambda\}) \neq 0$ and $\lambda \in \sigma_c(A)$ iff $P^A(\{\lambda\}) = 0$.

Proof. We give a proof here in order to make clear how little of the extensive material from Section XV.7 and XV.8 of [39] is really needed for the assertions (i) and (ii). Without loss of generality we can assume that $\lambda = 0$ since with A also $A - \lambda$ is a bounded spectral operator – with spectral measure given by $P^{A-\lambda}(E) = P^A(E + \lambda)$ for $E \in \mathcal{B}_{\mathbb{C}}$.

(i) Suppose $\lambda = 0 \in \sigma_p(A)$. We show that $P^A(\mathbb{C} \setminus \{0\})x = 0$ for every eigenvector x of A with eigenvalue 0 (which then implies $P^A(\{0\})x = P^A(\mathbb{C})x = x \neq 0$, as desired). Indeed, for every such eigenvector x and every closed subset E of $\mathbb{C} \setminus \{0\}$, the operator $A_E := A|_{P^A(E)X}$ is boundedly invertible (because $\sigma(A_E) \subset \overline{E} = E$) and so

$$P^A(E)x = A_E^{-1} A_E P^A(E)x = A_E^{-1} A P^A(E)x = A_E^{-1} P^A(E) Ax = 0.$$

Since $\mathbb{C} \setminus \{0\}$ is approximated from below by the closed subsets $E_n := \{z \in \mathbb{C} : |z| \geq 1/n\}$ of $\mathbb{C} \setminus \{0\}$, we obtain $P^A(\mathbb{C} \setminus \{0\})x = 0$, as claimed.

(ii) Suppose $P^A(\{\lambda\}) = P^A(\{0\}) = 0$. We show that $\operatorname{ran} A$ is dense in X (which together with (i) implies $0 \in \sigma_c(A)$, as desired). Indeed, if $\operatorname{ran} A$ was not dense in X, then there would exist a bounded operator $B \neq 0$ with $BA = 0$ by the Hahn–Banach theorem. It follows from this that, for every closed subset E of $\mathbb{C} \setminus \{0\}$,

$$BP^A(E)|_{P^A(E)X} = BA_E A_E^{-1} = BAA_E^{-1} = 0 \quad \text{and} \quad BP^A(E)|_{(1-P^A(E))X} = 0$$

and so $BP^A(E) = 0$. Since $\mathbb{C} \setminus \{0\}$ is approximated from below by the closed subsets $E_n := \{z \in \mathbb{C} : |z| \geq 1/n\}$ of $\mathbb{C} \setminus \{0\}$, we also obtain $BP^A(\mathbb{C} \setminus \{0\}) = 0$ and therefore $B = BP^A(\mathbb{C} \setminus \{0\}) + BP^A(\{0\}) = 0$. Contradiction!

Suppose finally that A is of finite type, that is, $A = S + N$ where $S = \int z \, dP^A(z)$ and $N = A - S$ is nilpotent. We show that $\sigma_r(A) = \emptyset$ by showing that whenever $\lambda \in \sigma(A)$ and $A - \lambda$ is injective, then $\lambda \in \sigma_c(A)$. So, let $\lambda \in \sigma(A)$ and let $A - \lambda$ be injective. Then the restriction $A_{\{\lambda\}} - \lambda = A|_{P^A(\{\lambda\})X} - \lambda$ to the subspace $P^A(\{\lambda\})X$ is injective as well and, on the other hand, this restriction $A_{\{\lambda\}} - \lambda = S_{\{\lambda\}} - \lambda + N_{\{\lambda\}} = N_{\{\lambda\}}$ is nilpotent. Consequently, $P^A(\{\lambda\})X = 0$ or $P^A(\{\lambda\}) = 0$ and hence $\lambda \in \sigma_c(A)$ by assertion (ii) above, as desired. We have thus shown that $\sigma_r(A) = \emptyset$ and by the same argument we see for every $\lambda \in \sigma(A)$ that if $P^A(\{\lambda\}) \neq 0$, then $\lambda \in \sigma_p(A)$ and that if $\lambda \in \sigma_c(A)$, then $P^A(\{\lambda\}) = 0$. ■

We have just seen that the spectral operators of finite type have empty residual spectrum, but there exist bounded spectral operators with $\sigma_r(A) \neq \emptyset$. (See, for instance, the example following Corollary XV.8.5 of [39].) In separable spaces, however, the residual spectrum and the point spectrum of a bounded spectral operator can at least not get too large: since

$$\sigma_p(A) \cup \sigma_r(A) \subset \{\lambda \in \sigma(A) : P^A(\{\lambda\}) \neq 0\}$$

by the above proposition, $\sigma_p(A)$ and $\sigma_r(A)$ must be countable if X is separable (Theorem XV.8.7). A simple consequence of this is that the left and right shift operators S_- and S_+ on $X = \ell^2(\mathbb{N})$ cannot be spectral (because they have $\sigma_p(S_-) = U_1(0) = \sigma_r(S_+)$ (Section 3.5)).

3.2 Spectrally related projections: associatedness and weak associatedness, (weak) semisimplicity

As was explained in Section 1.2.2, a first necessary preliminary step in general adiabatic theory is to identify a natural notion of spectral relatedness. In order to do so we introduce the following notion of associatedness (which is completely canonical) and weak associatedness (which – for non-normal, or at least, non-spectral operators – is not canonical).

Suppose $A : D(A) \subset X \to X$ is a densely defined closed linear map with $\rho(A) \neq \emptyset$, $\sigma \neq \emptyset$ is a compact isolated subset of $\sigma(A)$, λ a not necessarily isolated spectral value of A, and P a bounded projection in X. We then say, following [129], that *P is associated with A and σ* if and only if P commutes with A, $PD(A) = PX$ and

$$\sigma(A|_{PD(A)}) = \sigma \quad \text{whereas} \quad \sigma(A|_{(1-P)D(A)}) = \sigma(A) \setminus \sigma.$$

We say that *P is weakly associated with A and λ* if and only if P commutes with A, $PD(A) = PX$ and

$$A|_{PD(A)} - \lambda \text{ is nilpotent whereas } A|_{(1-P)D(A)} - \lambda \text{ is injective and}$$
$$\text{has dense range in } (1-P)X.$$

If above the order of nilpotence is at most m, we will often, more precisely, speak of P as being *weakly associated with A and λ of order m*. (It should be noticed that the above definition allows weakly associated projections to be zero, which however will be not relevant in our adiabatic theorems below.) Also, we call λ a *weakly semisimple eigenvalue of A* if and only if λ is an eigenvalue and there is a projection P weakly associated with A and λ of order 1. In this context, recall that λ is called a *semisimple eigenvalue of A* if and only if it is a pole of the resolvent map $(\,.\, - A)^{-1}$ of order 1 (which is then automatically an eigenvalue by (3.6) below). Also, a semisimple eigenvalue is called *simple* if and only if its geometric multiplicity is 1.

We point out that, for spectral values λ of a densely defined operator A with $\rho(A) \neq \emptyset$ and bounded projections P commuting with A, weak associatedness is a fairly natural notion of spectral relatedness. Indeed, it is more than natural to take spectral relatedness to mean at least that $A|_{PD(A)}$ is bounded with $\sigma(A|_{PD(A)}) = \{\lambda\}$ (or in other words, that $PD(A) = PX$ and $A|_{PD(A)} - \lambda$ is quasinilpotent) and that $\lambda \notin \sigma_p(A|_{(1-P)D(A)})$. (It is important to notice here that, if λ is non-isolated in $\sigma(A)$, then it must belong to $\sigma(A|_{(1-P)D(A)})$ by the closedness of spectra.) And then it is natural to furhter require that $A|_{PD(A)} - \lambda$ be even nilpotent (instead of only quasinilpotent) and that λ belong to the continuous (instead of the residual) spectrum of $A|_{(1-P)D(A)}$, which finally is nothing but the weak associatedness of P with A and λ.

3.2.1 Central facts about associatedness and weak associatedness

We now state some central facts about associatedness and weak associatedness, concerning the question of existence and uniqueness of (weakly) associated projections (for given operators A and spectral values λ) and the question of describing (in terms of A and λ) the subspaces into which a (weakly) associated projection decomposes the base space X. We will use these facts again and again and they play an important role in our adiabatic theorems. It should be pointed out that the stated facts about associatedness are completely well-known, while the stated facts about weak associatedness seem to be new (and are complementary to Corollary 2.2 of [77], which covers the case of isolated spectral values).

Theorem 3.2.1. *Suppose $A : D(A) \subset X \to X$ is a densely defined closed linear map with $\rho(A) \neq \emptyset$ and $\emptyset \neq \sigma \subset \sigma(A)$ is compact. If σ is isolated in $\sigma(A)$, then there exists a unique projection P associated with A and σ, namely*

$$P := \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} \, dz,$$

where γ is a cycle in $\rho(A)$ with indices $\mathrm{n}(\gamma, \sigma) = 1$ and $\mathrm{n}(\gamma, \sigma(A) \setminus \sigma) = 0$. If P is associated with A and $\sigma = \{\lambda\}$ and λ is a pole of $(\,.\, - A)^{-1}$ of order m, then

$$PX = \ker(A - \lambda)^k \quad \text{and} \quad (1 - P)X = \operatorname{ran}(A - \lambda)^k \tag{3.6}$$

for all $k \in \mathbb{N}$ with $k \geq m$.

Proof. See, for instance, [112] (Theorem 2.14 and Proposition 2.15) or [48] for detailed proofs of the existence and uniqueness statement and Theorem 5.8-A of [128] for a proof of (3.6). ■

Theorem 3.2.2. *Suppose $A : D(A) \subset X \to X$ is a densely defined closed linear map with $\rho(A) \neq \emptyset$ and $\lambda \in \sigma(A)$. If λ is non-isolated in $\sigma(A)$, then in general there exists no projection P weakly associated with A and λ, but if such a projection exists it is already unique. If P is weakly associated with A and λ of order m, then*

$$PX = \ker(A - \lambda)^k \quad \text{and} \quad (1 - P)X = \overline{\operatorname{ran}}(A - \lambda)^k \tag{3.7}$$

for all $k \in \mathbb{N}$ with $k \geq m$.

Proof. We first show that a projection P weakly associated with A and λ decomposes the space X according to (3.7), and from this we will conclude the existence and uniqueness statement.

So, let P be weakly associated with A and λ. We may clearly assume that $\lambda = 0$ because P, being weakly associated with A and λ, is also weakly associated with $A - \lambda$ and 0. Set $M := PX$ and $N := (1 - P)X$. We first show that $M = \ker A^k$ for all $k \geq m$. Since $A|_{PX} = A|_{PD}$ is nilpotent of order m, $A^k|_{PX} = (A|_{PX})^k = 0$ and hence $M = PX \subset \ker A^k$ for all $k \geq m$. And since $A|_{(1-P)D(A)}$ is injective,

$$A^k|_{(1-P)D(A^k)} = (A|_{(1-P)D(A)})^k$$

is injective as well and hence $\ker A^k \subset PX = M$ for all $k \in \mathbb{N}$. We now show that $N = \overline{\operatorname{ran}}\, A^k$ for all $k \geq m$. As $PX = \ker A^k$ for $k \geq m$, we have

$$\operatorname{ran} A^k = A^k PD(A^k) + A^k(1 - P)D(A^k) = (1 - P)A^k D(A^k) \subset (1 - P)X = N$$

and therefore $\overline{\operatorname{ran}}\, A^k \subset N$ for all $k \geq m$. It remains to show that the reverse inclusion $N \subset \overline{\operatorname{ran}}\, A^k$ holds true for all $k \in \mathbb{N}$ and this will be done by induction over k. Since $A|_{(1-P)D(A)}$ has dense range in $(1 - P)X = N$, the desired inclusion is clearly satisfied for $k = 1$. Suppose now that $N \subset \overline{\operatorname{ran}}\, A^k$ is satisfied for some arbitrary $k \in \mathbb{N}$. Since

$$\operatorname{ran} A|_{(1-P)D(A)} = A(1 - P)D(A) = A(z_0 - A)^{-1}N$$

and since $A(z_0 - A)^{-1}$ is a bounded operator for every $z_0 \in \rho(A)$, it then follows by the induction hypothesis that $A(z_0 - A)^{-1}N \subset \overline{\operatorname{ran}}\, A^{k+1}$ and hence

$$N = \overline{\operatorname{ran}}\, A|_{(1-P)D(A)} \subset \overline{\operatorname{ran}}\, A^{k+1},$$

which concludes the induction and hence the proof of (3.7).

With (3.7) at hand, we can now easily show that in general there exists no projection P weakly associated with A and λ, and that if such a projection exists it is already unique. We begin with the uniqueness statement. So, let P and Q be two projections weakly associated with A and λ of order m and n respectively, then

$$PX = \ker(A - \lambda)^m = \ker(A - \lambda)^n = QX,$$
$$(1 - P)X = \overline{\operatorname{ran}}(A - \lambda)^m = \overline{\operatorname{ran}}(A - \lambda)^n = (1 - Q)X$$

by virtue of (3.7) and therefore $P = Q$. In order to see the existence statement, choose $A := S_-$ on $X := \ell^2(\mathbb{N})$ and $\lambda := 0$ (S_- the left shift operator on $\ell^2(\mathbb{N})$) or alternatively $A := \operatorname{diag}(0, S_+)$ on $X := \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ and $\lambda := 0$ (S_+ the right shift operator on $\ell^2(\mathbb{N})$). It is then elementary to check that

$$\ker(A - \lambda)^k \subsetneq \ker(A - \lambda)^{k+1} \quad \text{resp.} \quad \overline{\operatorname{ran}}(A - \lambda)^k \supsetneq \overline{\operatorname{ran}}(A - \lambda)^{k+1}$$

for all $k \in \mathbb{N}$. In other words: the subspaces $\ker(A - \lambda)^k$ and $\overline{\operatorname{ran}}(A - \lambda)^k$ do not stop growing or shrinking, respectively. So, by virtue of (3.7), there cannot exist a projection weakly associated with A and λ. (See also (3.15) for an example where A is a spectral operator. Another – more interesting – example for the possible non-existence of weakly associated projections can be found at the beginning of Section 4.2.4). ■

We make some remarks which discuss certain converses of the above two theorems as well as the relation of associatedness and weak associatedness (and of semisimplicity and weak semisimplicity) in the case of an isolated spectral value.

1. It has been shown in the theorems above that associated and weakly associated projections P of a densely defined operator $A : D(A) \subset X \to X$ and certain spectral values $\lambda \in \sigma(A)$ yield decompositions of the space X into the closed subspaces given in (3.6) and (3.7). Conversely, such decompositions of X also yield associated and weakly associated projections: let $A : D(A) \subset X \to X$ be a densely defined operator with $\rho(A) \neq \emptyset$ and $\lambda \in \sigma(A)$.

(i) If P is a bounded projection such that

$$PX = \ker(A - \lambda)^m \quad \text{and} \quad (1 - P)X = \operatorname{ran}(A - \lambda)^m \tag{3.8}$$

for some $m \in \mathbb{N}$, then λ is isolated in $\sigma(A)$ and P is associated with A and λ, and furthermore, λ is a pole of $(\,.\, - A)^{-1}$ of order less than or equal to m.

(ii) If P is a bounded projection such that $PA \subset AP$ and

$$PX = \ker(A - \lambda)^m \quad \text{and} \quad (1 - P)X = \overline{\operatorname{ran}}(A - \lambda)^m \tag{3.9}$$

for some $m \in \mathbb{N}$, then P is weakly associated with A and λ of order less than or equal to m.

(See, for instance, Theorem 5.8-D of [128] for the proof of (i) – the proof of (ii) is not difficult. In case $m = 1$ in (3.9), the assumption $PA \subset AP$ is automatically satisfied.)

2. In the case of isolated spectral values λ of operators A as above, we have two notions of spectral relatedness (associated and weakly associated projections) and so the question arises how these two notions are related. If λ is a pole of $(\,.\, - A)^{-1}$, then associatedness and weak associatedness – as well as semisimplicity and weak semisimplicity – coincide: a projection P is then associated with A and λ if and only if it is weakly associated with A and λ. (Combine the preceding remark with the above theorems to see this equivalence.) If, however, λ is an essential singularity of $(\,.\, - A)^{-1}$, then associatedness and weak associatedness have nothing to do with each other: a projection P associated with A and λ can then not possibly be weakly associated with A and λ, and vice versa. (Indeed, if a projection P is both associated and weakly m-associated with A and λ, then

$$\begin{aligned} z \mapsto (z - A)^{-1} &= (z - A)^{-1}P + (z - A)^{-1}(1 - P) \\ &= \sum_{k=0}^{m-1} \frac{(A|_{PD(A)} - \lambda)^k}{(z - \lambda)^{k+1}}\, P + \big(z - A|_{(1-P)D(A)}\big)^{-1}(1 - P) \end{aligned}$$

has a pole of order m at λ.) A specific example of an operator A (on $X = L^2(I) \times L^2(I)$), where $\lambda = 0$ is an essential singularity of the resolvent and not only an associated projection P_1 but also a weakly associated projection P_2 exists, is given by

$$A := \operatorname{diag}(0, V) \quad \text{with} \quad (Vf)(t) := \int_0^t f(s)\, ds \quad (f \in L^2(I)). \tag{3.10}$$

3. If A is an operator as above with distinct spectral values $\lambda \neq \mu$ and if P is weakly associated with A and λ and Q is weakly associated with A and μ, then

$$PQ = 0 = QP. \tag{3.11}$$

An analogous statement for associated projections is well-known and easy to see, but we will not need that in the sequel. (In order to see (3.11), notice that

$$\sigma(A|_{PD(A)}) \subset \{\lambda\} \quad \text{and} \quad QX = \ker(A-\mu)^m \tag{3.12}$$

by the definition of weak associatedness and the above theorem. If now $x \in QX$, then

$$(A|_{PD(A)} - \mu)^m Px = P(A-\mu)^m x = P(A-\mu)^m Qx = 0$$

by virtue of (3.12b) and therefore $Px = 0$ by virtue of (3.12a) and $\mu \neq \lambda$. We have thus shown $PQ = 0$ and the other equality follows by symmetry.)

4. It is easy to see – with the help of Volterra operators and shift operators as building blocks – that the nilpotence, injectivity, and dense range requirements (encapsulated in the weak associatedness assumption) are all essential for the conclusion (3.7) of the above theorem.

3.2.2 Criteria for the existence of weakly associated projections

We have seen in the theorem above that for given operators A and spectral values λ, there will in general exist no projection weakly associated with A and λ. It is therefore important to have criteria for the existence of weakly associated projections. We present two such criteria: one for spectral operators and one for generators of bounded semigroups and spectral values on the imaginary axis.

In the case of spectral operators A one has the following convenient criterion for the existence of weakly associated projections. In particular, this criterion applies in case A is bounded spectral of finite type or (unbounded) spectral of scalar type.

Proposition 3.2.3. *Suppose that $A : D(A) \subset X \to X$ is a spectral operator with spectral measure P^A and $\lambda \in \sigma(A)$ such that for some bounded neighborhood σ of λ the bounded spectral operator $A|_{P^A(\sigma)X}$ is of finite type. Then there exists a (unique) projection P weakly associated with A and λ and it is given by $P = P^A(\{\lambda\})$.*

Proof. We often abbreviate $A_E := A|_{P^A(E)D(A)}$ for $E \in \mathcal{B}_{\mathbb{C}}$. It is clear from the definition of spectral operators that $P^A(\{\lambda\})$ commutes with A and that $P^A(\{\lambda\})D(A) = P^A(\{\lambda\})X$, so that we have only to establish the nilpotence, injectivity, and dense range condition from the definition of weak associatedness.

As a first step we show that $A|_{P^A(\{\lambda\})X} - \lambda = A_{\{\lambda\}} - \lambda$ is nilpotent. Since A_σ is a bounded spectral operator of finite type, we have $A_\sigma = S + N$ with $S = \int z \, dP^{A_\sigma}(z)$ and a nilpotent operator N (Theorem 3.1.2). So,

$$A_{\{\lambda\}} = S|_{P^A_{\{\lambda\}}X} + N|_{P^A_{\{\lambda\}}X} = \lambda + N|_{P^A_{\{\lambda\}}X}$$

and therefore $A_{\{\lambda\}} - \lambda$ is nilpotent, as desired.

As a second step we show that $A|_{(1-P^A(\{\lambda\}))D(A)} - \lambda = A_{\sigma(A)\setminus\{\lambda\}} - \lambda$ is injective with dense range in $(1 - P^A(\{\lambda\}))X = P^A(\sigma(A) \setminus \{\lambda\})X$. In order to do so, we have to treat the case where λ is isolated in $\sigma(A)$ and the case where λ is non-isolated in $\sigma(A)$ separately. Suppose first that λ is isolated in $\sigma(A)$. Then

$$\sigma(A_{\sigma(A)\setminus\{\lambda\}}) \subset \overline{\sigma(A) \setminus \{\lambda\}} = \sigma(A) \setminus \{\lambda\}$$

(because λ is isolated in $\sigma(A)$) and therefore $A_{\sigma(A)\setminus\{\lambda\}} - \lambda : D(A_{\sigma(A)\setminus\{\lambda\}}) \subset P^A(\sigma(A) \setminus \{\lambda\})X \to P^A(\sigma(A) \setminus \{\lambda\})X$ is bijective. In particular, it is injective with dense range in $P^A(\sigma(A) \setminus \{\lambda\})X$, as desired. Suppose now that λ is non-isolated in $\sigma(A)$. Then $A_{\sigma\setminus\{\lambda\}}$ is a bounded spectral operator with $\lambda \in \sigma(A_{\sigma\setminus\{\lambda\}})$ (because λ is non-isolated in $\sigma(A_\sigma)$) and with $P^{A_{\sigma\setminus\{\lambda\}}}(\{\lambda\}) = P^A(\{\lambda\})|_{P^A(\sigma\setminus\{\lambda\})X} = 0$. So, we have $\lambda \in \sigma_c(A_{\sigma\setminus\{\lambda\}})$ (Proposition 3.1.4) or, in other words,

$$A_{\sigma\setminus\{\lambda\}} - \lambda \text{ is injective with dense range in } P^A(\sigma \setminus \{\lambda\})X. \tag{3.13}$$

We also have $\sigma(A_{\sigma(A)\setminus\sigma}) \subset \overline{\sigma(A) \setminus \sigma} \subset \mathbb{C} \setminus \{\lambda\}$ (because σ is a neighborhood of λ) and therefore $\lambda \in \rho(A_{\sigma(A)\setminus\sigma})$ or, in other words,

$$A_{\sigma(A)\setminus\sigma} - \lambda : D(A_{\sigma(A)\setminus\sigma}) \subset P^A(\sigma(A) \setminus \sigma)X \to P^A(\sigma(A) \setminus \sigma)X \text{ is bijective.} \tag{3.14}$$

Combining now (3.13) and (3.14) and using that the direct sum decomposition $P^A(\sigma(A)\setminus \{\lambda\})X = P^A(\sigma(A) \setminus \sigma)X \oplus P^A(\sigma \setminus \{\lambda\})X$ yields a corresponding decomposition of the operator $A_{\sigma(A)\setminus\{\lambda\}}$, we easily conclude that $A_{\sigma(A)\setminus\{\lambda\}} - \lambda$ is injective with dense range in $P^A(\sigma(A) \setminus \{\lambda\})X$, as desired. ■

We remark that if, in the situation of the above proposition, λ is isolated, then $P = P^A(\{\lambda\})$ is also associated with A and λ as can be seen from (3.5) (or from the second remark after Theorem 3.2.2). We also remark that the finite type assumption of the above proposition is essential. Indeed, the operator A on $X := C(I) \times C(I)$ defined by

$$A := \operatorname{diag}(0, V) \quad \text{with} \quad (Vf)(t) := \int_0^t f(s)\, ds \quad (f \in C(I)) \tag{3.15}$$

is quasinilpotent and hence bounded spectral (Theorem 3.1.2), but there exists no projection weakly associated with A and $\lambda = 0$. (In order to see this, notice that $0 \in \sigma_r(V)$. So, if a weakly associated projection P existed, we would have

$$PX = \ker \operatorname{diag}(0, V^m) = C(I) \times 0 \quad \text{and} \quad (1 - P)X \subset \overline{\operatorname{ran}} \operatorname{diag}(0, V) \subsetneq 0 \times C(I)$$

for some $m \in \mathbb{N}$ by virtue of Theorem 3.2.2 and so $PX + (1 - P)X \subsetneq C(I) \times C(I) = X$. Contradiction!) Compare with the operator A from (3.10), which violates the finite type assumption as well, but nonetheless does have a weakly associated projection.

In the case of generators A of bounded semigroups and spectral value $\lambda \in i\mathbb{R}$, one has another criterion for the existence of weakly associated projections, which is due to [12] and will be used in the applied example of Section 4.2.4. It says the following: if $A : D(A) \subset X \to X$ is the generator of a bounded semigroup on a reflexive space X and $\lambda \in i\mathbb{R}$ such that the subspace

$$\ker(A - \lambda) + \overline{\operatorname{ran}}(A - \lambda)$$

is closed in X, then there exists a (unique) projection weakly associated with A and λ. (Combine Lemma 14 of [12] and the first remark following Theorem 3.2.2 to see this.) We point out that the assumption that X be reflexive is essential here. (See Example 5 or 6 of [12] or the example at the beginning of Section 4.2.4.)

3.2.3 Weak associatedness carries over to the dual operators

We close this section on spectral theory by noting that in reflexive spaces weak associatedness carries over to the dual maps – provided that some core condition is satisfied, which is the case for semigroup generators, for instance (Proposition II.1.8 of [41]). In the presented adiabatic theorem without spectral gap condition for reflexive spaces, this will be important. Associatedness carries over to dual maps as well, of course (Section III.6.6 of [67]) – but this will not be needed in the sequel.

Proposition 3.2.4. *Suppose $A : D(A) \subset X \to X$ is a densely defined closed linear map in the reflexive space X such that $\rho(A) \neq \emptyset$ and $D(A^k)$ is a core for A for all $k \in \mathbb{N}$. If P is weakly m-associated with A and $\lambda \in \sigma(A)$, then P^* is weakly m-associated with A^* and λ.*

Proof. We begin by showing – by induction over $k \in \mathbb{N}$ – the preparatory statement that

$$(A^k)^* = (A^*)^k \tag{3.16}$$

for all $k \in \mathbb{N}$, which might also be of independent interest (notice that $D(A^k)$ being a core for A is dense in X, so that $(A^k)^*$ is really well-defined). Clearly, (3.16) is true for $k = 1$ and, assuming that it is true for some arbitrary $k \in \mathbb{N}$, we now show that $(A^{k+1})^* = (A^*)^{k+1}$ holds true as well. It is easy to see that $(A^*)^{k+1} \subset (A^{k+1})^*$ and it remains to see that $D((A^{k+1})^*) \subset D((A^*)^{k+1})$. So let $x^* \in D((A^{k+1})^*)$. We show that

$$x^* \in D((A^k)^*) \quad \text{and} \quad (A^k)^* x^* \in D(A^*), \tag{3.17}$$

from which it then follows – by the induction hypothesis – that $x^* \in D((A^*)^{k+1})$ as desired. In order to prove that $x^* \in D((A^k)^*)$ we show that

$$x^* \in D((A^l)^*)$$

for all $l \in \{1, \dots, k\}$ – by induction over $l \in \{1, \dots, k\}$ and by working with suitable powers of $(A^* - z_0)^{-1} = ((A - z_0)^{-1})^*$, where z_0 is an arbitrary point of $\rho(A^*) = \rho(A) \neq \emptyset$ (Theorem III.5.30 of [67]). In the base step of the induction, notice that for all $y \in D(A)$

$$\begin{aligned}\big\langle (A^* - z_0)^{-k} (A^{k+1})^* x^*, y \big\rangle &= \big\langle x^*, A^{k+1} (A - z_0)^{-k} y \big\rangle \\ &= \big\langle x^*, (A - z_0) y \big\rangle + \sum_{i=0}^{k} \binom{k+1}{i} z_0^{k+1-i} \big\langle (A^* - z_0)^{-k+i} x^*, y \big\rangle,\end{aligned}$$

from which it follows that $x^* \in D((A-z_0)^*) = D(A^*)$. In the inductive step, assume that $x^* \in D(A^*), \dots, D((A^l)^*)$ for some arbitrary $l \in \{1, \dots, k-1\}$. Since for all $y \in D(A^{l+1})$

$$\begin{aligned}
\big\langle (A^* - z_0)^{-(k-l)} (A^{k+1})^* x^*, y \big\rangle &= \big\langle x^*, A^{k+1} (A - z_0)^{-(k-l)} y \big\rangle \\
&= \big\langle x^*, (A - z_0)^{l+1} y \big\rangle + \sum_{i=k-l+1}^{k} \binom{k+1}{i} z_0^{k+1-i} \big\langle x^*, (A - z_0)^{-(k-l)+i} y \big\rangle \\
&\quad + \sum_{i=0}^{k-l} \binom{k+1}{i} z_0^{k+1-i} \big\langle (A^* - z_0)^{-(k-l)+i} x^*, y \big\rangle,
\end{aligned}$$

it follows by the induction hypothesis of the l-induction and by applying the binomial formula to $(A - z_0)^{-(k-l)+i} y$ for $i \in \{k-l+1, \dots, k+1\}$ that $x^* \in D((A^{l+1})^*)$. So the l-induction is finished and it remains to show that $(A^k)^* x^* \in D(A^*)$. Since $D(A^{k+1})$ by assumption is a core for A, there is for every $y \in D(A)$ a sequence (y_n) in $D(A^{k+1})$ such that

$$\big\langle (A^k)^* x^*, Ay \big\rangle = \lim_{n\to\infty} \big\langle (A^k)^* x^*, Ay_n \big\rangle = \lim_{n\to\infty} \big\langle x^*, A^{k+1} y_n \big\rangle = \big\langle (A^{k+1})^* x^*, y \big\rangle.$$

It follows that $(A^k)^* x^* \in D(A^*)$ and this yields – together with the induction hypothesis of the k-induction – that $x^* \in D((A^*)^{k+1})$, which finally ends the proof the preparatory statement (3.16).

After this preparation we can now move on to the main part of the proof where we assume, without loss of generality, that $\lambda = 0$ and exploit the first remark after Theorem 3.2.2 to show that P^* is weakly m-associated with A^* and $\lambda = 0$. A^* is densely defined (due to the reflexivity of X (Theorem III.5.29 of [67])) with $\rho(A^*) = \rho(A) \neq \emptyset$ (Theorem III.5.30 of [67]) and

$$P^* A^* \subset (AP)^* \subset (PA)^* = A^* P^*$$

because $AP \supset PA$. Since $(A^m)^* = (A^*)^m$ by (3.16) and since $PX = \ker A^m$ and $(1-P)X = \overline{\operatorname{ran}} A^m$ (by Theorem 3.2.2), we further have

$$P^* X^* = \ker(1-P)^* = ((1-P)X)^\perp = (\overline{\operatorname{ran}} A^m)^\perp = \ker(A^m)^* = \ker(A^*)^m$$

and

$$(1-P^*)X^* = \ker P^* = (PX)^\perp = (\ker A^m)^\perp = (\ker(A^m)^{**})_\perp = \overline{\operatorname{ran}}(A^m)^* = \overline{\operatorname{ran}}(A^*)^m,$$

where in the fourth equality of the second line the closedness of A^m (following from $\rho(A) \neq \emptyset$) and the reflexivity of X have been used. (In the above relations, we denote by $U^\perp := \{x^* \in Z^* : \langle x^*, U^* \rangle = 0\}$ and $V_\perp := \{x \in Z : \langle V, x \rangle = 0\}$ the annihilators of subsets U and V of a normed space Z and its dual Z^*, respectively.) It is now clear from the first remark after Theorem 3.2.2 that P^* is weakly m-associated with A^* and $\lambda = 0$ and we are done. ■

3.3 Spectral gaps and continuity of set-valued maps

We continue by properly defining what exactly we mean by uniform and non-uniform spectral gaps. Suppose that $A(t) : D(A(t)) \subset X \to X$, for every t in some compact interval J, is a densely defined closed linear map and that $\sigma(t)$ is a compact subset of $\sigma(A(t))$ for every $t \in J$. We then speak of a *spectral gap for A and σ* if and only if $\sigma(t)$ is isolated in $\sigma(A(t))$ for every $t \in J$. Such a spectral gap for A and σ is called *uniform* if and only if $\sigma(\,.\,)$ is even uniformly isolated in $\sigma(A(\,.\,))$ in the sense that there is a t-independent constant $r_0 > 0$ such that

$$\overline{U}_{r_0}(\sigma(t)) \cap \sigma(A(t)) := \{z \in \mathbb{C} : \operatorname{dist}(z, \sigma(t)) \leq r_0\} \cap \sigma(A(t)) = \sigma(t)$$

for every $t \in I$. Also, we say that $\sigma(\,.\,)$ *falls into* $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ *at the point* $t_0 \in J$ if and only if there is a sequence (t_n) in J converging to t_0 such that

$$\operatorname{dist}(\sigma(t_n), \sigma(A(t_n)) \setminus \sigma(t_n)) \longrightarrow 0 \quad (n \to \infty).$$

It is clear that the set of points at which $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ is closed. Also, it follows by the compactness of J that a spectral gap for A and σ is uniform if and only if $\sigma(\,.\,)$ at no point falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$. The following proposition gives a criterion (in terms of some mild regularity conditions on $t \mapsto A(t)$, $\sigma(t)$, $P(t)$) for a spectral gap for A and σ to be even uniform. It is of some interest in the third remark at the beginning of Section 4.1.3. We refer to Section IV.2.4 and Theorem IV.2.25 of [67] for a definition and a characterization of *convergence* (and hence, continuity) *in the generalized sense* and to Section IV.3 of [67] for the definition of *upper and lower semicontinuity* of set-valued functions $t \mapsto \sigma(t)$.

Proposition 3.3.1. *Suppose that $A(t) : D(A(t)) \subset X \to X$ is a closed linear map for every t in a compact interval J and that $t \mapsto A(t)$ is continuous in the generalized sense. Suppose further that $\sigma(t)$ for every $t \in J$ is a compact and isolated subset of $\sigma(A(t))$ such that $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ at $t_0 \in J$, and let $t \mapsto \sigma(t)$ be upper semicontinuous at t_0. Finally, for every $t \in J$ let $P(t)$ be the projection associated with $A(t)$ and $\sigma(t)$. Then $t \mapsto P(t)$ is discontinuous at t_0 and*

$$\limsup_{n\to\infty} \big(\operatorname{rk} P(t_n)\big) \leq \operatorname{rk} P(t_0) - 1$$

for all sequences (t_n) such that $t_n \longrightarrow t_0$ and $\operatorname{dist}(\sigma(t_n), \sigma(A(t_n)) \setminus \sigma(t_n)) \longrightarrow 0$.

See [112] (Proposition 5.3) for a proof. Clearly, one also has the following converse of the above proposition: if $t \mapsto A(t)$ is continuous in the generalized sense as above and $t \mapsto \sigma(t)$ is even *continuous* (that is, upper and lower semicontinuous) then uniform isolatedness of $\sigma(\,.\,)$ in $\sigma(A(\,.\,))$ implies that $t \mapsto P(t)$ is continuous. (Use Theorem IV.3.15 of [67].)

3.4 Adiabatic evolutions and a trivial adiabatic theorem

As has been explained in Section 1.2.2, the principal goal of adiabatic theory is to establish the convergence (1.12) or, in other words, to show that the evolution systems U_ε for $\frac{1}{\varepsilon}A$ are, in some sense, approximately adiabatic w.r.t. P as $\varepsilon \searrow 0$. We say that an evolution system for a family A of linear operators $A(t) : D(A(t)) \subset X \to X$ is *adiabatic w.r.t. a family P of bounded projections $P(t)$ in X* if and only if $U(t,s)$ for every $(s,t) \in \Delta$ intertwines $P(s)$ with $P(t)$, more precisely:

$$P(t)U(t,s) = U(t,s)P(s) \tag{3.18}$$

for every $(s,t) \in \Delta$. Since the pioneering work [61] of Kato, the basic strategy in proving the convergence (1.12) has been to show that

$$U_\varepsilon(t) - V_\varepsilon(t) \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{3.19}$$

for every $t \in I$, where the V_ε are suitable comparison evolution systems that are adiabatic w.r.t. the family P of projections $P(t)$ related to the data A, σ. A simple way of obtaining adiabatic evolutions w.r.t. some given family P (independently observed by Kato in [61] and Daleckii–Krein in [27]) is described in the following important proposition.

Proposition 3.4.1 (Kato, Daleckii–Krein). *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is a densely defined closed linear map and $P(t)$ a bounded projection in X such that $P(t)A(t) \subset A(t)P(t)$ for every $t \in I$ and $t \mapsto P(t)$ is strongly continuously differentiable. If the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ exists on $D(A(t))$ for every $\varepsilon \in (0,\infty)$, then V_ε is adiabatic w.r.t. P for every $\varepsilon \in (0,\infty)$.*

Proof. Choose an arbitrary $(s,t) \in \Delta$ with $s \neq t$. It then follows by Lemma 2.1.3 and Lemma 2.1.5 that, for every $x \in D(A(s))$, the map

$$[s,t] \ni \tau \mapsto V_\varepsilon(t,\tau)P(\tau)V_\varepsilon(\tau,s)x$$

is continuous and right differentiable. Since $P(\tau)$ commutes with $A(\tau)$ and

$$P(\tau)P'(\tau)P(\tau) = 0 \tag{3.20}$$

for every $\tau \in I$ (which follows by applying P from the left and the right to $P' = P'P + PP'$), it further follows that the right derivative of this map is identically 0 and so (by Lemma 2.1.4) this map is constant. In particular,

$$P(t)V_\varepsilon(t,s)x - V_\varepsilon(t,s)P(s)x = V_\varepsilon(t,\tau)P(\tau)V_\varepsilon(\tau,s)x\big|_{\tau=s}^{\tau=t} = 0,$$

as desired. ∎

We now briefly discuss two situations where the conclusion of the adiabatic theorem is already trivially true.

Proposition 3.4.2. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is a densely defined closed linear map and $P(t)$ is a bounded projection in X such that the evolution system U_ε exists on $D(A(t))$ for every $\varepsilon \in (0, \infty)$ and such that $P(t)A(t) \subset A(t)P(t)$ for every $t \in I$ and $t \mapsto P(t)$ is strongly continuously differentiable.*

(i) If $P' = 0$, then U_ε is adiabatic w.r.t. P for every $\varepsilon \in (0, \infty)$ (in particular, the convergence (1.12) *holds trivially), and the reverse implication is also true.*

(ii) If there are $\gamma \in (0, \infty)$ and $M \in [1, \infty)$ such that for all $(s, t) \in \Delta$ and $\varepsilon \in (0, \infty)$

$$\|U_\varepsilon(t, s)\| \leq M e^{-\frac{\gamma}{\varepsilon}(t-s)}, \tag{3.21}$$

then $\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon)$ as $\varepsilon \searrow 0$, whenever the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ exists on $D(A(t))$ for every $\varepsilon \in (0, \infty)$.

Proof. (i) See, for instance, Section IV.3.2 of [75] for the reverse implication (differentiate the adiabaticity relation (3.18) with respect to the variable s) – the other implication is obvious from Proposition 3.4.1.

(ii) Since for $x \in D(A(0))$ one has (by Lemma 2.1.5, Lemma 2.1.3, and Lemma 2.1.4)

$$V_\varepsilon(t)x - U_\varepsilon(t)x = U_\varepsilon(t, s)V_\varepsilon(s)x\big|_{s=0}^{s=t} = \int_0^t U_\varepsilon(t, s)[P'(s), P(s)]V_\varepsilon(s)x \, ds$$

for every $t \in I$ and $\varepsilon \in (0, \infty)$, it follows with the help of Proposition 2.1.13 that

$$\|U_\varepsilon(t) - V_\varepsilon(t)\| \leq M^2 c \, e^{Mc} \, t \, e^{-\frac{\gamma}{\varepsilon}t}$$

for all $t \in I$ and $\varepsilon \in (0, \infty)$, where c denotes an upper bound of $s \mapsto \|[P'(s), P(s)]\|$. And from this the desired conclusion is obvious. ∎

Combining Proposition 3.4.2 (ii) with Example 4.1.5 one sees that adiabatic theory is interesting only if the evolution systems for $\frac{1}{\varepsilon}A$ are *only just* bounded w.r.t. $\varepsilon \in (0, \infty)$: if even the evolution for $\frac{1}{\varepsilon}(A + \gamma)$ is bounded in $\varepsilon \in (0, \infty)$ for some $\gamma > 0$, then adiabatic theory is trivial for A (by Proposition 3.4.2 (ii)), and if only the evolution for $\frac{1}{\varepsilon}(A - \gamma)$ is bounded in $\varepsilon \in (0, \infty)$ for some $\gamma > 0$, then adiabatic theory is generally impossible for A (by Example 4.1.5).

3.5 Standard examples

We will complement the adiabatic theorems of this thesis by examples in order to demonstrate, on the one hand, that the presented theorems are strictly more general than the previously known adiabatic theorems (positive examples) and that, on the other hand, some selected hypotheses of our theorems cannot be dispensed with (negative examples). We have made sure that in all positive examples the conclusion of the respective adiabatic theorem is not already trivially fulfilled in the sense that it does not already follow from the trivial adiabatic theorem presented above. (See Example 4.1.3 where this is once – and for all – explained in detail.) All examples will be of the following simple standard structure:

- $X = \ell^p(I_d)$ for some $p \in [1, \infty)$ and $d \in \mathbb{N} \cup \{\infty\}$ (where $I_d := \{1, \dots, d\}$ for $d \in \mathbb{N}$ and $I_\infty := \mathbb{N}$) or $X = L^p(X_0)$ for some $p \in [1, \infty)$ and some measure space $(X_0, \mathcal{A}, \mu)$ or X is a product of some of the aforementioned spaces (endowed with the sum norm)

- $A(t) = R(t)^{-1} A_0(t) R(t)$, where $A_0(t) : D \subset X \to X$ is a semigroup generator on X with t-independent dense domain D (chosen equal or unequal to X depending on whether we are in the case of time-independent or time-dependent domains), A_0 satisfies Condition 2.1.8, and $R(t) := e^{Ct}$ for some bounded operator C.

Condition 2.1.8 with $\omega = 0$ for A_0 ensures (by Lemma 2.1.7 and Corollary 2.1.10) that the hypotheses on A of the adiabatic theorems of Sections 4.1 to 5.1 are fulfilled. In some of our examples we will use the right or left shift operator S_+ and S_- on $\ell^p(I_\infty)$ defined by

$$S_+(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{and} \quad S_-(x_1, x_2, x_3, \dots) := (x_2, x_3, \dots).$$

Since $\|S_\pm\| \le 1$, it follows from the theorem of Hille–Yosida that $e^{i\vartheta} S_+ - 1$ and $e^{i\vartheta} S_- - 1$ generate contraction semigroups on $\ell^p(I_\infty)$ for $p \in [1, \infty)$ and $\vartheta \in \mathbb{R}$ (use a Neumann series expansion!). It is well-known (Example V.4.1 and V.4.2 of [129]) that $\sigma(S_\pm) = \overline{U}_1(0)$ for all $p \in [1, \infty)$, the fine structure of $\sigma(S_+)$ being given by

$$\begin{gathered}
\sigma_p(S_+) = \emptyset, \quad \sigma_c(S_+) = \emptyset, \quad \sigma_r(S_+) = \overline{U}_1(0) \quad (p = 1) \\
\sigma_p(S_+) = \emptyset, \quad \sigma_c(S_+) = \partial U_1(0), \quad \sigma_r(S_+) = U_1(0) \quad (p \in (1, \infty))
\end{gathered}$$

and the fine structure of $\sigma(S_-)$ being given by

$$\sigma_p(S_-) = U_1(0), \quad \sigma_c(S_-) = \partial U_1(0), \quad \sigma_r(S_-) = \emptyset \quad (p \in [1, \infty)).$$

Additionally, we will sometimes use multiplication operators M_f on $L^p(X_0)$ ($p \in [1, \infty)$) for some measurable function $f : X_0 \to \mathbb{C}$ and some σ-finite measure space $(X_0, \mathcal{A}, \mu)$ in which case, as is well-known (Proposition I.4.10 of [41]), one has

$$\sigma(M_f) = \operatorname{ess-ran} f := \big\{ z \in \mathbb{C} : \mu\big(f^{-1}(U_\varepsilon(z))\big) \neq 0 \text{ for all } \varepsilon > 0 \big\}$$

and, in particular (take μ to be the counting measure on $X_0 := I_d$),

$$\sigma\big(\operatorname{diag}((\lambda_n)_{n \in I_d})\big) = \sigma\big(M_{(\lambda_n)_{n \in I_d}}\big) = \overline{\{\lambda_n : n \in I_d\}}.$$

In quite some examples, we will work with families A of operators $A(t)$ in $X := \ell^p(I_d)$ whose spectra $\sigma(A(t))$ are singletons and whose nilpotent parts depend on t in the simplest possible way, namely via a scalar factor.

Condition 3.5.1. *$N \neq 0$ is a nilpotent operator in $X := \ell^p(I_d)$ (with $p \in [1, \infty)$ and $d \in \mathbb{N}$), $\lambda(t) \in \mathbb{C}$ and $\alpha(t) \in [0, \infty)$ for all $t \in I$, and there is an $r_0 > 0$ such that $-\operatorname{Re} \lambda(t) = |\operatorname{Re} \lambda(t)| \ge r_0 \alpha(t)$ for all $t \in I$.*

As is shown in the next lemma, this condition characterizes $(M,0)$-stability of families A of the simple type just described.

