H_∞-Control of Differential-Algebraic-Equation Systems

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Abstract

In this paper H_∞ control of high index and non-regular linear differential-algebraic-equation systems is addressed. Based on a generalization of the bounded real lemma (BRL) to index one systems, all linear output feedback controllers in standard, i.e. non-descriptor, state space form solving the H_∞ control problem can be characterized via biaffine matrix inequalities (BMIs). In a second step a congruence transformation and a subsequent change of variables show that certain linear matrix inequalities (LMIs) necessarily must hold in order to admit a solution of the H_∞ control problem. However, these conditions are not sufficient. Necessary and sufficient conditions for the existence of a controller solving the H_∞ control problem are derived as BMIs of reduced order compared to the original characterization via the BRL. The approach is illustrated by a simple example.

Keywords: State-space H_∞-control; Differential-algebraic systems; Descriptor systems; Linear matrix inequalities; Biaffine matrix inequalities; Bounded Real Lemma; Linearizing change of variables

1 Introduction

Differential-algebraic-equation (DAE) systems (sometimes also referred to as descriptor, singular or semistate systems) describe a broad class of systems which are not only of theoretical interest but also have great practical significance. Models of chemical processes, for example, typically consist of differential equations describing the dynamic balances of mass and energy while additional algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. [23, 18]. In mechanics DAE system descriptions, that are typically of index less or equal to three, result from holonomic and non-holonomic constraints [27]. Also in electronics and even in economics DAE descriptions are frequently encountered [19].

DAE systems are able to describe a system behaviour that cannot be captured by “non-descriptor” systems (i.e. systems governed only by differential equations) [33]. Therefore index reduction techniques (i.e. reduction of a DAEs to an ODEs [23]) necessarily are connected to a loss of information. Due to this fact in recent years much work has been focused on analysis and design techniques for DAE systems (see [3, 4, 5]). For linear systems many of the standard design techniques for non-descriptor systems have been extended to DAE systems. Based on a generalization of J-spectral factorization [14] also H_∞ controller design for DAE systems was established recently [31]. However the approach in [31] is restricted to the so called DGKF assumptions [7]. These assumptions, that are rather restrictive for practical applications, were overcome in [20] by means of a Riccati inequality approach. The controller synthesis in [20] essentially requires the solution of certain linear matrix inequalities. However, the controller is assumed to have the same structure as the plant, i.e. a DAE system. Although a subsequent normalization of the controller is possible this procedure is, from a practical point of view, not desirable since it requires a re-computation of perturbed synthesis LMIs and inversion of the possibly ill-conditioned results of the LMI computations.

In this paper we examine the possibility to directly synthesize H_∞ controllers in non-descriptor form. As indicated above this offers a number of advantages for practical applications, the most important one being a much simplified implementation of the controller.

The paper is structured as follows: Firstly the necessary background on DAE systems, LMIs and H_∞ control is summarized in Section 2. Then, in the main part of the paper, we derive a generalized version of the bounded real lemma (BRL). Based on this analysis result and an adaptation of the “linearizing change of variables” as proposed in [26], a necessary LMI condition for the existence of a non-descriptor suboptimal output feedback controller is stated. In contrast to the corresponding “classical” H_∞ control problem [26] the existence of a solution to this LMI is, however, not sufficient. A necessary and sufficient condition can be given as biaffine matrix inequality. This non-convex problem can be solved via iterative computation of LMI problems [13].

2 Background

2.1 Linear DAE systems

We consider linear, time-invariant DAE systems

$$E\dot{\boldsymbol{\xi}}(t) = A\boldsymbol{\xi}(t) + B\boldsymbol{w}(t), \qquad t \geq 0, \quad \boldsymbol{\xi}(0^-) = \boldsymbol{\xi}_0^- \tag{1}$$
$$\boldsymbol{z}(t) = C\boldsymbol{\xi}(t) + D\boldsymbol{w}(t).$$

with constant system matrices $E, A \in I\!R^{n_\xi \times n_\xi}$, $B \in I\!R^{n_\xi \times n_w}$, $C \in I\!R^{n_z \times n_\xi}$, and $D \in I\!R^{n_z \times n_w}$ and $n_\xi \geq \mathrm{rank}(E) =: r$. $\boldsymbol{\xi}(t) \in I\!R^{n_\xi}$ denotes the descriptor variables, $\boldsymbol{w}(t) \in I\!R^{n_w}$ the input variables, and $\boldsymbol{z}(t) \in I\!R^{n_z}$ the output variables. As a shorthand notation for system (1) we often write (E, A, B, C, D) (or (E, A, B, C) if $D = 0$).

Systems with representation (I, A, B, C, D) are said to be *non-descriptor systems*.

The concept of *system equivalence* will be used in the sequel:

Definition 2.1 *Two systems (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are said to be (system) equivalent, denoted by $(E, A, B, C, D) \sim (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, if there exist nonsingular transformation matrices $L, R \in I\!R^{n_\xi \times n_\xi}$ such that the equations*

$$\tilde{E} = LER \quad \tilde{A} = LAR \quad \tilde{B} = LB \quad \tilde{C} = CR \quad \tilde{D} = D$$

hold true (i.e. the two systems have the same input-output behaviour).

For notational convenience it is assumed in the following, that (1) describes a genuine ($0 < rank(E) < n_\xi$) DAE system. Although we do not require a representation of (1) with $\boldsymbol{\xi}$(t) split up into dynamic and algebraic variables, it is sometimes useful to transform (1) into a system equivalent *normalized SVD representation* [16], i.e. a representation with

$$\tilde{E} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{2}$$

In contrast to non-descriptor linear systems, (1) may have no solution, one solution or even multiple solutions for the same initial condition and input [21]. The solutions in general exhibit impulsive behaviour (i.e. are *generalized* solutions [6]) even if the input $\boldsymbol{w}(\cdot)$ is continuous [5]. For our purposes it will be necessary to characterize these properties in some detail:

Definition 2.2 *The system (E, A, B, C, D) and the associated matrix pencil $sE - A$ are said to be* regular *if the characteristic polynomial $p(s) := \det(sE - A)$ does not vanish identically in s. Otherwise it is called* singular.

Obviously regularity is invariant under system equivalence. Furthermore a regular system guarantees a unique solution of (1). On the other hand a singular system (1) always admits multiple solutions for the unforced ($\boldsymbol{w}(\cdot) \equiv \mathbf{0}$) homogeneous initial value problem [21]. Finally for regular systems (1) the transfer matrix

$$G(s) := C(sE - A)^{-1}B + D \tag{3}$$

is defined. The question of impulsive solutions of regular systems is usually studied in terms of the Weierstrass canonical form (WCF) of (E, A, B, C, D):

Theorem 2.1 *[11] Let (E, A, B, C, D) be regular. Then there exists an equivalent system $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \sim (E, A, B, C, D)$ with*

$$\tilde{E} = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\xi - r} \end{bmatrix} \tag{4}$$

where $J \in \mathbb{R}^{(n_\xi - r)\times(n_\xi - r)}$, $N \in \mathbb{R}^{r\times r}$ are matrices in Jordan canonical form and N is nilpotent.

