Murphy bases
for
endomorphism rings
of
tensor space

Mathias Werth

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Contents

Abstract

In the main part, a cellular basis for the Brauer algebra is constructed which has properties similar to the Murphy basis of the group algebra of the symmetric group. In particular, the restriction of a cell module to a certain subalgebra can be described easily via the combinatoric of this basis. Furthermore, the image of the Brauer algebra in the endomorphism ring of the (symplectic and orthogonal) tensor space has a cellular basis given by cosets (modulo annihilator) of these basis elements. This new basis also has properties similar to the Murphy basis. Finally, the (symplectic and orthogonal) tensor space possesses a filtration by cell modules of the image of the Brauer algebra in the endomorphism ring.

In a further part the constructions of the main part are applied to the walled Brauer algebra and its image in the endomorphism ring of mixed tensor space. It is shown, that known results about the walled Brauer algebra and its image in the endomorphism ring of mixed tensor space fit in this unified concept.

Acknowledgement

In my humble opinion one should not go on the journey of writing a thesis alone. Others can provide many things to bring everything to a good end: First, the possibility of departure, guidance to find a proper path, distraction to not always worry about the difficulties that may appear, and some final push and pulls to reach the goal. I want to express my gratitude for all the support I received in the past few years with all my heart.

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Deep gratitude I owe to both, my wife and my parents: Without you, I could have neither started nor ended this journey.

Zusammenfassung

Im Hauptteil wird eine zelluläre Basis für die Brauer Algebra konstruiert, die einige Eigenschaften hat, die ähnlich zu denen der Murphy Basis der Gruppenalgebra der Symmetrischen Gruppe sind. So ist die Einschränkung von Zellmoduln auf gewisse Unteralgebren leicht durch die Kombinatorik der Basis selbst beschreibbar. Desweiteren hat das Bild der Brauer Algebra im Endomorphismenring des (symplektischen und orthogonalen) Tensorraums eine zelluläre Basis, die aus Nebenklassen (modulo Annulator) besteht. Diese Basis des Endomorphismenrings hat ebenfalls Ähnlichkeiten mit der Murphy Basis. Zuletzt wird mit Hilfe dieser Basis eine Filtrierung des (symplektischen und orthogonalen) Tensorraums durch Zellmoduln des Bildes der Brauer Algebra im Endomorphismenring konstruiert.

In einem weiteren Teil werden die Konstruktionen des Hauptteils auf die rationale Brauer Algebra sowie deren Bild im Endomorphismenring des gemischten Tensorraums angewandt. Auf diese Weise werden bereits bekannte Ergebnisse für die rationale Brauer Algebra sowie deren Bild im Endomorphismenring des gemischten Tensorraums in einheitlicher Weise erneut gezeigt.

Danksagung

Das Erstellen einer Dissertation ist ein Abenteuer, das, meiner bescheidenen Meinung nach, nicht allein bewältigt werden sollte. Vieles, das zum guten Gelingen beiträgt, kann von Anderen kommen: Angefangen bei der Möglichkeit zum Aufbruch, über den Rat zum Finden des Weges, Ablenkung, um schwere Abschnitte zu überstehen, bis zum Ziehen und Schieben auf den letzten Metern bis zum Ziel. Für all die Unterstützung, die mir in den vergangenen Jahren zuteil wurde, möchte ich mich an dieser Stelle von Herzen bedanken.

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Ganz besonders möchte ich meiner Frau und meinen Eltern danken: Ohne Euch hätte ich dieses Abenteuer weder beginnen noch beenden können.

Introduction

How to read this thesis

This thesis is divided into 3 chapters. Each of these chapters tells the same story but with a different protagonist. The protagonists are

> the (group algebra of the) symmetric group,
>
> the Brauer algebra,
>
> the walled Brauer algebra.

The first section of each chapter defines the algebra in question. Some minor observations are stated and some basic concepts are introduced. In the second section the cellular structure is under investigation. One of the key features here is the fact, that all the algebras on the list admit a *Murphy* basis. These bases and their properties dominate the second section while other implications and features of the cellular structure have to take a back seat. The third section is about the algebra's action on the (mixed) tensor space. The algebra modulo the annihilator of the (mixed) tensor space give is isomorphic to a ring of endomorphisms. The centralizer of these endomorphisms is known in each case and one could also start the whole topic with these antagonists:

> the general linear group,
>
> the symplectic and orthogonal group,
>
> the general linear group (caveat: on the mixed tensor space!).

A lot is known about the interplay of these actions centralizing each other. Of course, some of the known facts are collected in order to use them for the main results:

> The Murphy bases and some (quasi-)idempotents that are forcefully inserted into the basis elements make it possible to construct explicit(!) Murphy bases for the endomorphism rings.

> These Murphy bases are used to construct bases of the (mixed) tensor space – bases that automatically give a filtration of the (mixed) tensor space by cell modules of the endomorphism ring.

Some final words on the three protagonists. In the case of the symmetric group the results are well known. So the first chapter is just a collection of results that are stated in such a way, that they are both, foundation and guideline for the other two chapters. The results concerning the walled Brauer algebra are more recent results. They can be found in an (not yet published) article of Friederike Stoll and the author. The way they are presented here differs slightly from the way they are presented in the article. This is done to give the three chapters a unified structure. The results concerning the Brauer algebra (with both antagonists) are new and the remainder of this thesis is used to present them in a pleasant way.

1. Symmetric groups

The symmetric group, and hence this chapter, is going to be the guiding example for this thesis. Having defined both the group, the group algebra, and some tools to study them in the first section, the second section introduces the Murphy basis of the group algebra of the symmetric group. Some properties of this basis are stated for use in other chapters. The third section deals with the action of the group algebra of the symmetric group on tensor-space.

Well, I could cut off the other thumb for a sense of symmetry.

(Julius Murphy Hibbert, M.D.)

1.1. Definitions

Algebraic and diagrammatic description. Throughout this chapter, r will denote a natural number and R a commutative unital ring.

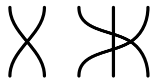
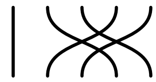
Denote by $\mathfrak{S}_r$ the *symmetric group* which acts on the set $\{1, 2, \dots, r\}$ from the right. Let s_i be the basic transposition $(i, i+1)$ and let $S = \{s_1, s_2, \dots, s_{r-1}\}$. Then $\mathfrak{S}_r$ is generated by the set S together with the relations

- $s_i^2 = 1,$
- $s_i s_j = s_j s_i \quad \text{for } |i - j| > 1,$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

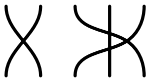
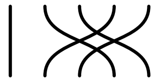
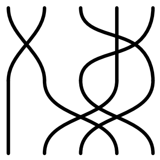
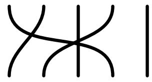
for all values of i and j where the above equations make sense.

To each permutation ω in $\mathfrak{S}_r$ a *diagram* can be assigned in the following way: Draw r dots in a top row and r dots in a bottom row. Enumerate the dots in each row with $1, 2, \dots, r$ from left to right. Connect dot i in the top row with dot $(i)\omega$ in the bottom row through an arc. The precise shape of the arcs is not relevant for the diagrams, i.e. the diagrams should be seen as representatives of homotopy classes fixing the dots in the top and bottom row. The multiplication of these diagrams is defined by concatenation, i.e. for two diagrams A and B the product $A \cdot B$ is obtained by putting the diagram A on top of the diagram B and identifying the dots in the bottom row of A with the dots in the top row of B. This definition is equivalent to the multiplication as composition of permutations.

1.1 Example. Let $r = 5$. The elements $(12)(35)$ and $(24)(35)$ in $\mathfrak{S}_5$ correspond to the diagrams

$(12)(35) =$  and $(24)(35) =$ .

Their product is

 $\cdot$  $=$  $=$  $= (142)$.

Let ω be a permutation in $\mathfrak{S}_r$ and let $s_{i_1}s_{i_2}\dots s_{i_k}$ ($s_{i_k} \in S$) be a word equal to ω. If $s_{i_1}s_{i_2}\dots s_{i_k}$ is of minimal length, then it is called a *reduced expression* for ω. In this case k is called the *length of* ω denoted by $l(\omega)$. Note that the length of a permutation also equals the minimal number of crossings in the corresponding diagram.

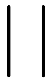
Embedding of subgroups. There are many ways to embed the group $\mathfrak{S}_{r-1}$ (or generally a symmetric group on less than r letters) as a subgroup of $\mathfrak{S}_r$. Throughout this thesis, $\mathfrak{S}_{r-1}$ should be seen as the subgroup of $\mathfrak{S}_r$ stabilizing r unless explicitly stated otherwise. This means that the generators $s_1, s_2, \dots, s_{r-2}$ are used to generate the subgroup $\mathfrak{S}_{r-1}$ of $\mathfrak{S}_r$. In this way the tower of groups

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \dots \subset \mathfrak{S}_{r-1} \subset \mathfrak{S}_r \subset \dots$$

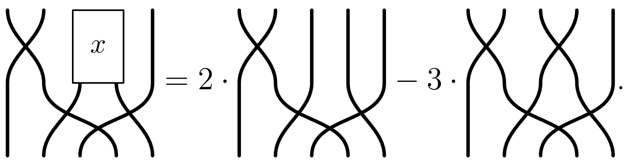
should be understood. In the same way there is a tower of algebras

$$R\mathfrak{S}_1 \subset R\mathfrak{S}_2 \subset \dots \subset R\mathfrak{S}_{r-1} \subset R\mathfrak{S}_r \subset \dots \tag{1.2}$$

A very convenient way to describe different ways to embed the algebras into each other uses the diagrammatic description. To do so, the following convention is fixed: Let x be an element of the group algebra of the symmetric group, i.e. x is a linear combination of permutations. A diagram d involving a box containing x represents the corresponding linear combination of diagrams obtained from d by replacing the box by the diagrams corresponding to permutations occurring in x with nonzero coefficients.

1.3 Example. Let $x = 2 \cdot$  $- 3 \cdot$  $\in R\mathfrak{S}_2$. Then

and



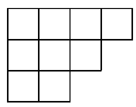
Partitions and tableaux. A finite sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of non-negative integers is called a *composition of* r if

$$|\lambda| := \sum_i \lambda_i = r.$$

If the numbers in the composition are non-increasing and non-zero, then λ is called a *partition of* r and denoted by $\lambda \vdash r$. The set of all partitions of r is denoted by $\mathrm{Par}(r)$. The unique partition of 0 is $\emptyset$. If k many consecutive parts of a composition are equal to i, this can be abbreviated by $i^{(k)}$.

1.4 Example. Both $(4, 3, 2)$ and $(2, 4, 3)$ are compositions of 9. The sequence $(4, 3, 2)$ is also a partition, but $(2, 4, 3)$ is not. Further, the composition $(2, 3, 3, 1)$ of 9 can also be written as $(2, 3^{(2)}, 1)$

To each composition λ of r, a diagram $[\lambda]$ called *Young diagram* is assigned. The diagram $[\lambda]$ is a collection of nodes as follows: In the first row of $[\lambda]$ there are λ_1-many nodes, in the second λ_2-many and so on. The nodes are represented by small boxes and the boxes are left-aligned in each row. Since this results in an obvious injection from the set of partitions of r to the set of Young diagrams with r nodes, both sets are loosely identified.

1.5 Example. The Young diagram  corresponds to the partition $(4, 3, 2)$ of 9.

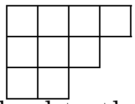
For any composition λ define the dual partition $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \dots)$ where λ'_i is the number of boxes in the ith column of $[\lambda]$. Note that this is always a partition since the boxes in $[\lambda]$ are left aligned. Moreover, λ'' is obtained from λ by ordering the parts of λ in increasing order. In particular, if λ is a partition of r, then $\lambda'' = \lambda$.

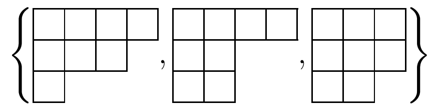
On the set of compositions of r an order called *dominance order* $\trianglerighteq$ is defined as follows:

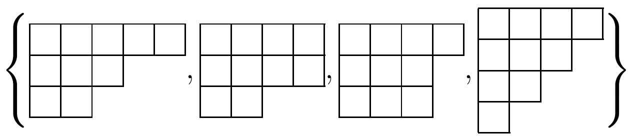
$$\lambda \trianglerighteq \mu \;\Leftrightarrow\; \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i, \quad \forall k.$$

For a partition λ of r, a node (inside $[\lambda]$) is called a *removable node* if $[\lambda]$ without this node is the Young diagram of a partition of $r-1$. In the same sense, a node (outside $[\lambda]$) is called an *addable node* if $[\lambda]$ together with this node is the Young diagram of a partition of $r+1$.

Let λ be a partition of r with t removable and p addable nodes. Define $\mathrm{Rem}(\lambda)$ to be the set of partitions of $r-1$ which can be obtained from λ by removing one node. The set $\mathrm{Rem}(\lambda) \subseteq \mathrm{Par}(r-1)$ is linearly ordered by the dominance order.Therefore, the elements of $\mathrm{Rem}(\lambda)$ can be enumerated such that $\mu^{(1)} \triangleright \mu^{(2)} \triangleright \cdots \triangleright \mu^{(t)}$ for $\mu^{(i)} \in \mathrm{Rem}(\lambda)$. Further, define $\mathrm{Add}(\lambda)$ to be the set of partitions of $r+1$ which can be obtained from λ by adding one node. The set $\mathrm{Add}(\lambda) \subseteq \mathrm{Par}(r+1)$ is also linearly ordered by the dominance order. Thus, the elements of $\mathrm{Add}(\lambda)$ can also be enumerated such that $\nu^{(1)} \triangleright \nu^{(2)} \triangleright \cdots \triangleright \nu^{(p)}$ for $\nu^{(j)} \in \mathrm{Add}(\lambda)$.

1.6 Example. The partition $\lambda = (4,3,2) =$  has 3 removable and 4 addable nodes. Therefore, the definitions above lead to the following sets which are already ordered by the dominance order $\trianglerighteq$:

$\mathrm{Rem}(\lambda) =$  ,

$\mathrm{Add}(\lambda) =$  .

Let λ be a composition. A λ-*tableau* $\mathfrak{t}$ is a map from the Young diagram $[\lambda]$ into a linearly ordered set. Usually the images of the nodes are filled inside the boxes they represent. Unless stated otherwise, the target set will be the set $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ with the natural order.

1.7 Example. Let $\lambda = (4,3,2)$. Then

$$\mathfrak{t} = \begin{array}{|c|c|c|c|} \hline 4 & 5 & 3 & 9 \\ \hline 1 & 7 & 2 \\ \cline{1-3} 8 & 6 \\ \cline{1-2} \end{array}$$

is a λ-tableau. For this tableau $\mathfrak{t}(1,1) = 4$, $\mathfrak{t}(2,1) = 1$, $\mathfrak{t}(2,3) = 2$ and so on.

For a λ-tableau $\mathfrak{t}$ one calls λ the *shape of* $\mathfrak{t}$, denoted by $\mathrm{shape}(\mathfrak{t})$.

A λ-tableau $\mathfrak{t}$ is called *row-semistandard* if the entries in each row do not decrease from left to right, and it is called *column-semistandard* if the entries in each column do not decrease from top to bottom. Further, it is called *row-standard* if the entries in each row increase from left to right and it is called *column-standard* if the entries in each column increase from top to bottom.

A λ-tableau $\mathfrak{t}$ is called *semistandard* if it is both column-standard and row-semistandard. The set of all semistandard tableaux of shape λ is denoted by $\mathrm{Tab}(\lambda)_n$ where the n indicates the target set $\{1, 2, \dots, n\}$.

A λ-tableau $\mathfrak{t}$ is called *standard* if λ is a partition, it is bijective on $\{1, 2, \dots, |\lambda|\}$ and it is both row- and column-standard. The set of standard tableaux of shape λ is denoted by $\mathrm{Std}(\lambda)$.

1.8 Example. Let $\lambda = (4, 3, 2)$. The tableau

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 9 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 5 & 7 \\ \cline{1-2} \end{array}$$

is a standard λ-tableau.

The set of λ-tableaux contains two elements which play a special role in the representation theory of the symmetric group. The first one is the *initial tableau* $\mathfrak{t}^\lambda$ in which the numbers $\{1, 2, \dots, r\}$ are filled in along the rows left to right and top to bottom. The second one is the *final tableau* $\mathfrak{t}_\lambda$ in which the numbers $\{1, 2, \dots, r\}$ are filled in along the columns top to bottom and left to right.

1.9 Example. Let $\lambda = (4, 3, 2) \vdash 9$, then

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2} \end{array} \quad \text{and} \quad \mathfrak{t}_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 3 & 6 \\ \cline{1-2} \end{array}.$$

Let λ and μ be two compositions of r and $\mathfrak{t}$ be a bijective λ-tableau. Define $\mu(\mathfrak{t})$ to be the λ-tableau in which the entries of $\mathfrak{t}$ are replaced by the index of the row in which they appear in $\mathfrak{t}^\mu$.

Side Note: If $\mathfrak{t}$ is a bijective semistandard λ-tableau, then it is standard. But not every row- and column-standard λ-tableau is standard, since it does not need to be bijective on the set $\{1, 2, \dots, |\lambda|\}$.

1.10 Example. Let $\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1&3&6&9\\ \hline 2&4&8 \\ \cline{1-3} 5&7 \\ \cline{1-2}\end{array}$ and $\mu = (2,3,2,2)$. Then $\mathfrak{t}^\mu = \begin{array}{|c|c|c|}\hline 1&2 \\ \hline 3&4&5\\ \hline 6&7 \\ \cline{1-2} 8&9 \\ \cline{1-2}\end{array}$. Thus,

$\mu(\mathfrak{t}) = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 1&2&4 \\ \cline{1-3} 2&3 \\ \cline{1-2}\end{array}$.

Action on tableaux. The symmetric group $\mathfrak{S}_r$ acts on the set of bijective λ-tableaux with entries in $\{1,2,\ldots,r\}$ from the right by permutation of entries. To each such tableau $\mathfrak{t}$, define two elements of the symmetric group: The element $d(\mathfrak{t})$ as the unique element such that $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$ and $b(\mathfrak{t})$ as the unique element such that $\mathfrak{t}_\lambda b(\mathfrak{t}) = \mathfrak{t}$.

1.11 Example. Let $\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1&3&6&9\\ \hline 2&4&8 \\ \cline{1-3} 5&7 \\ \cline{1-2}\end{array}$, then $d(\mathfrak{t}) = (23649785)$ and $b(\mathfrak{t}) = (354)(67)$.

For a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of r define the *Young subgroup* of $\mathfrak{S}_n$ as the row-stabilizer of $\mathfrak{t}^\lambda$. This subgroup is isomorphic to the direct product $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_k}$ and denoted by $\mathfrak{S}_\lambda$. Further, define the elements

$$x_\lambda := \sum_{\omega \in \mathfrak{S}_\lambda} \omega \quad \text{and} \quad y_\lambda := \sum_{\omega \in \mathfrak{S}_{\lambda'}} (-1)^{l(\omega)} \omega$$

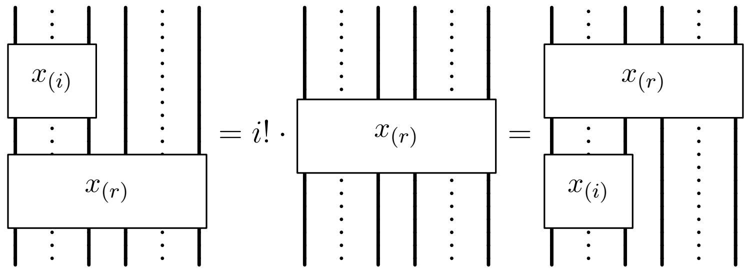
of the group algebra $R\mathfrak{S}_r$. Note that λ' is used in the definition of y_λ. Although this is highly non-standard it helps to smooth out some difficulties.

It is obvious that every ω in $\mathfrak{S}_r$ acts (via multiplication) as identity on $x_{(r)}$. Therefore, $x_{(r)}$ is central and generates the one dimensional two sided ideal in $\mathfrak{S}_r$ corresponding to the trivial character. Generators of this ideal and thus the element $x_{(r)}$ are called *symmetrizer* of $R\mathfrak{S}_r$. Similarly, every ω in $\mathfrak{S}_r$ acts (via multiplication) as $(-1)^{l(\omega)}$ on $y_{(1^{(r)})}$. Therefore, $y_{(1^{(r)})}$ is also central and generates the one dimensional two sided ideal in $\mathfrak{S}_r$ corresponding to the alternating character. Generators of this ideal and thus the element $y_{(1^{(r)})}$ are called *anti-symmetrizer* of $R\mathfrak{S}_r$. Analogously, x_λ is the symmetrizer of $R\mathfrak{S}_\lambda$ and y_λ is the anti-symmetrizer of $R\mathfrak{S}_{\lambda'}$.

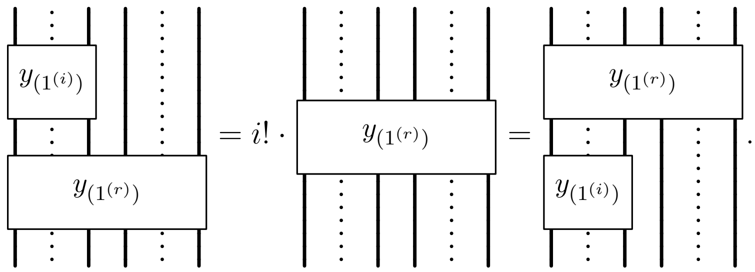
The following corollary is an easy consequence of these facts but it will prove to be useful.

Side Note: As a direct consequence of the definitions, $d(\mathfrak{t})b(\mathfrak{t})^{-1} = d(\mathfrak{t}_\lambda)$ and $b(\mathfrak{t})d(\mathfrak{t})^{-1} = b(\mathfrak{t}^\lambda)$. The element $d(\mathfrak{t}_\lambda)$ is the longest element in the set of distinguished double-coset-representatives of $\mathfrak{S}_\lambda \setminus \mathfrak{S}_r / \mathfrak{S}_{\lambda'}$.

1.12 Corollary. *Let $1 \leq i \leq r$ be a natural number. The following equations hold*



and



In particular, for every partition λ of r both x_λ and y_λ are quasi-idempotents, i.e. $x_\lambda \cdot x_\lambda = a \cdot x_\lambda$ and for some a in R.

1.2. Cellular structure

Cellular bases. The map $* : \mathfrak{S}_r \to \mathfrak{S}_r : \omega \mapsto \omega^{-1}$ extends by linearity to an algebra anti-isomorphism of $R\mathfrak{S}_r$. Note that for $\lambda \vdash r$, $x_\lambda^* = x_\lambda$ and $y_\lambda^* = y_\lambda$.

The following two theorems are due to Murphy [Mur92] and Graham and Lehrer [GL96]. The theorems provide the Murphy bases for the group algebras of the symmetric groups. These bases are cellular bases and have many remarkable properties. Some of these properties will be discussed in the remainder of this chapter. Both, the definition of cellular algebras and the notation used for cellular algebras can be found in the appendix.

1.13 Theorem (Murphy basis I)**.** *The set*

$$\mathfrak{N}_r^x := \{d(\mathfrak{s})^{-1} x_\lambda d(\mathfrak{t}) \mid \lambda \in \mathrm{Par}(r), \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

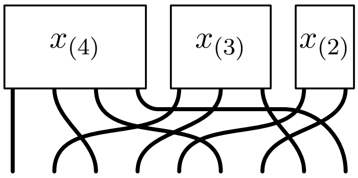
is an R-basis for $R\mathfrak{S}_r$. Moreover, $(\mathfrak{N}_r^x, \mathrm{Par}(r), \trianglerighteq)$ is a cellular basis and therefore $R\mathfrak{S}_r$ is a cellular algebra.

Note that cell modules are right modules throughout this thesis. The cell modules which arise from this cellular structure are called *Specht modules* (following Murphy

[Mur95] and Mathas [Mat99]) and are denoted by S_x^λ. The set $\{x_\lambda d(\mathfrak{t}) + S^{\rhd\lambda} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is an R-basis for the module S_x^λ, where $S^{\rhd\lambda}$ denotes the (two-sided) ideal in $R\mathfrak{S}_r$ generated by all x_μ with $\mu \rhd \lambda$. Note that in this way the *trivial* representation $S_x^{(r)}$ occurs as a submodule of $R\mathfrak{S}_r$ labelled by $\lambda = (r)$.

1.14 Example. Let $r = 9$ and $\lambda = (4,3,2)$. The tableau $\mathfrak{t} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 9 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 5 & 7 \\ \cline{1-2} \end{array}$ is an element of $\mathrm{Std}(\lambda)$ and so

$$x_\lambda d(\mathfrak{t}) = x_{(4,3,2)} d \left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 9 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 5 & 7 \\ \cline{1-2} \end{array} \right) =$$



is (a coset representative of) an R-basis element of the Specht module S_x^λ.

1.15 Theorem (Murphy basis II)**.** *The set*

$$\mathfrak{N}_r^y := \{b(\mathfrak{s})^{-1} y_\lambda b(\mathfrak{t}) \mid \lambda \in \mathrm{Par}(r), \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

is an R-basis for $R\mathfrak{S}_r$. Moreover, $(\mathfrak{N}_r^y, \mathrm{Par}(r), \trianglelefteq)$ is a cellular basis.

The cell modules which arise from this cellular structure can also be called *Specht modules* (following Dipper and James [DJ86]) and are denoted by S_y^λ. More often they are called dual Specht modules in the literature to distinguish them from the Specht modules defined earlier. The set $\{y_\lambda d(\mathfrak{t}) + S^{\lhd\lambda} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ is an R-basis for the module S_y^λ with $S^{\lhd\lambda}$ the (two-sided) ideal in $R\mathfrak{S}_r$ generated by all y_μ with $\mu \lhd \lambda$. Note that in this way the *alternating* representation $S_y^{(1^{(r)})}$ occurs as a submodule of $R\mathfrak{S}_r$ labelled by $\lambda = (1^{(r)})$.

1.16 Remark. The order on the set $\mathrm{Par}(r)$ is crucial for the cellular basis. Keep in mind that the order on $\mathrm{Par}(r)$ is reversed when switching between $\mathfrak{N}_r^x$ and $\mathfrak{N}_r^y$. Nevertheless, the choice is justified by the following observation:

1.17 Proposition ([Mur95], Theorem 5.3)**.** *For a partition λ of r the module S_x^λ is as an $R\mathfrak{S}_r$-module isomorphic to the dual of S_y^λ.*

1.18 Remark. For a field R of characteristic 0 the Specht modules are self dual and hence the modules S_x^λ and S_y^λ are isomorphic.

A crucial ingredient in the proof of Murphys basis theorem are the *Garnir*-tableaux and the *Garnir*-relations [Mur95, Definition 4.5 and Lemma 4.7] together with the dominance order on bijective, row-standard tableaux.

Let λ be a partition of n and $(i+1, j)$ be a node in the corresponding Young diagram (i.e. $\lambda_{i+1} \geq j$). Let $a = \mathfrak{t}^\lambda(i,j)$ and $b = \mathfrak{t}^\lambda(i+1,j)$. The (i,j)*th Garnir tableau of shape* λ is the bijective row-standard λ-tableau $\mathfrak{g}_{ij}$ in which all numbers less than a or greater than b occupy the same positions in $\mathfrak{t}^\lambda$ and in $\mathfrak{g}_{ij}$. The remaining numbers are inserted into the remaining boxes in increasing order from left to right, first along row $i+1$ and then in row i.

1.19 Example. Let $r = 9$ and $\lambda = (4,3,2)$. There are five Garnir tableaux of this shape.

$$\mathfrak{g}_{11} = \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 1&6&7\\\cline{1-3} 8&9\\\cline{1-2}\end{array}\;,\; \mathfrak{g}_{12} = \begin{array}{|c|c|c|c|}\hline 1&4&5&6\\\hline 2&3&7\\\cline{1-3} 8&9\\\cline{1-2}\end{array}\;,\; \mathfrak{g}_{13} = \begin{array}{|c|c|c|c|}\hline 1&2&6&7\\\hline 3&4&5\\\cline{1-3} 8&9\\\cline{1-2}\end{array}\;,\; \mathfrak{g}_{21} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 6&7&8\\\cline{1-3} 5&9\\\cline{1-2}\end{array}\;,\; \mathfrak{g}_{22} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&8&9\\\cline{1-3} 6&7\\\cline{1-2}\end{array}$$

The various steps in Murphy's proof would require additional definitions which are not needed for the remainder of this thesis. Therefore, just one of the final results is stated. Namely one of the results that proofs that the Murphy basis is a cellular basis:

1.20 Theorem ([Mur95, Theorem 4.18])**.** *Let* $\mathfrak{u}$ *and* $\mathfrak{v}$ *be two* μ*-tableaux for a composition* μ*, then* $d(\mathfrak{u})^{-1}x_\mu d(\mathfrak{v})$ *can be written as linear combination of elements in the set*

$$\{d(\mathfrak{s})^{-1}x_\lambda d(\mathfrak{t}) \mid \lambda \in \mathrm{Par}(r), \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \trianglerighteq \mu\}.$$

Restriction of Specht modules. Let λ be a partition of r and let $\mathfrak{t}$ be a λ-tableau. Let $0 \leq i \leq r$ be an integer, then $\mathfrak{t}{\downarrow_i}$ denotes the tableau one gets by removing all nodes with entries greater than i from $\mathfrak{t}$. Note that if $\mathfrak{t}$ is a standard tableau, then $\mathfrak{t}{\downarrow_i}$ is also a standard tableau and in particular, $\mathrm{shape}(\mathfrak{t}{\downarrow_i})$ is again a partition.

The dominance order on tableaux is defined as follows. For λ and μ partitions of r, let $\mathfrak{s}$ and $\mathfrak{t}$ be bijective row-standard λ- and μ-tableaux, respectively. The tableau $\mathfrak{t}$ dominates the tableau $\mathfrak{s}$ ($\mathfrak{t} \trianglerighteq \mathfrak{s}$) if for each $n \leq r$, $\mathfrak{t}{\downarrow_n} \trianglerighteq \mathfrak{s}{\downarrow_n}$.

The following theorem is well known and describes the behaviour of a Specht module under restriction in the tower of algebras (1.2) (cf. [Jam78, chapter 9]).

1.21 Theorem (Branching Rule for Restriction, preliminary formulation)**.** *Let* S_x^λ *be the Specht module for the symmetric group algebra* $R\mathfrak{S}_r$ *to the partition* $\lambda \vdash r$*. For every* $\mu^{(i)} \in \mathrm{Rem}(\lambda) = \{\mu^{(1)} \triangleright \mu^{(2)} \triangleright \cdots \triangleright \mu^{(t)}\}$ *define the* R*-submodule* M_i *of* $\mathrm{Res}^{R\mathfrak{S}_r}_{R\mathfrak{S}_{r-1}}(S_x^\lambda)$ *generated by the set*

$$\{x_\lambda d(\mathfrak{t}) + S^{\triangleright\lambda} \mid \mathfrak{t} \in \mathrm{Std}(\lambda), \mathrm{shape}(\mathfrak{t}{\downarrow_{r-1}}) \trianglerighteq \mu^{(i)}\}.$$

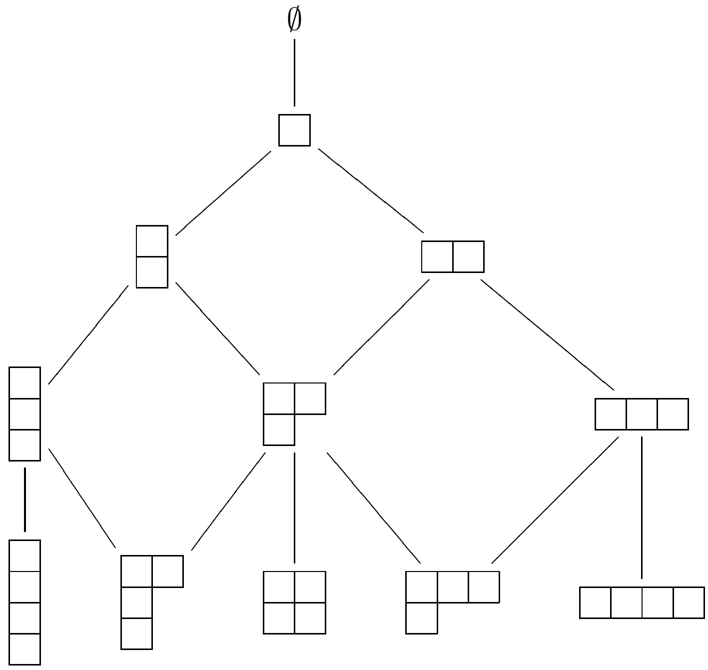


Figure 1.1.: The first four levels of the Bratelli-diagram associated to the symmetric group.

By the definition of the R-modules M_i the inclusions

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_t = \operatorname{Res}^{R\mathfrak{S}_r}_{R\mathfrak{S}_{r-1}} S^\lambda_x$$

hold. Actually, this is a filtration by $R\mathfrak{S}_{r-1}$-modules whose factors M_i/M_{i-1} have R-bases

$$\{x_\lambda d(\mathfrak{t}) + M_{i-1} \mid \mathfrak{t} \in \operatorname{Std}(\lambda), \operatorname{shape}(\mathfrak{t}{\downarrow_{r-1}}) = \mu^{(i)}\}.$$

The maps

$$M_i/M_{i-1} \to S^{\mu^{(i)}}_x \;:\; x_\lambda d(\mathfrak{t}) + M_{i-1} \mapsto x_{\mu^{(i)}} d(\mathfrak{t}{\downarrow_{r-1}}) + S^{\rhd\mu^{(i)}}$$

are isomorphisms of $R\mathfrak{S}_{r-1}$-modules.

1.22 Example. Let $\lambda = (4,3,2) \vdash 9$. Then λ has 3 removable nodes and $\operatorname{Rem}(\lambda) = \{(4,3,1) \rhd (4,2,2) \rhd (3,3,2)\}$. Therefore, M_1 is generated by all $x_{(4,3,2)}d(\mathfrak{t})$ with $\operatorname{shape}(\mathfrak{t}{\downarrow_8}) \trianglerighteq (4,3,1)$, M_2 is generated by all $x_{(4,3,2)}d(\mathfrak{t})$ with $\operatorname{shape}(\mathfrak{t}{\downarrow_8}) \trianglerighteq (4,2,2)$ and M_3 is generated by all $x_{(4,3,2)}d(\mathfrak{t})$ with $\operatorname{shape}(\mathfrak{t}{\downarrow_8}) \trianglerighteq (3,3,2)$.

Let $\mathfrak{t}$ be as in Example 1.14, then

$$\operatorname{shape}(\mathfrak{t}{\downarrow_8}) = \operatorname{shape}\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 9 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 5 & 7 \\ \cline{1-2}\end{array}{\downarrow_8}\right) = \operatorname{shape}\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & 8 \\ \hline 5 & 7 \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & \\ \cline{1-2}\end{array}.$$

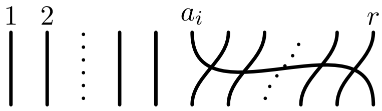
Consequently, the element $x_\lambda d(\mathfrak{t}) + M_2$ is a basis element of M_3/M_2. The isomorphism of Theorem 1.21 maps $x_\lambda d(\mathfrak{t}) + M_2$ on $x_{(3,3,2)}d(\mathfrak{t}{\downarrow_8}) + S^{\rhd(4,3,2)}$. See Figure 1.2 for the diagrammatic description of this.

1.23 Remark. To express the branching rule in a compact way the so-called *Bratelli-diagram associated to the symmetric group* is used. In row i all partitions of i are drawn. A partition λ in row i is connected to a partition μ in row $i+1$ iff $\mu \in \mathrm{Add}(\lambda)$. A truncated Bratelli-diagram is shown in Figure 1.1.

A *path* $\mathfrak{t}$ *(in the Bratelli-diagram associated to the symmetric group)* is a sequence of partitions $(\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$, such that $\lambda^{(i)} \in \mathrm{Add}(\lambda^{(i-1)})$. For such a path $\mathfrak{t} = (\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$, $\lambda^{(r)}$ is called its shape and r its length. It is an easy exercise to show that the set of standard λ-tableaux is in bijection with the set of paths of shape λ.

