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HOMOLOGIES AND ELATIONS IN COMPACT, CONNECTED PROJECTIVE PLANES

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In a compact, connected topological projective plane, let Ω be a closed Lie subgroup of the group of all axial collineations with a fixed axis A. We compare the set $\mathfrak{Z}\setminus A$ consisting of the centres of all non-identical homologies in Ω to orbits of the group $\Omega_{[A]}$ of all elations contained in Ω and of its connected component $\Theta = (\Omega_{[A]})^1$. It is shown that $\mathfrak{Z}\setminus A$ is the union of at most countably many Θ-orbits; moreover, $\mathfrak{Z}\setminus A$ turns out to be a single Θ-orbit whenever the connected component of Ω contains non-identical homologies. This result is analogous to a well-known theorem of André for finite planes. It has numerous consequences for the structure of collineation groups of compact, connected projective planes.

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topological projective plane   elation   homology

1. Introduction, results

Throughout this paper, we consider a compact, connected topological projective plane. Its point space will be denoted by $\mathbf{P}$.

Collineations fixing all the points of a line A are called *axial collineations* (or *perspectivities*) with *axis* A. For such collineations there is always a *centre*, i.e. a point such that all the lines through this point are invariant; this notion is dual to the notion of an axis. If, for an axial collineation α with axis A and centre z, the image $\alpha(p_0)$ of a single point p_0 different from the centre and not lying on the axis is known, then the images of all other points may be obtained by the geometric construction illustrated in Fig. 1. In particular, if α is not the identity, then α has no fixed point besides z and the points of A. Therefore, a non-identical perspectivity has a unique axis and a unique centre. A perspectivity with axis A and centre z is called an *elation* or a *homology* depending on whether $z \in A$ or $z \notin A$. The non-identical elations with axis A are just the perspectivities without any fixed point outside A. Therefore they form, together with the identity, a normal subgroup of the group of all collineations fixing A, and this elation group acts freely outside A.

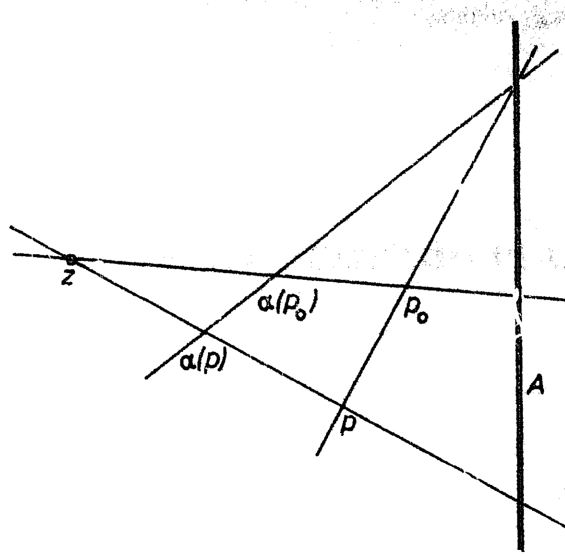

Fig. 1.

In a topological projective plane, where join and intersection are continuous, it is immediate from the construction in Fig. 1 that axial collineations are continuous. In the given compact, connected plane, we now consider, for a fixed line A, the group of all axial collineations with axis A, and we endow it with the compact-open topology.

The aim of this paper is to determine the configurations which are obtained by collecting the centres of all non-identical homologies belonging to a closed subgroup Ω of the group of all perspectivities with axis A. This will be carried out under the extra assumption that Ω is a Lie group. Probably, this is no restriction at all; some comments on this question will be made at the end of this section.

The following conventions will be used: For a point z the group of all perspectivities in Ω with centre z is denoted by $\Omega_{[z]}$;

$$\Omega_{[A]} = \bigcup_{a \in A} \Omega_{[a]}$$

denotes the normal subgroup of all elations in Ω. Let Ξ and Θ be the connected components of the identity in Ω and $\Omega_{[A]}$, respectively.

We are interested in the set

$$\mathfrak{Z} = \{z \in \mathbf{P};\ \Omega_{[z]} \neq \{\mathrm{id}\}\}$$

of all centres of non-identical elements in Ω. More generally, for a subset Ψ of Ω let

$$\mathfrak{Z}(\Psi) = \{z \in \mathbf{P};\ \Omega_{[z]} \cap \Psi \neq \{\mathrm{id}\}\}.$$

The results of this paper are mainly concerned with the set

$$\mathfrak{Z}\backslash A = \mathfrak{Z}(\Omega \backslash \Omega_{[A]})$$

of all centres of non-identical homologies in Ω.