Definition 2.3 *The index of nilpotence ν of N, i.e. $\nu := \min\{q | N^q = 0, q \in \mathbb{N}\}$ is said to be the* index *of the linear DAE system (E, A, B, C, D). Systems with $\nu \geq 2$ are called* high *index DAE systems.*

If (1) is in WCF, i.e.

$$\begin{bmatrix} \dot{\boldsymbol{\xi}}_1(t) \\ N\dot{\boldsymbol{\xi}}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1(t) \\ \boldsymbol{\xi}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \boldsymbol{w}(t), \qquad t \geq 0, \qquad \begin{matrix} \boldsymbol{\xi}_1(0^-) = \boldsymbol{\xi}_{10}^- \\ \boldsymbol{\xi}_2(0^-) = \boldsymbol{\xi}_{20}^- \end{matrix} \tag{5}$$

then the part $\boldsymbol{\xi}_1$ of the descriptor vector $\boldsymbol{\xi}^{\mathrm{T}} = [\boldsymbol{\xi}_1^{\mathrm{T}}, \boldsymbol{\xi}_2^{\mathrm{T}}]$ is governed by an ordinary differential equation while

$$\boldsymbol{\xi}_2(t) = -\sum_{i=0}^{\nu-1} \delta^{(i)}(t) N^{i+1} \boldsymbol{\xi}_{20}^- - \sum_{i=0}^{\nu-1} N^i \tilde{B}_2 \boldsymbol{w}^{(i)}(t), \qquad t \geq 0$$

solves the "algebraic part" in (5) [32] (with $\delta(t)$ the Dirac delta and superscript (i) the ith distributional derivative). We conclude that DAE systems will have no impulsive solutions (for all $\boldsymbol{w}(\cdot) \in L_2[0, \infty)$ and all initial conditions) iff their index is one.
Similar to non-descriptor systems the stability of *regular* DAE systems can be studied by means of the pencil $sE - A$:

Theorem 2.2 *[5] Let (E, A, B, C, D) be regular. The unforced ($\boldsymbol{w}(\cdot) \equiv \mathbf{0}$) system is asymptotically stable if and only if $\sigma(E, A) := \{s | s \in \mathbb{C}, \det(sE - A) = 0\} \subset \mathbb{C}^-$.*

On the other hand *singular* DAE systems are never asymptotically stable since the condition

$$\exists a, b > 0 : \quad \|\boldsymbol{\xi}(t)\| \leq ae^{-bt}\|\boldsymbol{\xi}(0)\| \quad \forall t \geq 0$$

of asymptotic stability is violated for homogeneous initial conditions (singular systems admit a non-trivial solution).

2.2 Matrix inequalities

If F is a quadratic matrix, the inequality $F < 0$ means that F is negative definite. The most important tool we will use later on to manipulate matrix inequalities is the Schur-complement formula [17] which says that the following three conditions are equivalent

$$\begin{aligned} a.) \quad & \begin{bmatrix} A & C \\ C^{\mathrm{T}} & B \end{bmatrix} < 0 \\ b.) \quad & A < 0, \quad B - C^{\mathrm{T}}A^{-1}B < 0 \\ c.) \quad & B < 0, \quad A - CB^{-1}C^{\mathrm{T}} < 0. \end{aligned} \tag{6}$$

Linear matrix inequalities are expressions of the form

$$F(\boldsymbol{x}) = F_0 + \sum_{i=1}^{N} x_i F_i < 0$$

with variables $(x_1, \ldots, x_N) =: \boldsymbol{x} \in I\!R^N$ and constant matrices $F_i = F_i^{\mathrm{T}} \in I\!R^{n\times n}, i = 0, \ldots N$. In control theory one often encounters problems in which the variables are matrices, for example the Lyapunov inequalities $A^{\mathrm{T}}X + XA < 0, X > 0$. It will be notationally more convenient to work directly with these inequalities rather than to transfer them to the form $F(\boldsymbol{x}) < 0$ with $\boldsymbol{x}$ denoting the independent matrix entries of X.
Three LMI-based problems are of central importance in this paper:

- The *feasibility problem*: Decide whether or not the LMI $F(\boldsymbol{x}) < 0$ is solvable and in the affirmative case find a solution.
- *Minimization* of a linear functional $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ under LMI constraints $F(\boldsymbol{x}) < 0$.
- The *generalized eigenvalue problem*: Minimize the maximum generalized eigenvalue of a pair of matrices $A(\boldsymbol{x})$, $B(\boldsymbol{x})$ subject to an LMI $C(\boldsymbol{x}) < 0$, i.e.

$$\text{Minimize } \lambda \text{ subject to } \begin{cases} A(\boldsymbol{x}) < \lambda B(\boldsymbol{x}) \\ B(\boldsymbol{x}) > 0 \\ C(\boldsymbol{x}) < 0. \end{cases} \tag{7}$$

The starting point for an efficient numerical treatment of these problems is the observation that the solution set of an LMI defines a convex set [2].
Due to the equivalence

$$F(\boldsymbol{x}) < 0,\; G(\boldsymbol{x}) < 0,\; \ldots \quad \Longleftrightarrow \quad \begin{bmatrix} F(\boldsymbol{x}) & & 0 \\ & G(\boldsymbol{x}) & \\ 0 & & \ddots \end{bmatrix} < 0 \tag{8}$$

also a collection of matrix inequalities leads to a convex solution set. We only mention that also LMIs $F(\boldsymbol{x}) < 0$ with additional linear restrictions $A\boldsymbol{x} = B$ define a convex set: With $\boldsymbol{x_0}$ denoting a special solution of the linear equation, i.e. $A\boldsymbol{x_0} = B$, and $\{\boldsymbol{x_i}\}$ being a basis of the kernel of A, the problem can be reformulated as $F(\boldsymbol{x_0} + \sum a_i \boldsymbol{x_i}) < 0$. Therefore we end up again with a convex problem (a LMI with new variables a_i).
A unified convex optimization based approach to numerical treatment of these and other problems can be found in [2, 22]. The important point for us is a proven polynomial time complexity of these problems and the existence of effective numerical tools tailored for LMI problems arising from control theoretic problem setups [10, 8].
We also will encounter *biaffine matrix inequalities* (BMIs [13]), i.e. expressions of the form

$$F(\boldsymbol{x}, \boldsymbol{y}) = F_0 + \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} x_i y_j F_{ij} < 0$$

with variables $(x_1, \ldots, x_{N_x}) =: \boldsymbol{x} \in I\!R^{N_x}$, $(y_1, \ldots, y_{N_y}) =: \boldsymbol{y} \in I\!R^{N_y}$, and constant matrices F_{ij} of appropriate dimensions. In general BMIs do not define convex solution sets [13]. However in many cases it is possible to find solutions by iterative solution of LMIs. We will take up this point again in Section 3.3.

2.3 Linear H_∞ control of non-descriptor systems

We consider the state space approach to linear H_∞ control of non-descriptor systems, a generalized plant Σ_I

$$\Sigma_I : \quad \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2\boldsymbol{u}(t) \\ \boldsymbol{z}(t) & = & C_1\boldsymbol{x}(t) + D_{11}\boldsymbol{w}(t) + D_{12}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) & = & C_2\boldsymbol{x}(t) + D_{21}\boldsymbol{w}(t) \end{array} \tag{9}$$

where $\boldsymbol{x}(t) \in I\!R^{n_x}$ denotes the state, $\boldsymbol{u}(t) \in I\!R^{n_u}$ the control input, $\boldsymbol{w} \in I\!R^{n_w}$ the disturbance input, $\boldsymbol{z} \in I\!R^{n_z}$ the external output, and $\boldsymbol{y} \in I\!R^{n_y}$ the measured output. A, B_i, C_i, and D_{ij} are constant matrices of appropriate dimension. Given a real number $\gamma > 0$ the control problem is to determine a dynamic output feedback controller K

$$K : \quad \begin{array}{rcl} \dot{\boldsymbol{\zeta}}(t) & = & A_K\boldsymbol{\zeta}(t) + B_K\boldsymbol{y}(t) \\ \boldsymbol{u}(t) & = & C_K\boldsymbol{\zeta}(t) + D_K\boldsymbol{y}(t) \end{array} \tag{10}$$

of (yet unspecified) order n_ζ (i.e. $\boldsymbol{\zeta} \in I\!R^{n_\zeta}$) such that the the H_∞ norm $\|G_{cl}\|_\infty$ of the closed loop transfer matrix $G_{cl} : \boldsymbol{w} \to \boldsymbol{z}$, $\boldsymbol{w}, \boldsymbol{z} \in H_2$ is strictly smaller than γ. Furthermore the controller has to guarantee internal stability of the closed loop system. H_∞ *optimal* control additionally is concerned with the determination of the smallest γ such that $\|G_{cl}\|_\infty < \gamma$ holds true. This problem setup typically arises from robust control system design [28].