Let λ be a partition of r. For every $\mu^{(i)} \in \mathrm{Rem}(\lambda)$ define the elements

$$d_{\lambda\to\mu^{(i)}} := s_{a_i}s_{a_i+1}\ldots s_{r-1} =$$



and

$$\bar{d}_{\lambda\to\mu^{(i)}} := \frac{1}{\mu^{(i)}!}d_{\lambda\to\mu^{(i)}}$$

where $\mu^{(i)}$ is obtained from λ by removing a node in the kth row, a_i is the number of boxes in the first k rows of λ and $\mu^{(i)}!$ is defined as $\mu_1^{(i)}!\mu_2^{(i)}!\mu_3^{(i)}!\cdots$. Of course, $\bar{d}_{\lambda\to\mu^{(i)}}$ is not an element of $R\mathfrak{S}_r$ for arbitrary rings R. So for the moment, it should be seen as an element of $\mathbb{Q}\mathfrak{S}_r$. Wherever it is used in the future, either it is multiplied with other elements such that the result is an element of $R\mathfrak{S}_r$ or the ring R is restricted such that $\frac{1}{\mu^{(i)}!}$ is an element of the ring.

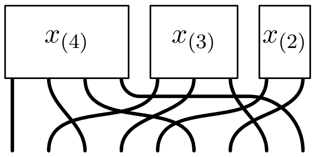
1.24 Remark. Let $\mathfrak{t}$ be the standard λ-tableau with entries in $\{1, 2, \ldots, r\}$ such that $\mathfrak{t}{\downarrow_{r-1}} = \mathfrak{t}^{\mu^{(i)}}$, i.e. $\mathfrak{t}$ has the entry r in the node, that is removed to obtain $\mu^{(i)}$, and the numbers $\{1, 2, \ldots, r-1\}$ are filled along the rows left to right and top to bottom. Then $d_{\lambda\to\mu^{(i)}} = d(\mathfrak{t})$.

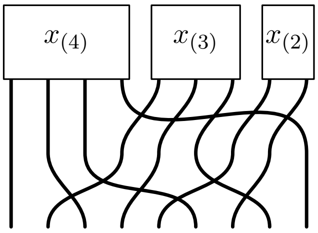
1.25 Remark. Let $\mathfrak{t}$ be a standard λ-tableau with $(\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ as corresponding path, then $d(\mathfrak{t}) = d_{\lambda^{(r)}\to\lambda^{(r-1)}}d_{\lambda^{(r-1)}\to\lambda^{(r-2)}}\cdots d_{\lambda^{(1)}\to\lambda^{(0)}}$.

1.26 Remark. Let λ be a partition of r and let $\mu \in \mathrm{Rem}(\lambda)$. As a direct consequence of Corollary 1.12 the equality

$$x_\lambda\bar{d}_{\lambda\to\mu}x_\mu = \frac{1}{\mu!}x_\lambda d_{\lambda\to\mu}x_\mu = x_\lambda d_{\lambda\to\mu}$$

holds. See Figure 1.3 for an example. This means that for rings R with $\frac{1}{\mu!} \notin R$ the element $x_\lambda\bar{d}_{\lambda\to\mu}x_\mu$ can be replaced by the element $x_\lambda d_{\lambda\to\mu} \in R\mathfrak{S}_r$.

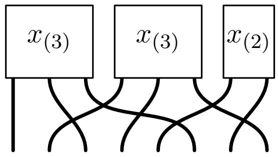
$x_\lambda d(\mathfrak{t}) =$ 

$=$ 

$d_{(4,3,2)\to(3,3,2)}$

$d(\mathfrak{t}\downarrow_8)$

hence

$x_\lambda d(\mathfrak{t}) + M_2 \mapsto$ 

$+\, S^{\rhd(3,3,2)}$

Figure 1.2.: Example to the isomorphism of Theorem 1.27. In the first row the element $d(\mathfrak{t}\downarrow_8)$ is embedded into $R\mathfrak{S}_9$ as described at the beginning of this chapter.

Shifting the perspective. With this in mind the branching rule (Theorem 1.21) may be reformulated in the following way:

1.27 Theorem (Branching Rule for Restriction)**.** *Let S_x^λ be the Specht module for the symmetric group $\mathfrak{S}_r$ to the partition $\lambda \vdash r$.*

The $R\mathfrak{S}_{r-1}$-submodules M_i in Theorem 1.21 are obtained as

$$M_i := \sum_{s \le j} (x_\lambda \bar{d}_{\lambda\to\mu^{(s)}} x_{\mu^{(s)}} + S^{\rhd\lambda}) R\mathfrak{S}_{r-1}.$$

Side Note: Let λ be a partition of r, μ an element of $\operatorname{Rem}(\lambda)$. In the way it is defined here, the Specht module S^λ is a quotient of the permutation module $M^\lambda = x_\lambda R\mathfrak{S}_r = \operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_r} \mathbf{1}$. The restriction of the permutation module to $R\mathfrak{S}_{r-1}$ decomposes as follows (Mackey decomposition):

$$\operatorname{Res}_{\mathfrak{S}_r}^{\mathfrak{S}_{r-1}} \operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_r} \mathbf{1} = \bigoplus_{\mathfrak{S}_\lambda \omega \mathfrak{S}_{r-1}} \operatorname{Ind}_{\mathfrak{S}_\lambda^\omega \cap \mathfrak{S}_{r-1}}^{\mathfrak{S}_{r-1}} \operatorname{Res}_{\mathfrak{S}_\lambda^\omega \cap \mathfrak{S}_{r-1}}^{\mathfrak{S}_\lambda} \mathbf{1}^\omega.$$

Every double coset $\mathfrak{S}_\lambda \omega \mathfrak{S}_{r-1}$ appearing in the decomposition equals a double coset $\mathfrak{S}_\lambda d_{\lambda\to\nu} \mathfrak{S}_{r-1}$ for some *composition* ν of $r-1$ obtained from λ by removing a node (and defining $d_{\lambda\to\nu}$ analogously).

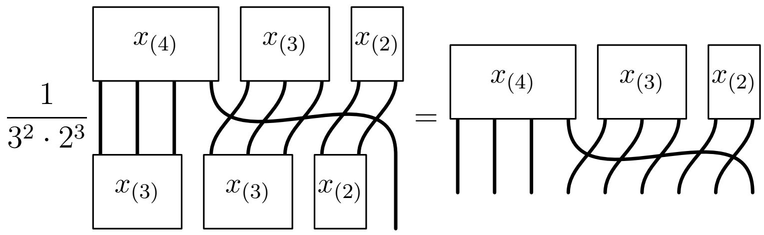


Figure 1.3.: Example to Remark 1.26.

The (inverse of the) isomorphism is given by

$$S_x^{\mu^{(i)}} \to M_i/M_{i-1} : x_{\mu^{(i)}} + S^{\rhd\mu^{(i)}} \mapsto x_\lambda \bar{d}_{\lambda\to\mu^{(i)}} x_{\mu^{(i)}} + M_{i-1}.$$

One way to think of paths (in the Bratelli-diagram) is that they describe the 'lineage' of the basis elements which they index in terms of the branching rule. To emphasise this 'lineage' of the basis elements even more, additional 'x_λ' can be inserted into the basis elements as shown in Figure 1.4.

To be precise: Let $\mathfrak{t} = (\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$, $\lambda^{(r)}$ be a path of shape λ in the Bratelli-diagram associated to the symmetric group. The element $d_\mathfrak{t} \in R\mathfrak{S}_r$ is defined as

$$d_\mathfrak{t} := \bar{d}_{\lambda^{(r)}\to\lambda^{(r-1)}} x_{\lambda^{(r-1)}} \bar{d}_{\lambda^{(r-1)}\to\lambda^{(r-2)}} x_{\lambda^{(r-2)}} \ldots x_{\lambda^{(1)}} \bar{d}_{\lambda^{(1)}\to\lambda^{(0)}}.$$

With this definition, the basis $\mathfrak{N}_r^x$ can be written as

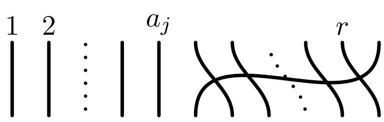
$$\mathfrak{N}_r^x = \{d_\mathfrak{s}^{-1} x_\lambda d_\mathfrak{t} \mid \lambda \in \mathrm{Par}(r), \mathfrak{s}, \mathfrak{t} \text{ paths of shape } \lambda\}. \tag{1.28}$$

This *is* exactly the same set as in Theorem 1.13 but it describes the 'lineage' in a more visible way. What does this mean? Every basis element contains all the (quasi-)idempotents occurring in the lineage of this basis element. This may give no additional information or properties here, but it is a useful thought for the investigation of the (walled) Brauer algebras in the following two chapters.

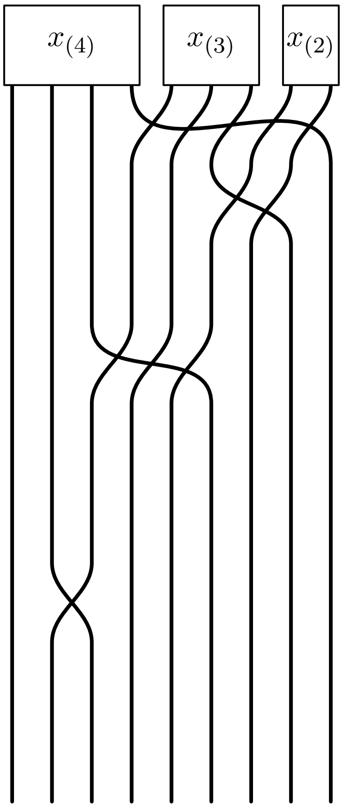
The rule for inducing Specht modules up is now formulated similarly to Theorem 1.27 which makes the rules for restriction and induction look even more symmetric to each other.

Let λ be a partition of r. For every $\nu^{(j)} \in \mathrm{Add}(\lambda)$ define the elements

$$d_{\lambda\to\nu^{(j)}} := s_{r+1,a_j} =$$



$x_\lambda d(\mathfrak{t}) =$



$= \frac{1}{3^4 \cdot 2^{13}} \cdot$

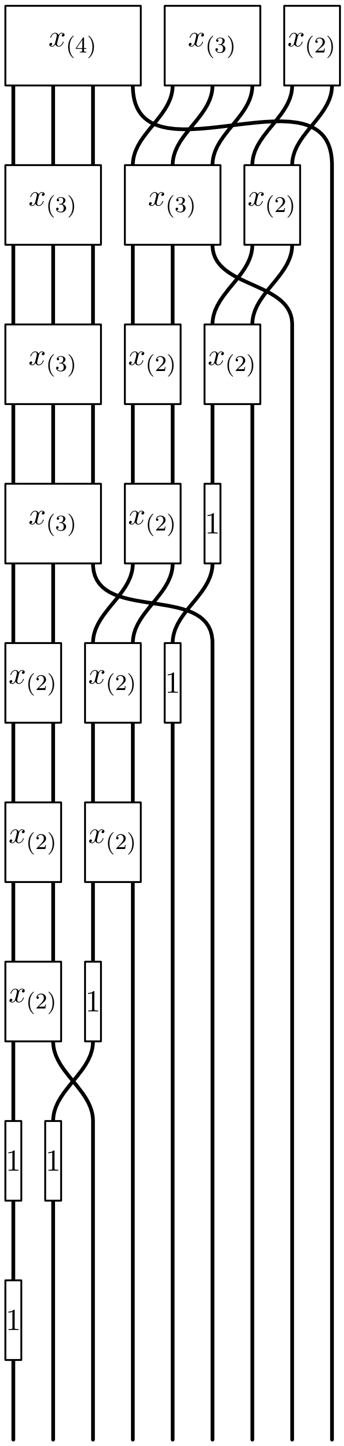


Figure 1.4.: The element of Example 1.14 artificially inflated and with additional 'x_λ' filled in. See also the paragraph following Theorem 1.27 for more details.

and

$$\bar{d}_{\lambda\to\nu^{(j)}} := \frac{1}{\lambda!} d_{\lambda\to\nu^{(j)}} := \frac{1}{\lambda_1!\lambda_2!\lambda_3!\cdots} d_{\lambda\to\nu^{(j)}}$$

where $a_j - 1$ is the number of boxes in the first k rows of λ and $\nu^{(j)}$ is obtained from λ by adding a node in the kth row.

A proof to the following theorem can also be found in [Jam78, chapter 9].

1.29 Theorem (Branching Rule for Induction)**.** *Let S_x^λ be the Specht module for the symmetric group $\mathfrak{S}_r$ to the partition $\lambda \vdash r$. For every $\nu^{(j)} \in \mathrm{Add}(\lambda)$ define the $R\mathfrak{S}_{r+1}$-submodules N_j of $\mathrm{Ind}_{R\mathfrak{S}_r}^{R\mathfrak{S}_{r+1}}(S_x^\lambda)$ as*

$$N_j := \sum_{s\leq j} (x_\lambda + S^{\rhd\lambda})\bar{d}_{\lambda\to\nu^{(s)}} x_{\nu^{(s)}} R\mathfrak{S}_{r+1}.$$

This leads to the filtration

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_p = \mathrm{Ind}_{R\mathfrak{S}_r}^{R\mathfrak{S}_{r+1}} S_x^\lambda$$

such that $N_j/N_{j-1} \cong S_x^{\nu^{(j)}}$ as $R\mathfrak{S}_{r+1}$-modules. The isomorphism is given by

$$S_x^{\nu^{(j)}} \to N_j/N_{j-1} : x_{\nu^{(j)}} + S^{\rhd\nu^{(j)}} \mapsto x_\lambda \bar{d}_{\lambda\to\nu^{(j)}} x_{\nu^{(j)}} + N_{j-1}.$$

1.3. Action on tensor space

Schur-Weyl-duality. Let V be an n-dimensional vector space over a field $\mathbb{F}$ with basis $\{v_1, v_2, \ldots, v_n\}$. The symmetric group $\mathfrak{S}_r$ acts on the r-fold *tensor space* $V^{\otimes r}$ from the right by place permutation. This action can be extended to the group algebra $\mathbb{F}\mathfrak{S}_r$. Hence, this extended action defines a map $\Psi : \mathbb{F}\mathfrak{S}_r^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}}(V^{\otimes r})$.

The general linear group acts diagonally on the r-fold tensor space. Thus, there is also a map $\Phi : \mathbb{F}\,\mathrm{GL}(V) \to \mathrm{End}_{\mathbb{F}}(V^{\otimes r})$. Moreover, the following statements hold:

1.30 Theorem.
1. *The left action of $\mathrm{GL}(V)$ on the r-fold tensor space commutes with the right action of $\mathbb{F}\mathfrak{S}_r$. Therefore, the maps Ψ and Φ read as follows*
   - $\Psi : \mathbb{F}\mathfrak{S}_r^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}(V^{\otimes r})$,
   - $\Phi : \mathbb{F}\,\mathrm{GL}(V) \to \mathrm{End}_{\mathbb{F}\mathfrak{S}_r}(V^{\otimes r})$.
2. *If $\mathbb{F}$ is an infinite field then the maps Ψ and Φ in 1. are surjective.*

3. *If $\mathbb{F}$ is an infinite field and $n \geq r$ then the map Ψ is injective and hence* $\mathbb{F}\mathfrak{S}_r^{\mathrm{op}} \cong \operatorname{End}_{\mathbb{F}\,\mathrm{GL}(V)}(V^{\otimes r})$.
4. *If $\mathbb{F} = \mathbb{C}$, then*
$$V^{\otimes r} \cong \bigoplus_{\substack{\lambda \vdash r \\ \lambda_1' \leq n}} \Delta_\lambda \otimes S^\lambda$$
as irreducible $(\mathbb{C}\,\mathrm{GL}(V), \mathbb{C}\mathfrak{S}_r)$-bimodules, where Δ_λ and S^λ are the irreducible $\mathbb{C}\,\mathrm{GL}(V)$- and $\mathbb{C}\mathfrak{S}_r$-modules associated to λ.

1.31 Remark. This double-centralizer property is well-known as Schur-Weyl-duality. The case $\mathbb{F} = \mathbb{C}$ was first proved by Schur in 1927 [Sch27]. The theorem is named after him and Weyl, who popularized it in 1939 [Wey39]. The generalisation for fields of arbitrary characteristic can be found in [CL74] by Carter and Lusztig.

The first corollary to the theorem is as sum formula relating the known dimensions of the irreducible $\mathbb{C}\,\mathrm{GL}(V)$- and $\mathbb{C}\mathfrak{S}_r$-modules to the dimension of the complex tensor space.

1.32 Corollary. *The equation*
$$\sum_{\lambda \vdash r} |\operatorname{Tab}(\lambda)_n| \cdot |\operatorname{Std}(\lambda)| = n^r$$
holds.

Proof. The last part of the theorem on Schur-Weyl-duality implies that
$$\dim_{\mathbb{C}} V^{\otimes r} = \sum_{\substack{\lambda \vdash r \\ \lambda_1' \leq n}} \dim_{\mathbb{C}}(\Delta_\lambda) \cdot \dim_{\mathbb{C}}(S^\lambda).$$
It is known that $\dim_{\mathbb{C}}(\Delta_\lambda) = |\operatorname{Tab}(\lambda)_n|$ and $\dim_{\mathbb{C}}(S^\lambda) = |\operatorname{Std}(\lambda)|$. Thus
$$n^r = \dim_{\mathbb{C}} V^{\otimes r} = \sum_{\lambda \vdash r} |\operatorname{Tab}(\lambda)_n| \cdot |\operatorname{Std}(\lambda)|.$$
□

The following theorem is due to Härterich [Här99] and gives a complete and explicit description of the annihilator.

1.33 Theorem. *If $n < r$, then the annihilator $\operatorname{Ann}_{\mathbb{F}\mathfrak{S}_r}(V^{\otimes r})$ is generated by the element $y_{(r-n,1^{(n)})}$. Moreover, the set*
$$\{b(\mathfrak{s})^{-1} y_\mu b(\mathfrak{t}) \mid \mu \in \operatorname{Par}(r),\ \mu \textit{ has more than } n \textit{ parts},\ \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mu)\} \subset \mathfrak{N}_r^y$$
is a basis of the annihilator.

1.34 Remark. Embedding $\mathbb{F}\mathfrak{S}_{n+1}$ into $\mathbb{F}\mathfrak{S}_r$ the element $y_{(1^{(n+1)})}$ equals $y_{(r-n,1^{(n)})}$.

1.35 Remark. In particular, the theorem above implies that the annihilator is an ideal in the cell chain and it consists of all cell blocks indexed by partitions with more than n parts. On the other hand there is the following description of the endomorphisms of $V^{\otimes r}$ which arise from the action of $\mathbb{F}\mathfrak{S}_r$.

1.36 Corollary. *Let* ann *be the annihilator* $\mathrm{Ann}_{\mathbb{F}\mathfrak{S}_r}(V^{\otimes r})$. *Then the set*

$$\{b(\mathfrak{s})^{-1}y_\mu b(\mathfrak{t}) + \mathrm{ann} \mid \mu \in \mathrm{Par}(r),\ \mu \textit{ has at most } n \textit{ parts},\ \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\mu)\}$$

is a basis of the endomorphism ring $\mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}(V^{\otimes r})$. *Moreover, this set is a cellular basis of* $\mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}(V^{\otimes r})$.

This is another remarkable property of the Murphy basis of the symmetric group. If the action on tensor space is considered, then the basis splits into two parts: one part generating the annihilator and one part being a cellular basis of the endomorphism ring.

Filtration of tensor space. At last, take a closer look on the tensor space itself and on a basis that has a couple of properties in common with the Murphy basis. In order to do so, define the weight of tableaux and vectors:

The weight of a tableau T is the composition $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ such that μ_i counts how often i appears in the tableau T, i.e. $\mu_i = |T^{-1}(i)|$. The set of all semistandard λ-tableaux of weight μ is denoted by $\mathrm{Tab}(\lambda)_{n,\mu}$. Denote for later reference the unique element of $\mathrm{Tab}(\lambda)_{n,\lambda}$ by T^λ.

1.37 Example. Let $\lambda = (4, 3, 2)$. Then

$$T^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array}.$$

Let $v = v_{i_1} \otimes \cdots \otimes v_{i_r}$ be a simple tensor in basis vectors of V. The weight of v is the composition $(\mu_1, \dots, \mu_n)$ of r into n parts such that μ_k counts how often k appears in $i_1, \dots, i_r$.

Denote by M^μ the subspace of $V^{\otimes r}$ generated by the basis vectors of weight μ. The subspace M^μ is called *weight space* and its elements *weight vectors* to weight μ. Clearly, the weight spaces are submodules of the tensor space as modules of the group algebra of the symmetric group $\mathbb{F}\mathfrak{S}_r$. It is well known that the tensor space has the *weight space decomposition*:

$$V^{\otimes r} = \bigoplus_\mu M^\mu$$

where the summation runs through all compositions μ of r into n parts.

Let λ be a partition of r and T a λ-tableau with entries in $\{1, \ldots, n\}$. Let $i_1, \ldots, i_r$ denote the sequence of entries of T read column by column top to bottom and left to right. Then the element v_T of the tensor space $V^{\otimes r}$ is defined as

$$v_T := v_{i_1} \otimes \cdots \otimes v_{i_r}.$$

Let $\mathfrak{t}$ further be a standard λ-tableau and define

$$v_{T\mathfrak{t}} := v_T y_\lambda b(\mathfrak{t}).$$

By definition T and v_T have the same weight and $v_{T\mathfrak{t}}$ is an element of the corresponding weight space.

The following observation will be used in the proof of Theorem 1.39.

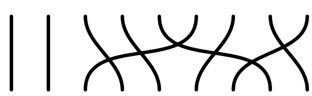
1.38 Lemma. *Let $\lambda \in \mathrm{Par}(r), \mathfrak{s} \in \mathrm{Std}(\lambda)$ and μ a composition of r into n parts. The following equation holds*

$$v_{T^\mu} b(\mathfrak{t}^\mu) b(\mathfrak{s})^{-1} = v_{\mu(\mathfrak{s})}.$$

Proof. The general proof can be deduced from the following example and the accompanying explanations.

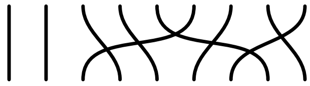
Let $\lambda = \ydiagram{4,3,2}$, $\mathfrak{s} = \begin{ytableau} 1 & 3 & 6 & 9 \\ 2 & 4 & 8 \\ 5 & 7 \end{ytableau}$ and $\mu = \ydiagram{1,2,1,3,2}$. Thus, $T^\mu = \begin{ytableau} 1 \\ 2 & 2 \\ 3 \\ 4 & 4 & 4 \\ 5 & 5 \end{ytableau}$ and $v_{T^\mu} = v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_2 \otimes v_4 \otimes v_5 \otimes v_4$. The element $b(\mathfrak{t}^\mu)$ is the element which permutes the entries in $\mathfrak{t}_\mu = \begin{ytableau} 1 \\ 2 & 6 \\ 3 \\ 4 & 7 & 9 \\ 5 & 8 \end{ytableau}$ to receive $\mathfrak{t}^\mu = \begin{ytableau} 1 \\ 2 & 3 \\ 4 \\ 5 & 6 & 7 \\ 8 & 9 \end{ytableau}$, i.e. $\mathfrak{t}_\mu b(\mathfrak{t}^\mu) = \mathfrak{t}^\mu$.

So,

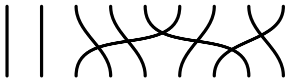
$b(\mathfrak{t}^\mu) =$  .

Side Note: Let μ and ν be compositions of r into n parts such that $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\mu_{(1)\omega}, \ldots, \mu_{(n)\omega})$ for some $\omega \in \mathfrak{S}_n$. Then $M^\mu \cong M^\nu$ as $\mathfrak{S}_r$-modules. Therefore, the investigation of weight spaces can be restricted to dominant weights, i.e. weights μ that are partitions of r into at most n parts.

Thus,

$$v_{T^\mu} b(\mathfrak{t}^\mu) = (v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_2 \otimes v_4 \otimes v_5 \otimes v_4) \cdot$$ 

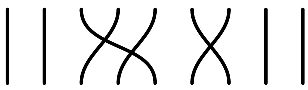
$v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_2 \otimes v_4 \otimes v_5 \otimes v_4$

$=$  $= v_1 \otimes v_2 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_4 \otimes v_4 \otimes v_5 \otimes v_5.$

In the last equation the third term visualizes the action of a permutation ω on a simple vector, i.e. the arcs connect the place on which an entry v_i is before ω is applied with the place on which it is afterwards.

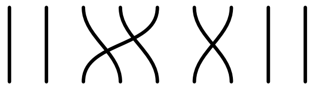
So in the element $v_{T^\mu} b(\mathfrak{t}^\mu)$ the indices are the entries of T^μ read *row by row*! This is no surprise, since this is what the element $b(\mathfrak{t}^\mu)$ does: Making the column-wise filled $\mathfrak{t}_\mu$ to a row-wise filled $\mathfrak{t}^\mu$.

Finally, the element $b(\mathfrak{s})$ permutes the entries in $\mathfrak{t}_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 7 & 9 \\ \hline 2 & 5 & 8 & \\ \cline{1-3} 3 & 6 & & \\ \cline{1-2} \end{array}$ to get $\mathfrak{s}$, i.e. $\mathfrak{t}_\lambda b(\mathfrak{s}) = \mathfrak{s}$. So,

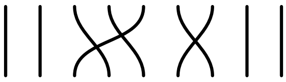
$b(\mathfrak{s}) =$  .

Thus,

$$v_{T^\mu} b(\mathfrak{t}^\mu) b(\mathfrak{s})^{-1}$$

$$= (v_1 \otimes v_2 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_4 \otimes v_4 \otimes v_5 \otimes v_5) \cdot$$ 

$v_1 \otimes v_2 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_4 \otimes v_4 \otimes v_5 \otimes v_5$

$=$  $= v_1 \otimes v_2 \otimes v_4 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_4 \otimes v_5 \otimes v_5 = v_{\mu(\mathfrak{s})},$

since $\mu(\mathfrak{s}) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 2 & 3 & 5 & \\ \cline{1-3} 4 & 4 & & \\ \cline{1-2} \end{array}$. So the equality holds as claimed. □

The following results are well known. Proofs and details can be found in [Mat99] and [Gre80] ([Gre07] resp.), for example.

Side Note: It is known that the weight spaces M^μ are in fact permutation modules. One gets the module M^μ by inducing the trivial $\mathfrak{S}_\mu$-module up to $\mathfrak{S}_r$. Since this correspondence is not true for the other algebras in question, the modules M^μ are just referenced as weight spaces in the text.

1.39 Theorem. *1. The set*

$$\left\{ v_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)_{n,\mu}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(r), \\ \mu \textit{ a composition of } r \textit{ into at most } n \textit{ parts} \end{array} \right\}$$

is a basis of $V^{\otimes r}$.

2. The set

$$\{v_{T\mathfrak{t}} \mid T \in \mathrm{Tab}(\lambda)_{n,\mu}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(r)\}$$

is a basis of M^μ.

Proof. This theorem follows in an elegant way from the Murphy basis and the Schur-Weyl duality over the complex numbers.

First, observe that $M^\mu = v_{T^\mu}\mathbb{F}\mathfrak{S}_r = v_{T^\mu}b(\mathfrak{t}^\mu)\mathbb{F}\mathfrak{S}_r$. Therefore, the set

$$\{v_{T^\mu}b(\mathfrak{t}^\mu)b(\mathfrak{s})^{-1}y_\lambda b(\mathfrak{t}) \mid \lambda \in \mathrm{Par}(r), \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

generates M^μ as a vector space. Let $\lambda \in \mathrm{Par}(r), \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$, then

$$v_{T^\mu}b(\mathfrak{t}^\mu)b(\mathfrak{s})^{-1}y_\lambda b(\mathfrak{t}) = v_{\mu(\mathfrak{s})}y_\lambda b(\mathfrak{t}) = v_{\mu(\mathfrak{s})\mathfrak{t}}.$$

Starting with a standard λ-tableau $\mathfrak{s}$, the λ-tableau $\mu(\mathfrak{s})$ is always column- and row-semistandard. But $v_{\mu(\mathfrak{s})\mathfrak{t}} = 0$ whenever $\mu(\mathfrak{s})$ is not column-standard, i.e. if it has repeated entries in a column, so only semistandard tableaux appear with this construction. The other way round, for every semistandard λ-tableau T there exists a standard λ-tableau $\mathfrak{t}$ such that $\mu(\mathfrak{t}) = T$. Thus, the set

$$\{v_{T\mathfrak{t}} \mid T \in \mathrm{Tab}(\lambda)_{n,\mu}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(r)\}$$

generates M^μ as a vector space and the set

$$\left\{ v_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)_{n,\mu}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(r), \\ \mu \text{ a composition of } r \text{ into at most } n \text{ parts} \end{array} \right\}$$

generates $V^{\otimes r}$ as a vector space. Since this set has at most $\sum_{\lambda\vdash r} |\mathrm{Tab}(\lambda)_n| \cdot |\mathrm{Std}(\lambda)| = n^r$ elements, it has exactly n^r elements. Thus, it is a basis of $V^{\otimes r}$.

□

For every $\nu \in \mathrm{Par}(r)$ define $V(\trianglelefteq \nu)$ and $V(\triangleleft \nu)$ as subspaces of $V^{\otimes r}$ generated by

$$\left\{ v_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)_{n,\mu}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(r), \lambda \trianglelefteq \nu, \\ \mu \text{ a composition of } r \text{ into at most } n \text{ parts} \end{array} \right\}$$

and

$$\left\{ v_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \operatorname{Tab}(\lambda)_{n,\mu}, \mathfrak{t} \in \operatorname{Std}(\lambda), \lambda \in \operatorname{Par}(r), \lambda \lhd \nu, \\ \mu \text{ a composition of } r \text{ into at most } n \text{ parts} \end{array} \right\}.$$

That these subspaces are $\mathbb{F}\mathfrak{S}_r$-submodules of the tensor space is an immediate consequence of the previous theorem. To see that they are also $\mathbb{F}\operatorname{GL}(V)$-submodules deduce the following lemma first:

1.40 Lemma. *Let λ be a partition of r and $\mathfrak{t}$ a standard λ-tableau. Let T be an arbitrary λ-tableau with entries in $\{1, \ldots, n\}$. The element $v_{T\mathfrak{t}}$ can be written as linear combination of elements $v_{T'\mathfrak{t}}$ for T' semi-standard and of some element in $V(\lhd \lambda)$.*

1.41 Example. To support the idea in the proof, the following example is given: Let $T = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 3&2\\\cline{1-2}\end{array}$. Then $\mu = (2, 1, 2)$ is the weight of T and $\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&3\\\cline{1-2}\end{array}$ is a (not the only) bijective tableau, such that $\mu(\mathfrak{s}) = T$. Thus, $v_T = v_{T^\mu} b(\mathfrak{t}^\mu) b(\mathfrak{s})^{-1}$.

Proof. By definition, the entries in the columns of T can be permuted at the cost of a sign for $v_{T\mathfrak{t}}$. This implies that $v_{T\mathfrak{t}}$ is zero if an entry occurs more than once in a column, as mentioned above. Therefore, assume that T is already column-standard.

If T has an entry in column i and row j that is less than the entry in column $i-1$ row j, then $T = \mu(\mathfrak{s})$ for some column-standard bijective tableau $\mathfrak{s}$ and μ the weight of T. Thus, $v_{T\mathfrak{t}} = v_{T^\mu} b(\mathfrak{t}^\mu) b(\mathfrak{s})^{-1} y_\lambda b(\mathfrak{t})$, where the tableau $\mathfrak{s}$ has an entry in column i and row j that is less than the entry in column $i-1$ row j. Using Garnir relations or rather Theorem 1.20, this inversion can be cancelled modulo lower terms. This proves the claim. $\square$

1.42 Lemma. *Let $\mathbb{F}$ be an infinite field. Let λ be a partition of r and $\mathfrak{t}$ a standard λ-tableau. Denote by $D(\lambda)_\mathfrak{t}$ the vector space generated by the set*

$$\{v_{T\mathfrak{t}} + V(\lhd \lambda) \mid T \in \operatorname{Tab}(\lambda)_n\}.$$

$D(\lambda)_\mathfrak{t}$ is an $\mathbb{F}\operatorname{GL}(V)$-module and as $\mathbb{F}\operatorname{GL}(V)$-module isomorphic to the dual Weyl module $D_{\lambda,\mathbb{F}}$ as defined in [Gre80].

Proof. Following [Gre80], there is a surjective $\mathbb{F}\operatorname{GL}(V)$-module homomorphism from $V^{\otimes r}$ to the dual Weyl module $D_{\lambda,\mathbb{F}}$ mapping the set $\{v_T \mid T \in \operatorname{Tab}_n(\lambda)\}$ on a basis of $D_{\lambda,\mathbb{F}}$. Since $V(\lhd \lambda)$ is contained in the kernel of this surjection ([Gre80, (5.2a)], there is also a surjective $\mathbb{F}\operatorname{GL}(V)$-module homomorphism from $V^{\otimes r}/V(\lhd \lambda)$ to $D_{\lambda,\mathbb{F}}$. Since the actions of $\mathbb{F}\operatorname{GL}(V)$ and $\mathbb{F}\mathfrak{S}_r$ commute, there is an injective $\mathbb{F}\operatorname{GL}(V)$-module homomorphism from $D(\lambda)_\mathfrak{t}$ to $V^{\otimes r}/V(\lhd \lambda)$ to $D_{\lambda,\mathbb{F}}$, that maps $v_{T\mathfrak{t}} + V(\lhd \lambda) = v_T y_\lambda b(\mathfrak{t}) + V(\lhd \lambda)$ on $v_T + V(\lhd \lambda)$. Combining the maps implies the result, since $D_{\lambda,\mathbb{F}}$ has dimension $|\operatorname{Tab}(\lambda)_n|$. $\square$

Now the following theorem is now an immediate consequence.

1.43 Theorem. *Let* λ *be a partition of* r. *Then*

$$V(\trianglelefteq \lambda)/V(\triangleleft \lambda) \cong \Delta(\lambda) \otimes S_y^{\lambda},$$

as $(\mathbb{F}\,\mathrm{GL}(V), \mathbb{F}\mathfrak{S}_r)$*-bimodules, where* S^{λ} *is the dual Specht module associated to* λ *and (for* $\mathbb{F}$ *an infinite field)* $\Delta(\lambda)$ *is the dual Weyl module.*

2. Brauer algebras

The Brauer algebra, and hence this chapter, is the main part of this thesis. The chapter is organized in the same way as the first chapter. The fascinating thing about this is that with the right point of view there is an analogue for all the results stated in the first chapter.

2.1. Definitions

Diagrammatic and algebraic description. Throughout this chapter, r will denote a natural number, R a commutative unital ring, and x an element of R.

A *Brauer diagram* is a graph with $2r$ vertices arranged in two rows, r in a top row and r in a bottom row. Each vertex is connected to precisely one other vertex by an edge. These edges are drawn inside of the rectangle which is given by the arrangement of the vertices. Two kinds of edges can be distinguished. *Horizontal* edges connect vertices in the same row and *vertical* edges connect vertices in different rows.

Let $\mathfrak{B}_r$ be the set of all (homotopy classes of) Brauer diagrams with $2r$ vertices. This set is a monoid with multiplication defined by concatenation of diagrams.

The *Brauer algebra* $B_r(x)$ is the R-algebra which is free as an R-module with the set $\mathfrak{B}_r$ as a basis. The multiplication is given by concatenation of diagrams, where closed cycles are deleted by multiplying with x (see Figure 2.1).

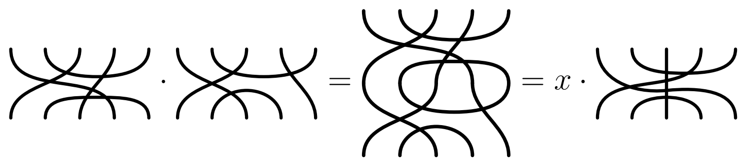


Figure 2.1.: Multiplication of two Brauer diagrams in $B_5(x)$.

