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Rows in Two-Dimensional Quasilattices

By

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Pentagonal Penrose tilings and their octagonal and dodecagonal counterparts represent simple models for two-dimensional quasicrystals. We investigate the properties of rows, which are the analogues to lattice lines of periodic lattices, by a method based on the projection formalism. It enables us to derive the vertex pattern and the mean vertex density of an arbitrary row as well as the mean vertex density of the corresponding family and its dependence on the row direction. The row separation and their quasiperiodic sequences are also being studied.

Penrosemuster und ihre okta- und dodekagonalen Verwandten stellen einfache Modelle für zweidimensionale Quasikristalle dar. Wir untersuchen die Eigenschaften von Atomketten, die den Gittergeraden in periodischen Gittern entsprechen, mit Hilfe eines auf dem Projektionsformalismus beruhenden Verfahrens. Wir können sowohl das Muster der Vertexpunkte und die mittlere Vertexdichte auf einer beliebigen Geraden als auch die über die gesamte Geradenschar gemittelten Vertexdichten und ihre Abhängigkeit von der Scharrichtung bestimmen. Ebenso berechnen wir die zwischen den Geraden einer Schar auftretenden Abstände und ihre quasiperiodische Abfolge.

1. Introduction

Shortly after the discovery of an icosahedral quasicrystal by Schechtman et al. [1], Bendersky [2] reported the observation of another quasicrystalline phase with fivefold symmetry. It is build up by periodically stacked planes, each plane being quasiperiodic. Later quasicrystals with eightfold and twelvefold symmetry were seen (Wang et al. [3] and Ishimasa et al. [4]).

These discoveries strengthened the interest in quasiperiodic tilings of the plane, which constitute basic models for two-dimensional quasicrystals and were first proposed by Penrose [5].

After the work of the de Bruijn [6], it has been shown by Kramer and Neri [7], Levine and Steinhardt [8], Duneau and Katz [9], and Kalugin et al. [10] that quasiperiodic structures can be obtained by a cut and projection method in a higher dimensional periodic lattice.

Katz and Duneau [11] had generalized the notion of lattice planes and lines to three-dimensional icosahedral quasilattices and had developed a method to investigate their properties by an appropriate cut and projection method. We extend this method to study quasilattice lines in two-dimensional quasilattices of eightfold and tenfold symmetry. Results for icosahedral quasilattices derived by Kupke and Trebin [12] are presented elsewhere.

Our interest in quasilattice geometry was stimulated by a computer simulation showing that channeling is feasible in quasicrystals, and by the following experimental confirmation by Carstanjen et al. [13]. Furthermore the knowledge of lattice planes and lines is essential for the understanding of other experiments such as electron transmission microscopy.

After repeating the cut and projection method for the Fibonacci sequence in one dimension we briefly describe the method for arbitrary quasilattices and then specialize to the cases

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of interest here. Results on the vertex pattern on an arbitrary row, its mean vertex density and properties of families of parallel rows are given for octagonal and decagonal quasilattices.

2. The Cut and Projection Method

2.1 A one-dimensional model

To establish the notations we briefly repeat the cut and projection method for one-dimensional quasicrystals (Fig. 1).

The quasilattice consists of two types of tiles, a long (l) and a short (s) interval. A tiling line $\mathbb{E}$ is drawn across a square lattice $\mathbb{Z}^2$ and a strip S is generated by shifting the unit cell $\gamma^{(2)}$ of $\mathbb{Z}^2$ along $\mathbb{E}$. The tiling is obtained by orthogonal projection of all lattice points inside the strip onto $\mathbb{E}$, the two intervals just being the projection of the two nonparallel edges of the unit cell. If the slope of the line $\mathbb{E}$ is irrational, the lattice and $\mathbb{E}$ have at most one point in common. Clearly the tiling is aperiodic. Projecting all lattice points onto a line perpendicular to $\mathbb{E}$ yields L′, a dense set of points in $\mathbb{E}'$. The projection of the strip onto $\mathbb{E}'$ is called acceptance domain $K = \pi'(S) = \pi'(\gamma^{(2)})$, π' being the projector onto $\mathbb{E}'$. A point in $K \cap L'$ uniquely corresponds to a point of the tiling. This correspondence can be extended to lines in two-dimensional quasilattices. It is the basis of our investigation.

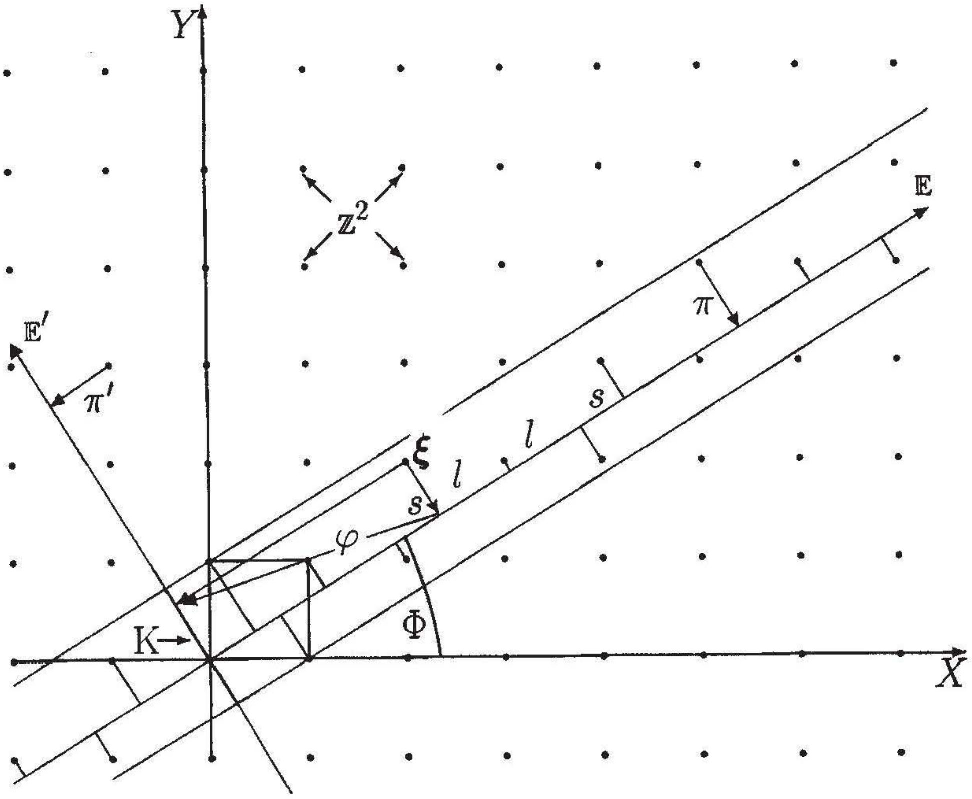

Fig. 1. A one-dimensional quasilattice T is obtained by the cut and projection method. Shifting the unit cell of $\mathbb{Z}^2$ along the tiling space $\mathbb{E}$ yields the strip, the projection of the unit cell on $\mathbb{E}'$ gives the acceptance domain K. The projection $\pi(\xi)$ of the lattice points inside the strip onto the tiling space $\mathbb{E}$ yields the vertex points of the tiling. The tiles consist of long (l) and short (s) intervals. The tiling is quasiperiodic if the slope is irrational ($\tan \Phi \notin \mathbb{Q}$). The image of a vertex point under the natural isomorphism φ is the projection of the lattice point ξ onto the orthogonal space $\mathbb{E}'$: $\varphi(\pi(\xi)) = \pi'(\xi)$

Each lattice point whose projection onto $\mathbb{E}$ is followed by a long (short) interval projects to a point above (below) $\mathbb{E}$ in $\mathbb{E}'$. As L′ fills $\mathbb{E}'$ densely and homogeneously, the ratio of the lengths of the two parts of K yields the relative probability of long and short intervals. This analysis can be extended to arbitrary finite subpatterns of the tiling by the polar calculus [11]. The knowledge of the lengths l and s and of their probabilities of occurrence allows us to calculate the mean *distance* between two points of the tiling. The mean *density* of vertex points is the inverse of the mean distance and is obtained directly as the quotient of the surface F_k of the acceptance domain K and the volume Ω of the unit cell of the lattice: $\varrho_V = F_K/\Omega$ (ELSER [14]).