Lemma 3.5.2. *Suppose that $N \neq 0$ is a nilpotent operator in $X := \ell^p(I_d)$ with $p \in [1,\infty)$ and $d \in \mathbb{N}$ and that $A(t) = \lambda(t)+\alpha(t)N$ for every $t \in I$, where $\lambda(t) \in \mathbb{C}$ and $\alpha(t) \in [0,\infty)$. Then A is $(M,0)$-stable for some $M \in [1,\infty)$ if and only if Condition 3.5.1 is satisfied.*

Proof. Suppose first that A is $(M,0)$-stable for some $M \in [1,\infty)$ and assume that $N = \operatorname{diag}(J_1,\dots,J_m)$ is in Jordan normal form with (decreasingly ordered) Jordan block matrices $J_1,\dots,J_m$ (notice that this assumption, by virtue of Lemma 2.1.7, does not restrict generality). We then show that $-\operatorname{Re}\lambda(t) = |\operatorname{Re}\lambda(t)| \geq \frac{1}{4M}\,\alpha(t)$ for every $t \in I$. It is clear by the $(M,0)$-stability of A that $\lambda(t) \in \sigma(A(t)) \subset \{\operatorname{Re} z \leq 0\}$ for every $t \in I$ and that the family $\tilde{A}$ with $\tilde{A}(t) := \operatorname{Re}\lambda(t) + \alpha(t)N$ is $(M,0)$-stable as well. If $\alpha(t) = 0$, then the desired inequality is trivial. If $\alpha(t) \neq 0$, then $\operatorname{Re}\lambda(t) < 0$ by the $(M,0)$-stability of A and therefore we get – computing $(\lambda - \tilde{A}(t))^{-1}e_2 = (\frac{\alpha(t)}{(\lambda-\operatorname{Re}\lambda(t))^2}, \frac{1}{\lambda-\operatorname{Re}\lambda(t)}, 0, 0, \dots)$ for $\lambda \in (0,\infty)$, setting $\lambda := |\operatorname{Re}\lambda(t)|$, and using the $(M,0)$-stability of $\tilde{A}$ – that

$$\frac{\alpha(t)}{4\,|\operatorname{Re}\lambda(t)|} \leq \left\| |\operatorname{Re}\lambda(t)| \left(|\operatorname{Re}\lambda(t)| - \tilde{A}(t)\right)^{-1} e_2 \right\| \leq M,$$

as desired. Suppose conversely that there is an $r_0 > 0$ such that $-\operatorname{Re}\lambda(t) = |\operatorname{Re}\lambda(t)| \geq r_0\alpha(t)$ for every $t \in I$. Then there is an $M = M_{r_0} \in [1,\infty)$ such that $\left\|e^{Ns}\right\| \leq M e^{r_0 s}$ for all $s \in [0,\infty)$ and thus

$$\left\| e^{A(t_n)s_n} \cdots e^{A(t_1)s_1} \right\| = e^{\operatorname{Re}\lambda(t_n)s_n} \cdots e^{\operatorname{Re}\lambda(t_1)s_1} \left\| e^{N(\alpha(t_n)s_n + \dots + \alpha(t_1)s_1)} \right\| \leq M$$

for all $s_1,\dots,s_n \in [0,\infty)$ and all $t_1,\dots,t_n \in I$ satisfying $t_1 \leq \dots \leq t_n$ (with arbitrary $n \in \mathbb{N}$), as desired. ∎

It should be noticed that Condition 3.5.1 does not already guarantee $(1,0)$-stability, however. Indeed, if for instance

$$A(t) := -\frac{t}{3} + t^2 N \quad \text{with} \quad N := \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix} \quad \text{in } X := \ell^p(I_d)$$

($p \in [1,\infty)$ and $2 \leq d \in \mathbb{N}$), then A is $(M,0)$-stable for some $M \in [1,\infty)$ by the above lemma, but not $(1,0)$-stable, because $A(1) = -\frac{1}{3} + N$ is not dissipative in $\ell^p(I_d)$ and hence (by the theorem of Lumer–Phillips) does not generate a contraction semigroup.

At some point (Example 4.2.8) the following simple lemma will be needed which, in essence, is the reason why adiabatic theory for multiplication operators $A(t) = M_{f_t}$ is typically uninteresting.

Lemma 3.5.3. *Suppose that $P(t)$ for every $t \in I$ is a bounded projection in $X := L^p(X_0)$ (where $(X_0, \mathcal{A}, \mu)$ is a measure space and $p \in [1,\infty)$) and that $P(t) = M_{\chi_{E_t}}$ for almost every $t \in I$, where $E_t \in \mathcal{A}$. If $t \mapsto P(t)$ is strongly continuously differentiable, then $t \mapsto P(t)$ is already constant.*

Proof. Set $I_0 := \{t \in I : P(t) = M_{\chi_{E_t}}\}$ and fix $t \in I_0$ and $f \in X$. We show that $P'(t)f = 0$, which by the density of I_0 in I and the strong continuity of $\tau \mapsto P'(\tau)$ implies the assertion. In order to do so, notice that there exists a sequence (h_n) with $h_n \neq 0$ and $t + h_n \in I_0$ such that

$$\frac{\chi_{E_{t+h_n}}(x) - \chi_{E_t}(x)}{h_n} f(x) = \Big(\frac{P(t+h_n)f - P(t)f}{h_n}\Big)(x) \longrightarrow \big(P'(t)f\big)(x) \quad (n \to \infty)$$

for almost all $x \in X_0$ with exceptional set $N = N_{t,f}$. Since

$$\frac{\chi_{E_{t+h_n}}(x) - \chi_{E_t}(x)}{h_n} \in \Big\{\frac{1}{h_n}, 0, -\frac{1}{h_n}\Big\},$$

the convergence can hold true only if for all $x \notin N$ one has $\big(\chi_{E_{t+h_n}}(x) - \chi_{E_t}(x)\big) f(x) = 0$ for n large enough. Consequently, $P'(t)f = 0$ as desired. ■

3.6 Some basic facts about quantum dynamical semigroups

We collect here some basic facts about quantum dynamical semigroups – in particular, dephasing quantum dynamical semigroups – which will be needed in Section 4.1.4 and 4.2.4. A *quantum dynamical semigroup* is, by definition, a strongly continuous semigroup (Φ_t) of bounded linear operators on $S^1(\mathfrak{h})$ (where $\mathfrak{h}$ is a generally non-separable Hilbert space) such that for every $t \in [0, \infty)$

(i) Φ_t is *trace-preserving*, which means that $\operatorname{tr}(\Phi_t(\rho)) = \operatorname{tr}(\rho)$ for all $\rho \in S^1(\mathfrak{h})$

(ii) Φ_t is *completely positive*, which means that for every $n \in \mathbb{N}$ the lifted map

$$\Phi_{t\,n} : M_n(\mathcal{S}) \subset M_n(\mathcal{A}) \to M_n(\mathcal{A}) \quad \text{with} \quad \Phi_{t\,n}((\rho_{ij})_{i,j}) := (\Phi_t(\rho_{ij}))_{i,j}$$

is positive, where $\mathcal{S} := S^1(\mathfrak{h})$ and $\mathcal{A} := L(\mathfrak{h})$ and where the notion of positivity is induced by the (unique!) C*-algebraic structure of the *-algebra $M_n(\mathcal{A}) \cong L(\mathfrak{h}^n)$ of n times n matrices with entries in $\mathcal{A}$.

Sometimes, condition (i) in the above definition is weakened to $\operatorname{tr}(\Phi_t(\rho)) \leq \operatorname{tr}(\rho)$ for all positive $\rho \in S^1(\mathfrak{h})$, and a quantum dynamical semigroup in our sense is then called conservative, but we will not use such more general semigroups in the sequel.

Complete positivity can be defined in the same way as above also for arbitrary subsets $\mathcal{S}$ of $\mathcal{A} := L(\mathfrak{h})$ and we will make use of this in Section 4.2.4 (with $\mathcal{S} := S^p(\mathfrak{h})$, the Schatten-$p$ class for $p \in (1, \infty)$). Since the complete positivity of $\Phi_t : \mathcal{S} \to \mathcal{S}$ is equivalent to that of the dual map $\Phi_t^* : \mathcal{A} \to \mathcal{A}$ with $\mathcal{S} := S^1(\mathfrak{h})$ and $\mathcal{A} := L(\mathfrak{h})$ (Chapter 2 of [74]), a quantum dynamical semigroup automatically is a semigroup of contractions. (Indeed, a completely positive map Φ^* on a unital C*-algebra $\mathcal{A}$ with $\Phi^*(1) \leq 1$ is automatically contractive. See [103] and [29] for this and many more properties of completely positive maps.)

When describing the evolution of open quantum systems, one is naturally led to the above notion of quantum dynamical semigroups: indeed, if we are given an open system (described by $\mathfrak{h}$) that initially is completely decoupled from the environment (described by $\mathfrak{k}$), then the initial states of the combined closed system $\mathfrak{h} \otimes \mathfrak{k}$ are given by $\rho \otimes \sigma_0$ and the states at time t by $U_t \rho \otimes \sigma_0 U_t^*$, where $\rho \in S^1(\mathfrak{h})$ and $\sigma_0 \in S^1(\mathfrak{k})$ is a density operator and (U_t) is the reversible (unitary) evolution of the combined closed system. Consequently, the evolved state $\Phi_t(\rho)$ of the open system at time t is given by

$$\Phi_t(\rho) = \operatorname{tr}_{\mathfrak{k}}(U_t \rho \otimes \sigma_0 U_t^*),$$

where $\operatorname{tr}_{\mathfrak{k}}$ denotes the partial trace which traces out the environment. And from this it follows that Φ_t is trace-preserving (by the definition of the partial trace) and that Φ_t is completely positive (by the work [73] of Kraus): indeed, it is shown there (see also [5] or [6]) that

$$\Phi_t(\rho) = \operatorname{tr}_{\mathfrak{k}}(U_t \rho \otimes \sigma_0 U_t^*) = \sum_{j \in J} B_j \rho B_j^* \qquad (\rho \in S^1(\mathfrak{h}))$$

with bounded operators B_j on $\mathfrak{h}$ satisfying $\sum_{j \in J} B_j^* B_j \leq 1$, and from this the complete positivity of Φ_t is simple. If moreover the open system is weakly coupled to the environment in a certain sense and σ_0 is an equilibrium state, one can also expect to have the semigroup property $\Phi_{t+s} = \Phi_t \Phi_s$ for $s, t \in [0, \infty)$ in an approximative sense. See, for instance, [28], [51], [37] for precise conditions.

As a simple but technically important prerequisite for understanding the structure of the generator of quantum dynamical semigroups, we remark: if B_j are bounded operators on $\mathfrak{h}$ for j in an arbitrary index set J such that

$$\sum_{j \in J} B_j^* B_j < \infty \tag{3.22}$$

in the sense that $\sum_{j \in F} B_j^* B_j \leq M < \infty$ for all finite subsets F of J and an F-independent constant M, then the series

$$\text{(i)} \sum_{j \in J} B_j^* B_j \rho, \quad \sum_{j \in J} \rho B_j^* B_j \quad \text{and} \quad \text{(ii)} \sum_{j \in J} B_j \rho B_j^*$$

converge in the norm of $S^1(\mathfrak{h})$ for every $\rho \in S^1(\mathfrak{h})$ and define bounded linear operators from $S^1(\mathfrak{h})$ to $S^1(\mathfrak{h})$. (In order to see (i) note that the net $(\sum_{j \in F} B_j^* B_j)$ is strongly convergent by the theorem of Vigier (Theorem 4.1.1 of [90]) and that $C_F \rho \longrightarrow C\rho$ and $\rho C_F^* \longrightarrow \rho C^*$ in the norm of $S^1(\mathfrak{h})$ for every $\rho \in S^1(\mathfrak{h})$ and every bounded and strongly convergent net (C_F) in $L(\mathfrak{h})$ with strong limit C. In order to see (ii) consult [73] or, more simply, note that

$$\Big\| \sum_{j \in F} B_j \rho B_j^* \Big\|_{S^1} = \operatorname{tr}\Big(\sum_{j \in F} B_j \rho B_j^* \Big) = \operatorname{tr}\Big(\sum_{j \in F} B_j^* B_j \rho \Big) \tag{3.23}$$

for positive $\rho \in S^1(\mathfrak{h})$ and finite subsets F of J, and apply (i).)

We can now recall the fundamental characterization [82] of the generators of special, namely norm continuous, quantum dynamical semigroups by Lindblad. It explicitly identifies the general structure of such generators. (Independently, Gorini, Kossakowski, Sudarshan [52] obtained this characterization in the special case of finite-dimensional $\mathfrak{h}$.)

Theorem 3.6.1 (Lindblad). *An operator Λ defined on $S^1(\mathfrak{h})$ is the generator of a norm continuous quantum dynamical semigroup (Φ_t) on $S^1(\mathfrak{h})$ if and only if*

$$\Lambda(\rho) = -i[H, \rho] + \sum_{j \in J} B_j \rho B_j^* - 1/2\{B_j^* B_j, \rho\} \qquad (\rho \in S^1(\mathfrak{h})) \tag{3.24}$$

for some bounded operators $H = H^*$ and B_j on $\mathfrak{h}$ satisfying $\sum_{j \in J} B_j^* B_j < \infty$. If $\mathfrak{h}$ is separable, then the index set J can be chosen to be countable.

Proof. We refer to [106] (Theorem 5.5) for a proof of the "only if" part, which smoothens the original proof in [82], and to [5] (Theorem 8.16) for a proof of the "if" part, which works just as well in the present situation of generally infinite-dimensional $\mathfrak{h}$ and is much simpler than the original proof in [82] (which makes use, among other things, of the Russo–Dye theorem). If $\mathfrak{h}$ is separable, then it has a countable orthonormal basis and therefore (3.22) implies that $B_j \neq 0$ only for countably many indices $j \in J$. ■

An analogous characterization of generators of general, merely strongly continuous, quantum dynamical semigroups is still missing, but there are partial results, of course. We will need the following simple sufficient condition, which in essence can be found in [29] (Lemma 5.5.1 and Theorem 5.5.2) (with complete positivity replaced by positivity).

Corollary 3.6.2. *Suppose $\Lambda : D(Z_0) \subset S^1(\mathfrak{h}) \to S^1(\mathfrak{h})$ is given by*

$$\Lambda(\rho) = Z_0(\rho) + \sum_{j \in J} B_j \rho B_j^* - 1/2\{B_j^* B_j, \rho\} \qquad (\rho \in D(Z_0)),$$

where Z_0 is the generator of the (weakly and hence strongly continuous) semigroup on $S^1(\mathfrak{h})$ defined by $e^{Z_0 t}(\rho) := e^{-iHt} \rho \, e^{iHt}$ with a generally unbounded self-adjoint operator H on $\mathfrak{h}$ and where B_j for $j \in J$ are bounded operators on $\mathfrak{h}$ with $\sum_{j \in J} B_j^* B_j < \infty$. Then

$$D(Z_0) = \big\{\rho \in S^1(\mathfrak{h}) : \rho D(H) \subset D(H) \text{ and } H\rho - \rho H \subset \sigma \text{ for a } \sigma \in S^1(\mathfrak{h})\big\}$$

with $Z_0(\rho)$ being the unique element σ of $S^1(\mathfrak{h})$ such that $-i(H\rho - \rho H) \subset \sigma$, and Λ is the generator of a quantum dynamical semigroup on $S^1(\mathfrak{h})$.

Proof. See [29] for a proof of the explicit description of Z_0 and its domain (Lemma 5.5.1). Z_0 is a semigroup generator on $S^1(\mathfrak{h})$ by definition and W defined by

$$W(\rho) := \sum_{j \in J} B_j \rho B_j^* - 1/2\{B_j^* B_j, \rho\} \qquad (\rho \in S^1(\mathfrak{h}))$$

is a bounded operator on $S^1(\mathfrak{h})$ by the remarks preceding the above theorem. And so the sum $\Lambda = Z_0 + W$ is the generator of a strongly continuous semigroup given by

$$e^{\Lambda t}(\rho) = \lim_{n\to\infty} \left(e^{Z_0 t/n} e^{Wt/n}\right)^n(\rho) \qquad (\rho \in S^1(\mathfrak{h})) \tag{3.25}$$

(Lie–Trotter). Since now $(e^{Z_0 t})$ is a quantum dynamical semigroup by its explicit form and (e^{Wt}) is a quantum dynamical semigroup by the above theorem (with $H = 0$), it follows from (3.25) that so is $(e^{\Lambda t})$ – because compositions and pointwise limits of completely positive maps are easily seen to be completely positive again. ■

See the works [30], [22] of Davies and of Chebotarev, Fagnola for much deeper sufficient conditions where the operators B_j are allowed to be unbounded as well. Also see the work [31] of Davies which shows that at least the generator Λ of a quantum dynamical semigroup that has a pure invariant state $|\xi\rangle\langle\xi|$ is of the form

$$\Lambda(\rho) = K\rho + \rho K^* + \sum_{j\in J} B_j \rho B_j^* \qquad (\rho \in D) \tag{3.26}$$

(generalizing (3.24)!) for some densely defined closed operators K, B_j on $\mathfrak{h}$ and some dense subspace D of $D(\Lambda)$.

We now specialize to an important special class of generators of quantum dynamical semigroups, the so-called dephasing generators. See [12], for instance. A generator as in the previous corallary is called *dephasing* if and only if

$$\ker Z_0^* \subset \ker \Lambda^*.$$

Since the dual operators Z_0^* and Λ^* by II.2.5 of [41] are the generators of the weak* continuous dual semigroups $((e^{Z_0 t})^*)$ and $((e^{\Lambda t})^*)$, dephasingness means precisely that every observable a conserved by $((e^{Z_0 t})^*)$ in the sense that

$$e^{iHt} a\, e^{-iHt} = (e^{Z_0 t})^*(a) = a \quad (t \in [0,\infty)) \qquad (\text{equivalently, } a \in \{H\}'),$$

is conserved by $((e^{\Lambda t})^*)$ as well. A more interesting and useful characterization is given by the following result, which in essence can be found in [12] (Proposition 17) (for bounded H and apparently for separable $\mathfrak{h}$).

Proposition 3.6.3. *Suppose Λ is as in the previous corollary.*

(i) Λ is dephasing if and only if B_j for every $j \in J$ lies in the double commutant $\{H\}''$ of H, which for separable $\mathfrak{h}$ is equal to

$$\{H\}'' = \{f(H) : f \ \text{bounded measurable function } \sigma(H) \to \mathbb{C}\}.$$

In particular, if Λ is dephasing, then B_j is a normal operator for every $j \in J$.

(ii) If Λ is dephasing, then $\ker \Lambda = \ker Z_0$ and, conversely, if $\ker \Lambda = \ker Z_0$ and the spectrum of H is pure point, then Λ is dephasing.

Proof. We have only to slightly modify the argument from [12], some care being necessary due to the unboundedness of the operators Z_0^* and Λ^* and the mere weak* continuity of the semigroups they generate. Since $\Lambda = Z_0 + W$ with W bounded, we have $\Lambda^* = Z_0^* + W^*$ (in particular, $D(\Lambda^*) = D(Z_0^*)$) and by II.2.5 of [41] we see that $\ker Z_0^* = \{H\}'$.

(i) Suppose that Λ is dephasing and let $a \in \{H\}'$. Then $a, a^*, a^*a \in \{H\}' = \ker Z_0^* \subset \ker \Lambda^*$ and therefore

$$0 = \Lambda^*(a^*a) - \Lambda^*(a^*)a - a^*\Lambda^*(a) = \sum_{j \in J} [B_j, a]^*[B_j, a], \tag{3.27}$$

where the second equality follows by straightforward computation from $\Lambda^* = Z_0^* + W^*$ together with the relation $Z_0^*(a^*a) = Z_0^*(a^*)a + a^*Z_0^*(a)$ and the explicit representation

$$W^*(a) = \sum_{j \in J} B_j^* a B_j - 1/2\{B_j^* B_j, a\}$$

for W^* (strong convergence by the theorem of Vigier). So, B_j commutes with $a \in \{H\}'$ for every $j \in J$, as desired. Suppose now that $B_j \in \{H\}''$ for every $j \in J$ and let $a \in \ker Z_0^*$. Then $B_j^* \in \{H\}''$ and $a \in \ker Z_0^* = \{H\}'$ and therefore $W^*(a) = 0$ by the above explicit representation for W^*. So, $\Lambda^*(a) = Z_0^*(a) + W^*(a) = 0$, as desired.

See [123] (Section X.2), [109] (Section 129) or [15] (Theorem 5.5.6) for a proof of the well-known explicit description of the double commutant of H in the case of separable $\mathfrak{h}$ (Riesz–Mimura). Separability is essential here as can be seen from an example in [123] (Section X.2). In order to see that every B_j is normal in case Λ is dephasing, notice that in this case

$$B_j, B_j^* \in \{H\}'' = \mathcal{A}'' = \overline{\mathcal{A}} \qquad \text{(closure w.r.t. the strong operator topology)}$$
$$\mathcal{A} := \{f(H) : f \ \text{ bounded measurable function } \ \sigma(H) \to \mathbb{C}\}$$

by the double commutant theorem of von Neumann and that the commutativity of the *-algebra $\mathcal{A}$ carries over to its strong closure $\overline{\mathcal{A}}$ by the density theorem of Kaplansky (for instance).

(ii) We will prove a refinement of the first implication of (ii) later on in the proof of Lemma 4.2.9 (iii). In fact, the first and second step of that proof will show that to obtain the inclusion $\ker \Lambda \subset \ker Z_0$ one only needs

$$\sum_{j \in J} B_j B_j^* = \sum_{j \in J} B_j^* B_j < \infty \tag{3.28}$$

which by (i) is a weaker requirement than dephasingness; dephasingness is needed only to obtain by (i) the reverse inclusion $\ker Z_0 \subset \ker \Lambda$.

We now prove the second implication of (ii). So, let the spectrum of H be pure point and let $\ker \Lambda = \ker Z_0$. Then there exists an orthonormal basis $\{e_i : i \in I\}$ of $\mathfrak{h}$ consisting of eigenvalues e_i of H and hence

$$P_F := \sum_{i \in F} \langle e_i, \, . \, \rangle \, e_i \qquad (F \text{ a finite subset of the index set } I)$$

belongs to $\{H\}' \cap S^1(\mathfrak{h}) = \ker Z_0 = \ker \Lambda$. We have to show in view of (i) that B_j for every $j \in J$ commutes with every $a \in \{H\}'$. In fact, it is sufficient to show this for every $a \in \{H\}' \cap S^1(\mathfrak{h})$ because an arbitrary element $a \in \{H\}'$ is strongly approximated by $a_F := P_F a \in \{H\}' \cap S^1(\mathfrak{h})$. So, let $a \in \{H\}' \cap S^1(\mathfrak{h})$. We then see in the same way as in (i) that $a, a^*, a^*a \in \{H\}' = \ker Z_0^*$ and hence

$$\Lambda^*(a^*a) - \Lambda^*(a^*)a - a^*\Lambda^*(a) = \sum_{j\in J} [B_j, a]^*[B_j, a]. \tag{3.29}$$

Since $a, a^* \in \{H\}' \cap S^1(\mathfrak{h}) = \ker Z_0 = \ker \Lambda$, one has

$$\operatorname{tr}(\Lambda^*(a^*)a) = \operatorname{tr}(a^*\Lambda(a)) = 0 \quad \text{and} \quad \operatorname{tr}(a^*\Lambda^*(a)) = \operatorname{tr}(\Lambda(a^*)a) = 0 \tag{3.30}$$

And since $P_F \in \ker \Lambda$ and $\sum_{j\in J}\{B_j^*B_j, a^*a\} \in S^1(\mathfrak{h})$, one has

$$\begin{aligned} 0 \le \sum_{i\in F} \big\langle e_i, \big(\sum_{j\in J} B_j^* a^* a B_j\big) e_i \big\rangle &= \operatorname{tr}(P_F \Lambda^*(a^*a)) + 1/2 \operatorname{tr}\big(P_F \sum_{j\in J}\{B_j^*B_j, a^*a\}\big) \\ &\longrightarrow 1/2 \operatorname{tr}\big(\sum_{j\in J}\{B_j^*B_j, a^*a\}\big), \end{aligned}$$

so that $\Lambda^*(a^*a) = \sum_{j\in J} B_j^* a^* a B_j - 1/2 \sum_{j\in J}\{B_j^*B_j, a^*a\}$ belongs to $S^1(\mathfrak{h})$ and

$$\operatorname{tr}(\Lambda^*(a^*a)) = \lim_F \operatorname{tr}(P_F \Lambda^*(a^*a)) = 0. \tag{3.31}$$

Combining now (3.29), (3.30), (3.31) we see that every B_j commutes with a, and we are done. ■

In the last implication of the above proposition, the assumption that H have pure point spectrum is essential. In order to see this (and various other things), we will need the following lemma.

Lemma 3.6.4. *Suppose Z_0 is the generator of the semigroup on $S^1(\mathfrak{h})$ defined by $e^{Z_0 t}(\rho) := e^{-iHt}\rho\, e^{iHt}$ and suppose $H : D(H) \subset \mathfrak{h} \to \mathfrak{h}$ is self-adjoint.*

(i) If $\sigma_p(H)$ is finite and each $\lambda \in \sigma_p(H)$ has finite multiplicity, then $\ker Z_0 = \{H\}' \cap S^1(\mathfrak{h})$ is finite-dimensional, more precisely

$$\ker Z_0 = \operatorname{span}\big\{ \langle e_{\lambda i}, .\rangle\, e_{\lambda j} : \lambda \in \sigma_p(H) \ \text{ and } \ i, j \in \{1, \dots, n_\lambda\}\big\},$$

where $\{e_{\lambda i} : i \in \{1, \dots, n_\lambda\}\}$ is an orthonormal basis of $\ker(H - \lambda)$ for every $\lambda \in \sigma_p(H)$. In particular, $\ker Z_0 = 0$ in case $\sigma_p(H) = \emptyset$.

(ii) If $\mathfrak{h}$ is infinite-dimensional, then $\{H\}'$ is infinite-dimensional.

Proof. (i) Clearly, $\ker Z_0 = \{H\}' \cap S^1(\mathfrak{h})$. As a preparation for the proof of the finite-dimensionality of $\ker Z_0$, we note that an element ρ of $S^1(\mathfrak{h})$ belongs to $\ker Z_0$ if and only if $\rho = \sum_{\lambda \in \sigma_p(H)} Q_{\{\lambda\}} \rho Q_{\{\lambda\}}$, where Q is the spectral measure of H. Indeed, if $\rho \in \ker Z_0$, then

$$\rho = 1/T \int_0^T e^{Z_0 t}(\rho)\, dt = 1/T \int_0^T e^{-iHt} \rho\, e^{iHt}\, dt \longrightarrow \sum_{\lambda \in \sigma_p(H)} Q_{\{\lambda\}} \rho Q_{\{\lambda\}}$$

w.r.t. the strong operator topology as $T \to \infty$ (Theorem 5.8 of [130]), and the converse implication is obvious. With this preparation at hand and the fact that $Q_{\{\lambda\}} = \sum_{i=1}^{n_\lambda} \langle e_{\lambda\, i}, \,.\, \rangle\, e_{\lambda\, i}$, we now see that for $\rho \in \ker Z_0$,

$$\rho = \sum_{\lambda \in \sigma_p(H)} Q_{\{\lambda\}} \rho Q_{\{\lambda\}} = \sum_{\lambda \in \sigma_p(H)} \sum_{i,j=1}^{n_\lambda} \langle e_{\lambda\, j}, \rho e_{\lambda\, i} \rangle \langle e_{\lambda\, i}, \,.\, \rangle\, e_{\lambda\, j}$$

belongs to $\operatorname{span}\big\{ \langle e_{\lambda\, i}, \,.\, \rangle\, e_{\lambda\, j} : \lambda \in \sigma_p(H) \text{ and } i, j \in \{1, \dots, n_\lambda\}\big\}$. We have thus proved the first of the asserted inclusions and the second inclusion is obvious.

(ii) In the case where $\sigma_p(H)$ is infinite or some $\lambda \in \sigma_p(H)$ has infinite multiplicity, there exists an infinite orthonormal system $\{\varphi_n : n \in \mathbb{N}\}$ consisting of eigenvalues of H and therefore the infinite subset

$$\{\rho_n : n \in \mathbb{N}\} \qquad (\rho_n := \langle \varphi_n, \,.\, \rangle\, \varphi_n)$$

of $\{H\}' \cap S^1(\mathfrak{h}) \subset \{H\}'$ is linearly independent, which proves the assertion. In the case where $\sigma_p(H)$ is finite and every $\lambda \in \sigma_p(H)$ has finite multiplicity, there exists an interval $J = [k, k+1]$ with $k \in \mathbb{Z}$ such that

$$\sigma(H) \cap J = \sigma(H) \cap [k, k+1]$$

is infinite. (If this was not so, then every spectral value $\lambda \in \sigma(H)$ would be isolated in $\sigma(H)$ and would hence be an eigenvalue of H. We would thus obtain $\sigma(H) = \sigma_p(H)$ and therefore $1 = Q_{\sigma(H)} = Q_{\sigma_p(H)} = \sum_{\lambda \in \sigma_p(H)} Q_{\{\lambda\}}$ would have finite rank. Contradiction!) We now show that the infinite subset

$$\{H_J^n : n \in \mathbb{N}\} \qquad (H_J := H Q_J)$$

of $\{H\}'$ is linearly independent, which proves the assertion. Indeed, if there was a (finite) linear combination

$$0 = \sum_{k=1}^n \alpha_k H_J^k = p(H_J) \qquad (p(\lambda) := \sum_{k=1}^n \alpha_k \lambda^k)$$

with $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ not all equal to 0, then the spectral mapping theorem would yield

$$p(\sigma(H_J)) = \sigma(p(H_J)) = \{0\}$$

so that $\sigma(H_J)$ and, a fortiori, $\sigma(H) \cap J$ would have to be finite. Contradiction! ■

With this lemma at hand, we can now convince ourselves that there exist non-dephasing generators Λ with $\ker \Lambda = \ker Z_0$ (as was claimed before the lemma).

Example 3.6.5. We choose a self-adjoint operator $H : D(H) \subset \mathfrak{h} \to \mathfrak{h}$ in an infinite-dimensional Hilbert space (with spectral measure denoted by Q) such that $\sigma_p(H)$ is finite and every $\lambda \in \sigma_p(H)$ has finite multiplicity, and we choose

$$B := \sum_{\lambda \in \sigma_p(H)} \beta_\lambda Q_{\{\lambda\}} + \beta \langle \psi, \,.\, \rangle \psi$$

where $\beta_\lambda \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus \{0\}$ and where $\psi = H\varphi / \|H\varphi\|$ and $\varphi \in M^\perp \setminus \{0\}$ with

$$M := Q_{\sigma_p(H)}\mathfrak{h} = \bigoplus_{\lambda \in \sigma_p(H)} \ker(H - \lambda).$$

It should be noticed that $M^\perp \setminus \{0\}$ is non-empty by the assumptions on the spectrum of H and by the infinite-dimensionality of $\mathfrak{h}$ and, moreover, that $H\varphi \neq 0$ because otherwise φ would be an eigenvector of H and would hence belong to M contradicting $\varphi \in M^\perp \setminus \{0\}$. It should also be noticed that B is a normal operator because $\psi = H\varphi / \|H\varphi\| \in HM^\perp \subset M^\perp$. We now define

$$\Lambda(\rho) := Z_0(\rho) + B\rho B^* - 1/2\{B^*B, \rho\} \qquad (\rho \in D(Z_0)),$$

where Z_0 is the generator of the semigroup in $\mathfrak{h}$ defined by $e^{Z_0 t}(\rho) := e^{-iHt} \rho \, e^{iHt}$. It is then clear that Λ is the generator of a quantum dynamical semigroup on $S^1(\mathfrak{h})$ (Theorem 3.6.1). With the help of Proposition 3.6.3 and Lemma 3.6.4 it also follows that

$$\ker \Lambda = \ker Z_0.$$

(Indeed, the inclusion $\ker \Lambda \subset \ker Z_0$ follows by the normality of B and by what has been remarked in the context of (3.28) in the proof of Proposition 3.6.3 (ii), and the reverse inclusion $\ker Z_0 \subset \ker \Lambda$ follows by the explicit description of $\ker Z_0$ given in Lemma 3.6.4 (i) and by the fact that every eigenvector of H with eigenvalue λ is an eigenvector of B with eigenvalue β_λ.) And finally,

$$HB \neq BH,$$

whence $B \notin \{H\}''$ and Λ is not dephasing (Proposition 3.6.3 (i)). (In order to see that H indeed does not commute with B, compute

$$HB\varphi = \beta \langle \psi, \varphi \rangle H\psi \quad \text{and} \quad BH\varphi = \beta \|H\varphi\| \psi.$$

In case $\langle \psi, \varphi \rangle = 0$, it follows that $HB\varphi - BH\varphi = -\beta \|H\varphi\| \psi \neq 0$ because $\beta \neq 0$. In case $\langle \psi, \varphi \rangle \neq 0$, it follows that $HB\varphi - BH\varphi = \beta \langle \psi, \varphi \rangle \big(H\psi - \|H\varphi\| / \langle \psi, \varphi \rangle \, \psi \big) \neq 0$ because otherwise ψ would be an eigenvector of H and would therfore belong to M. Contradiction!) ◄

4 Adiabatic theorems for operators with time-independent domains

4.1 Adiabatic theorems with spectral gap condition

After having provided the most important preliminaries in Chapter 3, we now prove an adiabatic theorem with uniform spectral gap condition (Section 4.1.1) and an adiabatic theorem with non-uniform spectral gap condition (Section 4.1.2) for general operators $A(t)$ with time-independent domains. In these theorems the considered spectral subsets $\sigma(t)$ are only assumed to be compact so that, even if they are singletons, they need not consist of eigenvalues: they are allowed to be singletons consisting of essential singularities of the resolvent. In [2], [12], [60] the case of poles is treated and in [2], [12] they are of order 1.

4.1.1 An adiabatic theorem with uniform spectral gap condition

We begin by proving an adiabatic theorem with uniform spectral gap condition by extending Abou Salem's proof from [2], which rests upon solving a suitable commutator equation.

Theorem 4.1.1. *Suppose $A(t) : D \subset X \to X$ for every $t \in I$ is a linear map such that Condition 2.1.8 is satisfied with $\omega = 0$. Suppose further that $\sigma(t)$ for every $t \in I$ is a compact subset of $\sigma(A(t))$, that $\sigma(\,.\,)$ at no point falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$, and that $t \mapsto \sigma(t)$ is continuous. And finally, for every $t \in I$, let $P(t)$ be the projection associated with $A(t)$ and $\sigma(t)$ and suppose that $I \ni t \mapsto P(t)$ is in $W_*^{2,1}(I, L(X))$. Then*

$$\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \quad (\varepsilon \searrow 0),$$

where U_ε and V_ε are the evolution systems for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$.

Proof. Since $\sigma(\,.\,)$ is uniformly isolated in $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ and $t \mapsto \sigma(t)$ is continuous, there is, for every $t_0 \in I$, a non-trivial closed interval $J_{t_0} \subset I$ containing t_0 and a cycle γ_{t_0} in $\rho(A(t_0))$ such that $\operatorname{ran}\gamma_{t_0} \subset \rho(A(t))$ and

$$\mathrm{n}(\gamma_{t_0}, \sigma(t)) = 1 \quad \text{and} \quad \mathrm{n}(\gamma_{t_0}, \sigma(A(t)) \setminus \sigma(t)) = 0$$

for all $t \in J_{t_0}$. We can now define

$$B(t)x := \frac{1}{2\pi i} \int_{\gamma_{t_0}} (z - A(t))^{-1} P'(t)(z - A(t))^{-1} x \, dz$$

for all $t \in J_{t_0}$, $t_0 \in I$ and $x \in X$. Since $\rho(A(t)) \ni z \mapsto (z - A(t))^{-1} P'(t)(z - A(t))^{-1} x$ is a holomorphic X-valued map (for all $x \in X$) and since the cycles γ_{t_0} and $\gamma_{t_0'}$ are homologous in $\rho(A(t))$ whenever t lies both in J_{t_0} and in $J_{t_0'}$, the path integral exists in X and does not depend on the special choice of $t_0 \in I$ with the property that $t \in J_{t_0}$. In other words, $t \mapsto B(t)$ is well-defined on I.

As a first preparatory step, we easily infer from the closedness of $A(t)$ that $B(t)X \subset D(A(t)) = D = Y$ and that

$$B(t)A(t) - A(t)B(t) \subset [P'(t), P(t)] \tag{4.1}$$

for all $t \in I$, which commutator equation will be essential in the main part of the proof. As a second preparatory step, we show that $t \mapsto B(t)$ is in $W^{1,1}_*(I, L(X,Y))$, which is not very surprising (albeit a bit technical). It suffices to show that $J_{t_0} \ni t \mapsto B(t)$ is in $W^{1,1}_*(J_{t_0}, L(X,Y))$ for every $t_0 \in I$. We therefore fix $t_0 \in I$. Since $\rho(A(t)) \ni z \mapsto (z - A(t))^{-1}$ is continuous w.r.t. the norm of $L(X,Y)$ for every $t \in J_{t_0}$, we see that $B(t)$ is in $L(X,Y)$ for every $t \in J_{t_0}$. We also see, by virtue of Lemma 2.1.2, that for every $z \in \operatorname{ran} \gamma_{t_0}$ the map $t \mapsto (z - A(t))^{-1} P'(t)(z - A(t))^{-1}$ is in $W^{1,1}_*(J_{t_0}, L(X,Y))$ and $t \mapsto C(t,z) = C_1(t,z) + C_2(t,z) + C_3(t,z)$ is a $W^{1,1}_*$-derivative of it, where

$$\begin{aligned} C_1(t,z) &= (z - A(t))^{-1} A'(t)(z - A(t))^{-1} P'(t)(z - A(t))^{-1}, \\ C_2(t,z) &= (z - A(t))^{-1} P''(t)(z - A(t))^{-1}, \\ C_3(t,z) &= (z - A(t))^{-1} P'(t)(z - A(t))^{-1} A'(t)(z - A(t))^{-1}, \end{aligned} \tag{4.2}$$

and A', P'' are arbitrary $W^{1,1}_*$-derivatives of A and P'. Since $t \mapsto C(t,z)$ is strongly measurable for all $z \in \operatorname{ran} \gamma_{t_0}$, it follows that

$$t \mapsto \frac{1}{2\pi i} \int_{\gamma_{t_0}} C(t,z)\, dz$$

is strongly measurable as well (as the strong limit of Riemann sums), and since $J_{t_0} \times \operatorname{ran} \gamma_{t_0} \ni (t,z) \mapsto (z - A(t))^{-1}$ is continuous w.r.t. the norm of $L(X,Y)$ and hence bounded, it follows by (4.2) that

$$t \mapsto \Big\| \frac{1}{2\pi i} \int_{\gamma_{t_0}} C(t,z)\, dz \Big\|_{X,Y}$$

has an integrable majorant. So $t \mapsto \frac{1}{2\pi i} \int_{\gamma_{t_0}} C(t,z)\, dz$ is in $W^{0,1}_*(J_{t_0}, L(X,Y))$ and one easily concludes that

$$B(t)x = B(t_0)x + \int_{t_0}^{t} \frac{1}{2\pi i} \int_{\gamma_{t_0}} C(\tau, z)x \, dz \, d\tau$$

for all $t \in J_{t_0}$ and $x \in X$, as desired.

After these preparations we can now turn to the main part of the proof. We fix $x \in D$ and let V_ε denote the evolution system for $\frac{1}{\varepsilon} A + [P', P]$ (which really exists due to Theorem 2.1.9). Then $s \mapsto U_\varepsilon(t,s)V_\varepsilon(s)x$ is in $W^{1,1}([0,t],X)$ (by Lemma 2.1.3 and Lemma 2.1.4) and we get, exploiting the commutator equation (4.1) for A and B, that

$$V_\varepsilon(t)x - U_\varepsilon(t)x = U_\varepsilon(t,s)V_\varepsilon(s)x\big|_{s=0}^{s=t} = \int_0^t U_\varepsilon(t,s)[P'(s),P(s)]V_\varepsilon(s)x\,ds$$
$$= \int_0^t U_\varepsilon(t,s)\big(B(s)A(s) - A(s)B(s)\big)V_\varepsilon(s)x\,ds$$

for all $t \in I$. Since for every $t \in I$ the maps $s \mapsto V_\varepsilon(s)\big|_Y$ and $s \mapsto U_\varepsilon(t,s)\big|_Y$ are continuously differentiable on $[0,t]$ w.r.t. the strong operator topology of $L(Y,X)$ (Lemma 2.1.5) and hence belong to $W_*^{1,1}([0,t],L(Y,X))$, and since $s \mapsto B(s)$ belongs to $W_*^{1,1}([0,t],L(X,Y))$, we can further conclude that $s \mapsto U_\varepsilon(t,s)B(s)V_\varepsilon(s)x$ is in $W^{1,1}([0,t],X)$ by Lemma 2.1.2, so that

$$V_\varepsilon(t)x - U_\varepsilon(t)x = \varepsilon \int_0^t U_\varepsilon(t,s)\Big(-\frac{1}{\varepsilon}A(s)B(s) + B(s)\frac{1}{\varepsilon}A(s)\Big)V_\varepsilon(s)x\,ds$$
$$= \varepsilon\, U_\varepsilon(t,s)B(s)V_\varepsilon(s)x\big|_{s=0}^{s=t} - \varepsilon \int_0^t U_\varepsilon(t,s)\big(B'(s) + B(s)[P'(s),P(s)]\big)V_\varepsilon(s)x\,ds$$

for all $t \in I$ and $\varepsilon \in (0,\infty)$, where B' denotes an arbitrary $W_*^{1,1}$-derivative of B. And from this, the conclusion of the theorem is obvious. ■

4.1.2 An adiabatic theorem with non-uniform spectral gap condition

We continue by proving an adiabatic theorem with non-uniform spectral gap condition where $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ at countably many points that, in turn, accumulate at only finitely many points. We do so by extending Kato's proof from [61] where finitely many eigenvalue crossings for skew self-adjoint $A(t)$ are treated.

Theorem 4.1.2. *Suppose $A(t) : D \subset X \to X$ for every $t \in I$ is a linear map such that Condition 2.1.8 is satisfied with $\omega = 0$. Suppose further that $\sigma(t)$ for every $t \in I$ is a compact subset of $\sigma(A(t))$, that $\sigma(\,.\,)$ at countably many points accumulating at only finitely many points falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$, and that $I \setminus N \ni t \mapsto \sigma(t)$ is continuous, where N denotes the set of those points where $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$. And finally, for every $t \in I \setminus N$, let $P(t)$ be the projection associated with $A(t)$ and $\sigma(t)$ and suppose that $I \setminus N \ni t \mapsto P(t)$ extends to a map (again denoted by P) in $W_*^{2,1}(I,L(X))$. Then*

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

where U_ε and V_ε are the evolution systems for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P',P]$.

Proof. We first prove the assertion in the case where $\sigma(\,.\,)$ at only finitely many points $t_1, \dots, t_m$ (ordered in an increasing way) falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$. So let $\eta > 0$. We partition the interval I as follows:

$$I = I_{0\,\delta} \cup J_{1\,\delta} \cup I_{1\,\delta} \cup \dots \cup J_{m\,\delta} \cup I_{m\,\delta},$$

where $J_{i\,\delta}$ for $i \in \{1, \dots, m\}$ is a relatively open subinterval of I containing t_i of length less than δ (which will be chosen in a minute) and where $I_{0\,\delta}, \dots, I_{m\,\delta}$ are the closed subintervals of I lying between the subintervals $J_{1\,\delta}, \dots, J_{m\,\delta}$. In the following, we set $t_{i\,\delta}^- := \inf I_{i\,\delta}$ and $t_{i\,\delta}^+ := \sup I_{i\,\delta}$ for $i \in \{0, \dots, m\}$, and we choose c so large that $\|P(s)\|$, $\|P'(s)\|$ and $\|[P'(s), P(s)]\| \le c$ for all $s \in I$. Since

$$\begin{aligned}\left\|V_\varepsilon(t, t_{i-1\,\delta}^+)x - U_\varepsilon(t, t_{i-1\,\delta}^+)x\right\| &= \left\| \int_{t_{i-1\,\delta}^+}^t U_\varepsilon(t,s)[P'(s), P(s)]V_\varepsilon(s, t_{i-1\,\delta}^+)x \, ds \right\| \\ &\le McMe^{Mc}\,\delta\,\|x\|\end{aligned}$$

for every $t \in J_{i\,\delta}$, $x \in D$ and $\varepsilon \in (0, \infty)$, we can achieve – by choosing δ small enough – that

$$\sup_{t \in J_{i\,\delta}} \left\|V_\varepsilon(t, t_{i-1\,\delta}^+) - U_\varepsilon(t, t_{i-1\,\delta}^+)\right\| < \frac{\eta}{\left(4M^2e^{2Mc}\right)^m} \tag{4.3}$$

for every $\varepsilon \in (0, \infty)$ and $i \in \{1, \dots, m\}$. And since $\sigma(\,.\,)\big|_{I_{i\,\delta}}$ at no point falls into $\big(\sigma(A(\,.\,))\setminus\sigma(\,.\,)\big)\big|_{I_{i\,\delta}}$, we conclude from the above adiabatic theorem with uniform spectral gap condition (applied to the restricted data $A|_{I_{i\,\delta}}$, $\sigma|_{I_{i\,\delta}}$, $P|_{I_{i\,\delta}}$) that there is an $\varepsilon_\delta \in (0, \infty)$ such that

$$\sup_{t \in I_{i\,\delta}} \left\|V_\varepsilon(t, t_{i\,\delta}^-) - U_\varepsilon(t, t_{i\,\delta}^-)\right\| < \frac{\eta}{\left(4M^2e^{2Mc}\right)^m} \tag{4.4}$$

for every $\varepsilon \in (0, \varepsilon_\delta)$ and $i \in \{0, \dots, m\}$. Combining the estimates (4.3) and (4.4) and using the product property from the definition of evolution systems, we readily conclude for every $i \in \{1, \dots, m\}$ that

$$\|V_\varepsilon(t) - U_\varepsilon(t)\| < \frac{\eta}{\left(4M^2e^{2Mc}\right)^{m-i}} \le \eta$$

for all $t \in I_{i-1\,\delta} \cup J_{i\,\delta} \cup I_{i\,\delta}$ and $\varepsilon \in (0, \varepsilon_\delta)$, and the desired conclusion follows.

We now prove the assertion in the case where $\sigma(\,.\,)$ at infinitely many points accumulating at only finitely many points $t_1, \dots, t_m$ (ordered in an increasing way) falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$. In order to do so, we partition I and choose δ as we did above. We then obtain the estimate (4.3) as above and the estimate (4.4) by realizing that $\sigma(\,.\,)\big|_{I_{i\,\delta}}$ at only finitely many points falls into $\big(\sigma(A(\,.\,)) \setminus \sigma(\,.\,)\big)\big|_{I_{i\,\delta}}$ (so that the case just proved can be applied). And from these estimates the conclusion follows as above. ■

It should be noticed that the hypotheses of the above adiabatic theorem allow the spectral subsets $\sigma(t)$ to be non-isolated in $\sigma(A(t))$ for all points $t \in N$, that is (by our definition of spectral gaps in Section 3.3), they also allow for some situations without spectral gap. It should also be noticed that, in the situation of the above theorem, one has $P(t)A(t) \subset A(t)P(t)$ for every $t \in I$ (although a priori this is clear only for $t \in I \setminus N$), which follows by a continuity argument. (Indeed, if $t_0 \in I$ then it can be approximated by a sequence (t_n) in $I \setminus N$. Since $t \mapsto (A(t_0)-1)(A(t)-1)^{-1}$ is norm continuous (by the $W_*^{1,1}$-regularity of $t \mapsto A(t)$ and Lemma 2.1.2), we see that for any $x \in D$

$$\begin{aligned} A(t_0)P(t_n)x &= (A(t_0)-1)(A(t_n)-1)^{-1}\, P(t_n)(A(t_n)-1)x + P(t_n)x \\ &\longrightarrow P(t_0)A(t_0)x \quad (n \to \infty) \end{aligned} \tag{4.5}$$

and therefore $P(t_0)A(t_0) \subset A(t_0)P(t_0)$ by the closedness of $A(t_0)$.) In particular, the evolution V_ε appearing in the above theorem really is adiabatic w.r.t. to P by Proposition 3.4.1, as it should be.