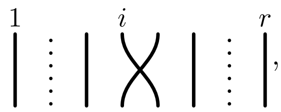
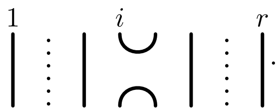
This description is accompanied by an embedding of the group algebra of the symmetric group $R\mathfrak{S}_r$ into $B_r(x)$, where the permutations are mapped to the corresponding Brauer diagrams without horizontal edges.

The Brauer algebra has also an algebraic presentation. Let $S = \{s_1, s_2, \dots, s_{r-1}\}$ and $E = \{e_1, e_2, \dots, e_{r-1}\}$. Then $B_r(x)$ is generated by the set $S \cup E$ together with the relations

- $s_i^2 = 1$,
- $s_i s_j = s_j s_i$ for $|i-j| > 1$,
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$,
- $e_i^2 = x e_i$,
- $e_i e_j = e_j e_i$ for $|i-j| > 1$,
- $e_i e_{i\pm 1} e_i = e_i$,
- $e_i s_i = s_i e_i = e_i$,
- $e_i s_j = s_j e_i$ for $|i-j| > 1$,
- $s_i e_{i+1} e_i = s_{i+1} e_i$,
- $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$

for all values of i and j where the above equations make sense.

The following map determines the isomorphism between the diagrammatic and algebraic presentation of the Brauer algebra $B_r(x)$ (cf. [Bra37] and [HW89])

$s_i \mapsto$ , $e_i \mapsto$ .

Ideals and subalgebras. The product of two Brauer diagrams never has fewer horizontal edges than each of the two Brauer diagrams. To be precise, if A and B are Brauer diagrams with a resp. b horizontal edges then the product $A \cdot B$ has at least $\max\{a, b\}$ horizontal edges. Therefore, the R-submodule of $B_r(x)$ generated by all Brauer diagrams with at least i horizontal edges is a two-sided ideal of $B_r(x)$. This ideal is denoted by $B_r^{(i)}(x)$. Consequently, the Brauer algebra has the following filtration

$$(0) \subset B_r^{(r/2)}(x) \subset \cdots \subset B_r^{(2)}(x) \subset B_r^{(1)}(x) \subset B_r(x). \tag{2.1}$$

Apart from being a subalgebra of the Brauer algebra the group algebra of the symmetric group is also an image of the Brauer algebra, namely $B_r(x)/B_r^{(1)}(x)$.

Again, if not stated otherwise the Brauer algebra $B_{r-1}(x)$ should be seen as the subalgebra of $B_r(x)$ in which every Brauer diagram has an edge connecting dot r in

Side Note: It would be sufficient to take just one of the e_i and the set S to generate the Brauer algebra. But in the way the generators and relations are presented, it can be seen that both, the group algebra of the symmetric group and the Temperley-Lieb algebra, are subalgebras of the Brauer algebra.

the top row with dot r in the bottom row. Therefore, there is the following tower of algebras

$$B_1(x) \subset B_2(x) \subset \dots \subset B_{r-1}(x) \subset B_r(x) \subset \dots \tag{2.2}$$

2.2. Cellular structure

First cellular bases. The cellular structure of the Brauer algebra can be considered as a refinement of filtration (2.1). That given, it is no surprise that some of the cell modules look like the cell modules of the symmetric group. To define the cellular structure some additional definitions are needed.

For every positive integer r define $\Lambda(r)$ to be the set of all pairs (λ, k) with $k \in \mathbb{N}$, $0 \leq k \leq r/2$ and $\lambda \in \mathrm{Par}(r-2k)$. The *dominance order* on partitions can be extended to the set $\Lambda(r)$ in two natural ways. Either partitions of smaller integers can be defined to dominate partitions of larger integers, or vice versa. To be precise: Let $(\lambda, k), (\mu, l)$ be two pairs in $\Lambda(r)$. Write $(\lambda, k) \trianglerighteq (\mu, l)$ if $k > l$ or $k = l$ and $\lambda \trianglerighteq \mu$. The set $\Lambda(r)$ together with this partial order will be denoted by $\Lambda_x(r)$.
Now the other order. Let $(\lambda, k), (\mu, l)$ be two pairs in $\Lambda(r)$. Write $(\lambda, k) \trianglerighteq (\mu, l)$ if $k < l$ or $k = l$ and $\lambda \trianglerighteq \mu$. The set $\Lambda(r)$ together with this partial order will be denoted by $\Lambda_y(r)$.

2.3 Example. The set $\Lambda(3)$ consists of the following elements

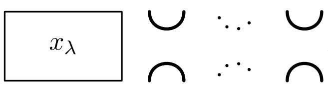
$$\Lambda(3) = \left\{ \left(\ydiagram{1}, 1\right), \left(\ydiagram{3}, 0\right), \left(\ydiagram{2,1}, 0\right), \left(\ydiagram{1,1,1}, 0\right) \right\}.$$

The sets $\Lambda(3)_x$ and $\Lambda(3)_y$ have the same elements but they are ordered as follows:

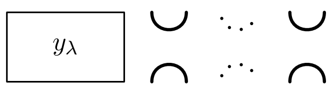
$$\Lambda(3)_x : \left(\ydiagram{1}, 1\right) \triangleright \left(\ydiagram{3}, 0\right) \triangleright \left(\ydiagram{2,1}, 0\right) \triangleright \left(\ydiagram{1,1,1}, 0\right)$$

$$\Lambda(3)_y : \left(\ydiagram{3}, 0\right) \triangleright \left(\ydiagram{2,1}, 0\right) \triangleright \left(\ydiagram{1,1,1}, 0\right) \triangleright \left(\ydiagram{1}, 1\right)$$

For every pair (λ, k) in $\Lambda(r)$ define two elements $x_{(\lambda,k)}$ and $y_{(\lambda,k)}$ as follows:

$x_{(\lambda,k)} := x_\lambda e^{(k)}_{r-1} =$ 

,

$y_{(\lambda,k)} := y_\lambda e^{(k)}_{r-1} =$ 

with $e_{r-1}^{(k)} := e_{r-2k+1}e_{r-2k+3}\dots e_{r-1}$.

To every pair (λ, k) in $\Lambda(r)$ attach the pair $(\lambda, (2^{(k)}))$ of partitions. The notation of tableaux can be extended naturally to these pairs in the following way. A (λ, k)-*tableau* $(\mathfrak{s}, \mathfrak{t})$ is a map from the pair $(\lambda, (2^{(k)}))$ to the set $\{1, 2, \dots, r\}$. Again the numbers are filled inside the boxes of the Young diagrams. A (λ, k)-tableau $(\mathfrak{s}, \mathfrak{t})$ is called *standard*, if it is bijective, $\mathfrak{s}$ is standard, $\mathfrak{t}$ is row-standard, and the entries in the first column of $\mathfrak{t}$ increase from top to bottom. The set of standard (λ, k)-tableaux is denoted by $\mathrm{Std}(\lambda, k)$. Further, in the set $\mathrm{Std}(\lambda, k)$ there is an analogue to the initial and to the final element in $\mathrm{Std}(\lambda)$. The *analogue to the initial tableau* is the pair $\left(\mathfrak{t}^{\lambda}, \mathfrak{t}^{\left(2^{(k)}\right)}\right)$, the *analogue to the final tableau* is the pair $\left(\mathfrak{t}_{\lambda}, \mathfrak{t}^{\left(2^{(k)}\right)}\right)$. Note that for both elements the second entry is $\mathfrak{t}^{\left(2^{(k)}\right)}$ with the numbers $r - 2k + 1, r - 2k + 2, \dots, r$ filled in along the rows left to right and top to bottom. Broadly speaking, this is because in the pairs $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda, k)$ the first entry controls the symmetric group part and the second entry controls the positions of the horizontal lines – and the starting position of the horizontal lines is the same for $x_{(\lambda,k)}$ and $y_{(\lambda,k)}$.

These two elements are again used to define elements $d(\mathfrak{s}, \mathfrak{t})$ and $b(\mathfrak{s}, \mathfrak{t})$. Again, the symmetric group acts on the set of bijective (λ, k)-tableaux from the right by permutation of entries. The element $d(\mathfrak{s}, \mathfrak{t})$ is the unique element in $\mathfrak{S}_r$ such that $\left(\mathfrak{t}^{\lambda}, \mathfrak{t}^{\left(2^{(k)}\right)}\right) d(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{s}, \mathfrak{t})$ and the element $b(\mathfrak{s}, \mathfrak{t})$ is the unique element in $\mathfrak{S}_r$ such that $\left(\mathfrak{t}_{\lambda}, \mathfrak{t}^{\left(2^{(k)}\right)}\right) b(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{s}, \mathfrak{t})$.

2.4 Example. Let $(\lambda, k) = (\ (3, 2, 1),\ 2)$ be a pair in $\Lambda(10)$. Then the analogue to the initial and the final element in $\mathrm{Std}(\lambda, k)$ are

$$\left(\mathfrak{t}^{\lambda}, \mathfrak{t}^{\left(2^{(k)}\right)}\right) = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 7&8\\\hline 9&10\\\hline\end{array} \right) \quad \text{and} \quad \left(\mathfrak{t}_{\lambda}, \mathfrak{t}^{\left(2^{(k)}\right)}\right) = \left(\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 7&8\\\hline 9&10\\\hline\end{array} \right).$$

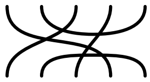
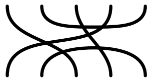
Further,

$$\left(\begin{array}{|c|c|c|}\hline 1&4&10\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 6&9\\\hline 7&8\\\hline\end{array} \right) \quad \text{and} \quad (\mathfrak{s}, \mathfrak{t}) = \left(\begin{array}{|c|c|c|}\hline 3&7&9\\\hline 5&10\\\cline{1-2} 8\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&6\\\hline 2&4\\\hline\end{array} \right).$$

are another two elements in $\mathrm{Std}(\lambda, k)$. So,

$$d(\mathfrak{s}, \mathfrak{t}) = (1\ 3\ 9\ 2\ 7)(4\ 5\ 10)(6\ 8) \quad \text{and} \quad b(\mathfrak{s}, \mathfrak{t}) = (1\ 3\ 8\ 6\ 9\ 2\ 5\ 10\ 4\ 7).$$

Let $* : \mathfrak{B}_r \to \mathfrak{B}_r$ be the map which sends every Brauer diagram to the diagram that one obtains by rotating it around a horizontal axis. The map $*$ can be extended to an R-linear anti-involution of the Brauer algebra $B_r(x)$. Further, the restriction of this map to $R\mathfrak{S}_r \subseteq B_r(x)$ coincides with the map $*$ defined in the previous chapter, so the notation should cause no confusion.

2.5 Example. Let $x =$  $\in \mathfrak{B}_5$, then $x^* =$ .

One can use any cellular basis of the group algebras of the symmetric groups to obtain a cellular basis of the Brauer algebra [KX01]. The following two bases are applications of this result. Enyang constructed the first of them explicitly [Eny04, Theorem 4.1].

2.6 Theorem. *The set*

$$\mathfrak{M}_r^x := \{d(\mathfrak{s}', \mathfrak{t}')^* x_{(\lambda,k)} d(\mathfrak{s}, \mathfrak{t}) \mid (\lambda, k) \in \Lambda_x(r), (\mathfrak{s}', \mathfrak{t}'), (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda, k)\}$$

is an R-basis for $B_r(x)$. Furthermore, $(\mathfrak{M}_r^x, \Lambda_x(r), \trianglerighteq)$ is a cellular basis and therefore $B_r(x)$ is a cellular algebra.

The cell modules which arise from this cellular basis are denoted by $C_x^{(\lambda,k)}$. The set $\{x_{(\lambda,k)} d(\mathfrak{s}, \mathfrak{t}) + C^{\triangleright(\lambda,k)} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda, k)\}$ is an R-basis for the module $C_x^{(\lambda,k)}$ with $C^{\triangleright(\lambda,k)}$ the (two-sided) ideal in $B_r(x)$ generated by all $x_{(\mu,l)} \in \Lambda_x(r)$ with $(\mu, l) \triangleright (\lambda, k)$.

2.7 Theorem. *The set*

$$\mathfrak{M}_r^y := \{b(\mathfrak{s}', \mathfrak{t}')^* y_{(\lambda,k)} b(\mathfrak{s}, \mathfrak{t}) \mid (\lambda, k) \in \Lambda_y(r), (\mathfrak{s}', \mathfrak{t}'), (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda, k)\}$$

is an R-basis for $B_r(x)$. Furthermore, $(\mathfrak{M}_r^y, \Lambda_y(r), \trianglelefteq)$ is a cellular basis and therefore $B_r(x)$ is a cellular algebra.

The cell modules which arise from this cellular basis are denoted by $C_y^{(\lambda,k)}$. The set $\{y_{(\lambda,k)} b(\mathfrak{s}, \mathfrak{t}) + C^{\triangleleft(\lambda,k)} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\lambda, k)\}$ is an R-basis for the module $C_y^{(\lambda,k)}$ with $C^{\triangleleft(\lambda,k)}$ the (two-sided) ideal in $B_r(x)$ generated by all $y_{(\mu,l)} \in \Lambda_y(r)$ with $(\mu, l) \triangleleft (\lambda, k)$.

2.8 Remark. Looking at $(\mathfrak{M}_r^x, \Lambda_x(r), \trianglerighteq)$ and $(\mathfrak{M}_r^y, \Lambda_y(r), \trianglelefteq)$ it may seem like an unnecessary complication to change both, the indexing set and the order on it. But this complication here will help to smooth out things later when the focus is on the endomorphism rings and their bases.

2.9 Remark. A useful way to think about the bases $\mathfrak{M}_r^x$, $\mathfrak{M}_r^y$ and their indexing sets is the concept of *iterated inflations* presented by König and Xi [KX01]: As mentioned at the beginning of this section, the cellular structure is a refinement of the filtration (2.2). Each quotient in the filtration (2.2) is isomorphic to the group algebra of a symmetric group tensored with two vector spaces, which correspond to the horizontal arcs in the diagrams. This is reflected precisely by the way the bases are built: a basis element of the group algebra of the symmetric group in the middle and horizontal arcs at the top and at the bottom. The position of these horizontal arcs is given by the tableaux of shape $(2^{(k)})$.

2.10 Remark. The bases $\mathfrak{M}_r^x$ and $\mathfrak{M}_r^y$ are no Murphy type bases in the sense that their corresponding cell module bases are not adapted to the restriction of cell modules to smaller Brauer algebras. They are also not adapted to the action of the Brauer algebras on tensor space. Thus, other bases have to be found.

Restriction of cell modules. The following theorem describes the behaviour of the cell modules under restriction in the tower (2.2). It is the analogue to Theorem 1.21. In order to state the theorem in a compact way, the following lemma and a definition are needed.

2.11 Lemma. *Let the pair (λ, k) be an element of $\Lambda_x(r)$, where λ has $t(\lambda)$ ($p(\lambda)$) removable (addable) nodes. With respect to the order on $\Lambda_x(r-1)$ the set*

$$\{(\mu, k) \mid \mu \in \mathrm{Rem}(\lambda)\} \cup \{(\nu, k-1) \mid \nu \in \mathrm{Add}(\lambda)\} \subseteq \Lambda_x(r-1)$$

is ordered linearly, i.e. $(\mu^{(1)}, k) \rhd (\mu^{(2)}, k) \rhd \ldots (\mu^{(t(\lambda))}, k) \rhd (\nu^{(1)}, k-1) \rhd (\nu^{(2)}, k-1) \rhd \ldots (\nu^{(p(\lambda))}, k-1)$.

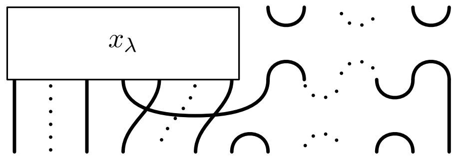
Let the pair (λ, k) be an element of $\Lambda_x(r)$ as above. For every $1 \leq j \leq t(\lambda)$ define the $B_{r-1}(x)$-submodule of $\mathrm{Res}^{B_r(x)}_{B_{r-1}(x)} C_x^{(\lambda,k)}$

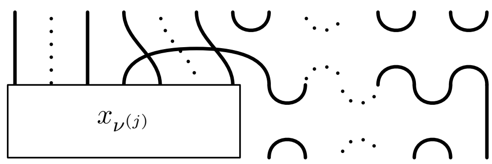
$$N_j := \sum_{s \leq j} \left(x_{(\lambda,k)} d_{\lambda \to \mu^{(s)}} e^{(k)}_{r-2} + C^{\rhd(\lambda,k)} \right) B_{r-1}(x)$$

and for every $1 \leq j \leq p(\lambda)$ define the $B_{r-1}(x)$-submodule of $\mathrm{Res}^{B_r(x)}_{B_{r-1}(x)} C_x^{(\lambda,k)}$

$$N_{t(\lambda)+j} := \sum_{s \leq j} \left(e^{(k)}_{r-1} d_{\lambda \to \nu^{(s)}} x_{\left(\nu^{(s)}, k-1\right)} + C^{\rhd(\lambda,k)} \right) B_{r-1}(x) + N_{t(\lambda)}.$$

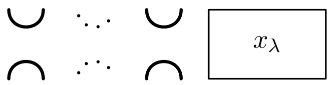
The diagrammatic description of the generators is as follows:

$x_{(\lambda,k)} d_{\lambda \to \mu^{(j)}} e^{(k)}_{r-2} =$ ,

$e^{(k)}_{r-1} d_{\lambda \to \nu^{(j)}} x_{\left(\nu^{(s)}, k-1\right)} =$ 

2.12 Remark. The definition of the N_j $(1 \leq j \leq t+p)$ is almost identical to the definition of the $N^{\mu^{(j)}}$ given by Enyang [Eny07]. There are two minor differences:

1. Throughout the article Enyang uses

   $x'_{(\lambda,k)} :=$ 

   instead of $x_{(\lambda,k)}$. With the concept of iterated inflations by König an Xi in mind, this causes no trouble.

2. The $N^{\mu^{(j)}}$ are defined to be the $B_{r-1}(x)$-submodules generated by the elements $x_{(\lambda,k)} d_{\lambda\to\mu^{(j)}} e^{(k)}_{r-2}$ and $e^{(k)}_{r-1} d_{\lambda\to\nu^{(j)}} x_{\left(\nu^{(j)},k-1\right)}$ respectively. Since they are included in each other (see the next theorem), these definitions are equivalent.

2.13 Theorem ([Eny07, Corollary 5.8])**.** *Let the pair (λ, k) be an element of $\Lambda_x(r)$ as above. The $B_{r-1}(x)$-submodules of $\operatorname{Res}^{B_r(x)}_{B_{r-1}(x)} C_x^{(\lambda,k)}$ defined above lead to the filtration*

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_{t(\lambda)+p(\lambda)} = \operatorname{Res}^{B_r(x)}_{B_{r-1}(x)} C_x^{(\lambda,k)}$$

such that $C_x^{\left(\mu^{(j)},k\right)} \cong N_j/N_{j-1}$ and $C_x^{\left(\nu^{(j)},k-1\right)} \cong N_{t(\lambda)+j}/N_{t(\lambda)+j-1}$ with the isomorphisms given by

$$N_j/N_{j-1} \to C_x^{\left(\mu^{(j)},k\right)} : x_{(\lambda,k)} d_{\lambda\to\mu^{(j)}} e^{(k)}_{r-2} + N_{j-1} \mapsto x_{\left(\mu^{(j)},k\right)} + C^{\rhd\left(\mu^{(j)},k\right)},$$

$$N_{t(\lambda)+j}/N_{t(\lambda)+j-1} \to C_x^{\left(\nu^{(j)},k-1\right)} : e^{(k)}_{r-1} d_{\lambda\to\nu^{(j)}} x_{\left(\nu^{(j)},k-1\right)} + N_{t(\lambda)+j-1} \mapsto x_{\left(\nu^{(j)},k-1\right)} + C^{\rhd\left(\nu^{(j)},k-1\right)}.$$

With Theorems 1.27 and 1.29 in mind this theorem can be reformulated in the following way:

2.14 Theorem. *Let the pair (λ, k) be an element of $\Lambda_x(r)$ as above. The $B_{r-1}(x)$-submodules N_j for $1 \leq j \leq t(\lambda)$ can equivalently be defined as*

$$N_j := \sum_{s\leq j} \left(x_{(\lambda,k)} \bar{d}_{\lambda\to\mu^{(s)}} x_{\left(\mu^{(s)},k\right)} + C^{\rhd(\lambda,k)} \right) B_{r-1}(x)$$

and for every $1 \leq j \leq p(\lambda)$ as

$$N_{t+j} := \sum_{s\leq j} \left(x_{(\lambda,k)} \bar{d}_{\lambda\to\nu^{(s)}} x_{\left(\nu^{(s)},k-1\right)} + C^{\rhd(\lambda,k)} \right) B_{r-1}(x) + N_{t(\lambda)}.$$

The (inverse of the) isomorphism can be given by

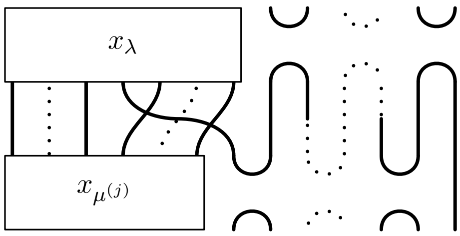
$$C_x^{\left(\mu^{(j)},k\right)} \to N_j/N_{j-1} : x_{\left(\mu^{(j)},k\right)} + C^{\rhd\left(\mu^{(j)},k\right)} \mapsto x_{(\lambda,k)} \bar{d}_{\lambda\to\mu^{(j)}} x_{\left(\mu^{(j)},k\right)} + N_{j-1},$$

$$C_x^{\left(\nu^{(j)},k-1\right)} \to N_{t(\lambda)+j}/N_{t(\lambda)+j-1} : x_{\left(\nu^{(j)},k-1\right)} + C^{\rhd\left(\nu^{(j)},k-1\right)} \mapsto x_{(\lambda,k)} \bar{d}_{\lambda\to\nu^{(j)}} x_{\left(\nu^{(j)},k-1\right)} + N_{t(\lambda)+j-1}.$$

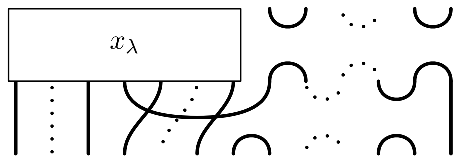
Proof. The submodules can be defined equivalently in that way, since the generators are in fact the same elements. This is best shown by the following small diagrammatic calculations:

$$x_{(\lambda,k)}\bar{d}_{\lambda\to\mu^{(j)}}x_{(\mu^{(j)},k)} = x_\lambda e^{(k)}_{r-1}\bar{d}_{\lambda\to\mu^{(j)}}x_{\mu^{(j)}}e^{(k)}_{r-2}$$

$$= \frac{1}{\mu^{(j)}!}$$



$=$

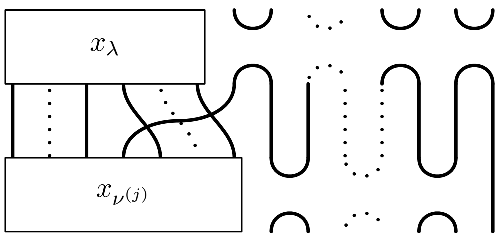


$$= x_\lambda e^{(k)}_{r-1}d_{\lambda\to\mu^{(j)}}e^{(k)}_{r-2}$$

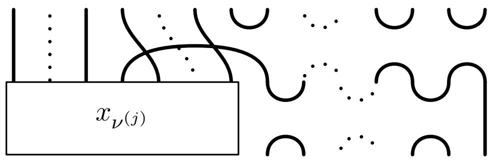
and

$$x_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x_{(\nu^{(j)},k-1)} = x_\lambda e^{(k)}_{r-1}\bar{d}_{\lambda\to\nu^{(j)}}x_{\nu^{(j)}}e^{(k-1)}_{r-2}$$

$$= \frac{1}{\lambda!}$$



$=$



$$= e^{(k)}_{r-1}d_{\lambda\to\nu^{(j)}}x_{\nu^{(j)}}e^{(k-1)}_{r-2}$$

This calculations also show how to deal with the fractions, i.e. substitute the left hand side with the right hand side if the fractions do not exist. The redefinition of the isomorphism follows in the same way. $\square$

Other cellular bases.

2.15 Remark. Theorem 2.14 can be used to construct another cellular basis for the Brauer algebra which should be seen as an analogue of the Murphy basis of the group algebra of the symmetric group as described in (1.28). The following definitions are needed to do so.

First define the *Bratelli diagram associated to the Brauer algebra.* Since the cell modules of $B_r(x)$ are indexed by the elements of $\Lambda(r)$, in row i of the diagram all elements of $\Lambda(i)$ are drawn. Two elements (λ,k) and (μ,l) in consecutive rows are connected if $C^{(\lambda,k)}$ is a subfactor in the filtration of $C^{(\mu,l)}$ as described in Theorem 2.14, i.e. $(\lambda,k)\in\Lambda(i)$ and $(\mu,l)\in\Lambda(i+1)$ are connected if either $k=l$ and $\mu\in\operatorname{Add}(\lambda)$ or $k+1=l$ and $\mu\in\operatorname{Rem}(\lambda)$. Therefore, a path $\mathfrak{t}$ (in the Bratelli-diagram associated to

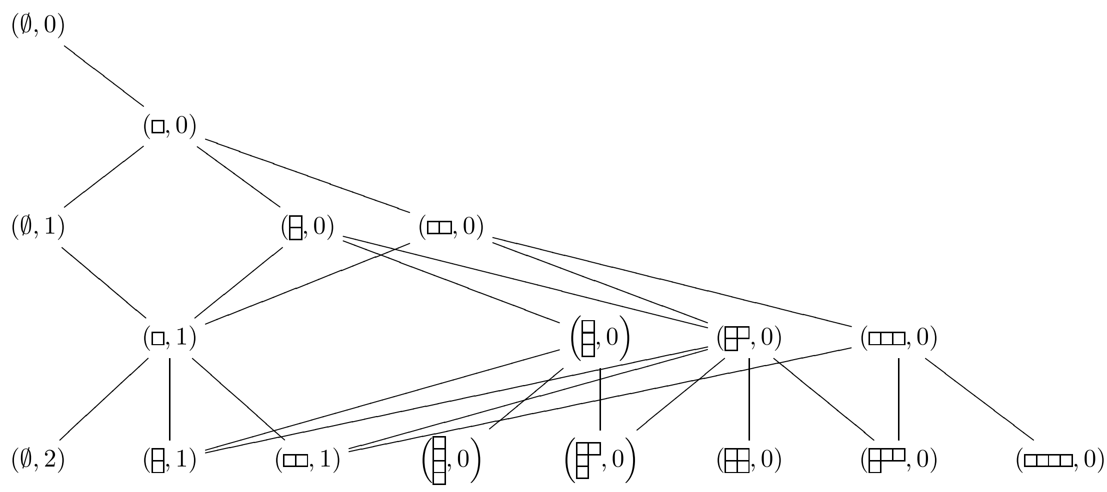


Figure 2.2.: The first four levels of the Bratelli-diagram associated to the Brauer algebra.

the Brauer algebra) is a sequence of pairs $((\emptyset,0),(\lambda^{(1)},k_1),(\lambda^{(2)},k_2),\ldots,(\lambda^{(r)},k_r))$, such that $(\lambda^{(i)},k_i)\in\Lambda(i)$ is connected to $(\lambda^{(i+1)},k_{i+1})\in\Lambda(i+1)$. For such a path $\mathfrak{t}=((\emptyset,0),(\lambda^{(1)},k_1),(\lambda^{(2)},k_2),\ldots,(\lambda^{(r)},k_r))$, the pair $(\lambda^{(r)},k_r)$ is called its *shape* and r its *length.*

For a path $\mathfrak{t}=((\emptyset,0)=(\lambda^{(0)},k_0),(\lambda^{(1)},k_1),\ldots,(\lambda^{(r)},k_r))$ of length r in the Bratteli diagram associated to the Brauer algebra, let

$$d_{\mathfrak{t}} := \bar{d}_{\lambda^{(r)}\to\lambda^{(r-1)}} x_{(\lambda^{(r-1)},k_{r-1})} \bar{d}_{\lambda^{(r-1)}\to\lambda^{(r-2)}} x_{(\lambda^{(r-2)},k_{r-2})} \cdots x_{(\lambda^{(1)},k_1)} \bar{d}_{\lambda^{(1)}\to\lambda^{(0)}}.$$

2.16 Lemma. *For r a positive integer and $(\lambda,k)\in\Lambda(r)$ the set*

$$\{x_{(\lambda,k)}d_{\mathfrak{t}} + C^{\rhd(\lambda,k)} \mid \mathfrak{t} \text{ a path of shape } (\lambda,k)\}$$

is an R-basis of the right-module $C_x^{(\lambda,k)}$ of $B_r(x)$.

Proof. This is proven by induction on r, where the case $r=1$ is clear. Assume that for all $(\mu,l)\in\Lambda(s)$ with $s<r$ the modules $C_x^{(\mu,l)}$ have a basis as above. Then in particular, every $C_x^{(\mu,l)}$ with (μ,l) connected to (λ,k) has such a basis. With the isomorphism given in Theorem 2.14 the proof is complete. □

2.17 Theorem. *For a positive integer r the set*

$$\mathfrak{B}_r^x := \{d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \text{ paths of shape } (\lambda,k),(\lambda,k)\in\Lambda_x(r)\}$$

is an R-basis for the Brauer algebra $B_r(x)$. Moreover, the following statements hold:

1. *Let $(\lambda, k) \in \Lambda(r)$, $\mathfrak{t}$ a path of shape (λ, k) and $b \in B_r(x)$. There exist $r_\mathfrak{u} \in R$ for $\mathfrak{u}$ a path of shape (λ, k), such that for all $\mathfrak{s}$ paths of shape (λ, k)*

$$d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t} b \equiv \sum_\mathfrak{u} r_\mathfrak{u} d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{u} \mod C^{\rhd(\lambda,k)},$$

with $C^{\rhd(\lambda,k)}$ the R-module freely generated by

$$\{d_\mathfrak{s}^* x_{(\mu,m)} d_\mathfrak{t} \mid \mathfrak{s}, \mathfrak{t} \text{ paths of shape } (\mu, m) \in \Lambda_x(r) \text{ with } (\mu, m) \rhd (\lambda, k)\}.$$

2. *For $d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t} \in \mathfrak{B}_r$ it holds $(d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t})^* = d_\mathfrak{t}^* x_{(\lambda,k)} d_\mathfrak{s}$.*

Thus $(\mathfrak{B}_r^x, \Lambda_x(r), \trianglerighteq)$ is a cellular basis.

Proof. Statement *1.* is a direct consequence of Lemma 2.16. Statement *2.* is true by the definition of the elements $d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t}$ and the algebra-anti-involution $*$.

That the set $\mathfrak{B}_r^x$ is an R-basis remains to be proven. The second statement together with Lemma 2.16 implies that the set

$$\{d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t} + C^{\rhd(\lambda,k)} \mid \mathfrak{s}, \mathfrak{t} \text{ paths of shape } (\lambda, k)\}$$

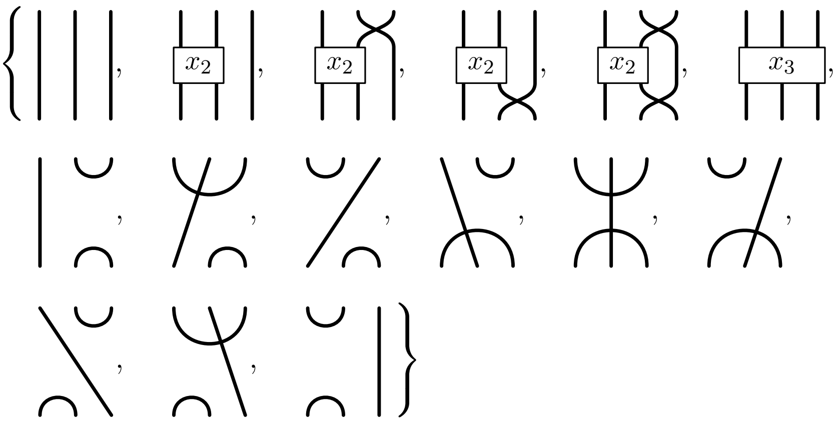
is an R-basis for the $B_n(x)/C^{\rhd(\lambda,k)}$-ideal generated by $x_{(\lambda,k)} + C^{\rhd(\lambda,k)}$. Induction on the order of $\Lambda_x(r)$ proves the claim. □

2.18 Remark. As can be seen in the proof of Theorem 2.14 the fractions in the '$d_\mathfrak{t}$s' vanish when the smaller box is 'consumed' by the bigger one. It is also possible to do it the other way round. If the smaller box 'consumes' the bigger one, the fraction also vanishes, but the sum over the coset representatives remains. Therefore, $\mathfrak{B}_r^x$ is an integral basis of the Brauer algebra.

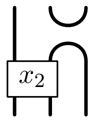
2.19 Remark. Since the group algebra of the symmetric group is a subalgebra of $B_r(x)$, one can ask to what extend bases of both algebra mirror this fact. As mentioned in the beginning of this chapter, the group basis of $R\mathfrak{S}_r$ is a subset of the basis of $B_r(x)$ given by Brauer diagrams. It is interesting (but it may not be surprising) that the Murphy basis $\mathfrak{N}_r^x$ of the group algebra of the symmetric group is also a subset of Enyang's basis of the Brauer algebra $\mathfrak{M}_r^x$ and of the Murphy-Enyang basis of the Brauer algebra $\mathfrak{B}_r^x$! The only thing needed to prove this is the fact that paths to $(\lambda, 0)$ are in bijection to paths to λ.

2.20 Example. To illustrate that in general $\mathfrak{B}_r^x \neq \mathfrak{M}_r^x$ take a look at the bases $\mathfrak{B}_3^x$ and $\mathfrak{M}_3^x$.

$\mathfrak{M}_3^x =$



The element $d^*_{((\emptyset,0),(\square,0),(\begin{smallmatrix}\square\\\square\end{smallmatrix},0),(\square,1))} x_{(\square,1)} d_{((\emptyset,0),(\square,0),(\square\square,0),(\square,1))} \in \mathfrak{B}_3^x$ is not in the set $\mathfrak{M}_3^x$, since its diagram is the following:



Clearly, Theorem 2.14, Lemma 2.16 and Theorem 2.17 can be stated and proven analogously when 'x' is replaced by 'y'. Only a definition corresponding to the definition of the $d_{\mathfrak{t}}$ is needed: For a path of length r in the Bratteli diagram associated to the Brauer algebra $\mathfrak{t} = ((\lambda^{(0)}, k_0), (\lambda^{(1)}, k_1), \ldots, (\lambda^{(r)}, k_r))$, let

$$b_{\mathfrak{t}} := \bar{b}_{\lambda^{(r)} \to \lambda^{(r-1)}} y_{(\lambda^{(r-1)}, k_{r-1})} \bar{b}_{\lambda^{(r-1)} \to \lambda^{(r-2)}} y_{(\lambda^{(r-2)}, k_{r-2})} \cdots y_{(\lambda^{(1)}, k_1)} \bar{b}_{\lambda^{(1)} \to \lambda^{(0)}}.$$

2.21 Theorem. *For a positive integer r the set*

$$\mathfrak{B}_r^y := \{ b_{\mathfrak{s}}^* y_{(\lambda,k)} b_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \text{ paths of shape } (\lambda, k), (\lambda, k) \in \Lambda_y(r) \}$$

is an R-basis for the Brauer algebra $B_r(x)$. Moreover, $(\mathfrak{B}_r^y, \Lambda_y(r), \trianglelefteq)$ is a cellular basis.

2.22 Remark. The bases $\mathfrak{B}_r^x$ and $\mathfrak{B}_r^y$ are Murphy-type bases for the Brauer algebras. What does this mean? By construction, the corresponding bases of the cell modules behave 'well' under restriction in the tower of algebras (2.2) as do the bases corresponding to $\mathfrak{M}_r^x$ and $\mathfrak{M}_r^y$ of the group algebra of the symmetric group. Therefore, these bases can be seen as generalizations of the Murphy bases of the symmetric group.