As we shall see later the sequence of points on any line of a planar quasilattice is constructed by a cut and projection method in a two-dimensional lattice just as above, with two generalizations: Neither it is necessary that the lattice is a square lattice nor that the acceptance domain is the projection of the unit cell.

2.2 *The cut and projection method in general*

In general we start with a discrete periodic lattice Λ embedded in Euclidian space $\mathbb{R}^n$ which is invariant under the action of the symmetry group under consideration (here C_8, C_{10v}). Euclidian space divides as a rule into two invariant subspaces $\mathbb{E}$ and $\mathbb{E}^\perp$ of dimension d and a lattice subspace U of dimension u such that $n = 2d + u$. $\mathbb{E}$ is the tiling or physical space, $\mathbb{E}' = \mathbb{E}^\perp \oplus \mathrm{U}$ is the orthogonal space. Let π, $\pi^\perp$, π', and π_U denote the orthogonal projection of the lattice points on $\mathbb{E}$, $\mathbb{E}^\perp$, $\mathbb{E}'$, and U, respectively. The projection $\mathrm{L} = \pi(\Lambda)$ and $\mathrm{L}^\perp = \pi^\perp(\Lambda)$ of the lattice Λ onto $\mathbb{E}$ and $\mathbb{E}^\perp$ yields dense point sets, whereas the projection onto U is discrete. Thus the set $\mathrm{L}' = \pi'(\Lambda)$ consists of densely filled hyperplanes perpendicular to U.

The quasilattice T is obtained by projecting all lattice points inside a strip S onto the physical space $\mathbb{E}$: $\mathrm{T} = \pi(\Lambda \cap \mathrm{S})$. S is generated by translating a subset K of $\mathbb{E}'$ along $\mathbb{E}$: $\mathrm{S} = \mathbb{E} + \mathrm{K}$. The projected pattern is discrete as long as K is bounded [11]. Because of the saturation [11] of the projection π' the equalities $\pi'(\mathrm{S} \cap \Lambda) = \pi'(\mathrm{S}) \cap \pi'(\Lambda) = \mathrm{K} \cap \mathrm{L}'$ are valid, and the strip method amounts to select all lattice points whose projection to $\mathbb{E}'$ falls into the acceptance domain K. A shift of K along U yields a different local isomorphism class (LI-class) of the tiling, see [15]. π is not necessarily injective whenever there is a nontrivial lattice subspace U. But the strip S can always be chosen such that the restriction $\pi_{|\mathrm{S}}$ is injective. Let us denote the lattice basic vectors by ε_i and their projections onto $\mathbb{E}$, $\mathbb{E}'$ etc. by e_i, e_i' etc. A generic lattice vector ξ is divided into components in $\mathbb{E}$, $\mathbb{E}^\perp$, and U: $\xi = \pi(\xi) + \pi^\perp(\xi) + \pi_\mathrm{U}(\xi) =: \xi^\| + \xi^\perp + \xi^\mathrm{U}$. There is a unique correspondence φ, the *natural isomorphism*, between the points of the pattern and the points of $\mathrm{K} \cap \mathrm{L}'$: $\varphi: \mathrm{T} \to \mathrm{K} \cap \mathrm{L}'$ with $\varphi = \pi' \circ \pi_{|\mathrm{S}}^{-1}$.

We now treat the two cases of interest separately:

Octagonal quasilattices as shown in Fig. 2 (bottom) are obtained by projecting points of a hypercubic lattice $\mathbb{Z}^4$. The acceptance domain is a regular octagon given by the projection of the unit cell into $\mathbb{E}'$ (Fig. 2, top left). The area of the surface of K is $F_\mathrm{K} = 1 + \sqrt{2}$. The four-dimensional volume of the unit cell Ω being unity this leads to a vertex density $\varrho_V = 1 + \sqrt{2}$ of the tiling. The pattern consists of two types of tiles, a square of surface $F = 1/2$ and a rhombus of $F = 1/4\sqrt{2}$, the latter being $\sqrt{2}$-times more frequent than the