4.1.3 Some remarks and examples

We begin with four remarks concerning the adiabatic theorems with uniform and non-uniform spectral gap condition alike.

1. In the special situation where $\sigma(t) = \{\lambda(t)\}$ and $\lambda(t)$ is a pole of the resolvent map $(\,.\, - A(t))^{-1}$ of order at most $m_0 \in \mathbb{N}$ for all $t \in I$, the operators $B(t)$ – used in the proof of the adiabatic theorems with spectral gap condition above to solve the commutator equation (4.1) – can be cast in a form, namely (4.6), which points the way to the solution of an appropriate (approximate) commutator equation in the adiabatic theorems without spectral gap condition below. Since $PP'P$, $\overline{P}P'\overline{P} = 0$ by (3.20) (where $\overline{P} := 1 - P$) and

$$(z - A(t))^{-1}P(t) = \frac{1}{z-\lambda(t)}\Big(1 - \frac{A(t)-\lambda(t)}{z-\lambda(t)}\Big)^{-1}P(t) = \sum_{k=0}^{m_0-1} \frac{(A(t)-\lambda(t))^k P(t)}{(z-\lambda(t))^{k+1}}$$

for every $z \in \rho(A(t))$ by Theorem 5.8-A of [128], we see that

$$\begin{aligned} B(t) = \sum_{k=0}^{m_0-1} \frac{1}{2\pi i}\int_{\gamma_t} \frac{\overline{R}(t,z)}{(z-\lambda(t))^{k+1}}\, dz\ P'(t)(A(t)-\lambda(t))^k P(t) \\ + \sum_{k=0}^{m_0-1} (A(t)-\lambda(t))^k P(t)P'(t)\ \frac{1}{2\pi i}\int_{\gamma_t} \frac{\overline{R}(t,z)}{(z-\lambda(t))^{k+1}}\, dz, \end{aligned}$$

and since the reduced resolvent map $z \mapsto \overline{R}(t,z) := (z - A(t)|_{\overline{P}(t)D(A(t))})^{-1}\overline{P}(t)$ is holomorphic on $\rho(A(t)) \cup \{\lambda(t)\}$, we further see – using Cauchy's theorem – that

$$\begin{aligned} B(t) = \sum_{k=0}^{m_0-1} \overline{R}(t,\lambda(t))^{k+1}P'(t)(\lambda(t)-A(t))^k P(t) \\ + \sum_{k=0}^{m_0-1} (\lambda(t)-A(t))^k P(t)P'(t)\overline{R}(t,\lambda(t))^{k+1}. \end{aligned} \tag{4.6}$$

2. In the even more special situation where $\sigma(t) = \{\lambda(t)\} \subset i\mathbb{R}$ and $\lambda(t)$ is a pole of the resolvent map $(\,.\, - A(t))^{-1}$, the hypotheses of the above adiabatic theorem with

uniform spectral gap condition become essentially – apart from regularity conditions – equivalent to the hypotheses of the respective adiabatic theorem (Theorem 9) of [12], and a similar equivalence holds true for the above adiabatic theorem with non-uniform spectral gap condition. Indeed, if $\sigma(t)$ for every $t \in I$ is a singleton consisting of a pole $\lambda(t)$ on the imaginary axis, then the order $m(t)$ of nilpotence of $A(t)|_{P(t)D} - \lambda(t)$ must be equal to 1, since otherwise the relation

$$\delta\big(\lambda(t) + \delta - A(t)\big)^{-1} P(t) = \sum_{k=0}^{m(t)-1} \frac{(A(t) - \lambda(t))^k}{\delta^k} P(t) \tag{4.7}$$

would yield the contradiction that the right hand side of (4.7) explodes as $\delta \searrow 0$ whereas the left hand side of (4.7) remains bounded as $\delta \searrow 0$ (by virtue of the $(M, 0)$-stability of A and by $\lambda(t) \in i\mathbb{R}$). And therefore (by Theorem 5.8-A of [128]) $P(t)X = \ker(A(t) - \lambda(t))$ and $(1 - P(t))X = \operatorname{ran}(A(t) - \lambda(t))$ as in [12].

3. It is obvious from the proof of Theorem 4.1.1 that the assumption that $\sigma(\,.\,)$ at no point fall into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ and that $t \mapsto \sigma(t)$ be continuous can be replaced by the weaker – but also less convenient – requirement that for each $t_0 \in I$ there be a non-trivial closed interval $J_{t_0} \subset I$ containing t_0 and a cycle γ_{t_0} such that $\operatorname{ran} \gamma_{t_0} \subset \rho(A(t))$ and

$$\mathrm{n}(\gamma_{t_0}, \sigma(t)) = 1 \quad \text{and} \quad \mathrm{n}(\gamma_{t_0}, \sigma(A(t)) \setminus \sigma(t)) = 0$$

for all $t \in J_{t_0}$. It can be shown that this weaker requirement still entails the upper semicontinuity of $t \mapsto \sigma(t)$ and hence – by Proposition 3.3.1 – it still ensures that $\sigma(\,.\,)$ at no point falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ or, in other words, that the spectral gap is uniform. (See Corollary 5.4 of [112] for a proof.) Consequently, if one adds to the thus weakened hypotheses the requirement that $t \mapsto \sigma(t)$ be lower semicontinuous (which – by Proposition 5.6 of [112] – is fulfilled if, for instance, $\sigma(t)$ is finite for every $t \in I$, and $0 \in \sigma(t)$ for all $t \in I$ or $0 \notin \sigma(t)$ for all $t \in I$), one arrives at an adiabatic theorem equivalent to the original one above (Theorem 4.1.1). Similar remarks hold for the case of non-uniform spectral gap (Theorem 4.1.2).

4. We finally remark that the above adiabatic theorems – along with the commutator equation method used in their proofs – can be extended to several subsets $\sigma_1(t), \dots, \sigma_r(t)$ of $\sigma(A(t))$: if A, σ_j, P_j for every $j \in \{1, \dots, r\}$ satisfy the hypotheses of the above adiabatic theorem with uniform or non-uniform spectral gap and if $\sigma_j(\,.\,)$ and $\sigma_l(\,.\,)$ for all $j \neq l$ fall into each other at only countably many points accumulating at only finitely many points, then there exists an evolution system V_ε, namely that for $\frac{1}{\varepsilon}A + K$ with

$$K(t) := \frac{1}{2} \sum_{j=1}^{r+1} [P_j'(t), P_j(t)] \quad \text{and} \quad P_{r+1}(t) := 1 - P(t) := 1 - \sum_{j=1}^{r} P_j(t), \tag{4.8}$$

which on the one hand is simultaneously adiabatic w.r.t. all the P_j by [61] and on the other hand well approximates the evolution system U_ε for $\frac{1}{\varepsilon}A$ in the sense that

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0).$$

In order to see this, one has only to observe that $B(t) := \frac{1}{2}\sum_{j=1}^{r+1} B_j(t)$ with

$$B_j := \frac{1}{2\pi i}\int_{\gamma_j} (z-A)^{-1}P_j'(z-A)^{-1}\,dz \quad (j \in \{1,\dots,r\})$$
$$B_{r+1} := \frac{1}{2\pi i}\int_{\gamma} (z-A)^{-1}P'(z-A)^{-1}\,dz \tag{4.9}$$
with $\gamma := \gamma_1 + \dots + \gamma_r$ $(\gamma_j = \gamma_{j\,t}$ as in the proofs above) and $P := P_1 + \dots + P_r$

solves the commutator equation $B(t)A(t) - A(t)B(t) \subset K(t)$ for all points t where no crossing takes place (because $[P_{r+1}', P_{r+1}] = [P', P]$) and then to proceed as in the proofs of the adiabatic theorems above. See also [20]. In the special case of skew self-adjoint operators $A(t)$ one can further refine the statement above: it is then possible to show – by further adapting the commutator equation method – that even the evolution system $\overline{V}_\varepsilon$ for $\frac{1}{\varepsilon}A + \overline{K}$ with

$$\overline{K}(t) := \frac{1}{2}\Big([(P_{r+1}^-)'(t), P_{r+1}^-(t)] + \sum_{j=1}^{r}[P_j'(t), P_j(t)] + [(P_{r+1}^+)'(t), P_{r+1}^+(t)]\Big)$$

well approximates the evolution system U_ε for $\frac{1}{\varepsilon}A$ – notice that $\overline{V}_\varepsilon$ is is not only adiabatic w.r.t. $P_{r+1} = P_{r+1}^- + P_{r+1}^+$ but also w.r.t. P_{r+1}^- and P_{r+1}^+ separately, where $P_{r+1}^\pm(t)$ are the spectral projections of $A(t)$ corresponding to the parts $\sigma^\pm(t)$ of the spectrum which on $i\mathbb{R}$ are located below resp. above all the compact parts $\sigma_1(t), \dots, \sigma_r(t)$. In order to see this, set

$$B_{r+1\,n}^\pm(t) := \frac{1}{2\pi i}\int_{\gamma_{n\,t}^\pm} (z-A(t))^{-1}(P_{r+1}^\pm)'(t)(z-A(t))^{-1}\,dz$$

where $\gamma_{n\,t}^\pm(\tau) := \pm\tau + c^\pm(t)$ for $\tau \in [-n,n]$ with points $c^\pm(t) \in i\mathbb{R}$ lying in the gap between $\sigma^\pm(t)$ and the rest of $\sigma(A(t))$ and depending continuously differentiably on t, and observe that (by the skew self-adjointness of $A(t)$)

$$P_{r+1\,n}^\pm(t)x := \frac{1}{2\pi i}\int_{\gamma_{n\,t}^\pm} (z-A(t))^{-1}x\,dz \longrightarrow P_{r+1}^\pm(t)x - \frac{1}{2}x \quad (n \to \infty)$$

and

$$\big\|B_{r+1\,n}(t)\big\|, \big\|B_{r+1\,n}'(t)\big\| \le \int_{-\infty}^{\infty} \frac{c}{\operatorname{dist}\big(\gamma_{n\,t}^\pm(\tau), \sigma(A(t))\big)^2}\,d\tau \le C < \infty \quad (n \in \mathbb{N}, t \in I).$$

A slightly less general general statement was first proven in [96] by a different method than the commutator equation technique indicated above.

We now move on to discuss some examples. In the first – very simple – example, $t \mapsto A(t)$ is only $W_*^{1,1}$-regular and only $(M,0)$-stable (without being strongly continuously differentiable or $(1,0)$-stable), which means that this example lies outside the scope of the previously known adiabatic theorems.

Example 4.1.3. Suppose A, σ, P with $A(t) = R(t)^{-1}A_0(t)R(t)$, $\sigma(t) = \{\lambda(t)\}$, $P(t) = R(t)^{-1}P_0R(t)$, and $R(t) = e^{Ct}$ are given as follows in $X := \ell^p(I_2) \times \ell^p(I_1)$ (where $p \in [1,\infty)$):

$$A_0(t) := \begin{pmatrix} \lambda(t) + \alpha(t)N & 0 \\ 0 & \mu(t) \end{pmatrix} \quad \text{and} \quad P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where λ, α, N satisfy Condition 3.5.1 and where $\mu(t) \in \{\operatorname{Re} z \le 0\}$ is such that $\lambda(\,.\,)$ falls into $\mu(\,.\,)$ at only countably many points accumulating at only finitely many points. Additionally, choose the family $\lambda + \alpha N$ to be not $(1,0)$-stable (which, by the discussion after Lemma 3.5.2, can easily be achieved), the functions $t \mapsto \lambda(t)$, $\alpha(t)$, $\mu(t)$ to be absolutely continuous without being continuously differentiable, and

$$C := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since A_0 by Lemma 3.5.2 is $(M_0,0)$-stable for some $M_0 \in [1,\infty)$, the rotated family A is $(M,0)$-stable for another $M \in [1,\infty)$ by Lemma 2.1.7. And since P_0 is obviously associated with $A_0(t)$ and $\sigma(t)$ for every $t \in I \setminus N$ (where N denotes the set of those points in I where $\lambda(\,.\,)$ falls into $\mu(\,.\,)$), the same is true for $P(t)$ and $A(t)$ instead of P_0, $A_0(t)$. So the hypotheses of the adiabatic theorem with non-uniform spectral gap condition (Theorem 4.1.2) are fulfilled and therefore

$$(1 - P(t))U_\varepsilon(t)P(0) \longrightarrow 0 \quad (\varepsilon \searrow 0)$$

uniformly in $t \in I$, but this does not already follow from the trivial adiabatic theorems above (Proposition 3.4.2 (i) and (ii)). Indeed, as $[P_0, C] \ne 0$, it follows that $P'(t) \ne 0$ for every $t \in I$. And if μ is chosen in such a way that $\mu(t_0) = 0$ for some $t_0 \in [0,1)$, then it follows that the (block diagonal!) evolution $U_{0\,\varepsilon}$ for $\frac{1}{\varepsilon}A_0$ for no $\gamma > 0$ satisfies the estimate (3.21) uniformly in $\varepsilon \in (0,\infty)$, whence by Proposition 2.1.13 the same is true for the evolution $\tilde{U}_{0\,\varepsilon}$ for $\frac{1}{\varepsilon}A_0 + C = \frac{1}{\varepsilon}A_0 + R'R^{-1}$ and hence (by the proof of Corollary 2.1.10) also for the evolution U_ε for $\frac{1}{\varepsilon}A$ that we are interested in. ◀

In the next example, the spectral subsets $\sigma(t) = \{\lambda(t)\}$ are singletons consisting of spectral values $\lambda(t) \in i\mathbb{R}$ of $A(t)$ that are not eigenvalues and, a fortiori, are not poles (but essential singularities) of $(\,.\, - A(t))^{-1}$. In particular, the adiabatic theorem with spectral gap condition from [12] cannot be applied here (also see Example 4 of [12]). We make use of the Volterra operator V in $L^2([0,1])$ defined by

$$(Vf)(t) := \int_0^t f(\tau)\, d\tau \quad (t \in I).$$

Since V is quasinilpotent (in fact, $\|V^n\| \le 1/\sqrt{(n+1)!}$ for all $n \in \mathbb{N}$) and both V and V^* are injective, it follows that $\sigma(V) = \{0\} = \sigma_c(V)$.

Example 4.1.4. Suppose A, σ, P with $A(t) = R(t)^{-1}A_0(t)R(t)$, $\sigma(t) = \{\lambda(t)\} := \{0\}$, $P(t) = R(t)^{-1}P_0R(t)$, and $R(t) = e^{Ct}$ are given as follows in $X := L^2(I) \times L^2(I)$:

$$A_0(t) := \begin{pmatrix} -V & 0 \\ 0 & a(t) + M_f \end{pmatrix} \quad \text{and} \quad P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where V is the Volterra operator defined above and where $f : I \to \mathbb{C}$ is a measurable function with $\text{ess-ran}\, f = [-1, 0]$. Additionally, suppose the function $t \mapsto a(t) \in (-\infty, 0]$ is absolutely continuous and falls into 0 at only countably many points accumulating at only finitely many points, and

$$C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $-V$ and $a(t) + M_f$ are dissipative in $L^2(I)$, the family A_0 is $(1, 0)$-stable and, by the unitarity of the rotation operators $e^{Ct} = R(t)$, the same goes for A. Also, since P_0 commutes with $A_0(t)$ and

$$\sigma(A_0(t)|_{P_0X}) = \sigma(-V) = \{0\} = \sigma(t),$$
$$\sigma(A_0(t)|_{(1-P_0)X}) = \sigma(a(t) + M_f) = a(t) + [-1, 0] = \sigma(A_0(t)) \setminus \sigma(t)$$

for every $t \in I \setminus N$, P_0 is associated with $A_0(t)$ and $\sigma(t)$ for every $t \in I \setminus N$ and hence the same holds true for $P(t)$ and $A(t)$ instead of P_0, $A_0(t)$. All other hypotheses of Theorem 4.1.2 are clear. ◄

We finally give a simple example showing that the conclusion of the above adiabatic theorems will, in general, fail if the evolution systems U_ε for $\frac{1}{\varepsilon}A$ are not bounded in ε (and hence A is not $(M, 0)$-stable).

Example 4.1.5. Suppose A, σ, P with $A(t) := R(t)^{-1}A_0(t)R(t)$, $\sigma(t) := \{\lambda(t)\}$ and $P(t) := R(t)^{-1}P_0R(t)$ are given as follows in $X := \ell^2(I_2)$:

$$A_0(t) := \begin{pmatrix} \lambda(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R(t) := e^{Ct} \quad \text{with} \quad C := 2\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $t \mapsto \lambda(t) \in [0, \infty)$ is absolutely continuous such that $\lambda(\,.\,)$ at only countably many points accumulating at only finitely many points falls into 0. Then all the hypotheses of Theorem 4.1.2 are fullfilled with the sole exception that A is not $(M, 0)$-stable (because $\sigma(A(t)) = \{0, \lambda(t)\}$ is contained in the closed left half-plane only for countably many $t \in I$) and, in fact, the conclusion of this theorems fails. Indeed, since

$$R(t) = e^{Ct} = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix},$$

we see that

$$A(t) = R(t)^{-1}A_0(t)R(t) = \lambda(t) \begin{pmatrix} \cos^2(2\pi t) & \cos(2\pi t)\sin(2\pi t) \\ \cos(2\pi t)\sin(2\pi t) & \sin^2(2\pi t) \end{pmatrix}$$

is a positive linear map (in the lattice sense) for all $t \in [0, t_0]$ with $t_0 := \frac{1}{4}$. And since $1 - P(t_0) = P_0$, we see (by the series expansion for U_ε) that

$$\begin{aligned}\|(1 - P(t_0))U_\varepsilon(t_0)P(0)e_1\| &= \big|\langle e_1, U_\varepsilon(t_0)e_1\rangle\big| = \langle e_1, U_\varepsilon(t_0)e_1\rangle \\ &\geq 1 + \frac{1}{\varepsilon}\int_0^{t_0} \langle e_1, A(\tau)e_1\rangle \, d\tau = 1 + \frac{1}{\varepsilon}\int_0^{t_0} \lambda(\tau)\cos^2(2\pi\tau)\, d\tau,\end{aligned}$$

which right hand side does not converge to 0 as $\varepsilon \searrow 0$, as desired. ◀

An example with non-diagonalizable $A(t)$ and $\sigma(A(t)) = \{0, i\}$ showing as well that the conclusion of the above adiabatic theorems will generally fail if the family A is not $(M, 0)$-stable can be found in Joye's paper [60] at the end of Section 1. A generic version of this (non-generic) example is given by the following data: $A(t) := R(t)^{-1}A_0(t)R(t)$, $\lambda(t) := 0$, $P(t) := R(t)^{-1}P_0R(t)$ in $X := \ell^2(I_3)$, where

$$A_0(t) = A_0 := \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \quad \text{and} \quad R(t) := e^{Ct} \quad \text{with} \quad C := \begin{pmatrix} 0 & 0 & 0 \\ ik & 0 & k \\ 1 & -1 & 0 \end{pmatrix}$$

for a parameter $k \in (-\infty, 0)$ and where P_0 is the orthogonal projection onto $\operatorname{span}\{e_1, e_2\}$.

4.1.4 Applied examples: quantum dynamical semigroups and neutron transport semigroups

We close the present section about adiabatic theorems with spectral gap condition by noting two applied examples: one to time-dependent generators of quantum dynamical semigroups and one to time-dependent generators of neutron transport semigroups. We thereby slightly extend corresponding examples from [12] and [112], respectively. (In the case of the second example, we correct a serious blunder in the irreducibility proof in Theorem 8.4 of [112]. See the proof of Lemma 4.1.7 (iii) below.) In both cases the considered operators are not skew self-adjoint (and not normal).

In our first example we apply the adiabatic theorem with (non-uniform) spectral gap to generators $A(t)$ of quantum dynamical semigroups on $X = S^1(\mathfrak{h})$,

$$A(t)\rho := Z_0(t)(\rho) + \sum_{j \in J} B_j(t)\rho B_j(t)^* - 1/2\{B_j(t)^*B_j(t), \rho\} \qquad (\rho \in D(Z_0(t))) \quad (4.10)$$
$$e^{Z_0(t)\tau}(\rho) := e^{-iH(t)\tau}\rho\, e^{iH(t)\tau}$$

with not necessarily bounded self-adjoint operators $H(t)$ and bounded operators $B_j(t)$ on $\mathfrak{h}$ satisfying

$$\sum_{j \in J} B_j(t)^*B_j(t) < \infty \qquad (t \in I). \quad (4.11)$$

Theorem 4.1.6. *Suppose that the operators $A(t)$ defined above have time-independent domain $D(Z_0(t)) = D$ and that $t \mapsto A(t)$ is in $W_*^{1,1}(I, L(Y,X))$, where Y is the space D endowed with the graph norm of $A(0)$. Suppose further that $\lambda(t) = 0 \in \sigma(A(t))$ for every $t \in I$, that $\lambda(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \{\lambda(\,.\,)\}$ at countably many points accumulating at only finitely many points, and that $I \setminus N \ni t \mapsto \lambda(t)$ is continuous, where N denotes the set of those points where $\lambda(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \{\lambda(\,.\,)\}$. Suppose finally that the projections $P(t)$ associated with $A(t)$ and $\lambda(t)$ for $t \in I \setminus N$ can be extended to a map $t \mapsto P(t)$ in $W_*^{2,1}(I, L(X))$ on the whole of I. Then*

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

where U_ε and V_ε are the evolution system for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$ on D.

Proof. We have only to notice that $A(t)$ generates a contraction semigroup in X for every $t \in I$ (Corollary 3.6.2), so that A is $(1,0)$-stable, and then to apply Theorem 4.1.2. ■

In our second example we apply the adiabatic theorem with (uniform) spectral gap condition to generators $A(t)$ of neutron transport semigroups on $X = L^2([-a,a] \times [-1,1])$,

$$A(t) = A_0(c(t)) - s(t), \tag{4.12}$$

where $c(t), s(t) \in (0, \infty)$ depend sufficiently smoothly on t and where $A_0(c) = A_0 + cB$,

$$(A_0\varphi)(x,\mu) := -\mu\,\partial_x \varphi(x,\mu) \quad \text{and} \quad (B\varphi)(x,\mu) := \frac{1}{2}\int_{[-1,1]} \varphi(x,\mu')\,d\mu' \tag{4.13}$$

for every $\varphi \in D(A_0)$ and every $\varphi \in X$, respectively. $D(A_0)$ is defined to be the subspace of (equivalence classes of) those φ in X that satisfy the following three conditions:

(i) $(-a,a) \ni x \mapsto \varphi(x,\mu)$ is weakly differentiable for almost every $\mu \in [-1,1]$,

(ii) $[-a,a] \times [-1,1] \ni (x,\mu) \mapsto -\mu\,\partial_x\varphi(x,\mu)$ is 2-integrable,

(iii) $\varphi(-a,\mu) = 0$ for almost all $\mu \in (0,1]$ and $\varphi(a,\mu) = 0$ for almost all $\mu \in [-1,0)$.

In the above and the following relations, $\partial_x\varphi(x,\mu)$ stands for $(\partial_x\varphi(\,.\,,\mu))(x)$ and $\partial_x\varphi(\,.\,,\mu)$ denotes the weak derivative of $(-a,a) \ni x \mapsto \varphi(x,\mu)$ (which exists for almost all $\mu \in [-1,1]$ by condition (i)). It should be noticed that the conditions (i) and (ii) imply that $x \mapsto \varphi(x,\mu)$ is absolutely continuous (Section 2.1.1) so that, in particular, the boundary conditions from condition (iii) are meaningful in the first place.

Such operators $A(t)$ as above arise in the description [78], [79], [80] of the transport of neutrons in an infinite homogeneous slab (extended between $x = -a$ and $x = a$) surrounded by vacuum under the assumptions that the neutrons interact only with the nuclei in the slab (but not with each other), that scattering (between neutrons and nuclei) is isotropic, and that the neutrons all have the same velocity (in absolute value). Indeed, these operators determine the distribution $\varphi(t_0)$ of the position and the direction of the neutron beam at every given time t_0 and for every given initial distribution φ_0 through the evolution equation

$$\varphi' = A(t)\varphi = A_0(c(t))\varphi - s(t)\varphi, \quad \varphi(0) = \varphi_0, \tag{4.14}$$

that is, if $t \mapsto \varphi(t) = \varphi(t, \,.\,, \,..\,) \in X$ is the solution of this evolution equation, then

$$\int_{E_1 \times E_2} \varphi(t, x, \mu)\, d(x, \mu)$$

is the number of neutrons that at time t are located at a position x in $E_1 \in \mathcal{B}_{[-a,a]}$ and move in a direction μ in $E_2 \in \mathcal{B}_{[-1,1]}$ (meaning that μ is the cosine of the angle between the direction of motion and the positive x-axis). In (4.14), the term $A_0\varphi$ corresponds to collisionless transport, $c(t)B\varphi$ corresponds to scattering, $s(t)\varphi$ to absorption (see Section VI.2 of [41]), and the scalars $c(t)$ and $s(t)$ (which are t-independent in [78], [79], [80]) have the following interpretation: $s(t)$ is the total cross section and $c(t)/s(t)$ is the average number of neutrons emerging from a collision of a neutron with a nucleus, that is,

$$c(t)/s(t) < 1, \quad c(t)/s(t) = 1, \quad c(t)/s(t) > 1$$

correspond to scattering and absorption, pure scattering, multiplication, respectively. See also [108] (Section 16.9).

It is easy to see that A_0 is densely defined ($C_c^\infty((-a, a) \times (-1, 1)) \subset D(A_0)$!) and dissipative (partial integration!) and it is also not difficult to see that $\lambda - A_0$ is surjective for every $\lambda \in (0, \infty)$ (use the same arguments as for (5.11) in [78]). So, A_0 is the generator of a contraction semigroup on X (Lumer–Phillips) and therefore the perturbed operator $A_0(c) = A_0 + cB$ for $c \in (0, \infty)$ is the generator of a quasi-contraction semigroup with

$$\left\| e^{A_0(c)\tau} \right\| \le \limsup_{n\to\infty} \left\| \left(e^{A_0\tau/n} e^{cB\tau/n} \right)^n \right\| \le \limsup_{n\to\infty} \left\| e^{cB\tau/n} \right\|^n \le e^{c\tau} \quad (\tau \in [0, \infty)), \tag{4.15}$$

where we used the theorem of Lie–Trotter and the fact that B is a non-zero orthogonal projection and hence has norm 1.

Lemma 4.1.7. *Suppose $c \in (0, \infty)$. Then*

(i) $\sigma_p(A_0(c))$ is a non-empty finite subset of the positive half-axis $(0, \infty)$,

$$\sigma_p(A_0(c)) = \{\beta_1(c), \ldots, \beta_{m_c}(c)\},$$

consisting of eigenvalues $\beta_1(c), \ldots, \beta_{m_c}(c)$ of finite geometric multiplicity, where $\beta_1(c), \ldots, \beta_{m_c}(c)$ are listed in decreasing order and counted according to geometric multiplicity. $\sigma_c(A_0(c)) = \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$ and $\sigma_r(A_0(c)) = \emptyset$.

(ii) $\beta_1(c), \ldots, \beta_{m_c}(c)$ are semisimple eigenvalues (so that, in particular, geometric and algebraic multiplicities coincide).

(iii) $\beta_1(c)$ is even a simple eigenvalue, that is, has (algebraic) multiplicity 1.

Proof. Assertion (i) is the content of the main theorem of [78] and assertion (ii) is stated in Lemma 2 of [79]. It should be remarked, however, that the proof of Lemma 2 of [79] is incomplete, for it does not rule out the possibility for the eigenvalues $\beta_1(c), \dots, \beta_{m_c}(c)$ to be essential singularities of $(\,.\, - A_0(c))^{-1}$. In order to do so, one can argue as follows: let $\beta_j \in \{\beta_1(c), \dots, \beta_{m_c}(c)\}$ and fix $g \in X = L^2([-a,a] \times [-1,1])$, write

$$u_\lambda := (\lambda - A_0(c))^{-1} g \quad \text{and} \quad \xi_\lambda := \int_{-1}^{1} u_\lambda(\,.\,, \mu')\, d\mu'$$

for $\lambda \in \{\operatorname{Re} z > 0\} \setminus \sigma_p(A_0(c))$, and remember from (5.9) of [78] that

$$\xi_\lambda = 2/c\,(2/c - L_\lambda)^{-1} G_\lambda = (1 - c/2L_\lambda)^{-1} G_\lambda \tag{4.16}$$

for every $\lambda \in \{\operatorname{Re} z > 0\} \setminus \sigma_p(A_0(c))$, where L_λ is the integral operator on $Y := L^2([-a,a])$ given by

$$(L_\lambda \varphi)(x) := \int_{-a}^{a} e_\lambda(x,y)\varphi(y)\, dy \quad \text{with} \quad e_\lambda(x,y) := \int_1^\infty \frac{e^{-\lambda|x-y|t}}{t}\, dt$$

and where G_λ is the element of Y from (5.10) of [78]. Since L_λ is compact (even Hilbert–Schmidt) on Y for every $\lambda \in \{\operatorname{Re} z > 0\}$ by the 2-integrability of e_λ and since $\{\operatorname{Re} z > 0\} \ni \lambda \mapsto L_\lambda$ is holomorphic by the formula preceding (3.4) and the formula (3.5) of [78], it follows from the holomorphic Fredholm theorem (Theorem VI.14 of [107]) that β_j is a pole of $\lambda \mapsto (1 - c/2L_\lambda)^{-1}$ of order n, say. It also follows from (5.11) of [78] that $\|G_\lambda\| \le 1/(\operatorname{Re}\lambda)\, \|g\|$ for $\lambda \in \{\operatorname{Re} z > 0\}$. So,

$$\|\xi_\lambda\| \le \frac{C}{|\lambda - \beta_j|^n} \frac{1}{\operatorname{Re}\lambda} \|g\| \le \frac{C'}{|\lambda - \beta_j|^n} \|g\| \tag{4.17}$$

for λ in a punctured neighborhood of β_j by virtue of (4.16). Since, finally,

$$\big\|(\lambda - A_0(c))^{-1} g\big\| = \|u_\lambda\| \le \frac{1}{\operatorname{Re}\lambda}\big(c^2/2\,\|\xi_\lambda\| + \|g\|\,\big) \tag{4.18}$$

by (5.7) of [78], it follows from (4.17) and (4.18) that β_j is a pole of $(\,.\, - A_0(c))^{-1}$ of order at most n, as desired.

We are left with assertion (iii) and we prove it with the help of a general Perron–Frobenius theorem (Theorem C-III.3.12 of [91]), which implies that the spectral bound of the generator A of an irreducible positive semigroup on a Banach lattice is an eigenvalue of algebraic multiplicity 1 provided only that the spectral bound is a pole of $(\,.\, - A)^{-1}$. Since the spectral bound of $A_0(c)$ is equal to $\beta_1(c)$ and $\beta_1(c)$ is a pole of the resolvent of $A_0(c)$ by (ii), we have only to show that $A_0(c)$ is the generator of an irreducible positive semigroup on the Banach lattice $X = L^2([-a,a] \times [-1,1])$. In doing so, we will repeatedly use the simple facts that

$$\begin{aligned}(\lambda - A_0(c))^{-1}\varphi &= (\lambda - A_0 - cB)^{-1}\varphi = \big(1 - c(\lambda - A_0)^{-1}B\big)^{-1}(\lambda - A_0)^{-1}\varphi \\ &= \sum_{n=1}^{\infty} c^n \big((\lambda - A_0)^{-1}B\big)^n (\lambda - A_0)^{-1}\varphi \end{aligned} \tag{4.19}$$

for all $\lambda > c$ and all $\varphi \in X$, and that

$$\big((\lambda - A_0)^{-1}\psi\big)(x,\mu) = \begin{cases} -\frac{1}{\mu}\int_x^a e^{\frac{\lambda}{\mu}(t-x)}\psi(t,\mu)\,dt, & (x,\mu) \in [-a,a] \times [-1,0) \\ \frac{1}{\mu}\int_{-a}^x e^{\frac{\lambda}{\mu}(t-x)}\psi(t,\mu)\,dt, & (x,\mu) \in [-a,a] \times (0,1] \end{cases} \tag{4.20}$$

for all $\lambda > 0$ and all $\psi \in X$.

In order to see that $A_0(c)$ generates a positive semigroup on X, we have only to show: if $\lambda > c$ and $\varphi \in X$ with $\varphi \geq 0$, then

$$\big((\lambda - A_0(c))^{-1}\varphi\big)(x,\mu) \geq 0 \quad \text{for almost every } (x,\mu) \in [-a,a] \times [-1,1] \tag{4.21}$$

(Theorem VI.1.8 of [41]). So, let $\lambda > c$ and $\varphi \in X$ with $\varphi \geq 0$. Since $(\lambda - A_0)^{-1}$ and B are positive operators in the lattice sense by (4.20), the desired relation (4.21) follows by (4.19).

In order to see that $A_0(c)$ generates even an irreducible semigroup on X, we have only to show: if $\lambda > c$ and $\varphi \in X$ with $\varphi \geq 0$ and $\varphi \neq 0$, then

$$\big((\lambda - A_0(c))^{-1}\varphi\big)(x,\mu) > 0 \quad \text{for almost every } (x,\mu) \in [-a,a] \times [-1,1] \tag{4.22}$$

(Definition C-III.3.1 of [91] in conjunction with the characterization of quasi-interior points in the special case of Lebesgue spaces from the very end of Section C-I.2 of [91]). So, let $\lambda > c$ and $\varphi \in X$ with $\varphi \geq 0$ and $\varphi \neq 0$. Since for such a φ all the summands in (4.19) are positive, that is,

$$\psi_n := \big((\lambda - A_0)^{-1}B\big)^n(\lambda - A_0)^{-1}\varphi \geq 0 \quad (n \in \mathbb{N} \cup \{0\}), \tag{4.23}$$

the desired strict positivity almost everywhere (4.22) will follow from the relation

$$\psi_2(x,\mu) > 0 \quad \text{for almost every } (x,\mu) \in [-a,a] \times [-1,1]. \tag{4.24}$$

We abbreviate $J := [-a,a]$ and $J' := [-1,1]$ and consider

$$J_0' := \big\{\mu \in J' : \varphi(\,.\,,\mu) \text{ does not vanish a.e.}\big\} = \Big\{\mu \in J' : \int_J \varphi(t,\mu)\,dt > 0\Big\}.$$

Since $\mu \mapsto \int_J \varphi(t,\mu)\,dt$ is measurable, the set J_0' is measurable, and since $\varphi \neq 0$ and

$$0 < \int_{J\times J'} \varphi(t,\mu)\,d(t,\mu) = \int_{J'}\int_J \varphi(t,\mu)\,dt\,d\mu = \int_{J_0'}\int_J \varphi(t,\mu)\,dt\,d\mu,$$

the set J_0' cannot be of measure 0. In particular, $J_{0-}' := J_0' \cap [-1,0)$ is a non-null set or $J_{0+}' := J_0' \cap (0,1]$ is a non-null set:

$$\lambda(J_{0-}') > 0 \qquad \text{or} \qquad \lambda(J_{0+}') > 0. \tag{4.25}$$

We will establish (4.24) in the case where J_{0-}' is non-null – the case where J_{0+}' is non-null is treated completely analogously. We consider, for every $\mu \in J_{0-}'$, the interval

$J_\mu := [-a, b_\mu)$ with b_μ being the supremum of the "essential support" of $\varphi(\,.\,,\mu)$, more precisely:

$$b_\mu := \inf\Big\{x \in J : \int_x^a \varphi(t,\mu)\,dt = 0\Big\}.$$

Since $\int_{-a}^a \varphi(t,\mu)\,dt > 0$ for $\mu \in J'_{0-}$, it follows that $b_\mu > -a$ and, in particular,

$$\lambda(J_\mu) = \lambda([-a,b_\mu)) > 0 \quad \text{for all } \mu \in J'_{0-}. \tag{4.26}$$

It also follows from (4.23) and (4.20) that

$$\psi_0(x,\mu) = -\frac{1}{\mu}\int_x^a e^{\frac{\lambda}{\mu}(t-x)}\varphi(t,\mu)\,dt > 0 \quad \text{for all } \mu \in J'_{0-} \text{ and } x \in J_\mu = [-a,b_\mu) \tag{4.27}$$

Combining these two inequalities, we conclude that for every $\mu \in J'$, the function $(B\psi_0)(\,.\,,\mu) = (B\psi_0)(\,.\,)$ (constant w.r.t. μ by the definition of B!) does not vanish almost everywhere. Indeed,

$$\int_J (B\psi_0)(t)\,dt = \frac{1}{2}\int_J\int_{J'} \psi_0(t,\mu')\,d\mu'\,dt \geq \frac{1}{2}\int_{J'_{0-}}\int_{J_{\mu'}} \psi_0(t,\mu')\,d\mu'\,dt > 0 \tag{4.28}$$

because of (4.27), (4.26), (4.25.a). Write α and β for the infimum and supremum of the "essential support" of $(B\psi_0)(\,.\,)$, that is,

$$\alpha := \sup\Big\{x \in J : \int_{-a}^x (B\psi_0)(t)\,dt = 0\Big\} \quad \text{and} \quad \beta := \inf\Big\{x \in J : \int_x^a (B\psi_0)(t)\,dt = 0\Big\}.$$

It then follows from (4.23) and (4.20) that

$$\psi_1(x,\mu) = -\frac{1}{\mu}\int_x^a e^{\frac{\lambda}{\mu}(t-x)}(B\psi_0)(t)\,dt > 0 \quad \text{for all } \mu < 0 \text{ and } x < \beta$$

and

$$\psi_1(x,\mu) = \frac{1}{\mu}\int_{-a}^x e^{\frac{\lambda}{\mu}(t-x)}(B\psi_0)(t)\,dt > 0 \quad \text{for all } \mu > 0 \text{ and } x > \alpha.$$

Since $\alpha < \beta$ by (4.28), we can conclude that

$$(B\psi_1)(x,\mu) = (B\psi_1)(x) = \frac{1}{2}\int_{-1}^1 \psi_1(x,\mu')\,d\mu' > 0$$

for every $x \in J = [-a,\beta) \cup (\alpha,a]$ and every $\mu \in J'$, and therefore the desired inequality (4.24) follows by (4.23) and (4.20). ∎

Lemma 4.1.8. *As c increases, the number m_c of eigenvalues of $A_0(c)$ (counted according to multiplicity) increases monotonically to ∞. Also,* $\operatorname{dom}\beta_n$ *for every $n \in \mathbb{N}$ is an unbounded open interval with*

$$\operatorname{dom}\beta_{n+1} \subset \operatorname{dom}\beta_n \subsetneq \operatorname{dom}\beta_1 = (0,\infty),$$

$c \mapsto \beta_n(c)$ *is continuous and strictly monotonically increasing, and*

$$\beta_n(c) \longrightarrow 0 \quad (c \searrow \inf\operatorname{dom}\beta_n) \quad \textit{and} \quad \beta_n(c) \longrightarrow \infty \quad (c \to \infty).$$

Proof. All the assertions follow from the arguments in [78] (Section 4). ■

With these lemmas at hand, we can now apply the adiabatic theorem with uniform spectral gap condtion to the operators $A(t)$ from (4.12) and the rightmost eigenvalue $\lambda(t)$ of $A(t)$ which is the only one to be physically significant (Section 5.1 of [89]).

Theorem 4.1.9. *Suppose $A(t) = A_0(c(t)) - s(t)$ for every $t \in I$ is as in (4.12) above and $\lambda(t) = \beta_1(c(t)) - s(t)$, where $c(t), s(t) \in (0, \infty)$ such that $s(t) \geq c(t)$ and $t \mapsto c(t), s(t)$ are continuously differentiable with absolutely continuous derivatives. Suppose further that $P(t)$ for every $t \in I$ is the projection associated with $A(t)$ and $\lambda(t)$. Then*

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \quad (\varepsilon \searrow 0),$$

where U_ε and V_ε are the evolution systems for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + [P', P]$ on $D = D(A_0)$.

Proof. Since $s(t) \geq c(t)$ for all t, the operators $A(t) = A_0(c(t)) - s(t)$ generate contraction semigroups on X by (4.15), and since $t \mapsto c(t), s(t)$ are continuously differentiable with absolutely continuous derivatives, $t \mapsto A(t)$ belongs to $W_*^{2,1}(I, L(Y, X))$ where Y is $D(A(0)) = D(A_0)$ endowed with the graph norm of $A(0)$. Since, moreover, $\beta_1(c) > \beta_2(c)$ for every $c \in (0, \infty)$ by Lemma 4.1.7 (iii) and since β_1, β_2 are continuous by Lemma 4.1.8, it follows that

$$\inf_{t \in I} \beta_1(c(t)) - \beta_2(c(t)) > 0,$$

whence $\lambda(\,.\,) = \beta_1(c(\,.\,)) - s(\,.\,)$ at no point falls into $\sigma(A(\,.\,)) \setminus \{\lambda(\,.\,)\}$, and that $t \mapsto \lambda(t)$ is continuous. And finally, it follows by the $W_*^{2,1}$-regularity of $t \mapsto A(t)$, the uniform isolatedness of $\lambda(\,.\,)$ in $\sigma(A(\,.\,))$, and the continuity of $t \mapsto \lambda(t)$ that

$$t \mapsto P(t) = \frac{1}{2\pi i} \int_{\gamma_t} (z - A(t))^{-1} \, dz$$

belongs to $W_*^{2,1}(I, L(X))$. So, the desired conclusion follows from Theorem 4.1.2. ■

If $s(t) > c(t)$ for all $t \in I$, the conclusion of the above theorem is already trivially satisfied: indeed, in this case there exists, by virtue of (4.15) above, a $\gamma > 0$ such that even $A(t) + \gamma$ is a contraction semigroup generator for every t and therefore (3.21) holds true with $M = 1$. If, however, $s(t) = c(t)$ for some $t \in I$, the conclusion of the above theorem does not seem to be already trivially satisfied: indeed, in this case there exists, by virtue of (4.29) below, no $\gamma > 0$ such that even $A(t) + \gamma$ is a contraction semigroup generator and therefore (3.21) does not seem to hold true for any $\gamma > 0$ and any $M \geq 1$. (I could not rigorously prove this, however: since $\lambda(t) = \beta_1(c(t)) - s(t)$ is probably strictly less than 0 in general even for t with $s(t) = c(t)$ and since $\lambda(t)$ is the growth bound of $A(t)$ (Theorem VI.1.15 of [41]), it might be that even for t with $s(t) = c(t)$ there exist $\gamma_t > 0$ and $M_t > 1$ such that $\left\| e^{A(t)\tau} \right\| \leq M_t e^{-\gamma_t \tau}$ for all $\tau \in [0, \infty)$, and therefore (3.21) cannot be ruled out a priori.) It remains to show that if $s(t) = c(t)$, then $A(t) + \gamma = A_0(c(t)) - c(t) + \gamma$ no $\gamma > 0$ is the generator of a contraction semigroup.

So, let $\gamma > 0$ and define $\varphi(x,\mu) := \varphi_0(x)$ for $(x,\mu) \in [-a,a] \times [-1,1]$ with an arbitrary $0 \neq \varphi_0 \in C_c^\infty((-a,a))$. It then follows that $\varphi \in D(A_0)$ and $B\varphi = \varphi$ and $\operatorname{Re}\langle \varphi, A_0\varphi\rangle = 0$, so that

$$\begin{aligned}\operatorname{Re}\langle \varphi, (A_0(c) - c + \gamma)\varphi\rangle &= \operatorname{Re}\langle \varphi, A_0\varphi\rangle + c\,\langle \varphi, B\varphi\rangle - (c-\gamma)\,\langle \varphi,\varphi\rangle \\ &= \gamma\,\|\varphi\|^2 > 0 \end{aligned} \tag{4.29}$$

and hence $A_0(c) - c + \gamma$ is not dissipative, as desired.

4.2 Adiabatic theorems without spectral gap condition

After having established general adiabatic theorems with spectral gap condition in Section 4.1, we can now prove an adiabatic theorem without spectral gap condition for general operators $A(t)$ with not necessarily weakly semisimple spectral values $\lambda(t)$: in Section 4.2.1 it appears in a qualitative version and in Section 4.2.2 in a quantitatively refined version, and both versions are applied to the special case of spectral operators. We thereby generalize the recent adiabatic theorems without spectral gap condition of Avron, Fraas, Graf, Grech from [12] and of Schmid from [112], which theorems – although independently obtained – are essentially the same (save for some regularity subtleties). In these theorems – which so far are the only ones to cover not necessarily skew self-adjoint operators $A(t)$ in the case without spectral gap – the considered eigenvalues $\lambda(t)$ are required to be weakly semisimple. Since, however, the eigenvalues of general operators are generally not weakly semisimple (Section 4.2.3) provides simple examples for this), it is natural to ask whether one can do without the requirement of weak semisimplicity (or, in other words, weak associatedness of order 1). And the theorems below show that one actually can: indeed, apart from a certain spectral marginality condition on $\lambda(t)$ (namely, $\lambda(t) + \delta e^{i\vartheta(t)} \in \rho(A(t))$) and a certain growth condition on the reduced resolvent of $A(t)$ (at the points $\lambda(t) + \delta e^{i\vartheta(t)}$), it suffices to require weak associatedness – which, at the beginning of Section 3.2, has been explained to be a fairly natural assumption.

4.2.1 A qualitative adiabatic theorem without spectral gap condition

We begin with a lemma that will be crucial in the proofs of the presented adiabatic theorems without spectral gap condition.

Lemma 4.2.1. *Suppose that $A : D(A) \subset X \to X$ is a densely defined closed linear map and that $\lambda \in \sigma(A)$ and $\delta_0 \in (0,\infty)$ and $\vartheta_0 \in \mathbb{R}$ such that $\lambda + \delta e^{i\vartheta_0} \in \rho(A)$ for all $\delta \in (0,\delta_0]$. Suppose finally that P is a bounded projection in X such that $PA \subset AP$ and*

$$(1-P)X \subset \overline{\operatorname{ran}}\,(A-\lambda)^{m_0}$$

for some $m_0 \in \mathbb{N}$, and that there is $M_0 \in (0,\infty)$ such that

$$\left\|\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}(1-P)\right\| \le \frac{M_0}{\delta}$$

for all $\delta \in (0,\delta_0]$. Then $\delta\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}(1-P)x \longrightarrow 0$ as $\delta \searrow 0$ for all $x \in X$.