$\mathfrak{Z}$ and $\mathfrak{Z}\setminus A$ are invariant under the normalizer of Ω in the collineation group of the plane, since for any collineation γ normalizing Ω we have

$$\gamma\Omega_{[z]}\gamma^{-1} = \Omega_{[\gamma(z)]} \tag{1}$$

as is easily verified.

In particular, $\mathfrak{Z}\setminus A$ is the union of orbits under the connected component Θ of the elation group $\Omega_{[A]}$. Our main result determines the configurations, which $\mathfrak{Z}\setminus A$ may possibly take on, by relating $\mathfrak{Z}\setminus A$ to the Θ-orbits contained in it:

1.1. Theorem. *Let Ω be a closed Lie subgroup of the group of all axial collineations with axis A. Then precisely one of the following assertions holds:*

(o) *$\mathfrak{Z}\setminus A$ is empty, i.e. Ω consists entirely of elations.*

(i) *$\mathfrak{Z}\setminus A$ is a point.*

(ii) *$\mathfrak{Z}\setminus A$ consists of more than one but at most countably many points. In this case, Ω is discrete and at most countable.*

(iii) *$\mathfrak{Z}\setminus A$ is a connected manifold of positive dimension. It is then an orbit of the elation group $\Omega_{[A]}$, which in this case must be connected. In particular, $\mathfrak{Z}\setminus A$ is closed in $\mathbb{P}\setminus A$ and homeomorphic to $\Omega_{[A]}$.*

(iv) *$\mathfrak{Z}\setminus A$ is the union of more than one, but at most countably many orbits under the connected component Θ of the elation group $\Omega_{[A]}$, and Θ is non-trivial. Each of these orbits is closed in $\mathbb{P}\setminus A$ and homeomorphic to Θ. In this case, the connected component Ξ of Ω contains elations only, so that $\Xi = \Theta$.*

This theorem will be proved in Sections 3 and 4. We continue the present section by making some comments on the theorem and by giving some typical applications of it.

Assertion (iii) of the theorem may be regarded as the generic case since it occurs whenever the connected component of Ω contains homologies for more than one centre. This case represents an analogy to the well-known result of André [1] that in finite planes $\mathfrak{Z}\setminus A$ is an orbit of $\Omega_{[A]}$. In compact, connected planes, however, this is not generally true; in Section 2 we shall consider counterexamples due to H. Salzmann.

The theorem says that $\mathfrak{Z}\setminus A$, if not empty, is the union of at most countably many orbits of Θ. Since these orbits are homeomorphic to Θ (see 3.1), we obtain rather precise information if we can narrow down the possibilities for the homeomorphism type of Θ. Now this is possible in broad generality; in fact one may prove (see 3.2 and 3.7):

1.2. Supplement. *Suppose that one of the following conditions is fulfilled:*

(a) *The point space $\mathbb{P}$ is of finite topological dimension.*

(b) *There are two non-identical elations with axis A and with different centres.*

(c) *There is a centre $z \in \mathbb{P}\setminus A$ for which the homology group $\Omega_{[z]}$ is not compact.*

Then Θ is homeomorphic to a euclidean space $\mathbb{R}^k$ $(k \geq 0)$.

It is conceivable that compact connected planes of infinite dimension do not exist (much in the same way as there are no locally compact topological vector spaces of infinite dimension). This would mean that the conclusion of the supplement holds in general.

Cases (iii) and (iv) of the theorem really are mutually exclusive, since Θ as a Lie group is always a manifold, and since a connected manifold M cannot be the disjoint union of at most countably many closed subspaces M_i which are also manifolds: In fact, we would have $\dim M_i < \dim M$ for all i (or else $M_i = M$ by the Brouwer invariance theorem) and therefore, by the countable sum theorem of dimension theory, $\dim(\bigcup_i M_i) = \max \dim M_i < \dim M$.

We now present some consequences of the theorem stated above. Generally speaking, it may be used to obtain information about collineation groups in the presence of homologies. As an example, we note the following corollary which may be derived from the theorem by exploiting the conjugation formula (1):

1.3. Corollary. *Let Γ be a closed Lie subgroup of the group of continuous collineations leaving the line A invariant, and denote its connected component by Δ.*

Assume further that Γ contains a non-identical homology with axis A whose centre $z \notin A$ is not a fixed point of Δ. Then the group $\Delta_{[A,A]}$ of elations in Δ with axis A is connected, and the orbit $\Delta(z)$ is equal to the orbit $\Delta_{[A,A]}(z)$ under this elation group; in particular, $\Delta(z)$ is closed in $\mathbb{P}\backslash A$ and homeomorphic to $\Delta_{[A,A]}$.

