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Research Note

Analytical Determination of the Group Velocity of an Arbitrary Lamb Wave from its Phase Velocity

Analytische Bestimmung der Gruppengeschwindigkeit einer beliebigen Lamb-Welle aus deren Phasengeschwindigkeit

Détermination analytique de la vitesse de groupe d'une onde de Lamb quelconque à partir de sa vitesse de phase

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Dedicated to my teacher Professor Dr. rer. nat. Ekkehart Kröner

The phase velocities c of Lamb waves in an isotropic free plate of thickness h obey the Rayleigh-Lamb frequency equations [1, 2]

$$\frac{4\alpha_1\alpha_2}{(1+\alpha_2^2)^2} = \left[\frac{\tanh(\pi\alpha_2 f/c)}{\tanh(\pi\alpha_1 f/c)}\right]^{\pm 1}, \tag{1}$$

$$\alpha_1 = \sqrt{1-(c/c_l)^2}, \quad \alpha_2 = \sqrt{1-c^2},$$

where the upper sign applies for symmetric modes and the lower sign for antisymmetric modes (c_l: velocity of longitudinal waves in the corresponding unbounded medium; f: frequency. As in [2] normalized quantities are used throughout this research note: All velocities are in units of c_t, the velocity of transversal waves in the unbounded medium; the frequency unit is equal to c_t/h).

The group velocity C of a wave can be computed from the phase velocity and the frequency derivative of the phase velocity:

$$C = \frac{c}{1-\dfrac{f}{c}\dfrac{\mathrm{d}c}{\mathrm{d}f}}, \tag{2}$$

a relation which follows from the kinematic definition of group velocity. It was argued in [2] that the derivative $\mathrm{d}c/\mathrm{d}f$ may be obtained analytically by means of an implicit differentiation of (1) with respect to f. This is indeed the case and the result is reported below. The analytical calculation is not too cumbersome unless one fails to introduce convenient abbreviations.

The result can be written in the form

$$\frac{f\,\mathrm{d}c}{c\,\mathrm{d}f} = \frac{\pm Y}{X_\pm \pm Z}, \tag{3}$$

where again the plus signs refer to the symmetric modes and the minus signs to the antisymmetric ones. The quantities on the right-hand side of eq. (3) are functions of f, c, and $c_l^2 = 2(1-\sigma)/(1-2\sigma)$, where σ denotes Poisson's ratio:

$$X_+ = T_1^2 X, \quad X_- = T_2^2 X,$$

$$T_1 = \tanh(\pi\alpha_1 f/c), \quad T_2 = \tanh(\pi\alpha_2 f/c),$$

$$X = \frac{4c^3}{\pi f A^2}\left[\frac{4\alpha_1^2\alpha_2^2}{A} - \alpha_1^2 - \left(\frac{\alpha_2}{c_l}\right)^2\right],$$

$$A = 1+\alpha_2^2 = 2-c^2;$$

$$Y = \alpha_1 T_1 \alpha_2^2 K_2 - \alpha_2 T_2 \alpha_1^2 K_1,$$

$$Z = \alpha_1 T_1 K_2 - \alpha_2 T_2 K_1,$$

$$K_1 = \cosh^{-2}(\pi\alpha_1 f/c),$$

$$K_2 = \cosh^{-2}(\pi\alpha_2 f/c).$$

If f, c and σ are real quantities, then $X_\pm$, Y and Z are also real, because they consist of sums of real terms. Both α_1 and α_2 may become imaginary leading to imaginary T_1 and T_2.

The analytical result has been checked against the numerical calculations for the quasi-longitudinal and the bending mode, which are shown in Figs. 5 and 11 of [2]. Since both the dispersion relation (1) and the relation (2) are true for arbitrary Lamb waves and no additional assumptions have been made during the derivation of (3), this equation is not only valid for the two fundamental modes but also for all other Lamb waves.

As pointed out in [2], the group velocity is useful for calculating the average intensity of a plate wave from the space-time average of the kinetic energy density. The phase velocity of a Lamb wave has to be calculated numerically still, but the numerical differentiation which was necessary so far in order to obtain the group velocity from (2) can now be avoided.

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