Proof. If $x \in \operatorname{ran}(A-\lambda)^{m_0}$, then $x = (\lambda - A)^{m_0} x_0$ for some $x_0 \in D(A^{m_0})$ and, by the assumed resolvent estimate,

$$\delta\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}\overline{P}x = \delta\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}\overline{P}\big(-\delta e^{i\vartheta_0}\big)^{m_0} x_0 \\ + \delta \sum_{k=1}^{m_0} \binom{m_0}{k} \big(\lambda + \delta e^{i\vartheta_0} - A\big)^{k-1}\big(-\delta e^{i\vartheta_0}\big)^{m_0-k}\overline{P}x_0 \longrightarrow 0$$

as $\delta \searrow 0$, where of course $\overline{P} := 1 - P$. And if $x \in X$, then $\overline{x} := \overline{P}x$ can be approximated arbitrarily well by elements y of $\operatorname{ran}(A-\lambda)^{m_0}$ and therefore

$$\delta\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}\overline{P}x = \delta\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}\overline{P}(\overline{x} - y) + \delta\big(\lambda + \delta e^{i\vartheta_0} - A\big)^{-1}\overline{P}y$$

can be made arbitrarily small for δ small enough by the assumed resolvent estimate and by what has just been shown. ■

With this lemma at hand, we can now prove the announced general adiabatic theorem without spectral gap condition for not necessarily weakly semisimple eigenvalues. Similarly to the works [11] of Avron and Elgart and [131] of Teufel its proof rests upon solving a suitable approximate commutator equation. In this undertaking the insights gained in Section 4.1, especially formula (4.6), will prove indispensable. (Alternatively, part (i) of the theorem could also – less elegantly – be based upon a suitable iterated partial integration argument, but part (ii) could not.)

Theorem 4.2.2. *Suppose $A(t) : D \subset X \to X$ for every $t \in I$ is a linear map such that Condition 2.1.8 is satisfied with $\omega = 0$. Suppose further that $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$, and that there are numbers $\delta_0 \in (0,\infty)$ and $\vartheta(t) \in \mathbb{R}$ such that $\lambda(t) + \delta e^{i\vartheta(t)} \in \rho(A(t))$ for all $\delta \in (0, \delta_0]$ and $t \in I$ and such that $t \mapsto \lambda(t)$ and $t \mapsto e^{i\vartheta(t)}$ are absolutely continuous. Suppose finally that $P(t)$ for every $t \in I$ is a bounded projection in X commuting with $A(t)$ such that $P(t)$ for almost every $t \in I$ is weakly associated with $A(t)$ and $\lambda(t)$, suppose there is an $M_0 \in (0,\infty)$ such that*

$$\Big\| \big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1}(1 - P(t)) \Big\| \le \frac{M_0}{\delta}$$

for all $\delta \in (0, \delta_0]$ and $t \in I$, let $\operatorname{rk} P(0) < \infty$ and suppose that $t \mapsto P(t)$ is strongly continuously differentiable.

(i) If X is arbitrary (not necessarily reflexive), then

$$\sup_{t \in I} \big\| \big(U_\varepsilon(t) - V_{0\,\varepsilon}(t)\big)P(0) \big\| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

where U_ε and $V_{0\,\varepsilon}$ are the evolution systems for $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}AP + [P', P]$ on X for every $\varepsilon \in (0,\infty)$.

(ii) If X is reflexive and $t \mapsto P(t)$ is norm continuously differentiable, then

$$\sup_{t \in I} \| U_\varepsilon(t) - V_\varepsilon(t) \| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

whenever the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ exists on D for every $\varepsilon \in (0,\infty)$.

Proof. We begin with some preparations which will be used in the proof of both assertion (i) and assertion (ii). As a first preparatory step, we show that $t \mapsto P(t)$ is in $W^{1,1}_*(I, L(X,Y))$ and conclude that $P(t)A(t) \subset A(t)P(t)$ for every $t \in I$ and that there is an $m_0 \in \mathbb{N}$ such that $P(t)X \subset \ker(A(t)-\lambda(t))^{m_0}$ for every $t \in I$. Since $P(t)$ for almost every $t \in I$ is weakly associated with $A(t)$ and $\lambda(t)$ and since

$$\dim P(t)X = \operatorname{rk} P(0)X < \infty$$

for every $t \in I$ (which equality is due to the continuity of $t \mapsto P(t)$ and Lemma VII.6.7 of [39]), there is a t-independent constant $m_0 \in \mathbb{N}$ – for instance, $m_0 := \operatorname{rk} P(0)$ – such that $P(t)$ is weakly associated of order m_0 with $A(t)$ and $\lambda(t)$ for almost every $t \in I$. In particular, it follows from Theorem 3.2.2 that

$$P(t)X \subset \ker(A(t)-\lambda(t))^{m_0} \quad \text{and} \quad (1-P(t))X \subset \overline{\operatorname{ran}}\,(A(t)-\lambda(t))^{m_0}$$

for almost every $t \in I$ (with exceptional set N). It now follows by the binomial formula that

$$P(t) = S_\delta(t)^{m_0}\big(A(t)-\lambda(t)-\delta e^{i\vartheta(t)}\big)^{m_0} P(t) = S_\delta(t) \sum_{k=0}^{m_0-1} \binom{m_0}{k} \big(-\delta e^{i\vartheta(t)}\big)^{m_0-k} \cdot$$
$$\cdot\, S_\delta(t)^{m_0-1-k}\, \big(1+\delta e^{i\vartheta(t)} S_\delta(t)\big)^k P(t)$$

for every $t \in I \setminus N$, where $S_\delta(t) := (A(t)-\lambda(t)-\delta e^{i\vartheta(t)}$. Since both sides of this equation depend continuously on $t \in I$, the equation holds for every $t \in I$, and since the right-hand side belongs to $W^{1,1}_*(I, L(X,Y))$ by Lemma 2.1.2, we also have

$$(t \mapsto P(t)) \in W^{1,1}_*(I, L(X,Y)). \tag{4.30}$$

With this regularity property at hand, it is now easy to see that the inclusions

$$P(t)A(t) \subset A(t)P(t) \quad \text{and} \quad P(t)X \subset \ker(A(t)-\lambda(t))^{m_0} \tag{4.31}$$

also hold for $t \in N$ (while they clearly hold for $t \in I \setminus N$). In order to see that (4.31.a) holds also for $t \in N$, notice that every such t is approximated by a sequence (t_n) in $I \setminus N$ and hence

$$P(t_n)x \longrightarrow P(t)x,$$
$$A(t)P(t_n)x = (A(t)-A(t_n))P(t_n)x + P(t_n)A(t_n)x \longrightarrow P(t)A(t)x$$

for every $x \in D(A(t)) = D$ by (4.30). So, (4.31.a) follows by the closedness of $A(t)$. (Alternatively, we could also have argued as in (4.5).) In order to see that (4.31.b) holds also for $t \in N$, notice that $\dim P(t)X = \operatorname{rk} P(0) < \infty$ and $P(t)A(t) \subset A(t)P(t)$ for every $t \in I$, so that $P(t)X = P(t)D(A(t))$ and $P(t)X \subset D(A(t)^{m_0})$ as well as

$$(A(t)-\lambda(t))^{m_0} P(t) = \big((A(t)-\lambda(t))P(t)\big)^{m_0}$$

for every $t \in I$. So, (4.31.b) follows by (4.30).

As a second preparatory step, we solve – in accordance with the proof of the adiabatic theorems with spectral gap condition – a suitable (approximate) commutator equation. Inspired by (4.6), we define the operators

$$B_{n\boldsymbol{\delta}}(t) := \sum_{k=0}^{m_0-1} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}(t)\Big) Q_n(t)(\lambda(t)-A(t))^k P(t) \\ + \sum_{k=0}^{m_0-1} (\lambda(t)-A(t))^k P(t) Q_n(t) \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}(t)\Big) \tag{4.32}$$

for $n \in \mathbb{N}$, $\boldsymbol{\delta} := (\delta_1, \dots, \delta_{m_0}) \in (0, \delta_0]^{m_0}$ and $t \in I$, where

$$\overline{R}_\delta(t) := R_\delta(t)\overline{P}(t) \quad \text{with} \quad R_\delta(t) := \big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1} \text{ and } \overline{P}(t) := 1 - P(t)$$

for $\delta \in (0, \delta_0]$, and where

$$Q_n(t) := \int_0^1 j_{\frac{1}{n}}(t-r) P'(r)\, dr.$$

In other words, Q_n is obtained from P' by mollification, whence $t \mapsto Q_n(t)$ is strongly continuously differentiable and $Q_n(t) \longrightarrow P'(t)$ as $n \to \infty$ w.r.t. the strong operator topology for $t \in (0,1)$ and

$$\sup\{\|Q_n(t)\| : t \in I, n \in \mathbb{N}\} \le \sup_{t\in I} \left\|P'(t)\right\|.$$

We now show that the operators $B_{n\boldsymbol{\delta}}(t)$ satisfy the approximate commutator equation

$$B_{n\boldsymbol{\delta}}(t)A(t) - A(t)B_{n\boldsymbol{\delta}}(t) + C_{n\boldsymbol{\delta}}(t) \subset [Q_n(t), P(t)] \tag{4.33}$$

with remainder terms $C_{n\boldsymbol{\delta}}(t)$ that will have to be suitably controlled below. Since

$$(\lambda - A)\Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big) = \Big(\prod_{1\le i\le k} \overline{R}_{\delta_i}\Big) - \delta_{k+1}e^{i\vartheta}\Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big) \supset \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big)(\lambda - A)$$

(the t-dependence being suppressed here and in the following lines for the sake of convenience), it follows that

$$(\lambda - A)B_{n\boldsymbol{\delta}} = \sum_{k=0}^{m_0-1} \Big(\prod_{1\le i\le k} \overline{R}_{\delta_i}\Big) Q_n (\lambda - A)^k P + \sum_{k=0}^{m_0-1} (\lambda - A)^{k+1} P Q_n \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big) - C^+_{n\boldsymbol{\delta}}$$

$$B_{n\boldsymbol{\delta}}(\lambda - A) \subset \sum_{k=0}^{m_0-1} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i}\Big) Q_n (\lambda - A)^{k+1} P + \sum_{k=0}^{m_0-1} (\lambda - A)^k P Q_n \Big(\prod_{1\le i\le k} \overline{R}_{\delta_i}\Big) - C^-_{n\boldsymbol{\delta}}$$

where we used the abbreviations

$$C^+_{n\boldsymbol{\delta}} := \sum_{k=0}^{m_0-1} \delta_{k+1} e^{i\vartheta} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i} \Big) Q_n (\lambda - A)^k P,$$

$$C^-_{n\boldsymbol{\delta}} := \sum_{k=0}^{m_0-1} (\lambda - A)^k P Q_n \, \delta_{k+1} e^{i\vartheta} \Big(\prod_{i=1}^{k+1} \overline{R}_{\delta_i} \Big). \tag{4.34}$$

Subtracting $B_{n\boldsymbol{\delta}}(\lambda - A)$ from $(\lambda - A)B_{n\boldsymbol{\delta}}$ and noticing that, by doing so, of all the summands not belonging to $C^+_{n\boldsymbol{\delta}}$, $C^-_{n\boldsymbol{\delta}}$ only

$$Q_n P - \Big(\prod_{i=1}^{m_0} \overline{R}_{\delta_i} \Big) Q_n (\lambda - A)^{m_0} P + (\lambda - A)^{m_0} P Q_n \Big(\prod_{i=1}^{m_0} \overline{R}_{\delta_i} \Big) - P Q_n = [Q_n, P]$$

remains (remember (4.31)), we see that

$$B_{n\boldsymbol{\delta}} A - A B_{n\boldsymbol{\delta}} \subset [Q_n, P] - C^+_{n\boldsymbol{\delta}} + C^-_{n\boldsymbol{\delta}}$$

which is nothing but (4.33) if one defines $C_{n\boldsymbol{\delta}} := C^+_{n\boldsymbol{\delta}} - C^-_{n\boldsymbol{\delta}}$.

As a third preparatory step we observe that $t \mapsto B_{n\boldsymbol{\delta}}(t)$ belongs to $W^{1,1}_*(I, L(X,Y))$ and estimate $B_{n\boldsymbol{\delta}}$ as well as $B'_{n\boldsymbol{\delta}}$. Since

$$t \mapsto (A(t) - \lambda(t))^k P(t) = \big((A(t) - \lambda(t)) P(t) \big)^k = P(t) \big((A(t) - \lambda(t)) P(t) \big)^k \tag{4.35}$$

is in $W^{1,1}_*(I, L(X,Y))$ by the first preparatory step the asserted $W^{1,1}_*(I, L(X,Y))$-regularity of $t \mapsto B_{n\boldsymbol{\delta}}(t)$ follows from Lemma 2.1.2. Additionally, there is a constant c such that

$$\sup_{t \in I} \big\| B_{n\boldsymbol{\delta}}(t) \big\| \le \sum_{k=1}^{m_0} c \Big(\prod_{i=1}^{k} \delta_i \Big)^{-1} \tag{4.36}$$

for all $\boldsymbol{\delta} \in (0, \delta_0]^{m_0}$ by the assumed resolvent estimate and the continuity of (4.35) just established. And since

$$\| R_\delta(t) \|_{X,X} \le \sum_{k=0}^{m_0-1} \frac{1}{\delta^{k+1}} \Big\| (A(t) - \lambda(t))^k P(t) \Big\|_{X,X} + \ \big\| \overline{R}_\delta(t) \big\|_{X,X} \le \frac{c}{\delta^{m_0}}$$

as well as

$$\big\| \overline{R}_\delta(t) \big\|_{X,Y} \le \big\| (A(t) - 1)^{-1} \big\|_{X,Y} \, \big\| (A(t) - 1) \overline{R}_\delta(t) \big\|_{X,X} \le \frac{c}{\delta}$$

for all $t \in I$ and all $\delta \in (0, \delta_0]$ (with another constant c) by the assumed resolvent estimate and the continuity of of (4.35) just established, it follows from Lemma 2.1.2 that there is a $W^{1,1}_*$-derivative $\overline{R}'_\delta$ of $t \mapsto \overline{R}_\delta(t)$ such that

$$\int_0^1 \big\| \overline{R}'_\delta(s) \big\| \, ds \le \frac{c}{\delta^{m_0+1}} \tag{4.37}$$

for all $\delta \in (0, \delta_0]$ (with yet another constant c) and, hence, that there is a $W^{1,1}_*$-derivative $B'_{n\,\boldsymbol{\delta}}$ of $t \mapsto B_{n\,\boldsymbol{\delta}}(t)$ such that

$$\int_0^1 \left\| B'_{n\,\boldsymbol{\delta}}(s) \right\| \, ds \le \sum_{k=1}^{m_0} c_n \Big(\prod_{i=1}^{k} \delta_i \Big)^{-(m_0+1)} \tag{4.38}$$

for all $\boldsymbol{\delta} \in (0, \delta_0]^{m_0}$ and some constant $c_n \in (0, \infty)$ depending on the supremum norm $\sup_{t \in I} \|Q'_n(t)\|$ of the strong derivative of $t \mapsto Q_n(t)$.

As a fourth and last preparatory step, we observe that for every $\varepsilon \in (0, \infty)$ the evolution system $V_{0\,\varepsilon}$ for $\frac{1}{\varepsilon}AP + [P', P]$ exists on X and is adiabatic w.r.t. P and satisfies the estimate

$$\|V_{0\,\varepsilon}(t,s)P(s)\| \le Mc\, e^{Mc(t-s)} \tag{4.39}$$

for all $(s,t) \in \Delta$, where c is an upper bound of $t \mapsto \|P(t)\|, \|P'(t)\|$. Indeed, $t \mapsto A(t)P(t)$ is strongly continuous (by the first preparatory step) and therefore the evolution system $V_{0\,\varepsilon}$ for $\frac{1}{\varepsilon}AP + [P', P]$ exists on X and (by virtue of (4.31.a) and Proposition 3.4.1) is adiabatic w.r.t. P for every $\varepsilon \in (0, \infty)$. It follows that for all $x \in X$ and $(s,t) \in \Delta$ the map $[s,t] \ni \tau \mapsto U_\varepsilon(t,\tau)V_{0\,\varepsilon}(\tau,s)P(s)x$ is continuous and right differentiable by Lemma 2.1.3 (use the adiabaticity of $V_{0\,\varepsilon}$ w.r.t. P and (4.31.b)) with bounded (even continuous) right derivative

$$\begin{aligned} \tau \mapsto \; & U_\varepsilon(t,\tau)\Big(\frac{1}{\varepsilon}A(\tau)P(\tau) - \frac{1}{\varepsilon}A(\tau) + [P'(\tau), P(\tau)]\Big)V_{0\,\varepsilon}(\tau,s)P(s)x \\ & = U_\varepsilon(t,\tau)P'(\tau)V_{0\,\varepsilon}(\tau,s)P(s)x \end{aligned}$$

(for the last equation, use the adiabaticity of $V_{0\,\varepsilon}$ w.r.t. P and (3.20)). So, by Lemma 2.1.4,

$$\begin{aligned} V_{0\,\varepsilon}(t,s)P(s)x - U_\varepsilon(t,s)P(s)x &= U_\varepsilon(t,\tau)V_{0\,\varepsilon}(\tau,s)P(s)x\big|_{\tau=s}^{\tau=t} \\ &= \int_s^t U_\varepsilon(t,\tau)P'(\tau)V_{0\,\varepsilon}(\tau,s)P(s)x \, d\tau \end{aligned} \tag{4.40}$$

for all $(s,t) \in \Delta$ and $x \in X$, and this integral equation, by the Gronwall inequality, yields the desired estimate for $V_{0\,\varepsilon}(t,s)P(s)$.

After these preparations we can now turn to the main part of the proof where the cases (i) and (ii) have to be treated separately. We first prove assertion (i). As has already been shown in (4.40),

$$\big(V_{0\,\varepsilon}(t) - U_\varepsilon(t)\big)P(0)x = U_\varepsilon(t,s)V_{0\,\varepsilon}(s)P(0)x\big|_{s=0}^{s=t} = \int_0^t U_\varepsilon(t,s)\, P'(s)\, V_{0\,\varepsilon}(s)P(0)x \, ds$$

so that, by rewriting the right hand side of this equation, we obtain

$$\begin{aligned} \big(V_{0\,\varepsilon}(t) - U_\varepsilon(t)\big)P(0)x &= \int_0^t U_\varepsilon(t,s)\, (P'(s) - Q_n(s))P(s)\, V_{0\,\varepsilon}(s)P(0)x \, ds \\ &\quad + \int_0^t U_\varepsilon(t,s)\, [Q_n(s), P(s)]\, V_{0\,\varepsilon}(s)P(0)x \, ds \end{aligned} \tag{4.41}$$

for all $t \in I$, $\varepsilon \in (0, \infty)$ and $x \in X$. Since $Q_n(s)P(s) \longrightarrow P'(s)P(s)$ for every $s \in (0,1)$ by the strong convergence of $(Q_n(s))$ to $P'(s)$ for $s \in (0,1)$ and by $\operatorname{rk} P(s) = \operatorname{rk} P(0) < \infty$ for $s \in I$, it follows by (4.39) and by the dominated convergence theorem that

$$\sup_{\varepsilon \in (0,\infty)} \sup_{t \in I} \left\| \int_0^t U_\varepsilon(t,s)\,(P'(s) - Q_n(s))P(s)\,V_{0\,\varepsilon}(s)P(0)\,ds \right\| \longrightarrow 0 \tag{4.42}$$

as $n \to \infty$. In view of (4.41) we therefore have to show that for each fixed $n \in \mathbb{N}$

$$\sup_{t \in I} \left\| \int_0^t U_\varepsilon(t,s)\,[Q_n(s), P(s)]\,V_{0\,\varepsilon}(s)P(0)\,ds \right\| \longrightarrow 0 \tag{4.43}$$

as $\varepsilon \searrow 0$. So let $n \in \mathbb{N}$ be fixed for the rest of the proof. Since $s \mapsto B_{n\,\boldsymbol{\delta}}(s)$ is in $W_*^{1,1}(I, L(X,Y))$ by the third preparatory step and since $[0,t] \ni s \mapsto U_\varepsilon(t,s)|_Y \in L(Y,X)$ as well as $s \mapsto V_{0\,\varepsilon}(s) \in L(X)$ are continuously differentiable w.r.t. the respective strong operator topologies, Lemma 2.1.2 yields that

$$[0,t] \ni s \mapsto U_\varepsilon(t,s)B_{n\,\boldsymbol{\delta}}(s)V_{0\,\varepsilon}(s)P(0)x$$

is the continuous representative of an element of $W^{1,1}([0,t], X)$ for every $x \in X$. With the help of the approximate commutator equation (4.33) of the second preparatory step, we therefore see that

$$\begin{aligned}
\int_0^t U_\varepsilon(t,s)\,[Q_n(s), P(s)]\,V_{0\,\varepsilon}(s)P(0)x\,ds &= \varepsilon \int_0^t U_\varepsilon(t,s)\Big(-\frac{1}{\varepsilon}A(s)B_{n\,\boldsymbol{\delta}}(s) \\
&\quad + B_{n\,\boldsymbol{\delta}}(s)\frac{1}{\varepsilon}A(s)\Big)V_{0\,\varepsilon}(s)P(0)x\,ds + \int_0^t U_\varepsilon(t,s)\,C^+_{n\,\boldsymbol{\delta}}(s)\,V_{0\,\varepsilon}(s)P(0)x\,ds \\
= \varepsilon\, U_\varepsilon(t,s)B_{n\,\boldsymbol{\delta}}(s)V_{0\,\varepsilon}(s)P(0)x\Big|_{s=0}^{s=t} &- \varepsilon \int_0^t U_\varepsilon(t,s)\Big(B'_{n\,\boldsymbol{\delta}}(s) + B_{n\,\boldsymbol{\delta}}(s)[P'(s), P(s)]\Big) \\
&\quad V_{0\,\varepsilon}(s)P(0)x\,ds + \int_0^t U_\varepsilon(t,s)\,C^+_{n\,\boldsymbol{\delta}}(s)\,V_{0\,\varepsilon}(s)P(0)x\,ds
\end{aligned} \tag{4.44}$$

for all $t \in I$, $\varepsilon \in (0,\infty)$, $x \in X$ and $\boldsymbol{\delta} \in (0, \delta_0]^{m_0}$. We now want to find functions $\varepsilon \mapsto \delta_{1\,\varepsilon}, \dots, \delta_{m_0\,\varepsilon}$ defined on a small interval $(0, \delta_0']$ and converging to 0 as $\varepsilon \searrow 0$ in such a way that, if they are inserted in the right hand side of (4.44), the desired convergence (4.43) follows. In view of the estimates (4.36), (4.38) and

$$\int_0^1 \left\|C^+_{n\,\boldsymbol{\delta}}(s)\right\|\,ds \le \sum_{k=1}^{m_0} c\Big(\prod_{1 \le i < k} \delta_i\Big)^{-1} \int_0^1 \left\|\delta_k \overline{R}_{\delta_k}(s)Q_n(s)P(s)\right\|\,ds, \tag{4.45}$$

we would like the functions $\varepsilon \mapsto \delta_{i\,\varepsilon}$ to converge to 0 so slowly that

$$\varepsilon\Big(\prod_{i=1}^k \delta_{i\,\varepsilon}\Big)^{-(m_0+1)} \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{4.46}$$

$$\Big(\prod_{1 \le i < k} \delta_{i\,\varepsilon}\Big)^{-1} \int_0^1 \left\|\delta_{k\,\varepsilon}\overline{R}_{\delta_{k\,\varepsilon}}(s)Q_n(s)P(s)\right\|\,ds \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{4.47}$$

for all $k \in \{1, \dots, m_0\}$. Since

$$\eta_n^+(\delta) := \int_0^1 \left\| \delta \overline{R}_\delta(s) Q_n(s) P(s) \right\| \, ds \longrightarrow 0 \quad (\delta \searrow 0) \tag{4.48}$$

by Lemma 4.2.1, by $\operatorname{rk} P(s) = \operatorname{rk} P(0) < \infty$ and by the dominated convergence theorem, such functions $\varepsilon \mapsto \delta_{i\,\varepsilon}$ really can be found. Indeed, define recursively

$$\delta_{m_0\,\varepsilon} := \varepsilon^{\frac{1}{(m_0+1)^2}} \quad \text{and} \quad \delta_{m_0-l\,\varepsilon} := \max \Big\{ \Big(\Big(\prod_{m_0-l+1 \le i < k} \delta_{i\,\varepsilon} \Big)^{-1} \eta_n^+(\delta_{k\,\varepsilon}) \Big)^{\frac{1}{2}} : \\ k \in \{m_0 - l + 1, \dots, m_0\} \Big\} \cup \Big\{ \varepsilon^{\frac{1}{(m_0+1)^2}} \Big\}$$

for $l \in \{1, \dots, m_0 - 1\}$. With the help of (4.48) it then successively follows, by proceeding from larger to smaller indices i, that $\delta_{i\,\varepsilon} \longrightarrow 0$ as $\varepsilon \searrow 0$ for all $i \in \{1, \dots, m_0\}$ (so that, in particular, $\delta_{i\,\varepsilon} \in (0, \delta_0]$ for small enough ε whence the expressions $\eta_n^+(\delta_{i\,\varepsilon})$ used in the recursive definition make sense for small ε in the first place) and that (4.46) and (4.47) are satisfied. Assertion (i) now follows from (4.41), (4.42), (4.44) by virtue of (4.36), (4.38), (4.45) and (4.39)

We now prove assertion (ii) and, for that purpose, additionally assume that X is reflexive and $t \mapsto P(t)$ is norm continuously differentiable. Analogously to (4.41) we obtain

$$\begin{aligned} \big(V_\varepsilon(t) - U_\varepsilon(t)\big)x = &\int_0^t U_\varepsilon(t,s) \left[P'(s) - Q_n(s), P(s)\right] V_\varepsilon(s) x \, ds \\ &+ \int_0^t U_\varepsilon(t,s) \left[Q_n(s), P(s)\right] V_\varepsilon(s) x \, ds \end{aligned} \tag{4.49}$$

for all $t \in I$, $\varepsilon \in (0, \infty)$ and $x \in D(A(0)) = D$. Since $Q_n(s) \longrightarrow P'(s)$ for every $s \in (0,1)$ by the additionally assumed norm continuous differentiability of $t \mapsto P(t)$, it follows by Proposition 2.1.13 and by the dominated convergence theorem that

$$\sup_{\varepsilon \in (0,\infty)} \sup_{t \in I} \left\| \int_0^t U_\varepsilon(t,s) \left[P'(s) - Q_n(s), P(s)\right] V_\varepsilon(s) \, ds \right\| \longrightarrow 0 \tag{4.50}$$

as $n \to \infty$. In view of (4.49) we therefore have to show that for each fixed $n \in \mathbb{N}$

$$\sup_{t \in I} \left\| \int_0^t U_\varepsilon(t,s) \left[Q_n(s), P(s)\right] V_\varepsilon(s) \, ds \right\| \longrightarrow 0 \tag{4.51}$$

as $\varepsilon \searrow 0$. So let $n \in \mathbb{N}$ be fixed for the rest of the proof. Again completely analogously to the proof of (i) it follows that

$$[0,t] \ni s \mapsto U_\varepsilon(t,s) B_{n\,\boldsymbol{\delta}}(s) V_\varepsilon(s) x$$

is the continuous representative of an element of $W^{1,1}([0,t],X)$ for every $x \in D(A(0)) = D$. With the help of the approximate commutator equation (4.33) of the second preparatory step, we therefore see that

$$\begin{aligned}\int_0^t U_\varepsilon(t,s)\,[Q_n(s),P(s)]\,V_\varepsilon(s)x\,ds &= \frac{1}{\varepsilon}\int_0^t U_\varepsilon(t,s)\Big(-\frac{1}{\varepsilon}A(s)B_{n\,\boldsymbol{\delta}}(s)\\ &\quad + B_{n\,\boldsymbol{\delta}}(s)\frac{1}{\varepsilon}A(s)\Big)V_\varepsilon(s)x\,ds\, + \int_0^t U_\varepsilon(t,s)\,C_{n\,\boldsymbol{\delta}}(s)\,V_\varepsilon(s)x\,ds\\ &= \varepsilon\, U_\varepsilon(t,s)B_{n\,\boldsymbol{\delta}}(s)V_\varepsilon(s)x\Big|_{s=0}^{s=t} - \varepsilon\int_0^t U_\varepsilon(t,s)\Big(B'_{n\,\boldsymbol{\delta}}(s) + B_{n\,\boldsymbol{\delta}}(s)[P'(s),P(s)]\Big)\\ &\quad V_\varepsilon(s)x\,ds + \int_0^t U_\varepsilon(t,s)\,C_{n\,\boldsymbol{\delta}}(s)\,V_\varepsilon(s)x\,ds \qquad (4.52)\end{aligned}$$

for all $t \in I$, $\varepsilon \in (0,\infty)$, $x \in D(A(0)) = D$ and $\boldsymbol{\delta} \in (0,\delta_0]^{m_0}$. In view of the estimates (4.36), (4.38), (4.45) and

$$\int_0^1 \left\|C^-_{n\,\boldsymbol{\delta}}(s)\right\|\,ds \le \sum_{k=1}^{m_0} c\Big(\prod_{1\le i<k}\delta_i\Big)^{-1}\int_0^1 \left\|P(s)Q_n(s)\delta_k\overline{R}_{\delta_k}(s)\right\|\,ds, \qquad (4.53)$$

we would now like to find functions $\varepsilon \mapsto \delta_{1\,\varepsilon},\dots,\delta_{m_0\,\varepsilon}$ defined on a small interval $(0,\delta'_0]$ and converging to 0 as $\varepsilon \searrow 0$ so slowly that (4.46), (4.47) and

$$\Big(\prod_{1\le i<k}\delta_{i\,\varepsilon}\Big)^{-1}\int_0^1 \left\|P(s)Q_n(s)\delta_{k\,\varepsilon}\overline{R}_{\delta_{k\,\varepsilon}}(s)\right\|\,ds \longrightarrow 0 \quad (\varepsilon\searrow 0) \qquad (4.54)$$

are satisfied for all $k \in \{1,\dots,m_0\}$. Why is it possible to find such functions $\varepsilon \mapsto \delta_{i\,\varepsilon}$? In essence, this is because of (4.48) and because

$$\eta^-_n(\delta) := \int_0^1 \left\|P(s)Q_n(s)\delta\overline{R}_\delta(s)\right\|\,ds \longrightarrow 0 \quad (\delta\searrow 0), \qquad (4.55)$$

which last convergence can be seen as follows: by virtue of Proposition 3.2.4, which applies by the additionally assumed reflexivity of X, $P(s)^*$ is weakly associated of order m_0 with $A(s)^*$ and $\lambda(s)$ for almost every $s \in I$, and therefore Lemma 4.2.1 together with $\operatorname{rk} P(s)^* = \operatorname{rk} P(s) < \infty$ yields the convergence

$$\left\|P(s)Q_n(s)\delta\overline{R}_\delta(s)\right\| = \left\|\delta\overline{R}_\delta(s)^*\,Q_n(s)^*P(s)^*\right\| \longrightarrow 0 \quad (\delta\searrow 0)$$

for almost every $s \in I$, from which (4.55) follows by the dominated convergence theorem. We now recursively define

$$\delta_{m_0\,\varepsilon} := \varepsilon^{\frac{1}{(m_0+1)^2}} \quad\text{and}\quad \delta_{m_0-l\,\varepsilon} := \max\Big\{\Big(\Big(\prod_{m_0-l+1\le i<k}\delta_{i\,\varepsilon}\Big)^{-1}\eta^+_n(\delta_{k\,\varepsilon})\Big)^{\frac{1}{2}},$$
$$\Big(\Big(\prod_{m_0-l+1\le i<k}\delta_{i\,\varepsilon}\Big)^{-1}\eta^-_n(\delta_{k\,\varepsilon})\Big)^{\frac{1}{2}} : k \in \{m_0-l+1,\dots,m_0\}\Big\} \cup \Big\{\varepsilon^{\frac{1}{(m_0+1)^2}}\Big\}$$

for $l \in \{1, \dots, m_0 - 1\}$. With the help of (4.48) and (4.55) it then successively follows, by proceeding from larger to smaller indices i, that $\delta_{i\,\varepsilon} \longrightarrow 0$ as $\varepsilon \searrow 0$ for all $i \in \{1, \dots, m_0\}$ and that (4.46), (4.47) and (4.54) are satisfied. Assertion (ii) now follows from (4.49), (4.50), (4.52) by virtue of (4.36), (4.38), (4.45), (4.53) and Proposition 2.1.13. ■

Some remarks, which in particular clarify the relation of the above theorem with the adiabatic theorem without spectral gap condition from [12] and [112], are in order.

1. Clearly, the adiabatic theorem above generalizes the adiabatic theorems without spectral gap condition from [12] (Theorem 11) and [112] (Theorem 6.4) which cover the less general case of weakly semisimple eigenvalues $\lambda(t)$ of (not necessarily skew self-adjoint) operators $A(t) : D \subset X \to X$ under less general regularity conditions. See Section 4.2.3 for simple examples where the previously known adiabatic theorems cannot be applied – and which show, moreover, that the adiabatic theorem above is by no means confined to spectral operators. In the special case where the eigenvalues $\lambda(t)$ from the above theorem lie on the imaginary axis $i\mathbb{R}$ for every $t \in I$, these eigenvalues are automatically weakly semisimple by the $(M, 0)$-stability hypothesis of the theorem and by the weak associatedness hypothesis. (Argue as in the second remark at the beginning of Section 4.1.3 to obtain that $P(t)$ is weakly associated of order 1 with $A(t)$ and $\lambda(t)$ for almost every t.) And so, the above adiabatic theorem – in the special case of purely imaginary eigenvalues – essentially reduces to the adiabatic theorems without spectral gap condition from [12] and [112]. A general adiabatic theory without spectral gap condition, however, should be able to cover more than just this special case, of course.

2. It can be seen from the proof of the above theorem: if the finite rank hypothesis on $P(0)$ is the only one to be violated, one still has the strong convergence

$$\sup_{t \in I} \left\| \big(U_\varepsilon(t) - V_{0\,\varepsilon}(t)\big) P(0) x \right\| \longrightarrow 0 \quad (\varepsilon \searrow 0) \quad \text{for every } x \in X, \tag{4.56}$$

provided $P(t)$ is even weakly associated of order 1 with $A(t)$ and $\lambda(t)$ for almost every $t \in I$. (In order to see this, notice that, under this extra condition, the inclusion $P(t)X \subset \ker(A(t) - \lambda(t))$ holds for every $t \in I$ by a closedness argument similar to the one in (4.5) and the ε-dependence of $V_{0\,\varepsilon}(s)P(0)$ is solely contained in a scalar factor,

$$V_{0\,\varepsilon}(s)P(0) = e^{\frac{1}{\varepsilon}\int_0^s \lambda(\tau)\,d\tau}\, W(s)P(0) \quad (s \in I),$$

where W denotes the evolution system for $[P', P]$.) See [12] (Theorem 11).

3. It can also be seen from the proof of the theorem above that one would obtain the same conclusion if – instead of requiring the weak associatedness of $P(t)$ with $A(t)$ and $\lambda(t)$ for almost every $t \in I$ – one only required the relations

$$P(t)X \subset \ker(A(t) - \lambda(t))^{m_0} \quad \text{and} \quad (1 - P(t))X \subset \overline{\operatorname{ran}}\,(A(t) - \lambda(t))^{m_0} \tag{4.57}$$

$$P(t)A(t) \subset A(t)P(t) \tag{4.58}$$

for almost every $t \in I$ and some $m_0 \in \mathbb{N}$. We point out, however, that one would nevertheless not obtain a truly more general theorem by thus modifying the hypotheses, because the projections $P(t)$ would then still be weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$. (In order to see this, notice that, for every $t \in I$ where (4.57) and (4.58) are satisfied, $(1 - P(t)) \ker(A(t) - \lambda(t))^{m_0} = 0$ by Lemma 4.2.1 and an expansion similar to (4.7) and that $P(t) \operatorname{ran}(A(t) - \lambda(t))^{m_0} = 0$ so that $P(t) \overline{\operatorname{ran}}\, (A(t) - \lambda(t))^{m_0} = 0$ as well. And then apply the first remark after Theorem 3.2.2.)

4. As in the case with spectral gap, the adiabatic theorem without spectral gap condition above can – along with the approximate commutator equation method used in its proof – be extended to several eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$: if A, λ_j, P_j for every $j \in \{1, \dots, r\}$ satisfy the hypotheses of part (ii) of the above adiabatic theorem and if $\lambda_j(\,.\,)$ and $\lambda_l(\,.\,)$ for all $j \neq l$ fall into each other at only countably many points accumulating at only finitely many points, then the evolution system V_ε for $\frac{1}{\varepsilon}A + K$ with K as in (4.8) is adiabatic w.r.t. all the P_j and well approximates the evolution system U_ε for $\frac{1}{\varepsilon}A$ in the sense that

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

provided V_ε exists on D. In order to see this (in the technically simpler case where the $t \mapsto P_j(t)$ are twice strongly continuously differentiable), one sets $B_{\boldsymbol{\delta}}(t) := \frac{1}{2} \sum_{j=1}^{r+1} B_{j\,\boldsymbol{\delta}}(t)$ where

$$B_{j\,\boldsymbol{\delta}} := \sum_{k=0}^{m_0-1} \Big(\prod_{i=1}^{k+1} \overline{R}_{j\,\delta_i} \Big) P_j' P_j (\lambda_j - A)^k + \sum_{k=0}^{m_0-1} (\lambda_j - A)^k P_j P_j' \Big(\prod_{i=1}^{k+1} \overline{R}_{j\,\delta_i} \Big) \quad (j \in \{1, \dots, r\})$$

$$\text{with} \quad \overline{R}_{j\,\delta} := (\lambda_j + \delta e^{i\vartheta_j} - A)^{-1}(1 - P_j) \quad \text{and where}$$

$$B_{r+1\,\boldsymbol{\delta}} := \sum_{j=1}^{r} B_{j\,\boldsymbol{\delta}} + \sum_{j \neq l} B_{j\,l},$$

$$\begin{aligned} B_{j\,l} := \sum_{i,k=0}^{m_0-1} \binom{k+i}{i} \frac{(-1)^i}{(\lambda_l - \lambda_j)^{k+i+1}} \Big(&(A - \lambda_j)^k P_j P_j' P_l (A - \lambda_l)^i \\ &+ (A - \lambda_l)^i P_l P_j' P_j (A - \lambda_j)^k \Big) \end{aligned}$$

and then one shows that the approximate commutator equation $B_{\boldsymbol{\delta}}(t)A(t) - A(t)B_{\boldsymbol{\delta}}(t) \subset K(t) - C_{\boldsymbol{\delta}}(t)$ with $C_{\boldsymbol{\delta}}(t) := \sum_{j=1}^{r} C_{j\,\boldsymbol{\delta}}(t)$ and

$$C_{j\,\boldsymbol{\delta}} := \sum_{k=0}^{m_0-1} \delta_{k+1} e^{i\vartheta_j} \Big(\prod_{i=1}^{k+1} \overline{R}_{j\,\delta_i} \Big) P_j' P_j (\lambda_j - A)^k - \sum_{k=0}^{m_0-1} (\lambda_j - A)^k P_j P_j' \, \delta_{k+1} e^{i\vartheta_j} \Big(\prod_{i=1}^{k+1} \overline{R}_{j\,\delta_i} \Big)$$

is satisfied for all points t where $P_j(t)$ is weakly associated with $A(t)$, $\lambda_j(t)$ for every $j \in \{1, \dots, r\}$ and where no crossing between two curves $\lambda_j(\,.\,)$ and $\lambda_l(\,.\,)$ takes place. In order to do so, one verifies that for all such t and all $j \neq l$

$$B_{j\,l}(t)A(t) - A(t)B_{j\,l}(t) \subset P_j(t)P_j'(t)P_l(t) - P_l(t)P_j'(t)P_j(t) = [P_j'(t), P_l(t)] \qquad (4.59)$$

and notices that $[P'_{r+1}, P_{r+1}] = [P', P]$ (for the second equality in (4.59), use that

$$P_j(t)P_l(t) = 0 = P_l(t)P_j(t)$$

which by the third remark after Theorem 3.2.2 follows from the weak associatedness of $P_j(t)$, $P_l(t)$ with $A(t)$ and $\lambda_j(t)$ resp. $\lambda_l(t)$ and from $\lambda_j(t) \neq \lambda_l(t)$). At first glance, the defining formula for $B_{r+1\,\boldsymbol{\delta}}$ might seem a bit mysterious, but in fact it can be guessed, just like the formulas for $B_{1\,\boldsymbol{\delta}}, \dots, B_{r\,\boldsymbol{\delta}}$, from the case with spectral gap: indeed, one obtains

$$B_{r+1} = B_{r+1\,\boldsymbol{\delta}}\big|_{\boldsymbol{\delta}=0}$$

by rewriting the formula for B_{r+1} from (4.9) with the help of Cauchy's theorem – in the special case of singletons $\sigma_j(t) = \{\lambda_j(t)\}$ consisting of poles of $(\,.\, - A(t))^{-1}$ of order at most m_0.

We close this section with a corollary tailored to the special situation of spectral operators. In this situation there are relatively simple and convenient criteria for the assumptions – in particular, the reduced resolvent estimate – of the above adiabatic theorem to be satisfied.

Corollary 4.2.3. *Suppose $A(t) : D \subset X \to X$ for every $t \in I$ is a spectral operator with spectral measure $P^{A(t)}$ such that Condition 2.1.8 is satisfied with $\omega = 0$ and such that $\sup_{t\in I} \sup_{E \in \mathcal{B}_{\mathbb{C}}} \left\|P^{A(t)}(E)\right\| < \infty$. Suppose further that $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$ such that the open sector*

$$\lambda(t) + \delta_0\, S_{(\vartheta(t)-\vartheta_0, \vartheta(t)+\vartheta_0)} := \left\{\lambda(t) + \delta e^{i\vartheta} : \delta \in (0, \delta_0), \vartheta \in (\vartheta(t) - \vartheta_0, \vartheta(t) + \vartheta_0)\right\}$$

of radius $\delta_0 \in (0,\infty)$ and angle $2\vartheta_0 \in (0,\pi)$ for every $t \in I$ is contained in $\rho(A(t))$ and such that $\operatorname{rk} P^{A(t)}(\{\lambda(t)\}) < \infty$ for almost every $t \in I$ and $t \mapsto \lambda(t)$, $e^{i\vartheta(t)}$ are absolutely continuous. Suppose finally that $A(t)|_{P^{A(t)}(\sigma(t))D}$ for every $t \in I$ is of scalar type for some punctured neighborhood

$$\sigma(t) := \sigma(A(t)) \cap \overline{U}_{r_0}(\lambda(t)) \setminus \{\lambda(t)\}$$

of $\lambda(t)$ in $\sigma(A(t))$ of radius $r_0 \in (0,\infty) \cup \{\infty\}$ and that $t \mapsto P^{A(t)}(\{\lambda(t)\})$ coincides almost everywhere with a strongly continuously differentiable map $t \mapsto P(t)$ and $t \mapsto P^{A(t)}(\tau(t))$ is continuous, where $\tau(t) := \sigma(A(t)) \setminus (\sigma(t) \cup \{\lambda(t)\})$. Then the conclusions (i) and (ii) of the preceding adiabatic theorem hold true.