Many solutions to the problem have been presented in recent years ([7, 30, 24, 29, 25, 15, 9] only to mention a few), we will merely sketch the methodically simplest approach via the bounded real lemma (BRL), i.e. via an LMI characterization of asymptotically stable non-descriptor systems with H_∞ norm smaller γ, since it will also be instrumental in our derivation for the DAE - case:

Proposition 2.1 *(Bounded Real Lemma) [1, 25] Consider a non-descriptor system* $(I, A_{cl}, B_{cl}, C_{cl}, D_{cl})$ *and the corresponding transfer matrix* $G_I(s) := C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}$. *Then* A_{cl} *is asymptotically stable, i.e.*$\{s|\det(sI - A_{cl}) = 0\} \subset \mathbb{C}^-$ *and* $\|G_I\|_\infty < \gamma$ *iff there exists a symmetric matrix* X *with*

$$\begin{bmatrix} A_{cl}^{\mathrm{T}}X + XA_{cl} & XB_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}}X & -\gamma I & D_{cl}^{\mathrm{T}} \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0, \qquad X > 0. \tag{11}$$

Viewing the system $(I, A_{cl}, B_{cl}, C_{cl}, D_{cl})$ as state space realization of the closed loop system (9), (10), the question whether a given controller solves the H_∞ control problem obviously can be checked by looking at the feasibility problem (11) for X. On the other hand for an unspecified controller (I, A_K, B_K, C_K, D_K) (11) constitutes *nonlinear* synthesis inequalities in A_K, B_K, C_K, D_K, and X (since $A_{cl} = A_{cl}(A_K, B_K, C_K, D_K)$, ...). The essential step towards an LMI characterization of the synthesis problem is based on a "Completion Lemma" for positive definite matrices [15, 9]; i.e. the existence of a positive definite matrix X can be characterized in terms of an LMI with the principle minors of X and X^{-1} as variables. With this result it is possible to separate the determination of (A_K, B_K, C_K, D_K), and X [15, 9]. The corresponding derivations also reveal that the necessary controller order n_ζ is less or equal to the system order n_x.

In the following section we will examine a way to reformulate the H_∞ problem in order to also capture the case of DAE systems.

3 The H_∞-control problem for linear DAE systems

Instead of (9) we consider a generalized plant Σ_E that is a descriptor system

$$\Sigma_E: \quad \begin{array}{rcrcrcr} E\dot{\boldsymbol{x}}(t) & = & A\boldsymbol{x}(t) & + & B_1\boldsymbol{w}(t) & + & B_2\boldsymbol{u}(t) \\ \boldsymbol{z}(t) & = & C_1\boldsymbol{x}(t) & + & D_{11}\boldsymbol{w}(t) & + & D_{12}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) & = & C_2\boldsymbol{x}(t) & + & D_{21}\boldsymbol{w}(t) & & \end{array} \tag{12}$$

with A, B_i, C_i, D_{ij} being constant matrices of equal dimension as in (9) and singular matrix E having the same dimension as A. Otherwise we use the same notation as in (9). As is clear from the discussion of DAE systems in Sect. 2 we have to take some care in "translating" the objectives of non-descriptor H_∞ control: Singular closed loop systems must be ruled out since they admit non-asymptotically stable solutions to the homogeneous initial value problem (besides the trivial solution, there exist solutions where some components of the descriptor vector are arbitrary). Impulsive solutions, i.e. unbounded non-differential modes, also do not fit in the concept of internal stability. In addition, considering implementation, we explicitly want the controller to be a non-descriptor system, i.e. we will not take a "state feedback plus DAE observer" structure into consideration. In summary we want to find a non-descriptor dynamic output feedback controller (10) such that:

1. The closed loop is a regular, asymptotically stable index one system. A system with these properties is said to be *admissible* [20].
2. The H_∞ norm $\|G_{cl}\|_\infty$ of the closed loop transfer matrix $G_{cl} : \boldsymbol{w} \in H_2 \mapsto \boldsymbol{z} \in H_2$ is strictly smaller than a prescribed real number $\gamma > 0$.

We stress the fact that the plant (12) itself is assumed to be neither regular nor any assumptions concerning the index have to be satisfied. Analogous to the BRL for non-descriptor systems we will now derive a simple answer to the question whether a given controller solves the H_∞ control problem.

3.1 The bounded real lemma for linear DAE systems

The main idea of this subsection is to provide an LMI based analysis result for linear DAE systems $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$, i.e. systems without direct feedthrough term ($D_{cl} = 0$). This setup implies certain inherent assumptions on the plant and/or controller matrices if $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ is interpreted as closed loop system description of (12), (10). However, we will see later on, that there is no advantage in taking the case $D_{cl} \neq 0$ into consideration.

Proposition 3.1 *A system* $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ *is admissible and*

$$\|G_{cl}\|_\infty < \gamma, \quad G_{cl}(s) := C_{cl}(sE_{cl} - A_{cl})^{-1}B_{cl} \tag{13}$$

iff there exists a matrix X *with*

$$E_{cl}^{\mathrm{T}}X = X^{\mathrm{T}}E_{cl} \geq 0 \tag{14}$$

$$\mathcal{B}(\gamma, X) := \begin{bmatrix} A_{cl}^{\mathrm{T}}X + XA_{cl} & X^{\mathrm{T}}B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}}X & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} < 0. \tag{15}$$

Remark. A solution X to (14), (15) is always regular due to the (1,1) entry in $\mathcal{B}(\gamma, X)$. If $E = I$, then "admissible" is equivalent to "asymptotically stable" and (14) implies that X is actually symmetric and positive definite. Therefore Proposition 3.1 contains the corresponding result for non-descriptor systems as special case.

Proof. *Sufficiency:* Assume (15) holds true for some matrix X. It follows

$$A_{cl}^{\mathrm{T}}X + X^{\mathrm{T}}A_{cl} < 0 \quad \text{with} \quad E_{cl}^{\mathrm{T}}X = X^{\mathrm{T}}E_{cl} \geq 0,$$

which in turn shows admissibility of $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ ([20], Lemma 2). Since $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ is regular the transfer matrix $G_{cl}(s) := C_{cl}(sE_{cl} - A_{cl})^{-1}B_{cl}$ is defined. Admissibility of $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ furthermore guarantees $G_{cl} \in \mathcal{H}_\infty$ and therefore $\|G_{cl}\|_\infty = \sup_\omega \overline{\sigma}(G_{cl}(j\omega))$. In order to establish the required norm bound we write $\mathcal{B}(\gamma, X)$ as

$$\mathcal{B}(\gamma, X) = \underbrace{\begin{bmatrix} A_{cl}^{\mathrm{T}}X + X^{\mathrm{T}}A_{cl} & X^{\mathrm{T}}B_{cl} & 0 \\ B_{cl}^{\mathrm{T}}X & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=:\, \mathcal{B}_l(X)} + \underbrace{\begin{bmatrix} 0 & 0 & C_{cl}^{\mathrm{T}} \\ 0 & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix}}_{=:\, \mathcal{B}_r(\gamma)}. \tag{16}$$

A congruence transformation of (16) then renders

$$\underbrace{\left[B_{cl}^{\mathrm{T}}(j\omega E_{cl} - A_{cl})^{-*}, I, \tfrac{1}{\gamma}T_E^*(j\omega)\right] \mathcal{B}_l(X) \begin{bmatrix} (j\omega E_{cl} - A_{cl})^{-1}B_{cl} \\ I \\ \tfrac{1}{\gamma}G_{cl}(j\omega) \end{bmatrix}}_{=0} +$$

$$\underbrace{\left[B_{cl}^{\mathrm{T}}(j\omega E_{cl} - A_{cl})^{-*}, I, \tfrac{1}{\gamma}T_E^*(j\omega)\right] \mathcal{B}_r(\gamma) \begin{bmatrix} (j\omega E_{cl} - A_{cl})^{-1}B_{cl} \\ I \\ \tfrac{1}{\gamma}G_{cl}(j\omega) \end{bmatrix}}_{= \frac{1}{\gamma}T_E^*(j\omega)G_{cl}(j\omega) - \gamma I} < 0$$

and therefore $\overline{\sigma}\left(G_{cl}(j\omega)\right) < \gamma$ for all ω, i.e. γ is a strict upper bound on $\|G_{cl}\|_\infty$.

