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A Uniqueness Condition for the Polyharmonic Equation in Free Space

P. Lesky Jr

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 10, D 5300 Bonn 1, West Germany

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Consider the polyharmonic wave equation $\partial_t^2 u + (-\Delta)^m u = f$ in $\mathbb{R}^n \times [0, \infty)$ with time-independent right-hand side. We study the asymptotic behaviour of $u(\mathbf{x}, t)$ as $t \to \infty$ and show that $u(\mathbf{x}, t)$ either converges or increases with order t^α or $\ln t$ as $t \to \infty$. In the first case we study the limit $u_0(\mathbf{x}) := \lim_{t\to\infty} u(\mathbf{x}, t)$ and give a uniqueness condition that characterizes u_0 among the solutions of the polyharmonic equation $(-\Delta)^m u = f$ in $\mathbb{R}^n$. Furthermore we prove in the case $2m \geqslant n$ that the polyharmonic equation has a solution satisfying the uniqueness condition if and only if f is orthogonal to certain solutions of the homogeneous polyharmonic equation.

1. Introduction

Consider the problem

$$\left.\begin{aligned} &\partial_t^2 u + (-\Delta)^m u = e^{-i\omega t} f \quad \text{in } \mathbb{R}^n \times [0, \infty), \\ &u(\mathbf{x}, 0) = \partial_t u(\mathbf{x}, 0) = 0 \qquad \text{in } \mathbb{R}^n, \end{aligned}\right\} \tag{1.1}$$

where $f \in C_0^\infty(\mathbb{R}^n)$, $\omega \geqslant 0$ and $\Delta := \partial_1^2 + \cdots \partial_n^2$. We are interested in the asymptotic behaviour of $u(\mathbf{x}, t)$ as $t \to \infty$. In the case $\omega > 0$ it has been shown by Eidus [2] that

1. If $m < n$, then the principle of limiting amplitude holds:

$$u(\mathbf{x}, t) = e^{-i\omega t} u_\omega(\mathbf{x}) + o(1) \quad \text{as } t \to \infty, \tag{1.2}$$

where

$$(-\Delta)^m u_\omega - \omega^2 u_\omega = f \quad \text{in } \mathbb{R}^n; \tag{1.3}$$

u_ω can be uniquely characterized by (1.3) and a suitable radiation condition.

2. If $m = n$, then

$$u(\mathbf{x}, t) = e^{-i\omega t} u_\omega(\mathbf{x}) + c_1 \int_{\mathbb{R}^n} f(\mathbf{x}')\mathrm{d}\mathbf{x}' + o(1) \quad \text{as } t \to \infty, \tag{1.4}$$

with a suitable constant $c_1 \neq 0$, where u_ω is a solution of (1.3).

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3. If $m > n$, then

$$u(\mathbf{x}, t) = t^{1-n/m} c_2 \int_{\mathbb{R}^n} f(\mathbf{x}')\mathrm{d}\mathbf{x}' + o(t^{1-n/m}) \quad \text{as } t \to \infty\,, \tag{1.5}$$

with $c_2 \neq 0$.

This shows that $u(\mathbf{x}, t)$ is unbounded as $t \to \infty$ if $\omega > 0$ and $m > n$. As pointed out in [6], similar resonance effects can be observed in the case $\omega = 0, m = 1, n = 1$ or $n = 2$. In section 2 we study (1.1) in the case $\omega = 0$ for arbitrary $m, n \in \mathbb{N}$. We discuss the asymptotic behaviour of the solution u as $t \to \infty$ and show:

1. If $2m < n$, then

$$u(\mathbf{x}, t) = u_0(\mathbf{x}) + o(1) \quad \text{as } t \to \infty \tag{1.6}$$

uniformly in every compact subset of $\mathbb{R}^n$, where u_0 satisfies the corresponding static equation

$$(-\Delta)^m u_0 = f \quad \text{in } \mathbb{R}^n. \tag{1.7}$$

2. If $2m \geqslant n$, then for odd n

$$u(\mathbf{x}, t) = \sum_{s=0}^{m-(n+1)/2} D_s t^{2-\frac{n+2s}{m}} \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{2s}\mathrm{d}\mathbf{x}' + u_0(\mathbf{x}) + o(1) \quad \text{as } t \to \infty, \tag{1.8}$$

and for even n

$$u(\mathbf{x}, t) = \sum_{s=0}^{m-1-n/2} D_s t^{2-\frac{n+2s}{m}} \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{2s}\mathrm{d}\mathbf{x}' + D^* \ln t \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{2m-n}\mathrm{d}\mathbf{x}' + u_0^*(\mathbf{x}) + o(1) \text{ as } t \to \infty \tag{1.9}$$

uniformly in every compact subset of $\mathbb{R}^n$, where u_0 and u_0^* are solutions of (1.7) and D_s and D^* are specified in (2.25) below.

Sections 3 and 4 deal with the polyharmonic equation (1.7) and with the solution u_0 determined by (1.6). Note that (1.6) holds also in the case $2m \geqslant n$ if f satisfies the condition

$$\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{2s}\mathrm{d}\mathbf{x}' = 0 \quad \text{for } s = 0, 1, \ldots, \left[m - \frac{n}{2}\right] \text{ and every } \mathbf{x} \in \mathbb{R}^n \tag{1.10}$$

($[r] := \max\{n \in \mathbb{N}_0 : n \leqslant r\}$), or, equivalently,

$$\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}'|^{2j}\mathbf{x}'^{\alpha}\mathrm{d}\mathbf{x}' = 0 \quad \text{for } j \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n \text{ with } j + |\alpha| \leqslant \left[m - \frac{n}{2}\right] \tag{1.11}$$

(compare (4.21), (4.22) in [3]; $|\alpha| := \alpha_1 + \cdots + \alpha_n$ for every multi-index $\alpha \in \mathbb{N}_0^n$, $\mathbf{x}'^{\alpha} := \mathbf{x}_1'^{\alpha_1} \ldots x_n'^{\alpha_n}$). We study the question of how u_0 can be characterized uniquely among the solutions of (1.7) by imposing a suitable asymptotic condition as $|\mathbf{x}| \to \infty$. The answer is easy in the case $2m < n$. Then

$$D^{\alpha}u_0(\mathbf{x}) := \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} u_0(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^{|\alpha|+1}}\right) \quad \text{as } |\mathbf{x}| \to \infty \tag{1.12}$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant 2m-1$, and there exists only one solution of (1.7) with this property. The same statement holds in the case $2m = n$ if f satisfies (1.11) (compare (2.32) and (4.4) below). If $2m > n$, then $u_0(\mathbf{x})$ may be unbounded as $|\mathbf{x}| \to \infty$, even if (1.11) is valid, as we shall see in an example at the beginning of Section 3. We shall show that u_0 is uniquely determined by (1.7) and the property

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u_0(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = o(R^{n-1}) \quad \text{as } R \to \infty \text{ for every } \mathbf{x}_0 \in \mathbb{R}^n. \tag{1.13}$$

Note that this condition is weaker than (1.12), so that u_0 satisfies (1.13) also in the case $2m \leqslant n$.

The verification of (1.13) yields as a further result the following statement on the solvability of (1.7) in the case $2m \geqslant n$: (1.7) has a solution satisfying (1.13) if and only if (1.11) holds. Condition (1.11) says that f is orthogonal to certain polynomial solutions of $(-\Delta)^m v = 0$ in $\mathbb{R}^n$.