The following corollary will be useful later.

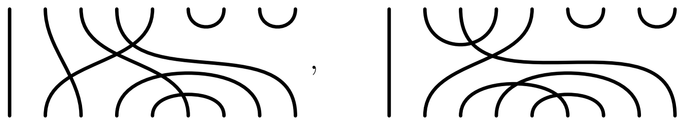


Figure 2.3.: Two elements of $e_8^{(2)}B_9(x)$

2.23 Corollary. *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_j)$ *be a partition. Let* λ^{+1} *denote the partition* $(\lambda_1 + 1, \lambda_2, \dots, \lambda_j)$ *and let* λ_{+1} *denote the partition* $(\lambda_1, \lambda_2, \dots, \lambda_j, 1)$*. For positive integers* r *and* l *with* $l < \frac{r}{2}$ *both*

$$\left\{ d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}} \;\middle|\; \begin{array}{l} \mathfrak{s}, \mathfrak{t} \textit{ paths of shape } (\lambda, k), (\lambda, k) \in \Lambda_x(r), \mathfrak{s} \textit{ ending with} \\ (\dots, (\lambda, k-l), (\lambda_{+1}, k-l), (\lambda, k-l+1), (\lambda_{+1}, k-l+1), \dots, (\lambda, k)) \end{array} \right\}$$

and

$$\left\{ b_{\mathfrak{s}}^* y_{(\lambda,k)} b_{\mathfrak{t}} \;\middle|\; \begin{array}{l} \mathfrak{s}, \mathfrak{t} \textit{ paths of shape } (\lambda, k), (\lambda, k) \in \Lambda_y(r), \mathfrak{s} \textit{ ending with} \\ (\dots, (\lambda, k-l), (\lambda^{+1}, k-l), (\lambda, k-l+1), (\lambda^{+1}, k-l+1), \dots, (\lambda, k)) \end{array} \right\}$$

are R*-bases for the* R*-module* $e_{r-1}^{(l)}B_r(x)$*.*

Proof. Obviously, both sets are subsets of $e_{r-1}^{(l)}B_r(x)$. To describe the cardinality of the sets the concept of iterated inflations is adapted. Every element in $e_{r-1}^{(l)}B_r(x)$ can be described as a permutation, a configuration of horizontal arcs in the top row and a configuration of horizontal arcs in the bottom row. Since l horizontal arcs in the top row are fixed, they need no separate description and the description of the elements in $e_{r-1}^{(l)}B_r(x)$ can be done as follows (see Figure 2.3):
An element of the symmetric group $\mathfrak{S}_{r-2f}$ $(0 \le f \le {}^{r-2l}/_2)$ in the middle, a top configuration as for the elements in $B_{r-2l}^{(f-2l)}(x)$ and a bottom configuration as for the elements in $B_r^{(f)}(x)$.

The elements of the symmetric group $\mathfrak{S}_{r-2f}$ can be substituted by elements of the Murphy bases, $\mathfrak{N}_{r-2f}^x$ or $\mathfrak{N}_{r-2f}^y$, without changing the cardinality.
Now count again: For every $(\lambda, k) \in \Lambda(r)$ $(k = l + f)$ there are $|C^{(\lambda,k-l)}|$ (top configuration) times $|C^{(\lambda,k)}|$ (bottom configuration) many elements in $B_r^{(f)}(x)$. But $|C^{(\lambda,k-l)}|$ equals the number of paths of shape $(\lambda, k-l)$ which equals the number of paths of shape (λ, k) ending with $(\dots, (\lambda, k-l), (\lambda_{+1}, k-l), (\lambda, k-l+1), (\lambda_{+1}, k-l+1), \dots, (\lambda, k))$ or $(\dots, (\lambda, k-l), (\lambda^{+1}, k-l), (\lambda, k-l+1), (\lambda^{+1}, k-l+1), \dots, (\lambda, k))$ respectively. And $|C^{(\lambda,k)}|$ equals the number of paths of shape (λ, k). This proves the claim. $\square$

2.3. Action on tensor space

This section deals with the action of the Brauer algebra on tensor space. It is divided into the symplectic and the orthogonal case since the two cases use different ingredients. The most visible difference is the appearance of symplectic tableaux on the one hand and orthogonal tableaux on the other hand. The ideas and results are more or less the same.

Let $\mathfrak{t}$ be a path to $(\lambda, k) \in \Lambda(r)$. Take the element $x_{(\lambda,k)}d_{\mathfrak{t}}$ and remove the horizontal arcs in the top row. This results in a linear combination of diagrams that have less dots in the top row than they have in the bottom row. These diagrams are called partial Brauer diagrams. In abuse of notation, the symbol $x_\lambda d_{\mathfrak{t}}$ shall represent this linear combination. The same convention is applied to $y_\lambda b_{\mathfrak{t}}$.

2.3.1. The symplectic case

Brauer-Schur-Weyl-duality. Let W be a $2n$-dimensional symplectic vector space over a field $\mathbb{F}$ with a symplectic form $\langle -, - \rangle$ and a symplectic basis $\{w_{\bar{1}}, w_1, w_{\bar{2}}, w_2, \dots, w_{\bar{n}}, w_n\}$, such that $\langle w_i, w_j \rangle = 0 = \langle w_{\bar{i}}, w_{\bar{j}} \rangle$ and $\langle w_i, w_{\bar{j}} \rangle = \delta_{i,j} = -\langle w_{\bar{i}}, w_j \rangle$.

Define a right-action of the Brauer algebra $B_r(-2n)$ on the r-fold tensor space $W^{\otimes r}$ in the following way. The elements s_i act by *signed place permutation*, i.e. by place permutation and multiplication with -1, and

$$\begin{aligned}&\big(w_{k_1} \otimes \cdots \otimes w_{k_{i-1}} \otimes w_{k_i} \otimes w_{k_{i+1}} \otimes w_{k_{i+2}} \otimes \cdots \otimes w_{k_r}\big) \cdot e_i \\ &:= -\langle w_{k_i}, w_{k_{i+1}} \rangle \sum_{l=1}^{n} w_{k_1} \otimes \cdots \otimes w_{k_{i-1}} \otimes (w_l \otimes w_{\bar{l}} - w_{\bar{l}} \otimes w_l) \otimes w_{k_{i+2}} \otimes \cdots \otimes w_{k_r}\end{aligned}$$

Hence, there is a map $\Psi : B_r(-2n)^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}}(W^{\otimes r})$.

The *symplectic similitude group* $\mathrm{GSp}(W)$ is defined as a subgroup of $\mathrm{GL}(W)$ in the following way:

$$\mathrm{GSp}(W) := \{g \in \mathrm{GL}(W) \mid \exists 0 \neq d \in \mathbb{F}, \text{ such that } \langle gv, gw \rangle = d\langle v, w \rangle \ \forall v, w \in U\}$$

The symplectic similitude group acts diagonally on the r-fold tensor space $W^{\otimes r}$. Hence there is a map $\Phi : \mathbb{F}\,\mathrm{GSp}(W) \to \mathrm{End}_{\mathbb{F}}(W^{\otimes r})$. Moreover, the following statements hold:

2.24 Theorem. *1. The left action of* $\mathrm{GSp}(W)$ *on the* r*-fold tensor space commutes with the right action of* $B_r(-2n)$. *Therefore, the maps* Ψ *and* Φ *are maps in the following sense:*

- $\Psi : B_r(-2n)^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$
- $\Phi : \mathbb{F}\,\mathrm{GSp}(W) \to \mathrm{End}_{B_r(-2n)}(W^{\otimes r})$

2. If $\mathbb{F}$ *is an infinite field then the maps* Ψ *and* Φ *in 1. are surjective.*

3. If $\mathbb{F}$ *is an infinite field and* $n \geq r$ *then the map* Ψ *is injective and hence* $B_r(-2n)^{\mathrm{op}} \cong \mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$.

4. If $\mathbb{F} = \mathbb{C}$, *then*

$$V^{\otimes r} = \bigoplus_{f=0}^{\frac{r}{2}} \bigoplus_{\substack{\lambda \vdash r-2f \\ \lambda'_1 \leq n}} \Delta_\lambda \otimes D^{\lambda'}$$

as irreducible $(\mathbb{F}\,\mathrm{GSp}(W), B_r(-2n))$*-bimodules, where* Δ_λ *and* $D^{\lambda'}$ *are the irreducible* $\mathbb{F}\,\mathrm{GSp}(W)$*- and* $B_r(-2n)$*-modules associated to* λ *and* λ'.

2.25 Remark. This double-centralizer property, which is often called Brauer-Schur-Weyl-duality or type C Schur-Weyl duality, is due to Brauer [Bra37] for the case $\mathbb{F} = \mathbb{C}$. For the other fields in question the results were proven by Dipper, Doty and Hu [DDH08].

Therefore, for infinite fields $B_r(-2n)/\ker(\Psi) \cong \mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$. The kernel of Ψ is the annihilator of the tensor space in the Brauer algebra. Dipper, Doty and Hu [DDH08] showed that the kernel of Ψ is rigid, i.e. its dimension does not depend on the choice of the field $\mathbb{F}$. Hence, the dimension of $\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$ does not depend on the choice of the (infinite) field $\mathbb{F}$ either.

Side Note: A natural question is, why the action of the general linear group is replaced by the action of the symplectic similitude group and not by the action of the symplectic group. If $\mathrm{Sp}(W)$ is used instead of $\mathrm{GSp}(W)$, most of the statements in Theorem 2.24 remain correct. The exception is the surjectivity of the map Φ, which needs not only an infinite field but an infinite field closed under square roots to stay true in general. See the introduction of [DDH08] for the computations implying this. Since the focus of this thesis is more on the Brauer algebra side, the usage of $\mathrm{GSp}(W)$ allows to state slightly more general results.

Wenzl [Wen88] gave a combinatorial description of the dimension of the endomorphism ring $\operatorname{End}_{\mathbb{C}\,\mathrm{GSp}(W)}(W^{\otimes r})$ which evolves around restriction and induction in the tower of Brauer algebras. This motivated him to give the following definition:

A Young diagram λ is called $(-2n)$-permissible if it contains at most n columns, i.e. $\lambda_1 \leq n$. Similarly, a path $\mathfrak{t} = ((\lambda^{(0)}, k_0), (\lambda^{(1)}, k_1), \ldots, (\lambda^{(r)}, k_r))$ in the Bratelli diagram is called $(-2n)$-permissible if every $\lambda^{(i)}$ is $(-2n)$-permissible.

With this definition the dimension of $\operatorname{End}_{\mathbb{C}\,\mathrm{GSp}(W)}(W^{\otimes r})$ can be expressed as follows:

2.26 Theorem ([Wen88, Theorem 3.4, Corollary 3.5])**.**

$$\dim \operatorname{End}_{\mathbb{C}\,\mathrm{GSp}(W)}(W^{\otimes r}) = \sum_{\substack{(\lambda,k)\in\Lambda(r) \\ \textit{with } \lambda \ (-2n)\textit{-permissible}}} n^2_{(\lambda,k)},$$

where $n_{(\lambda,k)}$ is the number of $(-2n)$-permissible paths to (λ, k).

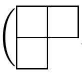
2.27 Remark. The Brauer algebra has a one dimensional two sided ideal corresponding to the *trivial representation*. Analogously to the group algebra of the symmetric group, the generators of this ideal are also called *symmetrizer*. A symmetrizer of the Brauer algebra $B_{n+1}(-2n)$ is the sum over all elements in $\mathfrak{B}_{n+1}$ (cf. [DHS13]).

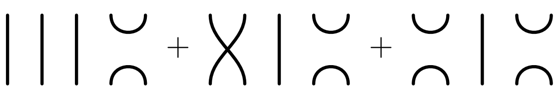
In the case $\mathbb{F} = \mathbb{C}$ a stronger version of the following proposition can be found in [Gav99, Theorem 3.7]. A proof for arbitrary fields can be found in [Hu08, Proposition 4.6].

2.28 Proposition. *Let W be the vector space described above and let r be a positive integer with $r > n$. The symmetrizer of the Brauer algebra $B_{n+1}(-2n)$ (embedded in $B_r(-2n)$) is an element of the annihilator of the r-fold tensor space.*

Basis for the endomorphism ring. This proposition motivates the following definitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of r define the direct product $\mathfrak{B}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_l}$ as submonoid of $\mathfrak{B}_r$ denoted by $\mathfrak{B}^{\mathrm{sp}}_\lambda$. Further define the elements

$$x^{\mathrm{sp}}_\lambda := \sum_{b\in\mathfrak{B}^{\mathrm{sp}}_\lambda} b \quad \text{and} \quad x^{\mathrm{sp}}_{(\lambda,k)} = x^{\mathrm{sp}}_\lambda e^{(k)}_{r-1}.$$

2.29 Example. Let $(\lambda, k) = \Big($  $, 1\Big) \in \Lambda(5)$, then

$x^{\mathrm{sp}}_{(\lambda,k)} =$ 

If $\lambda = (n+1, 1^{(r-n-1)})$ the element x_λ^{sp} is the symmetrizer of the Brauer algebra $B_{n+1}(-2n)$ (embedded in $B_r(-2n)$). Thus, whenever $\lambda_1 = n+1$, the element x_λ^{sp} has a factor equal to the symmetrizer of the Brauer algebra $B_{n+1}(-2n)$.

For $\mathfrak{t} = ((\lambda^{(0)}, k_0), (\lambda^{(1)}, k_1), \dots, (\lambda^{(r)}, k_r))$ a path of length r in the Bratteli diagram, let

$$d_{\mathfrak{t}}^{\mathrm{sp}} := \bar{d}_{\lambda^{(r)} \to \lambda^{(r-1)}} x^{\mathrm{sp}}_{(\lambda^{(r-1)}, k_{r-1})} \bar{d}_{\lambda^{(r-1)} \to \lambda^{(r-2)}} x^{\mathrm{sp}}_{(\lambda^{(r-2)}, k_{r-2})} \dots x^{\mathrm{sp}}_{(\lambda^{(1)}, k_1)} \bar{d}_{\lambda^{(1)} \to \lambda^{(0)}}.$$

2.30 Theorem. *Let r be a positive integer and let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) = 0$ *or* $\mathrm{char}(\mathbb{F}) \geq r$.

1. *The set*

$$\mathfrak{B}_r^{\mathrm{sp}} := \{ d_{\mathfrak{s}}^{\mathrm{sp}*} x^{\mathrm{sp}}_{(\lambda,k)} d_{\mathfrak{t}}^{\mathrm{sp}}. \mid \mathfrak{s}, \mathfrak{t} \text{ paths of shape } (\lambda, k), (\lambda, k) \in \Lambda(r) \}$$

   is an $\mathbb{F}$-basis for the Brauer algebra $B_r(x)$. Moreover, $(\mathfrak{B}_r^{\mathrm{sp}}, \Lambda_x(r), \trianglerighteq)$ is a cellular basis.

2. *The set*

$$\{ d_{\mathfrak{s}}^{\mathrm{sp}*} x^{\mathrm{sp}}_{(\lambda,k)} d_{\mathfrak{t}}^{\mathrm{sp}} \mid \mathfrak{s}, \mathfrak{t} \text{ paths of shape } (\lambda, k) \in \Lambda(r), \mathfrak{s} \text{ or } \mathfrak{t} \text{ not } (-2n)\text{-permissible} \}$$

   is an $\mathbb{F}$-basis for the annihilator $\mathrm{Ann}_{B_r(-2n)}(W^{\otimes r})$ *of tensor space in the Brauer algebra.*

3. *The set*

$$\mathfrak{C}_r^{\mathrm{sp}} := \{ d_{\mathfrak{s}}^{\mathrm{sp}*} x^{\mathrm{sp}}_{(\lambda,k)} d_{\mathfrak{t}}^{\mathrm{sp}} + \mathrm{ann} \mid \mathfrak{s}, \mathfrak{t} \ (-2n)\text{-permissible paths of shape } (\lambda, k) \in \Lambda(r) \}$$

   is an $\mathbb{F}$-basis for the endomorphism ring $\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$.
   Moreover, $(\mathfrak{C}_r^{\mathrm{sp}}, \Lambda_x(r), \trianglerighteq)$ is a cellular basis and therefore, $\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$ *is a cellular algebra.*

Proof. First note that $x^{\mathrm{sp}}_{(\lambda,k)} \equiv x_{(\lambda,k)} \mod B^{(k+1)}(x)$. Therefore, the $x_{(\lambda,k)}$s in the basis in Theorem 2.6 can be replaced by $x^{\mathrm{sp}}_{(\lambda,k)}$s.

Further note that as long as all the $\lambda!$ are invertible, the isomorphisms in Theorem 2.14 can be replaced as follows:

$$C_x^{(\mu^{(j)},k)} \to N_j/N_{j-1} : x_{(\mu^{(j)},k)} + C^{\triangleright(\mu^{(j)},k)} \mapsto x_{(\lambda,k)} \bar{d}_{\lambda \to \mu^{(j)}} x_{(\mu^{(j)},k)} + N_{j-1}$$

can be replaced by

$$C_x^{(\mu^{(j)},k)} \to N_j/N_{j-1} : x^{\mathrm{sp}}_{(\mu^{(j)},k)} + C^{\triangleright(\mu^{(j)},k)} \mapsto x^{\mathrm{sp}}_{(\lambda,k)} \bar{d}_{\lambda \to \mu^{(j)}} x^{\mathrm{sp}}_{(\mu^{(j)},k)} + N_{j-1}$$

since $x_{(\lambda,k)}\bar{d}_{\lambda\to\mu^{(j)}}x_{\left(\mu^{(j)},k\right)} + N_{j-1} \equiv x^{\mathrm{sp}}_{(\lambda,k)}\bar{d}_{\lambda\to\mu^{(j)}}x^{\mathrm{sp}}_{\left(\mu^{(j)},k\right)} + N_{j-1} \mod B^{(k)}_r(x)$.

$$C_x^{\left(\nu^{(j)},k-1\right)} \to N_{t+j}/N_{t+j-1} : x_{\left(\nu^{(j)},k\right)} + C^{\rhd\left(\nu^{(j)},k-1\right)} \mapsto x_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x_{\left(\nu^{(j)},k-1\right)} + N_{t+j-1}$$

can be replaced by

$$C_x^{\left(\nu^{(j)},k-1\right)} \to N_{t+j}/N_{t+j-1} : x^{\mathrm{sp}}_{\left(\nu^{(j)},k\right)} + C^{\rhd\left(\nu^{(j)},k-1\right)} \mapsto x^{\mathrm{sp}}_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x^{\mathrm{sp}}_{\left(\nu^{(j)},k-1\right)} + N_{t+j-1}$$

since $x_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x_{\left(\nu^{(j)},k-1\right)} + N_{t+j-1} \equiv x^{\mathrm{sp}}_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x^{\mathrm{sp}}_{\left(\nu^{(j)},k-1\right)} + V_{t+j-1} \mod N_t$. With this the first statement can be proved analogously to Theorem 2.17.

There is one snare one should be aware of:
The equalities

$$x_{(\lambda,k)}\bar{d}_{\lambda\to\mu^{(j)}}x_{\left(\mu^{(j)},k\right)} = x_{(\lambda,k)}\frac{1}{\mu^{(j)}!}d_{\lambda\to\mu^{(j)}}x_{\left(\mu^{(j)},k\right)} = x_{(\lambda,k)}d_{\lambda\to\mu^{(j)}}e^{(k)}_{r-2}$$

and

$$x_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x_{\left(\nu^{(j)},k-1\right)} = x_{(\lambda,k)}\frac{1}{\lambda!}d_{\lambda\to\nu^{(j)}}x_{\left(\nu^{(j)},k-1\right)} = e^{(k)}_{r-1}d_{\lambda\to\nu^{(j)}}x_{\left(\nu^{(j)},k-1\right)}$$

hold by Corollary 1.12. But in general

$$x^{\mathrm{sp}}_{(\lambda,k)}\bar{d}_{\lambda\to\mu^{(j)}}x^{\mathrm{sp}}_{\left(\mu^{(j)},k\right)} \neq x^{\mathrm{sp}}_{(\lambda,k)}d_{\lambda\to\mu^{(j)}}e^{(k)}_{r-2}$$

and

$$x^{\mathrm{sp}}_{(\lambda,k)}\bar{d}_{\lambda\to\nu^{(j)}}x^{\mathrm{sp}}_{\left(\nu^{(j)},k-1\right)} \neq e^{(k)}_{r-1}d_{\lambda\to\nu^{(j)}}x^{\mathrm{sp}}_{\left(\nu^{(j)},k-1\right)}.$$

See the appendix (B.2) for a supporting example. Therefore, the fractions cannot be cancelled out as they could in Theorem 2.17. The biggest denominator that appears, is hidden in the element $\bar{d}_{(r)\to(r-1)} = \frac{1}{(r-1)!}d_{(r)\to(r-1)}$. Thus, a field $\mathbb{F}$ containing the inverse of $(r-1)!$ is needed for this theorem.

Second statement: Let $\mathfrak{t}$ be a path of shape (λ,k), which is not $(-2n)$-permissible. By definition there is one element $\lambda^{(i)}$ in the path that has more than n columns. Since the elements next to each other in the path differ by only one box, $\lambda^{(i)}$ can be assumed to have exactly $n+1$ elements in the first row. Let $\mathfrak{s}$ be another path of shape (λ,k). The element $d^{\mathrm{sp}*}_{\mathfrak{s}}x^{\mathrm{sp}}_{(\lambda,k)}d^{\mathrm{sp}}_{\mathfrak{t}}$ contains the element $x^{\mathrm{sp}}_{(n+1)}$ as a factor. Thus, by Proposition 2.28 it is an element of the annihilator. So the set is a subset of the annihilator.

Since it is a subset of the basis $\mathfrak{B}^{\mathrm{sp}}_r$ it is also linearly independent. The cardinality of the set is

$$\dim B_r(-2n) - \sum_{\substack{(\lambda,k)\in\Lambda(r) \\ \text{with } \lambda\ (-2n)\text{-permissible}}} n^2_{(\lambda,k)},$$

where $n_{(\lambda,k)}$ is the number of $(-2n)$-permissible paths to (λ, k). So by Theorem 2.26 the cardinality equals the dimension of the annihilator.

The third statement is a direct consequence of the first and the second one. $\square$

2.31 Remark. The bases $\mathfrak{B}_r^{\mathrm{sp}}$ have both properties that were observed to hold for the Murphy bases of the symmetric groups. They behave well in the tower of algebras (2.2) and they split into a basis for the endomorphism ring and a basis for the annihilator. But there is a downside: These bases are not integral, i.e. they are bases only for fields of characteristic zero or of characteristic $\geq r$.

In [Hu08] Hu analysed the action of the symmetric group $\mathfrak{S}_{2r}$ on the Brauer algebra $B_r(-2n)$. By doing so, he deduced that the annihilator of the symplectic tensor space in the Brauer algebra has an integral basis. With this information the considerations done in the previous proof can be adapted to construct a cellular basis for the endomorphism ring $B_r(-2n)/\operatorname{ann}$.

2.32 Theorem. *Let r be a positive integer, $\mathbb{F}$ a field and W a $2n$-dimensional symplectic vector space over $\mathbb{F}$. The set*

$$\mathfrak{C}_r^{\mathrm{sp}} := \{d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}} + \operatorname{ann} \mid \mathfrak{s}, \mathfrak{t}\ (-2n)\text{-permissible paths of shape } (\lambda, k), (\lambda, k) \in \Lambda(r)\}$$

is an $\mathbb{F}$-basis for the endomorphism ring $\operatorname{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$. Moreover, the triple $(\mathfrak{C}_r^{\mathrm{sp}}, \Lambda_x(r), \trianglerighteq)$ is a cellular basis.

Proof. The theorem is almost proven. All it needs are a couple of auxiliary definitions to get the ingredients right.

For every path $\mathfrak{t} = ((\lambda^{(0)}, k_0), (\lambda^{(1)}, k_1), \ldots, (\lambda^{(r)}, k_r))$ with $(-2n)$-permissible shape that is not $(-2n)$-permissible there is a minimal i such that $\lambda^{(i)}$ is not $(-2n)$-permissible. With this i define the following element for those paths:

$$\begin{aligned}\tilde{d}_{\mathfrak{t}} := {} & \bar{d}_{\lambda^{(r)} \to \lambda^{(r-1)}} x_{(\lambda^{(r-1)}, k_{r-1})} \bar{d}_{\lambda^{(r-1)} \to \lambda^{(r-2)}} x_{(\lambda^{(r-2)}, k_{r-2})} \\ & \cdots \bar{d}_{\lambda^{(i+2)} \to \lambda^{(i+1)}} x_{(\lambda^{(i+1)}, k_{r-1})} \bar{d}_{\lambda^{(i+1)} \to \lambda^{(i)}} x^{\mathrm{sp}}_{(\lambda^{(i)}, k_{r-1})} \bar{d}_{\lambda^{(i)} \to \lambda^{(i-1)}} x_{(\lambda^{(i-1)}, k_{r-1})} \cdots \\ & \qquad x_{(\lambda^{(1)}, k_1)} d_{\lambda^{(1)} \to \lambda^{(0)}}.\end{aligned}$$

For a $(-2n)$-permissible path $\mathfrak{t}$ let $\tilde{d}_{\mathfrak{t}} := d_{\mathfrak{t}}$.

Now recall the basis $\mathfrak{B}_r^x = \{d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t}$ paths of shape $(\lambda, k), (\lambda, k) \in \Lambda_x(r)\}$. Divide this set into the following subsets:

$\mathfrak{set}_1 =$ all $d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}}$ where both $\mathfrak{s}$ and $\mathfrak{t}$ are $(-2n)$-permissible. (This implies the shape (λ, k) of the paths to be also $(-2n)$-permissible.)

$\mathfrak{set}_2 =$ all $d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}}$ where (λ, k) is not $(-2n)$-permissible.

$\mathfrak{set}_3 =$ all $d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t}$ where (λ, k) is $(-2n)$-permissible but $\mathfrak{s}$ or $\mathfrak{t}$ is not.

Now alter these sets in the following way:

$\mathfrak{set}_1$ stays unaltered.

$\mathfrak{set}_2$ each $d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t}$ is replaced by $d_\mathfrak{s}^* x_{(\lambda,k)}^{\mathrm{sp}} d_\mathfrak{t}$.

$\mathfrak{set}_3$ each $d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t}$ is replaced by $\tilde{d}_\mathfrak{s}^* x_{(\lambda,k)} \tilde{d}_\mathfrak{t}$.

Now one has to prove that the union of these three sets again is a basis for the Brauer algebra $B_r(-2n)$ (at least over fields with characteristic 0 or characteristic $\geq r$). This is again done by adapting Theorem 2.14, similarly to the proof of Theorem 2.30.
The altering of $\mathfrak{set}_2$ just needs the observation, that $x_{(\lambda,k)}^{\mathrm{sp}} \equiv x_{(\lambda,k)} \mod B_r^{(k+1)}(x)$. Hence, $d_\mathfrak{s}^* x_{(\lambda,k)} d_\mathfrak{t} \equiv d_\mathfrak{s}^* x_{(\lambda,k)}^{\mathrm{sp}} d_\mathfrak{t} \mod B_r^{(k+1)}(x)$.
To substitute the elements in $\mathfrak{set}_3$ by the new ones, one has to use both the isomorphism given in Theorem 2.17 and the isomorphism given in the proof of Theorem 2.30. This depends on the corresponding position in the path $\mathfrak{t}$, i.e. when the passage $\lambda^{(i+1)} \to \lambda^{(i)}$ is the first (=minimal index i) passage from a $(-2n)$-permissible tableau $\lambda^{(i)}$ to a tableau $\lambda^{(i+1)}$ that is not $(-2n)$-permissible, then use the altered isomorphism (Theorem 2.30)– otherwise use the original isomorphism (Theorem 2.17).

By construction, the second and the third set span the annihilator. Since the annihilator has an integral basis, the second and the third set can be exchanged with that integral basis. This results in an integral basis for the Brauer algebra $B_r(-2n)$. Thus, factoring out the annihilator proves the claim. □

2.33 Remark. Note that these bases for the endomorphism ring $\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$ are Murphy-type cellular bases. By construction, the corresponding bases of the cell modules behave well in a tower of endomorphism rings

$$\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes 1}) \subset \mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes 2}) \subset \dots \\ \subset \mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r-1}) \subset \mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r}) \subset \dots$$

Filtration of symplectic tensor space. In Theorem 2.24 the dimensions of the irreducible $\mathbb{F}\,\mathrm{GSp}(W)$-modules Δ_λ can be described in terms of symplectic semistandard tableaux. The target set for a *symplectic tableau* is the set $\mathcal{I} = \{\bar{1}, 1, \bar{2}, \dots, \bar{n}, n\}$ with the indicated linear order. The weight of a tableau T is the composition $\mu = (\mu_{\bar{1}}, \mu_1, \mu_{\bar{2}}, \dots)$ such that μ_i counts how often i ($i \in \mathcal{I}$) appears in the tableau T, i.e. $\mu_i = |T^{-1}(i)|$. The *symplectic weight of a tableau* T is obtained from its weight via the map

$$\pi\colon (\mu_{\bar{1}}, \mu_1, \dots, \mu_{\bar{n}}, \mu_n) \mapsto (\mu_1 - \mu_{\bar{1}}, \dots, \mu_n - \mu_{\bar{n}}).$$

Following King [Kin76] and Berele [Ber86] a symplectic tableau T is called *symplectic semistandard* if it is semistandard and the entries in row i are all greater or equal to $\bar{i}$ for all i. As before, denote by $\mathrm{Tab}(\lambda)_{\mathcal{I}}$ ($\mathrm{Tab}(\lambda)_{\mathcal{I},\mu}$) the set of all semistandard λ-tableaux (of weight μ) with entries in $\mathcal{I}$. The set of all *symplectic* semistandard tableaux (of *symplectic weight* μ) with target set $\mathcal{I}$ is denoted by $\mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}$ ($\mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I},\mu}$).

2.34 Remark. Note that the additional condition implies that λ is $(-2n)$-permissible iff symplectic semistandard λ-tableaux exist.

2.35 Example. The symplectic semistandard tableaux with $n = 2$ to the partitions of 3:

$\begin{array}{|c|c|c|}\hline \bar1&\bar1&\bar1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&\bar1&1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&\bar1&\bar2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&\bar1&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&1&1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&1&\bar2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&1&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&\bar2&\bar2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&\bar2&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1&2&2\\\hline\end{array}$,

$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&1&\bar2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&\bar2&\bar2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&\bar2&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar2&\bar2&\bar2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar2&\bar2&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar2&2&2\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}$,

$\begin{array}{|c|c|}\hline \bar1&\bar1\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&\bar1\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&1\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&1\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&\bar2\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&\bar2\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&2\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1&2\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&1\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&\bar2\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&\bar2\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&2\\\hline \bar2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$,

$\begin{array}{|c|c|}\hline \bar2&\bar2\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar2&2\\\hline 2\\\cline{1-1}\end{array}$.

Note that there is no symplectic semistandard tableau for $\lambda = \begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}$, since there is no possible entry for the third row.

The following is a corollary to both, Theorem 2.26 and the type C Schur-Weyl duality, and it uses the description of the dimensions of the irreducible $\mathbb{C}\,\mathrm{GSp}(W)$-modules by symplectic semistandard tableaux. Similar to the type A case, it relates the dimensions of the irreducible $\mathbb{C}\,\mathrm{GSp}(W)$- and $B_r(-2n)$-modules to the dimension of the symplectic tensor space.

2.36 Corollary. *The equation*

$$\sum_{\lambda\in\Lambda(r)} |\,\mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}| \cdot n_{(\lambda',k)} = (2n)^r$$

holds, where $n_{(\lambda',k)}$ is the number of $(-2n)$-permissible paths to (λ',k).

Proof. The last part of the theorem on Brauer-Schur-Weyl-duality implies that

$$\dim_{\mathbb{C}} W^{\otimes r} = \sum_{f=0}^{\frac{r}{2}} \sum_{\substack{\lambda\vdash r-2f\\ \lambda'_1\leq n}} \dim_{\mathbb{C}}(\Delta_\lambda)\cdot \dim_{\mathbb{C}}(D^{\lambda'}).$$

By King [Kin76] and Berele [Ber86] the dimension of (Δ_λ) is given by the number of symplectic semistandard tableaux of shape λ, i.e.

$$\dim_{\mathbb{C}}(\Delta_\lambda) = |\operatorname{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}|.$$

By Wenzl [Wen88] the dimension of the $D^{\lambda'}$ is given as the number of $(-2n)$-permissible paths to (λ', k), i.e.

$$\dim_{\mathbb{C}}(D^{\lambda'}) = n_{(\lambda',k)}.$$

Thus

$$(2n)^r = \dim_{\mathbb{C}} W^{\otimes r} = \sum_{\lambda \in \Lambda(r)} |\operatorname{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}| \cdot n_{(\lambda',k)}.$$

□

Recall that $\{w_{\bar{1}}, w_1, w_{\bar{2}}, w_2 \ldots, w_{\bar{n}}, w_n\}$ is the basis of W. Let $w = w_{i_1} \otimes \cdots \otimes w_{i_r} \in W^{\otimes r}$. Analogously to the type A case, the weight of w is the composition $(\mu_{\bar{1}}, \mu_1, \ldots, \mu_{\bar{n}}, \mu_n)$ of r into $2n$ parts such that μ_k counts how often k appears in $i_1, \ldots, i_r$. The symplectic weight of a vector is obtained from its weight via the map

$$\pi\colon (\mu_{\bar{1}}, \mu_1, \ldots, \mu_{\bar{n}}, \mu_n) \mapsto (\mu_1 - \mu_{\bar{1}}, \ldots, \mu_n - \mu_{\bar{n}}).$$

The symplectic tensor space has a decomposition similar to the weight space decomposition of the tensor space. To describe this decomposition define the symplectic weight space N^μ to the symplectic weight μ as the direct sum of all weight spaces M^λ, which have a weight λ that maps on μ under π. By construction, the symplectic weight spaces are modules for the Brauer algebra. As in the classical case the symplectic tensor space decomposes in the following way:

$$W^{\otimes r} = \bigoplus_{\mu} N^\mu$$

where μ runs through all symplectic weights with n parts.

Let λ be a partition of $r - 2k$ and T a λ-tableau with entries in $\mathcal{I}$. Similarly to the classical case, the element w_T of the tensor space is defined as

$$w_T := w_{i_1} \otimes \cdots \otimes w_{i_{r-2k}},$$

where again the sequence $i_1, \ldots, i_{r-2k}$ denotes the entries of T read top to bottom and column by column.

Let now $\mathfrak{t}$ denote a path to $(\lambda', k) \in \Lambda(r)$ and define

$$w_{T\mathfrak{t}} := w_T x_{\lambda'} d_{\mathfrak{t}} \in W^{\otimes r}.$$

Note that it is necessary for the following construction to transpose the partition in the 'Brauer algebra part'. Nevertheless, the elements $w_{T\mathfrak{t}}$ are *alternating* sums of simple tensors!

2.37 Example. Let $n = 7$, $r = 4$, $\lambda = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$, $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$, $\mathfrak{t} = ((\emptyset, 0), (\square, 0), (\square\square, 0))$ and $\mathfrak{s} = ((\emptyset, 0), (\square, 0), (\square\square, 0), (\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \end{array}, 0), (\square\square, 1))$. Then $w_T = w_1 \otimes w_2$,

$$w_{T\mathfrak{t}} = w_1 \otimes w_2 - w_2 \otimes w_1,$$

$$w_{T\mathfrak{s}} = \sum_{i=1}^{n} w_1 \otimes w_2 \otimes (w_i \otimes w_{\bar{i}} - w_{\bar{i}} \otimes w_i) - \sum_{i=1}^{n} w_2 \otimes w_1 \otimes (w_i \otimes w_{\bar{i}} - w_{\bar{i}} \otimes w_i).$$

2.38 Theorem. *1. The set*

$$\{w_{T\mathfrak{t}} \mid T \in \mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}, \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\lambda', k) \in \Lambda(r)\}$$

is a basis of $W^{\otimes r}$.

2. The set

$$\left\{w_{T\mathfrak{t}} \mid T \in \mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I},\mu}, \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\lambda', k) \in \Lambda(r)\right\}$$

is a basis of N^μ.