Necessity: The proof of necessity essentially is based on the fact that the system representation of a index one system in WCF is equivalent to a certain representation of a non-descriptor system. Therefore the application of the BRL is possible. After various algebraic manipulations (see Appendix A) it is possible to establish (14), (15). □

Proposition 3.1 represents a convenient tool for checking admissibility and H_∞ norm boundedness of singular systems $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ since it only requires the computation of the solution of the LMIs (14), (15), i.e. the solution of a feasibility problem. Especially it is not necessary to firstly establish the equivalence of the system at hand with an index one system and to check the norm bound in a second step. This will be the key to the solution of the corresponding synthesis problem in the following section.

For completeness it should be mentioned that, since γ enters affinely into (15), also the H_∞ norm computation for $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$, i.e. the determination of the smallest γ such that (14), (15) holds true, is a convex optimization problem.

3.2 The synthesis problem

After having established an LMI based analysis result we will now consider the corresponding synthesis problem.

3.2.1 Problem setup

We consider the generalized plant (12). Our general philosophy is to consider an unspecified controller, to formulate the closed loop equations, to plug them into the analysis result (i.e. Proposition 3.1), and to derive on this basis conditions for the controller matrices. Application of Proposition 3.1 requires a closed loop without direct feedthrough matrix, or, speaking in terms of the generalized plant (12), the matrices D_{ij} must be zero. Such an representation of the generalized plant allways can be achieved if the "new" descriptor variable vector $\boldsymbol{x}^{\mathrm{T}} := [\boldsymbol{x}^{\mathrm{T}}, \boldsymbol{\chi}_1^{\mathrm{T}}, \boldsymbol{\chi}_2^{\mathrm{T}}]$ with

$$\begin{aligned} \boldsymbol{\chi}_1(t) &:= D_{11}\boldsymbol{w}(t) + D_{12}\boldsymbol{u}(t) \\ \boldsymbol{\chi}_2(t) &:= D_{21}\boldsymbol{w}(t) \end{aligned} \tag{17}$$

is considered. By means of the auxiliary variables $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_2$ it is allways possible to reformulate (12) as

$$\begin{gathered} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{\chi}}_1(t) \\ \dot{\boldsymbol{\chi}}_2(t) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\chi}_1(t) \\ \boldsymbol{\chi}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ D_{11} \\ D_{21} \end{bmatrix} \boldsymbol{w}(t) + \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{z}(t) = \begin{bmatrix} C_1 & I & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\chi}_1(t) \\ \boldsymbol{\chi}_2(t) \end{bmatrix} \qquad \boldsymbol{y}(t) = \begin{bmatrix} C_2 & 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\chi}_1(t) \\ \boldsymbol{\chi}_2(t) \end{bmatrix}. \end{gathered} \tag{18}$$

Without loss of generality we therefore restrict our attention to the plant description

$$E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2\boldsymbol{u}(t), \quad \boldsymbol{z}(t) = C_1\boldsymbol{x}(t), \quad \boldsymbol{y}(t) = C_2\boldsymbol{x}(t). \tag{19}$$

In the course of the following discussion we will see, that the augmentation (18) of the descriptor vector not only does not alter the input output description but also has no impact on the necessary controller order.

In view of the BRL for DAE systems, for a given $\gamma > 0$ the H_∞ control problem consists in the determination of a non-descriptor output feedback controller

$$K: \quad \begin{array}{rcccl} \dot{\boldsymbol{\zeta}}(t) & = & A_K\boldsymbol{\zeta}(t) & + & B_K\boldsymbol{y}(t) \\ \boldsymbol{u}(t) & = & C_K\boldsymbol{\zeta}(t) & + & D_K\boldsymbol{y}(t) \end{array}, \qquad \boldsymbol{\zeta}(t) \in \mathbb{R}^{n_\zeta} \tag{20}$$

such that the closed loop system

$$E_{cl}\dot{\boldsymbol{\xi}}(t) = A_{cl}\boldsymbol{\xi}(t) + B_{cl}\boldsymbol{w}(t) \tag{21}$$

$$\boldsymbol{z}(t) = C_{cl}\boldsymbol{\xi}(t), \qquad \boldsymbol{\xi}(t) \in \mathbb{R}^{(n_x+n_\zeta)} \tag{22}$$

with

$$E_{cl} = \begin{bmatrix} E & 0 \\ 0 & I_{n_\zeta} \end{bmatrix}, \qquad A_{cl} = \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}, \tag{23}$$

$$B_{cl} = \begin{bmatrix} B_1 \\ 0_{n_\zeta \times n_w} \end{bmatrix}, \qquad C_{cl} = \begin{bmatrix} C_1 & 0_{n_z \times n_\zeta} \end{bmatrix} \tag{24}$$

allows for a solution X of the corresponding BRL inequalities

$$E_{cl}^{\mathrm{T}}X = X^{\mathrm{T}}E_{cl} \geq 0, \quad \begin{bmatrix} A_{cl}^{\mathrm{T}}X + X^{\mathrm{T}}A_{cl} & X^{\mathrm{T}}B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}}X & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} < 0. \tag{25}$$

As in the non-descriptor case (Section 2.3) these matrix inequalities are nonlinear in the unknowns $\{A_K, B_K, C_K, D_K, X\}$. Unfortunately the algebraic methods in [15, 9] do not apply since X is not necessarily symmetric nor positive definite. Nevertheless a modification of the "linearizing change of variables" in [26] can be used as follows in order to reduce the H_∞ control problem to an LMI feasibility problem. In order to facilitate the following discussion we assume without loss of generality that (19) is given in the system equivalent normalized SVD representation

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \quad C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \tag{26}$$

and the controller is assumed to be of full order, i.e. the controller order is at least equal to the number of dynamic descriptor variables of the plant ($n_\zeta \geq r := \mathrm{rank}(E)$). The existence of reduced order controllers (i.e. $n_\zeta < r$) can be examined along the same lines if the methods from [26] for the non-descriptor case are taken into account.

3.2.2 Linearizing Change of Variables

If there is a solution X to (25) it is necessarily non-singular and $E_{cl}^{\mathrm{T}} X = X^{\mathrm{T}} E_{cl}$ implies that X and X^{-1} can be written as

$$X = \begin{bmatrix} S_1 & 0 & N \\ S_3 & S_4 & P \\ N^{\mathrm{T}} & 0 & L \end{bmatrix}, \qquad X^{-1} = \begin{bmatrix} R_1 & 0 & M \\ R_3 & R_4 & Q \\ M^{\mathrm{T}} & 0 & K \end{bmatrix}, \tag{27}$$

$$S_1 = S_1^{\mathrm{T}}, \quad L = L^{\mathrm{T}}, \quad R_1 = R_1^{\mathrm{T}}, \quad K = K^{\mathrm{T}}, \tag{28}$$

with $S_1, R_1 \in I\!R^{r\times r}$, $L, K \in I\!R^{n_\zeta\times n_\zeta}$, $N, M \in I\!R^{r\times n_\zeta}$, $S_4, R_4 \in I\!R^{(n_x-r)\times(n_x-r)}$, $S_3, R_3 \in I\!R^{(n_x-r)\times r}$, and $P, Q \in I\!R^{(n_x-r)\times n_\zeta}$. The block partition (27) and $XX^{-1} = I$ and $XI = X$ leads to

$$X\Pi_1 = \Pi_2 \quad \text{with:} \quad \Pi_1 := \begin{bmatrix} R_1 & 0 & I_r \\ R_3 & R_4 & 0 \\ M^{\mathrm{T}} & 0 & 0 \end{bmatrix}, \qquad \Pi_2 := \begin{bmatrix} I_r & 0 & S_1 \\ 0 & I_{n_x-r} & S_3 \\ 0 & 0 & N^{\mathrm{T}} \end{bmatrix}. \tag{29}$$