The analysis in Section 2 is based on the spectral theory for unbounded self-adjoint operators. Most conclusions are analogous to some in [8] and [3]. Here we give only a short description of the main steps. In Section 3 we use Green's formula to derive an expansion of the form

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} v(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \sum_{j=0}^{m-1} c_j \Delta^j v(\mathbf{x}_0) R^{n-1+2j} \tag{1.14}$$

for every solution $v \in C^{2m}(\mathbb{R}^n)$ of the homogenous equation $(-\Delta)^m v = 0$, where $c_j \neq 0$ are suitable real constants. This shows that (1.7) has at most one solution with the property (1.13). A Taylor expansion yields that u_0 satisfies (1.13) if and only if (1.11) holds. This, together with (1.14), implies the above statement on the solvability of (1.7) for $2m \geqslant n$.

2. The time-dependent problem

We study the problem

$$\left.\begin{aligned} &\partial_t^2 u + (-\Delta)^m u = f && \text{in } \mathbb{R}^n \times [0, \infty), \\ &u(\mathbf{x}, 0) = \partial_t u(\mathbf{x}, 0) = 0 && \text{in } \mathbb{R}^n, \end{aligned}\right\} \tag{2.1}$$

with given $f \in C_0^\infty(\mathbb{R}^n)$. We require $u \in C^{2m}(\mathbb{R}^n \times [0, \infty))$ and

$$u(.\,, t) \in H_m(\mathbb{R}^n) \quad \text{for every } t \geqslant 0, \tag{2.2}$$

where $H_m(\mathbb{R}^n)$ denotes the mth Sobolev space. Then u is uniquely determined (compare the discussion in [3] in a related situation). We extend the operator $(-\Delta)^m$ to a positive self-adjoint operator in $L_2(\mathbb{R}^n)$ by setting

$$\left.\begin{aligned} &D(A) := \{U \in H_m(\mathbb{R}^n)\colon \Delta^m U \in L_2(\mathbb{R}^n)\}, \\ &AU \ := (-\Delta)^m U \quad \text{for } U \in D(A). \end{aligned}\right\} \tag{2.3}$$

Let $\{P_\lambda\}$ denote the (left continuous) spectral family of A. The functional calculus for unbounded self-adjoint operators and the elliptic regularity theory yield

$$u(\mathbf{x}, t) = \int_0^\infty \frac{1}{\lambda}(1 - \cos\sqrt{\lambda}t)\,\mathrm{d}(P_\lambda f(\mathbf{x})). \tag{2.4}$$

In order to obtain the asymptotic behaviour of $u(\mathbf{x}, t)$ as $t \to \infty$ we proceed as in [3], to which we refer for a more detailed presentation of the argument.

A modification of (3.11) in [3] yields the following representation of the resolvent $R_z = (A - zI)^{-1}$ of A:

$$R_z f(\mathbf{x}) = \frac{\mathrm{i}|z|^{\frac{\sigma+2}{2m}-1}}{4m(2\pi)^\sigma} \sum_{s=0}^{m-1} \mathrm{e}^{\mathrm{i}(\arg z + 2\pi s)(\frac{\sigma+2}{2m}-1)}$$

$$\times \int_{\mathbb{R}^n} \frac{f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^\sigma} H_\sigma^{(1)}(|\mathbf{x} - \mathbf{x}'||z|^{1/2m} \mathrm{e}^{\mathrm{i}(\arg z + 2\pi s)/2m}) \mathrm{d}\mathbf{x}', \tag{2.5}$$

where $\sigma = (n/2) - 1$ and

$$H_\sigma^{(1)}(\zeta) = J_\sigma(\zeta) + \mathrm{i}N_\sigma(\zeta) \quad (\zeta \in \mathbb{C}\setminus\{0\}) \tag{2.6}$$

denotes Hankel's function. By means of Stone's formula it follows that $P_\lambda f$ is continuous with respect to $\lambda \in \mathbb{R}$ and differentiable for $\lambda \neq 0$. In particular, we have

$$\frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} = \frac{1}{2m(2\pi)^{\sigma+1}} \lambda^{\frac{\sigma+2}{2m}-1} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^\sigma} J_\sigma(|\mathbf{x} - \mathbf{x}'|\lambda^{1/2m}) \mathrm{d}\mathbf{x}' \text{ for } \lambda > 0 \tag{2.7}$$

Note that $P_\lambda f = 0$ for $\lambda \leqslant 0$, since A is positive. Using

$$J_\sigma(\zeta) = \sum_{s=0}^{\infty} C_s \zeta^{2s+\sigma}, \tag{2.8}$$

with

$$C_s = \frac{(-1)^s}{2^{\sigma+2s} s! \Gamma(\sigma + s + 1)} \tag{2.9}$$

(compare [4]), we obtain for $\lambda \downarrow 0$

$$\frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} = \begin{cases} O(\lambda^{1/2m}) & \text{if } 2m < n, \\ \dfrac{1}{2m(2\pi)^{n/2}} \displaystyle\sum_{s=0}^{[m-n/2]} \frac{C_s}{\lambda^{1-(n+2s)/2m}} \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2s} \mathrm{d}\mathbf{x}' + O(\lambda^{1/2m}) & \text{if } 2m \geqslant n, \end{cases} \tag{2.10}$$

uniformly in every compact subset of $\mathbb{R}^n$. Note that

$$u(\mathbf{x}, t) = \int_0^\infty \frac{1}{\lambda}(1 - \cos\sqrt{\lambda}t) \frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} \mathrm{d}\lambda \tag{2.11}$$

and set

$$I_1(\mathbf{x}, t; \delta) := \int_0^\delta \frac{1}{\lambda}(1 - \cos\sqrt{\lambda}t) \frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} \mathrm{d}\lambda, \tag{2.12}$$

$$I_2(\mathbf{x}; \delta) := \int_\delta^\infty \frac{1}{\lambda} \frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} \mathrm{d}\lambda, \tag{2.13}$$

$$I_3(\mathbf{x}, t; \delta) := -\int_\delta^\infty \frac{\cos\sqrt{\lambda}t}{\lambda} \frac{\mathrm{d}P_\lambda f(\mathbf{x})}{\mathrm{d}\lambda} \mathrm{d}\lambda \tag{2.14}$$

($\delta > 0$). Let K be an arbitrary compact subset of $\mathbb{R}^n$. At first we study the case $2m \geqslant n$.

We insert (2.10) into (2.12) and obtain

$$I_1(\mathbf{x}, t; \delta) = \frac{1}{2m(2\pi)^{n/2}} \sum_{s=0}^{[m-n/2]} C_s I^*_{\beta_s}(t; \delta) \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2s} \mathrm{d}\mathbf{x}' + w_1(\mathbf{x}, t; \delta), \tag{2.15}$$

where

$$I^*_\beta(t; \delta) := \int_0^\delta \frac{1 - \cos\sqrt{\lambda} t}{\lambda^{1+\beta}} \mathrm{d}\lambda, \tag{2.16}$$

$$\beta_s := 1 - \frac{n + 2s}{2m} \tag{2.17}$$

and $w_1(\mathbf{x}, t; \delta) \to 0$ as $\delta \downarrow 0$ uniformly with respect to $(\mathbf{x}, t) \in K \times [0, \infty)$. In order to compute I^*_β, we substitute $\mu := \sqrt{\lambda} t$. This yields

$$I^*_\beta(t; \delta) = 2t^{2\beta} \int_0^{\sqrt{\delta} t} \frac{1 - \cos\mu}{\mu^{1+2\beta}} \mathrm{d}\mu. \tag{2.18}$$

If $\beta > 0$, it follows that

$$\begin{aligned} I^*_\beta(t; \delta) &= 2t^{2\beta} \left\{ \int_0^\infty \frac{1 - \cos\mu}{\mu^{1+2\beta}} \mathrm{d}\mu - \int_{\sqrt{\delta} t}^\infty \frac{\mathrm{d}\mu}{\mu^{1+2\beta}} + \int_{\sqrt{\delta} t}^\infty \frac{\cos\mu}{\mu^{1+2\beta}} \mathrm{d}\mu \right\} \\ &= t^{2\beta} \frac{\pi}{2\beta\Gamma(2\beta)\sin(\beta\pi)} - \frac{1}{\beta\delta^\beta} + W_1(t; \delta; \beta) \end{aligned} \tag{2.19}$$