The theorem is proved with the following three step strategy:
Step 1: Subdivide the weight modules into smaller pieces for which spanning sets are obtained (Lemma 2.40).
Step 2: Cancel out superfluous elements in these spanning sets (Lemma 2.43).
Step 3: Put all the remaining elements (for each piece of each weight space) together and compare the cardinality of this set with the dimension of the tensor space.

For the first step, recall the filtration of the Brauer algebra given in 2.1. Setting $\mathcal{TT}^{(k)}(N^\mu) := N^\mu \cdot B_r^{(k)}(-2n)$ gives a filtration of the weight-space N^μ:

$$\{0\} \subseteq \mathcal{TT}^{(\frac{r}{2})}(N^\mu) \subseteq \cdots \subseteq \mathcal{TT}^{(2)}(N^\mu) \subseteq \mathcal{TT}^{(1)}(N^\mu) \subseteq \mathcal{TT}^{(0)}(N^\mu) = N^\mu.$$

The factors $\mathcal{TT}^{(k)}(N^\mu)/\mathcal{TT}^{(k+1)}(N^\mu)$ are denoted by $\mathcal{HT}^{(k)}(N^\mu)$. The notation but not all of the definition for these sets is borrowed from [GW98] and the 'trace tensors' and 'harmonic tensors' defined there.

The second to last step in the proof of the following lemma is best understood by example.

2.39 Example. Let $r = 3$, $n = 3$ and $\mu = (-1, 0, 0)$. The weights λ mapping onto μ under π are $(2, 1, 0, 0, 0, 0)$, $(1, 0, 1, 1, 0, 0)$ and $(1, 0, 0, 0, 1, 1)$. The corresponding T^λ (without the empty rows in the second and third one) are $\begin{array}{|c|c|}\hline \bar{1} & \bar{1} \\ \hline 1 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|}\hline \bar{1} \\ \hline \bar{2} \\ \hline 2 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline \bar{1} \\ \hline \bar{3} \\ \hline 3 \\ \hline\end{array}$. The elements in $\mathcal{HT}^{(1)}(N^\mu)$ can thus be obtained by

$$(w_{\bar{1}} \otimes w_1 \otimes w_{\bar{1}}) B_r^{(1)}(-2n), \quad (w_{\bar{1}} \otimes w_{\bar{2}} \otimes w_2) B_r^{(1)}(-2n) \quad \text{or} \quad (w_{\bar{1}} \otimes w_{\bar{3}} \otimes w_3) B_r^{(1)}(-2n).$$

All these sets are the same. But the easiest way to obtain this set is to just start with $T = \boxed{\bar{1}}$. In this way one has to multiply $w_{\bar{1}}$ by $x_{((1),1)} d_{\mathfrak{t}}$ from the right, with $\mathfrak{t}$ being the three different paths to $(\boxed{1}, 1)$.

2.40 Lemma. *The factor $\mathcal{HT}^{(k)}(N^\mu)$ is generated as a vector space by the set*

$$\left\{ w_{T\mathfrak{t}} + \mathcal{TT}^{(k+1)}(N^\mu) \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)_{\mathcal{I},\nu}, \pi(\nu) = \mu, \\ \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\lambda', k), (\lambda', k) \in \Lambda(r) \end{array} \right\}.$$

Proof. Clearly, all (the coset representatives of) the vectors have weight μ.

Note that paths of shape $(\lambda, 0)$ are in bijection to paths of shape λ. Taking into account the definition of N^μ as sum of classical weight spaces M^ν (for $\pi(\nu) = \mu$) and the definition of the elements $w_{T\mathfrak{t}}$, one gets

$$\left\{ w_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)_{\mathcal{I},\nu}, \pi(\nu) = \mu, \\ \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\lambda', 0), (\lambda', 0) \in \Lambda(r) \end{array} \right\}$$

as a generating set for $N^\mu = \mathcal{TT}^{(0)}(N^\mu)$ and hence the claim for $k = 0$.

Now let $k = 1$. Let ν be a weight of $W^{\otimes r}$ that maps onto μ under π, i.e. M^ν is a summand of N^μ. Clearly, $w_{T^\nu} B_r^{(1)}(-2n) \subset \mathcal{TT}^{(1)}(N^\mu)$ and the union of the sets $w_{T^\nu} B_r^{(1)}(-2n)$ for all weights ν of $W^{\otimes r}$ that map onto μ under π is equal to $\mathcal{TT}^{(1)}(N^\mu)$. To get a better understanding of the set $w_{T^\nu} B_r^{(1)}(-2n)$, consider in a first step the basis given by Brauer diagrams. These are all Brauer diagrams with at least one horizontal arc in the top row. To be more precise only the Brauer diagrams with one horizontal arc in each row have to be considered, since products of w_{T^ν} with Brauer diagrams with more horizontal arcs are sums of products of w_{T^ν} with Brauer diagrams with one horizontal arc. The product of w_{T^ν} and such a Brauer diagram will be zero if the horizontal arc in the top row does not 'hit' a pair $(v_i, v_{\bar{i}})$. The precise position of this pair is not relevant for the outcome. So it would also be possible to start with $w_{T^{\nu'}} \otimes v_i \otimes v_{\bar{i}}$ where ν' is a weight of $W^{\otimes(r-2)}$ and is obtained from ν by setting $\nu'_i = \nu_i - 1$ and $\nu'_{\bar{i}} = \nu_{\bar{i}} - 1$ for one i and leaving the other parts unchanged. The union of the sets $(w_{T^{\nu'}} \otimes v_i \otimes v_{\bar{i}}) e_{r-1} B_r(-2n)$ for all ν' obtained from ν in the way described above equals the set $w_{T^\nu} B_r^{(1)}(-2n)$. On the other hand all the ν' have the same symplectic weight as ν. Corollary 2.23 describes a basis for $e_{r-1}B_r(-2n)$. With the considerations above only the basis elements with one horizontal arc in the top and bottom row need to be used. So,

$$\left\{ (w_{T^{\nu'}} \otimes v_i \otimes v_{\bar{i}}) d_{\mathfrak{s}}^* x_{(\lambda,k)} d_{\mathfrak{t}} \;\middle|\; \begin{array}{l} \mathfrak{s}, \mathfrak{t}\ (-2n)\text{-permissible paths of shape } (\lambda', 1), \\ \pi(\nu') = \mu,\ \ (\lambda', 1) \in \Lambda(r), \\ \mathfrak{s} \text{ ending with } (\dots, (\lambda', 0), (\lambda'_{+1}, 0), (\lambda', 1)) \end{array} \right\}$$

generates $\mathcal{TT}^{(1)}(N^\mu)$. Paths of shape $(\lambda', 1)$ ending with $\dots, (\lambda', 0), (\lambda'_{+1}, 0), (\lambda', 1))$ are in bijection to path of shape λ'. Therefore, with the ideas of Theorem 1.39:

$$\left\{ w_{T\mathfrak{t}} = w_T x_{(\lambda,k)} d_{\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)'_\nu, \pi(\nu') = \mu, \\ \mathfrak{t} \text{ a } (-2n)\text{-permissible paths of shape } (\lambda', 1), (\lambda', 1) \in \Lambda(r) \end{array} \right\}$$

generates $\mathcal{TT}^{(1)}(N^\mu)$.

The cases $k > 1$ can be done in the same way.

□

The lemma above gave spanning sets for the factors $\mathcal{HT}^{(k)}(N^\mu)$ which have a similar form as the elements in Theorem 2.38. The following lemmas show how to cancel out the superfluous elements.

The following lemma is an adaption of Lemma 2.1 in [Ber86].

2.41 Lemma. *Let λ be a partition of $r-2k$ and $\mathfrak{t}$ a $(-2n)$-permissible path to (λ', k). Let T_i ($i \in \{1, 2 \dots, n\}$) be a family of semistandard λ-tableaux of symplectic weight*

Side Note: In the proof of Lemma 2.40 there is one small trick concerning the difference between working with anti-symmetrizers (as in the type A case) and working with symmetrizers (as done here). Let μ be a weight and $\mathfrak{s}$ be a standard λ-tableau. With a side note stated above and Lemma 1.38 the following equation holds:

$$v_{T^\mu} b(\mathfrak{t}^\mu) d(\mathfrak{s})^{-1} = v_{T^\mu} b(\mathfrak{t}^\mu) b(\mathfrak{s})^{-1} b(\mathfrak{t}^\lambda) = v_{\mu(\mathfrak{s})} b(\mathfrak{t}^\lambda)$$

And the element $v_{\mu(\mathfrak{s})} b(\mathfrak{t}^\lambda)$ is the basis tensor corresponding to the transposed tableau of $\mu(\mathfrak{s})$.

μ, identical except for the entries in the two positions of the entries a and b in one column of $\mathfrak{t}_\lambda$, where T_i has the entries i and $\bar{i}$, respectively. Then

$$\sum_{i\in\mathcal{I}} w_{T_i\mathfrak{t}} \in \mathcal{TT}^{(k+1)}(N^\mu).$$

Proof. Let $\tilde{x}_{\lambda'}$ denote the element such that $(1+s_{a,b})\tilde{x}_{\lambda'} = x_{\lambda'}$. Then

$$\sum_{i\in\mathcal{I}} w_{T_i\mathfrak{t}} = \sum_{i\in\mathcal{I}} w_{T_i} x_{\lambda'} d_{\mathfrak{t}} = w_{T_1} e_{ab}\tilde{x}_{\lambda'} d_{\mathfrak{t}}$$

by the definition of the action of $s_{a,b}$ and e_{ab} and

$$w_{T_1} e_{ab}\tilde{x}_{\lambda'} d_{\mathfrak{t}} \in \mathcal{TT}^{(k+1)}(N^\mu),$$

since $e_{ab}\tilde{x}_{\lambda'} d_{\mathfrak{t}}$ has $k+1$ horizontal arcs in the bottom row. □

2.42 Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\{1,2\dots,n\}$, such that $\{1,2\dots,n\}$ is the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. Similarly to the lemma above define a family of tableaux in the following way: All tableaux in the family have fixed entries except for $2a$ many positions in the first column. For a subset $\mathcal{A}^*$ of $\{1,2\dots,n\}$ with size a the tableau $T_{\mathcal{A}^*}$ has the entries $\{\bar{i} \mid i\in\mathcal{A}^*\}$ in order in the first a of these positions and the entries $\{i \mid i\in\mathcal{A}^*\}$ in order in the other a positions. Let $\mathfrak{t}$ be a $(-2n)$-permissible path to (λ',k). Then*

$$\sum_{\substack{\mathcal{A}^*\subset\mathcal{A}\\ |\mathcal{A}^*|=a}} w_{T_{\mathcal{A}^*}\mathfrak{t}} \equiv (-1)^a \sum_{\substack{\mathcal{B}^*\subset\mathcal{B}\\ |\mathcal{B}^*|=a}} w_{T_{\mathcal{B}^*}\mathfrak{t}} \mod (\mathcal{TT}^{(k+1)}(N^\mu)).$$

Proof. This is proved by induction on a. Let $a=1$. By the previous lemma

$$\sum_{\substack{\mathcal{A}^*\subset\mathcal{A}\\ |\mathcal{A}^*|=1}} w_{T_{\mathcal{A}^*}\mathfrak{t}} + \sum_{\substack{\mathcal{B}^*\subset\mathcal{B}\\ |\mathcal{B}^*|=1}} w_{T_{\mathcal{B}^*}\mathfrak{t}} = \sum_{\substack{\mathcal{A}^*\subset\mathcal{A}\cup\mathcal{B}\\ |\mathcal{A}^*|=a=1}} w_{T_{\mathcal{A}^*}\mathfrak{t}} \in \mathcal{TT}^{(k+1)}(N^\mu).$$

This proves the claim for $a=1$.

Let now $a>1$ and assume $\lambda'=(2a)$. Denote by $\mathfrak{D}^{2a}_{(a,a)}$ the set of right coset representatives of $\mathfrak{S}_{(a,a)}$ in $\mathfrak{S}_{2a}$ such that $\mathfrak{S}_{(a,a)}\mathfrak{D}^{2a}_{(a,a)} = \mathfrak{S}_{2a}$.

$$\begin{aligned}
\sum_{\substack{\mathcal{A}^*\subset\mathcal{A}\\ |\mathcal{A}^*|=a}} w_{T_{\mathcal{A}^*}\mathfrak{t}} &= \sum_{\substack{\mathcal{A}^*\subset\mathcal{A}\\ |\mathcal{A}^*|=a}} w_{T_{\mathcal{A}^*}} x_{\lambda'} d_{\mathfrak{t}} \\
&= \sum_{\{i_1<i_2<\dots<i_a\}\subset\mathcal{A}} (w_{\bar{i}_1}\otimes w_{\bar{i}_2}\otimes\dots\otimes w_{\bar{i}_a}\otimes w_{i_1}\otimes w_{i_2}\otimes\dots\otimes w_{i_a}) x_{(2a)} d_{\mathfrak{t}} \\
&= \sum_{\{i_1<i_2<\dots<i_a\}\subset\mathcal{A}} ((w_{\bar{i}_1}\otimes w_{\bar{i}_2}\otimes\dots\otimes w_{\bar{i}_a}) x_{(a)} \otimes (w_{i_1}\otimes w_{i_2}\otimes\dots\otimes w_{i_a}) x_{(a)}) \sum_{\sigma\in\mathfrak{D}^{2a}_{(a,a)}} \sigma d_{\mathfrak{t}}
\end{aligned}$$

now expand the symmetrizer on the right but avoid signs by permuting entries on the left (remember that the permutations act by *signed* place permutation)

$$= \sum_{\substack{\{i_1<i_2<\cdots<i_a\}\subset\mathcal{A}\\ \omega\in\mathfrak{S}_a}} ((w_{\bar{i}_{(1)\omega}} \otimes w_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes w_{\bar{i}_{(a)\omega}})x_{(a)} \otimes w_{i_{(1)\omega}} \otimes w_{i_{(2)\omega}} \otimes \cdots \otimes w_{i_{(a)\omega}}) \sum_{\sigma\in\mathfrak{D}^{2a}_{(a,a)}} \sigma d_{\mathfrak{t}}$$

adding some zeros (remember that two occurences of a $w_{\bar{j}}$ in the simple tensor that is multiplied with $x_{(a)}$ will result in the summand beeing zero)

$$= \sum_{\substack{\{i_2<\cdots<i_a\}\subset\mathcal{A}\\ j\in\mathcal{A}\\ \omega\in\mathfrak{S}_{a-1}}} ((w_{\bar{j}} \otimes w_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes w_{\bar{i}_{(a)\omega}})x_{(a)} \otimes w_j \otimes w_{i_{(2)\omega}} \otimes \cdots \otimes w_{i_{(a)\omega}}) \sum_{\sigma\in\mathfrak{D}^{2a}_{(a,a)}} \sigma d_{\mathfrak{t}}$$

with Lemma 2.41

$$\equiv (-1) \sum_{\substack{\{i_2<\cdots<i_a\}\subset\mathcal{A}\\ j\in\mathcal{B}\\ \omega\in\mathfrak{S}_{a-1}}} ((w_{\bar{j}} \otimes w_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes w_{\bar{i}_{(a)\omega}})x_{(a)} \otimes w_j \otimes w_{i_{(2)\omega}} \otimes \cdots \otimes w_{i_{(a)\omega}}) \sum_{\sigma\in\mathfrak{D}^{2a}_{(a,a)}} \sigma d_{\mathfrak{t}}$$

with induction hypothesis

$$\equiv (-1)^a \sum_{\substack{\{i_2<\cdots<i_a\}\subset\mathcal{B}\\ j\in\mathcal{B}\\ \omega\in\mathfrak{S}_{a-1}}} ((w_{\bar{j}} \otimes w_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes w_{\bar{i}_{(a)\omega}})x_{(a)} \otimes w_j \otimes w_{i_{(2)\omega}} \otimes \cdots \otimes w_{i_{(a)\omega}}) \sum_{\sigma\in\mathfrak{D}^{2a}_{(a,a)}} \sigma d_{\mathfrak{t}}$$

deleting the zeros (i.e. every summand in which a $w_{\bar{j}}$ appears two times)

$$\begin{aligned} &= (-1)^a \sum_{\substack{\{i_1<i_2<\cdots<i_a\}\subset\mathcal{B}\\ \omega\in\mathfrak{S}_a}} ((w_{\bar{i}_{(1)\omega}} \otimes w_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes w_{\bar{i}_{(a)\omega}})x_{(a)} \\ &\quad \otimes w_{i_{(1)\omega}} \otimes w_{i_{(2)\omega}} \otimes \cdots \otimes w_{i_{(a)\omega}}) \sum_{\sigma\in\mathfrak{D}^{2a}_{(a,a)}} \sigma d_{\mathfrak{t}} \\ &= \sum_{\substack{\mathcal{B}^*\subset\mathcal{B}\\ |\mathcal{B}^*|=a}} w_{T_{\mathcal{B}^*}\mathfrak{t}} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)) \end{aligned}$$

The calculation for λ of arbitrary shape stays the same. The simple tensors just grow bigger. □

The following lemma shows that it is enough to use the vectors indexed by symplectic semistandard tableaux.

2.43 Lemma. *The factor $\mathcal{HT}^{(k)}(N^\mu)$ is generated as a vector space by the set*

$$\left\{ w_{T\mathfrak{t}} + \mathcal{T}\mathcal{T}^{(k+1)}(N^\mu) \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I},\mu}, \\ \mathfrak{t} \textit{ a } (-2n)\textit{-permissible path of shape } (\lambda', k), (\lambda', k) \in \Lambda(r) \end{array} \right\}.$$

Proof. Let T be a symplectic tableau that is semistandard but not symplectic semistandard, i.e. there is an entry $\bar{j}$ or j in row k with $k > j$. Without loss of generality this entry may be in the first column.

Define the following sets: Let $\mathcal{D}$ be the set $\{j+1, j+2, \ldots, n\}$. Let $\mathcal{A}$ be the set of all indices in $\{1, 2, \ldots, j\}$ for which $\bar{i}$ and i appear in the first column of T. Let $\mathcal{B}$ be the set of all indices in $\{1, 2, \ldots, j\}$ for which either $\bar{i}$ or i appears in the first column of T but not both. At last, let $\mathcal{C}$ be the set of indices in $\{1, 2, \ldots, j\}$ for which neither $\bar{i}$ nor i appears in the first column of T. Obviously, $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a disjoint union and therefore, $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = j$. Since T is column-standard, i.e. the first k entries in the first column of T are precisely the i and $\bar{i}$ in $\mathcal{A}$ and the i or $\bar{i}$ for $i \in \mathcal{B}$, it follows that $2|\mathcal{A}| + |\mathcal{B}| = k$. This implies $|\mathcal{A}| > |\mathcal{C}|$.

Therefore, with the notation of Lemma 2.42 the equation

$$\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{D} \\ |\mathcal{A}^*| = |\mathcal{A}|}} w_{T_{\mathcal{A}^*}\mathfrak{t}} \equiv (-1)^{|\mathcal{A}|} \sum_{\substack{\mathcal{C}^* \subset \mathcal{C} \\ |\mathcal{C}^*| = |\mathcal{A}|}} w_{T_{\mathcal{C}^*}\mathfrak{t}} = 0 \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu))$$

holds, since there exists no $T_{\mathcal{C}^*}$ with $\mathcal{C}^* \subset \mathcal{C}$ and $|\mathcal{C}^*| = |\mathcal{A}|$. Thus,

$$w_{T\mathfrak{t}} = w_{T_{\mathcal{A}}\mathfrak{t}} \equiv -\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{D} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} w_{T_{\mathcal{A}^*}\mathfrak{t}} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)).$$

By the definition of $\mathcal{B}$, there are multiple occurrences of the same entry in the first column of $T_{\mathcal{A}^*}$ if $\mathcal{A}^* \cap \mathcal{B} \neq \emptyset$. Hence

$$w_{T\mathfrak{t}} \equiv -\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \cup \mathcal{D} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} w_{T_{\mathcal{A}^*}\mathfrak{t}} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)).$$

Each term on the right-hand side of the equation is of higher weight than T (though they have the same symplectic weight!). $\square$

Theorem 2.38 follows with the following consideration: Putting all the spanning sets together gives the set

$$\{w_{T\mathfrak{t}} \mid T \in \operatorname{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}, \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\lambda', k), (\lambda', k) \in \Lambda(r)\}$$

as a spanning set of $W^{\otimes r}$. This set has at most $\sum_{\lambda \vdash r} |\operatorname{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}| \cdot n_{(\lambda', k)} = (2n)^r$ elements and thus it has exactly $(2n)^r$ elements and the theorem is proven.

For every $(\lambda, k) \in \Lambda(r)$ define $W(\trianglelefteq (\lambda, k))$ and $W(\triangleleft (\lambda, k))$ as subspaces of $W^{\otimes r}$ generated by

$$\left\{ w_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \operatorname{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}, \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\nu', l), (\nu', l) \in \Lambda_x(r), \\ (\nu', l) \trianglerighteq (\lambda', k) \end{array} \right\}$$

and

$$\left\{ w_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}, \mathfrak{t} \text{ a } (-2n)\text{-permissible path of shape } (\nu', l), (\nu', l) \in \Lambda_x(r), \\ (\nu', l) \rhd (\lambda', k) \end{array} \right\}.$$

The results about the ordinary tensor space together with the proofs above imply the following:

2.44 Theorem. *Let (λ, k) be a pair in $\Lambda(r)$. Then as $(\mathbb{F}\,\mathrm{GSp}(W), B_r(-2n))$-bimodules*

$$W(\unlhd (\lambda, k))/W(\lhd (\lambda, k)) \cong \Delta^{\mathrm{sp}}(\lambda) \otimes C^{(\lambda', k)},$$

where $\Delta^{\mathrm{sp}}(\lambda)$ is an $\mathbb{F}\,\mathrm{GSp}(W)$-module and a factor of the dual Weyl module $\Delta(\lambda)$ and $C^{(\lambda',k)}$ is the cell module of $\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$ associated to λ and (λ', k) respectively.

2.3.2. The orthogonal case

Brauer-Schur-Weyl-duality. For the remainder of this section let $\mathbb{F}$ be a field of characteristic other than 2.

Let U be an m-dimensional orthogonal vector space over a field $\mathbb{F}$ with a non-degenerate symmetric bilinear form $(-,-)$. Let $\{u_{\bar{1}}, u_1, u_{\bar{2}}, u_2 \dots, u_{\bar{n}}, u_n\}$ be the basis (of hyberbolic pairs) if $m = 2n$ and let $\{u_{\bar{1}}, u_1, u_{\bar{2}}, u_2 \dots, u_{\bar{n}}, u_n, u_0\}$ be the basis (of hyperbolic pairs and an anisotropic vector) if $m = 2n + 1$. Denote the index set in either case by $\mathcal{I}$. Let $\bar{\bar{i}} = i$ and let $\bar{0} = 0$. The basis and the bilinear form are supposed to satisfy $(u_i, u_{\bar{j}}) = \delta_{i,j}$.

Define a right-action of the Brauer algebra $B_r(m)$ on the r-fold tensor space $U^{\otimes r}$ in the following way. The elements s_i act by place permutation and

$$\begin{aligned}\big(u_{k_1} \otimes \cdots \otimes u_{k_{i-1}} \otimes u_{k_i} \otimes u_{k_{i+1}} \otimes u_{k_{i+2}} \otimes \cdots \otimes u_{k_r}\big) \cdot e_i \\ := (u_{k_i}, u_{k_{i+1}}) \sum_{l \in \mathcal{I}} u_{k_1} \otimes \cdots \otimes u_{k_{i-1}} \otimes u_l \otimes u_{\bar{l}} \otimes u_{k_{i+2}} \otimes \cdots \otimes u_{k_r}\end{aligned}$$

Hence, there is a map $\Psi : B_r(m)^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}}(U^{\otimes r})$.

The orthogonal similitude group $\mathrm{GO}(U)$ is defined as a subgroup of $\mathrm{GL}(U)$

$$\mathrm{GO}(U) := \{g \in \mathrm{GL}(U) \mid \exists 0 \neq d \in \mathbb{F}, \text{ such that } (gv, gw) = d(v, w)\ \forall v, w \in W\}$$

The orthogonal similitude group acts diagonally on the r-fold tensor space. Hence there is a map $\Phi : \mathbb{F}\,\mathrm{GO}(U) \to \mathrm{End}_{\mathbb{F}}(U^{\otimes r})$. Moreover, the following statements hold:

2.45 Theorem. *1. The left action of* $\mathrm{GO}(U)$ *on the* r*-fold tensor space commutes with the right action of* $B_r(m)$*. Therefore the maps* Ψ *and* Φ *are maps in the following sense:*

- $\Psi : B_r(m)^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$
- $\Phi : \mathbb{F}\,\mathrm{GO}(U) \to \mathrm{End}_{B_r(m)}(U^{\otimes r})$

2. If $\mathbb{F}$ *is an infinite field then the maps* Ψ *and* Φ *in 1. are surjective.*

3. If $\mathbb{F}$ *is an infinite field and* $m \geq r$ *then the map* Ψ *is injective and hence* $B_r(m)^{\mathrm{op}} \cong \mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$*.*

4. If $\mathbb{F} = \mathbb{C}$*, then*

$$U^{\otimes r} = \bigoplus_{f=0}^{\frac{r}{2}} \bigoplus_{\substack{\lambda \vdash r-2f \\ \lambda_1'+\lambda_2' \leq n}} \Delta_\lambda \otimes D^\lambda$$

as irreducible $(\mathbb{F}\,\mathrm{GO}(U), B_r(m))$*-bimodules, where* Δ_λ *and* D^λ *are the irreducible* $\mathbb{F}\,\mathrm{GO}(U)$*- and* $B_r(m)$*-modules associated to* λ*.*

2.46 Remark. This double-centralizer property, which is often called Brauer-Schur-Weyl-duality or type B/D Schur-Weyl duality, is due to Brauer [Bra37] for the case $\mathbb{F} = \mathbb{C}$. For the other fields in question the results were proven by Doty and Hu [DH09].

Therefore, for infinite fields $B_r(m)/\ker(\Psi) \cong \mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$. The kernel of Ψ is the annihilator of the tensor space in the Brauer algebra.

Doty and Hu [DDH08] showed that the kernel of Ψ is rigid, i.e. its dimension does not depend on the choice of the field $\mathbb{F}$. Hence, the dimension of $\mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$ also does not depend on the choice of the (infinite) field $\mathbb{F}$.

Wenzl [Wen88] gave a combinatorial description of the dimension of the endomorphism ring $\mathrm{End}_{\mathbb{C}\,\mathrm{GO}(U)}(U^{\otimes r})$ which evolves around restriction and induction in the tower of algebras in question. This motivated him to give the following definition:

A Young diagram λ is called m-permissible if its first two columns contain at most m boxes, i.e. $\lambda_1' + \lambda_2' \leq m$. Similarly, a path $\mathfrak{t} = ((\lambda^{(0)}, k_0), (\lambda^{(1)}, k_1), \ldots, (\lambda^{(r)}, k_r))$ in the Bratelli diagram is called m-permissible if every $\lambda^{(i)}$ is m-permissible.

With this definition the dimension of $\mathrm{End}_{\mathbb{C}\,\mathrm{GO}(U)}(U^{\otimes r})$ can be expressed as follows:

Side Note: The reason to favour $\mathrm{GSp}(W)$ over $\mathrm{Sp}(W)$ also applies here. Thus, $\mathrm{GO}(U)$ is used instead of $\mathrm{O}(U)$.

2.47 Theorem ([Wen88, Theorem 3.4, Corollary 3.5]).

$$\dim \operatorname{End}_{\mathbb{C}\,\mathrm{GO}(U)}(U^{\otimes r}) = \sum_{\substack{(\lambda,k)\in\Lambda(r)\\ \text{with } \lambda \text{ } m\text{-permissible}}} m^2_{(\lambda,k)},$$

where $m_{(\lambda,k)}$ is the number of m-permissible paths to (λ, k).

2.48 Remark. The anti-symmetrizer of the walled Brauer algebra $B_{m+1-s,s}(m)$ is the alternating sum over all elements in $\mathfrak{B}_{m+1-s,s}$ (cf. [Wer14]). Readers unfamiliar with walled Brauer diagrams and the walled Brauer algebra may jump to the beginning of the third chapter to get the needed definitions.

A stronger version of the following proposition can be found in [Gav99, Theorem 3.7] for the case $\mathbb{F} = \mathbb{C}$. A proof for arbitrary fields can be found in [LZ12, Lemma 9.1].

2.49 Proposition. *Let U be the vector space described above and let r be a positive integer with $r > m$. The anti-symmetrizer of the walled Brauer algebra $B_{m+1-s,s}(m)$ (embedded in $B_r(m)$) is an element of the annihilator of the r-fold tensor space.*

2.50 Remark. This is a bigger difference between the symplectic and the orthogonal case. In the symplectic case the *symmetrizer* of a smaller Brauer algebra lies in the annihilator. Here, in the orthogonal case, the *anti-symmetrizer* of a smaller *walled* Brauer algebra lies in the annihilator. While the transition from symmetrizer to anti-symmetrizer can be explained intuitively by the transition from *signed permutation of entries* to *permutation of entries*, the transition from Brauer algebra to walled Brauer algebra seems to have no such direct correspondent. However, the combinatorics of the dimension description 'seem to know' about that fact: In the symplectic case the permissibility depends only on one parameter in the Young diagrams, in the orthogonal case the permissibility depends on (the sum of) two parameters.

Basis for the endomorphsim ring. Propositon 2.49 motivates the following definitions. For a partition λ of r define the direct product $\mathfrak{B}_{\lambda_1,\lambda_2} \times \mathfrak{S}_{\lambda_3} \times \cdots \times \mathfrak{S}_{\lambda_k}$ as submonoid of $\mathfrak{B}_r$ denoted by $\mathfrak{B}^\circ_\lambda$. Further define the elements

$$y^\circ_\lambda := \sum_{b\in\mathfrak{B}^\circ_{\lambda'}} \operatorname{sign}(b)\cdot b \quad \text{ and } \quad y^\circ_{(\lambda,k)} = y^\circ_\lambda e^{(k)}_{r-1}.$$

Let $1 \leq s \leq \frac{m+1}{2}$ be an integer. If $\lambda = (r-m+1, 2^{(s-1)}, 1^{(m-2s+1)})$ the element y°_λ is the anti-symmetrizer of the walled Brauer algebra $B_{m+1-s,s}(m)$ (embedded in $B_r(m)$). Thus, whenever $\lambda_1 + \lambda_2 = m+1$ the element y°_λ has a factor equal to the anti-symmetrizer of the walled Brauer algebra $B_{m+1-s,s}(m)$.

For $\mathfrak{t} = ((\lambda^{(0)}, k_0), (\lambda^{(1)}, k_1), \ldots, (\lambda^{(r)}, k_r))$ a path of length r in the Bratteli diagram, let

$$b^{\mathrm{o}}_{\mathfrak{t}} := b_{\lambda^{(r)} \to \lambda^{(r-1)}} y^{\mathrm{o}}_{(\lambda^{(r-1)}, k_{r-1})} b_{\lambda^{(r-1)} \to \lambda^{(r-2)}} y^{\mathrm{o}}_{(\lambda^{(r-2)}, k_{r-2})} \cdots y^{\mathrm{o}}_{(\lambda^{(1)}, k_1)} b_{\lambda^{(1)} \to \lambda^{(0)}}.$$

The following two theorems are proved in exactly the same way as their symplectic counterparts. Only two adaptions have to be made: The 'y'-versions have to be used instead of the 'x'-versions and the $(-2n)$-permissibility has to be replaced by m-permissibility.

2.51 Theorem. *Let r be a positive integer and let $\mathbb{F}$ be a field with* $\operatorname{char}(\mathbb{F}) = 0$ *or* $2 \neq \operatorname{char}(\mathbb{F}) \geq r$.

1. *The set*

$$\mathfrak{B}^{\mathrm{o}}_r := \{ b^{\mathrm{o}*}_{\mathfrak{s}} y^{\mathrm{o}}_{(\lambda,k)} b^{\mathrm{o}}_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \textit{ paths of shape } (\lambda,k), (\lambda,k) \in \Lambda(r) \}$$

   is an $\mathbb{F}$-basis for the Brauer algebra $B_r(x)$. Moreover, $(\mathfrak{B}^{\mathrm{o}}_r, \Lambda_y(r), \trianglelefteq)$ is a cellular basis.

2. *The set*

$$\{ b^{\mathrm{o}*}_{\mathfrak{s}} y^{\mathrm{o}}_{(\lambda,k)} b^{\mathrm{o}}_{\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \textit{ paths of shape } (\lambda,k), \mathfrak{s} \textit{ or } \mathfrak{t} \textit{ not } m\textit{-permissible}, (\lambda,k) \in \Lambda(r) \}$$

   is an $\mathbb{F}$-basis for the annihilator $\operatorname{Ann}_{B_r(m)}(U^{\otimes} r)$ *of tensor space in the Brauer algebra.*

3. *The set*

$$\mathfrak{C}^{\mathrm{o}}_r := \{ b^{\mathrm{o}*}_{\mathfrak{s}} y^{\mathrm{o}}_{(\lambda,k)} b^{\mathrm{o}}_{\mathfrak{t}} + \mathrm{ann} \mid \mathfrak{s}, \mathfrak{t} \; m\textit{-permissible paths of shape } (\lambda,k), (\lambda,k) \in \Lambda(r) \}$$

   is an $\mathbb{F}$-basis for the endomorphism ring $\operatorname{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$.
   Moreover, $(\mathfrak{C}^{\mathrm{o}}_r, \Lambda_y(r), \trianglelefteq)$ is a cellular basis and therefore, $\operatorname{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$ *is a cellular algebra.*

2.52 Remark. The bases $\mathfrak{B}^{\mathrm{o}}_r$ have both properties that were observed to hold for the Murphy bases of the symmetric group. Both of them behave well in the tower of algebras (2.2) and they split into a basis for the endomorphism ring and a basis for the annihilator. But there is a downside: These bases are not integral, i.e. they are bases only for fields of characteristic zero or of characteristic $\geq r$.

2.53 Theorem. *Let r be a positive integer, $\mathbb{F}$ a field and U an n-dimensional orthogonal vector space over $\mathbb{F}$. The set*

$$\mathfrak{C}^{\mathrm{o}}_r := \{ b^{*}_{\mathfrak{s}} y_{(\lambda,k)} b_{\mathfrak{t}} + \mathrm{ann} \mid \mathfrak{s}, \mathfrak{t} \; m\textit{-permissible paths of shape } (\lambda,k), (\lambda,k) \in \Lambda(r) \}$$

is an $\mathbb{F}$-basis for the endomorphism ring $\operatorname{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$. *Moreover, $(\mathfrak{C}^{\mathrm{o}}_r, \Lambda_y(r), \trianglelefteq)$ is a cellular basis.*

2.54 Remark. Note, that these bases for the endomorphism ring $\mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$ are Murphy-type cellular algebras. By construction, the corresponding bases of the cell modules behave well in a tower of endomorphism rings

$$\begin{aligned}\mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes 1}) \subset \mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes 2}) \subset \dots \\ \subset \mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r-1}) \subset \mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r}) \subset \dots\end{aligned}$$

Filtration of orthogonal tensor space. The dimension of the Δ_λ in Theorem 2.45 can be described in terms of orthogonal semistandard tableaux. The target set for an orthogonal tableau is the set $\mathcal{I} = \{\bar{1}, 1, \bar{2}, 2, \dots, \bar{n}, n\}$ or $\mathcal{I} = \{\bar{1}, 1, \bar{2}, 2, \dots, \bar{n}, n, 0\}$ depending on the parity of m. The weight of a tableau T is the composition $\mu = (\mu_{\bar{1}}, \mu_1, \mu_{\bar{2}}, \dots)$ such that μ_i counts how often i appears in the tableau T, i.e. $\mu_i = |T^{-1}(i)|$. The orthogonal weight of a tableau T is obtained from its weight via the map:

$$\begin{aligned}\pi\colon (\mu_{\bar{1}}, \mu_1, \dots, \mu_{\bar{n}}, \mu_n) &\mapsto (\mu_1 - \mu_{\bar{1}}, \dots, \mu_n - \mu_{\bar{n}}), \quad \text{for } m \text{ even;} \\ \pi\colon (\mu_{\bar{1}}, \mu_1, \dots, \mu_{\bar{n}}, \mu_n, \mu_0) &\mapsto (\mu_1 - \mu_{\bar{1}}, \dots, \mu_n - \mu_{\bar{n}}), \quad \text{for } m \text{ odd.}\end{aligned}$$

Denote by α_i and β_i ($i = 1, 2, \dots, [\frac{n}{2}]$) the number of entries less than or equal to i in the first and second columns, respectively. Following King and Welsh [KW93] an orthogonal tableau T is called *orthogonal semistandard* if it is semistandard and for each $i = 1, 2, \dots, n$

1. $\alpha_i + \beta_i \leq 2i$;
2. if $\alpha_i + \beta_i = 2i$ with $\alpha_i > \beta_i$ and $T(\alpha_i, 1) = i$ and $T(\beta_i, 2) = \bar{i}$, then $T(\alpha_i - 1, 1) = \bar{i}$;
3. if $\alpha_i + \beta_i = 2i$ with $\alpha_i = \beta_i (= i)$ and $T(i, 1) = \bar{i}$ and $T(i, b) = i$ for some b, then $T(i - 1, b) = \bar{i}$.