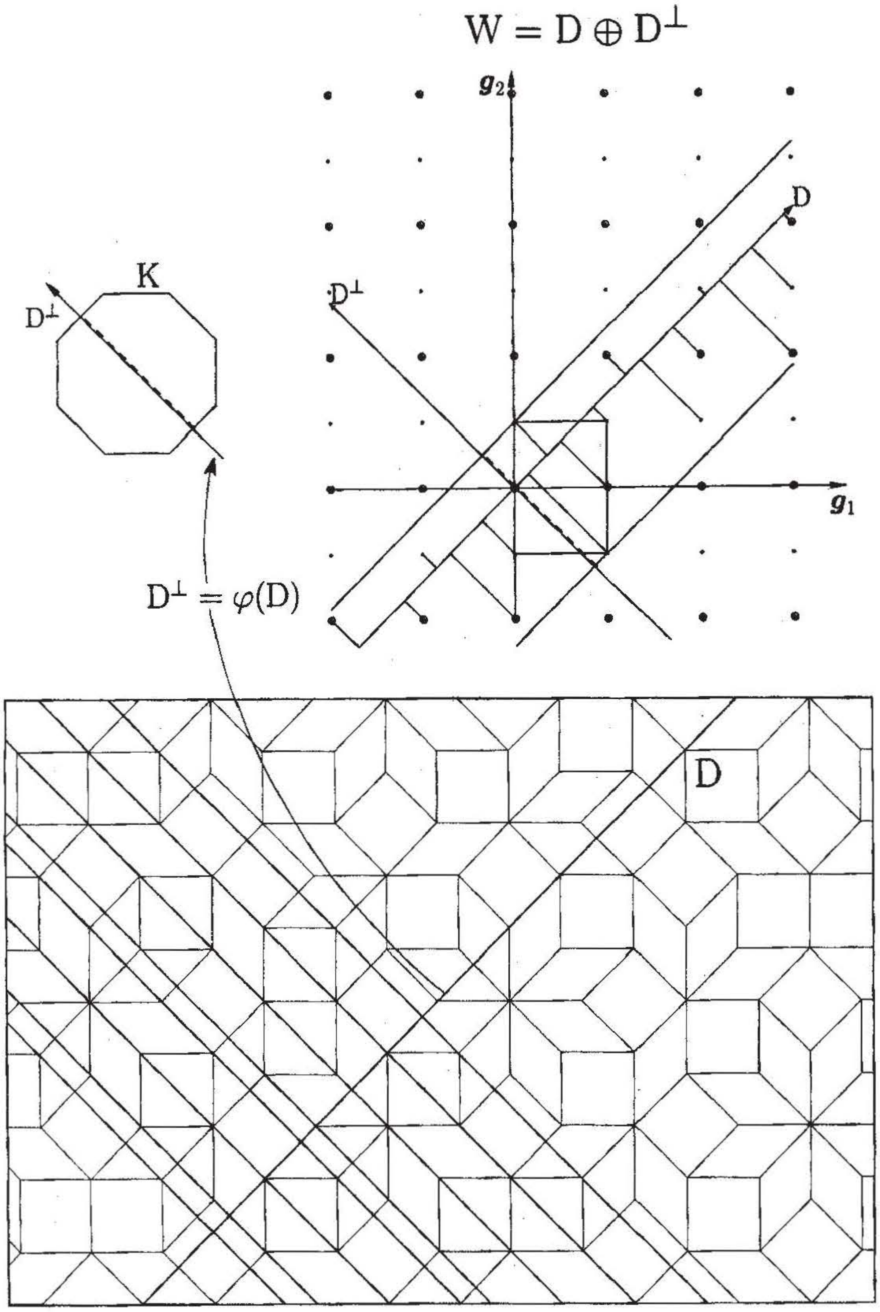

Fig. 2. Investigation of rows in an octagonal quasilattice. In the octagonal tiling (bottom) a line D and some members of a family of rows perpendicular to D are being displayed. The row $D \cap T$ of the tiling and the corresponding line $D^{\perp} = \varphi(D)$ in the orthogonal space (top left) span a lattice subspace $W = D \oplus D^{\perp}$, and the pattern $D \cap T$ is completely determined by the lattice $W \cap \mathbb{Z}^4$ and the strip $W \cap S$ (top right). To investigate the distances between the rows perpendicular to D one has to analyse the pattern $\pi_D(T)$, which leads to a strip method in the same plane W but with lattice $\pi_W(\mathbb{Z}^4)$ and strip $\pi_W(S)$. $W \cap \mathbb{Z}^4$ is a sublattice of $\pi_W(\mathbb{Z}^4)$, the additional points of the latter are marked with open circles. The projection onto D of the additional points inside the strip corresponds to the sections of D with the rows of the family not falling on a vertex point

square. Each vertex point belongs to one tile on the average, so the vertex density can be derived directly without using the argument of Elser [14], mentioned above. There is no invariant lattice subspace U, and L′ fills $\mathbb{E}'$ densely.

Decagonal quasilattices also consist of two types of tiles, an oblate rhombus ($F = \sin(2\pi/5)$) and a prolate one ($F = \sin(4\pi/5)$), as shown in Fig. 3. Here the oblate rhombus is

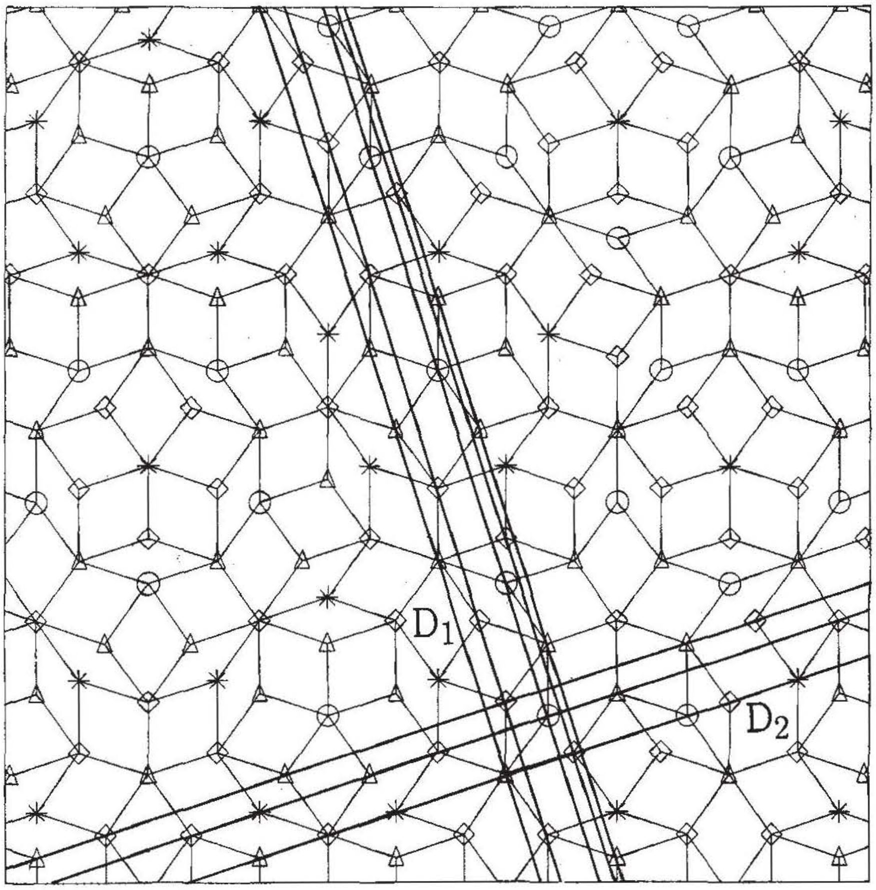

Fig. 3. In the above Penrose tiling every vertex point $P = \pi(\xi)$ is labeled with ○, △, ◇, ∗, depending on the layer of $K \cap L'$ into which the corresponding point $\varphi(P)$ falls (see also Fig. 4). Some members for two different families of rows are shown, parallel ($D_1 \parallel e_0$) and perpendicular ($D_2 \perp e_0$) to the basis vector e_0. Both cases discussed in the text are manifest: points from all four layers fall on the rows parallel to D_2 (case (a)), whereas only points from one layer lie on the rows parallel D_1 (case (b))

τ-times more frequent, $\tau = \frac{1}{2}(\sqrt{5} + 1)$ being the golden mean, and the mean vertex density of the tiling amounts to $\varrho_V = \tau\sqrt{\tau + 2} = \tan(2\pi/5)$. In the projection formalism one starts with a lattice $\mathbb{Z}^5$. The acceptance domain is given by the projection of the unit cell onto $\mathbb{E}'$, a rhombic icosahedron (Fig. 4, top). The orthogonal space contains a lattice subspace U, here spanned by the main diagonal $\boldsymbol{\delta} = \sum_i \boldsymbol{\varepsilon}_i$. Therefore L′ consists of equally spaced planes perpendicular to $\Delta = \mathbb{R} \cdot \boldsymbol{\delta}$ of separation $\|\boldsymbol{\delta}\|/5$. For the original Penrose pattern (Penrose-LI-class, Fig. 3) the acceptance domain contains four layers of pentagonal shape and two points of L′ (Fig. 4). There are five layers if the acceptance domain is shifted by γ and $\gamma \cdot \boldsymbol{\delta} \notin \mathbb{Z}$, leading to a generalized Penrose pattern.

3. Rows in Quasilattices

The natural generalization of lattice directions are the quasilattice directions spanned by vectors $\boldsymbol{d}$, which are integral linear combinations of the projected basis vectors of the lattice: $\boldsymbol{d} = \sum_{i=0}^{n} n_i \boldsymbol{e}_i$, $n_i \in \mathbb{Z}$. A straight line along a quasilattice direction is called a *row* if one or more vertex points fall on it. A set of parallel rows such that every vertex point of the