(compare integral 11c, section 1.1.3.4 in [1]) with

$$|W_1(t; \delta; \beta)| = \left| 2t^{2\beta} \int_{\sqrt{\delta} t}^\infty \frac{\cos\mu}{\mu^{1+2\beta}} \mathrm{d}\mu \right| \leqslant \frac{4}{\delta^{\beta+1/2} t}, \tag{2.20}$$

as an integration by parts shows. If $\beta = 0$, we obtain

$$\begin{aligned} I^*_0(t; \delta) &= 2\int_1^{\sqrt{\delta} t} \frac{1}{\mu} \mathrm{d}\mu + 2\int_0^1 \frac{1 - \cos\mu}{\mu} \mathrm{d}\mu - 2\int_1^\infty \frac{\cos\mu}{\mu} \mathrm{d}\mu + W_1(t; \delta; 0) \\ &= 2\ln t + \ln\delta + 2C_e + W_1(t; \delta; 0) \end{aligned} \tag{2.21}$$

(C_e denotes the Euler–Mascheroni constant; compare (3.67) in [7]). Setting

$$p_s(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2s} \mathrm{d}\mathbf{x}', \tag{2.22}$$

we obtain from (2.15), (2.19) and (2.21) for odd n

$$\begin{aligned} I_1(\mathbf{x}, t; \delta) = &\sum_{s=0}^{m-(n+1)/2} D_s t^{2\beta_s} p_s(\mathbf{x}) - \frac{1}{2m(2\pi)^{n/2}} \sum_{s=0}^{m-(n+1)/2} \frac{C_s}{\beta_s \delta^{\beta_s}} p_s(\mathbf{x}) \\ &+ w_1(\mathbf{x}, t; \delta) + W_2(\mathbf{x}, t; \delta), \end{aligned} \tag{2.23}$$

and for even n

$$\begin{aligned} I_1(\mathbf{x}, t; \delta) = & \sum_{s=0}^{m-1-n/2} D_s t^{2\beta_s} p_s(\mathbf{x}) + D^*(\ln t) p_{m-n/2}(\mathbf{x}) \\ & - \frac{1}{2m(2\pi)^{n/2}} \sum_{s=0}^{m-1-n/2} \frac{C_s}{\beta_s \delta^{\beta_s}} p_s(\mathbf{x}) \\ & + \frac{C_{m-n/2}}{2m(2\pi)^{n/2}} (\ln \delta + 2C_e) p_{m-n/2}(\mathbf{x}) \\ & + w_1(\mathbf{x}, t; \delta) + W_2(\mathbf{x}, t; \delta), \end{aligned} \tag{2.24}$$

where

$$\left. \begin{aligned} D_s &:= \frac{C_s}{2m(2\pi)^{n/2}} \frac{\pi}{2\beta_s \Gamma(2\beta_s) \sin(\beta_s \pi)} \quad \left(s = 0, 1, \ldots, \left[m - \frac{n}{2}\right]\right), \\ D^* &:= \frac{C_{m-n/2}}{m(2\pi)^{n/2}}, \end{aligned} \right\} \tag{2.25}$$

and

$$W_2(\mathbf{x}, t; \delta) := \frac{1}{2m(2\pi)^{n/2}} \sum_{s=0}^{[m-n/2]} C_s W_1(t; \delta; \beta_s) p_s(\mathbf{x}). \tag{2.26}$$

Now consider I_2 defined by (2.13). Note that

$$I_2(\mathbf{x}; \delta) = \lim_{\tau \downarrow 0} \left\{ R_{i\tau} f(\mathbf{x}) - \int_0^\delta \frac{1}{\lambda - i\tau} \frac{dP_\lambda f(\mathbf{x})}{d\lambda} d\lambda \right\}. \tag{2.27}$$

In order to study $R_{i\tau}$ as $\tau \downarrow 0$ we use (2.5), (2.6), (2.8) and

$$N_\sigma(\zeta) = \begin{cases} \displaystyle\sum_{s=0}^{\infty} C'_s \zeta^{2s-\sigma} & \left(\sigma + \frac{1}{2} \in \mathbb{N}_0\right), \\ \displaystyle\frac{2}{\pi} J_\sigma(\zeta) \left(C_e + \ln \frac{\zeta}{2}\right) + \sum_{s=0}^{\infty} C''_s \zeta^{2s+\sigma} + \sum_{s=0}^{\sigma-1} C'''_s \zeta^{2s-\sigma} & (\sigma \in \mathbb{N}_0), \end{cases} \tag{2.28}$$

where

$$\left. \begin{aligned} C'_s &= \frac{(-1)^{\sigma+s+1/2}}{2^{2s-\sigma} s! \Gamma(s+1-\sigma)}, \\ C''_s &= \frac{(-1)^{s+1}}{\pi 2^{\sigma+2s} s! (\sigma+s)!} \left(\sum_{r=1}^{s} \frac{1}{r} + \sum_{r=1}^{s+\sigma} \frac{1}{r} \right), \\ C'''_s &= -\frac{(\sigma - s - 1)!}{2^{2s-\sigma} \pi s!} \end{aligned} \right\} \tag{2.29}$$

(compare [4]). This, together with (2.10) and (2.27), implies that for odd n

$$I_2(\mathbf{x}, \delta) = u_0(\mathbf{x}) + \frac{1}{2m(2\pi)^{n/2}} \sum_{s=0}^{m-(n+1)/2} \frac{C_s}{\beta_s \delta^{\beta_s}} p_s(\mathbf{x}) + w_2(\mathbf{x}; \delta) \tag{2.30}$$

and for even n

$$I_2(\mathbf{x}, \delta) = u_0^*(\mathbf{x}) + \frac{1}{2m(2\pi)^{n/2}} \sum_{s=0}^{m-1-n/2} \frac{C_s}{\beta_s \delta^{\beta_s}} p_s(\mathbf{x})$$
$$- \frac{C_{m-n/2}}{2m(2\pi)^{n/2}} (\ln \delta + 2C_e) p_{m-n/2}(\mathbf{x}) + w_2(\mathbf{x}; \delta) \tag{2.31}$$

with $w_2(\mathbf{x}; \delta) \to 0$ as $\delta \downarrow 0$ uniformly in K, where

$$u_0(\mathbf{x}) := \begin{cases} -\dfrac{C'_{m-1}}{4(2\pi)^{n/2-1}} \displaystyle\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2m-n} d\mathbf{x}' & \text{if } 2m \geqslant n \text{ and } n \text{ odd}, \\ -\dfrac{C_{m-n/2}}{(2\pi)^{n/2}} \displaystyle\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2m-n} \ln|\mathbf{x} - \mathbf{x}'| d\mathbf{x}' & \text{if } 2m \geqslant n \text{ and } n \text{ even} \end{cases} \tag{2.32}$$

and

$$u_0^*(\mathbf{x}) := u_0(\mathbf{x}) - p_{m-n/2}(\mathbf{x}) \frac{1}{4(2\pi)^{n/2-1}} \left\{ \frac{2}{\pi} \left(C_e \frac{m-1}{m} - \ln 2 \right) C_{m-n/2} + C''_{m-n/2} \right\}. \tag{2.33}$$

Note that u_0 and u_0^* are solutions of (1.7). Since

$$I_3(\mathbf{x}, t; \delta) = o(1) \quad \text{as } t \to \infty \tag{2.34}$$

uniformly with respect to $\mathbf{x} \in K$, as a slight modification of the proof of Lemma 5.2 in [3] shows, we conclude from (2.23), (2.24), (2.30) and (2.31) that (1.8) and (1.9) hold uniformly in K. By (2.22), we have $u_0^* = u_0$ if f satisfies (1.11). In this case (1.8) and (1.9) reduce to (1.6).