Proof. We first observe that $P^{A(t)}(\{\lambda(t)\})$ is weakly associated with $A(t)$ and $\lambda(t)$ for every $t \in I$ where $\operatorname{rk} P^{A(t)}(\{\lambda(t)\}) < \infty$ by Proposition 3.2.3 and therefore $P(t)$ is weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$. Also, $\operatorname{rk} P(0) = \operatorname{rk} P(t) = \operatorname{rk} P^{A(t)}(\{\lambda(t)\}) < \infty$ for almost every $t \in I$ by the continuity of $t \mapsto P(t)$. We now verify the (reduced) resolvent estimate from the theorem above by showing that

$$\left\|\big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1} P^{A(t)}(\sigma(t))\right\| \le \frac{M_{0\,1}}{\delta} \tag{4.60}$$

$$\left\|\big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1} P^{A(t)}(\tau(t))\right\| \le M_{0\,2} \tag{4.61}$$

for every $t \in I$ and $\delta \in (0, \delta_0']$. Without loss of generality we may assume that $\lambda(t) \neq 0$ for all $t \in I$ (because otherwise we can choose $c \in i\mathbb{R}$ such that $\lambda(t) + c \neq 0$ for all t and consider the shifted data $A_c(t) := A(t) + c$, $\lambda_c(t) := \lambda(t) + c$ and $P_c(t) := P(t)$). In order to see (4.60) notice that

$$\big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1} P^{A(t)}(\sigma(t)) = \big(\lambda(t) + \delta e^{i\vartheta(t)} - A_\sigma(t)\big)^{-1} P_\sigma(t)$$

where $A_\sigma(t) := A(t)|_{P^{A(t)}(\sigma(t))D}$ and $P_\sigma(t) := P^{A(t)}(\sigma(t))$, and that, by the scalar type spectrality of $A_\sigma(t)$ and Theorem XVIII.2.11 of [39],

$$\begin{aligned}\big|\langle x^*, \big(\lambda(t) + \delta e^{i\vartheta(t)} - A_\sigma(t)\big)^{-1} P_\sigma(t) x\rangle\big| &\leq \int_{\sigma(A_\sigma(t))} \frac{1}{|\lambda(t) + \delta e^{i\vartheta(t)} - z|} \, d\big|P^{A_\sigma(t)}_{x^*, P_\sigma(t)x}\big|(z) \\ &\leq \frac{1}{\operatorname{dist}(\lambda(t) + \delta e^{i\vartheta(t)}, \sigma(A(t)))} \big|P^{A_\sigma(t)}_{x^*, P_\sigma(t)x}\big|(\mathbb{C}) \qquad (4.62)\end{aligned}$$

where $|P^{A_\sigma(t)}_{y^*,y}|$ denotes the total variation of the complex measure $E \mapsto P^{A_\sigma(t)}_{y^*,y}(E) := \big\langle y^*, P^{A_\sigma(t)}(E) y \big\rangle$ for $y \in P_\sigma(t)X$, $y^* \in (P_\sigma(t)X)^*$. Since, by $P^{A_\sigma(t)}(E) = P^{A(t)}(E)|_{P_\sigma(t)X}$ and Lemma III.1.5 of [39],

$$\big|P^{A_\sigma(t)}_{x^*, P_\sigma(t)x}\big|(\mathbb{C}) \leq 4 \sup_{E \in \mathcal{B}_\mathbb{C}} \big|\langle x^*, P^{A(t)}(E \cap \sigma(t)) x\rangle\big| \leq 4M' \, \|x^*\| \, \|x\|$$

for every $t \in I$ (where $M' := \sup_{t \in I} \sup_{E \in \mathcal{B}_\mathbb{C}} \big\|P^{A(t)}(E)\big\| < \infty$) and since, by the sector condition,

$$\operatorname{dist}\big(\lambda(t) + \delta e^{i\vartheta(t)}, \sigma(A(t))\big) \geq (\sin \vartheta_0)\, \delta$$

for every $t \in I$ and $\delta \in (0, \delta_0']$ (where δ_0' is chosen small enough), the desired estimate (4.60) follows from (4.62). In order to see (4.71) notice that, by $\lambda(t) \neq 0$ for $t \in I$,

$$\lambda(t) + \delta e^{i\vartheta(t)} \notin \sigma(\tilde{A}_\tau(t)) \subset \overline{\tau(t)} \cup \{0\} \subset \mathbb{C} \setminus \overline{U}_{r_0}(\lambda(t)) \cup \{0\}$$

for every $t \in I$ and $\delta \in [0, \delta_0']$ (where δ_0' is chosen small enough), and that

$$\big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1} P^{A(t)}(\tau(t)) = \big(\lambda(t) + \delta e^{i\vartheta(t)} - \tilde{A}_\tau(t)\big)^{-1} P_\tau(t),$$

where $\tilde{A}_\tau(t) := A(t) P^{A(t)}(\tau(t))$ and $P_\tau(t) := P^{A(t)}(\tau(t))$. (Also notice that in the case $r_0 = \infty$ there is nothing to show because then $\tau(t) = \emptyset$ for every $t \in I$.) We now show that $t \mapsto \tilde{A}_\tau(t)$ is continuous in the generalized sense. Since, for every fixed $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$,

$$\begin{aligned}(z - \tilde{A}_\tau(t))^{-1} &= \big(z - A(t)P_\tau(t)\big)^{-1} P_\tau(t) + \big(z - A(t)P_\tau(t)\big)^{-1}(1 - P_\tau(t)) \\ &= (z - A(t))^{-1} P_\tau(t) + \frac{1}{z}(1 - P_\tau(t)) \qquad (4.63)\end{aligned}$$

and since $(1-P_\tau(t))X = P^{A(t)}(\sigma(t) \cup \{\lambda(t)\})X \subset D(A(t)) = D$ by the boundedness of $\sigma(t) \cup \{\lambda(t)\} = \overline{U}_{r_0}(\lambda(t)) \cap \sigma(A(t))$, we obtain $(z - \tilde{A}_\tau(t_0))^{-1}X \subset D \subset D(\tilde{A}_\tau(t))$ and therefore

$$(z-\tilde{A}_\tau(t))^{-1} - (z-\tilde{A}_\tau(t_0))^{-1} = (z-\tilde{A}_\tau(t))^{-1}\big(\tilde{A}_\tau(t) - \tilde{A}_\tau(t_0)\big)(z-\tilde{A}_\tau(t_0))^{-1} \quad (4.64)$$

for every $t, t_0 \in I$. Since

$$\begin{aligned}\tilde{A}_\tau(t)(z-\tilde{A}_\tau(t_0))^{-1} = P_\tau(t)A(t)(z-\tilde{A}_\tau(t_0))^{-1} &\longrightarrow P_\tau(t_0)A(t_0)(z-\tilde{A}_\tau(t_0))^{-1} \\ &= \tilde{A}_\tau(t_0)(z-\tilde{A}_\tau(t_0))^{-1} \quad (t \to t_0)\end{aligned}$$

by the assumed continuity of $t \mapsto P_\tau(t)$ and the $W_*^{1,1}$-regularity of $t \mapsto A(t)$, and since $\sup_{t\in I} \|(z-\tilde{A}_\tau(t))^{-1}\| < \infty$ by (4.63) and the $(M,0)$-stability of A, it follows from (4.64) that $t \mapsto (z - \tilde{A}_\tau(t))^{-1}$ is continuous and therefore $t \mapsto \tilde{A}_\tau(t)$ is continuous in the generalized sense (Theorem IV.2.25 of [67]). In particular, $I \times [0, \delta_0'] \ni (t,\delta) \mapsto \big(\lambda(t) + \delta e^{i\vartheta(t)} - \tilde{A}_\tau(t)\big)^{-1}$ is continuous by Theorem IV.3.15 of [67], hence bounded, and the desired estimate (4.71) follows. Combining now (4.60) and (4.61) we obtain the desired (reduced) resolvent estimate

$$\left\|\big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1}(1-P(t))\right\| \le \frac{M_0}{\delta}$$

from the theorem above, because $1-P(t) = 1-P^{A(t)}(\{\lambda(t)\}) = P^{A(t)}(\sigma(t)) + P^{A(t)}(\tau(t))$ for almost every $t \in I$ and because the left hand side of the (reduced) resolvent estimate for fixed δ is continuous in t. ■

4.2.2 A quantitative adiabatic theorem without spectral gap condition

As a supplement to the qualitative adiabatic theorem above (Theorem 4.2.2), we note the following quantitative refinement. It implies that, if in the situation of the above theorem the maps $t \mapsto A(t), \lambda(t), e^{i\vartheta(t)}$ and $t \mapsto P(t)$ are even $W_*^{1,\infty}$- or $W_*^{2,\infty}$-regular respectively, then the rate of convergence (Lemma 4.2.1!) of the integrals

$$\begin{aligned}\eta^+(\delta) &:= \int_0^1 \left\|\delta\big(\lambda(s) + \delta e^{i\vartheta(s)} - A(s)\big)^{-1}P'(s)P(s)\right\| \, ds, \\ \eta^-(\delta) &:= \int_0^1 \left\|P(s)P'(s)\delta\big(\lambda(s) + \delta e^{i\vartheta(s)} - A(s)\big)^{-1}\right\| \, ds \qquad (4.65)\end{aligned}$$

yields a simple upper bound on the rate of convergence of $\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\|$ which we are interested in here. See [131] for an analogous result in the case of skew self-adjoint operators $A(t)$.

Theorem 4.2.4. *Suppose that $A(t)$, $\lambda(t)$, $P(t)$ are as in Theorem 4.2.2 with X not necessarily reflexive and that $t \mapsto A(t)$ is even in $W_*^{1,\infty}(I, L(Y,X))$, $t \mapsto \lambda(t), e^{i\vartheta(t)}$ are even Lipschitz and $t \mapsto P(t)$ is even in $W_*^{2,\infty}(I, L(X))$. Suppose further that $\eta : (0, \delta_0] \subset (0,1] \to (0,\infty)$ is a function such that $\eta(\delta) \longrightarrow 0$ as $\delta \searrow 0$ and*

$$\eta(\delta) \ge \delta \quad \textit{as well as} \quad \eta^\pm(\delta) \le \eta(\delta)$$

for all $\delta \in (0, \delta_0]$ with $\eta^{\pm}$ as above. Then there is a constant c such that

$$\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \le c\, \tilde{\eta}^{m_0}\big(\varepsilon^{2/(m_0(m_0+1))}\big) = c\,(\tilde{\eta} \circ \cdots \circ \tilde{\eta})\big(\varepsilon^{2/(m_0(m_0+1))}\big)$$

for ε sufficiently small, where $\tilde{\eta}(\delta) := \eta(\delta^{\frac{1}{2}})$.

Proof. We proceed as in the proof of the qualitative adiabatic theorem above, but now replace Q_n and Q_n' at any occurrence by P' and P''. We can then conclude from (4.49) and (4.52) (with the replacements just mentioned) that there is a constant c' such that

$$\sup_{t\in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \le\ c' \Bigg(\sum_{k=1}^{m_0} \varepsilon \Big(\prod_{j=1}^{k} \delta_j \Big)^{-1} + \sum_{k=1}^{m_0} \varepsilon \Big(\delta_k^{m_0+1} \prod_{j\neq k} \delta_j \Big)^{-1} \eta(\delta_k) + \sum_{k=1}^{m_0} \Big(\prod_{1\le j<k} \delta_j \Big)^{-1} \eta(\delta_k) \Bigg) \quad (4.66)$$

for all $\delta_1, \dots, \delta_{m_0} \in (0, \delta_0]$ and $\varepsilon \in (0, \infty)$. In this estimate the first, second, and third sum correspond to the $B_{\boldsymbol{\delta}}$-, $B'_{\boldsymbol{\delta}}$-, $C_{\boldsymbol{\delta}}$-terms in (4.52), respectively. See (4.36) and (4.45), (4.53) for the estimation of the $B_{\boldsymbol{\delta}}$-terms and $C_{\boldsymbol{\delta}}$-terms. In order obtain the upper bound for the $B'_{\boldsymbol{\delta}}$-terms, refine the estimate (4.38) on $\int_0^1 \|B'_{\boldsymbol{\delta}}(s)\|\, ds$ from the proof of the previous theorem by using the fact that

$$\operatorname*{ess-sup}_{s\in I} \big\| \big(A'(s) - \lambda'(s) - \delta\, r'(s)\big)(A(s) - 1)^{-1} \big\| \le c < \infty, \quad (4.67)$$

where the additional assumption that $t \mapsto A(t)$ and $t \mapsto \lambda(t), r(t) := e^{i\vartheta(t)}$ be even $W_*^{1,\infty}$-regular enters. It follows from this that the integral (from 0 to 1) of the critical terms in $B'_{\boldsymbol{\delta}}$, namely

$$\overline{R}_{\delta_1}(s) \cdots R_{\delta_l}(s)\big(A' - \lambda' - \delta_l\, r'\big)(s) \overline{R}_{\delta_l}(s) \cdots \overline{R}_{\delta_k}(s) P'(s) P(s), \quad (4.68)$$

$$P(s)P'(s)\, \overline{R}_{\delta_1}(s) \cdots \overline{R}_{\delta_l}(s) (A' - \lambda' - \delta_l\, r')(s) R_{\delta_l}(s) \cdots \overline{R}_{\delta_k}(s), \quad (4.69)$$

can be estimated by $(\delta_l^{m_0+1} \prod_{j\neq l} \delta_j)^{-1} \eta(\delta_l)$ for all $l \in \{1, \dots, k\}$, as desired. (In order to see this, insert in both of the above products (4.68) and (4.69) the identity operators $(A(s)-1)^{-1}(A(s)-1)$ behind $(A' - \lambda' - \delta_l\, r')(s)$, commute $(A(s)-1)\overline{R}_{\delta_l}(s)$ directly in front of $P'(s)P(s)$ and $\overline{R}_{\delta_l}(s)$ directly behind $P(s)P'(s)$ respectively, and then use (4.67) together with the fact that

$$\int_0^1 \big\|(A(s)-1)\overline{R}_\delta(s) P'(s) P(s)\big\|\, ds,\ \int_0^1 \big\|P(s)P'(s)\overline{R}_\delta(s)\big\|\, ds \le c\, \frac{\eta(\delta)}{\delta}$$

and that $\sup_{s\in I} \|R_\delta(s)\|, \sup_{s\in I} \|(A(s)-1)R_\delta(s)\| \le \frac{c}{\delta^{m_0}}$ for all sufficiently small $\delta \in (0, \delta_0]$.) We now recursively define

$$\delta_{m_0\, \varepsilon} := \varepsilon^{\frac{1}{m_0(m_0+1)}} \quad \text{and} \quad \delta_{m_0 - k\, \varepsilon} := \big(\eta(\delta_{m_0-k+1\, \varepsilon})\big)^{\frac{1}{2}}$$

for ε so small that $\delta_{m_0-k+1\,\varepsilon}$ lies in $(0,\delta_0]$ and for $k\in\{1,\dots,m_0-1\}$. (It should be noticed that $\delta_{m_0-k+1\,\varepsilon}\longrightarrow 0$ as $\varepsilon\searrow 0$ because $\eta(\delta)\longrightarrow 0$ and that $\delta_{m_0-k+1\,\varepsilon}$ therefore really lies in the domain $(0,\delta_0]$ of η for sufficiently small ε.) Since $\eta(\delta_{1\,\varepsilon})=\tilde{\eta}^{m_0}(\varepsilon^{2/(m_0(m_0+1))})$ and $\frac{1}{\delta_{k-1\,\varepsilon}}\eta(\delta_{k\,\varepsilon})=\delta_{k-1\,\varepsilon}\le\eta(\delta_{k-1\,\varepsilon})$ for $k\in\{2,\dots,m_0\}$, it follows by induction that

$$\Big(\prod_{1\le j<k}\delta_{j\,\varepsilon}\Big)^{-1}\eta(\delta_{k\,\varepsilon})\le\tilde{\eta}^{m_0}\big(\varepsilon^{2/(m_0(m_0+1))}\big) \tag{4.70}$$

and, in particular, $\eta(\delta_{k\,\varepsilon})\le\tilde{\eta}^{m_0}\big(\varepsilon^{2/(m_0(m_0+1))}\big)$ for all $k\in\{1,\dots,m_0\}$ and sufficiently small ε. Since $\delta_{m_0\,\varepsilon}\le\delta_{m_0-k+1\,\varepsilon}\le\delta_{m_0-k\,\varepsilon}$ for $k\in\{1,\dots,m_0-1\}$ and small ε, it further follows that

$$\begin{aligned}\varepsilon\Big(\prod_{j=1}^{k}\delta_{j\,\varepsilon}\Big)^{-1}\le\varepsilon\Big(\delta_{k\,\varepsilon}^{m_0+1}\prod_{j\ne k}\delta_{j\,\varepsilon}\Big)^{-1}\eta(\delta_{k\,\varepsilon})\le\varepsilon\Big(\prod_{j=1}^{m_0}\delta_{m_0\,\varepsilon}\Big)^{-(m_0+1)}\tilde{\eta}^{m_0}\big(\varepsilon^{2/(m_0(m_0+1))}\big)\\ =\tilde{\eta}^{m_0}\big(\varepsilon^{2/(m_0(m_0+1))}\big)\end{aligned} \tag{4.71}$$

for all $k\in\{1,\dots,m_0\}$ and sufficiently small ε. Combining (4.66), (4.70) and (4.71) we finally obtain the assertion. ■

1. Clearly, a function η as described in the above theorem exists under the hypotheses of the qualitative adiabatic theorem (Theorem 4.2.2) with X reflexive and $t\mapsto A(t)$ in $W_*^{1,\infty}(I,L(Y,X))$ and $t\mapsto\lambda(t),e^{i\vartheta(t)}$ Lipschitz continuous. In fact, one has only to define $\eta(\delta):=\max\{\eta^+(\delta),\eta^-(\delta),\delta\}$ with $\eta^\pm$ as in (4.65) and to remember that $\eta(\delta)\longrightarrow 0$ as $\delta\searrow 0$ by (4.48) and (4.55).

2. An inspection of the proof above – or, more precisely, of the arguments leading to (4.66) – shows that one obtains the same conclusion if one drops the finite rank hypothesis on $P(0)$ and, at the same time, replaces the hypothesis that $P(t)$ be weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t\in I$ by the condition that there exist an $m_0\in\mathbb{N}$ such that

$$P(t)A(t)\subset A(t)P(t)\quad\text{and}\quad P(t)X\subset\ker(A(t)-\lambda(t))^{m_0}$$

for every $t\in I$ (while leaving all other hypotheses of the above theorem unchanged).

We now specialize to the case of spectral operators $A(t)$ of scalar type and note the following quantitative adiabatic theorem tailored to scalar type spectral operators $A(t)$ whose spectral measures $P^{A(t)}$ are Hölder continuous in t around $\lambda(t)$ in some sense (which, in particular, means that in a punctured neighborhood of $\lambda(t)$ there is no more eigenvalue of $A(t)$). It generalizes a result for skew self-adjoint $A(t)$ of Avron and Elgart (Corollary 1 in [11]) and a refinement of it due to Teufel (Remark 1 in [131]) and slightly improves the rates of convergence given there.

Corollary 4.2.5. *Suppose $A(t): D\subset X\to X$ for every $t\in I$ is a spectral operator of scalar type (with spectral measure $P^{A(t)}$) such that Condition 2.1.8 is satisfied with $\omega=0$*

and such that $\sup_{t\in I}\sup_{E\in\mathcal{B}_\mathbb{C}}\left\|P^{A(t)}(E)\right\|<\infty$. *Suppose further that* $\lambda(t)$ *for every* $t\in I$ *is an eigenvalue of* $A(t)$ *such that the open sector*

$$\lambda(t)+\delta_0\,S_{(\vartheta(t)-\vartheta_0,\vartheta(t)+\vartheta_0)}:=\left\{\lambda(t)+\delta e^{i\vartheta}:\delta\in(0,\delta_0),\vartheta\in(\vartheta(t)-\vartheta_0,\vartheta(t)+\vartheta_0)\right\}$$

of radius $\delta_0\in(0,\infty)$ *and angle* $2\vartheta_0\in(0,\pi)$ *for every* $t\in I$ *is contained in* $\rho(A(t))$ *and such that* $t\mapsto\lambda(t)$, $e^{i\vartheta(t)}$ *are absolutely continuous. Suppose finally that* $P(t)$ *for every* $t\in I$ *is a bounded projection in* X *such that* $P(t)=P^{A(t)}(\{\lambda(t)\})$ *for almost every* $t\in I$ *and* $t\mapsto P(t)$ *is in* $W^{2,1}_*(I,L(X))$, *and suppose that* $P^{A(t)}$ *is Hölder continuous locally around* $\lambda(t)$ *with exponent* $\alpha\in(0,1]$ *uniformly in* $t\in I$ *in the following sense: there is a constant* $c_0\in(0,\infty)$ *such that*

$$\left\|P^{A(t)}(E)x\right\|\le c_0\,\lambda(E)^{\frac{\alpha}{2}}\;\|x\|$$

for all $x\in X$ *and for all* $t\in I$ *and* $E\in\mathcal{B}_\mathbb{C}$ *that are contained in the punctured neighborhood* $U_{r_0}(\lambda(t))\setminus\{\lambda(t)\}$ *of* $\lambda(t)$ *(with* r_0 *independent of* t*). Then there exists a constant* $c\in(0,\infty)$ *such that*

$$\sup_{t\in I}\|U_\varepsilon(t)-V_\varepsilon(t)\|\le c\,\varepsilon^{\frac{\alpha}{2(1+\alpha)}}$$

for small enough $\varepsilon\in(0,\infty)$, *where* V_ε *denotes the evolution system for* $\frac{1}{\varepsilon}A+[P',P]$.

Proof. We first show that there exists a function $\eta:(0,\delta_0']\to(0,\infty)$ such that $\eta(\delta)\longrightarrow 0$ as $\delta\searrow 0$ and

$$\eta(\delta)\ge\delta\quad\text{and}\quad\left\|\delta\overline{R}_\delta(t)\right\|=\left\|\delta\big(\lambda(t)+\delta e^{i\vartheta(t)}-A(t)\big)^{-1}(1-P(t))\right\|\le\eta(\delta)\qquad(4.72)$$

for all $\delta\in(0,\delta_0']$ and $t\in I$ (with a suitable δ_0'). In fact, it is sufficient to prove (4.72) for all t in the set $I\setminus N$ of those t where $P(t)=P^{A(t)}(\{\lambda(t)\})$, because this set $I\setminus N$ is dense in I by assumption and because the left-hand side of the second inequality in (4.72) is continuous in t by assumption. We observe that for every $t\in I\setminus N$

$$\left|\left\langle x^*,\delta\overline{R}_\delta(t)x\right\rangle\right|\le\int_{\sigma(A(t))\setminus\{\lambda(t)\}}\frac{\delta}{|\lambda(t)+\delta e^{i\vartheta(t)}-z|}\,d\big|P^{A(t)}_{x^*,x}\big|(z),$$

where $\big|P^{A(t)}_{x^*,x}\big|$ denotes the total variation of $E\mapsto P^{A(t)}_{x^*,x}(E):=\left\langle x^*,P^{A(t)}(E)x\right\rangle$ (use the scalar type spectrality of $A(t)$ and Theorem XVIII.2.11 of [39]). We then divide the punctured spectrum $\sigma(A(t))\setminus\{\lambda(t)\}$ into the parts

$$\sigma_{1\,r_\delta}(t):=\sigma(A(t))\cap U_{r_\delta}(\lambda(t))\setminus\{\lambda(t)\}\quad\text{and}\quad\sigma_{2\,r_\delta}(t):=\sigma(A(t))\cap\mathbb{C}\setminus U_{r_\delta}(\lambda(t))$$

of those spectral values that are close to $\lambda(t)$ resp. far from $\lambda(t)$, where $r_\delta:=\delta^\gamma$ and $\gamma\in(0,1)$ will be chosen in (4.74) below. Since, by Lemma III.1.5 of [39],

$$\big|P^{A(t)}_{x^*,x}\big|(E)\le 4\sup_{F\in\mathcal{B}_E}\left|\langle x^*,P^{A(t)}(F)P^{A(t)}(E)x\rangle\right|\le 4M'\,\|x^*\|\,\left\|P^{A(t)}(E)x\right\|$$

for every $t \in I$ and $E \in \mathcal{B}_\mathbb{C}$ (where $M' := \sup_{t \in I} \sup_{F \in \mathcal{B}_\mathbb{C}} \left\| P^{A(t)}(F) \right\| < \infty$) and since, by the assumed sector condition,

$$\operatorname{dist}\left(\lambda(t) + \delta e^{i\vartheta(t)}, \sigma(A(t))\right) \geq (\sin \vartheta_0)\, \delta$$

for every $t \in I$ and $\delta \in (0, \delta_0']$ (where δ_0' is chosen small enough), there are positive constants c_1, c_2 such that

$$\int_{\sigma_1 r_\delta(t)} \frac{\delta}{|\lambda(t) + \delta e^{i\vartheta(t)} - z|}\, d\left|P^{A(t)}_{x^*,x}\right|(z) \leq \frac{1}{\sin \vartheta_0} \left|P^{A(t)}_{x^*,x}\right|\left(\dot{U}_{r_\delta}(\lambda(t))\right) \leq c_1 \delta^{\alpha\gamma} \|x^*\| \|x\|$$

as well as

$$\int_{\sigma_2 r_\delta(t)} \frac{\delta}{|\lambda(t) + \delta e^{i\vartheta(t)} - z|}\, d\left|P^{A(t)}_{x^*,x}\right|(z) \leq \frac{\delta}{r_\delta - \delta} \left|P^{A(t)}_{x^*,x}\right|(\mathbb{C}) \leq c_2 \delta^{1-\gamma} \|x^*\| \|x\|$$

for every $x \in X$, $x^* \in X^*$, $\delta \in (0, \delta_0']$ and $t \in I$, where $\dot{U}_{r_\delta}(\lambda(t)) := U_{r_\delta}(\lambda(t)) \setminus \{\lambda(t)\}$ of course. Consequently,

$$\left\| \delta \overline{R}_\delta(t) \right\| \leq c_1\, \delta^{\alpha\gamma} + c_2\, \delta^{1-\gamma} \leq \max\{c_1, c_2\}\, \delta^{\min\{\alpha\gamma, 1-\gamma\}} = c_0'\, \delta^{\beta(\gamma)} \tag{4.73}$$

for every $t \in I \setminus N$ and $\delta \in (0, \delta_0']$ (notice that $\beta(\gamma) := \min\{\alpha\gamma, 1-\gamma\}$, for given γ, is the best – that is, biggest – possible exponent in the second inequality above). And as $\gamma \mapsto \beta(\gamma)$ is maximal at $\gamma_0 := \frac{1}{1+\alpha}$, we choose

$$\gamma := \gamma_0, \quad \beta := \beta(\gamma_0) = \frac{\alpha}{1+\alpha}, \quad \eta(\delta) := c_0'\, \delta^\beta = c_0'\, \delta^{\frac{\alpha}{1+\alpha}}, \tag{4.74}$$

thereby obtaining (4.72) (first for all $t \in I \setminus N$ and then for all $t \in I$).

With (4.72) at hand, we can now show the desired conclusion in essentially the same way as in the proof of the previous theorem (but for the convenience of the reader, we give a self-contained argument). Indeed, since $A(t)$ is a spectral operator of scalar type and $P(t) = P^{A(t)}(\{\lambda(t)\})$ for almost every $t \in I$, the projection $P(t)$ for almost every $t \in I$ is weakly associated of order 1 with $A(t)$ and $\lambda(t)$ (Proposition 3.2.3) and so

$$P(t)A(t) \subset A(t)P(t) = \lambda(t)P(t)$$

holds for every $t \in I$ by the closedness argument in (4.5). We can therefore conclude from (4.49) and (4.52) (with Q_n and Q_n' replaced by P' and P'' at any occurrence and with $m_0 = 1$) and from (4.72) that there is a constant c' such that

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \leq c' \Big(\varepsilon\, \frac{1}{\delta} + \varepsilon\, \frac{1}{\delta^2}\, \eta(\delta) + \eta(\delta) \Big) \tag{4.75}$$

for all $\varepsilon \in (0, \infty)$ and $\delta \in (0, \delta_0']$ with η as in (4.74) above. Choosing now $\delta_\varepsilon := \varepsilon^{\frac{1}{2}}$ we immediately get the desired conclusion from (4.75) and (4.74). (In order to see (4.75), notice that the first and third term on the right-hand side of (4.75) are upper bounds for the $B_{\boldsymbol{\delta}}$-terms and $C_{\boldsymbol{\delta}}$-terms in (4.52) by virtue of (4.72). And to see that the middle term in (4.75) is an upper bound for the $B'_{\boldsymbol{\delta}}$-terms in (4.52), argue as in the proof of the previous theorem, but notice that now it is sufficient to have instead of (4.67) a δ-independent bound on the integral of $s \mapsto (A'(s) - \lambda'(s) - \delta r'(s))(A(s) - 1)^{-1}$ because now we cannot only estimate the integral of $s \mapsto \overline{R}_\delta(s)$ but by (4.72) even its supremum.) ■

4.2.3 Some examples

We begin with two examples where $\lambda(t)$ is an eigenvalue of $A(t)$ that is allowed to be non-isolated and non-weakly-semisimple for every $t \in I$. In particular, these examples cannot be dealt with by way of the previously known adiabatic theorems. In the first example, $A(t)$ is a spectral operator whereas in the second it is not (by Theorem XV.3.10 and XV.8.7 of [39] and by the spectral structure of the right shift S_+ (Section 3.5)).

Example 4.2.6. Suppose A, λ, P with $A(t) = R(t)^{-1}A_0(t)R(t)$, $P(t) = R(t)^{-1}P_0R(t)$, and $R(t) = e^{Ct}$ are given as follows in $X := \ell^p(I_d) \times \ell^p(I_\infty)$ (where $p \in [1, \infty)$ and $d \in \mathbb{N}$):

$$A_0(t) := \begin{pmatrix} \lambda(t) + \alpha(t)N & 0 \\ 0 & \operatorname{diag}\big((\lambda_n)_{n\in\mathbb{N}}\big) \end{pmatrix} \quad \text{and} \quad P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\lambda(t) \in (-\infty, 0]$, $\alpha(t)$, N are such that Condition 3.5.1 is satisfied and where $(\lambda_n)_{n\in\mathbb{N}}$ is an enumeration of $[-1, 0] \cap \mathbb{Q}$ such that $\lambda(t) \notin \{\lambda_n : n \in \mathbb{N}\}$ for almost every $t \in I$. Additionally, suppose $t \mapsto \lambda(t)$ and $t \mapsto \alpha(t)$ are absolutely continuous and C is the right shift operator on $\ell^p(I_d) \times \ell^p(I_\infty) \cong \ell^p(I_\infty)$:

$$C(z_1, \dots, z_d, z_{d+1}, \dots) := (0, z_1, \dots, z_{d-1}, z_d, \dots).$$

Then $t \mapsto A(t)$ is in $W^{1,\infty}_*(I, L(X))$ and $t \mapsto A_0(t)$ is $(M_0, 0)$-stable (by Lemma 3.5.2), so that A is $(M, 0)$-stable for some $M \in [1, \infty)$ by Lemma 2.1.7. Since $A_0(t)|_{P_0X} - \lambda(t)$ is nilpotent of order at most $m_0 := \dim \ell^p(I_d) = d$ for every $t \in I$ and since $A_0(t)|_{(1-P_0)X} - \lambda(t)$ is injective and has dense range in $(1 - P_0)X$ (because $\lambda(t) \notin \{\lambda_n : n \in \mathbb{N}\}$) for almost every $t \in I$, P_0 is weakly associated of order m_0 with $A_0(t)$ and $\lambda(t)$, whence the same is true for $A(t)$, P instead of $A_0(t)$ and P_0. And finally, the resolvent estimate of Theorem 4.2.2 is clearly fulfilled if we choose $\vartheta(t) := \frac{\pi}{2}$ for all $t \in I$. All other hypotheses of Theorem 4.2.2 (i) are obvious. ◀

In the above example, we have chosen C to be the right shift operator on $X = \ell^p(I_d) \times \ell^p(I_\infty)$ in order to make sure that the example cannot be reduced to a finite-dimensional subspace: there is no finite-dimensional subspace M of X such that

$$M \supset P(0)X \text{ and } A(t)M \subset M \text{ as well as } P(t)M \subset M$$

for every $t \in I$. (Clearly, if given data A, λ, P can be reduced to a finite-dimensional subspace M and satisfy the hypotheses of Theorem 4.2.2 (i), it suffices to prove the respective statement for the reduced data A^M, λ, P^M given by

$$A^M(t) := A(t)|_M \quad \text{and} \quad P^M(t) := P(t)|_M \quad (t \in I).$$

And this, in turn, can typically already be done with the help of the adiabatic theorem with non-uniform spectral gap condition of Section 4.1: by the finite-dimensionality of M, $\lambda(t)$ is isolated in $\sigma(A^M(t))$ for every $t \in I$ and, by Theorem 3.2.2 and the first remark following it, $P^M(t)$ is the projection associated with $A^M(t)$ and $\lambda(t)$ for almost every $t \in I$.) In order to see the claimed irreducibility of the example above, we assume, on the contrary, that A, λ, P as given by the example can be reduced to a finite-dimensional subspace M of X. Then $R(t)M$ is invariant under $A_0(t)$ for every $t \in I$ and hence (by finite-dimensional spectral theory applied to $A_0^{|}(t) := A_0(t)|_{R(t)M}$)

$$R(t)M \quad = \bigoplus_{\lambda \in \sigma_p(A_0^{|}(t))} \bigcup_{k \in \mathbb{N}} \ker(A_0^{|}(t) - \lambda)^k \quad \subset \bigoplus_{\lambda \in \sigma_p(A_0^{|}(t))} \bigcup_{k \in \mathbb{N}} \ker(A_0(t) - \lambda)^k,$$

which latter space (by the special choice of $A_0(t)$) for every $t \in I$ contains only vectors with finitely many non-zero components. Consequently, the same is also true for the vectors in $R(t)M$ for every $t \in I$. We now obtain the desired contradiction by observing that the vector $R(t)v = e^{Ct}v$, for every $t \neq 0$ and every vector $0 \neq v \in X$ with only finitely many non-zero components, has infinitely many non-zero components because

$$e^{Ct}e_i = \big(0, \dots, 0, 1, \frac{t}{1!}, \frac{t^2}{2!}, \frac{t^3}{3!}, \dots\big)$$

for $i \in \mathbb{N}$ (where the entry 1 is in the ith place).

Example 4.2.7. Suppose A, λ, P with $A(t) = R(t)^{-1}A_0(t)R(t)$, $P(t) = R(t)^{-1}P_0R(t)$, and $R(t) = e^{Ct}$ are given as follows in $X := \ell^p(I_d) \times \ell^p(I_\infty)$ (where $p \in (1, \infty)$ and $d \in \mathbb{N}$):

$$A_0(t) := \begin{pmatrix} \lambda(t) + \alpha(t)N & 0 \\ 0 & S_+ - 1 \end{pmatrix} \quad \text{and} \quad P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\lambda(t) \in \partial U_1(-1)$, $\alpha(t)$, N are such that Condition 3.5.1 is satisfied. Additionally, $t \mapsto \lambda(t)$ and $t \mapsto \alpha(t)$ are absolutely continuous and C is the bounded linear map in $\ell^p(I_d) \times \ell^p(I_\infty) \cong \ell^p(I_\infty)$ given by

$$C(z_1, \dots, z_d, z_{d+1}, \dots) := (0, \dots, 0, z_{d+1}, -z_d, 0, \dots),$$

where in the vector on the right z_{d+1}, $-z_d$ appear in the dth and $(d+1)$th place. Since $\lambda(t) \in \partial U_1(-1) = \sigma_c(S_+ - 1)$ for every $t \in I$ because $p \neq 1$ (Section 3.5), P_0 is weakly associated with $A_0(t)$ and $\lambda(t)$ and therefore the same goes for $A_0(t)$, P_0 replaced by $A(t)$ and $P(t)$. Also, if for every $t \in I$ we choose $\vartheta(t)$ such that $\lambda(t) = -1 + e^{i\vartheta(t)}$, then the resolvent estimate of Theorem 4.2.2 holds true because

$$\left\| \big(\lambda(t) + \delta e^{i\vartheta(t)} - A_0(t)\big)^{-1}(1 - P_0) \right\| \leq \left\| \big(1 + \delta - e^{-i\vartheta(t)}S_+\big)^{-1} \right\| \leq \frac{1}{\delta}$$

for every $t \in I$ and $\delta \in (0, \infty)$ (Section 3.5). ◀

Just like the first example, the example above cannot be reduced to a finite-dimensional subspace M. Indeed, assuming that A, λ, P from above could be reduced, we obtain that $R(t)M$ for every $t \in I$ is invariant under $A_0(t)$ and

$$\emptyset \neq \sigma(A_0(t)|_{R(t)M}) = \sigma_p(A_0(t)|_{R(t)M}) \subset \sigma_p(A_0(t)) = \{\lambda(t)\}$$

(as $\sigma_p(S_+ - 1) = \emptyset$ (Section 3.5)), so that (by finite-dimensional spectral theory)

$$R(t)M = \ker(A_0(t)|_{R(t)M} - \lambda(t))^{\dim M} \subset \ker(A_0(t) - \lambda(t))^{\dim M} = P_0X \subset M$$

for all $t \in I$. Consequently, $R(t)P_0X \subset R(t)M \subset P_0X$. In other words, P_0X is invariant under $R(t) = e^{Ct}$ for every $t \in I$, which is obviously not true by the choice of C in the above example. Contradiction! We point out that the above example really works only in case $p \neq 1$. In fact, if $p = 1$ then $\lambda(t) \in \partial U_1(-1) = \sigma_r(S_+ - 1)$ (Section 3.5) from which it follows by the block structure of $A_0(t)$ that

$$\ker(A_0(t) - \lambda(t))^k \subset \ell^p(I_d) \times 0,$$
$$\overline{\operatorname{ran}}\,(A_0(t) - \lambda(t))^k \subset \ell^p(I_d) \times \overline{\operatorname{ran}}\,(S_+ - 1 - \lambda(t))^k \subsetneq \ell^p(I_d) \times \ell^p(I_\infty)$$

and hence $\ker(A_0(t) - \lambda(t))^k + \overline{\operatorname{ran}}\,(A_0(t) - \lambda(t))^k \neq X$ for every $k \in \mathbb{N}$ and $t \in I$. Consequently, the adiabatic theorems proven above cannot be applied.

In our last example we show that the conclusion of the adiabatic theorem without spectral gap condition may fail if the regularity assumption on P is the only one to be violated.

Example 4.2.8. Set $A(t) := M_{f_t}$ in $X := L^p(\mathbb{R})$ (for some $p \in [1, \infty)$), where

$$f_t := f_0(\,.\, + t) \quad \text{with} \quad 0 \neq f_0 \in C_c^1(\mathbb{R}, i\mathbb{R}),$$

$\lambda(t) := 0$ and $P(t) := M_{\chi_{E_t}}$ with $E_t := \{f_t = 0\}$. Then all the assumptions of the adiabatic theorem without spectral gap condition – in the version for projections of infinite rank (second remark after Theorem 4.2.2) – are satisfied with the sole exception that $t \mapsto P(t)$ is not strongly continuously differentiable (by Lemma 3.5.3). And indeed, the conclusion of the adiabatic theorem already fails: as the $A(t)$ are pairwise commuting and $t \mapsto f_t(x)$ is Riemann integrable for every $x \in \mathbb{R}$, one has

$$\big(U_\varepsilon(t, s)g\big)(x) = \Big(e^{\frac{1}{\varepsilon}\int_s^t A(\tau)\,d\tau}\, g\Big)(x) = e^{\frac{1}{\varepsilon}\int_s^t f_\tau(x)\,d\tau}\, g(x)$$

for almost every $x \in \mathbb{R}$ and therefore (by $f_0(\mathbb{R}) \subset i\mathbb{R}$)

$$\|(1 - P(t))U_\varepsilon(t)P(0)g\|^p = \int \big|(1 - \chi_{E_t}(x))\chi_{E_0}(x)g(x)\big|^p\, dx$$

for every $t \in I$, $\varepsilon \in (0, \infty)$ and $g \in X$. Since the right hand side of this equation does not depend on $\varepsilon \in (0, \infty)$ and since for every $t \in (0, 1]$ there is a $g \in X$ such that this right hand side does not vanish, the conclusion of the adiabatic theorem without spectral gap – more precisely, the weaker assertion that $\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)g\| \longrightarrow 0$ for all $g \in X$ – fails. ◀

It should be pointed out that the failure of both the assumptions and the conclusion of the adiabatic theorems without spectral gap condition presented above is a quite typical phenomenon in the case where $A(t) = M_{f_t}$ in $X = L^p(X_0)$ for some $p \in [1, \infty)$ and some

σ-finite measure space $(X_0, \mathcal{A}, \mu)$. Indeed, if $A(t) = M_{f_t}$ in $X = L^p(X_0)$ for measurable functions $f_t : X_0 \to \{\operatorname{Re} z \le 0\}$ such that $D(M_{f_t}) = D$ for all $t \in I$, if $\lambda(t)$ is an eigenvalue of $A(t)$, and if $P(t)$ for almost every $t \in I$ (with exceptional set N) is weakly associated with $A(t)$ and $\lambda(t)$, then

$$P(t) = M_{\chi_{\{f_t = \lambda(t)\}}} = M_{\chi_{E_t}} \text{ for every } t \in I \setminus N$$

by Theorem 3.2.2, and therefore the following holds true. As soon as $I \setminus N \ni t \mapsto P(t)$ is not constant, the assumptions of the adiabatic theorem without spectral gap (Theorem 4.2.2) must fail (because then $I \setminus N \ni t \mapsto P(t) = M_{\chi_{E_t}}$ cannot extend to a strongly continuously differentiable map by Lemma 3.5.3). And as soon as, in addition, the maps f_t are $i\mathbb{R}$-valued and $t \mapsto f_t g \in X$ is continuous for all $g \in D$, the conclusion of Theorem 4.2.2, or more precisely, of its corollary

$$\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\| \longrightarrow 0 \quad \text{and} \quad \sup_{t \in I} \|P(t)U_\varepsilon(t)(1 - P(0))\| \longrightarrow 0,$$

must fail as well. (In order to see this, one gathers from Theorem 2.3 of [101] (and its proof) that the evolution system U_ε for $\frac{1}{\varepsilon}A$ exists on D and can be strongly approximated by finite products of operators of the form $e^{M_{f_\tau}\sigma}$ with $\tau \in I$ and $\sigma \in [0, \infty)$, and infers from this that for arbitrary $g \in X$

$$\Big|(1 - \chi_{E_t}(x))\big(U_\varepsilon(t)\chi_{E_0}g\big)(x) - \chi_{E_t}(x)\big(U_\varepsilon(t)(1 - \chi_{E_0})g\big)(x)\Big| = \big|\chi_{E_t}(x) - \chi_{E_0}(x)\big|\big|g(x)\big|$$

for almost every $x \in X_0$, whence

$$\big\|(1 - P(t))U_\varepsilon(t)P(0)g - P(t)U_\varepsilon(t)(1 - P(0))g\big\| = \|P(t)g - P(0)g\|$$

for all $t \in I \setminus N$, $\varepsilon \in (0, \infty)$. Since the right hand side of this equation does not depend on $\varepsilon \in (0, \infty)$ and since $I \setminus N \ni t \mapsto P(t)$ is not constant, there is a $t \in (0, 1]$ and a $g \in X$ such that $(1 - P(t))U_\varepsilon(t)P(0)g$ and $P(t)U_\varepsilon(t)(1 - P(0))g$ do not both converge to 0 as $\varepsilon \searrow 0$.)

4.2.4 An applied example: quantum dynamical semigroups

In this section we apply the adiabatic theorem without spectral gap condition from Section 4.2.1 to not necessarily dephasing generators $A(t)$ of quantum dynamical semigroups

$$A(t)\rho := Z_0(t)(\rho) + \sum_{j \in J} B_j(t)\rho B_j(t)^* - 1/2\{B_j(t)^*B_j(t), \rho\} \qquad (\rho \in D(Z_0(t))) \quad (4.76)$$
$$e^{Z_0(t)\tau}(\rho) := e^{-iH(t)\tau}\rho\, e^{iH(t)\tau}$$

with not necessarily bounded self-adjoint operators $H(t)$ (Corollary 3.6.2), thereby extending a result of Avron, Fraas, Graf, Grech from [12] (Theorem 22) for dephasing generators $A(t)$ with bounded $H(t)$.

We cannot, however, expect to apply the adiabatic theorem directly to these operators $A(t)$ on the natural space $X = S^1(\mathfrak{h})$, because its crucial assumption that for almost every t there exist a projection weakly associated with $A(t)$ and $\lambda(t)$ is not satisfied in many cases. In fact, if for example $A := \Lambda$ is dephasing with Λ as in Corollary 3.6.2 and such that

- the self-adjoint operator $H : D(H) \subset \mathfrak{h} \to \mathfrak{h}$ has finite point spectrum $\sigma_p(H)$ and each $\lambda \in \sigma_p(H)$ has finite multiplicity, and
- the base space $\mathfrak{h}$ is infinite-dimensional,

then there exists no projection weakly associated with A and $\lambda = 0$, although the sum $\ker A + \overline{\operatorname{ran}} A$ is closed in X. See also the special case of this example from [12] (Example 6) where $A = Z_0$ and $\ker A = 0$. It should be noticed, however, that a trivial kernel $\ker A = 0$ is not a typical situation in which one would like to apply the adiabatic theorem without spectral gap. (In order to see the claims made above, notice that

$$\ker A = \ker \Lambda = \ker Z_0$$

is finite-dimensional by Proposition 3.6.3 ($A = \Lambda$ is dephasing!) and by Lemma 3.6.4, and that

$$(\overline{\operatorname{ran}} A)^\perp = (\overline{\operatorname{ran}} \Lambda)^\perp = \ker \Lambda^* \supset \ker Z_0^* = \{H\}'$$

is infinite-dimensional by the dephasingness of $A = \Lambda$ and by Lemma 3.6.4. If there existed a projection weakly associated with A and $\lambda = 0$, then

$$S^1(\mathfrak{h}) = X = \ker A \oplus \overline{\operatorname{ran}} A = N \oplus R \qquad (N := \ker A \text{ and } R := \overline{\operatorname{ran}} A)$$

by the same argument as in the first remark after our adiabatic theorem without spectral gap condition (Theorem 4.2.2). So, on the one hand $X/R \cong N$ and hence $(X/R)^*$ would be finite-dimensional, but on the other hand $(X/R)^* \cong R^\perp$ (Theorem III.10.2 of [25]) would be infinite-dimensional. Contradiction! In order to see that the subspace $\ker A + \overline{\operatorname{ran}} A$ is closed, appeal to Proposition III.4.3 of [25].)

We therefore extend, following [12], the operators $A(t)$ of (4.76) from the non-reflexive space $S^1(\mathfrak{h})$ to the reflexive spaces $S^p(\mathfrak{h})$ with $p \in (1, \infty)$, in which one has the criterion for weak associatedness due to [12] mentioned in the second part of Section 3.2.2. In order for such an extension to be possible – and to yield generators of completely positive contraction semigroups on $S^p(\mathfrak{h})$ – we need to require a bit more than in Corollary 3.6.2 (namely (4.77) below), but dephasingness is not necessary.

Lemma 4.2.9. *Suppose that $\tilde{Z}_0$ is the generator of the (weakly and hence strongly continuous) semigroup on $S^p(\mathfrak{h})$ defined by $e^{\tilde{Z}_0 t}(\rho) := e^{-iHt} \rho\, e^{iHt}$, where $p \in (1, \infty)$ and H is self-adjoint on $\mathfrak{h}$, and that B_j are bounded operators on $\mathfrak{h}$ for $j \in J$ satisfying*

$$\sum_{j \in J} B_j^* B_j = \sum_{j \in J} B_j B_j^* < \infty, \tag{4.77}$$

where J is an arbitrary index set. Then the following statements hold true:

(i) the series

$$\sum_{j\in J} B_j^* B_j \rho, \quad \sum_{j\in J} \rho B_j^* B_j, \quad \sum_{j\in J} B_j \rho B_j^*$$

converge in the norm of $S^p(\mathfrak{h})$ for every $\rho \in S^p(\mathfrak{h})$ and define bounded linear operators from $S^p(\mathfrak{h})$ to $S^p(\mathfrak{h})$,

(ii) the operator $A : D(\tilde{Z}_0) \subset S^p(\mathfrak{h}) \to S^p(\mathfrak{h})$ defined by

$$A(\rho) := \tilde{Z}_0(\rho) + \sum_{j\in J} B_j \rho B_j^* - 1/2\{B_j^* B_j, \rho\} \qquad (\rho \in D(\tilde{Z}_0))$$

is the generator of a completely positive contraction semigroup on $S^p(\mathfrak{h})$ and its domain is given by

$$D(\tilde{Z}_0) = \big\{\rho \in S^p(\mathfrak{h}) : \rho D(H) \subset D(H) \text{ and } H\rho - \rho H \subset \sigma \text{ for a } \sigma \in S^p(\mathfrak{h})\big\},$$

where $\tilde{Z}_0(\rho)$ is the unique element σ of $S^p(\mathfrak{h})$ with $-i(H\rho - \rho H) \subset \sigma$,

(iii) there exists a unique projection P weakly associated with A and $\lambda = 0$ provided that $\ker A$ is finite-dimensional in $S^p(\mathfrak{h})$ or that $p = 2$ (with $\ker A$ of arbitrary dimension).