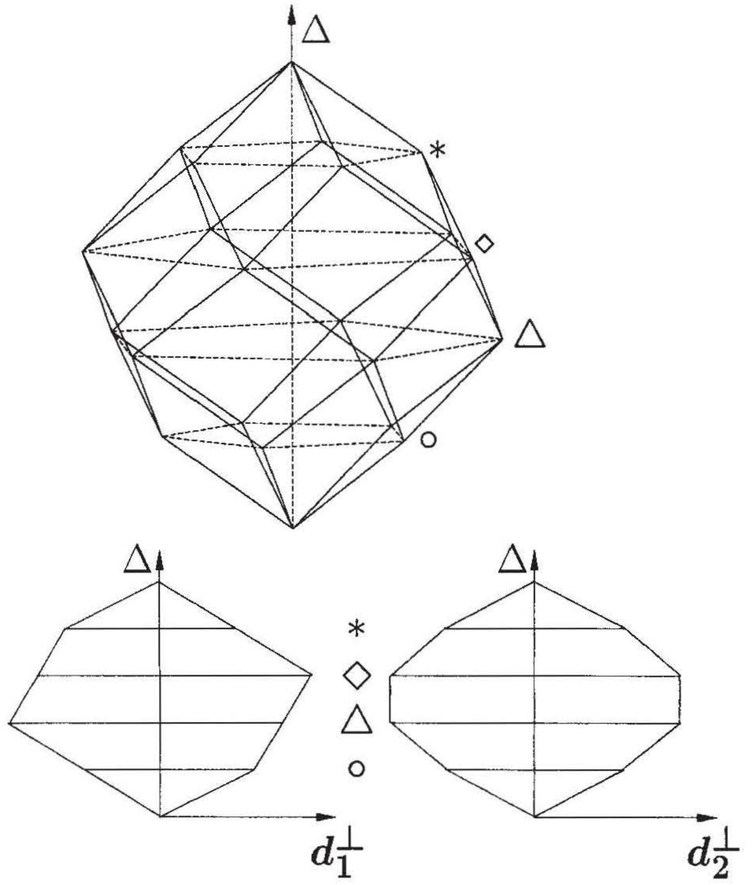

Fig. 4. The acceptance domain K of the Penrose tiling is a rhombic icosahedron containing for layers of the projected points L′. Two sections through $K \cap \bar{L}'$ ($\bar{L}'$ is the closure of L′) are shown, one parallel (left side, $\boldsymbol{d}_1^\perp = \boldsymbol{e}_i^\perp$) and the other perpendicular (right side, $\boldsymbol{d}_2^\perp = \boldsymbol{e}_{i+2}^\perp - \boldsymbol{e}_{i+3}^\perp$) to one of the basis vectors $\boldsymbol{e}_i^\perp$. The main diagonal Δ gives the second direction of the cutting plane. The different layers are labeled with ○, △, ◇, * in the same way as in Fig. 3 and 6. On the left side all four line segments of the section with $K \cap \bar{L}'$ contain points of L′ (case (a)), whereas this is true for only one segment on the right side (case (b))

infinite tiling falls on one of them will henceforth simply be referred to as a *family* of rows. The direction $\boldsymbol{d} = \sum_{i=0}^{n} n_i \boldsymbol{e}_i$ of the rows is labelled by its "Miller indices" $(n_1 n_2 \ldots n_n)$. Notice that different indices can belong to the same direction.

We will now investigate the properties of such families and for this purpose we extend a method developed by Katz and Duneau [11] for icosahedral quasilattices. It is based on the strip method.

A unique point $\boldsymbol{x}'$ in the acceptance domain belongs to each vertex point $\boldsymbol{x}$ of the tiling T via the natural isomorphism: $\varphi(\boldsymbol{x}) = \boldsymbol{x}' \in K \cap L'$. To an arbitrary quasilattice direction $\boldsymbol{d}$ corresponds a unique direction $\boldsymbol{d}^\perp \in \boldsymbol{E}^\perp$. If there is a nontrivial lattice subspace U of dimension u however, those points in $K \cap L'$ which are related to the vertex points on a line $D = \mathbb{R} \cdot \boldsymbol{d} \in \mathbb{E}$, i.e. the points $\boldsymbol{x} \in D \cap T$, may lie in different planes of L′. In this case it is a $(u + 1)$-dimensional volume $D^\perp \oplus U$ in the orthogonal space which belongs to the row $D \cap T$: $\boldsymbol{x} \in D \cap T \xrightarrow{\varphi} \boldsymbol{x}' \in (D^\perp \oplus U) \cap (K \cap L')$. Not every section of $D^\perp \oplus U$ with $K \cap \bar{L}'$ ($\bar{L}'$ denotes the closure of L′) necessarily contains points of L′, depending on the direction under consideration, a fact which complicates the analysis of tilings obtained by projection with nonvanishing U. The number of vertex points on a given row clearly is equal to the number of points in $(D^\perp \oplus U) \cap (K \cap L')$, and because the nonempty sections are filled homogeneously, this number is proportional to their lengths.

A quasilattice direction D and the corresponding volume $D^\perp \oplus U$ span a $(u + 2)$-dimensional *lattice* space $V = D^\perp \oplus U \oplus D$ with $V \cap \mathbb{E}' = D^\perp \oplus U$ and $V \cap \mathbb{E} = D$. (This is not necessarily the case for an arbitrary (non-quasilattice) direction D.) The row

$D \cap T$ can therefore be derived by a cut and projection method in V with lattice $\Lambda_V = V \cap \Lambda$ and strip $S_V = V \cap S$ where D is the tiling space, and the acceptance domain is given by $(D^\perp \oplus U) \cap (K \cap L')$. So $D \cap T$ can be investigated by the same methods as used for the investigation of the tiling itself. Furthermore the lattice points of Λ_V are all situated in two-dimensional planes W_i perpendicular to U, whose number equals the number of layers of $K \cap L'$. This fact reduces the analysis to one or more two-dimensional problems. It allows us to determine the exact positions of vertex points on a given line D as well as the different intervals and their probability of occurrence. Also the mean separation $\bar{d}$ of strings can be calculated and leads directly to the mean string density $\bar{\varrho} = \varrho_V \cdot \bar{d}$ through the vertex density ϱ_V. $\bar{\varrho}$ is the mean number of vertex points per unit length on the rows of a family.

From one row to the next only the lengths of the sections $(D^\perp \oplus U) \cap (K \cap L')$ do change, so that the variation of the string density over a family of rows is completely determined by the shape of K. On the contrary, the differences in the mean string density from one direction to another are mainly caused by the different unit cells of Λ_V.

The projection of the pattern T onto a row D, $\pi_D(T)$, is also calculated by a strip-projection technique, as it correspond to the projection of all points of $K \cap L'$ onto $D^\perp \oplus U$. So to every point of $\pi_D(T)$ corresponds a point of $\pi_{(D_\perp \oplus U)}(K \cap L')$, and the pattern is obtained by a strip method in the volume $V = D^\perp \oplus U \oplus D$ with lattice $\Lambda = \pi_V(\Lambda)$ and strip $S_V = \pi_V(S)$. Clearly $D \cap T$ is a subpattern of $\pi_D(T)$.

The separations between the points of $\pi_D(T)$ are those between the rows of a family perpendicular to D. The acceptance domain $\pi_{(D_\perp \oplus U)}(K \cap L')$ is bounded for any quasilattice direction and therefore leads to a discrete pattern. Hence the members of a family always have discrete separations, justifying the notion of rows as generalisation of lattice lines.

We now treat the two cases of interest separately and give results on the vertex density of rows, its variation with the members of the family, the separations between these members and their respective probability of occurrence.

3.1 Rows in the octagonal tiling

As the projected points L' fill the orthogonal space $\mathbb{E}'$ densely, a unique line $D^\perp = D'$ corresponds to each quasilattice line D in the octagonal tiling, and the pattern $D \cap T$ ($\pi_D(T)$) can be obtained by a strip method in the plane $W = D \oplus D'$ with lattice $W \cap \mathbb{Z}^4$ ($\pi_W(\mathbb{Z}^4)$) and strip $W \cap S$ ($\pi_W(S)$), as outlined in Fig. 2. The occupation density ϱ on a given row D is given by the section of $D' \cap K$ and the area of the unit cell F_U of $W \cap \mathbb{Z}^4$: $\varrho = |D' \cap K|/F_U$. As F_U is equal for all rows of a family, the shape of K completely determines the possible values of ϱ and the distribution over the members of a family can be derived by inspecting Fig. 5: Part a) shows the acceptance domain K with two possible sections $D' \cap K$ denoted A and B. The lengths n of all possible sections for different (but still parallel) lines D' are shown in part b). As the sections of all D' with $\tilde{x}'$ are densely and homogeneously distributed on $\tilde{x}'$, the number of lines D' with density n between n_0 and $n_0 + \Delta n$ is proportional to the intervals $\Delta\tilde{x}'_1$ and $\Delta\tilde{x}'_2$. In the limit $\Delta n \to 0$ the probability $p(n)\,dn$ of finding a row with occupation densities between n and $n + dn$ is given by the inverse of $|dn/d\tilde{x}'|$, drawn in part c) of the figure. To get the occupation densities ϱ in the quasilattice one has to divide the values n by the corresponding surface F_U of the unit cell.