Now we study the case $2m < n$. Let K be an arbitrary compact subset of $\mathbb{R}^n$. By (2.10) and (2.12) we obtain

$$I_1(\mathbf{x}; t; \delta) \to 0 \quad \text{as } \delta \downarrow 0 \tag{2.35}$$

uniformly with respect to $(\mathbf{x}, t) \in K \times [0, \infty)$. Taking into account that $R_{i\tau} f(\mathbf{x}) \to u_0(\mathbf{x})$ as $\tau \downarrow 0$ for $2m < n$ with

$$u_0(\mathbf{x}) := \frac{\Gamma(n/2 - m)}{\pi^{n/2} 4^m (m-1)!} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{n-2m}} d\mathbf{x}' \quad (2m < n), \tag{2.36}$$

we conclude from (2.27) that

$$I_2(\mathbf{x}; \delta) = u_0(\mathbf{x}) + o(1) \quad \text{as } \delta \downarrow 0 \tag{2.37}$$

uniformly in K. Thus it follows by (2.34) that (1.6) holds uniformly in K, where u_0 is given by (2.36). In particular, u_0 is a solution of (1.7). Thus we have verified the following Theorem:

Theorem 2.1. *Let $u \in C^{2m}(\mathbb{R}^n \times [0, \infty))$ be the unique solution of* (2.1), (2.2). *Then the following statements hold*:

1. *If $2m < n$, then* (1.6) *holds uniformly in every compact subset of $\mathbb{R}^n$, and u_0 is given by* (2.36).
2. *If $2m \geqslant n$, then the asymptotic behaviour of u as $t \to \infty$ is given by the estimates* (1.8) *and* (1.9), *which hold uniformly in every compact subset of $\mathbb{R}^n$. If, in addition, f satisfies* (1.11), *then* (1.6) *holds uniformly in every compact subset of $\mathbb{R}^n$; in this case u_0 is given by* (2.32).

3. The polyharmonic equation

3.1. *An example*

Assume that $2m \geqslant n$ and that $f \in C_0^\infty(\mathbb{R}^n)$ satisfies (1.11). Consider the solution u_0 of (1.7) given by (2.32). In order to find a condition that singles out u_0 among the solutions of (1.7), we study first the special case $m = n = 3$. Since

$$|\mathbf{x} - \mathbf{x}'|^3 = |\mathbf{x}|^3\left(1 + \frac{|\mathbf{x}'|^2 - 2\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2}\right)^{3/2}$$

$$= |\mathbf{x}|^3 \sum_{j=0}^{3} \binom{3/2}{j}\left(\frac{|\mathbf{x}'|^2 - 2\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2}\right)^j + O\left(\frac{1}{|\mathbf{x}|}\right) \quad \text{as } |\mathbf{x}| \to \infty, \tag{3.1}$$

we obtain by (1.11) and (2.32)

$$u_0(\mathbf{x}) = -\frac{C_2'}{4\sqrt{2\pi}} \int_{\mathbb{R}^3} f(\mathbf{x}')\left\{\frac{3}{2|\mathbf{x}|}[(\mathbf{x}\cdot\mathbf{x}')^2 - \mathbf{x}\cdot\mathbf{x}'|\mathbf{x}'|^2] + \frac{(\mathbf{x}\cdot\mathbf{x}')^3}{2|\mathbf{x}|^3}\right\}\mathrm{d}\mathbf{x}'$$

$$+ O\left(\frac{1}{|\mathbf{x}|}\right) \quad \text{as } |\mathbf{x}| \to \infty. \tag{3.2}$$

This formula shows that u_0 is unbounded as $|\mathbf{x}| \to \infty$ if for example one of the integrals $\int_{\mathbb{R}^n} f(\mathbf{x}') x_i'^2 \, \mathrm{d}\mathbf{x}' (i = 1, 2, 3)$ does not vanish. In particular, we have

$$u_0(\mathbf{x}) = O(|\mathbf{x}|) \quad \text{as } |\mathbf{x}| \to \infty. \tag{3.3}$$

This asymptotic condition does not suffice for the unique characterization of u_0, since also $u(\mathbf{x}) = u_0(\mathbf{x}) + \mathbf{c}\cdot\mathbf{x} + d$ with $\mathbf{c} \in \mathbb{R}^3$, $d \in \mathbb{R}$ is a further solution of (1.7) with the property (3.3). In order to characterize u_0 uniquely, note that

$$\int_{|\mathbf{x}| = R} u_0(\mathbf{x}) \, \mathrm{d}S_{\mathbf{x}} = O(R) \quad \text{as } R \to \infty,$$

since

$$\int_{\mathbb{R}^3} f(\mathbf{x}')\left\{\int_{|\mathbf{x}| = R} (\mathbf{x}\cdot\mathbf{x}')^2 \, \mathrm{d}S_{\mathbf{x}}\right\}\mathrm{d}\mathbf{x}' = \int_{|\mathbf{x}| = R} x_1^2 \, \mathrm{d}S_{\mathbf{x}} \int_{\mathbb{R}^3} f(\mathbf{x}')|\mathbf{x}'|^2 \, \mathrm{d}\mathbf{x}' = 0$$

by (1.11). Moreover, it can be shown in the same way that

$$\int_{|\mathbf{x} - \mathbf{x}_0| = R} u_0(\mathbf{x}) \, \mathrm{d}S_{\mathbf{x}} = O(R) \quad \text{as } R \to \infty \text{ for every } \mathbf{x}_0 \in \mathbb{R}^3. \tag{3.4}$$

Note that $u(\mathbf{x}) = u_0(\mathbf{x}) + \mathbf{c}\cdot\mathbf{x}$ satisfies the asymptotic estimate in (3.4) for $\mathbf{x}_0 = 0$. On the other hand, u_0 is the only function of the form $u(\mathbf{x}) = u_0(\mathbf{x}) + \mathbf{c}\cdot\mathbf{x} + d$, that satisfies (3.4) for every $\mathbf{x}_0 \in \mathbb{R}^3$. In the following we prove that u_0 is uniquely characterized by (1.7) and (3.4) in the general case.

3.2. *The uniqueness proof*

We prove:

Lemma 3.1. *If* $v \in C^{2m}(\mathbb{R}^n)$ *satisfies*

$$(-\Delta)^m v = 0 \quad \text{in } \mathbb{R}^n \tag{3.5}$$

and

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} v(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = o(R^{n-1}) \quad \textit{as } R \to \infty \textit{ for every } \mathbf{x}_0 \in \mathbb{R}^n, \tag{3.6}$$

then $v = 0$ *in* $\mathbb{R}^n$.