R_4 is non-singular since X^{-1} is, and without loss of generality (see Appendix B) N, M can be assumed to have full row rank. Therefore Π_1 (and Π_2) are column full rank matrices and the non-singular congruence transformation

$$\Pi_1^{\mathrm{T}} E^{\mathrm{T}} X \Pi_1 = \Pi_1^{\mathrm{T}} X^{\mathrm{T}} E \Pi_1 \geq 0 \tag{30}$$

$$\begin{bmatrix} \Pi_1^{\mathrm{T}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^{\mathrm{T}} X + X^{\mathrm{T}} A_{cl} & X^{\mathrm{T}} B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}} X & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \Pi_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \tag{31}$$

of (25) is possible. Such a congruence transformation has been proposed in [26] in order to reveal the essentially affine structure of the underlying matrix inequalities. In the DAE case, we additionally have to take care of the nonlinear condition $S_4 R_4 = I_{n_x-r}$ that follows from $XX^{-1} = I$ and (27). This problem is circumvented by the special structure of Π_1, Π_2 (29): the congruence transformation (30) based on this matrices leads to inequalities (33), (34) where S_4 no more appears and therefore $S_4 R_4 = I_{n_x-r}$ does not affect the affine structure. With

$$\tilde{M} := \left[M^{\mathrm{T}}, 0_{n_\zeta\times(n_x-r)} \right], \qquad \boldsymbol{R} := \begin{bmatrix} R_1 & 0 \\ R_3 & R_4 \end{bmatrix}, \qquad \boldsymbol{S_l} := \begin{bmatrix} S_1 \\ S_3 \end{bmatrix}$$

and the following change of controller variables

$$\begin{aligned} \hat{\boldsymbol{A}}_{\boldsymbol{K}} &:= N A_K \tilde{M} + N B_K C_2 \boldsymbol{R} + \boldsymbol{S_l}^{\mathrm{T}} B_2 C_K \tilde{M} + \boldsymbol{S_l}^{\mathrm{T}} (A + B_2 D_K C_2) \boldsymbol{R} \\ \hat{\boldsymbol{B}}_{\boldsymbol{K}} &:= N B_K + \boldsymbol{S_l}^{\mathrm{T}} B_2 D_K \\ \hat{\boldsymbol{C}}_{\boldsymbol{K}} &:= C_K \tilde{M} + D_K C_2 \boldsymbol{R} \\ \hat{\boldsymbol{D}}_{\boldsymbol{K}} &:= D_K \end{aligned} \tag{32}$$

the inequalities (25) become

$$\begin{pmatrix} \boldsymbol{R_1} & I_r \\ I_r & \boldsymbol{S_1} \end{pmatrix} > 0 \tag{33}$$

$$\begin{bmatrix} A\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}A^{\mathrm{T}} + B_2\hat{\boldsymbol{C}}_{\boldsymbol{K}} + \left(B_2\hat{\boldsymbol{C}}_{\boldsymbol{K}}\right)^{\mathrm{T}} & \left(A + B_2\hat{\boldsymbol{D}}_{\boldsymbol{K}}C_2\right)\begin{bmatrix} I_r \\ 0 \end{bmatrix} + \hat{\boldsymbol{A}}_{\boldsymbol{K}}^{\mathrm{T}} & B_1 & \boldsymbol{R}^{\mathrm{T}}C_1^{\mathrm{T}} \\ [I_r, 0]\left(A + B_2\hat{\boldsymbol{D}}_{\boldsymbol{K}}C_2\right)^{\mathrm{T}} + \hat{\boldsymbol{A}}_{\boldsymbol{K}} & \begin{array}{c}\left(\boldsymbol{S_l}^{\mathrm{T}}A + \hat{\boldsymbol{B}}_{\boldsymbol{K}}C_2\right)\begin{bmatrix} I_r \\ 0 \end{bmatrix} + \\ + [I_r, 0]\left(\boldsymbol{S_l}^{\mathrm{T}}A + \hat{\boldsymbol{B}}_{\boldsymbol{K}}C_2\right)^{\mathrm{T}}\end{array} & \boldsymbol{S_l}^{\mathrm{T}}B_1 & [I_r, 0]C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} & B_1^{\mathrm{T}}\boldsymbol{S_l} & -\gamma I & 0 \\ C_1\boldsymbol{R} & C_1\begin{bmatrix} I_r \\ 0 \end{bmatrix} & 0 & -\gamma I \end{bmatrix} < 0. \tag{34}$$

In the above formulas Strict inequality in (33) is due to the fact that the "$\geq$" in (25) only stems from the singularity of E_{cl} (see appendix B for details). Due to the preceding remarks the solvability of the inequalities (33), (34) constitutes a *necessary* condition for the existence of a suboptimal H_∞ controller. Clearly (33), (34) are affine inequalities in the matrix variables R, S_l, $\tilde{A}_K$, $\tilde{B}_K$, $\tilde{C}_K$, $\tilde{D}_K$ (highlighted with boldface), i.e. (33), (34) are necessary LMI conditions for the existence of a solution to the synthesis problem.

However these conditions are not sufficient since the matrix $\tilde{M}$ in (32) is singular, i.e. (32) does not constitute a surjection between A_K and $\hat{A}_K$ and between C_K and $\hat{C}_K$.

The idea behind the following refinement of the change of controller variables (32) is to separate $\hat{A}_K$, $\hat{C}_K$ into the invertible part ($\hat{A}'_K$, $\hat{C}'_K$), i.e. the part corresponding to the matrix M^{T} from $\tilde{M} = [M^{\mathrm{T}}, 0]$, and a remainder ($\varrho_1$, ϱ_2):

$$\begin{aligned} \hat{A}_K &= NA_K\tilde{M} + S_l^{\mathrm{T}}B_2C_K\tilde{M} + \hat{B}_KC_2R + S_l^{\mathrm{T}}AR \\ &= \hat{A}'_K\begin{bmatrix} I_r & 0 \end{bmatrix} + \underbrace{\left(\hat{B}_KC_2 + S_l^{\mathrm{T}}A\right)\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}R_4\begin{bmatrix} 0 & I_{n-r} \end{bmatrix}}_{=:\ \varrho_1(\hat{B}_K, S_l, R_4)} \\ \hat{C}_K &= \hat{C}'_K\begin{bmatrix} I_r & 0 \end{bmatrix} + \underbrace{\hat{D}_KC_2\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}R_4\begin{bmatrix} 0 & I_{n-r} \end{bmatrix}}_{=:\ \varrho_2(\hat{D}_K, R_4)} \end{aligned} \tag{35}$$

with

$$\begin{aligned} \hat{A}'_K &:= NA_KM + S_l^{\mathrm{T}}B_2C_KM + \left(\hat{B}_KC_2 + S_l^{\mathrm{T}}A\right)R_l \\ \hat{C}'_K &:= C_KM + D_KC_2R_l \end{aligned} \qquad R_l := \begin{bmatrix} R_1 \\ R_3 \end{bmatrix}. \tag{36}$$

Since the remainders ϱ_1, ϱ_2 are biaffine in the matrix variables the inequality (34) becomes a *biaffine matrix inequality* (BMI) in R, S_l, $\tilde{A}'_K$, $\tilde{B}_K$, $\tilde{C}'_K$, $\tilde{D}_K$ if (32) is refined via (35), (36).

Obviously the existence of a solution of this BMI is a necessary condition for the existence of a γ suboptimal H_∞ controller.

In order to show that a solution of this BMI (together with (33)) is also sufficient we only have to reverse our argument: Assume a solution $\{R, S_l, \tilde{A}'_K, \tilde{B}_K, \tilde{C}'_K, \tilde{D}_K\}$ exists. Then we have to determine matrices M, N with full row rank such that $X\Pi_1 = \Pi_2$ holds true. Due to the block partition of this equation (29) we must have

$$MN^{\mathrm{T}} = I - R_1 S_1. \tag{37}$$

From (33) we derive $S_1 > 0$, $R_1 - S_1^{-1} > 0$ with the Schur-complement formula (6). Therefore the right hand side $I - R_1 S_1 = -(R_1 - S_1^{-1})S_1$ of (37) is non-singular, hence a full rank decomposition is possible. With full row rank matrices M, N the equations (32), (36) for $\hat{D}_K$, $\hat{C}'_K$, $\hat{B}_K$, and $\hat{A}'_K$ can be solved for the controller matrices $\{D_K, C_K, B_K, A_K\}$. The matrices Π_1, Π_2 are non-singular, therefore $X := \Pi_2\Pi_1^{-1}$ surely solves (29), hence the congruence transformation (30), (31) can be reversed and $\{A_K, B_K, C_K, D_K, X\}$ actually solves the BRL, i.e. the synthesis problem.