The same distribution is shown for the [1$\bar{1}$00] and the [2100] directions in part d) and e).

Due to the simple geometric shape of K, the probabilities are always constant on intervals, and there is always a finite maximum occupation density ϱ_{max} (in some directions occurring

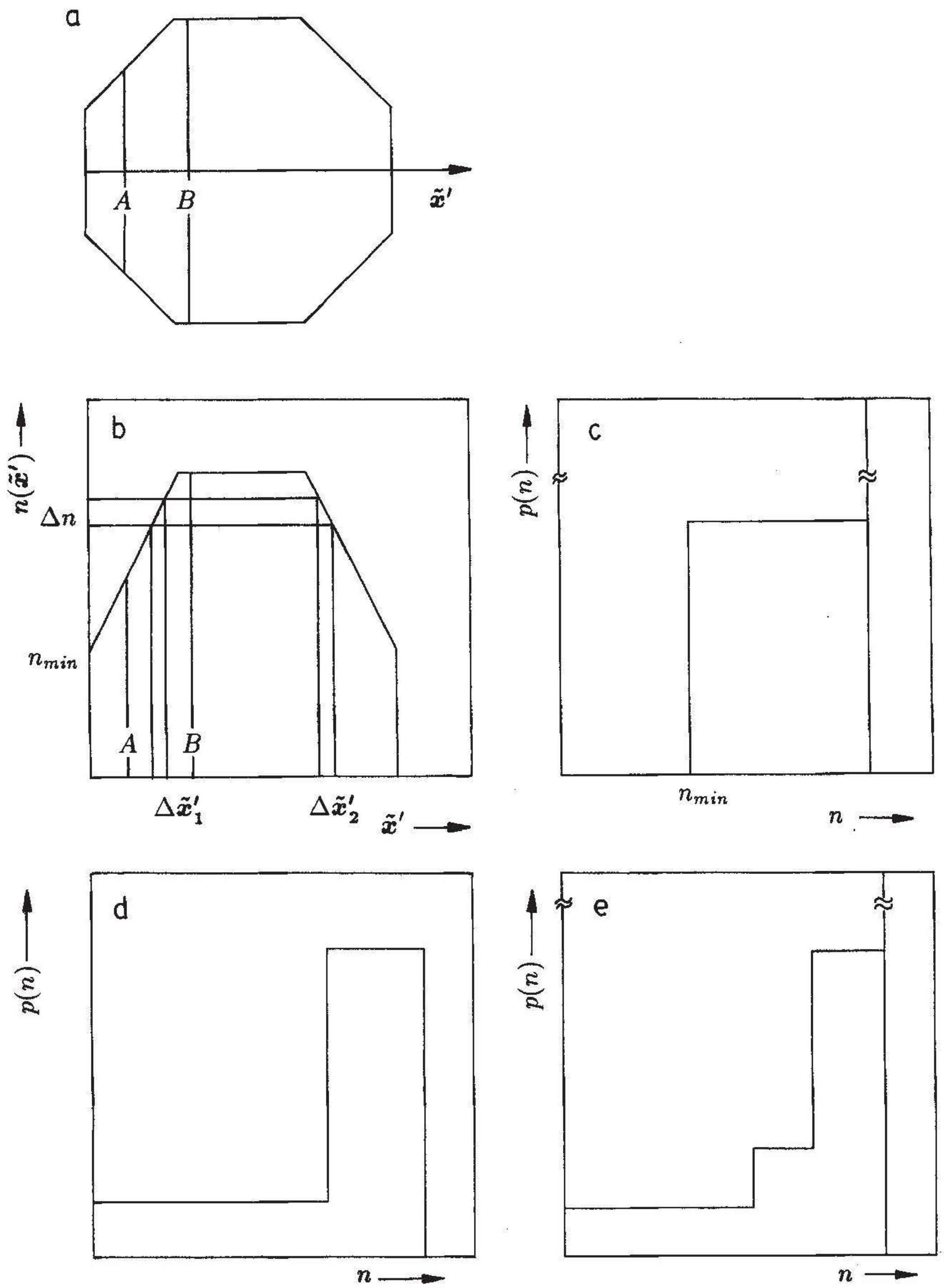

Fig. 5. Probability distribution of the occupation densities in octagonal tilings. The height of the acceptance domain K (part a)) over the line $\tilde{x}'$ is displayed in part b). $\tilde{x}'$ is perpendicular to the line $D' = \varphi(D)$, D being the direction under investigation, here the [1000] direction. From this picture the probability distribution $p(n)$ can be derived, part (c), see text for further explanations. $\int_n^{n+\Delta n} p(n')\,dn'$ is the probability to find an occupation density in the interval $(n, n + \Delta n)$. A minimal occupation density $n_{\min} > 0$ exists in this particular direction, and the maximal density occurs infinitely many times. On the bottom (parts d), e)) the probability distribution in [1$\bar{1}$00] and [2100] direction are being displayed

just once, in others infinitely often). With the exception of the [1000] direction the densities are densely distributed between 0 and ϱ_{max}, each density occurring at most twice. The distribution $p(n) = p(F_U \varrho)$ yields the mean vertex density $\bar{\varrho}$, which can be calculated independently from the mean separation $\bar{d}$ of the rows via the equation $\bar{\varrho} = \varrho_V \bar{d}$.

The distances between the rows are being read off the projected pattern $\pi_D(T)$, with D perpendicular to the rows. But two mutually perpendicular directions are equivalent due to the eightfold symmetry. D has always irrational slope with respect to the projected lattice $\pi_W(\mathbb{Z}^4)$, and therefore the sequence of separations always is quasiperiodic. As the acceptance domain K_W is bounded the separations have a nonzero value.

Table 1
Row separations in the octagonal tiling for the [1000], [1$\bar{1}$00], and [2100] direction

	1000	1$\bar{1}$00	2100
$a =$	$\frac{1}{2}\sqrt{2}n = 0.5$	$(\sqrt{2}-1)\,n = 0.38268$	$\frac{1}{2}\sqrt{2}n = 0.17870$
$b =$	$\frac{1}{2}(2-\sqrt{2})\,n = 0.20711$	$\frac{1}{2}(2-\sqrt{2})\,n = 0.27059$	$(\sqrt{2}-1)\,n = 0.10468$
$c =$	–	$\frac{1}{2}(3\sqrt{2}-4)\,n = 0.11209$	$\frac{1}{2}(2-\sqrt{2})\,n = 0.07402$
$p(a) =$	$\frac{1}{2}\sqrt{2} = 0.70711$	$\frac{1}{2}(\sqrt{2}-1) = 0.20711$	$\frac{1}{14}(5\sqrt{2}-6) = 0.07651$
$p(b) =$	$\frac{1}{2}(2-\sqrt{2}) = 0.2929$	$\frac{1}{4}(4-\sqrt{2}) = 0.64645$	$\frac{1}{7}(4-\sqrt{2}) = 0.36940$
$p(c) =$	–	$\frac{1}{4}(2-\sqrt{2}) = 0.14645$	$\frac{3}{14}(4-\sqrt{2}) = 0.55409$
$\bar{d}$	$\sqrt{2}-1 = 0.41421$	$\frac{1}{2}\frac{2-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}$	$\frac{2\sqrt{2}-1}{7\sqrt{17}}\sqrt{5-2\sqrt{2}} = 0.09336$
$\lVert \boldsymbol{d} \rVert$	$\sqrt{2}$	$\sqrt{4-2\sqrt{2}}$	$\sqrt{10+4\sqrt{2}}$
$\boldsymbol{g}_1$	1000	1$\bar{1}$00	2100
$\boldsymbol{g}_2$	010$\bar{1}$	0011	121$\bar{2}$
F_U	$\sqrt{2}$	2	$\sqrt{34}$
$\sqrt{2}\,\lvert\pi_{D'}(K)\rvert$	$1+\sqrt{2}$	$\sqrt{4+2\sqrt{2}}$	$\frac{1+2\sqrt{2}}{\sqrt{5-2\sqrt{2}}}$
$\bar{\varrho}$	1	$\frac{1}{4}\sqrt{4+2\sqrt{2}}$	$\frac{\sqrt{2}(1+\sqrt{2})\sqrt{5-2\sqrt{2}}}{(1+2\sqrt{2})\sqrt{34}}$