Remark. In the case $n = 1$ the integral in (3.6) has to be understood in the sense

$$\int_{|x-x_0|=R} v(x)\,\mathrm{d}S_x := v(x_0 + R) + v(x_0 - R). \tag{3.7}$$

Proof. Let $\mathbf{x}_0 \in \mathbb{R}^n$ be fixed and assume that $R > 0$. First we derive a representation of $\int_{|x-x_0|=R} g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}}$ for $g \in C^2(\mathbb{R}^n)$. We set $B_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n\colon\ \varepsilon < |\mathbf{x}-\mathbf{x}_0| < R\}$ for $0 < \varepsilon < R$. In the case $n \geqslant 3$ we use $\Delta_{\mathbf{x}}(1/|\mathbf{x}-\mathbf{x}_0|^{n-2}) = 0$ for $\mathbf{x} \neq \mathbf{x}_0$ and conclude from Green's formula that

$$\int_{B_\varepsilon} \frac{\Delta g(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^{n-2}}\,\mathrm{d}\mathbf{x} = \int_{\partial B_\varepsilon} \left\{ \frac{1}{|\mathbf{x}-\mathbf{x}_0|^{n-2}} \frac{\partial g(\mathbf{x})}{\partial \mathbf{n}} - g(\mathbf{x}) \frac{\partial}{\partial \mathbf{n}_{\mathbf{x}}} \frac{1}{|\mathbf{x}-\mathbf{x}_0|^{n-2}} \right\} \mathrm{d}S_{\mathbf{x}}, \tag{3.8}$$

where $\mathbf{n}$ denotes the normal unit vector on ∂B_ε pointing into the exterior of B_ε. Letting $\varepsilon \downarrow 0$, we obtain by the theorem of Gauss

$$\begin{aligned}
&\int_{|\mathbf{x}-\mathbf{x}_0| \leqslant R} \frac{\Delta g(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^{n-2}}\,\mathrm{d}\mathbf{x} \\
&= \frac{1}{R^{n-2}} \int_{|\mathbf{x}-\mathbf{x}_0|=R} \frac{\partial g(\mathbf{x})}{\partial \mathbf{n}}\,\mathrm{d}S_{\mathbf{x}} + \frac{n-2}{R^{n-1}} \int_{|\mathbf{x}-\mathbf{x}_0|=R} g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} - (n-2)\Gamma_n g(\mathbf{x}_0) \\
&= \frac{1}{R^{n-2}} \int_{|\mathbf{x}-\mathbf{x}_0| \leqslant R} \Delta g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} + \frac{n-2}{R^{n-1}} \int_{|\mathbf{x}-\mathbf{x}_0|=R} g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} - (n-2)\Gamma_n g(\mathbf{x}_0)
\end{aligned} \tag{3.9}$$

($\Gamma_n :=$ surface measure of the unit sphere in $\mathbb{R}^n$), and hence

$$\begin{aligned}
\int_{|\mathbf{x}-\mathbf{x}_0|=R} g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} &= \Gamma_n R^{n-1} g(\mathbf{x}_0) - \frac{R}{n-2} \int_{r=0}^{R} \left\{ \int_{|\mathbf{x}-\mathbf{x}_0|=r} \Delta g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} \right\} \mathrm{d}r \\
&\quad + \frac{R^{n-1}}{n-2} \int_{r=0}^{R} \frac{1}{r^{n-2}} \left\{ \int_{|\mathbf{x}-\mathbf{x}_0|=r} \Delta g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} \right\} \mathrm{d}r.
\end{aligned} \tag{3.10}$$

This formula holds also in the case $n = 1$ with $\Gamma_1 := 2$. In fact, integrating by parts twice, we obtain

$$\begin{aligned}
&\int_{B_\varepsilon} |\mathbf{x}-\mathbf{x}_0| g''(\mathbf{x})\,\mathrm{d}x \\
&\quad = R\{g'(x_0 + R) - g'(x_0 - R)\} - \varepsilon\{g'(x_0 + \varepsilon) - g'(x_0 - \varepsilon)\} \\
&\qquad - g(x_0 + R) - g(x_0 - R) + g(x_0 + \varepsilon) + g(x_0 - \varepsilon) \\
&\quad = R \int_{x_0 - R}^{x_0 + R} g''(x)\,\mathrm{d}x - \int_{|x-x_0|=R} g(x)\,\mathrm{d}S_x + 2g(x_0) + o(1) \quad \text{as } \varepsilon \downarrow 0,
\end{aligned} \tag{3.11}$$

and from this and (3.7), (3.10) follows.

In the case $n = 2$ we use $\Delta_{\mathbf{x}} \ln|\mathbf{x} - \mathbf{x}_0| = 0$ for $\mathbf{x} \neq \mathbf{x}_0$. As above Green's formula and the theorem of Gauss yield

$$\int_{|\mathbf{x}-\mathbf{x}_0| \leqslant R} \ln|\mathbf{x} - \mathbf{x}_0| \cdot \Delta g(\mathbf{x})\,\mathrm{d}\mathbf{x}$$
$$= (\ln R) \int_{|\mathbf{x}-\mathbf{x}_0| \leqslant R} \Delta g(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{1}{R} \int_{|\mathbf{x}-\mathbf{x}_0| = R} g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} + \Gamma_2 g(\mathbf{x}_0), \tag{3.12}$$

and therefore

$$\int_{|\mathbf{x}-\mathbf{x}_0| = R} g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \Gamma_2 R g(\mathbf{x}_0) + R \ln R \int_{r=0}^{R} \left\{ \int_{|\mathbf{x}-\mathbf{x}_0| = r} \Delta g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} \right\} \mathrm{d}r$$
$$- R \int_{r=0}^{R} (\ln r) \left\{ \int_{|\mathbf{x}-\mathbf{x}_0| = r} \Delta g(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} \right\} \mathrm{d}r. \tag{3.13}$$

Now we set $g := \Delta^{m-k} v$ and compute $\int_{|\mathbf{x}-\mathbf{x}_0| = R} \Delta^{m-k} v(\mathbf{x})\mathrm{d}S_{\mathbf{x}}$. Taking into account that v satisfies (3.5), we have for $k = 1$ by (3.10) and (3.13), respectively,

$$\int_{|\mathbf{x}-\mathbf{x}_0| = R} \Delta^{m-1} v(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \Gamma_n R^{n-1} \Delta^{m-1} v(\mathbf{x}_0).$$

If $n \neq 2$, then we obtain by (3.10) and induction with respect to k

$$\int_{|\mathbf{x}-\mathbf{x}_0| = R} \Delta^{m-k} v(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \Gamma_n R^{n-1} \Delta^{m-k} v(\mathbf{x}_0) + \sum_{j=1}^{k-1} c_{kj}(n) \Delta^{m-k+j} v(\mathbf{x}_0) R^{n-1+2j} \tag{3.14}$$

with suitable constants $c_{kj}(n) \in \mathbb{R}\backslash\{0\}$. If $n = 2$, then (3.13) and induction yield also (3.14), since

$$(R \ln R) \int_{r=0}^{R} r^j\,\mathrm{d}r - R \int_{r=0}^{R} (\ln r) r^j\,\mathrm{d}r = \frac{R^{j+2}}{(j+1)^2} \quad (j = 0, 1, \ldots).$$

Thus we have in $\mathbb{R}^n$ (with arbitrary $n \in \mathbb{N}$)

$$\int_{|\mathbf{x}-\mathbf{x}_0| = R} v(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \Gamma_n R^{n-1} v(\mathbf{x}_0) + \sum_{j=1}^{m-1} c_{mj}(n) \Delta^j v(\mathbf{x}_0) R^{n-1+2j} \tag{3.15}$$

for every solution $v \in C^{2m}(\mathbb{R}^n)$ of (3.5). This and (3.6) imply $v(\mathbf{x}_0) = 0$, which proves Lemma 3.1.