The preceding derivations are summarized in the following theorem:

Theorem 3.1 *The H_∞ control problem to find a γ suboptimal non-descriptor output feedback controller (20) for linear high-index DAE systems (19) has a solution only if the LMIs (33), (34) are feasible. Iff the LMI/BMIs*

$$\alpha(R_1, S_1) := \begin{bmatrix} R_1 & I \\ I & S_1 \end{bmatrix} > 0,$$

$$\beta(S_l, R_l, R_4, \hat{A}'_K, \hat{B}_K, \hat{C}'_K, \hat{D}_K) := \begin{bmatrix} \Psi_{11} & \Psi_{12} & B_1 & R^{\mathrm{T}}C_1^{\mathrm{T}} \\ \Psi_{12}^{\mathrm{T}} & \Psi_{22} & S_l^{\mathrm{T}} & B_1 \\ B_1^{\mathrm{T}} & B_1^{\mathrm{T}}S_l & -\gamma I & 0 \\ C_1 R & B_1^{\mathrm{T}} & 0 & -\gamma I \end{bmatrix} < 0 \tag{38}$$

with:

$$\Psi_{11} := AR + R^{\mathrm{T}}A^{\mathrm{T}} + \tag{39}$$
$$B_2\left(\hat{C}'_K\left[I_r, 0\right] + D_K C_2\begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix} R_4\left[0, I_{n-r}\right]\right) + \left(\hat{C}'_K\left[I_r, 0\right] + D_K C_2\begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix} R_4\left[0, I_{n-r}\right]\right)^{\mathrm{T}} B_2^{\mathrm{T}}$$

$$\Psi_{12} := \left(A + B_2\hat{D}_K C_2\right)\begin{bmatrix} I_r \\ 0\end{bmatrix} + \begin{bmatrix} I_r \\ 0\end{bmatrix}\hat{A}'^{\mathrm{T}}_K + \begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix} R_4^{\mathrm{T}}\left[0, I_{n-r}\right]\left(C_2^{\mathrm{T}}\hat{B}_K^{\mathrm{T}} + A^{\mathrm{T}}S_l\right)$$

$$\Psi_{22} := \left(S_l^{\mathrm{T}}A + \hat{B}_K C_2\right)\begin{bmatrix} I_r \\ 0\end{bmatrix} + [I_r, 0]\left(S_l^{\mathrm{T}}A + \hat{B}_K C_2\right)^{\mathrm{T}}$$

admit a solution $\{R, S_l, \tilde{A}'_K, \tilde{B}_K, \tilde{C}'_K, \tilde{D}_K\}$ the H_∞ control problem is solvable. A non-descriptor state space description (I, A_K, B_K, C_K, D_K) of the controller can be computed from (32), (36).

Remark. The "smallest" full rank decomposition (37) corresponds to $n_\zeta = r$, i.e. the controller order is equal to the number of dynamic descriptor variables of the plant. Therefore the complexity of the output feedback controller is independent of a preceding augmentation of the descriptor vector of the plant in order to guarantee that the plant has no direct feedthrough terms (18), i.e. there is no loss of generality in assuming the generalized plant structure (19).

3.3 Controller Computation

In the preceding section we derived sufficient LMI/BMI conditions for the H_∞ control problem. Of course also the BRL for DAE systems is a sufficient LMI/BMI condition. However biaffinity in (38) essentially is connected to the matrix R_4, i.e. a submatrix of X^{-1}. On the other hand the BRL constitutes a biaffine condition for the whole matrix X. Since also all structural information on X is apparent in the LMI/BMI condition (38) we will substantially simplify controller computation if we base calculations on Theorem 3.1.

The main point in an algorithmic procedure for controller computation therefore is the question how BMIs can be solved. Recently [12, 13] it has been shown that BMIs arising in control problems where the bilinear terms are "small" compared to the size of the matrix which should be made negative definite can efficiently be addressed by a sequence of generalized eigenvalue problems (i.e. LMI problems). For a given BMI $\beta(X, Y) < 0$ with unknown matrix variables X, Y and some initial guess X_0 the following procedure is iterated:

$$\begin{aligned} \text{Step i:} \quad & Y_i := \arg\min_{Y} \lambda \quad \text{subject to } \beta(X_i, Y) < \lambda I \\ \text{Step i+1:} \quad & X_{i+1} := \arg\min_{X} \lambda \quad \text{subject to } \beta(X, Y_i) < \lambda I. \end{aligned} \tag{40}$$

If λ becomes negative, a solution of the BMI is found. We pay a prize, however: By construction the sequence of generalized eigenvalues λ is obviously non-increasing but due to non-convexity of the BMI problem the iteration process may converge to a $\lambda > 0$ even if the BMI has a solution.

We are now able to sum up the necessary steps from a DAE system (12) up to the determination of the controller matrices:

- Perform a SVD decomposition of E: $E =: U\Sigma V^{\mathrm{T}}$ (with U, V unitary and $\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$ diagonal). Define transformation matrices $L := U^{\mathrm{T}}$, $R := V \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & I_{n_x - r} \end{bmatrix}$ and compute the system equivalent SVD representation

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, LAR, L\left[B_1, B_2\right], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} R, \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \right) \sim \left(E, A, \left[B_1, B_2\right], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \right).$$

For sake of simplicity of notation we denote the transformed system matrices as before (i.e. $A := LAR, \dots$).

- Introduce additional descriptor variables as in (18) in order to get a system description wit $D_{ij} = 0$.

We are now in the situation that Theorem 3.1 applies. As adaptation of the conceptual algorithm (40) for the solution of BMI problems we propose the following scheme for the solution of (38):

- Solve the necessary LMI conditions (33,34) and use the computed matrix R_4 as initialization for the iteration process.

- Perform the iteration

$$\begin{aligned} \text{Step i:}\quad & \{R_{li}, S_{li}, \hat{A}'_{K_i}, \hat{B}_{K_i}, \hat{C}'_{K_i}, \hat{D}_{K_i}\} := \arg\min_{\{R_l, S_l, \hat{A}'_K, \hat{B}_K, \hat{C}'_K, \hat{D}_K\}} \ \lambda \\ & \text{subject to} \ \ \alpha(R_1, S_1) > 0, \quad \beta(R_l, S_l, R_{4i}, \hat{A}'_K, \hat{B}_K, \hat{C}'_K, \hat{D}_K) < \lambda I \\ \text{Step i+1:}\quad & \{R_{li+1}, R_{4i+1}, \hat{A}'_{K_{i+1}}, \hat{C}'_{K_{i+1}}\} := \arg\min_{\{R_l, R_4, \hat{A}'_K, \hat{C}'_K\}} \ \lambda \\ & \text{subject to} \ \ \alpha(R_1, S_{1i}) > 0, \quad \beta(R_l, R_4, S_{li}, \hat{A}'_K, \hat{B}_{K_i}, \hat{C}'_K, \hat{D}_{K_i}) < \lambda I \end{aligned} \tag{41}$$

  until the difference between two subsequent generalized eigenvalues is less than a prescribed tolerance.