The values are given for a tiling obtained through projection from the hypercubic lattice $\mathbb{Z}^4$. So the length of the edges of the tiles is $\sqrt{1/2}$. $n = \lVert \boldsymbol{d} \rVert$ denotes the norm of the vector $\boldsymbol{d}$ spanning the line under consideration. a, b, c label the different separations, $p(a)$ etc. their respective probabilities. The bottom part of the table contains a compilation of data for the investigated directions. $\bar{d}$ is the mean distance between the rows of a family, $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ span the lattice $\pi_W(\mathbb{Z}^4)$ in the plane $W = D \oplus D'$. F_U is the area of the surface of the unit cell of $\pi_W(\mathbb{Z}^4)$, $F_K = 1 + \sqrt{2}$ that of the acceptance domain K. The length of the projection of K onto D′ is $\lvert\pi_{D'}(K)\rvert$. The mean vertex density $\bar{\varrho}$ on the rows of a family is given by $\bar{\varrho} = F_K/(\lvert\pi_{D'}(K)\rvert\, F_U)$ and related to the mean distances $\bar{d}$ between the rows via $\bar{\varrho} = \varrho_V \bar{d}$, where the mean vertex density ϱ_V of the tiling is given by $\varrho_V = 1 + \sqrt{2}$.

Of particular interest are the distances between the rows in the [1000] direction: There are only two of them and they build a so-called "Octonacci"-sequence, given by the substitution law A → AAB and B → A. In the other directions we always find three different distances, the length of the largest being the sum of the length of the other two. The values of the separations and their probabilities, calculated with the polar calculus as outlined in Section 2.1, are listed in Table 1 together with the mean separations $\bar{d}$.

3.2 *Rows in decagonal quasilattices*

Due to the nontrivial subspace U = Δ spanned by the main diagonal $\boldsymbol{\delta}$ of the cubic unit cell the analysis is here somewhat more complicated than in the octagonal case. We only investigated the original Penrose tiling, with $\gamma \cdot \boldsymbol{\delta} \in \mathbb{Z}$. In this case $K \cap L'$ consists of four layers with pentagonal shape.

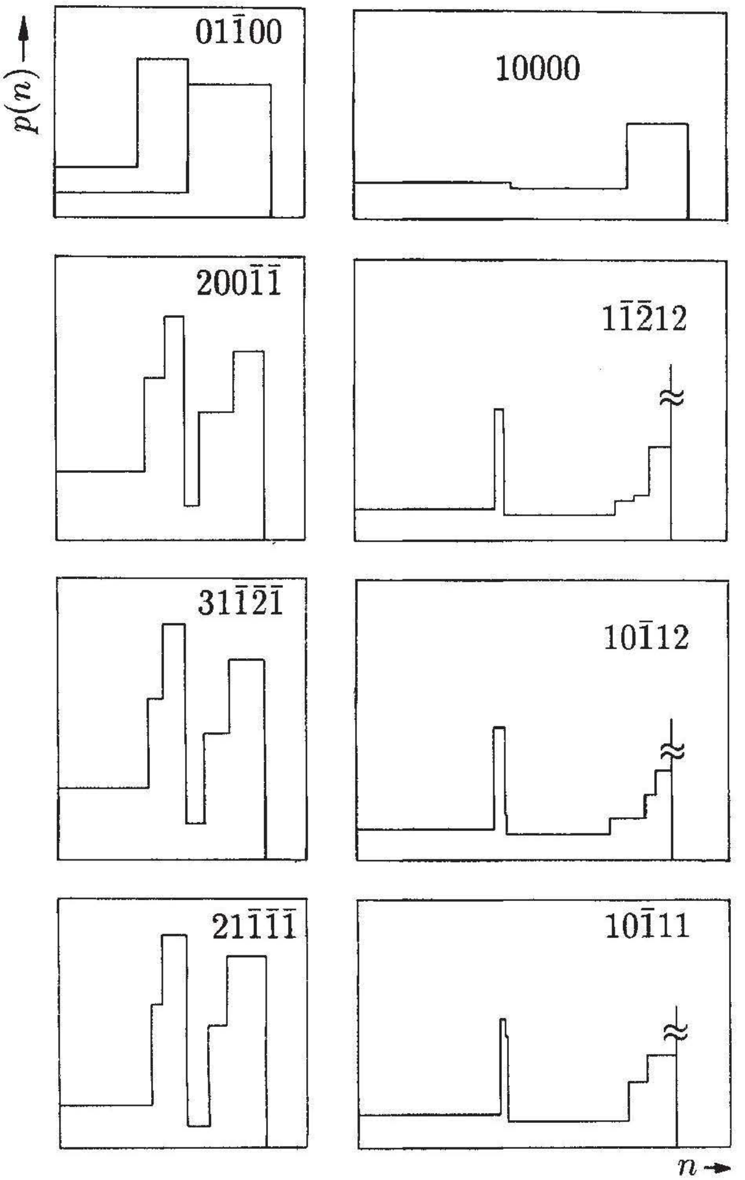

Fig. 6. Probability distribution of the occupation density for some families of rows in a Penrose pattern. The direction of the family is indicated by its Miller indices. This distribution can be derived analogously to octagonal quasilattices