3.3. *The existence of the solution*

Lemma 3.1 implies that the problem

$$\left.\begin{aligned} (-\Delta)^m u &= f && \text{in } \mathbb{R}^n, \\ \int_{|\mathbf{x}-\mathbf{x}_0| = R} u(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} &= o(R^{n-1}) && \text{as } R \to \infty \text{ for every } \mathbf{x}_0 \in \mathbb{R}^n \end{aligned}\right\} \tag{3.16}$$

admits at most one solution $u \in C^{2m}(\mathbb{R}^n)$. If $2m < n$, then the function u_0 defined by (2.36) is the solution of (3.16). In fact, under the assumption $f \in C_0^1(\mathbb{R}^n)$ we have $u_0 \in C^{2m}(\mathbb{R}^n)$,

$$u_0(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right) \quad \text{as } |\mathbf{x}| \to \infty \tag{3.17}$$

and $(-\Delta)^m u_0 = f$ in $\mathbb{R}^n$, which follows from

$$(-\Delta)^{m-1}u_0(\mathbf{x}) = \frac{\Gamma(n/2-1)}{4\pi^{n/2}}\int_{\mathbb{R}^n}\frac{f(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{n-2}}\,\mathrm{d}\mathbf{x}'.$$

In the following we suppose that $2m \geqslant n$ and that $f\in C_0^1(\mathbb{R}^n)$ satisfies (1.11) and therefore (1.10). It is our aim to prove that the function u_0 given by (2.32) is the solution of (3.16). As above we have $u_0\in C^{2m}(\mathbb{R}^n)$ and $(-\Delta)^m u_0 = f$ in $\mathbb{R}^n$. Hence it suffices to verify the infinity condition in (3.16). For the sake of simplicity, we set

$$D(n,m) := \begin{cases} -\dfrac{C'_{m-1}}{4(2\pi)^{n/2-1}} & \text{if } n \text{ is odd,} \\ -\dfrac{C_{m-n/2}}{(2\pi)^{n/2}} & \text{if } n \text{ is even.}\end{cases} \tag{3.18}$$

First we study the case $n=1$. Let $x_0\in\mathbb{R}$ be fixed. We choose an $a>0$ such that $f(x)=0$ for $|x|>a$. For $R>\max\{a-x_0,\,a+x_0\}$ we obtain from (2.32)

$$u_0(x_0\pm R) = D(1,m)\sum_{j=0}^{2m-1}\binom{2m-1}{j}(\pm 1)^j R^{2m-1-j}\int_{-a}^{a} f(x')(x_0-x')^j\,\mathrm{d}x'$$

and

$$\begin{aligned}\int_{|x-x_0|=R} u_0(x)\,\mathrm{d}S_x &= u_0(x_0+R)+u_0(x_0-R)\\ &= 2D(1,m)\sum_{j=0}^{m-1}\binom{2m-1}{2j}R^{2m-1-2j}\int_{-a}^{a} f(x')(x_0-x')^{2j}\,\mathrm{d}x'.\end{aligned} \tag{3.19}$$

This implies by (1.10)

$$\int_{|x-x_0|=R} u_0(x)\,\mathrm{d}S_x = 0 \quad \text{for } R>\max\{a-x_0,\,a+x_0\}. \tag{3.20}$$

Hence u_0 is the solution of (3.16).

Now we study the case $n\geqslant 3$, n odd. By (2.32) and (3.18) we have with $\mathbf{z} := \mathbf{x}-\mathbf{x}_0$

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} \mathbf{u}_0(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = D(n,m)\int_{\mathbb{R}^n} f(\mathbf{x}')\left\{\int_{|\mathbf{z}|=R}|\mathbf{z}-(\mathbf{x}'-\mathbf{x}_0)|^{2m-n}\,\mathrm{d}S_{\mathbf{z}}\right\}\mathrm{d}\mathbf{x}'. \tag{3.21}$$

We use the expansion

$$\begin{aligned}|\mathbf{z}-\mathbf{z}'|^{2m-n} &= |\mathbf{z}|^{2m-n}\left(1+\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)^{(2m-n)/2}\\ &= |\mathbf{z}|^{2m-n}\sum_{j=0}^{2m-n}\binom{m-n/2}{j}\left(\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)^j + O\left(\frac{1}{|\mathbf{z}|}\right)\\ &= |\mathbf{z}|^{2m-n}\sum_{j=0}^{2m-n}\binom{m-n/2}{j}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{|\mathbf{z}'|}{|\mathbf{z}|}\right)^{j+k}\left(\frac{-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|\,|\mathbf{z}'|}\right)^{j-k} + O\left(\frac{1}{|\mathbf{z}|}\right)\end{aligned}$$

as $|\mathbf{z}|\to\infty$. Substituting $l := j+k$ in the inner sum, we obtain

$$|\mathbf{z}-\mathbf{z}'|^{2m-n} = \sum_{j=0}^{2m-n}\sum_{l=j}^{2j} d_{jl}|\mathbf{z}|^{2m-n-l}|\mathbf{z}'|^l\left(\frac{-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|\,|\mathbf{z}'|}\right)^{2j-l} + O\left(\frac{1}{|\mathbf{z}|}\right) \tag{3.22}$$

as $|\mathbf{z}| \to \infty$ with $d_{jl} = d_{jl}(n, m) := \binom{m - n/2}{j}\binom{j}{l-j}$. Inserting (3.22) into (3.21), we have

$$\begin{aligned}\int_{|\mathbf{x}-\mathbf{x}_0|=R} & u_0(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} \\ &= D(n,m)\sum_{j=0}^{2m-n}\sum_{l=j}^{2j} d_{jl}R^{2m-n-l}\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}'-\mathbf{x}_0|^l \\ &\quad\times\left\{\int_{|\mathbf{z}|=R}\left(\frac{-2\mathbf{z}\cdot(\mathbf{x}'-\mathbf{x}_0)}{|\mathbf{z}|\,|\mathbf{x}'-\mathbf{x}_0|}\right)^{2j-l}\mathrm{d}S_{\mathbf{z}}\right\}\mathrm{d}\mathbf{x}' + O(R^{n-2}) \qquad (3.23)\end{aligned}$$

as $R \to \infty$. Note that the inner integral does not depend on $\mathbf{x}'$:

$$\int_{|\mathbf{z}|=R}\left(\frac{-2\mathbf{z}\cdot(\mathbf{x}'-\mathbf{x}_0)}{|\mathbf{z}|\,|\mathbf{x}'-\mathbf{x}_0|}\right)^{2j-l}\mathrm{d}S_{\mathbf{z}} = \alpha(j,l,n)R^{n-1}, \qquad (3.24)$$

with

$$\alpha(j,l,n) := \int_{|\mathbf{z}|=1}(-2z_1)^{2j-l}\mathrm{d}S_{\mathbf{z}}. \qquad (3.25)$$

Thus we obtain

$$\begin{aligned}\int_{|\mathbf{x}-\mathbf{x}_0|=R} & u_0(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} \\ &= D(n,m)\sum_{j=0}^{2m-n}\sum_{l=j}^{2j} d_{jl}\alpha(j,l,n)R^{2m-1-l}\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}'-\mathbf{x}_0|^l\,\mathrm{d}\mathbf{x}' \\ &\quad + O(R^{n-2}) \quad \text{as } R\to\infty. \qquad (3.26)\end{aligned}$$

Note that it suffices to restrict the inner summation in (3.26) to even indices l with $l \leqslant 2m - n$, since $\alpha(j, l, n) = 0$ for odd l by (3.25) and $R^{2m-1-l} = O(R^{n-2})$ as $R \to \infty$ if $l \geqslant 2m - n + 1$. We substitute $k := l/2$ in (3.26) and change the order of the summations. Taking into account that n is odd, we conclude that

$$\begin{aligned}\int_{|\mathbf{x}-\mathbf{x}_0|=R} u_0(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} &= D(n,m)\sum_{k=0}^{m-(n+1)/2}\beta_k(n,m)R^{2m-1-2k} \\ &\quad\times\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}'-\mathbf{x}_0|^{2k}\,\mathrm{d}\mathbf{x}' + O(R^{n-2}) \quad \text{as } R\to\infty, \qquad (3.27)\end{aligned}$$

with

$$\beta_k(n,m) := \sum_{j=k}^{2k} d_{j,2k}\alpha(j,2k,n) = \sum_{j=k}^{2k}\binom{m-n/2}{j}\binom{j}{2k-j}\alpha(j,2k,n). \qquad (3.28)$$

Since we have assumed that f satisfies (1.10) it follows from (3.27) that

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u_0(\mathbf{x})\mathrm{d}S_{\mathbf{x}} = O(R^{n-2}) \quad \text{as } R\to\infty. \qquad (3.29)$$

This shows that u_0 is the solution of (3.16) if $n \geqslant 3$, n odd.