If Γ even contains a non-trivial connected subgroup of homologies with axis A, then $\Gamma(z) = \Delta(z)$, and all elations of Γ with axis A are contained in Δ.

For the special case where the given homologies are reflections (i.e. of order two), results of this type may be proved directly, using well known techniques of generating elations from reflections. In the proof of the theorem above, we make use of such a specialized result (lemma 4.6), of course with an independent proof.

One of the features of the theorem is that it exhibits many elations. This leads to characterizations of translation planes by conditions which postulate the existence of many homologies. The standard condition of this kind would be to ask that every point not on A should be the centre of a non-identical homology (see [5, p. 104, corollary 1] for such a result in finite projective planes). In compact connected planes, this condition may be weakened by topological methods:

1.4. Corollary. *In addition to the assumptions of the theorem, suppose that Ω contains, for each point z in a subset of second Baire category in $\mathbb{P}\backslash A$, a non-identical homology with centre z. Then the elation group $\Omega_{[A]}$ is transitive on $\mathbb{P}\backslash A$.*

Indeed, under these assumptions the set $\mathfrak{Z}\backslash A$ is uncountable; by the theorem above, it is therefore the union of at most countably many closed subsets of $\mathbb{P}\backslash A$ each of which is an orbit under the connected elation group Θ. By category reasons, one of these closed orbits must contain an open subset. This orbit then is both closed and open in $\mathbb{P}\backslash A$, and since $\mathbb{P}\backslash A$ is connected [7], Θ is transitive on $\mathbb{P}\backslash A$.

In our theorem, a special case of particular importance arises if the centres of the collineations in Ω all lie on a fixed line L different from A. For this case, partial results are already available in the literature (see [8] and [12]). However, these results have been developed for rather special situations only and by using arguments which do not lend themselves to generalizations. Therefore it might be worthwhile to write down explicitly what our theorem says in this special case. Note that in this situation the intersection point $a = A \wedge L$ is the centre of all elations in Ω. Every orbit $\Theta(z)$ of a point $z \in \mathbb{P}\backslash A$ under the connected component Θ of $\Omega_{[A]}$ is therefore a subset of the line $z \vee a$ joining z to a. If, for the sake of concreteness, we integrate some hypothesis of the type considered in Supplement 1.2, then $\Theta(z)$ is homeomorphic to a euclidean space $\mathbb{R}^k$. In particular, if $k > 0$, $\Theta(z)$ is not closed in the compact projective line $z \vee a$. However, $\Theta(z)$ is a closed subset of $\mathbb{P}\backslash A$ (see 3.1) and is therefore closed in $z \vee a\backslash\{a\}$; in other words, $\Theta(z) \cup \{a\}$ is the one-point compactification of $\Theta(z) \approx \mathbb{R}^k$, that is, a k-sphere. Thus we have obtained:

1.5. Corollary. *Let A and L be two different lines, and let Λ be a closed Lie subgroup of the group of all axial collineations whose axis is A and whose centres lie on L. Suppose either that $\mathbb{P}$ is of finite topological dimension, or that for some $z \in \mathbb{P}\backslash A$ the homology group $\Lambda_{[z]}$ is not compact.*

Then for the set $\mathfrak{Z}(\Lambda)$ of all the centres of non-identical elements in Λ one of the following descriptions is valid:

(i) *$\mathfrak{Z}(\Lambda)$ is empty or a point.*

(ii) *$\mathfrak{Z}(\Lambda)$ consists of more than one but at most countably many points, and Λ is discrete and at most countable.*

(iii) *$\mathfrak{Z}(\Lambda)$ is a sphere of positive dimension consisting of the point $a = A \wedge L$ and of an orbit under the group $\Lambda_{[a]}$ of elations in Λ (which in this case must be connected).*

(iv) *$\mathfrak{Z}(\Lambda)$ is a bouquet of more than one but at most countably many spheres of equal positive dimension. In this case, the connected component Λ^1 of the identity in Λ contains only elations, and each sphere of the bouquet consists of the point $a = A \wedge L$ and of an orbit under this elation group Λ^1.*

Paralleling Corollary 1.4, one also gets the following result about transitivity properties of the elation group in the presence of many homologies with centres on a fixed line L:

1.6. Corollary. *Let A and L be two different lines, and let Λ be a closed Lie subgroup of the group of all axial collineations whose axis is A and whose centres lie on L. Suppose further than Λ contains, for each point z in a subset of second Baire category in L, a non-identical homology with centre z and axis A. Then the elation group $\Lambda_{[a]}$ ($a = A \wedge L$) is transitive on $L\backslash\{a\}$.*

Corollaries 1.4 and 1.6 together may be used to formulate characterizations of the classical planes over the real numbers, the complex numbers, the quaternions and the Cayley numbers in terms of existence of sufficiently many axial collineations.