The second and the third condition are sometimes called protection conditions, since a certain i has to be 'protected' by an $\bar{i}$ above. The set of all *orthogonal semistandard* tableaux (of *orthogonal weight* μ) with target set $\mathcal{I}$ is denoted by $\mathrm{Tab}(\lambda)^{\mathrm{o}}_{\mathcal{I}}$ ($\mathrm{Tab}(\lambda)^{\mathrm{o}}_{\mathcal{I},\mu}$).

2.55 Remark. Note that if m is even, the first condition implies that λ is m-permissible iff orthogonal semistandard λ-tableaux exist. If m is odd there exist orthogonal semistandard λ-tableaux for λ not m-permissible.

Proof. Let λ be a m-permissible partition, i.e. $\lambda'_1 + \lambda'_2 \leq m$. Filling the first λ'_1 elements of $\{\bar{1}, 1, \bar{2}, 2, \dots\}$ in the first column, the next λ'_2 elements in the second one and then filling the empty boxes with the entries that are filled in to the left gives a orthogonal semistandard tableau. Obviously it is semistandard, the first additional condition is met and the other two additional conditions do not trigger.

Let $m = 2n$ be even and let T be a orthogonal semistandard tableau. Since $\lambda'_1 + \lambda'_2 = \alpha_n + \beta_n \le 2n = m$, the shape of T is m-permissible.

For $m = 3$, the tableau $T = \begin{array}{|c|c|}\hline \bar{1} & 0 \\ \hline 1 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}$ is orthogonal semistandard but its shape is not m-permissible. ☐

2.56 Example. The orthogonal semistandard tableaux with $m = 4$ to the partitions of 3:

$\begin{array}{|c|c|c|}\hline \bar1 & \bar1 & \bar1 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1 & \bar1 & \bar2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1 & \bar1 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1 & \bar2 & \bar2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1 & \bar2 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar1 & 2 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 1 & \bar2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & \bar2 & \bar2 \\ \hline\end{array}$,

$\begin{array}{|c|c|c|}\hline 1 & \bar2 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar2 & \bar2 & \bar2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar2 & \bar2 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline \bar2 & 2 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline\end{array}$,

$\begin{array}{|c|c|}\hline \bar1 & \bar1 \\ \hline \bar2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & \bar1 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & \bar2 \\ \hline 1 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & \bar2 \\ \hline \bar2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & \bar2 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & 2 \\ \hline \bar2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & \bar2 \\ \hline \bar2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & \bar2 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$,

$\begin{array}{|c|c|}\hline \bar2 & \bar2 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$,

$\begin{array}{|c|}\hline \bar1 \\ \hline 1 \\ \hline \bar2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline \bar1 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline \bar1 \\ \hline \bar2 \\ \hline 2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline \bar2 \\ \hline 2 \\ \hline\end{array}$.

In complete analogy with the symplectic case, the following corollary to the type B/D Schur-Weyl duality relates the dimensions of the irreducible $\mathbb{C}\,\mathrm{GO}(U)$- and $B_r(m)$-modules to the dimension of the orthogonal tensor space.

2.57 Corollary. *The equation*

$$\sum_{\lambda \in \Lambda(r)} |\operatorname{Tab}(\lambda)^{\mathrm{o}}_{\mathcal{I}}| \cdot m_{(\lambda,k)} = m^r$$

holds, where $m_{(\lambda,k)}$ is the number of m-permissible paths to (λ, k).

Recall that $\{u_{\bar1}, u_1, u_{\bar2} \dots\}$ is the basis of U. Let $u = u_{i_1} \otimes \cdots \otimes u_{i_r} \in U^{\otimes r}$. The orthogonal weight of u is obtained from its weight via the map:

$$\pi \colon (\mu_{\bar1}, \mu_1, \dots, \mu_{\bar n}, \mu_n) \mapsto (\mu_1 - \mu_{\bar1}, \dots, \mu_n - \mu_{\bar n}), \quad \text{for } m \text{ even;}$$
$$\pi \colon (\mu_{\bar1}, \mu_1, \dots, \mu_{\bar n}, \mu_n, \mu_0) \mapsto (\mu_1 - \mu_{\bar1}, \dots, \mu_n - \mu_{\bar n}), \quad \text{for } m \text{ odd.}$$

The orthogonal tensor space has a decomposition similar to the weight space decomposition of the tensor space. To describe this decomposition define the orthogonal weight space N^μ to the orthogonal weight μ as the direct sum of all weight spaces M^λ, which have a weight λ that maps on μ under π. By construction, the orthogonal weight spaces are modules for the Brauer algebra. As in the classical case the tensor space decomposes in the following way:

$$U^{\otimes r} = \bigoplus_{\mu} N^\mu$$

where μ runs through all orthogonal weights with n parts.

Let λ be a partition of $r-2k$ and T a λ-tableau with entries in $\mathcal{I}$. As before, the element u_T of the tensor space is defined as

$$u_T := u_{i_1} \otimes \cdots \otimes u_{i_{r-2k}},$$

where again the sequence $i_1, \ldots, i_{r-2k}$ denotes the entries of T read top to bottom and column by column.

Let now $\mathfrak{t}$ denote a path to $(\lambda, k) \in \Lambda(r)$ and define

$$u_{T\mathfrak{t}} := u_T y_\lambda b_\mathfrak{t} \in U^{\otimes r}.$$

The following theorem is an analogue to Theorem 1.39.

2.58 Theorem. *1. The set*

$$\left\{u_{T\mathfrak{t}} \mid T \in \mathrm{Tab}(\lambda)^{\mathrm{o}}_{\mu}, \mathfrak{t} \text{ a } m\text{-permissible path of shape } (\lambda,k), (\lambda,k) \in \Lambda(r)\right\}$$

is a basis of $U^{\otimes r}$.

2. The set

$$\left\{u_{T\mathfrak{t}} \mid T \in \mathrm{Tab}(\lambda)^{\mathrm{o}}_{\mathcal{I},\mu}, \mathfrak{t} \text{ a } m\text{-permissible path of shape } (\lambda,k), (\lambda,k) \in \Lambda(r)\right\}$$

is a basis of N^μ.

The theorem is proved with the same three step strategy as in the symplectic case. The steps itself differ only slightly from the symplectic case. Of course, the main difference is the combinatoric. This results in a longer second step (Lemmas 2.62, 2.63, 2.64), since more cases of semistandard but not orthogonal semistandard tableaux have to be considered.

The following lemma, the first step in the proof, follows in the same way as the corresponding lemma in the symplectic case.

2.59 Lemma. *The factor $\mathcal{HT}^{(k)}(N^\mu)$ is generated as a vector space by the set*

$$\left\{ u_{T\mathfrak{t}} + \mathcal{TT}^{(k+1)}(N^\mu) \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)_{\mathcal{I},\nu}, \pi(\nu) = \mu, \\ \mathfrak{t} \textit{ a m-permissible path of shape } (\lambda, k), (\lambda, k) \in \Lambda(r) \end{array} \right\}.$$

Since $\mathrm{Tab}(\lambda)^{\circ}_{\mathcal{I},\mu} \subseteq \{T \in \mathrm{Tab}(\lambda)_{\mathcal{I},\nu} \mid \pi(\nu) = \mu\}$, it remains to show, how the vectors associated to the semistandard but not orthogonal semistandard tableaux can be expressed as linear combinations of vectors associated to orthogonal semistandard tableaux. This is done adapting the arguments of King and Welsh [KW93] in the following five lemmas.

2.60 Lemma (Lemma 3.3 of [KW93])**.** *Let λ be a partition of $r-2k$ and $\mathfrak{t}$ a m-permissible path to (λ, k). Let T_i, for $i \in \mathcal{I}$, be a family of semistandard λ-tableaux of orthogonal weight μ identical except for the entries in the two positions of the entries a and b in $\mathfrak{t}_\lambda$, where T_i has the entries i and $\bar{i}$, respectively. Then*

$$\sum_{i \in \mathcal{I}} u_{T_i\mathfrak{t}} \in \mathcal{TT}^{(k+1)}(N^\mu).$$

Proof.

$$\sum_{i \in \mathcal{I}} u_{T_i\mathfrak{t}} = \sum_{i \in \mathcal{I}} u_{T_i} y_\lambda b_\mathfrak{t} = u_{T_1} e_{ab} y_\lambda b_\mathfrak{t} \in \mathcal{TT}^{(k+1)}(N^\mu),$$

since $e_{ab} y_\lambda b_\mathfrak{t}$ has $k+1$ horizontal arcs in the bottom row. □

The following lemma is the adaption of an argument in the proof of Lemma 3.6 in [KW93] which is an adaption of an argument in [Ber86] itself. Therefore, the proof given here is very similar to the proof of Lemma 2.42.

2.61 Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{I}$, such that $\mathcal{I}$ is the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. Similarly to the lemma above define a family of tableaux in the following way: All tableaux in the family have fixed entries except for a many positions in the first column and a many positions in the second column. For a subset $\mathcal{A}^*$ of $\mathcal{I}$ with size a the tableau $T_{\mathcal{A}^*}$ has the entries $\{\bar{i} \mid i \in \mathcal{A}^*\}$ in order in the first column and the entries $\{i \mid i \in \mathcal{A}^*\}$ in order in the second column. Let $\mathfrak{t}$ be a m-permissible path to (λ, k). Then the following equation holds*

$$\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \\ |\mathcal{A}^*| = a}} u_{T_{\mathcal{A}^*}\mathfrak{t}} = (-1)^a \sum_{\substack{\mathcal{B}^* \subset \mathcal{B} \\ |\mathcal{B}^*| = a}} u_{T_{\mathcal{B}^*}\mathfrak{t}} \mod (\mathcal{TT}^{(k+1)}(N^\mu)).$$

Proof. This is proved by induction on a. Let $a = 1$. By the previous lemma the equation

$$\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \\ |\mathcal{A}^*|=1}} u_{T_{\mathcal{A}^*}\mathfrak{t}} + \sum_{\substack{\mathcal{B}^* \subset \mathcal{B} \\ |\mathcal{B}^*|=1}} u_{T_{\mathcal{B}^*}\mathfrak{t}} = \sum_{\substack{\mathcal{A}^* \subset \mathcal{I} = \mathcal{A} \cup \mathcal{B} \\ |\mathcal{A}^*|=1}} u_{T_{\mathcal{A}^*}\mathfrak{t}} \in \mathcal{TT}^{(k+1)}(N^\mu)$$

holds. Proving the claim for $a = 1$.

Let now $a > 1$ assume $\lambda = (a, a)$.

$$\begin{aligned}
\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \\ |\mathcal{A}^*|=a}} u_{T_{\mathcal{A}^*}\mathfrak{t}} &= \sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \\ |\mathcal{A}^*|=a}} u_{T_{\mathcal{A}^*}} y_\lambda b_\mathfrak{t} \\
&= \sum_{\{i_1 < i_2 < \cdots < i_a\} \subset \mathcal{A}} (u_{\bar{i}_1} \otimes u_{\bar{i}_2} \otimes \cdots \otimes u_{\bar{i}_a}) y_a \otimes (u_{i_1} \otimes u_{i_2} \otimes u_{i_3} \otimes \cdots \otimes u_{i_a}) y_a b_\mathfrak{t}
\end{aligned}$$

now expand the anti-symmetrizer on the right but avoid signs by permuting entrys on the left

$$= \sum_{\substack{\{i_1 < i_2 < \cdots < i_a\} \subset \mathcal{A} \\ \omega \in \mathfrak{S}_a}} (u_{\bar{i}_{(1)\omega}} \otimes u_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes u_{\bar{i}_{(a)\omega}}) y_a \otimes u_{i_{(1)\omega}} \otimes u_{i_{(2)\omega}} \otimes \cdots \otimes u_{i_{(a)\omega}} b_\mathfrak{t}$$

adding some zeros

$$= \sum_{\substack{\{i_2 < \cdots < i_a\} \subset \mathcal{A} \\ j \in \mathcal{A} \\ \omega \in \mathfrak{S}_{a-1}}} (u_{\bar{j}} \otimes u_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes u_{\bar{i}_{(a)\omega}}) y_a \otimes u_j \otimes u_{i_{(2)\omega}} \otimes \cdots \otimes u_{i_{(a)\omega}} b_\mathfrak{t}$$

with Lemma 2.60

$$\equiv (-1) \sum_{\substack{\{i_2 < \cdots < i_a\} \subset \mathcal{A} \\ j \in \mathcal{B} \\ \omega \in \mathfrak{S}_{a-1}}} (u_{\bar{j}} \otimes u_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes u_{\bar{i}_{(a)\omega}}) y_a \otimes u_j \otimes u_{i_{(2)\omega}} \otimes \cdots \otimes u_{i_{(a)\omega}} b_\mathfrak{t}$$

with induction hypothesis

$$\equiv (-1)^a \sum_{\substack{\{i_2 < \cdots < i_a\} \subset \mathcal{B} \\ j \in \mathcal{B} \\ \omega \in \mathfrak{S}_{a-1}}} (u_{\bar{j}} \otimes u_{\bar{i}_{(2)\omega}} \otimes \cdots \otimes u_{\bar{i}_{(a)\omega}}) y_a \otimes u_j \otimes u_{i_{(2)\omega}} \otimes \cdots \otimes u_{i_{(a)\omega}} b_\mathfrak{t}$$

deleting the zeros gives

$$= \sum_{\substack{\mathcal{B}^* \subset \mathcal{B} \\ |\mathcal{B}^*|=a}} u_{T_{\mathcal{B}^*}\mathfrak{t}} \qquad \mod (\mathcal{TT}^{(k+1)}(N^\mu))$$

The calculation for λ of arbitrary shape stays the same. □

2.62 Lemma (Lemma 3.6 of [KW93]). *Let λ be a partition of $r - 2k$ and $\mathfrak{t}$ a m-permissible path to (λ, k). Let T be a semistandard tableau which is not orthogonal semistandard in that $\alpha_j + \beta_j > 2j$ for some $1 \leq j \leq n$. Then modulo $\mathcal{TT}^{(k+1)}(N^\mu)$ the vector $v_{T\mathfrak{t}}$ can be written as a linear combination of vectors of higher weight.*

Proof. Define the following sets: $\mathcal{J} := \{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{j}, j\}$ and $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$. Divide $\mathcal{J}$ further into the following subsets. Let $\mathcal{A}$ be the set of all indices $i \in \mathcal{J}$ for which i appears in the first column of T and $\bar{i}$ appears in the second column of T. Let $\mathcal{B}$ be the set of all indices $i \in \mathcal{J}$ for which either i appears in the first column of T or $\bar{i}$ appears in the second column of T but not both. At last, let $\mathcal{C}$ be the set of indices $i \in \mathcal{J}$ for which neither i appears in the first column of T nor $\bar{i}$ appears in the second column of T. Obviously, $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \mathcal{J}$ is a disjoint union and therefore, $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = 2j$. Since $\alpha_j + \beta_j > 2j$, it follows that $2|\mathcal{A}| + |\mathcal{B}| > 2j$. This implies $|\mathcal{A}| > |\mathcal{C}|$.

Therefore, with the notation of Lemma 2.61

$$\sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}|}} u_{T_{\mathcal{A}^*}\mathfrak{t}} \equiv (-1)^{|\mathcal{A}|} \sum_{\substack{\mathcal{C}^* \subset \mathcal{C} \\ |\mathcal{C}^*| = |\mathcal{A}|}} u_{T_{\mathcal{C}^*}\mathfrak{t}} = 0 \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)).$$

Thus

$$u_{T\mathfrak{t}} = u_{T_{\mathcal{A}}\mathfrak{t}} \equiv - \sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} u_{T_{\mathcal{A}^*}\mathfrak{t}} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)).$$

By the definition of $\mathcal{B}$ there are repetitions in the first or second column of $T_{\mathcal{A}^*}$ if $\mathcal{A}^* \cap \mathcal{B} \neq \emptyset$. Hence

$$u_{T\mathfrak{t}} \equiv - \sum_{\substack{\mathcal{A}^* \subset \mathcal{A} \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} u_{T_{\mathcal{A}^*}\mathfrak{t}} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)).$$

Each term on the right-hand side of the equation is of higher weight than T (though they have the same orthogonal weight!). □

2.63 Lemma (Lemma 3.7 of [KW93])**.** *Let T be a semistandard tableau which is not orthogonal semistandard in that $\alpha_j + \beta_j = 2j$ for some j with $\alpha_j > \beta_j$ and an unprotected j occurs in the first column in that $T(\alpha_j, 1) = j$, $T(\beta_j, 2) = \bar{j}$ and $T(\alpha_j - 1, 1) \neq \bar{j}$. Then modulo $\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)$ the vector $v_{T\mathfrak{t}}$ can be written as linear combination of vectors corresponding to dominating tableaux and vectors of higher weight.*

Proof. Define $\mathcal{J}$, $\mathcal{K}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as before. By assumption j is an element of $\mathcal{A}$ and $\bar{j}$ is an element of $\mathcal{C}$. Since $\alpha_j + \beta_j = 2j$, it follows that $2|\mathcal{A}| + |\mathcal{B}| = 2j$, which implies $\mathcal{A} = \mathcal{C}$. Define now $\mathcal{A}' = \mathcal{A} \cup \{\bar{j}\}$ and $\mathcal{C}' = \mathcal{C} \setminus \{\bar{j}\}$. Since $|\mathcal{C}'| < |\mathcal{A}|$, as before

$$u_{T\mathfrak{t}} = u_{T_{\mathcal{A}}\mathfrak{t}} \equiv - \sum_{\substack{\mathcal{A}^* \subset \mathcal{A}' \cup \mathcal{B} \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} u_{T_{\mathcal{A}^*}\mathfrak{t}} = - \sum_{\substack{\mathcal{A}^* \subset \mathcal{A}' \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} u_{T_{\mathcal{A}^*}\mathfrak{t}} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^\mu)).$$

All but one term on the right-hand side of the equation are of higher weight than T. The only exception is the term $u_{T_{\mathcal{A}^*}\mathfrak{t}}$ with $\mathcal{A}^* = \mathcal{A} \cup \{\bar{j}\} \setminus \{j\}$. But $T_{\mathcal{A}^*}$ dominates T. $\square$

2.64 Lemma (Lemma 3.8 of [KW93])**.** *Let T be a semistandard tableau which is not orthogonal semistandard in that $\alpha_j + \beta_j = 2j$ for some j with $\alpha_j = \beta_j = j$ and an unprotected j occurs in the bth column ($b \geq 2$) in that $T(j,1) = \bar{j}$, $T(j,b) = j$ and $T(j-1,b) \neq \bar{j}$. Then modulo $\mathcal{TT}^{(k+1)}(N^\mu)$ the vector $v_{T\mathfrak{t}}$ can be written as linear combination of vectors corresponding to dominating tableaux and vectors of higher weight.*

Proof. Define $\mathcal{J}$, $\mathcal{K}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as before but use the bth column instead of the second. The following equation holds as above

$$u_{T\mathfrak{t}} = v_{T_{\mathcal{A}}\mathfrak{t}} \equiv - \sum_{\substack{\mathcal{A}^* \subset \mathcal{A}' \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A}}} u_{T_{\mathcal{A}^*}\mathfrak{t}} \qquad \mathrm{mod}\ (\mathcal{TT}^{(k+1)}(N^\mu)).$$

Again, all but one term on the right-hand side of the equation are of higher weight than T. The tableau $T_{\mathcal{A}^*}$ with $\mathcal{A}^* = \mathcal{A} \cup \{\bar{j}\} \setminus \{j\}$ has an inversion, since $\bar{j}$ occurs to the right of j in row j. Using Garnir relations one can rewrite $v_{T_{\mathcal{A}^*}\mathfrak{t}}$ as follows

$$u_{T_{\mathcal{A}^*}\mathfrak{t}} = u_{T_{\mathcal{A}}\mathfrak{t}} + h,$$

where h is a linear combination of $u_{T'\mathfrak{t}'}$ where T' dominates T and $\mathfrak{t}'$ dominates or is equal to $\mathfrak{t}$. Combining these two equations implies

$$u_{T\mathfrak{t}} = u_{T_{\mathcal{A}}\mathfrak{t}} \equiv \frac{1}{2} \left(- \sum_{\substack{\mathcal{A}^* \subset \mathcal{A}' \cup \mathcal{K} \\ |\mathcal{A}^*| = |\mathcal{A}| \\ \mathcal{A}^* \neq \mathcal{A} \\ \mathcal{A}^* \neq \mathcal{A} \cup \{\bar{j}\} \setminus \{j\}}} u_{T_{\mathcal{A}^*}\mathfrak{t}} - h \right) \qquad \mathrm{mod}\ (\mathcal{TT}^{(k+1)}(N^\mu)).$$

And the terms on the right hand side of the equation are either of higher weight than T or dominate T. $\square$

Now, the following lemma and thus Theorem 2.58 is immediate.

2.65 Lemma. *The factor $\mathcal{HT}^{(k)}(N^\mu)$ is generated as a vector space by the set*

$$\left\{ u_{T\mathfrak{t}} + \mathcal{TT}^{(k+1)}(N^\mu) \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\lambda)^{\circ}_{\mu}, \\ \mathfrak{t} \textit{ a m-permissible path of shape } (\lambda, k), (\lambda, k) \in \Lambda(r) \end{array} \right\}.$$

For every $(\lambda, k) \in \Lambda(r)$ define $U(\trianglelefteq (\lambda, k))$ and $U(\lhd (\lambda, k))$ as subspaces of $U^{\otimes r}$ generated by

$$\left\{ u_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\nu)^{\circ}_{\mathcal{I}}, \mathfrak{t} \text{ a } m\text{-permissible path of shape } (\nu, l), \\ (\nu, l) \in \Lambda_y(r), (\nu, l) \trianglelefteq (\lambda, k) \end{array} \right\}$$

and

$$\left\{ u_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} T \in \mathrm{Tab}(\nu)^{\circ}_{\mathcal{I}}, \mathfrak{t} \text{ a } m\text{-permissible path of shape } (\nu, l), \\ (\nu, l) \in \Lambda_y(r), (\nu, l) \lhd (\lambda, k) \end{array} \right\}$$

The results about the ordinary tensor space together with the proofs above imply the following:

2.66 Theorem. *Let (λ, k) be a pair in $\Lambda(r)$. Then*

$$U(\trianglelefteq (\lambda, k))/U(\lhd (\lambda, k)) \cong \Delta^{\circ}(\lambda) \otimes C^{(\lambda,k)},$$

as $(\mathbb{F}\,\mathrm{GO}(U), B_r(m))$-bimodules, where $\Delta^{\circ}(\lambda)$ is a $\mathbb{F}\,\mathrm{GO}(U)$-module and a factor of the dual Weyl module $\Delta(\lambda)$ and $C^{(\lambda,k)}$ is the cell module of $\mathrm{End}_{\mathbb{F}\,\mathrm{GO}(U)}(U^{\otimes r})$ associated to λ and (λ, k) respectively.

3. Walled Brauer algebras

This chapter deals with the walled Brauer algebra. The chapter is organized in the same way as the previous chapter. The main results are already covered in [SW14] but they are presented here in a quite different way. This is done to point out that the walled Brauer algebra and the mixed tensor space can be dealt with in the same way as the Brauer algebra and the symplectic/orthogonal tensor space could.

3.1. Definitions

Throughout this chapter, r and s will denote natural numbers, R a commutative unital ring, and x an element of R.

Diagrammatic and algebraic description. A Brauer diagram with $2(r+s)$ vertices is called a (r,s)-*walled Brauer diagram* if the following holds: No vertex in the set of the r leftmost vertices in the top row and the s rightmost vertices in the bottom row is connected to another vertex in the same set. This implies that the same holds for the set of the s rightmost vertices in the top row and the r leftmost vertices in the bottom row.

Another way to describe (r,s)-walled Brauer diagrams is to imagine a vertical wall between the rth and $(r+1)$th vertex in the top and bottom row. This wall has to be crossed by horizontal edges but it must not be crossed by vertical ones. This justifies the term *walled* Brauer diagram.

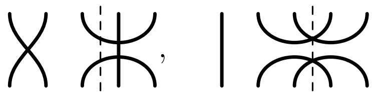


Figure 3.1.: Two $(3,2)$-walled Brauer diagrams (with dashed wall).

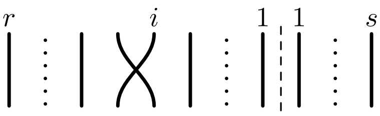
Let $\mathfrak{B}_{r,s}$ be the set of all (r,s)-walled Brauer diagrams with the wall between the rth and $(r+1)$th vertices. This set is a submonoid of $\mathfrak{B}_{r+s}$. Following the definition of [BCH$^+$94] the subalgebra of $B_{r+s}(x)$ generated by the walled Brauer diagrams is called the *walled Brauer algebra* $B_{r,s}(x)$. In fact, the walled Brauer diagrams form an R-basis of $B_{r,s}(x)$.

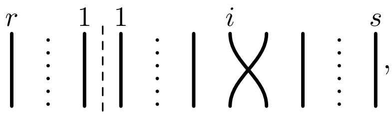
As for the Brauer algebra there is also an algebraic description for the walled Brauer algebra. Let $S' = \{s'_1, s'_2, \dots, s'_{r-1}\}$, $S = \{s_1, s_2, \dots, s_{s-1}\}$ and $E = \{e\}$. Then $B_{r,s}(x)$ is generated by the set $S \cup S' \cup E$ together with the relations

- $s_i'^2 = 1$
- $s'_i e = e s'_i$ for $i > 1$
- $s'_i s'_j = s'_j s'_i$ for $|i-j| > 1$
- $s'_i s'_{i+1} s'_i = s'_{i+1} s'_i s'_{i+1}$
- $s_i^2 = 1$
- $s_i e = e s_i$ for $i > 1$
- $s_i s_j = s_j s_i$ for $|i-j| > 1$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- $s_i s'_j = s'_j s_i$
- $e^2 = xe$
- $e s_1 e = e$
- $e s'_1 e = e$
- $e s_1 s'_1 e = s_1 s'_1 e s_1 s'_1 e$
- $e s_1 s'_1 e = e s_1 s'_1 e s_1 s'_1$

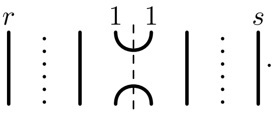
for all values of i and j where the above equations make sense.

The following map determines the isomorphism between the diagrammatic and the algebraic presentation of the walled Brauer algebra $B_{r,s}(x)$

$s'_i \mapsto$ 

, $s_i \mapsto$ 

,

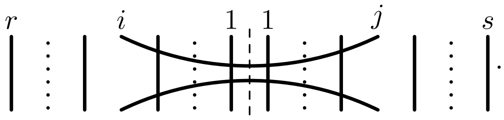
$e \mapsto$ 

.

Let $1 \le i \le r$ and $1 \le j \le r$. Define

$$e_{ij} = s'_{i-1} s'_{i-2} \dots s'_1 s_{j-1} s_{j-2} \dots s_1 e s'_1 s'_2 \dots s'_{i-1} s_1 s_2 \dots s_{j-1}.$$

So,

$e_{ij} =$ 

.

Ideals and subalgebras. The filtration (2.1) of the Brauer algebra leads to a filtration of the walled Brauer algebra. Define $B^{(i)}_{r,s}(x) := B_{r,s}(x) \cap B^{(i)}_{r+s}(x)$. Then

$$(0) \subset B^{(\min\{r,s\})}_{r,s}(x) \subset \dots \subset B^{(2)}_{r,s}(x) \subset B^{(1)}_{r,s}(x) \subset B_{r,s}(x) \tag{3.1}$$

is this filtration.

The group algebra $R(\mathfrak{S}_r \times \mathfrak{S}_s)$ is both a subalgebra and an image of the walled Brauer algebra $B_{r,s}(x)$, namely $B_{r,s}(x)/B^{(1)}_{r,s}(x)$.

Again, if not stated otherwise the walled Brauer algebra $B_{r,s-1}(x)$ (and $B_{r-1,s}(x)$ resp.) is embedded as the subalgebra of $B_{r,s}(x)$ in which every Brauer diagram has an edge connecting dot s in the right side of the top row with dot s in the right side of the bottom row (dot r in the left side of the top row with dot r in the left side of the bottom rom). In this way a lot of different towers of walled Brauer algebras can be constructed. The following tower of algebras is the one, in which the behaviour of the cell modules shall be investigated.

$$B_{1,0}(x) \subset B_{2,0}(x) \subset \cdots \subset B_{r-1,0}(x) \subset B_{r,0}(x) \subset B_{r,1}(x) \subset \cdots \subset B_{r,s}(x) \subset \ldots \tag{3.2}$$

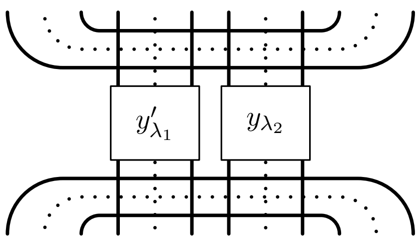
3.2. Cellular structure

First cellular bases. The cellular structure of the walled Brauer algebra can analogously to the Brauer algebra case be considered as a refinement of filtration (3.1).

For every ordered pair of non-negative integers (r, s) define $\Lambda(r, s)$ to be the set of all triples (λ, μ, k) with $\lambda \in \mathrm{Par}(r-k)$, $\mu \in \mathrm{Par}(s-k)$ and $0 \leq k \leq \min\{r, s\}$. The *dominance order* on partitions can be extended to the set $\Lambda(r, s)$ in the following way. Let (λ, μ, k), (ν, ξ, l) be two triples in $\Lambda(r, s)$. Write $(\lambda, \mu, k) \trianglerighteq (\nu, \xi, l)$ if $k < l$ or $k = l$ and $\lambda \trianglerighteq \nu$ and $\mu \trianglerighteq \xi$.

For every triple (λ, μ, k) in $\Lambda(r, s)$ define the element $y_{(\lambda,\mu,k)}$ as follows:

$$y_{(\lambda_1,\lambda_2,k)} := y'_{\lambda_1} y_{\lambda_2} e^{(k)}_{r,s} =$$



with $e^{(k)}_{r,s} := e_{r-k+1,s-k+1} e_{r-k+2,s-k+2} \cdots e_{r,s}$.

To every triple (λ, μ, k) in $\Lambda(r, s)$ attach the quadruple $(\lambda, 1^{(k)}, \mu, 1^{(k)})$ of partitions. The notation of tableaux is extended to these quadruples in the following way. A (λ, μ, k)-*tableau* $(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ is a pair of bijective maps; the first of them maps from the pair $(\lambda, 1^{(k)})$ to the set $\{1, 2, \ldots, r\}$, the second of them maps from the pair $(\mu, 1^{(k)})$ to the set $\{1, 2, \ldots, s\}$. Again the numbers are filled inside the boxes of the Young diagrams. A (λ, μ, k)-tableau $(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ is called *standard*, if $\mathfrak{s}$, $\mathfrak{t}$ and $\mathfrak{u}$ are standard. The set of standard (λ, μ, k)-tableaux is denoted by $\mathrm{Std}(\lambda, \mu, k)$.

The group $\mathfrak{S}_r \times \mathfrak{S}_s$ acts on the set of (λ, μ, k)-tableaux from the right. For a (λ, μ, k)-tableaux $(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ the subgroup $\mathfrak{S}_r \times \{\mathrm{id}\}$ acts on the pair $(\mathfrak{s}, \mathfrak{t})$ by permutation of entries, the subgroup $\{\mathrm{id}\} \times \mathfrak{S}_s$ acts on the pair $(\mathfrak{u}, \mathfrak{v})$ by permutation of entries. Further, there is again an analogue to the final element in $\mathrm{Std}(\lambda)$. In abuse of notation, it is the quadruple $\left(\mathfrak{t}_\lambda, \mathfrak{t}^{\left(1^{(k)}\right)}, \mathfrak{t}_\mu, \mathfrak{t}^{\left(1^{(k)}\right)}\right)$. In the two tableaux $\mathfrak{t}^{\left(1^{(k)}\right)}$ the numbers $r-k+1, r-k+2, \ldots, r$ and $s-k+1, s-k+2, \ldots, s$ respectively are inserted. This is once again used to define the unique element $b(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in \mathfrak{S}_r \times \mathfrak{S}_s$ such that $\left(\mathfrak{t}_\lambda, \mathfrak{t}^{\left(1^{(k)}\right)}, \mathfrak{t}_\mu, \mathfrak{t}^{\left(1^{(k)}\right)}\right) b(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = (\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v})$.

3.3 Example. Let $(\lambda, \mu, k) = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}, 3 \right)$ be a triple in $\Lambda(9, 7)$. The analogue to the final tableau is

$$\left(\mathfrak{t}_\lambda, \mathfrak{t}^{\left(1^{(3)}\right)}, \mathfrak{t}_\mu, \mathfrak{t}^{\left(1^{(3)}\right)}\right) = \left(\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 7 \\ \hline 8 \\ \hline 9 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 5 \\ \hline 6 \\ \hline 7 \\ \hline \end{array} \right)$$

Further,

$$(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = \left(\begin{array}{|c|c|c|}\hline 3 & 6 & 8 \\ \hline 4 & 9 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 5 \\ \hline 3 \\ \hline 7 \\ \hline \end{array} \right)$$

is another element in $\mathrm{Std}(\lambda, \mu, k)$. So,

$$b(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = (\ (137)(2468)(59)\ ,\ (36)\ ) \in \mathfrak{S}_9 \times \mathfrak{S}_7$$

As in the Brauer algebra case, one can use any cellular basis of the group algebra of the symmetric group to obtain a cellular basis of the walled Brauer algebra. The following theorem [Eny03, Theorem 6.13] is an application of this result.

3.4 Theorem. *The set*

$$\mathfrak{M}^y_{r,s} := \left\{ b(\mathfrak{s}', \mathfrak{t}', \mathfrak{u}', \mathfrak{v}')^* y_{(\lambda,\mu,k)} b(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \;\middle|\; \begin{array}{l} (\lambda, \mu, k) \in \Lambda(r, s), \\ (\mathfrak{s}', \mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in \mathrm{Std}(\lambda, \mu, k) \end{array} \right\}$$

is an R-basis for $B_{r,s}(x)$. Furthermore, $(\mathfrak{M}^y, \Lambda(r, s), \trianglelefteq)$ is a cellular basis and therefore $B_{r,s}(x)$ is a cellular algebra.

The cell modules which arise from this cellular basis are denoted by $C^{(\lambda,\mu,k)}$. The set $\{y_{(\lambda,\mu,k)} b(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) + C^{\triangleleft(\lambda,\mu,k)} \mid (\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in \mathrm{Std}(\lambda, \mu, k)\}$ is an R-basis for the module $C^{(\lambda,\mu,k)}$ with $C^{\triangleleft(\lambda,\mu,k)}$ the (two-sided) ideal in $B_{r,s}(x)$ generated by all $y_{(\rho,\xi,l)} \in \Lambda(r, s)$ with $(\rho, \xi, l) \triangleleft (\lambda, \mu, k)$.