Proof. We begin by noting the following interpolation theorem which will be used in the proof of (i) and (ii). If C is a linear map from $F(\mathfrak{h})$ (finite rank operators) to $S^1(\mathfrak{h})$ such that

$$\|C(\rho)\|_{S^1} \leq M_1 \|\rho\|_{S^1} \quad \text{and} \quad \|C(\rho)\|_{S^\infty} \leq M_\infty \|\rho\|_{S^\infty}$$

for all $\rho \in F(\mathfrak{h})$, then there exists a unique bounded operator $\tilde{C} : S^p(\mathfrak{h}) \to S^p(\mathfrak{h})$ extending C, and additionally one has

$$\big\|\tilde{C}(\rho)\big\|_{S^p} \leq M_1^{1/p} M_\infty^{1-1/p} \|\rho\|_{S^p}$$

for all $\rho \in S^p(\mathfrak{h})$. (We point out that this result, which for separable spaces $\mathfrak{h}$ can be found in Appendix IX.4 of [107] or in [142], is valid for non-separable $\mathfrak{h}$ as well. Indeed,

$$\|\rho\|_{(S^1,S^\infty)_\theta} = \|\rho\|_{S^p} \quad (\theta := 1/p \in (0,1))$$

for all $\rho \in F(\mathfrak{h})$ by the same arguments as those for Theorem 2.2.7 of [142] and therefore $(S^1(\mathfrak{h}), S^\infty(\mathfrak{h}))_\theta = S^p(\mathfrak{h})$ by the density of the embeddings

$$F(\mathfrak{h}) \hookrightarrow S^p(\mathfrak{h}) \quad \text{and} \quad F(\mathfrak{h}) \hookrightarrow S^1(\mathfrak{h}) \cap S^\infty(\mathfrak{h}) \hookrightarrow (S^1(\mathfrak{h}), S^\infty(\mathfrak{h}))_\theta$$

(Exercise 2.1.3 (4) of [85]), where $S^1(\mathfrak{h}) \cap S^\infty(\mathfrak{h})$ is endowed with the usual norm of intersections of interpolation pairs (Introduction of [85]). Apply now the complex interpolation theorem of Calderón–Lions (Theorem 2.1.6 of [85]) to the bounded extension

$C : S^1(\mathfrak{h}) + S^\infty(\mathfrak{h}) \to S^1(\mathfrak{h}) + S^\infty(\mathfrak{h})$ of C, where $S^1(\mathfrak{h}) + S^\infty(\mathfrak{h}) = S^\infty(\mathfrak{h})$ is endowed with the usual norm of sums of interpolation pairs (Introduction of [85]).)

(i) Since by assumption the net $(\sum_{j\in F} B_j^* B_j)$ with F running through all finite subsets of J is bounded, it is strongly convergent by the theorem of Vigier (Theorem 4.1.1 of [90]). So, $(\sum_{j\in F} B_j^* B_j \rho)$ and $(\sum_{j\in F} \rho B_j^* B_j)$ are convergent in the norm of $S^p(\mathfrak{h})$ for all $\rho \in S^p(\mathfrak{h})$, for if (C_F) is a bounded strongly convergent net in $L(\mathfrak{h})$ with strong limit C, then $C_F\rho \longrightarrow C\rho$ and $\rho\, C_F^* \longrightarrow \rho\, C^*$ in the norm of $S^p(\mathfrak{h})$ for every $\rho \in S^p(\mathfrak{h})$. In particular,

$$\Big\| \sum_{j\in J} B_j^* B_j \rho \Big\|_{S^p}, \quad \Big\| \sum_{j\in J} \rho B_j^* B_j \Big\|_{S^p} \le \Big\| \sum_{j\in J} B_j^* B_j \Big\| \, \|\rho\|_{S^p} \qquad (\rho \in S^p(\mathfrak{h})).$$

Since $\sum_{j\in J} B_j^* B_j < \infty$ and $\sum_{j\in J} B_j B_j^* < \infty$ by our assumption (4.77), we see that

$$\Big\| \sum_{j\in F} B_j \rho B_j^* \Big\|_{S^1} \le 2 \Big\| \sum_{j\in J} B_j^* B_j \Big\| \, \|\rho\|_{S^1} \quad \text{and} \quad \Big\| \sum_{j\in F} B_j^* \sigma B_j \Big\|_{S^1} \le 2 \Big\| \sum_{j\in J} B_j B_j^* \Big\| \, \|\sigma\|_{S^1}$$

for all finite subsets F and all $\rho, \sigma \in S^1(\mathfrak{h})$ by decomposing the real and imaginary part of ρ into their positive and negative parts and by then applying (3.23). We also see by duality that

$$\Big\| \sum_{j\in F} B_j \rho B_j^* \Big\|_{S^\infty} \le 2 \Big\| \sum_{j\in J} B_j B_j^* \Big\| \, \|\rho\|_{S^\infty}$$

for all $\rho \in S^\infty(\mathfrak{h})$. So, by the interpolation theorem from the beginning of the proof,

$$\Big\| \sum_{j\in F} B_j \rho B_j^* \Big\|_{S^p} \le 2\, C_0 \, \|\rho\|_{S^p} \qquad (C_0 := \| \sum_{j\in J} B_j^* B_j \| = \| \sum_{j\in J} B_j B_j^* \|) \tag{4.78}$$

for all $\rho \in S^p(\mathfrak{h})$ and all finite subsets F. It now follows that $(\sum_{j\in F} B_j \rho B_j^*)$ is a Cauchy net, hence convergent, in $S^p(\mathfrak{h})$ for all $\rho \in S^p(\mathfrak{h})$ by (3.23) and the dense and continuous embeddedness of $S^1(\mathfrak{h})$ in $S^p(\mathfrak{h})$. In particular, (4.78) yields

$$\Big\| \sum_{j\in J} B_j \rho B_j^* \Big\|_{S^p} \le 2 \Big\| \sum_{j\in J} B_j^* B_j \Big\| \, \|\rho\|_{S^p} \qquad (\rho \in S^p(\mathfrak{h})).$$

(ii) Since $\sum_{j\in J} B_j^* B_j = \sum_{j\in J} B_j B_j^* < \infty$ by our assumption (4.77), we see that

$$W = (W')^*|_{S^1(\mathfrak{h})}, \tag{4.79}$$

where $W(\rho) := \sum_{j\in J} B_j \rho B_j^* - 1/2\{B_j^* B_j, \rho\}$ and $W'(\rho) := \sum_{j\in J} B_j^* \rho B_j - 1/2\{B_j B_j^*, \rho\}$ for $\rho \in S^1(\mathfrak{h})$. We also see from Theorem 3.6.1 (with $H = 0$) that

$$\big\| e^{Wt}(\rho) \big\|_{S^1} \le \|\rho\|_{S^1} \qquad \text{and} \qquad \big\| e^{W't}(\sigma) \big\|_{S^1} \le \|\sigma\|_{S^1}$$

for all $\rho, \sigma \in S^1(\mathfrak{h})$ and $t \in [0, \infty)$, whence by (4.79) and duality

$$\big\| e^{Wt}(\rho) \big\|_{S^\infty} = \big\| (e^{W't})^*(\rho) \big\|_{S^\infty} \le \|\rho\|_{S^\infty}$$

for all $\rho \in F(\mathfrak{h})$. So, by the interpolation theorem from the beginning of the proof, the unique bounded extension $\tilde{}(e^{Wt}) : S^p(\mathfrak{h}) \to S^p(\mathfrak{h})$ of e^{Wt} is a contraction and by (i) is given by $\tilde{}(e^{Wt}) = e^{\tilde{W}t}$ with

$$\tilde{W}(\rho) := \sum_{j \in J} B_j \rho B_j^* - 1/2\{B_j^* B_j, \rho\} \qquad (\rho \in S^p(\mathfrak{h})). \tag{4.80}$$

With the help of the theorem of Lie–Trotter, it now follows that $A = \tilde{Z}_0 + \tilde{W}$ generates a contraction semigroup on $S^p(\mathfrak{h})$, and with the same arguments as for Theorem 3.6.1 and Corollary 3.6.2, it follows that e^{At} for every $t \in [0, \infty)$ is a completely positive operator from $S^p(\mathfrak{h})$ to $L(\mathfrak{h})$ in the sense indicated after the definition of quantum dynamical semigroups from Section 3.6. And finally, the explicit description of the domain and the action of $\tilde{Z}_0$ can be proved in the same way as Lemma 5.5.1 of [29].

(iii) Suppose first that $p \in (1, \infty)$ and $\ker A$ is finite-dimensional in $X = S^p(\mathfrak{h})$. We then see that $\ker A + \overline{\operatorname{ran}} A$ is closed in X (Proposition III.4.3 of [25]) and hence, by Section 3.2.2 and the reflexivity of $S^p(\mathfrak{h})$, that there exists a unique projection P weakly associated with A and $\lambda = 0$.

Suppose now that $p = 2$. We then show in a similar way as in [12] that $\ker A$ is orthogonal to $\operatorname{ran} A$ w.r.t. the scalar product of $S^2(\mathfrak{h})$. It then follows that $\ker A + \overline{\operatorname{ran}} A$ is closed in X and hence, by Section 3.2.2 and the reflexivity of $S^2(\mathfrak{h})$, that there exists a unique projection P weakly associated with A and $\lambda = 0$.

As a first step, we show that every $a \in \ker A$ commutes with B_j, B_j^* for every $j \in J$. So let $a \in \ker A$, then $a, a^* \in D(\tilde{Z}_0)$ and $a^* \in \ker A$ because

$$\tilde{Z}_0(a^*) = \tilde{Z}_0(a)^* \quad \text{and} \quad \tilde{W}(a^*) = \tilde{W}(a)^*.$$

Also, $a^* a \in D(Z_0) = D(\Lambda)$ (where $\Lambda := A|_{S^1(\mathfrak{h})}$) and

$$Z_0(a^* a) = \tilde{Z}_0(a^*) a + a^* \tilde{Z}_0(a)$$

because $e^{Z_0 t}(a^* a) = e^{\tilde{Z}_0 t}(a)^* e^{\tilde{Z}_0 t}(a)$. Consequently,

$$\Lambda(a^* a) = \Lambda(a^* a) - A(a^*) a - a^* A(a) = W(a^* a) - \tilde{W}(a^*) a - a^* \tilde{W}(a) \tag{4.81}$$

and by a straightforward computation [82], [12] this is equal to

$$\begin{aligned} \sum_{j \in J} B_j a^* a B_j^* - B_j a^* B_j^* a + a^* B_j^* B_j a - a^* B_j a B_j^* &= \sum_{j \in J} B_j a^* [a, B_j^*] + a^* B_j [B_j^*, a] \\ &= \sum_{j \in J} [a, B_j^*]^* [a, B_j^*]. \end{aligned} \tag{4.82}$$

In the first equality, the assumption $\sum_{j \in J} B_j^* B_j = \sum_{j \in J} B_j B_j^*$ was used again, and all the series converge in $S^1(\mathfrak{h})$ (for first term on the left-hand side, this follows by the remark preceding Theorem 3.6.1 and for the other three terms on the left-hand side this follows by (i) and the Hölder inequality $\|c\,d\|_{S^1} \leq \|c\|_{S^2}\,\|d\|_{S^2}$). Since $(e^{\Lambda t})$ as a quantum dynamical semigroup is trace-preserving, it follows that

$$0 = \operatorname{tr}(\Lambda(a^*a)) = \operatorname{tr}\Big(\sum_{j\in J}[a,B_j^*]^*[a,B_j^*]\Big) = \sum_{j\in J}\operatorname{tr}([a,B_j^*]^*[a,B_j^*])$$

So, $a \in \ker A$ commutes with all B_j^* and, since $a^* \in \ker A$ as well, a^* commutes with all B_j^* by the same argument, which concludes the proof of the first step.

As a second step, we show that every $a \in \ker A$ commutes with H or, in other words, that $\ker A \subset \ker \tilde{Z}_0$. So let $a \in \ker A$, then a commutes with B_j and B_j^* for all $j \in J$ by the first step and therefore

$$\tilde{W}(a) = \sum_{j\in J} B_j a B_j^* - 1/2\{B_j B_j^*, a\} = 0$$

(note that in the first equality, the assumption (4.77) was used once again). So, $\tilde{Z}_0(a) = A(a) - \tilde{W}(a) = 0$, as desired.

As a third and last step, we can finally show that $\ker A$ is orthogonal to $\operatorname{ran} A$. Indeed, if $a \in \ker A$ and $b = A(b_0) \in \operatorname{ran} A$, then

$$\langle a, A(b_0)\rangle = \operatorname{tr}(a^*A(b_0)) = \operatorname{tr}(a^*\tilde{Z}_0(b_0)) + \operatorname{tr}(a^*\tilde{W}(b_0)).$$

Since $a^* \in \ker A$, a^* commutes with all B_j, B_j^* by the first step and hence $a^*\tilde{W}(b_0) = \tilde{W}(a^*b_0) = W(a^*b_0)$; moreover, a^* commutes with H by the second step and hence $a^*b_0 \in D(Z_0) = D(\Lambda)$ with $Z_0(a^*b_0) = a^*\tilde{Z}_0(b_0)$. Consequently,

$$\langle a, A(b_0)\rangle = \operatorname{tr}(Z_0(a^*b_0) + W(a^*b_0)) = \operatorname{tr}(\Lambda(a^*b_0)) = 0$$

because $(e^{\Lambda t})$ is trace-preserving, as desired. ■

We make some remarks which, among other things, give sufficient conditions for some of the assumptions of the above lemma.

1. Clearly, a sufficient condition for the equality in (4.77) to hold is the normality of all the operators B_j, which one has, for instance, if $\Lambda = A|_{S^1(\mathfrak{h})}$ is dephasing (Proposition 3.6.3) or if $\Lambda = A|_{S^1(\mathfrak{h})}$ is of the simple, clearly non-dephasing, form discussed at the end of Section 5 of [82] (Gauß and Poisson semigroup generators).

2. If $p \in (1,2]$ in the above lemma, then $\ker A \subset \ker \tilde{Z}_0$. (Indeed, this follows by inspecting the first and second step of the proof of part (iii) of the lemma above.) Combined with Lemma 3.6.4 (i) (which obviously carries over to $S^p(\mathfrak{h})$) this yields a simple sufficient condition for the finite-dimensionality of $\ker A$ in $S^p(\mathfrak{h})$, namely: if A is as in the lemma above with $p \in (1,2]$ and if the self-adjoint operator H has finite point spectrum $\sigma_p(H)$ and each $\lambda \in \sigma_p(H)$ has finite multiplicity, then $\ker A$ is finite-dimensional in $S^p(\mathfrak{h})$.

3. If $p = 2$ and $\ker A = \ker \tilde{Z}_0$, then the projection P weakly associated with A and $\lambda = 0$ is orthogonal (by the orthogonality of the subspaces $\ker A$ and $\overline{\operatorname{ran}} A$ in $S^2(\mathfrak{h})$ just proved in the lemma above) and hence, by $\ker \tilde{Z}_0 = \{H\}' \cap S^2(\mathfrak{h})$, is given explicitly as

$$P\rho = \sum_{\mu \in \sigma_p(H)} Q^H_{\{\mu\}} \rho \, Q^H_{\{\mu\}} \qquad (\rho \in S^2(\mathfrak{h})),$$

where Q^H denotes the spectral measure of H (Theorem 5.8 of [130] or [26]). (See the discussion at the very end of [12].) A sufficient, but not necessary, condition for the equality $\ker A = \ker \tilde{Z}_0$ in $S^2(\mathfrak{h})$ is the dephasingness of $\Lambda = A|_{S^1(\mathfrak{h})}$. See the example below.

With the above lemma at hand, we can now apply the adiabatic theorem without spectral gap condition to operators $A(t)$ of the form

$$A(t)\rho := \tilde{Z}_0(t)(\rho) + \sum_{j \in J} B_j(t)\rho B_j(t)^* - 1/2\{B_j(t)^* B_j(t), \rho\} \qquad (\rho \in D(\tilde{Z}_0(t))) \quad (4.83)$$

$$e^{\tilde{Z}_0(t)\tau}(\rho) := e^{-iH(t)\tau} \rho \, e^{iH(t)\tau}$$

on the reflexive space $X = S^p(\mathfrak{h})$ $(p \in (1, \infty))$ with self-adjoint operators $H(t)$ on $\mathfrak{h}$ and bounded operators $B_j(t)$ on $\mathfrak{h}$ satisfying

$$\sum_{j \in J} B_j(t)^* B_j(t) = \sum_{j \in J} B_j(t) B_j(t)^* < \infty \qquad (t \in I). \quad (4.84)$$

Theorem 4.2.10. *Suppose that the operators $A(t)$ defined above have time-independent domain $D(\tilde{Z}_0(t)) = D$ and that $t \mapsto A(t)$ is in $W^{1,1}_*(I, L(Y, X))$, where Y is the space D endowed with the graph norm of $A(0)$. Suppose further that $\lambda(t) = 0$ is an eigenvalue of $A(t)$ for every $t \in I$ and, finally, that either*

(i) $\ker A(t)$ is finite-dimensional for almost every $t \in I$ or (ii) $p = 2$,

and that there is a null set in I such that the projections $P(t)$ weakly associated with $A(t)$ and $\lambda(t)$ for t outside that null set can be extended to a continuously differentiable map $t \mapsto P(t)$ on the whole of I. Then

$$\sup_{t \in I} \|(U_\varepsilon(t) - V_{0\,\varepsilon}(t))P(0)\rho\| \longrightarrow 0 \quad (\varepsilon \searrow 0)$$

for every $\rho \in X = S^p(\mathfrak{h})$, where $V_{0\,\varepsilon} = W$ is the evolution system for $\frac{1}{\varepsilon}AP + [P', P] = [P', P]$ on X.

Proof. We have only to notice that $A(t)$ generates a contraction semigroup in X for every $t \in I$ (Lemma 4.2.9 (ii)), that the projections $P(t)$ weakly associated with $A(t)$ and $\lambda(t)$ really exist for almost every $t \in I$ (Lemma 4.2.9 (iii)), and then to apply the second remark after Theorem 4.2.2. ■

Clearly, the above theorem is a generalization of the respective result (Theorem 22) from [12] for dephasing generators $A(t)$ of quantum dynamical semigroups on $X = S^2(\mathfrak{h})$ with bounded $H(t)$. Interestingly, these types of generators are normal operators on $S^2(\mathfrak{h})$, that is, $A(t)^*A(t) = A(t)A(t)^*$ (as can be easily verified using Proposition 3.6.3 (i)). In particular, the adiabatic theorem for normal operators would be sufficient for the result from [12].

We conclude with a simple example of generators $A(t)$ in $X = S^2(\mathfrak{h})$ satisfying the assumptions of the above theorem without being dephasing (or normal).

Example 4.2.11. We choose a self-adjoint operator H and a normal operator B on an infinite-dimensional Hilbert space $\mathfrak{h}$ in the same way as in Example 3.6.5 and, in addition, we take H to be bounded. We then define $A(t)$ for every $t \in I$ on $X := S^2(\mathfrak{h})$ through

$$A(t)\rho := \tilde{Z}_0(t)(\rho) + B(t)\rho B(t)^* - 1/2\{B(t)^*B(t), \rho\} \qquad (\rho \in S^2(\mathfrak{h}))$$
$$\tilde{Z}_0(t)(\rho) := -i[H(t), \rho],$$

where $H(t) := R(t)^{-1}HR(t)$ and $B(t) := R(t)^{-1}BR(t)$ with $R(t) := e^{iCt}$ and C a bounded self-adjoint operator on $\mathfrak{h}$. Clearly, $A(t)$ for every $t \in I$ is of the form considered in the theorem above and $D(A(t)) = X$ is time-independent while $t \mapsto A(t)$ is $W^{1,1}_*$-regular. It is also clear that

$$\ker A(t) \subset \ker \tilde{Z}_0(t) \quad \text{and} \quad \ker \tilde{Z}_0(t) \subset \ker A(t)$$

by the second remark after Lemma 4.2.9 and by the choice of H and B, respectively. (Indeed, for the second inclusion use the explicit description of $\ker \tilde{Z}_0(t)$ from the S^2-version of Lemma 3.6.4 (i) and the fact that every eigenvector of H with eigenvalue μ is an eigenvector of B with eigenvalue β_μ.) So, by the third remark after Lemma 4.2.9, the projection $P(t)$ weakly associated with $A(t)$ and $\lambda(t) = 0$ is explicitly given by

$$P(t)\rho = \sum_{\mu \in \sigma_p(H(t))} Q^{H(t)}_{\{\mu\}} \rho \, Q^{H(t)}_{\{\mu\}} = \sum_{\mu \in \sigma_p(H)} R(t)^{-1} Q^H_{\{\mu\}} R(t) \, \rho \, R(t)^{-1} Q^H_{\{\mu\}} R(t)$$

for every $t \in I$, where $Q^{H(t)}$ and Q^H denote the spectral measures of $H(t)$ and H. In particular, $t \mapsto P(t)$ is continuously differentiable. So all the assumptions of the above theorem are satisfied, but $A(t)$ is non-dephasing for every t because

$$H(t)B(t) \neq B(t)H(t)$$

by Example 3.6.5. Incidentally, $A(t) = \tilde{Z}_0(t) + \tilde{W}(t)$ is also non-normal on X for every t because $\tilde{Z}_0(t)$ is skew self-adjoint and $\tilde{W}(t)$ is self-adjoint, but $\tilde{Z}_0(t)$ does not commute with $\tilde{W}(t)$. (Indeed, for $\rho := \langle \psi(t), \,.\, \rangle \, \psi(t)$ with $\psi(t) := R(t)^{-1}\psi$ one computes

$$\tilde{Z}_0(t)\big(\tilde{W}(t)(\rho)\big) = 0 \neq \tilde{W}(t)\big(\tilde{Z}_0(t)(\rho)\big),$$

where the last inequality can be seen as follows: apply $\tilde{W}(t)\big(\tilde{Z}_0(t)(\rho)\big)$ to the vector $\eta := -\langle \psi(t), H(t)\psi(t) \rangle \, \psi(t) + H(t)\psi(t)$ and notice that $\langle \psi(t), \eta \rangle = 0$ and $\eta \in \ker B(t) = \ker B(t)^*$ and $\langle H(t)\psi(t), \eta \rangle \neq 0$ because $H(t)\psi(t)$ and $\psi(t)$ are linearly independent). ◀

5 Adiabatic theorems for operators with time-dependent domains

5.1 Adiabatic theorems for general operators with time-dependent domains

In this section we extend the adiabatic theorems (with and without spectral gap condition) for time-independent domains of Section 4.1 and 4.2 to the case of operators $A(t)$ with time-dependent domains – by slightly modifying the proofs of the respective adiabatic theorems for time-independent domains. Striving for such an extension is fairly natural because the requirement of constant domains is rather restrictive – just think of differential operators $A(t)$ with (fully) time-dependent boundary conditions. We will see in Section 5.2 that the adiabatic theorems of this section allow one to almost effortlessly derive adiabatic theorems for operators $A(t) = iA_{a(t)}$ defined by symmetric sesquilinear forms $a(t)$. All the theorems of this section are generalizations of the respective adiabatic theorems for time-independent domains if in these latter theorems all $W_*^{n,1}$-regularity requirements are strengthened to n times strong continuous differentiability requirements. We will need the following very natural condition on A, which will take the role of Condition 2.1.8.

Condition 5.1.1. *$A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is a densely defined closed linear map such that, for every $\varepsilon \in (0, \infty)$, there is an evolution system U_ε for $\frac{1}{\varepsilon}A$ on $D(A(t))$ and there is a constant $M \in [1, \infty)$ such that $\|U_\varepsilon(t, s)\| \le M$ for all $(s, t) \in \Delta$ and $\varepsilon \in (0, \infty)$.*

We point out that there is a huge number of papers establishing the existence of evolution systems U for a given family A of linear maps $A(t)$ on $D(A(t))$ as, for instance, [63], [71], [124], [64], [44], [4]. See the survey article [118] for many more references. Instead of working with evolution systems on the spaces $Y_t = D(A(t))$ as in Condition 5.1.1, one could also prove adiabatic theorems operating – as in [65] or [66] – with evolution systems for A on certain subspaces Y of the intersection of all $D(A(t))$, but then one would have to impose various invariance conditions on the subspace Y, such as the $A(t)$-admissibiltity of Y, the invariance

$$(z - A(t))^{-1} Y \subset Y \tag{5.1}$$

for $z \in \operatorname{ran} \gamma_t$ (case with spectral gap) or $z \in \{\lambda(t) + \varepsilon e^{i\vartheta(t)} : \varepsilon \in (0, \varepsilon_0]\}$ (case without spectral gap), and the invariance of Y under $P(t)$ and $P'(t)$. Such invariance conditions, however, are difficult to verify in practice: (5.1), for instance, is clear only for complex numbers z with sufficiently large positive real part (Proposition 2.3 of [65]).

5.1.1 Adiabatic theorems with spectral gap condition

We will need the following condition depending on $m \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ (number of points at which $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$).

Condition 5.1.2. *$A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is a linear map such that Condition 5.1.1 is satisfied. $\sigma(t)$ for every $t \in I$ is a compact subset of $\sigma(A(t))$, $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ at exactly m points that accumulate at only finitely many points, and $I \setminus N \ni t \mapsto \sigma(t)$ is continuous, where N denotes the set of those m points at which $\sigma(\,.\,)$ falls into $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$.*

$$J_{t_0} \ni t \mapsto (z - A(t))^{-1} \text{ is SOT-continuously differentiable for all } z \in \operatorname{ran}\gamma_{t_0},$$
$$\operatorname{ran}\gamma_{t_0} \ni z \mapsto \frac{d}{dt}(z - A(t))^{-1} \text{ is SOT-continuous for all } t \in J_{t_0},$$
$$\sup_{(t,z) \in J_{t_0} \times \operatorname{ran}\gamma_{t_0}} \left\| \frac{d}{dt}(z - A(t))^{-1} \right\| < \infty$$

for every $t_0 \in I \setminus N$, where the cycle γ_{t_0} and the non-trivial closed interval $J_{t_0} \ni t_0$ are chosen such that $\operatorname{ran}\gamma_{t_0} \subset \rho(A(t))$ and $\operatorname{n}(\gamma_{t_0}, \sigma(t)) = 1$ and $\operatorname{n}(\gamma_{t_0}, \sigma(A(t)) \setminus \sigma(t)) = 0$ for every $t \in J_{t_0}$. And finally, $P(t)$ is the projection associated with $A(t)$ and $\sigma(t)$ for every $t \in I \setminus N$ and $I \setminus N \ni t \mapsto P(t)$ extends to a twice SOT-continuously differentiable map on the whole of I.

With this condition at hand, we can now formulate an adiabatic theorem with uniform spectral gap condition ($m = 0$) and non-uniform spectral gap condition ($m \in \mathbb{N} \cup \{\infty\}$) for time-dependent domains.

Theorem 5.1.3. *Suppose $A(t)$, $\sigma(t)$, $P(t)$ for $t \in I$ are such that Condition 5.1.2 is satisfied with $m = 0$ or $m \in \mathbb{N} \cup \{\infty\}$, respectively. Then*

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \text{ resp. } o(1) \quad (\varepsilon \searrow 0),$$

whenever the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ exists on $D(A(t))$ for all $\varepsilon \in (0, \infty)$.

Proof. We have only to prove the theorem in the case of a uniform spectral gap ($m = 0$), since the theorem in the case of a non-uniform spectral gap ($m \in \mathbb{N} \cup \{\infty\}$) then follows in the same way as Theorem 4.1.2 followed from Theorem 4.1.1. In order to do so, we must only slightly modify the proof of Theorem 4.1.1. We define the operators $B(t)$ as in the proof of that theorem (where now γ_{t_0} and J_{t_0} are given by Condition 5.1.2), take over the first preparatory step of that proof, and easily show – instead of what has been shown in the second preparatory step – that $t \mapsto B(t)$ is SOT-continuously differentiable. (It has to be used for this last statement that Condition 5.1.2 implies

$$\sup_{(t,z) \in J_{t_0} \times \operatorname{ran}\gamma_{t_0}} \left\| (z - A(t))^{-1} \right\| < \infty$$

which can be seen as in the proof of (5.6) below.) We can then almost literally take over the main part of the proof of Theorem 4.1.1: the only thing that has to be changed is that the $W^{1,\infty}$-regularity of

$$[0,t] \ni s \mapsto U_\varepsilon(t,s)B(s)V_\varepsilon(s)x$$

for $x \in D(A(0))$ can no longer be deduced from Lemma 2.1.2, but has to be inferred from Lemma 2.1.3 and Lemma 2.1.4, and that Proposition 2.1.13 has to be invoked for an ε-independent bound on V_ε. ■

In general, the existence of the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ on $D(A(t))$ does not seem to be guaranteed under – the fairly general – Condition 5.1.2. (In view of Proposition 2.1.13 one would, of course, like to define V_ε as a perturbation series and show that $[s,1] \ni t \mapsto V_\varepsilon(t,s)y$ for every $y \in D(A(s))$ is a continuously differentiable solution to the initial value problem $x' = \frac{1}{\varepsilon}A(t)x + [P'(t), P(t)]x$, $x(s) = y$, but this is not clear in general.) It is therefore good to know that under Condition 5.1.2 with $m = 0$ one has at least the following statement:

$$\sup_{t\in I} \|(1-P(t))U_\varepsilon(t)P(0)\|, \quad \sup_{t\in I} \|P(t)U_\varepsilon(t)(1-P(0))\| = O(\varepsilon) \tag{5.2}$$

as $\varepsilon \searrow 0$, which follows from the adiabatic theorem of higher order (Theorem 5.1.7 (i) and (iii) with degree of regularity $n = 1$) below. It should be pointed out, however, that Theorem 5.1.3 itself – operating with the evolution systems for $\frac{1}{\varepsilon}A + [P', P] = \frac{1}{\varepsilon}A_{0\,\varepsilon} + K_{0\,\varepsilon} \neq \frac{1}{\varepsilon}A_{1\,\varepsilon} + K_{1\,\varepsilon}$ as comparison evolutions – is not contained in Theorem 5.1.7.

5.1.2 Adiabatic theorems without spectral gap condition

We now prove an adiabatic theorem without spectral gap condition for time-dependent domains in which case we have to explicitly require the differentiability of the resolvent as well as an estimate on the derivative of the resolvent, which two things are no longer automatically satisfied as they were in the case of time-independent domains.

Theorem 5.1.4. *Suppose $A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is a linear map such that Condition 5.1.1 is satisfied. Suppose further that $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$, and that there are numbers $\delta_0 \in (0,\infty)$ and $\vartheta(t) \in \mathbb{R}$ such that $\lambda(t) + \delta e^{i\vartheta(t)} \in \rho(A(t))$ for all $\delta \in (0, \delta_0]$ and $t \in I$ and such that $t \mapsto \lambda(t)$ and $t \mapsto e^{i\vartheta(t)}$ are continuously differentiable and $t \mapsto \big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1}$ is SOT-continuously differentiable. Suppose finally that $P(t)$ for every $t \in I$ is a bounded projection in X commuting with $A(t)$ such that $P(t)$ for almost every $t \in I$ is weakly associated with $A(t)$ and $\lambda(t)$ and that*

$$P(t)X \subset \ker(A(t) - \lambda(t))^{m_0}$$

for every $t \in I$ (and some $m_0 \in \mathbb{N}$). Additionally, suppose that there are $M_0, M_0' \in (0, \infty)$ such that

$$\Big\| \big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1}(1 - P(t))\Big\| \le \frac{M_0}{\delta},$$

$$\Big\| \frac{d}{dt}\Big(\big(\lambda(t) + \delta e^{i\vartheta(t)} - A(t)\big)^{-1}(1 - P(t))\Big)\Big\| \le \frac{M_0'}{\delta^{m_0+1}}$$

for all $\delta \in (0, \delta_0]$ and $t \in I$, let $\operatorname{rk} P(0) < \infty$ and let $t \mapsto P(t)$ be SOT-continuously differentiable. Then

$$\sup_{t \in I} \big\| \big(U_\varepsilon(t) - V_{0\,\varepsilon}(t)\big)P(0)\big\| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

where $V_{0\,\varepsilon}$ for every $\varepsilon \in (0, \infty)$ denotes the evolution system for $\frac{1}{\varepsilon}AP + [P', P]$ on X. If, in addition, X is reflexive and $t \mapsto P(t)$ norm continuously differentiable, then

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0),$$

whenever the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ exists on $D(A(t))$ for every $\varepsilon \in (0, \infty)$.

Proof. We have only to slightly modify the proof of the respective theorem for operators with time-independent domains (Theorem 4.2.2) and we begin by describing the necessary modifications in the preparatory steps of that proof: the first preparatory step can be dropped, while the second remains valid without change because the relations (4.31) (which are the only facts from the first preparatory step that are used in the second) now hold true for every $t \in I$ by assumption. In particular, $B_{n\,\boldsymbol{\delta}}$ and $C_{n\,\boldsymbol{\delta}}$ are defined in exactly the same way as before. Instead of the third preparatory step, we now show that $t \mapsto B_{n\,\boldsymbol{\delta}}(t)$ is SOT-continuously differentiable and estimate $B_{n\,\boldsymbol{\delta}}$ and $B_{n\,\boldsymbol{\delta}}'$ accordingly. Indeed, since $P(t)X \subset \ker(A(t) - \lambda(t))^{m_0} \subset D(A(t)^{m_0})$ for every $t \in I$, we see that $(A(t) - \lambda(t))P(t)$ is a bounded linear map in X for every $t \in I$ and, by the binomial formula,

$$\begin{aligned} t \mapsto (A(t) - \lambda(t))P(t) &= (A(t) - \lambda(t))S_\delta(t)\; S_\delta(t)^{m_0 - 1}\big(A(t) - \lambda(t) - \delta e^{i\vartheta(t)}\big)^{m_0} P(t) \\ &= \big(1 + \delta e^{i\vartheta(t)} S_\delta(t)\big) \sum_{k=0}^{m_0 - 1} \binom{m_0}{k} \big(-\delta e^{i\vartheta(t)}\big)^{m_0 - k} \cdot \\ &\qquad \cdot S_\delta(t)^{m_0 - 1 - k}\, \big(1 + \delta e^{i\vartheta(t)} S_\delta(t)\big)^k P(t) \end{aligned} \tag{5.3}$$

is SOT-continuously differentiable by the assumed SOT-continuous differentiability of $t \mapsto S_\delta(t) := \big(A(t) - \lambda(t) - \delta e^{i\vartheta(t)}\big)^{-1}$. So,

$$t \mapsto (A(t) - \lambda(t))^k P(t) = ((A(t) - \lambda(t))P(t))^k \tag{5.4}$$

for $k \in \{1, \dots, m_0\}$ is SOT-continuously differentiable as well and the desired SOT-continuous differentiability of $t \mapsto B_{n\,\boldsymbol{\delta}}(t)$ follows. It also follows from this and the assumed resolvent estimates that the estimate (4.36) for $B_{n\,\boldsymbol{\delta}}$ remains true and that the estimate (4.38) for $B'_{n\,\boldsymbol{\delta}}$ can be improved to

$$\sup_{t\in I} \left\| B'_{n\,\boldsymbol{\delta}}(t) \right\| \leq \sum_{k=1}^{m_0} c_n \Big(\prod_{i=1}^{k} \delta_i \Big)^{-(m_0+1)}. \tag{5.5}$$

Since by the SOT-continuous differentiability of (5.4) also the map $t \mapsto A(t)P(t)$ is SOT-continuously differentiable, the fourth preparatory step remains true as an inspection of the proof of that step shows (in particular, only the existence and boundedness of U_ε are used there).

In the main part of the proof of the respective theorem for operators with time-independent domains, only one thing has to be changed, namely: the fact that

$$[0,t] \ni s \mapsto U_\varepsilon(t,s)B_{n\,\boldsymbol{\delta}}(s)V_{0\,\varepsilon}(s)P(0)x \quad \text{resp.} \quad [0,t] \ni s \mapsto U_\varepsilon(t,s)B_{n\,\boldsymbol{\delta}}(s)V_\varepsilon(s)x$$

is the continuous representative of an element in $W^{1,1}([0,t],X)$ for all $x \in X$ resp. all $x \in D(A(0))$ can no longer be deduced from Lemma 2.1.2 but has to be inferred from Lemma 2.1.3 and Lemma 2.1.4 (using that $s \mapsto B_{n\,\boldsymbol{\delta}}(s)$ is SOT-continuously differentiable with $B_{n\,\boldsymbol{\delta}}(s)X \subset D(A(s))$ for every $s \in I$ and that the SOT-derivative is bounded by virtue of (5.5)). ■

Similarly, one sees that the variants of the adiabatic theorem without spectral gap condition of Section 4.2 carry over to the case of time-dependent domains as well, provided their hypotheses are adapted in a similar way as above.

5.1.3 An adiabatic theorem of higher order

In this subsection we extend the adiabatic theorem of higher order of Joye and Pfister from [59] to the case of general operetors $A(t)$ with possibly time-dependent domains – mainly for the sake of completeness and in order to make clear the relation to the basic adiabatic theorem with spectral gap (Theorem 5.1.3). We will use the elegant iterative scheme of [59] which we briefly recall (in a slightly modified form).

Suppose $A(t) : D(A(t)) \subset X \to X$ is a densely defined closed linear map and γ_t is a cycle in $\mathbb{C}$ for every $t \in J$, where J is a compact interval, and let $\varepsilon \in (0,\infty)$ and $n \in \mathbb{N}$. Then $A_{0\,\varepsilon}$, $P_{0\,\varepsilon}$, $K_{0\,\varepsilon}$ are called *well-defined w.r.t.* γ_t *(*$t \in J$*)* if and only if $\operatorname{ran}\gamma_t \subset \rho(A_{0\,\varepsilon}(t))$ for all $t \in J$, where $A_{0\,\varepsilon}(t) := A(t)$, and $J \ni t \mapsto P_{0\,\varepsilon}(t)$ is WOT-continuously differentiable, where

$$P_{0\,\varepsilon}(t) := \frac{1}{2\pi i} \int_{\gamma_t} (z - A_{0\,\varepsilon}(t))^{-1}\, dz.$$

In this case $K_{0\,\varepsilon}$ is defined by $K_{0\,\varepsilon}(t) := [P'_{0\,\varepsilon}(t), P_{0\,\varepsilon}(t)]$. And, for general $n \in \mathbb{N}$, $A_{n\,\varepsilon}$, $P_{n\,\varepsilon}$, $K_{n\,\varepsilon}$ are called *well-defined w.r.t.* γ_t *(*$t \in J$*)* if and only if $A_{n-1\,\varepsilon}$, $P_{n-1\,\varepsilon}$, $K_{n-1\,\varepsilon}$

are well-defined w.r.t. γ_t $(t \in J)$, $\operatorname{ran}\gamma_t \subset \rho(A_{n\,\varepsilon}(t))$ for all $t \in J$, where $A_{n\,\varepsilon}(t) := A(t) - \varepsilon K_{n-1\,\varepsilon}(t)$, and $J \ni t \mapsto P_{n\,\varepsilon}(t)$ is WOT-continuously differentiable, where

$$P_{n\,\varepsilon}(t) := \frac{1}{2\pi i}\int_{\gamma_t} (z - A_{n\,\varepsilon}(t))^{-1}\,dz.$$

In this case $K_{n\,\varepsilon}$ is defined by $K_{n\,\varepsilon}(t) := [P'_{n\,\varepsilon}(t), P_{n\,\varepsilon}(t)]$.

We will need the following conditions depending on $n \in \mathbb{N} \cup \{\infty\}$ (degree of regularity) in the adiabatic theorem of higher order below.

Condition 5.1.5. *$A(t) : D(A(t)) \subset X \to X$ for every $t \in I$ is a densely defined closed linear map. $\sigma(t)$ for every $t \in I$ is a compact and isolated subset of $\sigma(A(t))$, there is an $r_0 > 0$ such that $U_{r_0}(\sigma(t)) \setminus \sigma(t) \subset \rho(A(t))$ for all $t \in I$, and $t \mapsto \sigma(t)$ is continuous. For every $t_0 \in I$, there are positive constants a_{t_0}, b_{t_0}, c_{t_0} such that*

$$J_{t_0} \ni t \mapsto (z - A(t))^{-1} \text{ is } n \text{ times WOT-continuously differentiable for all } z \in \operatorname{ran}\gamma_{t_0},$$

$$\operatorname{ran}\gamma_{t_0} \ni z \mapsto \frac{d^l}{dt^l}(z - A(t))^{-1} \text{ is SOT-continuous for all } t \in J_{t_0}, l \in \{1, \dots, n\},$$

$$\sup_{(t,z) \in J_{t_0} \times \operatorname{ran}\gamma_{t_0}} \left\| \frac{d^l}{dt^l}(z - A(t))^{-1} \right\| \le a_{t_0} c_{t_0}^l \frac{l!}{(1+l)^2} \text{ for all } l \in \{1, \dots, n\},$$

where γ_{t_0} is a cycle in $\overline{U}_{\frac{4r_0}{7}}(\sigma(t_0)) \setminus U_{\frac{3r_0}{7}}(\sigma(t_0))$ with

$$\mathrm{n}\left(\gamma_{t_0}, U_{\frac{3r_0}{7}}(\sigma(t_0))\right) = 1 \quad \text{and} \quad \mathrm{n}\left(\gamma_{t_0}, \mathbb{C} \setminus \overline{U}_{\frac{4r_0}{7}}(\sigma(t_0))\right) = 0$$

and where $J_{t_0} \subset I$ is a non-trivial closed interval containing t_0 such that $\sigma(t) \subset U_{\frac{r_0}{7}}(\sigma(t_0))$ and $\sigma(t_0) \subset U_{\frac{r_0}{7}}(\sigma(t))$ for all $t \in J_{t_0}$. And finally, $P(t)$ for every $t \in I$ is the projection associated with $A(t)$ and $\sigma(t)$, $t \mapsto P(t)$ is $n+1$ times WOT-continuously differentiable and

$$\sup_{t \in J_{t_0}} \left\| \frac{d^l}{dt^l}[P'(t), P(t)] \right\| \le b_{t_0} c_{t_0}^l \frac{l!}{(1+l)^2} \text{ for all } l \in \{0, 1, \dots, n\} \text{ and } t_0 \in I.$$

In the special case of time-independent domains $D(A(t)) = D$, one easily sees – using the remark before Lemma 2.1.2 – that the requirements on the resolvent of $A(t)$ in Condition 5.1.5 are fulfilled for an $n \in \mathbb{N}$ if, for instance, $t \mapsto A(t)x$ is n times weakly continuously differentiable for all $x \in D$. And they are fulfilled for $n = \infty$ if, for instance, there is an open neighbourhood U_I of I in $\mathbb{C}$ such that, for every $x \in D$, $t \mapsto A(t)x$ extends to a holomorphic map on U_I (Cauchy inequalities!).

Lemma 5.1.6 (Joye–Pfister). *(i) Suppose that Condition 5.1.5 is satisfied for an $n \in \mathbb{N}$. Then there is an $\varepsilon^* > 0$ such that $A_{n\,\varepsilon}$, $P_{n\,\varepsilon}$, $K_{n\,\varepsilon}$ are well-defined w.r.t. γ_t $(t \in I)$ for every $\varepsilon \in (0, \varepsilon^*]$. Furthermore,*

$$\sup_{t \in I} \|K_{n\,\varepsilon}(t) - K_{n-1\,\varepsilon}(t)\| = O(\varepsilon^n) \quad (\varepsilon \searrow 0).$$

(ii) Suppose that Condition 5.1.5 is satisfied for $n = \infty$. Then there is an $\varepsilon^* > 0$ and for every $\varepsilon \in (0, \varepsilon^*]$ there is a natural number $n^*(\varepsilon) \in \mathbb{N}$ such that $A_{n^*(\varepsilon)\,\varepsilon}$, $P_{n^*(\varepsilon)\,\varepsilon}$, $K_{n^*(\varepsilon)\,\varepsilon}$ are well-defined w.r.t. γ_t $(t \in I)$ for every $\varepsilon \in (0, \varepsilon^*]$. Furthermore, there is a constant $g \in (0, \infty)$ such that

$$\sup_{t \in I} \big\| K_{n^*(\varepsilon)\,\varepsilon}(t) - K_{n^*(\varepsilon)-1\,\varepsilon}(t) \big\| = O\big(e^{-\frac{g}{\varepsilon}}\big) \quad (\varepsilon \searrow 0).$$

Proof. We begin with some general preparatory considerations from which both part (i) and part (ii) will easily follow. Suppose (for the entire proof) that Condition 5.1.5 is satisfied for $n = 1$ and fix $t_0 \in I$ for the moment. We have

$$J_{t_0} \times C_{t_0} := J_{t_0} \times \overline{U}_{\frac{5r_0}{7}}(\sigma(t_0)) \setminus U_{\frac{2r_0}{7}}(\sigma(t_0)) \subset\subset \big\{(t,z) \in J_{t_0} \times \mathbb{C} : z \in \rho(A(t))\big\} =: U_{t_0}$$

and $U_{t_0} \ni (t,z) \mapsto (z - A(t))^{-1}$ is continuous, because $J_{t_0} \ni t \mapsto A(t)$ is continuous in the generalized sense due to the WOT-continuous differentiability of $J_{t_0} \ni t \mapsto (z - A(t))^{-1}$ (Theorem IV.3.15 of [67]). Consequently, $\sup_{(t,z) \in J_{t_0} \times C_{t_0}} \big\|(z - A(t))^{-1}\big\| < \infty$, whence we can (and will) assume w.l.o.g. that

$$\sup_{(t,z) \in J_{t_0} \times C_{t_0}} \big\|(z - A(t))^{-1}\big\| \leq a_{t_0}. \tag{5.6}$$

We now define $\varepsilon^*_{t_0}$ and $n^*_{t_0}(\varepsilon)$ just like in Joye and Pfister's paper [59], that is,

$$\begin{aligned} \varepsilon^*_{t_0} &:= \max\Big\{\varepsilon \in \big(0, \frac{1}{2a_{t_0}b_{t_0}}\big) : \sum_{k=1}^{\infty} \big(2\alpha^2 a_{t_0} b_{t_0} \frac{\varepsilon}{1 - 2a_{t_0}b_{t_0}\varepsilon}\big)^k \leq \alpha\Big\}, \\ n^*_{t_0}(\varepsilon) &:= \Big\lfloor \frac{1}{e c_{t_0} d_{t_0}\, \varepsilon} \Big\rfloor \text{ for } \varepsilon \in (0, \infty), \end{aligned} \tag{5.7}$$

where α and d_{t_0} are defined by equation (2.30) and equation (2.50) of [59]. (In particular, $\varepsilon^*_{t_0}$ and $n^*_{t_0}(\varepsilon)$ only depend on γ_{t_0}, a_{t_0}, b_{t_0} and c_{t_0}.) We now show by finite induction over k: whenever Condition 5.1.5 is satisfied for a certain $n' \in \mathbb{N}$, then the following holds true for all $\varepsilon \in (0, \varepsilon^*_{t_0}]$ and all $k \in \{1, \dots, n^*_{t_0}(\varepsilon, n')\}$ with $n^*_{t_0}(\varepsilon, n') := \min\{n^*_{t_0}(\varepsilon), n'\}$:

(a) $A_{k\,\varepsilon}$, $P_{k\,\varepsilon}$, $K_{k\,\varepsilon}$ are well-defined w.r.t. γ_t $(t \in J_{t_0})$ and $J_{t_0} \ni t \mapsto K_{k\,\varepsilon}(t)$ is $n^*_{t_0}(\varepsilon, n') - k$ times WOT-continuously differentiable

(b) $\sup_{t \in J_{t_0}} \Big\| K^{(l)}_{k\,\varepsilon}(t) - K^{(l)}_{k-1\,\varepsilon}(t) \Big\| \leq b_{t_0} c_{t_0}^{k+l} d_{t_0}^k \varepsilon^k \frac{(k+l)!}{(1+l)^2}$ for all $l \in \mathbb{N} \cup \{0\}$ with the property that $k + l \leq n^*_{t_0}(\varepsilon, n')$

(c) $\sup_{t \in J_{t_0}} \Big\| K^{(l)}_{k\,\varepsilon}(t) \Big\| \leq 2 b_{t_0} c_{t_0}^l \frac{l!}{(1+l)^2}$ for all $l \in \mathbb{N} \cup \{0\}$ with $k + l \leq n^*_{t_0}(\varepsilon, n')$.