We close this exposition of results by a remark concerning the hypothesis that the collineation groups under consideration be Lie groups. This precaution might be superfluous; it is generally conjectured that the group G_c of all continuous collineations of a compact, connected plane is always a Lie group. For at most four-dimensional planes this has been proved (see [9, 4.1], [11, 3.9]). It is also known to be true for special types of planes, e.g. translation planes (where G_c is a linear group, cf. [4, 3.2]). In general, at present we know only that G_c is locally compact and has a countable basis ([6, 2.9], see also [13, *]).

2. A counterexample

The following family of examples is due to H. Salzmann [8]. For fixed $a \in \mathbb{R}\setminus\{0\}$ with $|a|<1$ put

$$q(u, v) = a \cos 2\pi u \cos 2\pi v,$$

$$p(x, y) = \int_0^x \int_0^y q(u, v)\, dv\, du,$$

and define, for $s, x \in \mathbb{R}$, a new multiplication $\circ$ by

$$s \circ x = sx + p(s, x).$$

Salzmann states in [8, § 4] that one gets an affine plane with point set $\mathbb{R}^2$ by taking as lines the subsets $\{(x, s \circ x + t); x \in \mathbb{R}\}$ for $s, t \in \mathbb{R}$ and the verticals $\{(x_0, y); y \in \mathbb{R}\}$ for $x_0 \in \mathbb{R}$. It follows from general results [10, 2.12 and 7.17] that the corresponding projective plane, which is obtained by adjoining a line A at infinity, is a compact connected topological plane.

Furthermore Salzmann proves that the group of translations (elations whose axis is the line A at infinity) of this plane consists of the transformations

$$(x, y) \mapsto (x + n, y + t) \quad \text{with } n \in \mathbb{Z}, t \in \mathbb{R}.$$

Using the identity

$$s \circ (-x) = -(s \circ x), \tag{2}$$

which follows easily from the symmetry properties of $q(u, v)$, we show that the transformation

$$(x, y) \mapsto (-x, -y)$$

is a homology with centre $(0, 0)$ and axis A: Indeed, it maps the point $(x, s \circ x + t)$ to $(-x, -(s \circ x) - t)$, which by (2) is equal to $(-x, s \circ (-x) - t)$; so the line $(x, s \circ x + t)$ with slope s through $(0, t)$ is mapped onto the line with the same slope s through $(0, -t)$.

By composition with the translation $(x, y) \mapsto (1+x, y)$ one obtains another homology with axis A, namely

$$(x, y) \mapsto (1-x, -y);$$

the centre of this homology is the fixed point $(\frac{1}{2}, 0)$.

This centre is not contained in the orbit $\mathbb{Z} \times \mathbb{R}$ of the centre $(0, 0)$ under the translation group; thus the set of all centres of non-identical homologies with axis A contains more than one orbit of the group of all elations with axis A.

3. Preparations for the proof

In this section, we study the structure of groups of elations and of groups of homologies with fixed axis and centre, and the orbits of such groups. Most of the facts collected here are part of the folklore on the subject; nevertheless they cannot be found ready to use in the literature.

3.1. *Let Φ be a closed subgroup of the group of elations with axis A. Then the orbits of Φ are closed in $\mathbb{P} \backslash A$ and homeomorphic to Φ.*