Restriction of cell modules. The following theorem describes the behaviour of the cell modules under restriction in the tower (3.2). It is formulated as an analogue of Theorem 1.27.

3.5 Theorem. *Let $(\lambda_1, \lambda_2, k)$ be an element of $\Lambda(r, s)$. With respect to the order on $\Lambda(r, s-1)$ the set*

$$\{(\lambda_1, \mu, k) \mid \mu \in \mathrm{Rem}(\lambda_2)\} \cup \{(\nu, \lambda_2, k-1) \mid \nu \in \mathrm{Add}(\lambda_1)\} \subseteq \Lambda(r, s-1)$$

is ordered linearly, i.e. $(\lambda_1, \mu^{(1)}, k) \rhd (\lambda_1, \mu^{(2)}, k) \rhd \cdots \rhd (\lambda_1, \mu^{(t)}, k) \rhd (\nu^{(1)}, \lambda_2, k-1) \rhd (\nu^{(2)}, \lambda_2, k-1) \rhd \cdots \rhd (\nu^{(p)}, \lambda_2, k-1)$.

Define $B_{r,s-1}(x)$-submodules of $\mathrm{Res}^{B_{r,s}(x)}_{B_{r,s-1}(x)} C^{(\lambda,\mu,k)}$ *as*

$$V_j := \sum_{s \leq j} \left(y_{(\lambda_1,\lambda_2,k)} \bar{b}'_{\lambda_2 \to \mu^{(s)}} y_{\left(\lambda_1,\mu^{(s)},k\right)} + C^{\rhd(\lambda_1,\lambda_2,k)} \right) B_{r,s-1}(x)$$

and

$$V_{t+j} := \sum_{s \leq j} \left(y_{(\lambda_1,\lambda_2,k)} \bar{b}_{\lambda_1 \to \nu^{(s)}} y_{\left(\nu^{(s)},\lambda_2,k-1\right)} + C^{\rhd(\lambda_1,\lambda_2,k)} \right) B_{r,s-1}(x) + V_t$$

This leads to the filtration

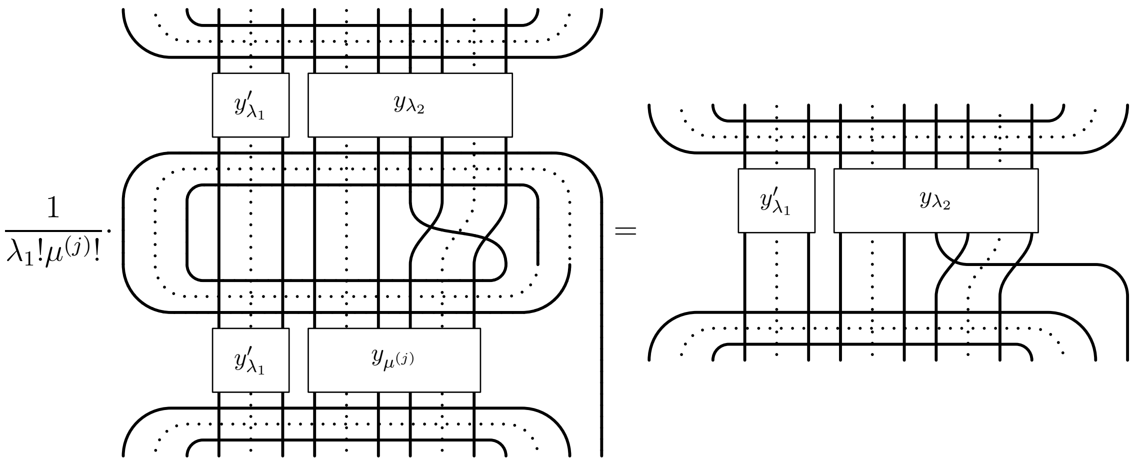
$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{t+p} = \mathrm{Res}^{B_{r,s}(x)}_{B_{r,s-1}(x)} C^{(\lambda_1,\lambda_2,k)}$$

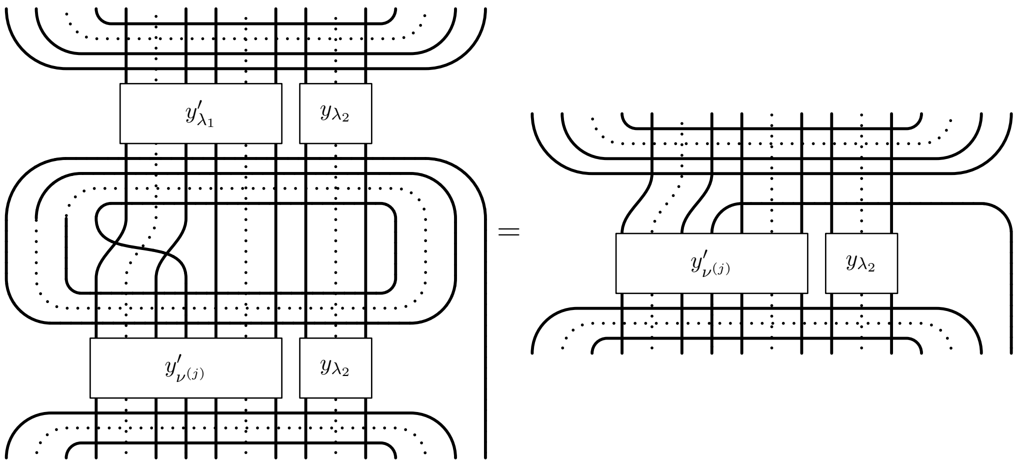
such that $V_j/V_{j-1} \cong C^{\left(\lambda_1,\mu^{(j)},k\right)}$ *and* $V_{t+j}/V_{t+j-1} \cong C^{\left(\nu^{(j)},\lambda_2,k-1\right)}$ *with the isomorphisms given by*

$$C^{\left(\lambda_1,\mu^{(j)},k\right)} \to V_j/V_{j-1} : y_{\left(\lambda_1,\mu^{(j)},k\right)} + C^{\rhd\left(\lambda_1,\mu^{(j)},k\right)} \mapsto y_{(\lambda_1,\lambda_2,k)} \bar{b}'_{\lambda \to \mu^{(j)}} y_{\left(\mu^{(j)},\lambda_2,k\right)} + V_{j-1},$$

$$C^{\left(\nu^{(j)},\lambda_2,k-1\right)} \to V_{t+j}/V_{t+j-1} : y_{\left(\nu^{(j)},\lambda_2,k\right)} + C^{\rhd\left(\nu^{(j)},\lambda_2,k-1\right)} \mapsto y_{(\lambda_1,\lambda_2,k)} \bar{b}_{\lambda \to \nu^{(j)}} y_{\left(\nu^{(j)},\lambda_2,k-1\right)} + V_{t+j-1}.$$

Proof. First take a closer look on the generators of the submodules.



$\frac{1}{\nu^{(j)}!\lambda_2!}\cdot$ 

The isomorphism $V_j/V_{j-1} \cong C^{\left(\lambda_1,\mu^{(j)},k\right)}$ can almost be deduced from Theorem 1.27 and the isomorphism $V_{t+j}/V_{t+j-1} \cong C^{\left(\nu^{(j)},\lambda_2,k-1\right)}$ can almost be deduced from Theorem 1.29. It remains to consider the action of e_{ij} on the various generators. But with the diagrams at hand, this is done quickly. There are three different cases that may appear:

1. The horizontal arc 'hits' the same horizontal arc on the left hand side and on the right hand side: In this case the product is just x times the generator.
2. The horizontal arc 'hits' the box on the left hand side and the box on the right hand side: In this case the product will be zero modulo $B_{r,s}^{(k+1)}(x)$, since this results in more horizontal arcs in the bottom row (or the rightmost arc is no longer a horizontal one, which is zero modulo V_t).
3. In all other cases the element e_{ij} acts as a permutation.

□

Other cellular bases. The theorem above is again the main ingredient to construct an alternative cellular basis. Analogously to the Brauer algebra case, the following definitions are needed:

First define the *Bratelli diagram associated to the walled Brauer algebra*. The cell modules of $B_{r,s}(x)$ are indexed by the elements of $\Lambda(r,s)$. Since the special tower (3.2) is the foundation for this section and this tower depends on the parameter r (unlike in the symmetric group or Brauer algebra case, where the towers did not have such a dependency), the Bratelli diagram will also depend on it in the following way: In row $i \leq r$ of the diagram all elements of $\Lambda(i,0)$ are drawn and in row $i > r$ of the diagram all elements of $\lambda(r, i-r)$ are drawn. Two elements (λ_1,λ_2,k) and (μ_1,μ_2,l) in consecutive rows are connected if $C^{(\lambda_1,\lambda_2,k)}$ is a subfactor in the filtration of $C^{(\mu_1,\mu_2,l)}$ as described in Theorem 3.5. Therefore, *a path*

$\mathfrak{t}$ *(in the Bratelli-diagram associated to the walled Brauer algebra)* is a sequence of triples $((\emptyset,\emptyset,0),(\lambda^{(1)},\emptyset,k_1),(\lambda^{(2)},\emptyset,k_2),\ldots,(\lambda^{(r)},\emptyset,k_r)$, $(\lambda^{(r+1)},\mu^{(r+1)},k_{r+1}),\ldots,$ $(\lambda^{(r+s)},\mu^{(r+s)},k_{r+s}))$, such that

- for $1 \leq i \leq r$ the triple $(\lambda^{(i)},\emptyset,0)$ is an element of $\Lambda(i,0)$, such that $\lambda^{(i)} \in \mathrm{Add}(\lambda^{(i-1)})$;
- for $r < i$ the triple $(\lambda^{(i)},\mu^{(i)},k_i)$ is an element of $\Lambda(r,i-r)$, such that either $\lambda^{(i)} \in \mathrm{Rem}(\lambda^{(i-1)})$, $\mu^{(i)} = \mu^{(i-1)}$ and $k_i = k_{i-1}+1$ or $\lambda^{(i)} = \lambda^{(i-1)}$, $\mu^{(i)} \in \mathrm{Add}(\mu^{(i-1)}$ and $k_i = k_{i-1}$.

As before, call $r+s$ the length of such a path and $(\lambda^{(r+s)},\mu^{(r+s)},k_{r+s})$ its shape.

For a path of length $r+s$ in the Bratteli diagram associated to the walled Brauer algebra $\mathfrak{t} = ((\emptyset,\emptyset,0)$, $(\lambda^{(1)},\emptyset,k_1)$, $(\lambda^{(2)},\emptyset,k_2)$, $\ldots$, $(\lambda^{(r)},\emptyset,k_r)$, $(\lambda^{(r+1)},\mu^{(r+1)},k_{r+1})$, $\ldots$, $(\lambda^{(r+s)},\mu^{(r+s)},k_{r+s}))$, let

$$\begin{aligned}
b_{\mathfrak{t}} :=& \bar{b}_{\left(\lambda_1^{(r+s)},\lambda_2^{(r+s)}\right)\to\left(\lambda_1^{(r+s-1)},\lambda_2^{(r+s-1)}\right)} y_{\left(\lambda_1^{(r+s-1)},\lambda_2^{(r+s-1)},k_{r+s-1}\right)}\\
&\cdot \bar{b}_{\left(\lambda_1^{(r+s-1)},\lambda_2^{(r+s-1)}\right)\to\left(\lambda_1^{(r+s-2)},\lambda_2^{(r+s-2)}\right)} y_{\left(\lambda_1^{(r+s-2)},\lambda_2^{(r+s-2)},k_{r+s-2}\right)}\\
&\cdots\\
&\cdot \bar{b}_{\left(\lambda_1^{(2)},\lambda_2^{(2)}\right)\to\left(\lambda_1^{(1)},\lambda_2^{(1)}\right)} y_{\left(\lambda_1^{(1)},\lambda_2^{(1)},k_1\right)},
\end{aligned}$$

where

$$\bar{b}_{\left(\lambda_1^{(i)},\lambda_2^{(i)}\right)\to\left(\lambda_1^{(i-1)},\lambda_2^{(i-1)}\right)} := \begin{cases} \bar{b}_{\lambda_1^{(i)}\to\lambda_1^{(i-1)}}, & \text{if } \lambda_1^{(i)} \neq \lambda_1^{(i-1)},\\ \bar{b}_{\lambda_2^{(i)}\to\lambda_2^{(i-1)}}, & \text{if } \lambda_2^{(i)} \neq \lambda_2^{(i-1)}.\end{cases}$$

3.6 Theorem. *For positive integers r and s the set*

$$\mathfrak{B}^y_{r,s} := \{b^*_{\mathfrak{s}} y_{(\lambda_1,\lambda_2,k)} b_{\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \text{ paths of shape } (\lambda_1,\lambda_2,k), (\lambda_1,\lambda_2,k) \in \Lambda(r,s)\}$$

is an R-basis for the walled Brauer algebra $B_{r,s}(x)$. Moreover, $(\mathfrak{B}^y_{r,s},\Lambda(r,s),\trianglelefteq)$ is a cellular basis.

3.7 Remark. This Murphy type basis for the walled Brauer algebra equals the cellular basis presented by Stoll and the author in [SW14]. The isomorphism between the indexing sets can also be found there.

3.3. Action on mixed tensor space

This section deals with the action of the walled Brauer algebra on mixed tensor space.

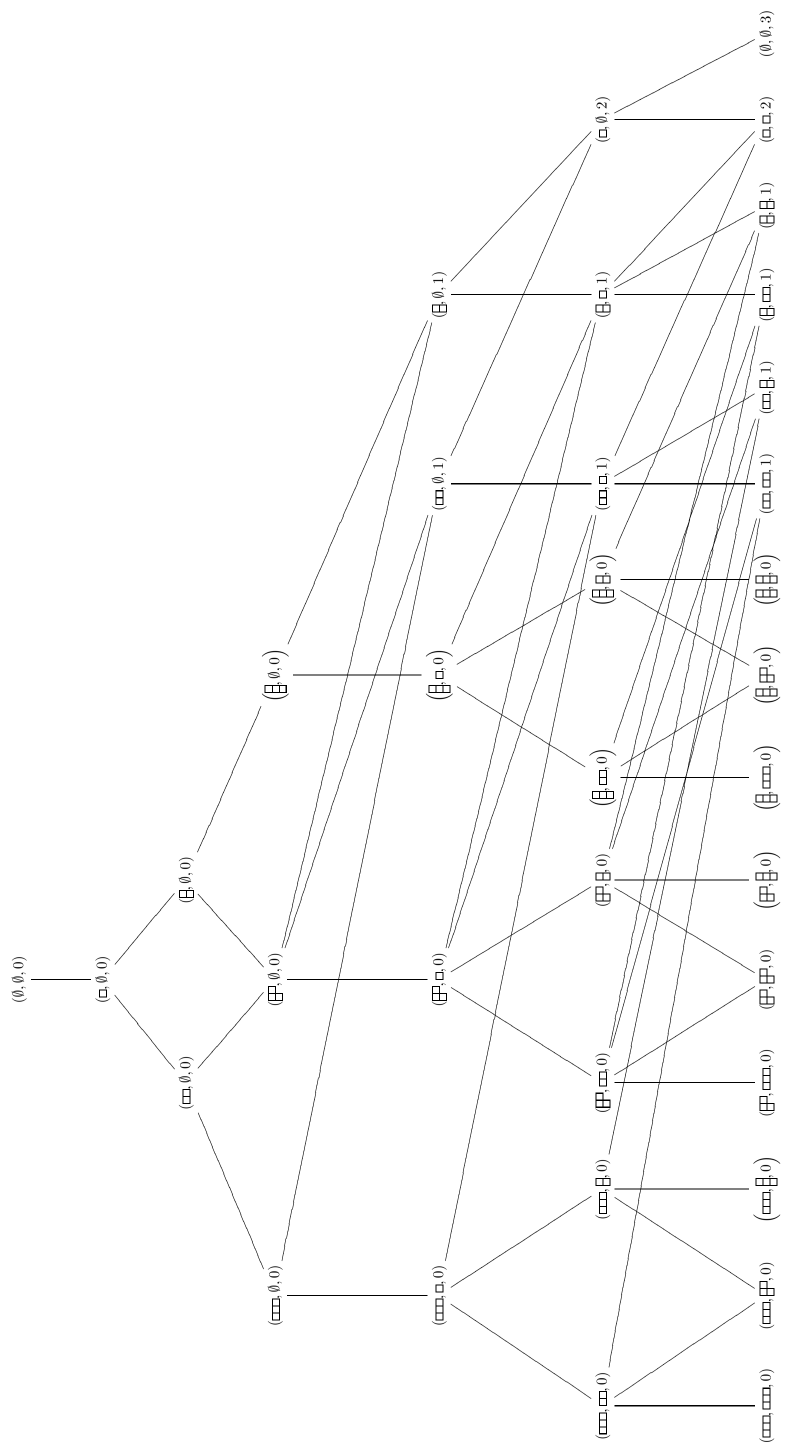


Figure 3.2.: The first 6 levels of the Bratelli-diagram associated to the walled Brauer algebras with $r = 3$ (for a better readability it is put in landscape format).

Rational Schur-Weyl-duality. Let V be an n-dimensional vector space over a field $\mathbb{F}$ with basis $\{v_1, v_2 \dots, v_n\}$, and let V^* be its dual space with dual basis $\{v_1^*, v_2^* \dots, v_n^*\}$, such that $v_i^*(v_j) = \delta_{i,j}$. Define a right-action of the walled Brauer algebra $B_{r,s}(n)$ on the mixed tensor space $(V^*)^{\otimes r} \otimes V^{\otimes s}$ in the following way. The elements s_i' and s_i act by place permutation and

$$\left(v_{k_r'}^* \otimes \dots \otimes v_{k_2'}^* \otimes v_{k_1'}^* \otimes v_{k_1} \otimes v_{k_2} \otimes \dots \otimes v_{k_s}\right) \cdot e$$
$$:= v_{k_1'}^*(v_{k_1}) \sum_{l=1}^{n} v_{k_r'}^* \otimes \dots \otimes v_{k_2'}^* \otimes v_l^* \otimes v_l \otimes v_{k_2} \otimes \dots \otimes v_{k_s}$$

Hence, there is a map $\Psi : B_{r,s}(n)^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}}((V^*)^{\otimes r} \otimes V^{\otimes s})$. The general linear group acts diagonally on the mixed tensor space. Hence, there is a map $\Phi : \mathbb{F}\,\mathrm{GL}(V) \to \mathrm{End}_{\mathbb{F}}((V^*)^{\otimes r} \otimes V^{\otimes s})$. Moreover, the following statements hold:

3.8 Theorem. *1. The left action of* $\mathrm{GL}(V)$ *on the mixed tensor space commutes with the right action of* $B_{r,s}(n)$*. Therefore, the maps* Ψ *and* Φ *are maps in the following sense:*

- $\Psi : B_{r,s}(n)^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s})$
- $\Phi : \mathbb{F}\,\mathrm{GL}(V) \to \mathrm{End}_{B_{r,s}(n)}((V^*)^{\otimes r} \otimes V^{\otimes s})$

2. If $\mathbb{F}$ *is an infinite field then the maps* Ψ *and* Φ *in 1. are surjective.*

3. If $\mathbb{F}$ *is an infinite field and* $n \geq (r+s)$ *then the map* Ψ *is injective and hence* $B_{r,s}(n)^{\mathrm{op}} \cong \mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s})$.

3.9 Remark. Parts of this double-centralizer property were first proven by Benkart, Chakrabarti, Halverson, Leduc, Lee, and Stroomer [BCH+94] in the case $\mathbb{F} = \mathbb{C}$. A complete proof of the result and the proof for the other fields in question can be found in [DDS13, DDS14] by Dipper, Doty and Stoll.

Therefore, for infinite fields $B_{r,s}(n)/\ker(\Psi) \cong \mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s})$.

Due to Dipper, Doty and Stoll [DDS14] is the following result.

3.10 Proposition. *There is an isomorphism between* $\mathbb{F}\mathfrak{S}_{r+s}$ *and* $B_{r,s}(n)$*, that maps the annihilator of the tensor space in* $\mathbb{F}\mathfrak{S}_{r+s}$ *bijectively on the annihilator of the mixed tensor space in* $B_{r,s}(n)$*. In particular, the dimension of the annihilator of the mixed tensor space is as well independent of the field* $\mathbb{F}$*.*

Following Stembridge [Ste87], the dimension of $\mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s})$ can be described analogously to the description of the dimensions of the other endomorphism rings in the previous chapters. The following definition is needed to do so:

A pair of Young diagrams (λ, μ) is called n-permissible if it contains at most n boxes in the first columns of the diagrams, i.e. $\lambda'_1 + \mu'_1 \leq n$. Similarly, a path $\mathfrak{t} = ((\emptyset, \emptyset, 0), (\lambda^{(1)}, \emptyset, k_1), (\lambda^{(2)}, \emptyset, k_2), \ldots, (\lambda^{(r)}, \emptyset, k_r), (\lambda^{(r+1)}, \mu^{(r+1)}, k_{r+1}), \ldots, (\lambda^{(r+s)}, \mu^{(r+s)}, k_{r+s}))$ in the Bratelli diagram is called n-permissible if every pair $(\lambda^{(i)}, \mu^{(i)})$ is n-permissible.

With this definition the dimension of $\mathrm{End}_{\mathbb{C}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s})$ can be expressed as follows:

3.11 Theorem ([Ste87]). *1.*

$$\dim \mathrm{End}_{\mathbb{C}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s}) = \sum_{\substack{(\lambda,\mu,k)\in\Lambda(r,s) \\ \text{with } (\lambda,\mu,k)\ n\text{-permissible}}} n^2_{(\lambda,\mu,k)},$$

where $n_{(\lambda,\mu,k)}$ is the number of n-permissible paths to (λ, μ, k).

2.

$$V^{\otimes r} \otimes V^{*\otimes s} \cong \bigoplus_{\substack{(\lambda,\mu,k)\in\Lambda(r,s) \\ \text{with } (\lambda,\mu,k)\ n\text{-permissible}}} V_{(\lambda,\mu)}^{\oplus n_{\lambda,\mu,k}}$$

as $\mathbb{C}\,\mathrm{GL}(V)$-modules, where $n_{(\lambda,\mu,k)}$ is the number of n-permissible paths to (λ, μ, k) and $V_{(\lambda,\mu)}$ is the rational simple $\mathbb{C}\,\mathrm{GL}(V)$-module associated to (λ, μ).

3.12 Remark. The walled Brauer algebra has a one dimensional two sided ideal corresponding to the alternating representation. Analogously to the other cases, the generators of this ideal are called anti-symmetrizer. Define the sign of a diagram as the parity of the number of crossings plus the number of horizontal arcs in one row. This definition continues the definition of the sign of a permutation. The anti-symmetrizer of the walled Brauer algebra $B_{n+1-s,s}(n)$ is the alternating sum over all elements in $\mathfrak{B}_{n+1-s,s}$ (cf. [Wer14]).

The following proposition is a direct consequence of the above mentioned isomorphism between $\mathbb{F}\mathfrak{S}_{r+s}$ and $B_{r,s}(n)$.

3.13 Proposition ([DDS14]). *Let V be the vector space described above and let r be a positive integer with $r > n$. The anti-symmetrizer of the walled Brauer algebra $B_{n+1-s,s}(n)$ (embedded in $B_{r,s}(n)$) is an element of the annihilator of the mixed tensor space.*

Basis for the endomorphism ring. This proposition motivates the following definitions. For a pair of partitions (λ, μ) of $r + s$ define the direct product

$$\mathfrak{S}'_{\lambda_k} \times \cdots \times \mathfrak{S}'_{\lambda_2} \times \mathfrak{B}_{\lambda_1,\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\lambda_k}$$

as submonoid of $\mathfrak{B}_{r,s}$ denoted by $\mathfrak{B}_{\lambda,\mu}$. Further define the elements

$$y^{\mathrm{m}}_{\lambda,\mu} := \sum_{b\in\mathfrak{B}_{\lambda',\mu'}} \operatorname{sign}(b)\cdot b \quad \text{ and } \quad y^{\mathrm{m}}_{(\lambda,\mu,k)} = y^{\mathrm{m}}_{\lambda,\mu} e^{(k)}_{r,s}$$

as analogues to the elements $x^{\mathrm{sp}}_{\lambda}$, $x^{\mathrm{sp}}_{(\lambda,k)}$, y^{o}_{λ} and $y^{\mathrm{o}}_{(\lambda,k)}$ before. Thus, whenever $\lambda'_1 + \mu'_1 = n+1$ the element $y^{\mathrm{m}}_{\lambda,\mu}$ has a factor equal to the anti-symmetrizer of the walled Brauer algebra $B_{\lambda'_1,\mu'_1}(n)$. Therefore, the element $y^{\mathrm{m}}_{\lambda,\mu}$ is an element of the annihilator of mixed tensor space in this setting.

For $\mathfrak{t} = ((\emptyset,\emptyset,0), (\lambda^{(1)},\emptyset,k_1), (\lambda^{(2)},\emptyset,k_2), \ldots, (\lambda^{(r)},\emptyset,k_r), (\lambda^{(r+1)},\mu^{(r+1)},k_{r+1}), \ldots, (\lambda^{(r+s)},\mu^{(r+s)},k_{r+s}))$ a path of length $r+s$ in the Bratteli diagram, let

$$\begin{aligned} b^{\mathrm{m}}_{\mathfrak{t}} :=& \bar{b}_{\left(\lambda_1^{(r+s)},\lambda_2^{(r+s)}\right)\to\left(\lambda_1^{(r+s-1)},\lambda_2^{(r+s-1)}\right)} y^{\mathrm{m}}_{\left(\lambda_1^{(r+s-1)},\lambda_2^{(r+s-1)},k_{r+s-1}\right)} \\ &\cdot \bar{b}_{\left(\lambda_1^{(r+s-1)},\lambda_2^{(r+s-1)}\right)\to\left(\lambda_1^{(r+s-2)},\lambda_2^{(r+s-2)}\right)} y^{\mathrm{m}}_{\left(\lambda_1^{(r+s-2)},\lambda_2^{(r+s-2)},k_{r+s-2}\right)} \\ &\cdots \\ &\cdot \bar{b}_{\left(\lambda_1^{(2)},\lambda_2^{(2)}\right)\to\left(\lambda_1^{(1)},\lambda_2^{(1)}\right)} y^{\mathrm{m}}_{\left(\lambda_1^{(1)},\lambda_2^{(1)},k_1\right)}, \end{aligned}$$

The following theorems follow in the same way as the corresponding theorems in the previous chapter.

3.14 Theorem. *Let r, s be positive integers and let $\mathbb{F}$ be a field with* $\operatorname{char}(\mathbb{F}) = 0$ *or* $\operatorname{char}(\mathbb{F}) > r, s$.

1. *The set*

$$\mathfrak{B}^{\mathrm{m}}_{r,s} := \{b^{\mathrm{m}*}_{\mathfrak{s}} y^{\mathrm{m}}_{(\lambda,\mu,k)} b^{\mathrm{m}}_{\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \text{ paths of shape } (\lambda,\mu,k), (\lambda,\mu,k)\in\Lambda(r,s)\}$$

   is an $\mathbb{F}$-basis for the walled Brauer algebra $B_{r,s}(x)$. Moreover, $(\mathfrak{B}^{\mathrm{m}}_{r,s}, \Lambda(r,s), \trianglelefteq)$ is a cellular basis.

2. *The set*

   $\{b^{\mathrm{m}*}_{\mathfrak{s}} y^{\mathrm{m}}_{(\lambda,\mu,k)} b^{\mathrm{m}}_{\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t}$ *paths of shape* (λ,μ,k), $\mathfrak{s}$ *or* $\mathfrak{t}$ *not n-permissible,* $(\lambda,\mu,k)\in\Lambda_r\}$

   is an $\mathbb{F}$-basis for the annihilator $\operatorname{ann}_{B_{r,s}(n)}((V^*)^{\otimes r}\otimes V^{\otimes s})$ *of mixed tensor space in the walled Brauer algebra.*

3. *The set*

   $\mathfrak{C}^{\mathrm{m}}_{r,s} := \{b^{\mathrm{m}*}_{\mathfrak{s}} y^{\mathrm{m}}_{(\lambda,\mu,k)} b^{\mathrm{m}}_{\mathfrak{t}} + \operatorname{ann} \mid \mathfrak{s},\mathfrak{t}$ *n-permissible paths of shape* $(\lambda,\mu,k)\in\Lambda(r,s)\}$

   is an $\mathbb{F}$-basis for the endomorphism ring $\operatorname{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r}\otimes V^{\otimes s})$. *Moreover,* $(\mathfrak{C}^{\mathrm{m}}_{r,s}, \Lambda(r,s), \trianglelefteq)$ *is a cellular basis and therefore,* $\operatorname{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r}\otimes V^{\otimes s})$ *is a cellular algebra.*

3.15 Theorem. *Let r and s be positive integers, $\mathbb{F}$ a field, V a n-dimensional vector space over $\mathbb{F}$, and V^* its dual space. The set*

$$\mathfrak{C}^{\mathrm{m}}_{r,s} := \{b_{\mathfrak{s}}{}^* y^{\mathrm{m}}_{(\lambda,\mu,k)} b_{\mathfrak{t}} + \mathrm{ann} \mid \mathfrak{s}, \mathfrak{t} \text{ } n\text{-permissible paths of shape } (\lambda,\mu,k), (\lambda,\mu,k) \in \Lambda(r,s)\}$$

is an $\mathbb{F}$-basis for the endomorphism ring $\mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}((V^*)^{\otimes r} \otimes V^{\otimes s})$.
Moreover, $(\mathfrak{C}^{\mathrm{m}}_{r,s}, \Lambda(r,s), \trianglelefteq)$ is a cellular basis.

Filtration of mixed tensor space. The dimension of the $V_{(\lambda,\mu)}$ in Theorem 3.11 can be described in terms of rational semistandard tableaux. A *rational* (λ,μ)-*tableau* is a pair (S,T) of a λ-tableau S with target set $\{1', 2', \ldots, n'\}$ and a μ-tableau T with target set $\{1, 2, \ldots, n\}$. The weight of a rational (λ,μ)-tableau (S,T) is the composition $(\nu_{1'}, \ldots, \nu_{n'}, \nu_1, \ldots, \nu_n)$ such that ν_i counts how often i appears in the pair (S,T), i.e. $\nu_i = |(S,T)^{-1}(i)|$. The rational weight of a rational (λ,μ)-tableau (S,T) is obtained from its weight via the map

$$\pi\colon (\nu_{1'}, \ldots, \nu_{n'}, \nu_1, \ldots, \nu_n) \mapsto (\nu_1 - \nu_{1'}, \ldots, \nu_n - \nu_{n'}).$$

Denote by α_i and β_i ($i = 1, 2, \ldots, n$) the number of entries less than or equal to i' and i in the first column of S and T, respectively. Following Stembridge [Ste87] a rational tableau (S,T) is called *rational semistandard* if both tableaux S and T are semistandard and for each $i = 1, 2, \ldots, n$ the equality $\alpha_i + \beta_i \leq i$ holds. The set of all rational semistandard tableaux (of rational weight ν) is denoted by $\mathrm{Rat}(\lambda,\mu)$ ($\mathrm{Rat}(\lambda,\mu)_\nu$).

3.16 Remark. Note that the additional condition implies that (λ,μ) is n-permissible iff rational semistandard (λ,μ)-tableaux exist.

Further, the additional condition is almost the same as the first additional condition for orthogonal semistandard tableaux. For the orthogonal tableaux the amount of 'small' entries in the *first two columns* is restricted, for the rational tableaux the amount of 'small' entries in the *two first columns* is restricted. This fact will be useful later.

The following is a corollary to Proposition 2.26 and uses the description of the dimensions of the irreducible rational $\mathbb{C}\,\mathrm{GL}(V)$-modules by rational semistandard tableaux. Similar to the other cases it relates the dimensions of the irreducible $\mathbb{C}\,\mathrm{GL}(V)$- and $B_{r,s}(n)$-modules to the dimension of the mixed tensor space.

3.17 Corollary. *The equation*

$$\sum_{(\lambda,\mu,k)\in\Lambda(r,s)} |\,\mathrm{Rat}(\lambda,\mu)| \cdot n_{(\lambda,\mu,k)} = n^{r+s}$$

holds, where $n_{(\lambda,\mu,k)}$ is the number of n-permissible paths to (λ,μ,k).

Recall the bases $\{v_1, v_2 \ldots, v_n\}$ and $\{v_1^*, v_2^* \ldots, v_n^*\}$ of V and V^*. Let $v = v_{i'_r}^* \otimes \cdots \otimes v_{i'_1}^* \otimes v_{i_1} \otimes \cdots \otimes v_{i_s} \in (V^*)^{\otimes r} \otimes V^{\otimes s}$. The weight of v is the composition $(\nu_{1'}, \ldots, \nu_{n'}, \nu_1, \ldots, \nu_n)$ of $r + s$ into 2n parts such that $\nu_{i'}$ counts how often v_i^* appears in v and ν_i counts how often v_i appears in v. The rational weight of a vector in $(V^*)^{\otimes r} \otimes V^{\otimes s}$ is obtained from its weight via the map

$$\pi\colon (\nu_{1'}, \ldots, \nu_{n'}, \nu_1, \ldots, \nu_n) \mapsto (\nu_1 - \nu_{1'}, \ldots, \nu_n - \nu_{n'}).$$

One can described the rational weight space decomposition of the mixed tensor space as follows. Define the rational weight space N^μ to weight μ as the direct sum of all weight spaces M^λ, which have a weight λ that maps on μ under π. By construction, the rational weight spaces are modules for the walled Brauer algebra. Then as in the other cases the mixed tensor space decomposes in the following way:

$$V^{*\otimes r} \otimes V^{\otimes s} = \bigoplus_\mu N^\mu$$

where μ runs through all rational weights of size n.

Let (λ, μ) be a pair of partitions of $r - k$ and $s - k$. Let (S, T) be a rational (λ, μ)-tableau. Let $i'_1, \ldots, i'_{r-k}$ and $i_1, \ldots, i_{s-k}$ denote the sequence of entries of S and T resp. read top to bottom and column by column. Then the element $v_{(S,T)}$ of the mixed tensor space $(V^*)^{\otimes r} \otimes V^{\otimes s}$ is defined as

$$v_{(S,T)} := v_{i'_{r-k}}^* \otimes \cdots \otimes v_{i'_1}^* \otimes v_{i_1} \otimes \cdots \otimes v_{i_{r-k}}.$$

Let further be $\mathfrak{t}$ a path to (λ, μ, k) and define

$$v_{(S,T)\mathfrak{t}} := v_{(S,T)} y_{(\lambda,\mu)} b_{\mathfrak{t}}.$$

The following theorem is an analogue to Theorem 1.39.

3.18 Theorem. *1. The set*

$$\left\{ v_{(S,T)\mathfrak{t}} \;\middle|\; \begin{array}{l} (S,T) \in Rat(\lambda,\mu), \mathfrak{t} \text{ a n-permissible path of shape } (\lambda,\mu,k), \\ (\lambda,\mu,k) \in \Lambda(r,s) \end{array} \right\}$$

is a basis of $(V^*)^{\otimes r} \otimes V^{\otimes s}$.

2. The set

$$\left\{ v_{(S,T)\mathfrak{t}} \;\middle|\; \begin{array}{l} (S,T) \in Rat(\lambda,\mu)_\nu, \mathfrak{t} \text{ a n-permissible path of shape } (\lambda,\mu,k), \\ (\lambda,\mu,k) \in \Lambda(r,s) \end{array} \right\}$$

is a basis of N^ν.