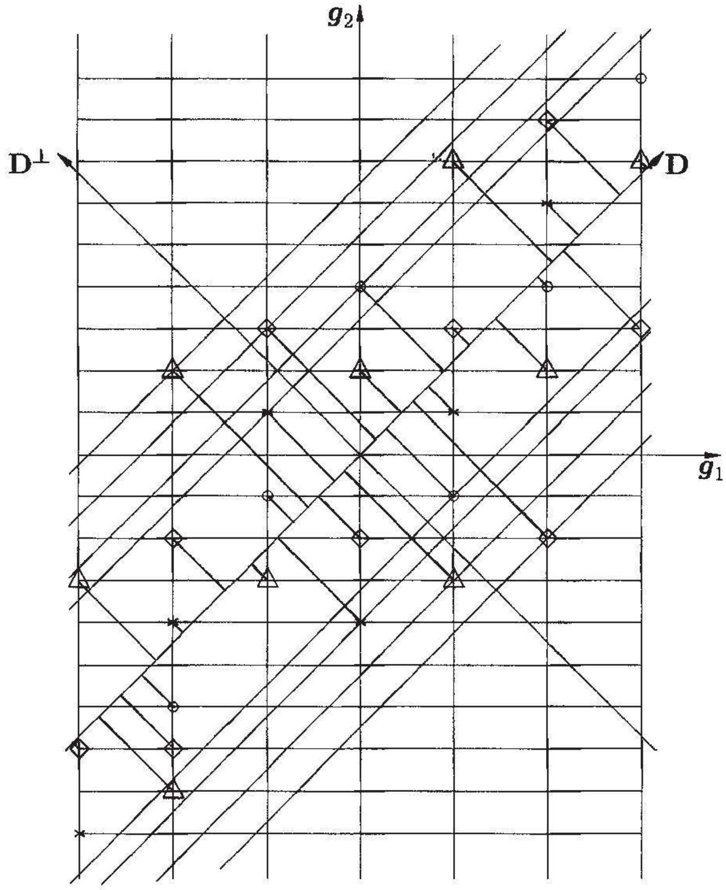

Fig. 7. Investigation of separations between rows in a Penrose tiling. The four planes W_i in the volume V spanned by D, $D^{\perp}$, and Δ are shown (rows perpendicular e_0, i.e. D $\parallel$ e_0). The lattice points in the planes W_i which fall into the strips $\pi_V(S) \cap W_i$ are marked with $\circ$, $\triangle$, $\diamond$, and $*$, according to the number i of the plane W_i. The width of the strip $\pi_V(S) \cap W_i$ is the length of the line segment of Fig. 4. Notice that not every intersection of the lattice lines corresponds to lattice points of $\pi_V(\mathbb{Z}^5)$! Fig. 4, bottom left, shows a section through the planes W_i parallel to Δ. The layers with index 0 and 5 consists of one point only. The planes W_i are spanned by $g_1 = (1111\bar{4})$ and $g_2 = (1\bar{1}\bar{1}10)$

The points in the sections $(D^{\perp} \oplus \Delta) \cap (K \cap L')$ correspond to the points of a row $D \cap T$. There are two different cases, depending on the direction: Either all sections of $(D^{\perp} \oplus \Delta) \cap (K \cap \bar{L}')$ contain points of L′ (case (a)) or only one (case (b)). In the latter case only points of one layer fall on a given row via φ^{-1}, the next parallel row containing points of another single layer, as can be seen in Fig. 3, where the vertex points are marked according to the layer of L′ where its partners lie (following an idea of Ishihara and Yamamoto [16]). If case (a) holds for a direction D, case (b) is realized for the perpendicular direction, and vice versa. Apart from that fact the determination of occupation densities and probability distributions for a family is analogous to the octagonal case, the distribution of the probabilities of the vertex densities is shown in Fig. 6. Case (a) holds for the directions of which the distributions are displayed on the left, the perpendicular directions are shown on the right. As in the octagonal case every density from zero to a maximum occurs, but the probability distribution is more complicated due to the four layers of $K \cap L'$.

The same two cases (a) and (b) occur in the analysis of the projected pattern $\pi_D(T)$. The lattice points in the lattice $\Lambda_V = \pi_V(\mathbb{Z}^5)$ fall into four lattices $\Lambda_i = \Lambda_V \cap W_i$. The projection of these four lattices onto one plane W parallel W_i is shown in Fig. 7 for the $[01\bar{1}00]$ direction, for which case (a) holds. The calculation of the separations and of their probabilities with the polar calculus can be very time consuming, but the mean separation $\bar{d}$ can be derived directly from the lengths of the layers of the acceptance domain in the planes W_i and the surface of the unit cells of Λ_i. In case (b) one obtain identical lattices in all four planes W_i and the calculations are relatively simple. The mean separation $\bar{d}$ is given in Table 2 for the directions investigated. The values of $\bar{d}$ reflect the symmetry of the tiling,

Table 2
Mean separations $\bar{d}$ between rows in Penrose tilings

angle	Miller-indices	mean distance	relative occupation density (%)
0.000°	00001	0.256872	100
4.386°	$200\bar{1}\bar{1}$	0.054078	21
6.645°	$1\bar{1}\bar{2}12$	0.044134	17
8.268°	$21\bar{1}\bar{1}\bar{1}$	0.053674	21
9.732°	$10\bar{1}12$	0.073876	29
11.355°	$31\bar{1}\bar{2}\bar{1}$	0.032065	12
13.614°	$10\bar{1}11$	0.074433	29
18.000°	$01\bar{1}00$	0.186628	73

$\bar{d}$ is proportional to the mean occupation density $\bar{\varrho}$ on the rows, which is given in percent for the directions investigated. The absolute values are presented for a tiling projected from the lattice $\mathbb{Z}^5$. The angles listed below are those between the quasilattice line D and the bases vector e_0. Due to the symmetries $\bar{d}(36^\circ + \varphi) = \bar{d}(\varphi)$ and $\bar{d}(18^\circ - \varphi) = \bar{d}(18^\circ + \varphi)$ (compare Fig. 9) we list only values in the range from 0° to 18°.

Table 3
Distances between rows in the [00001] and $[01\bar{1}00]$ direction

distances	probabilities
[00001] direction $\mathrel{\hat=}$ 0°	
$a = \frac{1}{20}\sqrt{5+\sqrt{5}}\,(5-\sqrt{5}) = 0.37176$	$p(a) = \frac{1}{4}(\sqrt{5}-1) = 0.30902$
$b = \frac{1}{20}\sqrt{5+\sqrt{5}}\,(3\sqrt{5}-5) = 0.22975$	$p(b) = \frac{1}{2} = 0.5$
$c = \frac{1}{10}\sqrt{5+\sqrt{5}}\,(5-2\sqrt{5}) = 0.14200$	$p(c) = \frac{1}{4}(3-\sqrt{5}) = 0.19098$
$\bar{d} = \frac{5}{8}(3-\sqrt{5})/\lVert \boldsymbol{d} \rVert = \frac{1}{8}\sqrt{5+\sqrt{5}}\,(3-\sqrt{5}) = 0.256872$	
$[01\bar{1}00]$ direction $\mathrel{\hat=}$ 18°	
$a = \sqrt{\frac{2}{5}}\,\frac{1}{2} = 0.31623$	$p(a) = \sqrt{5} - 2 = 0.23607$
$b = \sqrt{\frac{2}{5}}\,\frac{1}{4}(\sqrt{5}-1) = 0.19544$	$p(b) = \frac{1}{4}(17 - 7\sqrt{5}) = 0.33688$
$c = \sqrt{\frac{2}{5}}\,\frac{1}{4}(3-\sqrt{5}) = 0.12079$	$p(c) = \frac{1}{4}(\sqrt{5}-1) = 0.30902$
$\sqrt{\frac{2}{5}}\,\frac{1}{4}(2\sqrt{5}-4) = 0.07465$	$p(d) = \frac{1}{2}(\sqrt{5}-2) = 0.11803$
$\bar{d} = \frac{5}{4}(\sqrt{5}-2)/\lVert \boldsymbol{d} \rVert = \sqrt{\frac{2}{5}}\,\frac{5}{4}(\sqrt{5}-2) = 0.186628$	

The separations are labelled with a, b, c, and d, their respective probabilities with $p(a)$ etc. $\boldsymbol{d}$ is a vector spanning the projection line D, $\lVert \boldsymbol{d} \rVert$ denotes its norm. The absolute values are valid for a tiling obtained through projection from the hypercubic lattice $\mathbb{Z}^5$, the length of the edges of the tiles is therefore $\sqrt{2/5}$. The sequence of distances is displayed graphically in Fig. 8