Finally we assume that n is even. By (2.32) and (3.18) we have with $\mathbf{z} := \mathbf{x} - \mathbf{x}_0$

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u_0(\mathbf{x})\,dS_{\mathbf{x}}$$
$$= D(n,m)\int_{\mathbb{R}^n} f(\mathbf{x}')\left\{\int_{|\mathbf{z}|=R} |\mathbf{z}-(\mathbf{x}'-\mathbf{x}_0)|^{2m-n}\ln|\mathbf{z}-(\mathbf{x}'-\mathbf{x}_0)|\,dS_{\mathbf{z}}\right\}d\mathbf{x}'. \quad (3.30)$$

It holds that

$$|\mathbf{z}-\mathbf{z}'|^{2m-n}\ln|\mathbf{z}-\mathbf{z}'| = (\ln|\mathbf{z}|)|\mathbf{z}-\mathbf{z}'|^{2m-n} + \frac{1}{2}|\mathbf{z}-\mathbf{z}'|^{2m-n}\ln\frac{|\mathbf{z}-\mathbf{z}'|^2}{|\mathbf{z}|^2}$$

and

$$|\mathbf{z}-\mathbf{z}'|^{2m-n} = |\mathbf{z}|^{2m-n}\sum_{j=0}^{2m-n}\binom{m-n/2}{j}\left(\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)^j$$

with

$$\binom{m-n/2}{j} := 0 \text{ for } j \geqslant m-n/2+1.$$

A Taylor expansion yields

$$|\mathbf{z}-\mathbf{z}'|^{2m-n}\ln\frac{|\mathbf{z}-\mathbf{z}'|^2}{|\mathbf{z}|^2}$$
$$= |\mathbf{z}|^{2m-n}\left(1+\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)^{m-n/2}\ln\left(1+\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)$$
$$= |\mathbf{z}|^{2m-n}\sum_{j=0}^{2m-n} c_j\left(\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)^j + O\left(\frac{1}{|\mathbf{z}|}\right) \quad \text{as } |\mathbf{z}|\to\infty$$

with suitable real constants c_j. Thus we have

$$|\mathbf{z}-\mathbf{z}'|^{2m-n}\ln|\mathbf{z}-\mathbf{z}'|$$
$$= |\mathbf{z}|^{2m-n}\sum_{j=0}^{2m-n}\left\{\binom{m-n/2}{j}\ln|\mathbf{z}| + \frac{c_j}{2}\right\}\left(\frac{|\mathbf{z}'|^2-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}|^2}\right)^j + O\left(\frac{1}{|\mathbf{z}|}\right) \quad (3.31)$$

as $|\mathbf{z}|\to\infty$. By the argument leading to (3.22) it follows that

$$|\mathbf{z}-\mathbf{z}'|^{2m-n}\ln|\mathbf{z}-\mathbf{z}'|$$
$$= \sum_{j=0}^{2m-n}\sum_{l=j}^{2j}\{d_{jl}\ln|\mathbf{z}| + d'_{jl}\}|\mathbf{z}|^{2m-n-l}|\mathbf{z}'|^l\left(\frac{-2\mathbf{z}\cdot\mathbf{z}'}{|\mathbf{z}||\mathbf{z}'|}\right)^{2j-l} + O\left(\frac{1}{|\mathbf{z}|}\right) \quad (3.32)$$

as $|\mathbf{z}|\to\infty$, where

$$d'_{jl} := \frac{c_j}{2}\binom{j}{l-j}.$$

Inserting (3.32) into (3.30), we obtain by (3.24)

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u_0(\mathbf{x})\,dS_{\mathbf{x}} = D(n,m)\sum_{j=0}^{2m-n}\sum_{l=j}^{2j}\{d_{jl}\ln R + d'_{jl}\}\alpha(j,l,n)R^{2m-1-l}$$
$$\times \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}'-\mathbf{x}_0|^l\,d\mathbf{x}' + O(R^{n-2}) \quad \text{as } R\to\infty. \quad (3.33)$$

As in (3.26) it suffices to restrict the inner summation in (3.33) to even indices l with $l \leqslant 2m - n$. Setting $k := l/2$ and changing the order of the summations we conclude that

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u_0(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = D(n,m) \sum_{k=0}^{m-n/2} \{\beta_k(n,m)\ln R + \beta_k'(n,m)\} R^{2m-1-2k} \times \int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}'-\mathbf{x}_0|^{2k}\mathrm{d}\mathbf{x}' + O(R^{n-2}) \quad \text{as } R \to \infty. \tag{3.34}$$

with

$$\beta_k'(n,m) := \sum_{j=k}^{2k} d_{j,2k}'\alpha(j,2k,n). \tag{3.35}$$

From (3.34) and (1.10) it follows (3.29). Therefore the function u_0 is the solution of (3.16) in the case of even n. Hence we have proved:

Theorem 3.1 *Let $f \in C_0^1(\mathbb{R}^n)$. Furthermore assume that $2m < n$ or that $2m \geqslant n$ and f satisfies* (1.11). *Then problem* (3.16) *has a unique solution $u \in C^{2m}(\mathbb{R}^n)$, which is given by* (2.32).

3.4. *An alternative theorem*

In the case $2m < n$, problem (3.16) has a solution for every $f \in C_0^1(\mathbb{R}^n)$ by Theorem 3.1. In the case $2m \geqslant n$ we prove the following alternative:

Theorem 3.2. *Assume that $2m \geqslant n$ and that $f \in C_0^1(\mathbb{R}^n)$. Then*:

1. *If f satisfies* (1.11), *then problem* (3.16) *has a uniquely determined solution $u \in C^{2m}(\mathbb{R}^n)$.*
2. *If* (1.11) *is not valid, then* (3.16) *has no solution $u \in C^{2m}(\mathbb{R}^n)$.*

Proof. It suffices to prove part 2 of the theorem, since part 1 is contained in Theorem 3.1. We suppose that $2m \geqslant n$ and that $u \in C^{2m}(\mathbb{R}^n)$ is a solution of (3.16). We show that f satisfies (1.11).

We set $v := u - u_0$, where u_0 is given by (2.32). Let $\mathbf{x}_0 \in \mathbb{R}^n$ be fixed. Note that (3.15) holds, since v is a solution of the homogenous equation (3.5). We combine (3.15) with (3.19) in the case $n = 1$, with (3.27) in the case $n \geqslant 3$, n odd and with (3.34) in the case of even n. Then we obtain for odd n

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \int_{|\mathbf{x}-\mathbf{x}_0|=R} \{v(\mathbf{x}) + u_0(\mathbf{x})\}\,\mathrm{d}S_{\mathbf{x}} = \sum_{k=0}^{m-1} \gamma_k R^{n-1+2k} + \sum_{k=0}^{m-(n+1)/2} \gamma_k' R^{2m-1-2k} + O(R^{n-2}) \tag{3.36}$$

and for even n

$$\int_{|\mathbf{x}-\mathbf{x}_0|=R} u(\mathbf{x})\,\mathrm{d}S_{\mathbf{x}} = \sum_{k=0}^{m-1} \gamma_k'' R^{n-1+2k} + \sum_{k=0}^{m-n/2} \gamma_k'(\ln R) R^{2m-1-2k} + O(R^{n-2}) \tag{3.37}$$

as $R \to \infty$; here the constants $\gamma_k \in \mathbb{R}$ depend on v and $\gamma_k'' \in \mathbb{R}$ depend on v and f, since the first sum in (3.37) contains a part of the sum in (3.34). Furthermore,

$$\gamma'_k = D(n, m)\beta_k(n, m)\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2k}d\mathbf{x}' \quad \text{for } 0 \leqslant k \leqslant \left[m - \frac{n}{2}\right] \tag{3.38}$$

with $\beta_k(n, m)$ defined by (3.28) if $n \geqslant 2$ and by

$$\beta_k(1, m) := 2\binom{2m-1}{2k} \quad (0 \leqslant k \leqslant m - 1) \tag{3.39}$$

(compare (3.19)).