Proof. This can be proven with the steps used for the orthogonal case. The only difference is that the Lemmas 2.63 and 2.64 are not needed to do so. □

For every $(\lambda, \mu, k) \in \Lambda(r, s)$ define $V(\trianglelefteq (\lambda, \mu, k))$ and $V(\triangleleft (\lambda, \mu, k))$ as subspaces of $V^{*\otimes r} \otimes V^{\otimes s}$ generated by

$$\left\{ v_{(S,T)\mathfrak{t}} \;\middle|\; \begin{array}{l} (S,T) \in Rat(\lambda, \mu), \mathfrak{t} \text{ a } n\text{-permissible path of shape } (\lambda, \mu, k), \\ (\lambda, \mu, k) \in \Lambda(r, s), (\nu, \rho, l) \trianglelefteq (\lambda, \mu, k) \end{array} \right\}$$

and

$$\left\{ w_{T\mathfrak{t}} \;\middle|\; \begin{array}{l} (S,T) \in Rat(\lambda, \mu), \mathfrak{t} \text{ a } n\text{-permissible path of shape } (\lambda, \mu, k), \\ (\lambda, \mu, k) \in \Lambda(r, s), (\nu, \rho, l) \triangleleft (\lambda, \mu, k) \end{array} \right\}.$$

The results about the ordinary tensor space together with the statements above imply the following:

3.19 Theorem. *Let (λ, μ, k) be a triple in $\Lambda(r, s)$. Then*

$$V(\trianglelefteq (\lambda, \mu, k))/W(\triangleleft (\lambda, \mu, k)) \cong V_{(\lambda,\mu)} \otimes C^{(\lambda,\mu,k)},$$

as $(\mathbb{F}\,\mathrm{GL}(V), B_{r,s}(n))$-bimodules, where $\Delta(\lambda, \mu)$ is a $\mathbb{F}\,\mathrm{GL}(V)$-module and $C^{(\lambda,\mu,k)}$ is the cell module of $\mathrm{End}_{\mathbb{F}\,\mathrm{GL}(V)}(V^{*\otimes r} \otimes V^{\otimes s})$ associated to (λ, μ) and (λ, μ, k) respectively.

A. Cellular algebras

This chapter deals with an introduction to the theory of cellular algebras as defined by Graham and Lehrer in 1996. The definition of and some results on cellular algebras are stated. These results clarify why it is useful to investigate the cellular structure of an algebra. If not stated otherwise all definitions and results are due to Graham and Lehrer. The notation of their celebrated paper [GL96] is almost completely adopted.

Throughout this chapter, R will denote an unital commutative ring.

A.1 Definition. A *cellular algebra* over R is an associative (unital) algebra A, together with *cell datum* $(\Lambda, M, C, *)$ where

(C1) Λ is a partially ordered set (poset) with order '$\leq$' and for each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of 'tableaux of type λ') such that $C : \amalg_{\lambda\in\Lambda} M(\lambda)\times M(\lambda) \to A$ is an injective map whose image is an R-basis of A.

(C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear anti-involution of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for any $a \in A$ we have

$$aC^\lambda_{S,T} \equiv \sum_{S'\in M(\lambda)} r_a(S', S)C^\lambda_{S',T} \qquad \bmod A(< \lambda)$$

where $r_a(S', S) \in R$ is independent of T and where $A(< \lambda)$ is the R-submodule generated by $\{C^\mu_{S'',T''} \mid \mu < \lambda,\ S'', T'' \in M(\mu)\}$.

A.2 Remark. If $*$ is applied to (C3), one obtains

(C3') $$C^\lambda_{T,S}a^* \equiv \sum_{S'\in M(\lambda)} r_a(S', S)C^\lambda_{T,S'} \qquad \bmod A(< \lambda) \qquad (\forall a \in A,\ S, T \in M(\lambda)).$$

A.3 Remark. To emphasize the partial order on Λ and simultaneously shorten the notation, the following convention is used in this thesis: For a cellular algebra A with cell datum $(\Lambda, M, C, *)$ as above and $\mathfrak{B} = \{C^\lambda_{S,T} \mid \lambda \in \Lambda,\ S, T \in M(\lambda)\}$ the basis arising from this cell datum, the triple $(\mathfrak{B}, \Lambda, \leq)$ is called a *cellular basis* of A.

A.4 Lemma. *Let $\lambda \in \Lambda$ and $a \in A$. Then for any elements $S_1, S_2, T_1, T_2 \in M(\lambda)$ it holds*

$$C^\lambda_{S_1,T_1} a C^\lambda_{S_2,T_2} = \phi_a(T_1, S_2) C^\lambda_{S_1,T_2}$$

where $\phi_a(T_1, S_2) \in R$ depends only on a, T_1 and S_2 (i.e. is independent of T_2, S_1).

A.5 Remark (Specialisation). If $\sigma : R \to R'$ is a homomorphism of commutative rings, then R' becomes an R-module in the obvious way. The R'-algebra $A^\sigma := R' \otimes_R A$ is called the *specialisation* of A at σ. If A is a cellular algebra with cell datum $(\Lambda, M, *, C)$ then A^σ is a cellular algebra with essentially the same cell datum, with $*$ and C being modified in the obvious way (i.e. $(r' \otimes a)^* := r' \otimes a^*$ and $C(S, T) = 1_{R'} \otimes C^\lambda_{S,T}$ for $S, T \in M(\lambda)$).

A.6 Definition. For each $\lambda \in \Lambda$ define the right A-module $W(\lambda)$ as follows: $W(\lambda)$ is a free R-module with basis $\{C_S \mid S \in M(\lambda)\}$ and (C3) allows to define an A-action by

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S'} \qquad (a \in A, S \in M(\lambda))$$

where $r_a(S', S)$ is the element of R defined in (C3). It is called the *cell representation* of A corresponding to $\lambda \in \Lambda$.

A.7 Remark. Graham and Lehrer note that the definition of the cell representation is a consequence of (C3) and one can analogously define a right action on the set $\{C_S \mid S \in M(\lambda)\}$ using (C3'). They denote the left module by $W(\lambda)$ and the right module with $W(\lambda)^*$.

A.8 Remark. As a difference to the work of Graham and Lehrer the right A-modules will be called the *cell modules* in this thesis. Moreover, they will be defined as subfactors of the algebra A itself with a basis that corresponds to this. To be precise, instead of taking $\{C_S \mid S \in M(\lambda)\}$ as a basis for the cell modules, the cell module is the module with vector space basis $\{C^\lambda_{S',S} + A(< \lambda) \mid S \in M(\lambda)\}$ for some fixed $S' \in M(\lambda)$.

A.9 Definition. Let $\lambda \in \Lambda$. The elements $\phi_1 \in R$ defined in Lemma A.4 can be used to define a bilinear form $\phi_\lambda : W(\lambda) \times W(\lambda) \to R$ by extending $\phi_\lambda(C_S, C_T) = \phi_1(S, T)$, $S, T \in M(\lambda)$ bilinearly.

For the remainder of this section assume that R is a field and all modules are finite dimensional. Further fix a cell datum $(\Lambda, M, C, *)$.

A.10 Definition. For $\lambda \in \Lambda$, define $\mathrm{rad}(\lambda) := \{x \in W(\lambda) \mid \phi_\lambda(x, y) = 0 \ \forall y \in W(\lambda)\}$

A.11 Proposition. *Let $\lambda \in \Lambda$. Then*

1. $\operatorname{rad}(\lambda)$ *is an* A*-submodule of* $W(\lambda)$

and if $\phi_\lambda \neq 0$*, then*

2. *the quotient* $W(\lambda)/\operatorname{rad}(\lambda)$ *is absolutely irreducible;*
3. $\operatorname{rad}(\lambda)$ *is the radical of the* A*-module* $W(\lambda)$ *(i.e. the minimal submodule with semisimple quotient).*

A.12 Definition. Denote the (absolutely irreducible) A-module $W(\lambda)/\operatorname{rad}(\lambda)$ for $\lambda \in \Lambda$ with $\phi_\lambda \neq 0$ by L_λ.

The following two theorems give a hint why it is worth studying the cellular structure of an algebra.

A.13 Theorem. *Let* $\Lambda_0 := \{\lambda \in \Lambda \mid \phi_\lambda \neq 0\}$*. Then the set* $\{L_\lambda \mid \lambda \in \Lambda_0\}$ *is a complete set of (representatives of isomorphism classes of) absolutely irreducible* A*-modules.*

A.14 Theorem. *The following are equivalent*

1. *The algebra* A *is semisimple.*
2. *The non-zero cell representations* $W(\lambda)$ *are irreducible and pairwise inequivalent.*
3. *The form* ϕ_λ *is nondegenerate for each* $\lambda \in \Lambda$*.*

B. Examples

In this chapter some examples are collected.

B.1 Example (to Lemma 2.61)**.** Let $n = 6$ and $a = 3$, then $\mathcal{I} = \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3\}$. To support the readability of the exchange process, fix the following convention $\mathcal{I} = \{\bar{1} = 6, 1 = \bar{6}, \bar{2} = 5, 2 = \bar{5}, \bar{3} = 4, 3 = \bar{4}\}$. To avoid that most of the interesting stuff happens in the indices, the calculations will be made directly on the tableaux with the convention that a tableau T represents $v_T y_\lambda$ with $\lambda = \operatorname{shape}(T)$. Let

$$T := \begin{ytableau} \bar{1} & 1 \\ \bar{2} & 2 \\ \bar{3} & 3 \end{ytableau}.$$

Then the proof of the lemma reads as follows. First, expand the second column stabilizer and permute the first column in each case to avoid the signs:

$$\begin{aligned}
\begin{ytableau} \bar{1} & 1 \\ \bar{2} & 2 \\ \bar{3} & 3 \end{ytableau}
&= \begin{ytableau} \bar{1} & 1 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
- \begin{ytableau} \bar{1} & 2 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
- \begin{ytableau} \bar{1} & 1 & 3 & 2 \\ \bar{2} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} \bar{1} & 3 & 1 & 2 \\ \bar{2} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} \bar{1} & 2 & 3 & 1 \\ \bar{2} \\ \bar{3} \end{ytableau}
- \begin{ytableau} \bar{1} & 3 & 2 & 1 \\ \bar{2} \\ \bar{3} \end{ytableau} \\
&= \begin{ytableau} \bar{1} & 1 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} \bar{2} & 2 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} \bar{1} & 1 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
+ \begin{ytableau} \bar{3} & 3 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
+ \begin{ytableau} \bar{2} & 2 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
+ \begin{ytableau} \bar{3} & 3 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
\end{aligned}$$

Second, add a couple of zeros ($a! \cdot (a-1)$ many, to be precise). The entries causing them to be zero are marked gray:

$$\begin{aligned}
= \; & \begin{ytableau} \bar{1} & 1 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{2} & 2 & 2 & 3 \\ *(gray) \bar{2} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{3} & 3 & 2 & 3 \\ \bar{2} \\ *(gray) \bar{3} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{1} & 1 & 1 & 3 \\ *(gray) \bar{1} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} \bar{2} & 2 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{3} & 3 & 1 & 3 \\ \bar{1} \\ *(gray) \bar{3} \end{ytableau} \\
+ \; & \begin{ytableau} \bar{1} & 1 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{2} & 2 & 3 & 2 \\ \bar{3} \\ *(gray) \bar{2} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{3} & 3 & 3 & 2 \\ *(gray) \bar{3} \\ \bar{2} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{1} & 1 & 1 & 2 \\ *(gray) \bar{1} \\ \bar{2} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{2} & 2 & 1 & 2 \\ \bar{1} \\ *(gray) \bar{2} \end{ytableau}
+ \begin{ytableau} \bar{3} & 3 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau} \\
+ \; & \begin{ytableau} *(gray) \bar{1} & 1 & 3 & 1 \\ \bar{3} \\ *(gray) \bar{1} \end{ytableau}
+ \begin{ytableau} \bar{2} & 2 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{3} & 3 & 3 & 1 \\ *(gray) \bar{3} \\ \bar{1} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{1} & 1 & 2 & 1 \\ \bar{2} \\ *(gray) \bar{1} \end{ytableau}
+ \begin{ytableau} *(gray) \bar{2} & 2 & 2 & 1 \\ *(gray) \bar{2} \\ \bar{1} \end{ytableau}
+ \begin{ytableau} \bar{3} & 3 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
\end{aligned}$$

Third, rewrite the tableaux according to Lemma 2.61:

$$
= -\begin{ytableau} \bar{4} & 4 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{4} & 4 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{4} & 4 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
$$

Fourth, rearrange the sum and add more gray zeros ($\frac{a!}{2} \cdot (a-2) \cdot a$ many this time):

$$
= -\begin{ytableau} \bar{4} & 4 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 3 & 3 \\ *(gray) \bar{3} \\ *(gray) \bar{3} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 2 & 2 \\ *(gray) \bar{2} \\ *(gray) \bar{2} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{4} & 4 & 1 & 1 \\ *(gray) \bar{1} \\ *(gray) \bar{1} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{4} & 4 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{5} & 5 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 3 & 3 \\ *(gray) \bar{3} \\ *(gray) \bar{3} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 2 & 2 \\ *(gray) \bar{2} \\ *(gray) \bar{2} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{5} & 5 & 1 & 1 \\ *(gray) \bar{1} \\ *(gray) \bar{1} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{5} & 5 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{6} & 6 & 1 & 3 \\ \bar{1} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 2 & 3 \\ \bar{2} \\ \bar{3} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 3 & 3 \\ *(gray) \bar{3} \\ *(gray) \bar{3} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 1 & 2 \\ \bar{1} \\ \bar{2} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 2 & 2 \\ *(gray) \bar{2} \\ *(gray) \bar{2} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 3 & 2 \\ \bar{3} \\ \bar{2} \end{ytableau}
$$

$$
-\begin{ytableau} \bar{6} & 6 & 1 & 1 \\ *(gray) \bar{1} \\ *(gray) \bar{1} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 2 & 1 \\ \bar{2} \\ \bar{1} \end{ytableau}
-\begin{ytableau} \bar{6} & 6 & 3 & 1 \\ \bar{3} \\ \bar{1} \end{ytableau}
$$

Fifth, rewrite the tableaux according to Lemma 2.61:

$$
= \begin{ytableau} \bar{4} & 4 & 4 & 3 \\ \bar{4} \\ \bar{3} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 5 & 3 \\ \bar{5} \\ \bar{3} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 6 & 3 \\ \bar{6} \\ \bar{3} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 4 & 2 \\ \bar{4} \\ \bar{2} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 5 & 2 \\ \bar{5} \\ \bar{2} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 6 & 2 \\ \bar{6} \\ \bar{2} \end{ytableau}
$$

$$
+\begin{ytableau} \bar{4} & 4 & 4 & 1 \\ \bar{4} \\ \bar{1} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 5 & 1 \\ \bar{5} \\ \bar{1} \end{ytableau}
+\begin{ytableau} \bar{4} & 4 & 6 & 1 \\ \bar{6} \\ \bar{1} \end{ytableau}
$$

$$
+\begin{ytableau} \bar{5} & 5 & 4 & 3 \\ \bar{4} \\ \bar{3} \end{ytableau}
+\begin{ytableau} \bar{5} & 5 & 5 & 3 \\ \bar{5} \\ \bar{3} \end{ytableau}
+\begin{ytableau} \bar{5} & 5 & 6 & 3 \\ \bar{6} \\ \bar{3} \end{ytableau}
+\begin{ytableau} \bar{5} & 5 & 4 & 2 \\ \bar{4} \\ \bar{2} \end{ytableau}
+\begin{ytableau} \bar{5} & 5 & 5 & 2 \\ \bar{5} \\ \bar{2} \end{ytableau}
+\begin{ytableau} \bar{5} & 5 & 6 & 2 \\ \bar{6} \\ \bar{2} \end{ytableau}
$$

$$+\begin{ytableau} \bar5 & 5 & 4 & 1 \\ \bar4 \\ \bar1 \end{ytableau} + \begin{ytableau} \bar5 & 5 & 5 & 1 \\ \bar5 \\ \bar1 \end{ytableau} + \begin{ytableau} \bar5 & 5 & 6 & 1 \\ \bar6 \\ \bar1 \end{ytableau}$$

$$+\begin{ytableau} \bar6 & 6 & 4 & 3 \\ \bar4 \\ \bar3 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 5 & 3 \\ \bar5 \\ \bar3 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 6 & 3 \\ \bar6 \\ \bar3 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 4 & 2 \\ \bar4 \\ \bar2 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 5 & 2 \\ \bar5 \\ \bar2 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 6 & 2 \\ \bar6 \\ \bar2 \end{ytableau}$$

$$+\begin{ytableau} \bar6 & 6 & 4 & 1 \\ \bar4 \\ \bar1 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 5 & 1 \\ \bar5 \\ \bar1 \end{ytableau} + \begin{ytableau} \bar6 & 6 & 6 & 1 \\ \bar6 \\ \bar1 \end{ytableau}$$

Sixth, rewrite the tableaux according to Lemma 2.61 (no need to add zeros this time!):

$$= -\begin{ytableau} \bar4 & 4 & 4 & 6 \\ \bar4 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 5 & 6 \\ \bar5 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 6 & 6 \\ \bar6 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 4 & 5 \\ \bar4 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 5 & 5 \\ \bar5 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 6 & 5 \\ \bar6 \\ \bar5 \end{ytableau}$$

$$-\begin{ytableau} \bar4 & 4 & 4 & 4 \\ \bar4 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 5 & 4 \\ \bar5 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 6 & 4 \\ \bar6 \\ \bar4 \end{ytableau}$$

$$-\begin{ytableau} \bar5 & 5 & 4 & 6 \\ \bar4 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 5 & 6 \\ \bar5 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 6 & 6 \\ \bar6 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 4 & 5 \\ \bar4 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 5 & 5 \\ \bar5 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 6 & 5 \\ \bar6 \\ \bar5 \end{ytableau}$$

$$-\begin{ytableau} \bar5 & 5 & 4 & 4 \\ \bar4 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 5 & 4 \\ \bar5 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 6 & 4 \\ \bar6 \\ \bar4 \end{ytableau}$$

$$-\begin{ytableau} \bar6 & 6 & 4 & 6 \\ \bar4 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 5 & 6 \\ \bar5 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 6 & 6 \\ \bar6 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 4 & 5 \\ \bar4 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 5 & 5 \\ \bar5 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 6 & 5 \\ \bar6 \\ \bar5 \end{ytableau}$$

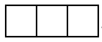
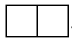
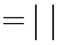
$$-\begin{ytableau} \bar6 & 6 & 4 & 4 \\ \bar4 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 5 & 4 \\ \bar5 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 6 & 4 \\ \bar6 \\ \bar4 \end{ytableau}$$

Almost done; delete the zeros:

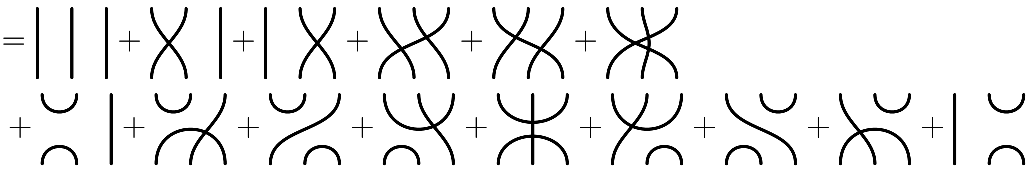
$$= -\begin{ytableau} \bar4 & 4 & 5 & 6 \\ \bar5 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar4 & 4 & 6 & 5 \\ \bar6 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 4 & 6 \\ \bar4 \\ \bar6 \end{ytableau} - \begin{ytableau} \bar5 & 5 & 6 & 4 \\ \bar6 \\ \bar4 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 4 & 5 \\ \bar4 \\ \bar5 \end{ytableau} - \begin{ytableau} \bar6 & 6 & 5 & 4 \\ \bar5 \\ \bar4 \end{ytableau}$$

Finally, merge the second column back together:

$$\begin{ytableau} \bar1 & 1 \\ \bar2 & 2 \\ \bar3 & 3 \end{ytableau} = -\begin{ytableau} \bar4 & 4 \\ \bar5 & 5 \\ \bar6 & 6 \end{ytableau} \qquad \mod (\mathcal{T}\mathcal{T}^{(k+1)}(N^{\mu}))$$

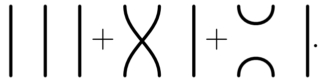
B.2 Example (to Theorem 2.30)**.** Let $\lambda =$ , $\mu =$ . Then $d_{\lambda\to\mu} =$ ,

$$x_\lambda^{\mathrm{sp}} =$$



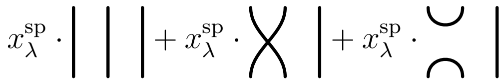
and

$$x_\mu^{\mathrm{sp}} =$$

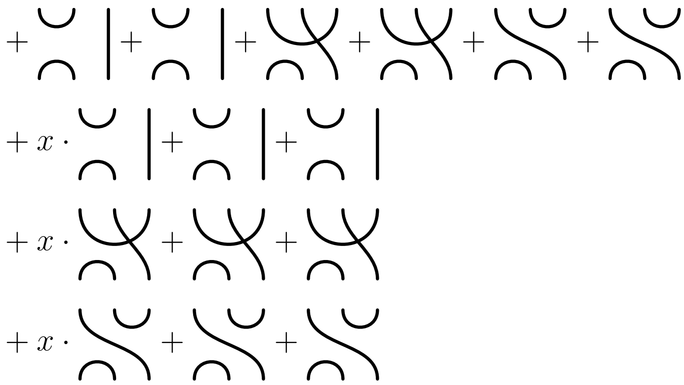


Now the product is

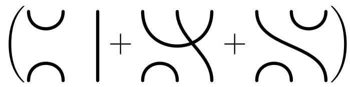
$$x_\lambda^{\mathrm{sp}} d_{\lambda\to\mu} x_\mu^{\mathrm{sp}} =$$



$$= x_\lambda^{\mathrm{sp}} + x_\lambda^{\mathrm{sp}}$$



$$= 2 \cdot x_\lambda^{\mathrm{sp}} + (4 + x)$$



Thus,

$$\frac{1}{2} x_\lambda^{\mathrm{sp}} d_{\lambda\to\mu} x_\mu^{\mathrm{sp}} \neq x_\lambda^{\mathrm{sp}} d_{\lambda\to\mu}.$$

B.3 Example. For the symplectic weight $\mu = (0,0,0)$ an examination of the symplectic weight space $N^\mu \subset U^{\otimes 4}$ is given:
Since in this case $n = 3$, all partitions of 4 but $\ydiagram{1,1,1,1}$ are (-6)-permissible. Thus, only symplectic semistandard tableaux of shape $\ydiagram{4}$, $\ydiagram{3,1}$, $\ydiagram{2,2}$ and $\ydiagram{2,1,1}$ have to be considered (and only these exist) for the 'top layer', i.e. $\mathcal{HT}^{(0)}(N^\mu)$. To $\ydiagram{4}$ there exist the 6 following symplectic semistandard tableaux of weight μ:

$$\begin{ytableau} \bar{1} & \bar{1} & 1 & 1 \end{ytableau}\,,\ \begin{ytableau} \bar{1} & 1 & \bar{2} & 2 \end{ytableau}\,,\ \begin{ytableau} \bar{1} & 1 & \bar{3} & 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & \bar{2} & 2 & 2 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 2 & \bar{3} & 3 \end{ytableau}\,,\ \begin{ytableau} \bar{3} & \bar{3} & 3 & 3 \end{ytableau}$$

To $\ydiagram{3,1}$ there exist the 9 following symplectic semistandard tableaux of weight μ:

$$\begin{ytableau} \bar{1} & 1 & \bar{2} \\ 2 \end{ytableau}\,,\ \begin{ytableau} \bar{1} & 1 & 2 \\ \bar{2} \end{ytableau}\,,\ \begin{ytableau} \bar{1} & 1 & \bar{3} \\ 3 \end{ytableau}\,,\ \begin{ytableau} \bar{1} & 1 & 3 \\ \bar{3} \end{ytableau}\,,\ \begin{ytableau} \bar{2} & \bar{2} & 2 \\ 2 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 2 & \bar{3} \\ 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 2 & 3 \\ \bar{3} \end{ytableau}\,,\ \begin{ytableau} \bar{2} & \bar{3} & 3 \\ 2 \end{ytableau}\,,\ \begin{ytableau} \bar{3} & \bar{3} & 3 \\ 3 \end{ytableau}$$

To $\ydiagram{2,2}$ there exist the 6 following symplectic semistandard tableaux of weight μ:

$$\begin{ytableau} \bar{1} & 1 \\ \bar{2} & 2 \end{ytableau}\,,\ \begin{ytableau} \bar{1} & 1 \\ \bar{3} & 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & \bar{2} \\ 2 & 2 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 2 \\ \bar{3} & 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & \bar{3} \\ 2 & 3 \end{ytableau}\,,\ \begin{ytableau} \bar{3} & \bar{3} \\ 3 & 3 \end{ytableau}$$

To $\ydiagram{2,1,1}$ there exist the four following symplectic semistandard tableaux of weight μ:

$$\begin{ytableau} \bar{1} & 1 \\ \bar{3} \\ 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 2 \\ \bar{3} \\ 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & \bar{3} \\ 2 \\ 3 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 3 \\ 2 \\ \bar{3} \end{ytableau}$$

For the middle layer $(\mathcal{HT}^{(1)}(N^\mu))$ both partitions of 2 are (-6)-permissible. To $\ydiagram{1,1}$ there exist the three following symplectic semistandard tableaux of weight μ:

$$\begin{ytableau} \bar{1} & 1 \end{ytableau}\,,\ \begin{ytableau} \bar{2} & 2 \end{ytableau}\,,\ \begin{ytableau} \bar{3} & 3 \end{ytableau}$$

To $\ydiagram{2}$ there exist the two following symplectic semistandard tableaux of weight μ:

$$\begin{ytableau} \bar{2} \\ 2 \end{ytableau}\,,\ \begin{ytableau} \bar{3} \\ 3 \end{ytableau}$$

For the bottom layer $(\mathcal{HT}^{(2)}(N^\mu))$ there is only the empty partition with the unique empty tableau. This tableau has weight μ.

For the following paths, see also Figure 2.2. For paths of shape a partition of 4 the zeros are omitted. There is one (-6)-permissible path to $\left(\ydiagram{1,1,1,1}, 0\right)$:

$$\left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{3}, \ydiagram{4}\right)$$

There are three (-6)-permissible paths to $\left(\ydiagram{2,1,1}, 0\right)$:

$$\left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,1,1}\right), \ \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,1,1}\right), \ \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1,1}, \ydiagram{2,1,1}\right)$$

There are two (-6)-permissible paths to $\left(\ydiagram{2,2}, 0\right)$:

$$\left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,2}\right), \ \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,2}\right)$$

There are three (-6)-permissible paths to $\left(\ydiagram{3,1}, 0\right)$:

$$\left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{3}, \ydiagram{3,1}\right), \ \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{3,1}\right), \ \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{3,1}\right)$$

There are six (-6)-permissible paths to $\left(\ydiagram{1,1}, 1\right)$:

$$\left((\emptyset, 0), (\ydiagram{1}, 0), (\ydiagram{2}, 0), \left(\ydiagram{2,1}, 0\right), \left(\ydiagram{1,1}, 1\right)\right), \ \left((\emptyset, 0), (\ydiagram{1}, 0), \left(\ydiagram{1,1}, 0\right), \left(\ydiagram{2,1}, 0\right), \left(\ydiagram{1,1}, 1\right)\right),$$
$$\left((\emptyset, 0), (\ydiagram{1}, 0), \left(\ydiagram{1,1}, 0\right), \left(\ydiagram{1,1,1}, 0\right), \left(\ydiagram{1,1}, 1\right)\right), \ \left((\emptyset, 0), (\ydiagram{1}, 0), (\ydiagram{2}, 0), (\ydiagram{1}, 1), \left(\ydiagram{1,1}, 1\right)\right),$$
$$\left((\emptyset, 0), (\ydiagram{1}, 0), \left(\ydiagram{1,1}, 0\right), (\ydiagram{1}, 1), \left(\ydiagram{1,1}, 1\right)\right), \ \left((\emptyset, 0), (\ydiagram{1}, 0), (\emptyset, 1), (\ydiagram{1}, 1), \left(\ydiagram{1,1}, 1\right)\right)$$

There are six (-6)-permissible paths to $(\ydiagram{2}, 1)$:

$$((\emptyset, 0), (\ydiagram{1}, 0), (\ydiagram{2}, 0), (\ydiagram{3}, 0), (\ydiagram{2}, 1)), \ \left((\emptyset, 0), (\ydiagram{1}, 0), (\ydiagram{2}, 0), \left(\ydiagram{2,1}, 0\right), (\ydiagram{2}, 1)\right),$$
$$\left((\emptyset, 0), (\ydiagram{1}, 0), \left(\ydiagram{1,1}, 0\right), \left(\ydiagram{2,1}, 0\right), (\ydiagram{2}, 1)\right), \ ((\emptyset, 0), (\ydiagram{1}, 0), (\ydiagram{2}, 0), (\ydiagram{1}, 1), (\ydiagram{2}, 1)),$$
$$\left((\emptyset, 0), (\ydiagram{1}, 0), \left(\ydiagram{1,1}, 0\right), (\ydiagram{1}, 1), (\ydiagram{2}, 1)\right), \ ((\emptyset, 0), (\ydiagram{1}, 0), (\emptyset, 1), (\ydiagram{1}, 1), (\ydiagram{2}, 1))$$

There are three (-6)-permissible paths to $(\emptyset, 2)$:

$$((\emptyset,0)\,,(\square,0)\,,(\square\square,0)\,,(\square,1)\,,(\emptyset,2))\,,\ \left((\emptyset,0)\,,(\square,0)\,,\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},0\right),(\square,1)\,,(\emptyset,2)\right),$$
$$((\emptyset,0)\,,(\square,0)\,,(\emptyset,1)\,,(\square,1)\,,(\emptyset,2))$$

Combining the indexing sets gives indices for 90 basis elements, verifying the index sets.

The definition of the symplectic weight space N^μ uses the weight spaces M^λ for weights λ that map onto μ under π, i.e. for all

$$\lambda \in \left\{ \begin{array}{l} (2,2,0,0,0,0),(0,0,2,2,0,0),(0,0,0,0,2,2), \\ (1,1,1,1,0,0),(1,1,0,0,1,1),(0,0,1,1,1,1) \end{array} \right\}.$$

Hence, the symplectic tensor space N^ν decomposes as a $B_4(-6)/B_4^{(1)}(-6) \cong \mathbb{F}\mathfrak{S}_4$-module into a direct sum of these M^λ. Obviously, this decomposition is not a decomposition as $B_4(-6)$-modules. The weight spaces are 'bound together' by the trace tensors, i.e. images of the elements e_i.

The following phenomenon can be observed in this context: The modules M^λ described above can also be found as submodules of the module $\mathcal{HT}^{(0)}(N^\mu) = \mathcal{TT}^{(0)}(N^\mu)/\mathcal{TT}^{(1)}(N^\mu)$. Similarly, for each weight $\nu \in \{(1,1,0,0,0,0),(0,0,1,1,0,0),(0,0,0,0,1,1)\}$ the inflation of the weight module M^ν to a $B_4^{(1)}(-6)/B_4^{(2)}(-6)$-module can be found as submodule of $\mathcal{HT}^{(1)}(N^\mu) = \mathcal{TT}^{(1)}(N^\mu)/\mathcal{TT}^{(2)}(N^\mu)$. (These modules are isomorphic as $B_4^{(1)}(-6)/B_4^{(2)}(-6)$-modules to the module generated by $x_\nu e_3$.)

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Nomenclature

$\mathrm{Add}(\lambda)$	The set of Young diagrams obtained from λ by adding one node, page 6
$\mathfrak{B}_\lambda^{\mathrm{sp}}$	A submonoid of $\mathfrak{B}_r$ similar to the Young subgroup $in\mathfrak{S}_r$, page 39
$\mathfrak{B}_r$	The set of Brauer diagrams with $2r$ vertices, page 25
$\mathfrak{B}_r^x$	'Murphy-Enyang' basis of the Brauer algebra, anti-symmetrizer version, page 33
$\mathfrak{B}_r^y$	'Murphy-Enyang' basis of the Brauer algebra, anti-symmetrizer version, page 35
$\mathfrak{B}_{r,s}$	The set of (r,s)-walled Brauer diagrams, page 65
$\mathfrak{B}_{r,s}^y$	Murphy type basis of the walled Brauer algebra, page 71
$\mathfrak{E}_r^{\mathrm{sp}}$	Murphy basis of the endomorphism ring $\mathrm{End}_{\mathbb{F}\,\mathrm{GSp}(W)}(W^{\otimes r})$, page 42
$\mathrm{GL}(V)$	The general linear group of the vector space V, page 17
$\mathrm{GO}(U)$	The orthogonal similitude group of the orthogonal space U, page 52
$\mathrm{GSp}(W)$	The symplectic similitude group of the symplectic space W, page 37
$\Lambda(r)$	The set of all pairs (λ,k) with $\lambda \in \mathrm{Par}(r-2k)$, page 27
$\Lambda_x(r)$	The set $\Lambda(r)$ with the 'symmetrizer' partial order, page 27
$\Lambda_y(r)$	The set $\Lambda(r)$ with the 'anti-symmetrizer' partial order, page 27
$\mathfrak{M}_r^x$	Enyang's basis of the Brauer algebra, symmetrizer version, page 29
$\mathfrak{M}_r^y$	Enyang's basis of the Brauer algebra, anti-symmetrizer version, page 29
$\mathfrak{M}_{r,s}^x$	Enyang's basis of the walled Brauer algebra, page 68

$\mathfrak{N}_r^x$ The Murphy basis of the symmetric group, symmetrizer version, page 9

$\mathfrak{N}_r^y$ The Murphy basis of the symmetric group, anti-symmetrizer version, page 10

$\mu(\mathfrak{t})$, page 7

$\mathrm{Par}(r)$ The set of partitions of r, page 5

$\mathrm{Rem}(\lambda)$ The set of Young diagrams obtained from λ by removing one node, page 6

$\mathfrak{S}_\lambda$ The Young subgroup, page 8

$\mathfrak{S}_r$ The symmetric group on r letters, page 3

$\mathrm{Std}(\lambda)$ The set of standard tableaux of shape λ, page 7

$\mathrm{Std}(\lambda, \mu, k)$ The set of standard (λ, μ, k)-tableaux, page 68

$\mathrm{Std}(\lambda, k)$ The set of standard (λ, k)-tableaux, page 28

$\mathrm{Tab}(\lambda)^{\mathrm{o}}_{\mathcal{I},\mu}$ The set of all orthogonal semistandard tableaux of shape λ and orthogonal weight μ, page 56

$\mathrm{Tab}(\lambda)^{\mathrm{o}}_{\mathcal{I}}$ The set of all orthogonal semistandard tableaux of shape λ, page 56

$\mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I},\mu}$ The set of all symplectic semistandard tableaux of shape λ and symplectic weight μ, page 44

$\mathrm{Tab}(\lambda)^{\mathrm{sp}}_{\mathcal{I}}$ The set of all symplectic semistandard tableaux of shape λ, page 44

$\mathrm{Tab}(\lambda)_n$ The set of semistandard tableaux of shape λ, page 7

$\mathrm{Tab}(\lambda)_{n,\mu}$ The set of semistandard tableaux of shape λ and weight μ, page 19

$\mathfrak{t}^\lambda$ The initial tableau of shape λ, page 7

$\mathfrak{t}_\lambda$ The final tableau of shape λ, page 7

$b(\mathfrak{t})$ The element in the symmetric group such that $\mathfrak{t}_\lambda b(\mathfrak{t}) = \mathfrak{t}$, page 8

$B_r(x)$ The Brauer algebra, page 25

$B_r^{(i)}(x)$ Ideals in the Brauer algebra, page 26

$C^{(\lambda,\mu,k)}$	Cell modules of the walled Brauer algebra, page 68
$C_x^{(\lambda,k)}$	Cell modules of the Brauer algebra, symmetrizer version, page 29
$C_y^{(\lambda,k)}$	Cell modules of the Brauer algebra, anti-symmetrizer version, page 29
$d(\mathfrak{t})$	The element in the symmetric group such that $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$, page 8
M^μ	The weight space to weight μ, page 19
S_x^λ	Specht modules of the symmetric group, page 10
S_y^λ	Dual Specht modules of the symmetric group, page 10
T^λ	The unique element in $\mathrm{Tab}(\lambda)_{n,\lambda}$, page 19
x_λ	The symmetrizer of $R\mathfrak{S}_\lambda$, page 8
y_λ	The anti-symmetrizer of $R\mathfrak{S}_{\lambda'}$, page 8