- If the last computed generalized eigenvalue is negative the LMI/BMI (38) is solved. The controller order can be fixed as r (see the last remark) and matrices $N, M \in \mathbb{R}^{r \times r}$ can be computed as full rank factorization

$$MN^{\mathrm{T}} := I - R_1 S_1. \tag{42}$$

- Compute the controller matrices A_K, B_K, C_K, D_K as

$$\begin{aligned} D_K &= \hat{D}_K \\ C_K &= \hat{C}'_K M^{-1} - D_K C_2 R_l M^{-1} \\ B_K &= N^{-1} \hat{B}_K - N^{-1} S_l^{\mathrm{T}} B_2 D_K \\ A_K &= N^{-1} \left(\hat{A}'_K - \left(\hat{B}_K C_2 + S_l^{\mathrm{T}} A \right) R_l - S_l^{\mathrm{T}} B_2 C_K M \right) M^{-1} \end{aligned} \tag{43}$$

  or alternatively determine $X = \Pi_2 \Pi_1^{-1}$ and solve the BRL LMI for A_K, B_K, C_K, and D_K.

3.3.1 A numerical example

We consider the following example with index $\nu = 2$ from [31]

$$E = \left[\begin{array}{ccc:cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \; A = \left[\begin{array}{ccc:cc} -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right], \; B_1 = \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hdashline 1 \\ 0 \end{array}\right], \; B_2 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ \hdashline 0 \\ 1 \end{array}\right] \tag{44}$$

$$C_1 = \left[\begin{array}{ccc:cc} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right], \qquad C_2 = \left[\begin{array}{ccc:cc} 1 & 0 & 1 & 1 & 0 \end{array}\right]. \tag{45}$$

The original problem as discused in [31] has three descriptor variables and direct feedthrough terms $D_{12} = [0, 1]^{\mathrm{T}}$ and $D_{21} = 1$ (see Eq. (12)). In (44), (45) new descriptor variables are already introduced in order to transform the system to the form (19). This transformation is indicated by the block-partioning in (44), (45). Since (44) is given in normalized SVD representation, the iteration (41) directly can be applied. All necessary computations are realized within the LMI Control Toolbox [10]. In this example (with $\gamma = 1$) matrices S_l, R_l, R_4, $\hat{A}'_K$, $\hat{B}_K$, $\hat{C}'_K$, and $\hat{D}_K$, such that the closed loop system BRL (38) holds true, are determined after two iterations. The controller order is chosen to be two, i.e. equal to the number of dynamic descriptor variables in (44). The freedom in (42) can be exploited in order to guarantee well-conditioned matrices M, N. Then the controller matrices

$$A_K = \begin{bmatrix} -3.629 & 2.617 \\ 3.948 & -10.137 \end{bmatrix}, \quad B_K = \begin{bmatrix} 2.094 \\ -4.294 \end{bmatrix}, \quad C_K = \begin{bmatrix} 0.210 & -0.291 \end{bmatrix}, \quad D_K = \begin{bmatrix} -0.550 \end{bmatrix}. \tag{46}$$

are computed via (43). The main advantage of our approach is that we directly get (46), i.e. an non-descriptor controller while the approach in [31] can lead to a DAE controller.
By construction the H_∞ norm of the closed loop (with this controller) is strictly less than one. The actual H_∞ norm of the closed loop can be determined by minimization of γ subject to the BRL conditions (14), (15). Since the controller matrices are known this is again an LMI problem. The H_∞ norm of the closed loop system is computed as $\|G_{cl}\|_\infty \approx 0.808$.

4 Conclusion

We considered H_∞ control of DAE systems via non-descriptor controllers. A necessary condition for the existence of suboptimal non-descriptor H_∞ controllers is given by LMI conditions. These conditions, however, are not sufficient. Necessary and sufficient conditions are derived as BMI/LMI conditions. The result even applies to singular descriptor systems. The resulting controller can be easily implemented and renders the closed loop to be an index one system, i.e. the closed loop operator is not only H_∞ norm bounded but also impulse-free. Controller computation is possible by means of standard LMI tools. The proposed procedure successfully is applied to an high index DAE system from literature.

A Proof of the BRL for DAE systems (necessity)

Let $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ be admissible and $\|G_{cl}\|_\infty < \gamma$. We have to show the existence of a matrix X such that (14) and (15) hold. The following arguments will be constructive in the sense, that they provide a simple method to compute such a matrix X. First we note that admissibility of $(E_{cl}, A_{cl}, B_{cl}, C_{cl})$ implies the existence of an equivalent system

$$(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_r & 0 \\ 0 & I_{n_\xi - r} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right) \tag{47}$$

in Weierstrass canonical form where $A_r \in I\!R^{r\times r}$ ($r := \mathrm{rank}(E_{cl})$), is a stable matrix. The transformed system is given as

$$\begin{aligned} \dot{\boldsymbol{\xi}}_1(t) &= A_r \boldsymbol{\xi}_1(t) + B_1 \boldsymbol{w}(t) \\ \mathbf{0} &= \quad \boldsymbol{\xi}_2(t) + B_2 \boldsymbol{w}(t) \\ \boldsymbol{z}(t) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1(t) \\ \boldsymbol{\xi}_2(t) \end{bmatrix} . \end{aligned} \tag{48}$$

Therefore the transfer matrix $G_{cl}(s) = C_{cl}(sE_{cl} - A_{cl})^{-1} B_{cl}$ of the original system can be expressed as

$$G_{cl}(s) = C_1 \left(sI - A_r\right)^{-1} B_1 - C_2 B_2 \tag{49}$$

and the standard BRL (Proposition 2.1) can be used to characterize the H_∞-norm bound γ:

$$\exists X_1 > 0, \quad X_1 \in I\!R^{r\times r} : \quad \begin{bmatrix} A_r^{\mathrm{T}} X_1 + X_1 A_r & X_1 B_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}} X_1 & -\gamma I & -B_2^{\mathrm{T}} C_2^{\mathrm{T}} \\ C_1 & -C_2 B_2 & -\gamma I \end{bmatrix} < 0. \tag{50}$$

Without loss of generality we assume that C_2 has at least as much rows as columns, i.e. $n_z \geq n_\xi - r$. Otherwise C_2 and C_1 are augmented by rows with zeros. This augmentation obviously does not affect the validity of (50). As a consequence a column full rank matrix $\tilde{C}_2$ of the same size as C_2 exists, i.e.

$$\tilde{C}_2 \in I\!R^{n_z \times (n_\xi - r)} \quad , \quad \mathrm{rank}\, \tilde{C}_2 = n_\xi - r, \tag{51}$$

and we can perform the following non-singular transformation of (50):

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ \frac{1}{\gamma}C_1 & \frac{1}{\gamma}\tilde{C}_2 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_r^{\mathrm{T}}X_1 + X_1A_r & X_1B_1 & C_1^{\mathrm{T}} \\ B_1^{\mathrm{T}}X_1 & -\gamma I & -B_2^{\mathrm{T}}C_2^{\mathrm{T}} \\ C_1 & -C_2B_2 & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ \frac{1}{\gamma}C_1 & \frac{1}{\gamma}\tilde{C}_2 & 0 \end{bmatrix} \quad < \quad 0. \tag{52}$$

The resulting matrix inequality

$$\begin{bmatrix} A_r^{\mathrm{T}}X_1 + X_1A_r + \frac{1}{\gamma}C_1^{\mathrm{T}}C_1 & 0 & X_1B_1 - \frac{1}{\gamma}C_1^{\mathrm{T}}C_2B_2 \\ 0 & -\frac{1}{\gamma}\tilde{C}_2^{\mathrm{T}}\tilde{C}_2 & -\frac{1}{\gamma}\tilde{C}_2^{\mathrm{T}}C_2B_2 \\ B_1^{\mathrm{T}}X_1 - \frac{1}{\gamma}B_2^{\mathrm{T}}C_2^{\mathrm{T}}C_1 & -\frac{1}{\gamma}B_2^{\mathrm{T}}C_2^{\mathrm{T}}\tilde{C}_2 & -\gamma I \end{bmatrix} < 0$$