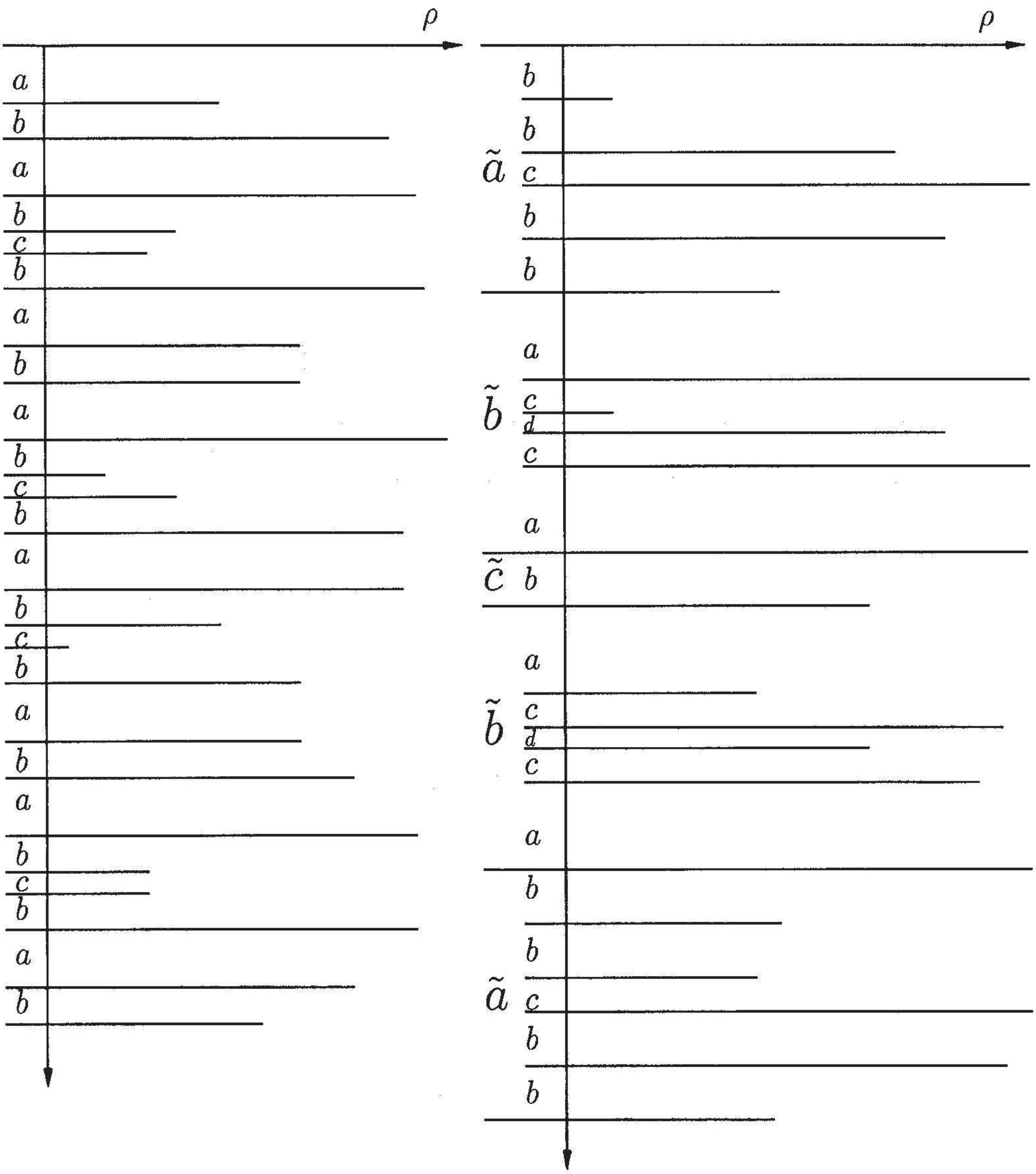

Fig. 8. Separations between the rows of the family parallel (left part) and perpendicular (right part) to one of the basis vectors e_i. Three different distances a, b, and c occur in the case shown on the left-hand side, every second distance is b, a, and c appear according to a Fibonacci sequence. After regrouping the distances as shown on the right-hand side, the occurrence of these groups is governed by the same building law: every second sequence is $\hat{b} = acdca$ and $\tilde{a} = bbcbb$, and $\tilde{c} = b$ follow a Fibonacci sequence. The length of the horizontal line segments is proportional to the occupation density ϱ of the row. The values of the separations are given in Table 3

that is $\bar{d}(\varphi) = \bar{d}(36° + \varphi)$. But for the original Penrose pattern (i.e. $\gamma \cdot \boldsymbol{\delta} \in \mathbb{Z}$) a further symmetry exists, resulting from the symmetry of the layers of the acceptance domain: $\bar{d}(18° + \varphi) = \bar{d}(18° - \varphi)$. This symmetry has been observed experimentally by Carstanjen et al. [13] and justifies our restriction to this special LI-class. The values for the separations

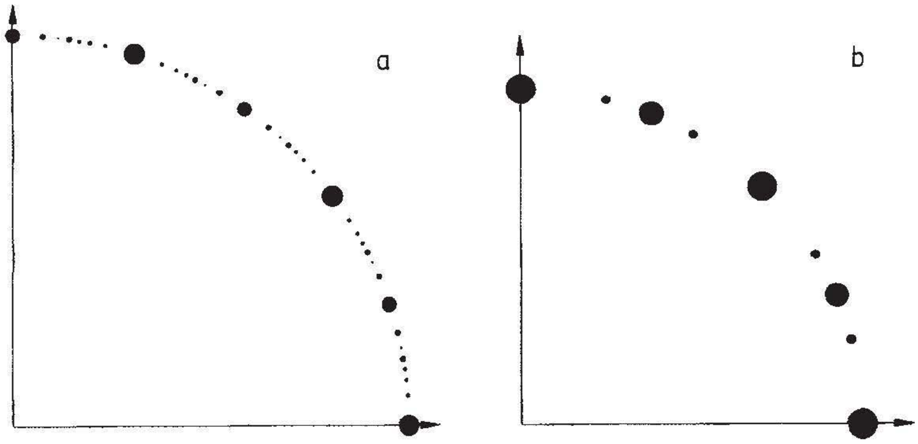

Fig. 9. Directional dependence of the mean vertex density on a family of rows in a) a pentagonal Penrose pattern and b) an octagonal quasilattice. The radius of the points is proportional to the density and their location on the unit circle indicates the direction of the corresponding family. Only a 90° section of the full circle is being displayed. See also Table 2

itself are listed only for the two most symmetric directions in Table 3. In [10000] direction the separations a and c are arranged according to a Fibonacci sequence, every second separation is b (Fig. 8). The sequence in $[01\bar{1}00]$ direction can be related to the same sequence as above by regrouping the separations as indicated in the figure. In Fig. 9 the directional dependence of the mean vertex density on a family of rows in a pentagonal Penrose pattern and an octagonal quasilattice is shown.

4. Conclusions

We have generalized a method of Katz and Duneau [11] developed for the investigation of rows in icosahedral quasilattices to quasilattices having several layers in the acceptance domain.

By this method we have investigated the properties of rows in octagonal and pentagonal quasilattices. Such rows are well-defined. We have calculated the separations between parallel rows, the vertex pattern, and the occupation density of individual rows as well as the mean occupation density of a whole family for some highly symmetric directions. Especially the latter should be of great help for the interpretation of channeling experiments with decagonal quasicrystals, as the Lindhard theory [17] predicts good channeling properties for rows with high occupation density.

For a direct comparison with experimental results one should investigate more realistic models of quasicrystals such as that by Steurer and Kuo [18] or Burkov [19]. They can also be investigated by our method, because the atomic hypersurfaces of different atoms (Al and Cu/Co in this model) correspond to the different layers of $L' \cap K$ in the cut and projection method. The two cases discussed in the Penrose pattern have interesting consequences: Case (a) leads to mixed strings, that is strings containing both Al and Cu/Co, whereas case (b) gives rise to strings containing only one sort of atoms.

Our method can also be applied to dodecagonal quasilattices, but due to the existence of at least twelve layers in the acceptance domain the calculations are rather tedious.

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