Suppose that Condition 5.1.5 is satisfied for a certain $n' \in \mathbb{N}$ and fix $\varepsilon \in (0, \varepsilon^*_{t_0}]$. Set $k = 1$ for the induction basis. We have only to prove assertion (a) since assertions (b) and (c) can be gathered from the proof of Proposition 2.1 of [59]. It is obvious that $A_{0\,\varepsilon}$,

$P_{0\,\varepsilon}$, $K_{0\,\varepsilon}$ are well-defined w.r.t. γ_t ($t \in J_{t_0}$) and that $t \mapsto K_{0\,\varepsilon}(t) = [P'(t), P(t)]$ is n' times WOT-continuously differentiable. Since, for $z \in C_{t_0}$ and $t \in J_{t_0}$,

$$(z - A_{1\,\varepsilon}(t)) = \big(1 + \varepsilon K_{0\,\varepsilon}(t)(z - A(t))^{-1}\big)(z - A(t))$$
$$\text{and } \left\|\varepsilon K_{0\,\varepsilon}(t)(z - A(t))^{-1}\right\| \le \varepsilon b_{t_0} \left\|(z - A(t))^{-1}\right\| \le \varepsilon_{t_0}^* b_{t_0} a_{t_0} < \frac{1}{2}$$

(remember the estimate for $K_{0\,\varepsilon} = [P', P]$ from Condition 5.1.5, the estimate for the resolvent of A from (5.6), and the definition of $\varepsilon_{t_0}^*$ in (5.7)), we see that

$$\operatorname{ran}\gamma_t \subset \overline{U}_{\frac{4r_0}{7}}(\sigma(t)) \setminus U_{\frac{3r_0}{7}}(\sigma(t)) \subset \overline{U}_{\frac{5r_0}{7}}(\sigma(t_0)) \setminus U_{\frac{2r_0}{7}}(\sigma(t_0)) = C_{t_0} \subset \rho(A_{1\,\varepsilon}(t))$$

for all $t \in J_{t_0}$. And since

$$\mathrm{n}\left(\gamma_t, U_{\frac{2r_0}{7}}(\sigma(t_0))\right) = 1 = \mathrm{n}\left(\gamma_{t_0}, U_{\frac{2r_0}{7}}(\sigma(t_0))\right),$$
$$\mathrm{n}\left(\gamma_t, \mathbb{C} \setminus \overline{U}_{\frac{5r_0}{7}}(\sigma(t_0))\right) = 0 = \mathrm{n}\left(\gamma_{t_0}, \mathbb{C} \setminus \overline{U}_{\frac{5r_0}{7}}(\sigma(t_0))\right)$$

and $C_{t_0} \subset \rho(A_{1\,\varepsilon}(t))$ for all $t \in J_{t_0}$, the cycles γ_t and γ_{t_0} are homologous in $\rho(A_{1\,\varepsilon}(t))$ for $t \in J_{t_0}$, so that

$$\begin{aligned} J_{t_0} \ni t \mapsto P_{1\,\varepsilon}(t) &= \frac{1}{2\pi i} \int_{\gamma_t} (z - A_{1\,\varepsilon}(t))^{-1}\, dz \\ &= \frac{1}{2\pi i} \int_{\gamma_{t_0}} (z - A(t))^{-1} \big(1 + \varepsilon K_{0\,\varepsilon}(t)(z - A(t))^{-1}\big)^{-1}\, dz \end{aligned}$$

is n' times WOT-continuously differentiable. (In order to see this, use the product rule and inverses rule for WOT-continuous differentiability from the remark before Lemma 2.1.2 as well as Condition 5.1.5.) Consequently, $A_{1\,\varepsilon}$, $P_{1\,\varepsilon}$, $K_{1\,\varepsilon}$ are well-defined w.r.t. γ_t ($t \in J_{t_0}$) and $t \mapsto K_{1\,\varepsilon}(t)$ is $n' - 1$ times (in particular, $n_{t_0}^*(\varepsilon, n') - 1$ times) WOT-continuously differentiable.

Choose now $k \in \{2, \dots, n_{t_0}^*(\varepsilon, n')\}$ and assume that assertions (a), (b), (c) are true for $k - 1$. We then have to show that they are also true for k. As above we have only to establish (a) since (b) and (c) can then be derived as in the proof of Proposition 2.1 of [59], as a close inspection of that proof shows. And in order to prove (a) we can proceed essentially as above: just use assertion (c) for $k - 1$ to get the estimate

$$\sup_{(t,z) \in J_{t_0} \times C_{t_0}} \left\|\varepsilon K_{k-1\,\varepsilon}(t)(z - A(t))^{-1}\right\| \le 2 b_{t_0} a_{t_0} \varepsilon_{t_0}^* < 1$$

and continue as above, thereby concluding the inductive proof of (a), (b), (c).

Choosing finitely many points $t_1, \dots, t_m \in I$ such that $J_{t_1} \cup \dots \cup J_{t_m} = I$, and setting

$$\varepsilon^* := \min\{\varepsilon_{t_1}^*, \dots, \varepsilon_{t_m}^*\} \text{ and } n^*(\varepsilon) := \min\{n_{t_1}^*(\varepsilon), \dots, n_{t_m}^*(\varepsilon)\}, \tag{5.8}$$

we find – in virtue of the above preparations – that, for every $\varepsilon \in (0, \varepsilon^*]$, the following holds true: whenever Condition 5.1.5 is fulfilled for an $n' \in \mathbb{N}$, then $A_{k\,\varepsilon}$, $P_{k\,\varepsilon}$, $K_{k\,\varepsilon}$ are well-defined w.r.t. γ_t ($t \in I$) and

$$\sup_{t\in I} \|K_{k\,\varepsilon}(t) - K_{k-1\,\varepsilon}(t)\| \le bc^k d^k \varepsilon^k k! \tag{5.9}$$

for every $k \in \{1, \dots, n^*(\varepsilon, n')\}$, where b, c, d are obtained by taking the maximum of the corresponding quantities for the points $t_1, \dots, t_m$ and $n^*(\varepsilon, n') := \min\{n^*(\varepsilon), n'\}$.

Suppose now as in (i) that Condition 5.1.5 is satisfied for an $n \in \mathbb{N}$. Since $n^*(\varepsilon) \longrightarrow \infty$ as $\varepsilon \searrow 0$, we can assume w.l.o.g. that $n^*(\varepsilon, n) = n$ for all $\varepsilon \in (0, \varepsilon^*]$ and therefore assertion (i) follows from (5.9). Suppose finally as in (ii) that Condition 5.1.5 is satisfied for $n = \infty$. Since for every $\varepsilon \in (0, \varepsilon^*]$ there is $n' \in \mathbb{N}$ such that $n^*(\varepsilon, n') = n^*(\varepsilon)$ and since Condition 5.1.5 is satisfied, in particular, for this n', assertion (ii) follows from (5.9) with the help of Stirling's formula (see, for instance, the proof of Theorem 2.1 of [59] or of Theorem 1b of [99]). ■

After these preparations we can now prove the announced adiabatic theorem of higher order extending Theorem 2.1 of [59] where skew self-adjoint operators $A(t)$ are considered that analytically depend on t and have time-independent domains.

Theorem 5.1.7. *Suppose $A(t)$, $\sigma(t)$, $P(t)$ for $t \in I$ are such that Condition 5.1.1 is satisfied and Condition 5.1.5 with WOT replaced by SOT is satisfied for an $n \in \mathbb{N}$ or $n = \infty$, respectively. Then*

(i) $\sup_{t\in I} \|P_\varepsilon(t) - P(t)\| = O(\varepsilon)$ *as $\varepsilon \searrow 0$, where, for all $\varepsilon \in (0, \varepsilon^*]$ and $t \in I$, $P_\varepsilon(t) := P_{n\,\varepsilon}(t)$ in case $n \in \mathbb{N}$ and $P_\varepsilon(t) := P_{n^*(\varepsilon)\,\varepsilon}(t)$ in case $n = \infty$ (and where ε^* and $n^*(\varepsilon)$ are defined as in (5.8) of the lemma above).*

(ii) Whenever the evolution system V_ε for $\frac{1}{\varepsilon}A_{n\,\varepsilon} + K_{n\,\varepsilon}$ resp. $\frac{1}{\varepsilon}A_{n^*(\varepsilon)\,\varepsilon} + K_{n^*(\varepsilon)\,\varepsilon}$ exists on $D(A(t))$ for all $\varepsilon \in (0, \varepsilon^*]$, then V_ε is adiabatic w.r.t. P_ε and for a suitable constant $g \in (0, \infty)$

$$\sup_{t\in I} \|V_\varepsilon(t) - U_\varepsilon(t)\| = O(\varepsilon^n) \text{ resp. } O\big(e^{-\frac{g}{\varepsilon}}\big) \quad (\varepsilon \searrow 0).$$

(iii) Additionally, one has – the existence of V_ε being irrelevant here – that

$$\sup_{t\in I} \|(1 - P_\varepsilon(t))U_\varepsilon(t)P_\varepsilon(0)\|,$$
$$\sup_{t\in I} \|P_\varepsilon(t)U_\varepsilon(t)(1 - P_\varepsilon(0))\| = O(\varepsilon^n) \text{ resp. } O\big(e^{-\frac{g}{\varepsilon}}\big) \quad (\varepsilon \searrow 0).$$

Proof. (i) Set $A_\varepsilon(t) := A_{n\,\varepsilon}(t)$ and $K_\varepsilon^-(t) := K_{n-1\,\varepsilon}(t)$ in case $n \in \mathbb{N}$ and $A_\varepsilon(t) := A_{n^*(\varepsilon)\,\varepsilon}(t)$ and $K_\varepsilon^-(t) := K_{n^*(\varepsilon)-1\,\varepsilon}(t)$ in case $n = \infty$ (for $t \in I$ and $\varepsilon \in (0, \varepsilon^*]$). As was shown in the proof of the above lemma, the cycles γ_t and γ_{t_i} are homologous in $\rho(A_\varepsilon(t))$

for every $t \in J_{t_i}$ (where $t_1, \dots, t_m$ are points of I chosen as in the definition of ε^* and $n^*(\varepsilon)$ in (5.8)) and every $\varepsilon \in (0, \varepsilon^*]$, whence

$$\begin{aligned} P_\varepsilon(t) - P(t) &= \frac{1}{2\pi i} \int_{\gamma_{t_i}} (z - A_\varepsilon(t))^{-1} - (z - A(t))^{-1} \, dz \\ &= -\frac{1}{2\pi i} \int_{\gamma_{t_i}} (z - A_\varepsilon(t))^{-1} \, \varepsilon K_\varepsilon^-(t) \, (z - A(t))^{-1} \, dz \end{aligned}$$

for all $t \in J_{t_i}$ and $\varepsilon \in (0, \varepsilon^*]$. Also, it was shown in the proof of the above lemma that for all $\varepsilon \in (0, \varepsilon^*]$ and all $i \in \{1, \dots, m\}$ one has $\sup_{(t,z) \in J_{t_i} \times \operatorname{ran} \gamma_{t_i}} \left\| (z - A(t))^{-1} \right\| \le a_{t_i}$, $\sup_{t \in J_{t_i}} \|K_\varepsilon^-(t)\| \le 2b_{t_i}$, and

$$\begin{aligned} \left\| (z - A_\varepsilon(t))^{-1} \right\| &\le \left\| (z - A(t))^{-1} \right\| \left\| \big(1 + \varepsilon K_\varepsilon^-(t)(z - A(t))^{-1}\big)^{-1} \right\| \\ &\le a_{t_i} \sum_{m=0}^{\infty} (\varepsilon 2 b_{t_i} a_{t_i})^m \le \frac{a_{t_i}}{1 - 2a_{t_i} b_{t_i} \varepsilon_{t_i}^*} < \infty \end{aligned}$$

for all $(t, z) \in J_{t_i} \times \operatorname{ran} \gamma_{t_i}$. Assertion (i) is now clear (notice that for this assertion Condition 5.1.1 is not needed – only Condition 5.1.5 in its original WOT version is used).

(ii) Set $K_\varepsilon^+(t) := [P_\varepsilon'(t), P_\varepsilon(t)]$ for $t \in I$ and $\varepsilon \in (0, \varepsilon^*]$ and suppose that the evolution system V_ε for $\frac{1}{\varepsilon} A_\varepsilon + K_\varepsilon^+$ exists on $D(A(t))$. Since for every $x \in D(A(0))$ the map $[0, t] \ni s \mapsto U_\varepsilon(t, s) V_\varepsilon(s) x$ is continuous and right differentiable (by Lemma 2.1.3) and since the right derivative $s \mapsto U_\varepsilon(t, s)\big(K_\varepsilon^+(s) - K_\varepsilon^-(s)\big) V_\varepsilon(s) x$ is bounded, it follows from Lemma 2.1.4 that

$$\begin{aligned} V_\varepsilon(t) x - U_\varepsilon(t) x &= U_\varepsilon(t, s) V_\varepsilon(s) x \big|_{s=0}^{s=t} \\ &= \int_0^t U_\varepsilon(t, s) \big(K_\varepsilon^+(s) - K_\varepsilon^-(s)\big) V_\varepsilon(s) x \, ds \end{aligned} \tag{5.10}$$

for all $t \in I$. And from this, in turn, we conclude the desired estimates – using the estimates for $K_\varepsilon^+ - K_\varepsilon^-$ from Lemma 5.1.6 and applying Proposition 2.1.13 (ii). It remains to show that V_ε is adiabatic w.r.t. P_ε. As, by assumption, Condition 5.1.5 is satisfied with WOT replaced by SOT (up to now, the unaltered Condition 5.1.5 was sufficient), $t \mapsto P_\varepsilon(t)$ is continuously differentiable not only w.r.t. WOT but also w.r.t. SOT. And therefore, the adiabaticity of V_ε follows from Propostion 3.4.1.

(iii) Arguing as in the adiabaticity proof above, we get for every $x \in D(A(0))$ and every $t \in I$ that

$$\begin{aligned} P_\varepsilon(t) U_\varepsilon(t) x - U_\varepsilon(t) P_\varepsilon(0) x &= U_\varepsilon(t, s) P_\varepsilon(s) U_\varepsilon(s) x \big|_{s=0}^{s=t} \\ &= \int_0^t U_\varepsilon(t, s) \Big(P_\varepsilon'(s) - \frac{1}{\varepsilon} \big(A(s) P_\varepsilon(s) - P_\varepsilon(s) A(s) \big) \Big) U_\varepsilon(s) x \, ds. \end{aligned}$$

Since $A_\varepsilon(s)$ commutes with $P_\varepsilon(s)$ for $s \in I$ and since $A = A_\varepsilon + \varepsilon K_\varepsilon^-$, we have

$$\begin{aligned} P_\varepsilon'(s) - \frac{1}{\varepsilon}\big(A(s)P_\varepsilon(s) - P_\varepsilon(s)A(s)\big) &\subset P_\varepsilon'(s) - [K_\varepsilon^-(s), P_\varepsilon(s)] \\ = P_\varepsilon'(s) - [K_\varepsilon^+(s), P_\varepsilon(s)] + [K_\varepsilon^+(s) - K_\varepsilon^-(s), P_\varepsilon(s)] &= [K_\varepsilon^+(s) - K_\varepsilon^-(s), P_\varepsilon(s)] \end{aligned}$$

for every $s \in I$, and the desired conclusion follows with the help of Lemma 5.1.6. ■

It is obvious from the definition of Joye and Pfister's iterative scheme that $P_\varepsilon(t) = P(t)$ for all t in the (possibly empty) set $I \setminus \operatorname{supp} P'$, and therefore it follows from Theorem 5.1.7 (iii) that

$$\sup_{t \in I \setminus \operatorname{supp} P'} \|(1 - P(t))U_\varepsilon(t)P(0)\|,$$

$$\sup_{t \in I \setminus \operatorname{supp} P'} \|P(t)U_\varepsilon(t)(1 - P(0))\| = O(\varepsilon^n) \text{ resp. } O\big(e^{-\frac{g}{\varepsilon}}\big) \quad (\varepsilon \searrow 0).$$

A result similar to Theorem 5.1.7 could have been proved with the help of a method developed by Nenciu in [99] – this can easily be gathered from the exposition in Section 7 of [112]. We have chosen Joye and Pfister's method since it is easier to remember and effortlessly transferred to the case of several compact isolated subsets $\sigma_1(t), \dots, \sigma_m(t)$ of $\sigma(A(t))$ where each is uniformly isolated in $\sigma(A(t))$ and uniformly isolated from each of the others.

We finally comment on a recent superadiabatic-type theorem by Joye from [60] dealing with time-independent domains and several spectral subsets $\sigma_i(t)$. It allows for a generalization of Condition 5.1.1 at the cost of a specialization of Condition 5.1.5 and states the following (where we confine ourselves, for the sake of notational simplicity, to the case of only one spectral subset $\sigma_i(t) = \sigma(t)$): if – and what follows is a special case of Condition 5.1.5 – there is an open neighbourhood U_I of I such that $t \mapsto A(t)x$ for every $x \in D$ extends to a holomorphic map on U_I and if $\sigma(t) = \{\lambda(t)\}$ for every $t \in I$ for a uniformly isolated spectral value $\lambda(t)$ of $A(t)$ of finite algebraic multiplicity (hence an eigenvalue) such that $t \mapsto \lambda(t)$ is continuous, then it suffices for the conclusion of Theorem 5.1.7 to hold that – instead of Condition 5.1.1 – $\lambda(t)$ lie in the left closed complex half-plane and $A(t)\overline{P}(t)$ generate a contraction semigroup on X for every $t \in I$ (where $\overline{P} := 1 - P$). So, in the above-mentioned special case of Condition 5.1.5 the boundedness requirement on U_ε from Condition 5.1.1 is not necessary for assertions (i), (ii) and (iii) of Theorem 5.1.7. It is, however, necessary for the convergences

$$\sup_{t \in I} \|(1 - P(t))U_\varepsilon(t)P(0)\|, \quad \sup_{t \in I} \|P(t)U_\varepsilon(t)(1 - P(0))\| \longrightarrow 0 \quad (\varepsilon \searrow 0)$$

with the originally given projections $P(t)$, which we are primarily interested in here. See the example at the end of Section 1 of [60] for a proof of this necessity statement. Also, it should be remarked that the above-mentioned special requirements (analyticity and finite algebraic multiplicity) of Joye's theorem from [60] are really essential for the proof in [60]. Indeed, this proof essentially rests upon the following estimate for the evolution system $V_{0\,\varepsilon}$ for $\frac{1}{\varepsilon}A_{0\,\varepsilon} + K_{0\,\varepsilon} = \frac{1}{\varepsilon}A + [P', P]$ on D

$$\sup_{(s,t)\in\Delta} \|V_{0\,\varepsilon}(t,s)\| \le c\, e^{c/\varepsilon^{\beta}} \quad (\varepsilon \in (0,\varepsilon^*]) \tag{5.11}$$

with constants $\beta \in (0,1)$ and $c \in (0,\infty)$ (Proposition 6.1 of [60]), which then – by the usual perturbation argument (Proposition 2.1.13) – yields the estimates

$$\sup_{(s,t)\in\Delta} \|U_{\varepsilon}(t,s)\|\,, \ \sup_{(s,t)\in\Delta} \|V_{\varepsilon}(t,s)\| \le c'\, e^{c'/\varepsilon^{\beta}} \quad (\varepsilon \in (0,\varepsilon^*]) \tag{5.12}$$

from which, in turn, by the integral representation (5.10) and the exponential decay of $K_{\varepsilon}^{+} - K_{\varepsilon}^{-}$ from Lemma 5.1.6 (analyticity requirement!), the conclusion of Theorem 5.1.7 finally follows. And the fundamental estimate (5.11) rests upon a result on the growth (in ε) of the evolution system for analytic families $\frac{1}{\varepsilon}N$ of nilpotent operators $N(t)$ on finite-dimensional spaces (Proposition 4.1 of [60]), which proposition (by the analyticity and finite algebraic multiplicity requirement!) can be applied to the nilpotent endomorphisms

$$N(t) := W(t)^{-1}(A(t) - \lambda(t))W(t)\big|_{P(0)X}$$

of the finite-dimensional space $P(0)X$ that analytically depend on t. W denotes the evolution system for $[P', P]$ on X exactly intertwining the subspaces $P(t)X$.

5.1.4 An example with time-dependent domains

We confine ourselves to an example illustrating the adiabatic theorem without spectral gap condition. In this example, a differential operator of the simplest kind occurs, namely $B = B_p : W^{1,p}(\mathbb{R}) \subset L^p(\mathbb{R}) \to L^p(\mathbb{R})$ with $Bf := \partial f$ (weak derivative). Since B is the generator of the (left) translation group T on $L^p(\mathbb{R})$ (which is given by $T(t)f := f(\,.\,+t)$ for $t \in \mathbb{R}$), one has $\sigma(B) \subset i\mathbb{R}$, and since for every $\lambda \in i\mathbb{R}$ the function g, defined by

$$g(t) := \frac{e^{\lambda t}}{t^{\alpha}}\,\chi_{[1,\infty)}(t) \quad (t \in \mathbb{R})$$

with arbitrary $\alpha \in (\frac{1}{p}, 1 + \frac{1}{p}]$, belongs to $L^p(\mathbb{R})$ but not to the range of $B - \lambda$, one even has $\sigma(B) = i\mathbb{R}$ for $p \in [1,\infty)$. Additionally, since $B_q^* = -B_{q^*}$ for every $q \in [1,\infty)$ with dual exponent q^* and since $\sigma_p(B_q) = \emptyset$ for $q \in [1,\infty)$ and $\sigma_p(B_q) = i\mathbb{R}$ for $q = \infty$, one obtains the following fine structure of the spectrum of B:

$$\sigma_p(B) = \emptyset, \quad \sigma_c(B) = \emptyset, \quad \sigma_r(B) = i\mathbb{R} \quad (p = 1)$$
$$\sigma_p(B) = \emptyset, \quad \sigma_c(B) = i\mathbb{R}, \quad \sigma_r(B) = \emptyset \quad (p \in (1,\infty)).$$

Example 5.1.8. Suppose A, λ, P with $A(t) = R(t)^{-1}A_0(t)R(t)$, $P(t) = R(t)^{-1}P_0R(t)$, and $R(t) = e^{Ct}$ are given as follows in $X := \ell^p(I_d) \times L^p(\mathbb{R})$ (where $p \in (1,\infty)$ and $d \in \mathbb{N}$):

$$A_0(t) := \begin{pmatrix} \lambda(t) + \alpha(t)N & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\lambda(t) \in \{\operatorname{Re} z \leq 0\}$, $\alpha(t)$, N are such that Condition 3.5.1 is satisfied and where B is the differentiation operator on $L^p(\mathbb{R})$ defined above, so that, in particular,

$$D(A_0(t)) = D := \ell^p(I_d) \times W^{1,p}(\mathbb{R}).$$

Additionally, suppose $t \mapsto \lambda(t)$, $\alpha(t)$ are continuously differentiable and C is the bounded linear map in $\ell^p(I_d) \times L^p(\mathbb{R})$ given by

$$C := \begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix} \quad \text{with} \quad C_0(x_1, \dots, x_d) := x_d f_0$$

for an arbitrary fixed $0 \neq f_0 \in L^p(\mathbb{R})$. Since (by $p \neq 1$) the spectrum of $A_0(t)|_{(1-P_0)D} = B$ is purely continuous for every $t \in I$, P_0 is weakly associated with $A_0(t)$ and P_0, and hence the same is true for $A(t)$ and $P(t)$ instead of $A_0(t)$, P_0. Since, moreover, B generates a contraction group (not only a semigroup) in $L^p(\mathbb{R})$, the resolvent estimates of Theorem 5.1.4 are satisfied with $\vartheta(t) := \pi$. ◀

It follows in the same way as after Example 4.2.7 that A, λ, P of the above example cannot be reduced to a finite-dimensional subspace and that our adiabatic theorem without spectral gap condition does not apply if $p = 1$ in the example above. Also, it should be noticed that the domains $D(A(t)) = e^{-Ct}D$ of the above $A(t)$ really are time-dependent – more precisely, one has $D(A(t_1)) \neq D(A(t_2))$ for $t_1 \neq t_2$ – if only f_0 is chosen to lie not in $W^{1,p}(\mathbb{R})$. Indeed, if under this condition on f_0 one has the twofold representation

$$(x, f - t_1 x_d \, f_0) = e^{-Ct_1}(x, f) = e^{-Ct_2}(y, g) = (y, g - t_2 y_d \, f_0)$$

for some (x, f), $(y, g) \in \ell^p(I_d) \times W^{1,p}(\mathbb{R}) = D$ with $x_d \neq 0$, then t_1 must be equal to t_2.

5.2 Adiabatic theorems for operators defined by symmetric sesquilinear forms

After having established general adiabatic theorems for time-dependent domains in Section 5.1, we now apply these theorems to obtain – as simple corollaries – adiabatic theorems for operators $A(t) = iA_{a(t)}$ defined by densely defined closed symmetric sesquilinear forms $a(t)$ with time-independent form domain – such as, for instance, Schrödinger operators $A(t)$ with time-dependent potentials $V(t)$ belonging to the Rollnik class. In particular, the theorem of Section 5.2.3 contains the adiabatic theorem of Bornemann from [17] as a special case.

5.2.1 Some notation and preliminaries

We start by recording the basic conditions (depending on a regularity parameter $n \in \mathbb{N} \cup \{\infty\}$) that shall be imposed on the sesquilinear forms $a(t)$ in the adiabatic theorems of this section.

Condition 5.2.1. *$a(t)$ for every $t \in I$ is a symmetric sesquilinear form on the Hilbert space H^+ (with norm $\| \,.\, \|^+$ and scalar product $\langle \,.\, , .. \rangle^+$) which is densely and continuously embedded into H (with norm $\| \,.\, \|$ and scalar product $\langle \,.\, , .. \rangle$). There is a number $m \in (0, \infty)$ such that*

$$\langle \,.\, , .. \rangle_t^+ := a(t)(\,.\, , ..) + m \, \langle \,.\, , .. \rangle$$

is a scalar product on H^+ and such that the induced norm $\| \,.\, \|_t^+$ is equivalent to $\| \,.\, \|^+$ for every $t \in I$. And finally, $t \mapsto a(t)(x, y)$ is n times continuously differentiable for all $x, y \in H^+$.

In Condition 5.2.1, the requirement that $\langle \,.\, , .. \rangle_t^+$ be a scalar product on H^+ whose norm $\| \,.\, \|_t^+$ is equivalent to $\| \,.\, \|^+$ for every $t \in I$ could be reformulated in an equivalent way by saying that there is $m \in (0, \infty)$ such that $a(t)(\,.\, , ..) + m \, \langle \,.\, , .. \rangle$ is $\| \,.\, \|^+$-bounded and $\| \,.\, \|^+$-coercive. It is well-known that under Condition 5.2.1 there is, for every $t \in I$, a unique self-adjoint operator $A_{a(t)} : D(A_{a(t)}) \subset H \to H$ such that

$$D(A_{a(t)}) \subset H^+ \quad \text{and} \quad \big\langle x, A_{a(t)} y \big\rangle = a(t)(x, y)$$

for every $x \in H^+$ and $y \in D(A_{a(t)})$ (Theorem VI.2.1 and Theorem VI.2.6 of [67] or Theorem 10.1.2 of [14]). As usual, we denote – in the situation of Condition 5.2.1 – by H^- the space of $\| \,.\, \|^+$-continuous conjugate linear functionals $H^+ \to \mathbb{C}$, which obviously is a Hilbert space w.r.t. the norms

$$f \mapsto \|f\|^- := \sup_{\|x\|^+ = 1} |f(x)| \quad \text{and} \quad f \mapsto \|f\|_t^- := \sup_{\|x\|_t^+ = 1} |f(x)| \quad (t \in I).$$

And by $j : H \to H^-$ we denote the injective continuous linear map with $j(x) := \langle \,.\, , x \rangle \in H^-$ for $x \in H$.

We continue by citing the fundamental theorem of Kisyński (Theorem 8.1 of [71]) giving sufficient conditions for the well-posedness of the initial value problems corresponding to A on $D(A(t))$, where $A(t) = i A_{a(t)}$ with symmetric sesquilinear forms $a(t)$ with constant domain. Similar theorems on well-posedness can be proved for the case of operators $A(t) = -A_{a(t)}$ defined by sectorial sesquilinear forms $a(t)$ with time-independent form domain. See, for instance, Fujie and Tanabe's article [44] (Theorem 3.1) or Kato and Tanabe's article [64] (Theorem 7.3).

Theorem 5.2.2 (Kisyński)**.** *Suppose $a(t)$ for every $t \in I$ is a sesquilinear form such that Condition 5.2.1 is satisfied with $n = 2$ and set $A(t) := i A_{a(t)}$ for $t \in I$. Then there is a unique evolution system U for A on $D(A(t))$ and $U(t, s)$ is unitary in H for every $(s, t) \in \Delta$.*

In particular, this theorem guarantees that the basic Condition 5.1.1 of the general adiabatic theorems for time-dependent domains is satisfied if Condition 5.2.1 is with $n = 2$. When it comes to verifying the other conditions of the general adiabatic theorems discussed in Section 5.1, the following lemma will be important.

Lemma 5.2.3. *Suppose that Condition 5.2.1 is satisfied for a certain $n \in \mathbb{N}$ and, for every $t \in I$, denote by $\tilde{A}_0(t)$ the bounded linear map $H^+ \to H^-$ extending $A_0(t) := A_{a(t)}$, that is, $\tilde{A}_0(t)x := a(t)(\,.\,, x)$ for $x \in H^+$. Then the following holds true:*

(i) $t \mapsto \tilde{A}_0(t)$ is n times WOT-continuously differentiable.

(ii) If for a certain $z \in \mathbb{C}$ the operator $A_0(t) - z : D(A_0(t)) \subset H \to H$ is bijective for all $t \in J_0$ (a non-trivial subinterval of I), then so is $\tilde{A}_0(t) - zj : H^+ \to H^-$ and

$$(A_0(t) - z)^{-1}x = (\tilde{A}_0(t) - zj)^{-1}j(x)$$

for all $t \in J_0$ and $x \in H$. In particular, $J_0 \ni t \mapsto (A_0(t) - z)^{-1}$ is n times WOT-continuously differentiable.

Proof. (i) We have only to show that $t \mapsto F(\tilde{A}_0(t)x)$ is n times continuously differentiable for every $x \in H^+$ and every $F \in (H^-)^*$, because H^- is reflexive. Since the canonical conjugate linear map

$$H^+ \ni y \mapsto i(y) \in (H^-)^* \quad \text{with} \quad i(y)(f) := f(y) \quad \text{for } f \in H^-$$

is surjective by the reflexivity of H^+, the claim is obvious from the continuous differentiability requirement in Condition 5.2.1.

(ii) We fix $t \in I$ and show that

$$\rho(A_0(t)) \subset \rho(A_0^-(t)), \tag{5.13}$$

where $A_0^-(t) : j(H^+) \subset H^- \to H^-$ is defined by $A_0^-(t)j(x) = \tilde{A}_0(t)x$ for $x \in H^+$. Since $A_0^-(t)$ is self-adjoint in $(H^-, \|\,.\,\|_t^-)$, it follows that $\mathbb{C} \setminus \mathbb{R} \subset \rho(A_0^-(t))$ and that

$$(A_0^-(t) - z)^{-1}j(x) = j\big((A_0(t) - z)^{-1}x\big) \tag{5.14}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$ and $x \in H$. It therefore remains to prove that $\rho(A_0(t)) \cap \mathbb{R} \subset \rho(A_0^-(t))$. So let $z \in \rho(A_0(t)) \cap \mathbb{R}$. Then there is $\delta > 0$ such that $(z - 2\delta, z + 2\delta) \subset \rho(A_0(t))$, from which it follows by Stone's formula (applied to both $A_0(t)$ and $A_0^-(t)$) and by (5.14) that

$$0 = j\Big(P_{(z-\delta,z+\delta)}x + \frac{1}{2}P_{\{z-\delta,z+\delta\}}x\Big) = \Big(P^-_{(z-\delta,z+\delta)} + \frac{1}{2}P^-_{\{z-\delta,z+\delta\}}\Big)j(x)$$

for all $x \in H$, where P and P^- denote the spectral measure of $A_0(t)$ and $A_0^-(t)$, respectively. It follows (by the density of $j(H)$ in H^-) that $P^-_{(z-\delta,z+\delta)} = 0$ and hence $z \in \rho(A_0^-(t))$. So (5.13) is established and the desired conclusion ensues. ■

5.2.2 Adiabatic theorems with spectral gap condition

We will need the following condition depending on a parameter $m \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ for the adiabatic theorem with spectral gap condition below.

Condition 5.2.4. $A(t) = iA_{a(t)}$ *for* $t \in I$, *where the sesquilinear forms* $a(t)$ *satisfy Condition 5.2.1 with* $n = 2$. $\sigma(t)$ *for every* $t \in I$ *is a compact subset of* $\sigma(A(t))$, $\sigma(\,.\,)$ *falls into* $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$ *at exactly* m *points that accumulate at only finitely many points, and* $I \setminus N \ni t \mapsto \sigma(t)$ *is continuous, where* N *denotes the set of those* m *points at which* $\sigma(\,.\,)$ *falls into* $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$. $P(t)$ *for every* $t \in I \setminus N$ *is the projection associated with* $A(t)$ *and* $\sigma(t)$ *and* $I \setminus N \ni t \mapsto P(t)$ *extends to a twice SOT-continuously differentiable map (again denoted by* P*) on the whole of* I.

In view of Lemma 5.2.3 it is now very easy to derive the following adiabatic theorem with uniform ($m = 0$) or non-uniform ($m \in \mathbb{N} \cup \{\infty\}$) spectral gap condition from the corresponding general adiabatic theorem with spectral gap condition (Theorem 5.1.3).

Theorem 5.2.5. *Suppose* $A(t)$, $\sigma(t)$, $P(t)$ *for* $t \in I$ *are as in Condition 5.2.4 with* $m = 0$ *or* $m \in \mathbb{N} \cup \{\infty\}$, *respectively. Then*

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \text{ resp. } o(1) \quad (\varepsilon \searrow 0),$$

whenever the evolution system V_ε *for* $\frac{1}{\varepsilon}A + [P', P]$ *exists on* $D(A(t))$ *for every* $\varepsilon \in (0, \infty)$.

Proof. Choose, for every $t_0 \in I \setminus N$, non-trivial closed intervals J_{t_0} and cycles γ_{t_0} as in Condition 5.1.2 (which is possible by the relative openness of $I \setminus N$ in I). In virtue of Lemma 5.2.3 it is then clear that Condition 5.1.2 is fulfilled, and the assertion follows from Theorem 5.1.3. ■

If the existence of the evolution V_ε for $\frac{1}{\varepsilon}A + [P', P]$ cannot be ensured, one still has the remark after Theorem 5.1.3. In the case of uniform spectral gap, the existence of V_ε is guaranteed if, for instance, Condition 5.2.1 is fulfilled with $n = 3$, since then $I \ni t \mapsto P(t)$ is thrice WOT-continuously differentiable (by Lemma 5.2.3 (ii)) so that the symmetric sesquilinear forms $\frac{1}{\varepsilon}a(t) + b(t) = \frac{1}{\varepsilon}a(t) - i\,\langle\,.\,, [P'(t), P(t)]\,..\,\rangle$ corresponding to $\frac{1}{\varepsilon}A(t) + [P'(t), P(t)]$ satisfy Condition 5.2.1 with $n = 2$ and Theorem 5.2.2 can be applied.

We finally note conditions under which the general adiabatic theorem of higher order (Theorem 5.1.7) can be applied to the case of operators $A(t)$ defined by symmetric sesquilinear forms.

Condition 5.2.6. *Suppose that* $A(t) = iA_{a(t)}$ *for* $t \in I$ *where the sesquilinear forms* $a(t)$ *satisfy Condition 5.2.1 with a certain* $n \in \mathbb{N} \setminus \{1\}$ *or with* $n = \infty$, *respectively. In the latter case suppose further that there is an open neighbourhood* U_I *of* I *in* $\mathbb{C}$ *and for each* $w \in U_I$ *there is a* $\|\,.\,\|^+$*-bounded sesquilinear form* $\tilde{a}(w)$ *on* H^+ *such that* $\tilde{a}(t) = a(t)$ *for* $t \in I$ *and that* $U_I \ni w \mapsto \tilde{a}(w)(x, y)$ *is holomorphic for every* $x, y \in H^+$. *Suppose moreover that* $\sigma(t)$ *for every* $t \in I$ *is an isolated compact subset of* $\sigma(A(t))$, *that* $\sigma(\,.\,)$ *at no point falls into* $\sigma(A(\,.\,)) \setminus \sigma(\,.\,)$, *and that* $t \mapsto \sigma(t)$ *is continuous. And finally, suppose* $P(t)$ *for every* $t \in I$ *is the projection associated with* $A(t)$ *and* $\sigma(t)$ *and* $t \mapsto P(t)$ *is* $n+1$ *times times SOT-continuously differentiable.*

It is not difficult (albeit a bit technical) to show that under Condition 5.2.6 the hypotheses of Theorem 5.1.7 are really satisfied. (In the case $n = \infty$ define $\tilde{A}_0(w)$ by $\tilde{A}_0(w)x := \tilde{a}(w)(\,.\,,x)$ for $x \in H^+$. Then $\tilde{A}_0(w)$ is a bounded linear map $H^+ \to H^-$ and $U_I \ni w \mapsto \tilde{A}_0(w) \in L(H^+, H^-)$ is WOT-holomorphic and hence holomorphic w.r.t. the norm operator topology. A simple perturbation argument and Cauchy's inequality (in conjunction with the formula in Lemma 5.2.3 (ii)) then yield estimates of the desired kind.)

5.2.3 An adiabatic theorem without spectral gap condition

In the adiabatic theorem without spectral gap condition below, the following condition will be used.

Condition 5.2.7. *$A(t) = iA_{a(t)}$ for $t \in I$ where the sesquilinear forms $a(t)$ satisfy Condition 5.2.1 with $n = 2$. $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$ such that $t \mapsto \lambda(t)$ is continuous. And $P(t)$ for every $t \in I$ is an orthogonal projection in H such that $P(t)$ is weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$, $\operatorname{rk} P(0) < \infty$ and $t \mapsto P(t)$ is SOT-continuously differentiable.*

While in the case with spectral gap Lemma 5.2.3 was sufficient, we need another – well-expected – lemma in the case without spectral gap.

Lemma 5.2.8. *Suppose that Condition 5.2.7 is satisfied and that, in addition, $t \mapsto \lambda(t)$ is continuously differentiable. Then $t \mapsto (\lambda(t) + \delta - A(t))^{-1}$ is SOT-continuously differentiable for every $\delta \in (0, \infty)$ and there is an $M_0' \in (0, \infty)$ such that*

$$\left\| \frac{d}{dt}(\lambda(t) + \delta - A(t))^{-1} \right\| \leq \frac{M_0'}{\delta^2}$$

for all $t \in I$ and $\delta \in (0, 1]$.

Proof. Set $A_0(t) := A_{a(t)} = -iA(t)$ and $\lambda_0(t) := -i\lambda(t)$ and let $\tilde{A}_0(t) : H^+ \to H^-$ be the bounded extension of $A_0(t)$. Since by Lemma 5.2.3 $t \mapsto \tilde{A}_0(t)$ is twice WOT- and, in particular, once SOT-continuously differentiable and $t \mapsto \lambda_0(t)$ is continuously differentiable, it follows that

$$t \mapsto \big(A_0(t) - (\lambda_0(t) - i\delta)\big)^{-1} = \big(\tilde{A}_0(t) - (\lambda_0(t) - i\delta)j\big)^{-1} j$$

is SOT-continuously differentiable for every $\delta \in (0, \infty)$ and that

$$\begin{aligned} &\frac{d}{dt}\big(A_0(t) - (\lambda_0(t) - i\delta)\big)^{-1} \\ &\qquad = \big(\tilde{A}_0(t) - (\lambda_0(t) - i\delta)j\big)^{-1} \big(\lambda_0'(t)j - \tilde{A}_0'(t)\big) \big(\tilde{A}_0(t) - (\lambda_0(t) - i\delta)j\big)^{-1} j \end{aligned} \tag{5.15}$$

for $t \in I$ and $\delta \in (0, \infty)$. We therefore show that there is a constant $c_0' \in (0, \infty)$ such that

$$\left\| \big(\tilde{A}_0(t) - (\lambda_0(t) - i\delta)j\big)x \right\|_t^- \geq \frac{\delta}{c_0'} \left\| x \right\|_t^+ \tag{5.16}$$

for all $x \in H^+$, $t \in I$ and $\delta \in (0,1]$. In order to do so we observe the following simple fact: if instead of j the natural isometric isomorphism

$$j_t^+ : (H^+, \| \,.\, \|_t^+) \to (H^-, \| \,.\, \|_t^-) \text{ with } j_t^+(x) := \langle \,.\, , x \rangle_t^+ \text{ for } x \in H^+$$

occurred in (5.16), this assertion would be trivial. We are therefore led to express j in terms of j_t^+: by the definition of the scalar product $\langle \,.\,, \,..\, \rangle_t^+$ in Condition 5.2.1, we have

$$j = \frac{1}{m}\big(\tilde{A}_0(t) - j_t^+\big)$$

for all $t \in I$, so that

$$\tilde{A}_0(t) - (\lambda_0(t) - i\delta) j = \frac{m + \lambda_0(t) - i\delta}{m}\Big(\tilde{A}_0(t) - \frac{\lambda_0(t) - i\delta}{m + \lambda_0(t) - i\delta} j_t^+\Big).$$

Since for all $x \in H^+$ with $\|x\|_t^+ = 1$

$$\begin{aligned}\Big\|\Big(\tilde{A}_0(t) - \frac{\lambda_0(t) - i\delta}{m + \lambda_0(t) - i\delta} j_t^+\Big)x\Big\|_t^- &\geq \Big|a(t)(x,x) - \frac{\lambda_0(t) - i\delta}{m + \lambda_0(t) - i\delta}\big(j_t^+(x)\big)(x)\Big| \\ &\geq \Big|\operatorname{Im}\Big(\frac{\lambda_0(t) - i\delta}{m + \lambda_0(t) - i\delta}\Big)\Big| = \frac{m\delta}{|m + \lambda_0(t) - i\delta|^2},\end{aligned}$$

it follows that

$$\Big\|\big(\tilde{A}_0(t) - (\lambda_0(t) - i\delta)j\big)x\Big\|_t^- \geq \Big|\frac{m + \lambda_0(t) - i\delta}{m}\Big| \frac{m\delta}{|m + \lambda_0(t) - i\delta|^2} \, \|x\|_t^+ \geq \frac{\delta}{c_0'} \, \|x\|_t^+$$

for all $x \in H^+$ and all $t \in I$, $\delta \in (0,1]$, where $c_0' := m + \|\lambda\|_\infty + 1$. So (5.16) is proven and it follows that

$$\Big\|\big(\tilde{A}_0(t) - (\lambda_0(t) - i\delta)j\big)^{-1}\Big\|_{H^-, H^+} \leq \frac{c_0'}{\delta} \tag{5.17}$$

for all $t \in I$ and $\delta \in (0,1]$, because the equivalence of the norms $\| \,.\, \|_t^+$ with $\| \,.\, \|$ required in Condition 5.2.1 is uniform w.r.t. t by Lemma 7.3 of [71]. In view of (5.15) and (5.17) the asserted estimate is now clear. ■

With this lemma at hand, it is now simple to derive the following adiabatic theorem without spectral gap condition which generalizes an adiabatic theorem of Bornemann (Theorem IV.1 of [17]). See the discussion below for a detailed comparison of these results.