Note that in (3.36) the exponents $n - 1 + 2k$ in the first sum are even and that the exponents $2m - 1 - 2k$ in the second sum are odd. Since u is supposed to satisfy the asymptotic condition in (3.16) and since $2m \geqslant n$, it follows from (3.36) and (3.37), respectively, that $\gamma_k = \gamma'_k = 0$ if n is odd and $\gamma''_k = \gamma'_k = 0$ if n is even. Since $D(n, m) \neq 0$ for every $n, m \in \mathbb{N}$ by (3.18), (2.9) and (2.29), we have to show that $\beta_k(n, m) \neq 0$ for $0 \leqslant k \leqslant [m - (n/2)]$. Then (3.38) and $\gamma'_k = 0$ for $0 \leqslant k \leqslant [m - (n/2)]$ imply that f satisfies (1.10) and therefore (1.11).

If $n = 1$, we have $\beta_k(n, m) \neq 0$ by (3.39). In the case $n \geqslant 2$ we consider (3.28). Note that by (3.25)

$$\alpha(j, 2k, n) = \int_{|\mathbf{z}|=1} (-2z_1)^{2j-2k}dS_{\mathbf{z}} = 2^{2j-2k+1}\pi^{(n-1)/2}\frac{\Gamma(j-k+\frac{1}{2})}{\Gamma(j-k+\frac{n}{2})}. \tag{3.40}$$

We set

$$\delta_j(k, n, m) := \pi^{(n-1)/2}2^{2j-2k+1}\frac{\Gamma(j-k+\frac{1}{2})}{\Gamma(j-k+\frac{n}{2})}\binom{m-n/2}{j}\binom{j}{2k-j} \tag{3.41}$$

$$\left(\text{with } \binom{m-n/2}{j} := 0 \text{ if } n \text{ is even and } j \geqslant m + 1 - n/2\right).$$

Then we have by (3.28) and (3.40)

$$\beta_k(n, m) = \sum_{j=k}^{2k} \delta_j(k, n, m). \tag{3.42}$$

It holds that

$$\delta_{j+1}(k+1, n, m+1) = \frac{m+1-\frac{n}{2}}{k+1}\left\{\delta_j(k, n, m) + \frac{m-\frac{n}{2}}{\pi}\delta_{j-1}(k, n+2, m)\right\}$$

for $k + 1 \leqslant j \leqslant 2k$ and

$$\delta_{k+1}(k+1, n, m+1) = \frac{m+1-\frac{n}{2}}{k+1}\delta_k(k, n, m),$$

$$\delta_{2k+2}(k+1, n, m+1) = \frac{m+1-\frac{n}{2}}{k+1}\frac{m-\frac{n}{2}}{\pi}\delta_{2k}(k, n+2, m).$$

Hence it follows that

$$\beta_{k+1}(n, m+1) = \frac{m+1-\frac{n}{2}}{k+1}\left\{\beta_k(n, m) + \frac{m-\frac{n}{2}}{\pi}\beta_k(n+2, m)\right\}. \tag{3.43}$$

Taking into account that

$$\beta_0(n, m) = 2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \tag{3.44}$$

for $n, m \in \mathbb{N}$, we obtain by induction

$$\beta_k(n, m) = 2\frac{\pi^{n/2}}{\Gamma(\frac{n+2k}{2})}\frac{(m-1)!}{(m-k-1)!}\binom{m-n/2}{k} \neq 0 \tag{3.45}$$

for $0 \leqslant k \leqslant [m - (n/2)]$ $(n \geqslant 2)$. This concludes the proof of Theorem 3.2.

4. Remarks

1. Assume that $f \in C_0^\infty(\mathbb{R}^n)$. Then the problems (2.1), (2.2) and (3.16) are related: the solution $u(\mathbf{x}, t)$ of (2.1), (2.2) converges to a limit $u_0(\mathbf{x})$ as $t \to \infty$ if and only if (3.16) has a solution. In this case the limit u_0 is the unique solution of (3.16).
2. The alternative Theorem 3.2 says that (3.16) has a solution if and only if $f \in C_0^1(\mathbb{R}^n)$ is orthogonal to the polynomial solutions of $(-\Delta)^m v = f$ in $\mathbb{R}^n$ given by

$$p_{j\alpha}(\mathbf{x}) := |\mathbf{x}|^{2j}\mathbf{x}^\alpha \text{ with } j \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n,\ j + |\alpha| \leqslant \left[m - \frac{n}{2}\right]. \tag{4.1}$$

If $2m < n$, then the set of the polynomials (4.1) is empty in agreement with the fact that problem (3.16) has a unique solution for every $f \in C_0^1(\mathbb{R}^n)$ in this case. If $m = 1$ and $n \leqslant 2$, then $p(\mathbf{x}) = 1$ is the only polynomial of the form (4.1). Thus the polynomials (4.1) can be considered as a generalization of the standing wave 1, introduced by Morgenröther and Werner [5] in the special case $m = 1$, to equations of arbitrary order $2m$. The polynomials (4.1) occur in the resonance terms in (1.8) and (1.9), since

$$\begin{aligned}\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{2s}\,d\mathbf{x}' &= \sum_{j+k+l=s} \frac{s!}{j!\,k!\,l!}\int_{\mathbb{R}^n} f(\mathbf{x}')|\mathbf{x}|^{2j}(-2\mathbf{x}\cdot\mathbf{x}')^k|\mathbf{x}'|^{2l}\,d\mathbf{x}' \\ &= \sum_{j+k+l=s}\frac{s!}{j!\,k!\,l!}\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n} f(\mathbf{x}')p_{l\alpha}(\mathbf{x}')\,d\mathbf{x}'\,p_{j\alpha}(\mathbf{x})\end{aligned}$$

$$\left(s = 0, 1, \ldots, \left[m - \frac{n}{2}\right],\ c_\alpha \in \mathbb{R}\right). \tag{4.2}$$

3. From Lemma 3.1 it follows that the problem

$$\left.\begin{aligned}(-\Delta)^m u &= f \quad \text{in } \mathbb{R}^n,\\ u(\mathbf{x}) &= o(1) \quad \text{as } |\mathbf{x}| \to \infty\end{aligned}\right\} \tag{4.3}$$

has at most one solution. In the case $m = 1$ this result is a well known consequence of the maximum principle. Note that the maximum principle does not hold in the case $m > 1$, as the solution $p(\mathbf{x}) = -|\mathbf{x}|^2$ of $(-\Delta)^m p = 0$ shows.

If $2m < n$, then problem (4.3) has a solution, which is given by (2.36). In the case $2m = n$, (4.3) has a solution if and only if $f \in C_0^1(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} f(\mathbf{x}')\,d\mathbf{x}' = 0$.

This follows from (2.32), the asymptotic estimate

$$\int_{\mathbb{R}^n} f(\mathbf{x}') \ln|\mathbf{x} - \mathbf{x}'| \, d\mathbf{x}' = \ln|\mathbf{x}| \int_{\mathbb{R}^n} f(\mathbf{x}') \, d\mathbf{x}' + O\left(\frac{1}{|\mathbf{x}|}\right) \quad \text{as } |\mathbf{x}| \to \infty \tag{4.4}$$

and the second part of Theorem 3.2. If $2m > n$, then (4.3) may have no solution, even if f satisfies (1.11), as the example at the beginning of Section 3 shows.

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