can be rewritten as

$$\begin{bmatrix} A_r^{\mathrm{T}}X_1 + X_1A_r & -\frac{1}{\gamma}C_1^{\mathrm{T}}C_2 & X_1B_1 - \frac{1}{\gamma}C_1^{\mathrm{T}}C_2B_2 \\ -\frac{1}{\gamma}C_2^{\mathrm{T}}C_1 & -\frac{1}{\gamma}\left(\tilde{C}_2^{\mathrm{T}}\tilde{C}_2 + C_2^{\mathrm{T}}C_2\right) & -\frac{1}{\gamma}\tilde{C}_2^{\mathrm{T}}C_2B_2 \\ B_1^{\mathrm{T}}X_1 - \frac{1}{\gamma}B_2^{\mathrm{T}}C_2^{\mathrm{T}}C_1 & -\frac{1}{\gamma}B_2^{\mathrm{T}}C_2^{\mathrm{T}}\tilde{C}_2 & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \\ 0 \end{bmatrix}\begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix} < 0$$

and Schur-complemented (6) into

$$\begin{bmatrix} A_r^{\mathrm{T}}X_1 + X_1A_r & -\frac{1}{\gamma}C_1^{\mathrm{T}}C_2 & X_1B_1 - \frac{1}{\gamma}C_1^{\mathrm{T}}C_2B_2 & C_1^{\mathrm{T}} \\ -\frac{1}{\gamma}C_2^{\mathrm{T}}C_1 & -\frac{1}{\gamma}\left(\tilde{C}_2^{\mathrm{T}}\tilde{C}_2 + C_2^{\mathrm{T}}C_2\right) & -\frac{1}{\gamma}\tilde{C}_2^{\mathrm{T}}C_2B_2 & C_2^{\mathrm{T}} \\ B_1^{\mathrm{T}}X_1 - \frac{1}{\gamma}B_2^{\mathrm{T}}C_2^{\mathrm{T}}C_1 & -\frac{1}{\gamma}B_2^{\mathrm{T}}C_2^{\mathrm{T}}\tilde{C}_2 & -\gamma I & 0 \\ C_1 & C_2 & 0 & -\gamma I \end{bmatrix} < 0. \tag{53}$$

Besides the rank condition (51) we now impose a further restriction on $\tilde{C}_2$: the inequality (53) should remain valid if the matrix C_2 is replaced by $\tilde{C}_2$ in the upper left 3×3 sub-block matrix[1]. We end up with

$$\begin{bmatrix} \begin{bmatrix} A_r & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} X' + X'^{\mathrm{T}} \begin{bmatrix} A_r & 0 \\ 0 & I \end{bmatrix} & X'^{\mathrm{T}}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} & \begin{bmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{bmatrix} \\ \begin{bmatrix} B_1^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix} X' & -\gamma I & 0 \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} & 0 & -\gamma I \end{bmatrix} < 0 \tag{54}$$

where X' is defined as follows:

$$X' := \begin{bmatrix} X_1 & 0 \\ -\frac{1}{\gamma}\tilde{C}_2^{\mathrm{T}}C_1 & -\frac{1}{\gamma}\tilde{C}_2^{\mathrm{T}}\tilde{C}_2 \end{bmatrix}. \tag{55}$$

Clearly a preceding augmentation of C_1 and C_2 can be canceled in the third row and column of (54) without invalidating the inequality. If the transformation to WCR is performed by non-singular matrices L, R (i.e. $\tilde{E} = LER$, $\tilde{A} = LAR$) it is easy to check that $X := L^{\mathrm{T}}X'R^{-1}$ fulfills the inequalities (14) and (15) of the original problem.

[1] Due to strict inequality in (52) such a $\tilde{C}_2$ always exists. Provided that $\tilde{C}_2 := C_2$ not already suffices (51) one simply can take $\tilde{C}_2$ as C_2 with small perturbations in the main diagonal entries.

B Non-singularity of N, M

We have

$$E_{cl} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0_{n_x-r} & 0 \\ 0 & 0 & I_{n_\zeta} \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} S_1 & 0 & N \\ S_3 & S_4 & P \\ N^{\mathrm{T}} & 0 & L \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} R_1 & 0 & M \\ R_3 & R_4 & Q \\ M^{\mathrm{T}} & 0 & K \end{bmatrix} \tag{56}$$

with dimensions as in Section 3.2.2. Due to $E_{cl}^{\mathrm{T}}X = X^{\mathrm{T}}E_{cl}$ any solution X of the BRL inequalities (25) necessarily can be partitioned as in (56). In order to establish the congruence transformation (30), (31) we have to show that the feasibility of the BRL inequalities (25) implies the existence of a solution X such that the sub-matrix N in (56) has full row rank. From the first BRL inequality $E_{cl}^{\mathrm{T}}X \geq 0$ we get $X^{-\mathrm{T}}E_{cl}^{\mathrm{T}}XX^{-1} = X^{-\mathrm{T}}E_{cl}^{\mathrm{T}} \geq 0$ by congruence transformation, or explicitly, using the block partition (56) of X and X^{-1}:

$$E_{cl}^{\mathrm{T}}X = \begin{bmatrix} S_1 & 0 & N \\ 0 & 0_{n_x-r} & 0 \\ N^{\mathrm{T}} & 0 & L \end{bmatrix} \geq 0, \qquad X^{-\mathrm{T}}E_{cl}^{\mathrm{T}} = \begin{bmatrix} R_1 & 0 & M \\ 0 & 0_{n_x-r} & 0 \\ M^{\mathrm{T}} & 0 & K \end{bmatrix} \geq 0. \tag{57}$$

Multiplication renders $E_{cl}^{\mathrm{T}}XX^{-\mathrm{T}}E_{cl}^{\mathrm{T}} = E_{cl}$ or essentially

$$\underbrace{\begin{bmatrix} S_1 & N \\ N^{\mathrm{T}} & L \end{bmatrix}}_{=:\,\Upsilon_1} \underbrace{\begin{bmatrix} R_1 & M \\ M^{\mathrm{T}} & K \end{bmatrix}}_{=:\,\Upsilon_2} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n_\zeta} \end{bmatrix}. \tag{58}$$

Therefore Υ_1, Υ_2 are non-singular and in view of (57) positive definite. Due to this analysis the *non-strict* BRL inequality $E_{cl}^{\mathrm{T}}X \geq 0$ can be replaced by the *strict* inequality

$$\begin{bmatrix} S_1 & N \\ N^{\mathrm{T}} & L \end{bmatrix} > 0, \tag{59}$$

i.e. all inequality restrictions on N due to the BRL (25) are *strict*. With $N \in \mathbb{R}^{r \times n_\zeta}$ and $n_\zeta \geq r$ (full order controller case) it is now always possible to perturb N such that the perturbed matrix has full row rank and the BRL inequalities (25) remain valid. From (29) it follows that a solution X with full row rank sub-matrix N has an inverse X^{-1} with a full row rank sub-matrix M.

As an important consequence we can strengthen the "$\geq$" in (30) in the following sense: We have $\Pi_1^{\mathrm{T}}X^{\mathrm{T}}E_{cl}\Pi_1 = \Pi_1^{\mathrm{T}}X^{\mathrm{T}}E_{cl}E_{cl}E_{cl}\Pi_1 = \Pi_1 E_{cl}^{\mathrm{T}}XE_{cl}E_{cl}\Pi_1 \geq 0$ or essentially

$$\begin{bmatrix} R_1 & M \\ I_r & 0 \end{bmatrix} \underbrace{\begin{bmatrix} S_1 & N \\ N^{\mathrm{T}} & L \end{bmatrix}}_{=\,\Upsilon_1} \begin{bmatrix} R_1 & I_r \\ M^{\mathrm{T}} & 0 \end{bmatrix} \geq 0. \tag{60}$$

Since Υ_1 is positive definite and N, M can be assumed to be full row rank matrices, (60) can be interpreted as congruence transformation of a positive definite matrix, i.e. the inequality (60) actually is strict. With

$$\begin{aligned} S_1R_1 + NM^{\mathrm{T}} &= I & \qquad S_1M + NK &= 0 \\ N^{\mathrm{T}}R_1 + LM^{\mathrm{T}} &= 0 & \qquad N^{\mathrm{T}}M + LK &= I \end{aligned}$$

from the block partition of $XX^{-1} = I$, (60) finally becomes

$$\begin{bmatrix} R_1 & I_r \\ I_r & S_1 \end{bmatrix} > 0. \tag{61}$$

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