Theorem 5.2.9. *Suppose $A(t)$, $\lambda(t)$, $P(t)$ for $t \in I$ are such that Condition 5.2.7 is satisfied. Then*

$$\sup_{t \in I} \big\|\big(U_\varepsilon(t) - V_{0\,\varepsilon}(t)\big)P(0)\big\| \longrightarrow 0 \quad \textit{and} \quad \sup_{t \in I} \big\|P(t)\big(U_\varepsilon(t) - V_{0\,\varepsilon}(t)\big)\big\| \longrightarrow 0$$

as $\varepsilon \searrow 0$, where $V_{0\,\varepsilon}$ denotes the evolution system for $\frac{1}{\varepsilon}AP + [P', P] = \frac{1}{\varepsilon}\lambda P + [P', P]$ for every $\varepsilon \in (0, \infty)$. If, in addition, $t \mapsto P(t)$ is thrice WOT-continuously differentiable, then the evolution system V_ε for $\frac{1}{\varepsilon}A + [P', P]$ exists on $D(A(t))$ for every $\varepsilon \in (0, \infty)$ and

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| \longrightarrow 0 \quad (\varepsilon \searrow 0).$$

Proof. We have to verify the hypotheses of the general adiabatic theorem without spectral gap condition for time-dependent domains (Theorem 5.1.4) with $m_0 = 1$. In view of Lemma 5.2.8 it remains to establish three small things, namely the inclusions $P(t)H \subset \ker(A(t) - \lambda(t))$ and $P(t)A(t) \subset A(t)P(t)$ for every $t \in I$ (from Theorem 5.1.4) and the continuous differentiability of $t \mapsto \lambda(t)$ (from Theorem 5.1.4 and from Lemma 5.2.8). We know by assumption that $P(t)H = \ker(A(t) - \lambda(t)) = \ker(A_0(t) - \lambda_0(t))$ for almost every $t \in I$ so that $P(t)H \subset D(A_0(t)) \subset H^+$ and

$$0 = j\big((A_0(t) - \lambda_0(t))P(t)x\big) = \big(A_0^-(t) - \lambda_0(t)\big)j(P(t)x)$$

for all $x \in H$ and almost every $t \in I$ (where $A_0(t)$, $\lambda_0(t)$ are defined as in the proof of Lemma 5.2.8 and where $A_0^-(t)$ is the self-adjoint operator in $(H^-, \|\,.\,\|_t^-)$ from the proof of Lemma 5.2.3). Applying the closedness argument after Theorem 4.1.2 to the closed operator $iA_0^-(t) : j(H^+) \subset H^- \to H^-$ (with time-independent domain!), we see that $j(P(t)H) \subset j(H^+)$ and

$$0 = \big(A_0^-(t) - \lambda_0(t)\big)j(P(t)x) = a(t)(\,.\,, P(t)x) - \lambda_0(t)\,\langle\,.\,, P(t)x\rangle$$

for all $x \in H$ and every (not only almost every) $t \in I$. In particular, for every $t \in I$,

$$0 = a(t)(y, P(t)x) - \lambda_0(t)\,\langle y, P(t)x\rangle = \langle (A_0(t) - \lambda_0(t))y, P(t)x\rangle$$

for $y \in D(A_0(t))$ and $x \in H$, so that

$$P(t)H \subset \ker(A_0(t) - \lambda_0(t))^* = \ker(A(t) - \lambda(t)) \tag{5.18}$$

for every $t \in I$, as desired. In other words, $A(t)P(t) = \lambda(t)P(t)$ for every $t \in I$ and therefore we also obtain

$$P(t)A(t) = -P(t)^*A(t)^* \subset -\big(A(t)P(t)\big)^* = \lambda(t)P(t) = A(t)P(t)$$

for every $t \in I$, as desired. Since, finally, for every $t_0 \in I$ there is a neighbourhood $J_{t_0} \subset I$ and an $x_0 \in H$ such that $P(t)x_0 \neq 0$ for $t \in J_{t_0}$, it follows from (5.18) that

$$\frac{1}{\lambda(t) - 1} = \frac{\big\langle P(t)x_0, (A(t) - 1)^{-1}P(t)x_0\big\rangle}{\langle P(t)x_0, P(t)x_0\rangle}$$

for every $t \in J_{t_0}$, from which in turn it follows (by Lemma 5.2.3) that $t \mapsto \lambda(t)$ is continuously differentiable, as desired.

According to what has been said at the beginning of the proof, it is now clear that Lemma 5.2.8 can be applied and that the hypotheses of the first part of Theorem 5.1.4

are satisfied. Since the evolution system U_ε is unitary (by Theorem 5.2.2) and $V_{0\,\varepsilon}$ is unitary as well, we see by obviously modifying the proof of Theorem 5.1.4 that

$$\sup_{(s,t)\in I^2} \big\| \big(U_\varepsilon(t,s) - V_{0\,\varepsilon}(t,s)\big) P(s) \big\| \longrightarrow 0 \quad (\varepsilon \searrow 0), \tag{5.19}$$

where $U_\varepsilon(t,s) := U_\varepsilon(s,t)^{-1} = U_\varepsilon(s,t)^*$ and $V_{0\,\varepsilon}(t,s) := V_{0\,\varepsilon}(s,t)^{-1} = V_{0\,\varepsilon}(s,t)^*$ for $(s,t) \in I^2$ with $s > t$. Since

$$\big\| P(t)\big(U_\varepsilon(t) - V_{0\,\varepsilon}(t)\big) \big\| = \big\| \big(U_\varepsilon(0,t) - V_{0\,\varepsilon}(0,t)\big) P(t) \big\|$$

for $t \in I$ (take adjoints), the first two of the asserted convergences follow from (5.19).

Suppose finally that $t \mapsto P(t)$ is thrice WOT-continuously differentiable. Then the symmetric sesquilinear forms $\frac{1}{\varepsilon} a(t) + b(t) = \frac{1}{\varepsilon} a(t) - i \,\langle\, .\,, [P'(t), P(t)] \,..\, \rangle$ corresponding to the operators $\frac{1}{\varepsilon} A(t) + [P'(t), P(t)]$ satisfy Condition 5.2.1 with $n = 2$ and therefore the evolution system V_ε for $\frac{1}{\varepsilon} A + [P, P]$ exists on $D(A(t))$ for every $\varepsilon \in (0, \infty)$ by Theorem 5.2.2. Also, $t \mapsto P(t)$ is obviously norm continuously differentiable and so the hypotheses of the second part of Theorem 5.1.4 are satisfied, which gives the last convergence. ■

What are the differences between the above theorem and Bornemann's adiabatic theorem of [17]? While in Theorem IV.1 of [17] $\lambda(t)$ is required to belong to the discrete spectrum of $A(t)$ (and hence to be an isolated eigenvalue) for every $t \in I$, in the above theorem it is only required that $\lambda(t)$ has finite multiplicity for almost every $t \in I$: the eigenvalues $\lambda(t)$ are allowed to have infinite multiplicity on a set of measure zero and, moreover, they are allowed to be non-isolated in $\sigma(A(t))$ for every $t \in I$. Also, the regularity conditions on A and P of the above theorem are slightly weaker than those of Theorem IV.1: for instance, $t \mapsto \tilde{A}_0(t)$ is required to be twice continuously differentiable w.r.t. the norm operator topology in [17] whereas above it is only required that $t \mapsto a(t)(x, y)$ be twice continuously differentiable for $x, y \in H^+$ (or equivalently (Lemma 5.2.3), that $t \mapsto \tilde{A}_0(t)$ be twice WOT-continuously differentiable). And finally, the statement of the theorem above is more general than the conclusion of Theorem IV.1 in [17] which says that, for all $x \in H^+$ (and hence for all $x \in H$) and uniformly in $t \in I$,

$$\begin{aligned} \langle U_\varepsilon(t)x, P(t)U_\varepsilon(t)x\rangle &= \langle U_\varepsilon(t)x, P(t)U_\varepsilon(t)x - U_\varepsilon(t)P(0)x\rangle + \langle x, P(0)x\rangle \\ &\longrightarrow \langle x, P(0)x\rangle \quad (\varepsilon \searrow 0). \end{aligned}$$

6 Adiabatic switching of linear perturbations

6.1 Introduction and assumptions

Adiabatic switching of (linear) perturbations has a long tradition in quantum physics. Since the famous work [46] of Gell-Mann and Low, it has been used, for instance, to relate – by what is now known as the Gell-Mann and Low formula – the eigenstates of a perturbed system, described by $A_0 + V$, to the eigenstates of the unperturbed system, described by A_0. Adiabatic switching, in this context, means that $A_0 = \underline{A}(0)$ is infinitely slowly deformed into $\underline{A}(1) = A_0 + V$ in the following sense: one chooses a switching function $\kappa : (-\infty, 0] \to [0, 1]$ vanishing at $-\infty$ and taking the value 1 at 0 and then passes – more and more slowly – from $A_0 = \underline{A}(\kappa(-\infty))$ via

$$\{\infty\} \cup (-\infty, 0] \ni s \mapsto \underline{A}(\kappa(\varepsilon s)) = A_0 + \kappa(\varepsilon s)\, V$$

to $\underline{A}(\kappa(0)) = A_0 + V$ by making the slowness parameter $\varepsilon \in (0, \infty)$ smaller and smaller. A rigorous – and non-perturbative – proof of the Gell-Mann and Low formula for non-degenerate and isolated eigenvalues $\underline{\lambda}(\kappa)$ of $\underline{A}(\kappa) = A_0 + \kappa V$ has been given by Nenciu and Rasche in [97]. It is based on the adiabatic theorem with spectral gap condition. In a recent paper [20] of Brouder, Panati, Stoltz, the Gell-Mann and Low theorem has been extended to the case of degenerate isolated eigenvalues – again by using the adiabatic theorem with spectral gap condition. In this chapter, we further extend the Gell-Mann and Low theorem to the case of non-isolated degenerate eigenvalues.

Condition 6.1.1. *$\underline{A}(\kappa) := A_0 + \kappa V$ for $\kappa \in [0, 1]$, where $A_0 : D \subset H \to H$ is a skew self-adjoint operator in the Hilbert space H and where V is a skew symmetric operator in H that is A_0-bounded with relative bound less than 1. $\underline{\lambda}_1(\kappa), \dots, \underline{\lambda}_r(\kappa)$ for every $\kappa \in [0, 1]$ are eigenvalues of $\underline{A}(\kappa)$, such that $\kappa \mapsto \underline{\lambda}_j(\kappa)$ is continuously differentiable for every $j \in \{1, \dots, r\}$ and such that there are only finitely many crossing points between the curves $\kappa \mapsto \underline{\lambda}_j(\kappa)$, that is, for all $j, l \in \{1, \dots, r\}$ with $j \neq l$ the map $\kappa \mapsto \underline{\lambda}_j(\kappa) - \underline{\lambda}_l(\kappa)$ has only finitely many zeroes. And finally, $\underline{P}_1(\kappa), \dots, \underline{P}_r(\kappa)$ for every $\kappa \in [0, 1]$ are orthogonal projections in H, such that $\kappa \mapsto \underline{P}_j(\kappa)$ is twice strongly continuously differentiable, $0 \neq \operatorname{rk} \underline{P}_j(0) < \infty$, and $\underline{P}_j(\kappa)$ is the spectral projection of $\underline{A}(\kappa)$ corresponding to $\lambda_j(\kappa)$ for every $\kappa \in [0, 1] \setminus N$ with some exceptional set N.*

Condition 6.1.2. *$\kappa : (-\infty, 0] \to [0, 1]$ is a non-decreasing twice continuously differentiable (switching) function such that*

(i) $\kappa(t) \longrightarrow \kappa(-\infty) = 0$ $(t \to -\infty)$ *and* $\kappa(0) = 1$

(ii) κ *and* κ' *are integrable on* $(-\infty, 0]$.

Suppose that $\underline{A}$, $\underline{\lambda}_1, \dots, \underline{\lambda}_r$, $\underline{P}_1, \dots, \underline{P}_r$ satisfy Condition 6.1.1 and that κ is as in Condition 6.1.2 such that the exceptional set $\{t \in (-\infty, 0] : \kappa(t) \in N\}$ is a null set, and define

$$A(t) := \underline{A}(\kappa(t)), \quad \lambda_j(t) := \underline{\lambda}_j(\kappa(t)), \quad P_j(t) := \underline{P}_j(\kappa(t))$$

for $t \in (-\infty, 0]$ and $j \in \{1, \dots, r\}$, along with

$$\underline{K}(\kappa) := \frac{1}{2}\sum_{j=1}^{r+1}[\underline{P}_j'(\kappa), \underline{P}_j(\kappa)] \quad \text{and} \quad K(t) := \frac{1}{2}\sum_{j=1}^{r+1}[P_j'(t), P_j(t)] = \kappa'(t)\underline{K}(\kappa(t)), \tag{6.1}$$

where $\underline{P}_{r+1}(\kappa) := 1 - \underline{P}_1(\kappa) - \dots - \underline{P}_r(\kappa)$ for $\kappa \in [0,1]$ and $P_{r+1}(t) := 1 - P_1(t) - \dots - P_r(t)$ for $t \in (-\infty, 0]$. It then follows by a standard result of Kato (Theorem 6.1 of [65]) on non-autonomous linear evolution equations that the initial value problems

$$x' = A(\varepsilon s)x, \; x(s_0) = y_0 \quad \text{and} \quad x' = A(\varepsilon s)x + \varepsilon K(\varepsilon s)x, \; x(s_0) = y_0$$

are well-posed on D in the sense of Section VI.9 of [41] for each value of the slowness parameter $\varepsilon \in (0, \infty)$ or, equivalently (substitute $t = \varepsilon s$), that the initial value problems

$$x' = \frac{1}{\varepsilon}A(t)x, \; x(t_0) = y_0 \quad \text{and} \quad x' = \frac{1}{\varepsilon}A(t)x + K(t)x, \; x(t_0) = y_0$$

are well-posed on D. In other words, the evolution systems U_ε, V_ε for the families $\frac{1}{\varepsilon}A$ and $\frac{1}{\varepsilon}A + K$ exist on D and, by the skew self-adjointness of $\frac{1}{\varepsilon}A(t)$ and $K(t)$ for $t \in (-\infty, 0]$, the evolution operators $U_\varepsilon(t,s)$, $V_\varepsilon(t,s)$ are unitary for all

$$(s,t) \in \Delta_{(-\infty,0]} := \{(s,t) \in (-\infty, 0]^2 : s \le t\}.$$

V_ε is an auxiliary evolution system and it is well-known – see [61] for instance – to be adiabatic w.r.t. all the P_j, that is, it exactly intertwines the subspaces $P_j(s)H$ and $P_j(t)H$:

$$V_\varepsilon(t,s)P_j(s) = P_j(t)V_\varepsilon(t,s) \tag{6.2}$$

for all $(s,t) \in \Delta_{(-\infty,0]}$ and $j \in \{1, \dots, r+1\}$. Additionally U_ε^I, V_ε^I, defined by

$$U_\varepsilon^I(t,s) := e^{-A_0 t/\varepsilon}U_\varepsilon(t,s)e^{A_0 s/\varepsilon} \quad \text{and} \quad V_\varepsilon^I(t,s) := e^{-A_0 t/\varepsilon}V_\varepsilon(t,s)e^{A_0 s/\varepsilon} \tag{6.3}$$

for $(s,t) \in \Delta_{(-\infty,0]}$, are the evolution systems for $\frac{1}{\varepsilon}A^I$ and $\frac{1}{\varepsilon}A^I + K^I$ on D, where

$$A^I(t) := \kappa(t)\, e^{-A_0 t/\varepsilon}\, V\, e^{A_0 t/\varepsilon}\big|_D \quad \text{and} \quad K^I(t) := e^{-A_0 t/\varepsilon}K(t)e^{A_0 t/\varepsilon}. \tag{6.4}$$

(In order to see that the derivative of $t \mapsto U_\varepsilon^I(t,s)x$ for $x \in D$ really is continuous – as is required in the definition of evolution systems – use that $t \mapsto U_\varepsilon(t,s)|_Y$ is strongly continuous in $L(Y,Y)$ (Theorem 6.1 (f) of [65]) and that $V|_Y$ is in $L(Y,H)$, where Y denotes the space D endowed with the graph norm of A_0.) In the above formulas, the superindex I refers to the interaction picture, of course.

6.2 Adiabatic switching and a Gell-Mann and Low theorem without spectral gap condition

We can now state and prove a Gell-Mann and Low theorem without spectral gap condition, where the eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$ of $A(t) = \underline{A}(\kappa(t))$ are allowed to be non-isolated in $\sigma(A(t))$ for every $t \in (-\infty, 0]$ – as long as they stay isolated from *each other* except for finitely many crossing points. Its proof rests upon the variant of the adiabatic theorem without spectral gap condition for several eigenvalues $\lambda_1(t), \dots, \lambda_r(t)$. See the fourth remark after the general adiabatic theorem without spectral gap condition (Theorem 4.2.2) from Section 4.2. It seems that even for the case of skew self-adjoint operators this variant of the adiabatic theorem for several eigenvalues is new.

Theorem 6.2.1. *Suppose $\underline{A}$, $\underline{\lambda}_1, \dots, \underline{\lambda}_r$, $\underline{P}_1, \dots, \underline{P}_r$ are as in Condition 6.1.1 and that κ is as in Condition 6.1.2 and define*

$$A(t) := \underline{A}(\kappa(t)), \quad \lambda_j(t) := \underline{\lambda}_j(\kappa(t)), \quad P_j(t) := \underline{P}_j(\kappa(t))$$

for $t \in \{-\infty\} \cup (-\infty, 0]$ and $j \in \{1, \dots, r\}$. Suppose further that for all $j, l \in \{1, \dots, r\}$ with $j \neq l$ the map $\{-\infty\} \cup (-\infty, 0] \ni t \mapsto \lambda_j(t) - \lambda_l(t)$ has only finitely many zeroes and that the exceptional set

$$\big\{t \in \{-\infty\} \cup (-\infty, 0] : \kappa(t) \in N\big\}$$

where the P_j are allowed to differ from the spectral projection of A corresponding to λ_j, is a null set (remember Condition 6.1.1 for the definition of N). Then

$$\frac{U_\varepsilon^I(0, -\infty)x}{\langle x', U_\varepsilon^I(0, -\infty)x\rangle} \longrightarrow \frac{W(0, -\infty)x}{\langle x', W(0, -\infty)x\rangle} \in \ker(A(0) - \lambda_j(0)) \quad (\varepsilon \searrow 0)$$

for all $x \in P_j(-\infty)H$ and $x' \in H$ such that $\langle x', W(0, -\infty)x\rangle \neq 0$. In the above relations W denotes the evolution system for K, where $K(t)$ for $t \in (-\infty, 0]$ is defined as in (6.1).

Proof. We proceed in three steps following the lines of proof of [20]. As a first simple step observe that the limit

$$W(0, -\infty) := \lim_{t \to -\infty} W(0, t),$$

employed in the very formulation of the theorem, exists w.r.t. the norm operator topology of H and that, likewise, the limits

$$U_\varepsilon^I(0, -\infty)x := \lim_{t \to -\infty} U_\varepsilon^I(0, t)x \quad \text{and} \quad V_\varepsilon^I(0, -\infty)x := \lim_{t \to -\infty} V_\varepsilon^I(0, t)x$$

exist for every $x \in H$. Indeed, by virtue of (6.1),

$$\big\|W(0, t) - W(0, t')\big\| = \left\| \int_t^{t'} W(0, \tau)K(\tau)\, d\tau \right\| \leq \left| \int_t^{t'} c\, \kappa'(\tau)\, d\tau \right| \longrightarrow 0 \quad (t, t' \to -\infty),$$

and similarly, using the relative boundedness of V w.r.t. A_0 and the density of D in H, one sees the existence of the other limits.

As a second step we show that the assertion holds true at least for $V_\varepsilon^I(0,-\infty)$ instead of $U_\varepsilon^I(0,-\infty)$, more precisely,

$$\frac{V_\varepsilon^I(0,-\infty)x}{\langle x', V_\varepsilon^I(0,-\infty)x\rangle} = \frac{W(0,-\infty)x}{\langle x', W(0,-\infty)x\rangle} \in \ker(A(0)-\lambda_j(0)) \tag{6.5}$$

for every $x \in P_j(-\infty)H$ and every $x' \in H$ such that $\langle x', W(0,-\infty)x\rangle \neq 0$. So choose and fix vectors x and x' as above – notice that such vectors always exist by $\operatorname{rk} P_j(0) \neq 0$ and by the unitarity of $W(0,-\infty)$. Since

$$P_j(t)H \subset \ker(A(t)-\lambda_j(t)) \tag{6.6}$$

for every $t \in \{-\infty\} \cup (-\infty, 0]$ (use a continuity argument to extend this inclusion from $\{-\infty\} \cup (-\infty,0] \setminus \kappa^{-1}(N)$ to all of $\{-\infty\} \cup (-\infty,0]$) and since V_ε is adiabatic w.r.t. P_j by (6.2), it follows that

$$V_\varepsilon(s,t)P_j(t) = e^{1/\varepsilon \int_t^s \lambda_j(\tau)\, d\tau}\, W(s,t)P_j(t)$$

for all $(t,s) \in \Delta_{(-\infty,0]}$, in other words: the ε-dependence of $V_\varepsilon(s,t)P_j(t)$ is solely contained in a scalar factor. Consequently,

$$\begin{aligned} V_\varepsilon^I(0,t)x = V_\varepsilon(0,t)e^{1/\varepsilon \lambda_j(-\infty)t}x = e^{1/\varepsilon \int_t^0 \lambda_j(\tau)-\lambda_j(-\infty)\, d\tau}\, & W(0,t)P_j(t)x \\ & + e^{1/\varepsilon\, \lambda_j(-\infty)t}\, V_\varepsilon(0,t)\big(P_j(-\infty)-P_j(t)\big)x, \end{aligned}$$

from which it follows with the help of

$$\big|\lambda_j(\tau)-\lambda_j(-\infty)\big| = \big|\underline{\lambda}_j(\kappa(\tau)) - \underline{\lambda}_j(0)\big| \le \big\|\underline{\lambda}_j'\big\|_\infty\, \kappa(\tau) \quad (\tau \in (-\infty,0])$$

and the integrability of κ that

$$V_\varepsilon^I(0,-\infty)x = e^{1/\varepsilon \int_{-\infty}^0 \lambda_j(\tau)-\lambda_j(-\infty)\, d\tau}\, W(0,-\infty)P_j(-\infty)x \tag{6.7}$$

for every $\varepsilon \in (0,\infty)$. We now see that the equality in (6.5) holds true, and the element relation in (6.5) follows by the adiabaticity of W w.r.t. P_j and by (6.6).

As a third – core – step resting upon the adiabatic theorem without spectral gap condition, we show that

$$V_\varepsilon^I(0,-\infty)x - U_\varepsilon^I(0,-\infty)x \longrightarrow 0 \quad (\varepsilon \searrow 0) \tag{6.8}$$

for every $x \in P_j(-\infty)H$, which then yields the convergence

$$\frac{V_\varepsilon^I(0,-\infty)x}{\langle x', V_\varepsilon^I(0,-\infty)x\rangle} - \frac{U_\varepsilon^I(0,-\infty)x}{\langle x', U_\varepsilon^I(0,-\infty)x\rangle} \longrightarrow 0 \quad (\varepsilon \searrow 0)$$

for every $x \in P_j(-\infty)H$ and every $x' \in H$ such that $\langle x', W(0,-\infty)x\rangle \neq 0$, and hence – by virtue of (6.5) – the desired conclusion. So let $x \in P_j(-\infty)H$ be fixed. Since

$$V_\varepsilon^I(0,t) - U_\varepsilon^I(0,t) = \big(V_\varepsilon(0,t) - U_\varepsilon(0,t)\big)e^{A_0 t/\varepsilon}$$

and

$$V_\varepsilon(0,t) - U_\varepsilon(0,t) = \big(V_\varepsilon(0,t_0) - U_\varepsilon(0,t_0)\big)V_\varepsilon(t_0,t) + U_\varepsilon(t_0,t)\big(V_\varepsilon(t_0,t) - U_\varepsilon(t_0,t)\big)$$

for every $t_0 \in (-\infty, 0]$ and every $t \in (-\infty, t]$, we see by the unitarity of $e^{A_0 t/\varepsilon}$, $V_\varepsilon(t_0,t)$, $U_\varepsilon(t_0,t)$ that

$$\begin{aligned}&\big\|V_\varepsilon^I(0,-\infty)x - U_\varepsilon^I(0,-\infty)x\big\| \\ &\qquad \le \big\|V_\varepsilon(0,t_0) - U_\varepsilon(0,t_0)\big\|\,\|x\| + \limsup_{t\to-\infty}\big\|V_\varepsilon(t_0,t) - U_\varepsilon(t_0,t)\big\|\,\|x\| \qquad (6.9)\end{aligned}$$

for every $t_0 \in (-\infty, 0]$. So, the desired convergence (6.8) will be established provided we can show first that

$$\limsup_{t\to-\infty}\big\|V_\varepsilon(t_0,t) - U_\varepsilon(t_0,t)\big\| \longrightarrow 0 \quad (t_0 \to -\infty) \qquad (6.10)$$

uniformly in $\varepsilon \in (0,\infty)$, and second that

$$\big\|V_\varepsilon(0,t_0) - U_\varepsilon(0,t_0)\big\| \longrightarrow 0 \quad (\varepsilon \searrow 0) \qquad (6.11)$$

for every fixed $t_0 \in (-\infty, 0]$. In order to see (6.10) we have only to notice that

$$\big\|V_\varepsilon(t_0,t) - U_\varepsilon(t_0,t)\big\| \le \int_t^{t_0} \|K(s)\|\, ds \le c\int_t^{t_0} \kappa'(s)\, ds$$

for all $\varepsilon \in (0,\infty)$ and all $t_0, t \in (-\infty, 0]$ with $t \le t_0$ and to recall that κ' is integrable. In order to see (6.11) we have only to observe that the variant of the adiabatic theorem for several eigenvalue curves $\lambda_1, \dots, \lambda_r$ (fourth remark after Theorem 4.2.2) can be applied here. Since, however, the arguments for this variant of the adiabatic theorem were only sketched above, we provide here a detailed proof of (6.11). Choose and fix $t_0 \in (-\infty, 0]$ for the rest of the proof and, for every $s \in [t_0, 0]$ that is not a zero of any of the functions $\lambda_j - \lambda_l$ with $j \neq l$, define

$$B_\delta(s) := \frac{1}{2}\sum_{j=1}^{r+1} B_{j\,\delta}(s) \quad \text{and} \quad C_\delta(s) := \sum_{j=1}^{r} C_{j\,\delta}(s)$$

for $\delta \in (0,\infty)$, where

$$B_{j\,\delta} := P_j P_j' \overline{R}_{j\,\delta} + \overline{R}_{j\,\delta} P_j' P_j \quad \text{and} \quad C_{j\,\delta} := \delta \overline{R}_{j\,\delta} P_j' P_j - P_j P_j' \delta \overline{R}_{j\,\delta},$$
$$\overline{R}_{j\,\delta} := (\lambda_j + \delta - A)^{-1}(1 - P_j)$$

for $j \in \{1, \dots, r\}$ and where

$$B_{r+1\,\delta} := \sum_{j=1}^{r} B_{j\,\delta} + \sum_{j=1}^{r} \sum_{l \neq j} B_{j\,l} \quad \text{with} \quad B_{j\,l} := P_l \frac{P_l'}{\lambda_j - \lambda_l} P_j + P_j \frac{P_l'}{\lambda_j - \lambda_l} P_l.$$

It then follows that $[0, t_0] \setminus Z \ni s \mapsto B_\delta(s)$ is strongly continuously differentiable with $B_\delta(s)H \subset D$ (where Z denotes the finite set of zeroes of the functions $\lambda_j - \lambda_l$ on $[t_0, 0]$) and that

$$B_{j\,\delta}A - AB_{j\,\delta} \subset [P_j', P_j] - C_{j\,\delta}$$

for $j \in \{1, \dots, r\}$ as well as

$$\begin{aligned} B_{r+1\,\delta}A - AB_{r+1\,\delta} &\subset \sum_{j=1}^{r} [P_j', P_j] - C_{j\,\delta} + \sum_{j=1}^{r} \sum_{l \neq j} B_{j\,l}A - AB_{j\,l} \\ &= \big[P_1' + \dots + P_r', P_1 + \dots + P_r\big] - C_\delta = [P_{r+1}', P_{r+1}] - C_\delta \end{aligned}$$

because $B_{j\,l}A - AB_{j\,l} \subset [P_l', P_j]$ for all $j, l \in \{1, \dots, r\}$ with $j \neq l$ and because P_{r+1} was defined as $1 - P_1 - \dots - P_r$ after (6.1). Consequently, the approximate commutator equation

$$B_\delta(s)A(s) - A(s)B_\delta(s) + C_\delta(s) \subset \frac{1}{2} \sum_{j=1}^{r+1} [P_j'(s), P_j(s)] = K(s) \tag{6.12}$$

is satisfied for all $s \in [t_0, 0] \setminus Z$ and $\delta \in (0, \infty)$.

In the special case where Z is empty (no crossings between the λ_j in $[t_0, 0]$), it further follows that there is a constant $c \in (0, \infty)$ such that

$$\sup_{s \in [t_0, 0]} \|B_\delta(s)\| \leq \frac{c}{\delta} \quad \text{and} \quad \sup_{s \in [t_0, 0]} \big\|B_\delta'(s)\big\| \leq \frac{c}{\delta^2}$$

for all $\delta \in (0, \infty)$. And therefore, as

$$\begin{aligned} V_\varepsilon(0, t_0) - U_\varepsilon(0, t_0) &= \int_{t_0}^{0} U_\varepsilon(0, s)K(s)V_\varepsilon(s, t_0)\, ds = \varepsilon\, U_\varepsilon(0, s)B_\delta(s)V_\varepsilon(s, t_0)\Big|_{s=t_0}^{s=0} \\ &\quad - \varepsilon \int_{t_0}^{0} U_\varepsilon(0, s)\Big(B_\delta'(s) + B_\delta(s)K(s)\Big)V_\varepsilon(s, t_0)\, ds + \int_{t_0}^{0} U_\varepsilon(0, s)C_\delta(s)V_\varepsilon(s, t_0)\, ds \end{aligned}$$

for every $t \in (-\infty, t_0]$ (by the commutator equation (6.12) and the fundamental theorem of calculus), we obtain the estimate

$$\|V_\varepsilon(0, t_0) - U_\varepsilon(0, t_0)\| \leq c\, \frac{\varepsilon}{\delta} + c\, \frac{\varepsilon}{\delta^2} + \int_{t_0}^{0} \|C_\delta(s)\|\, ds \tag{6.13}$$

for every $\delta \in (0, \infty)$ and $\varepsilon \in (0, \infty)$. Since by assumption $P_j(s)$ for almost every $s \in [t_0, 0]$ is the spectral projection of $A(s)$ onto $\{\lambda_j(s)\}$, it follows by a standard argument of Avron and Elgart in the adiabatic theory without spectral gap condition – see [11] or [131], for instance – that

$$\int_{t_0}^{0} \|C_\delta(s)\| \, ds \longrightarrow 0 \quad (\delta \searrow 0) \tag{6.14}$$

and, hence, the desired convergence (6.11) in the special case of empty Z follows from (6.13) by setting $\delta = \delta_\varepsilon := \varepsilon^{1/3}$ and letting ε tend to 0.

In the general case where Z is finite (finitely many crossings between the λ_j in $[t_0, 0]$), one achieves the desired convergence (6.11) in the same way as in the adiabatic theorem with non-uniform spectral gap condition (Theorem 4.1.2): one decomposes the interval $[t_0, 0]$ into small neighbourhoods around the points of Z and into compact subintervals containing no points of Z, where the neighbourhoods are chosen so small that their contribution to the left hand side of (6.11) becomes small uniformly in $\varepsilon \in (0, \infty)$ and where then ε, in the same way as in the above special case of empty Z, is chosen so small that the contribution of the compact intervals to the left hand side of (6.11) becomes small as well. ■

In the special case where $\operatorname{supp} \kappa$ is compact, the proof above gets even simpler because in that case one has

$$W(0, -\infty) = W(0, t_0),$$
$$U_\varepsilon^I(0, -\infty)x = U_\varepsilon^I(0, t_0)x, \quad V_\varepsilon^I(0, -\infty)x = V_\varepsilon^I(0, t_0)x$$

for $t_0 := \inf \operatorname{supp} \kappa$, so that the first and second step of the above proof become trivial.

With the above theorem at hand, we can now also extend a formula for the energy shift from [53] to the more general situation of not necessarily isolated eigenvalues; this formula expresses the energy shift $\lambda_j(0) - \lambda_j(-\infty)$ as a limit of logarithmic derivatives of certain transition functions.

Corollary 6.2.2. *Suppose that the assumptions of Theorem 6.2.1 are satisfied. Then the energy shift $\lambda_j(0) - \lambda_j(-\infty)$ can be expressed as a limit of logarithmic derivatives of certain transition functions, more precisely,*

$$\lambda_j(0) - \lambda_j(-\infty) = \lim_{\varepsilon \searrow 0} \varepsilon \frac{d}{dt} \log \left\langle x', U_\varepsilon^I(t, -\infty)x \right\rangle \Big|_{t=0} \tag{6.15}$$

for all $x, x' \in P_j(-\infty)H$ with $\langle x', W(0, -\infty)x \rangle \in \mathbb{C} \setminus (-\infty, 0]$. In the above equation, log *denotes the principal branch of the complex logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$.*

Proof. We fix $j \in \{1, \dots, r\}$ and assume $x, x' \in P_j(-\infty)H$ with $\langle x', W(0, -\infty)x \rangle \in \mathbb{C} \setminus (-\infty, 0]$ (notice that existence of such vectors x, x' is not claimed in the statement of the corollary – they exist iff the spaces $P_j(-\infty)H$ and $P_j(0)H$ are not orthogonal to each other). We also set

$$f_\varepsilon(t) := \left\langle x', U_\varepsilon^I(t, -\infty)x \right\rangle \quad \text{and} \quad g_\varepsilon(t) := \left\langle x', V_\varepsilon^I(t, -\infty)x \right\rangle \tag{6.16}$$

for $t \in [-1,0]$ and $\varepsilon \in (0,\infty)$ (notice that the existence of the limits $W(0,-\infty)$, $U_\varepsilon^I(t,-\infty)$, $V_\varepsilon^I(t,-\infty)$ in the strong sense has already been shown in the first step of the proof of the previous theorem).

As a first step we show that the function $f_\varepsilon : [-1,0] \to \mathbb{C}$ is differentiable with derivative at 0 given by

$$f_\varepsilon'(0) = -\frac{1}{\varepsilon} \left\langle Vx', U_\varepsilon^I(0,-\infty)x \right\rangle \tag{6.17}$$

for every $\varepsilon \in (0,\infty)$. In order to do so, we consider the pointwise approximants $f_{\varepsilon n} : [-1,0] \to \mathbb{C}$ to f_ε defined by

$$f_{\varepsilon n}(t) := \left\langle x', U_\varepsilon^I(t,-n)x \right\rangle \qquad (n \in \mathbb{N}) \tag{6.18}$$

and convince ourselves that they are differentiable and that the sequence $(f_{\varepsilon n}')$ of their derivatives is uniformly convergent as $n \to \infty$. Since U_ε^I is the evolution system for $\frac{1}{\varepsilon}A^I$ on D with A^I given by (6.4) and since $x \in P_j(-\infty)H \subset \ker(A(-\infty) - \lambda_j(-\infty)) \subset D$, the function $f_{\varepsilon n}$ is differentiable for every $\varepsilon \in (0,\infty)$ and every $n \in \mathbb{N}$ with

$$\begin{aligned} f_{\varepsilon n}'(t) &= \frac{1}{\varepsilon} \left\langle x', A^I(t)\, U_\varepsilon^I(t,-n)x \right\rangle = \frac{\kappa(t)}{\varepsilon} \left\langle x', e^{-A_0 t/\varepsilon}\, V e^{A_0 t/\varepsilon}\, U_\varepsilon^I(t,-n)x \right\rangle \\ &= -\frac{\kappa(t)}{\varepsilon} \left\langle e^{-A_0 t/\varepsilon}\, V e^{A_0 t/\varepsilon}\, x', U_\varepsilon^I(t,-n)x \right\rangle \end{aligned} \tag{6.19}$$

for $t \in [-1,0]$. In the last equality it was used that V is skew symmetric and that $x' \in P_j(-\infty)H \subset \ker(A(-\infty) - \lambda_j(-\infty)) \subset D \subset D(V)$. Since, moreover,

$$\begin{aligned} \sup_{t\in[-1,0]} \left\| U_\varepsilon^I(t,-n)x - U_\varepsilon^I(t,-m)x \right\| &= \sup_{t\in[-1,0]} \left\| \int_{-n}^{-m} U_\varepsilon^I(t,\tau)\, \frac{\kappa(\tau)}{\varepsilon}\, e^{-A_0\tau/\varepsilon}\, V e^{A_0\tau/\varepsilon} x \, d\tau \right\| \\ &\le \frac{1}{\varepsilon} \left\| V(A_0 - 1)^{-1} \right\| \left| \int_{-n}^{-m} \kappa(\tau)\, d\tau \right| \left\| (A_0 - 1)x \right\| \longrightarrow 0 \end{aligned}$$

as $m, n \to \infty$, it follows in view of (6.19) that

$$\begin{aligned} \sup_{t\in[-1,0]} \left| f_{\varepsilon n}'(t) - f_{\varepsilon m}'(t) \right| \le \frac{1}{\varepsilon} \left\| V(A_0 - 1)^{-1} \right\| \left\| (A_0 - 1)x' \right\| \cdot \\ \cdot \sup_{t\in[-1,0]} \left\| U_\varepsilon^I(t,-n)x - U_\varepsilon^I(t,-m)x \right\| \longrightarrow 0 \end{aligned}$$

as $m, n \to \infty$. So, the pointwise limit f_ε of the functions $f_{\varepsilon n}$ is differentiable with derivative given by $f_\varepsilon'(t) = \lim_{n\to\infty} f_{\varepsilon n}'(t)$ for $t \in [-1,0]$. In particular, $f_\varepsilon'(0)$ is given by (6.17) in virtue of (6.19).

As a second step we show that $f_\varepsilon(0) \neq 0$ for ε small enough and that

$$\varepsilon\, f_\varepsilon'(0)/f_\varepsilon(0) \longrightarrow \lambda_j(0) - \lambda_j(-\infty) \quad (\varepsilon \searrow 0). \tag{6.20}$$

Since $|g_\varepsilon(0)| = |\langle x', W(0,-\infty)x\rangle| \neq 0$ for all $\varepsilon \in (0,\infty)$ by virtue of (6.7) and since $f_\varepsilon(0) - g_\varepsilon(0) \longrightarrow 0$ as $\varepsilon \searrow 0$ by virtue of (6.8), we see that indeed $f_\varepsilon(0) \neq 0$ for ε small enough. With the help of (6.17) and the previous theorem it then follows that

$$\varepsilon\, f_\varepsilon'(0)/f_\varepsilon(0) = -\frac{\langle Vx', U_\varepsilon^I(0,-\infty)x\rangle}{\langle x', U_\varepsilon^I(0,-\infty)x\rangle} \longrightarrow -\frac{\langle Vx', W(0,-\infty)x\rangle}{\langle x', W(0,-\infty)x\rangle} \qquad (\varepsilon \searrow 0). \tag{6.21}$$

Write now $V = A(0) - A(-\infty)$ and recall that $x' \in P_j(-\infty)H \subset \ker(A(-\infty) - \lambda_j(-\infty))$ and that $W(0,-\infty)x \in P_j(0)H \subset \ker(A(0) - \lambda_j(0))$ to obtain

$$\begin{aligned}\big\langle Vx', W(0,-\infty)x\big\rangle &= \big\langle A(0)x', W(0,-\infty)x\big\rangle - \big\langle A(-\infty)x', W(0,-\infty)x\big\rangle \\ &= \big(\lambda_j(-\infty) - \lambda_j(0)\big)\,\big\langle x', W(0,-\infty)x\big\rangle\,. \end{aligned}\tag{6.22}$$

Combining (6.21) and (6.22) we then arrive at the asserted convergence (6.20)

As a third step we show that the derivative $(\log\circ f_\varepsilon)'(0)$ exists precisely for those $\varepsilon \in (0,\infty)$ for which $f_\varepsilon(0) \in \mathbb{C}\setminus(-\infty,0]$, and that 0 is an accumulation point of the set

$$E := \{\varepsilon \in (0,\infty) : f_\varepsilon(0) \in \mathbb{C}\setminus(-\infty,0]\} \tag{6.23}$$

of admissible values of ε. (It should be noticed that, as $(\log\circ f_\varepsilon)'(0)$ exists only for $\varepsilon \in E$, the accumulation point property of 0 is necessary in order for the limit $\lim_{\varepsilon\searrow 0}(\log\circ f_\varepsilon)'(0)$ to make sense in the first place.) Since $f_\varepsilon(0) \in \mathbb{C}\setminus(-\infty,0]$ for $\varepsilon \in E$ and since $f_\varepsilon(t) \longrightarrow f_\varepsilon(0)$ as $t \nearrow 0$ by the first step, we see that

$$f_\varepsilon(t) \in \mathbb{C}\setminus(-\infty,0] = \operatorname{dom}(\log) \quad (t \in (t_{0\,\varepsilon}, 0])$$

for every $\varepsilon \in E$. So, for $\varepsilon \in E$, the function $(\log\circ f_\varepsilon)|_{(t_{0\,\varepsilon},0]}$ is well-defined and differentiable and, in particular, the derivative at 0,

$$(\log\circ f_\varepsilon)'(0) = f_\varepsilon'(0)/f_\varepsilon(0), \tag{6.24}$$

exists. Conversely, for $\varepsilon \notin E$, the point $f_\varepsilon(0)$ does not belong to $\mathbb{C}\setminus(-\infty,0] = \operatorname{dom}(\log)$ and so $(\log\circ f_\varepsilon)'(0)$ does not exist. We have thus shown that $(\log\circ f_\varepsilon)'(0)$ exists precisely for $\varepsilon \in E$ and it remains to show that 0 is an accumulation point of E. We know from (6.7) that

$$g_\varepsilon(0) = \big\langle x', V_\varepsilon^I(0,-\infty)x\big\rangle = e^{i\varphi_0/\varepsilon}\, z_0,$$
$$i\varphi_0 := \int_{-\infty}^0 \lambda_j(\tau) - \lambda_j(-\infty)\, d\tau \in i\mathbb{R} \quad \text{and} \quad z_0 := \big\langle x', W(0,-\infty)x\big\rangle \in \mathbb{C}\setminus(-\infty,0].$$

In case $\varphi_0 = 0$, we have $f_\varepsilon(0) - z_0 = f_\varepsilon(0) - g_\varepsilon(0) \longrightarrow 0$ as $\varepsilon \searrow 0$ by virtue of (6.8) and therefore $f_\varepsilon(0) \in \mathbb{C}\setminus(-\infty,0]$ for ε small enough. So, 0 is an accumulation point of E in the case $\varphi_0 = 0$. In case $\varphi_0 \neq 0$, consider the set

$$\begin{aligned}E_{\vartheta_0} :=&\{\varepsilon \in (0,\infty) : \arg g_\varepsilon(0) \notin (-\vartheta_0 + \pi, \pi + \vartheta_0)\} \\ =&\{\varepsilon \in (0,\infty) : \varphi_0/\varepsilon + \arg z_0 \notin (-\vartheta_0 + \pi, \pi + \vartheta_0) + 2\pi\mathbb{Z}\}\end{aligned}\tag{6.25}$$

for an arbitrary angle $\vartheta_0 \in (0, \pi/2)$ and choose $\varepsilon_0 > 0$ in such a way that

$$|f_\varepsilon(0) - g_\varepsilon(0)| < |z_0|/2 \,\sin\vartheta_0 \tag{6.26}$$

for all $\varepsilon \in (0, \varepsilon_0]$ (which is possible by virtue of (6.8)). It is clear that 0 is an accumulation point of E_{ϑ_0} and thus also of $E_{\vartheta_0} \cap (0, \varepsilon_0]$. It is also easy to see that $E_{\vartheta_0} \cap (0, \varepsilon_0] \subset E$. Indeed, if $\varepsilon \in E_{\vartheta_0} \cap (0, \varepsilon_0]$, then

$$\operatorname{dist}(g_\varepsilon(0), (-\infty, 0]) \geq |z_0| \sin\vartheta_0 \quad \text{and} \quad |f_\varepsilon(0) - g_\varepsilon(0)| < |z_0|/2 \,\sin\vartheta_0$$

by virtue of (6.25) and (6.26), respectively, and therefore

$$\operatorname{dist}(f_\varepsilon(0), (-\infty, 0]) \geq |z_0|/2 \,\sin\vartheta_0 > 0,$$

which implies $\varepsilon \in E$, of course. So, 0 is an accumulation point of E also in the case $\varphi_0 \neq 0$, which concludes our third step.

Combining now (6.21) and (6.24) (and bearing in mind that 0 is an accumulation point of E), we finally obtain the desired conclusion and the proof is finished. ■

In physics, the switching function is almost always chosen to be the exponential function: $\kappa(t) = e^t$ for $t \in [0, \infty)$. And for that special choice of κ an alternative formula for the energy shift can be deduced from the corollary above, namely:

$$\lambda_j(0) - \lambda_j(-\infty) = \lim_{\varepsilon \searrow 0} \varepsilon \frac{d}{d\mu} \Big(\log \big\langle x', (U_\varepsilon^\mu)^I(0, -\infty)x \big\rangle \Big)\Big|_{\mu=1}, \tag{6.27}$$

where U_ε^μ is the evolution system for $\frac{1}{\varepsilon}A^\mu$ on D with $A^\mu(t) := A_0 + \mu\,\kappa(t)V = A_0 + \mu\, e^t\, V$ for $t \in (-\infty, 0]$ and $\mu \in (0, 1]$ and where

$$(U_\varepsilon^\mu)^I(t, s) := e^{A_0 t/\varepsilon}\, U_\varepsilon^\mu(t, s)\, e^{A_0 s/\varepsilon} \qquad ((s, t) \in \Delta_{(-\infty, 0]}).$$

It seems that (6.27) is much more frequently used in the physics literature than (6.15). See, for instance, [43]. In order to deduce (6.27) from the corollary above, one has only to notice that $A^\mu(t) = A_0 + \mu\, e^t\, V = A(t + \log\mu)$ for all $t \in (-\infty, 0]$ and $\mu \in (0, 1]$. So,

$$U_\varepsilon^\mu(t, s) = U_\varepsilon(t + \log\mu, s + \log\mu) \qquad ((s, t) \in \Delta_{(-\infty, 0]}) \tag{6.28}$$

and therefore one sees for vectors $x, x' \in P_j(-\infty)H \subset \ker(A_0 - \lambda_j(-\infty))$ that

$$\begin{aligned} \big\langle x', (U_\varepsilon^\mu)^I(0, -n)x \big\rangle &= \Big\langle x', e^{A_0(\log\mu)/\varepsilon}\, U_\varepsilon^I(\log\mu, -n + \log\mu)\, e^{-A_0(\log\mu)/\varepsilon} x \Big\rangle \\ &= \big\langle x', U_\varepsilon^I(\log\mu, -n + \log\mu)x \big\rangle \end{aligned}$$

for all $\mu \in (0, 1]$ and $n \in \mathbb{N}$. Consequently,

$$\big\langle x', (U_\varepsilon^\mu)^I(0, -\infty)x \big\rangle = \big\langle x', U_\varepsilon^I(\log\mu, -\infty)x \big\rangle = f_\varepsilon(\log\mu) \tag{6.29}$$

for all $\mu \in (0, 1]$ with f_ε defined as in (6.16), so that the corollary above and its proof yield the desired alternative formula (6.27) for the energy shift.

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