For the proof, we may use sequence arguments since $\mathbb{P}$ has a countable basis for its topology [10, 7.10]. Let $z \in \mathbb{P} \backslash A$, and assume that $p \in \mathbb{P} \backslash A$ is an accumulation point of the orbit $\Phi(z)$; then there is a sequence (φ_n) in Φ with $\lim \varphi_n(z) = p$. Since the line A is compact, we may assume that the centres $a_n \in A$ of the elations φ_n converge to $a \in A$. Let z' be any point of $\mathbb{P} \backslash A$ not contained in the lines $z \vee a_n$ $(n \in \mathbb{N})$ and $z \vee a$ joining z to a_n and to a. Since $\varphi_n(z')$ may be constructed geometrically as

$$(((z \vee z') \wedge A) \vee \varphi_n(z)) \wedge (z' \vee a_n),$$

it is clear that $\varphi_n(z')$ converges to the point

$$p' = (((z \vee z') \wedge A) \vee p)) \wedge (z' \vee a).$$

As is shown in [6, Lemma 3.7, p. 264], one can therefore construct an axial collineation φ with axis A mapping z to p and z' to p'. This axial collineation is uniquely determined, and is the limit of the φ_n in the compact-open topology. Since Φ is closed, φ is contained in Φ; therefore we have $p \in \Phi(z)$, and the orbits of Φ are closed in $\mathbb{P} \backslash A$. Moreover, this argument proves the following: If, for a fixed $z \in \mathbb{P} \backslash A$, the images of z under a sequence of elations $\varphi_n \in \Phi$ converge to $\varphi(z)$ for $\varphi \in \Phi$, then every subsequence of (φ_n) contains a sub-subsequence which converges to φ, so that φ is in fact the limit of the whole sequence (φ_n). This shows that the inverse of the continuous bijective map $\Phi \to \Phi(z) : \varphi \mapsto \varphi(z)$ is also continuous and therefore is a homeomorphism.

3.2. *Suppose that there are two non-identical elations with axis A and with different* *centres. Then a closed connected Lie subgroup Φ of the group of elations with axis A is isomorphic to a vector group* $\mathbb{R}^k$.

This follows at once from the well-known basic fact that in this situation all elations with axis A commute and have the same order, finite or infinite (see for instance [5, p. 97 Theorem 4.14]).

3.3. *Let Φ be a closed connected Lie subgroup of the group of elations with axis A, and let $\omega \neq \mathrm{id}$ be a homology with axis A which normalizes Φ. Then the set $\mathfrak{Z}(\Phi\omega)$ of centres of the perspectivities in the coset $\Phi\omega$ is the orbit $\Phi(z)$ of the centre z of ω.*

Proof. Either we have $\Phi = \Phi_{[a]}$ for some $a \in A$, or Φ contains elations with different centres and is a vector group by 3.2. In the latter case, Φ is the union of its one-parameter subgroups. A one-parameter subgroup Φ_1, being a homomorphic image of $\mathbb{R}$, contains a dense subgroup which is locally cyclic (e.g. the image of $\mathbb{Q} \subseteq \mathbb{R}$) so that any two of its elements have the same centre. By continuity, the same is true for all elements of Φ_1. Thus, each one-parameter subgroup is contained in $\Phi_{[a]}$ for some $a \in A$. This implies that the subgroups $\Phi_{[a]}$ are connected. Furthermore, as a consequence of the conjugation formula (1), they too are normalized by ω.

Thus, for any $a \in A$, the subgroup $\Psi = \Phi_{[a]}$ shares the properties which were required of Φ. We now prove $\mathfrak{Z}(\Psi\omega) = \Psi(z)$; the same conclusion for Φ will then follow from $\Phi = \bigcup_{a \in A} \Phi_{[a]}$.

For $\psi \in \Psi$, the point $\psi(z)$ is the centre of the conjugate $\psi\omega\psi^{-1} \in \Psi\omega\Psi$. Since Ψ is normalized by ω, we have $\Psi\omega\Psi = \Psi\Psi\omega \subseteq \Psi\omega$; so $\Psi(z) \subseteq \mathfrak{Z}(\Psi\omega)$. Since the orbit $\Psi(z)$ is closed in $\mathbb{P}\backslash A$ and homeomorphic to Ψ by 3.1, it remains to be shown that $\mathfrak{Z}(\Psi\omega)$ is homeomorphic to Ψ as well. For then, it follows from the Brouwer invariance theorem that $\Psi(z) \cong \Psi$ is also open in $\mathfrak{Z}(\Psi\omega) \cong \Psi$, so that indeed $\Psi(z) = \mathfrak{Z}(\Psi\omega)$ because Ψ is connected.

In order to show $\mathfrak{Z}(\Psi\omega) \cong \Psi$, let $L = z \vee a$ and choose any point $p \in \mathbb{P}\backslash(L \cup A)$ (as indicated in Fig. 2). The orbit of $p' = \omega(p)$ under Ψ is contained in the line $M = p' \vee a$, because Ψ consists of elations with centre a. For $\psi \in \Psi$, the line L, which passes

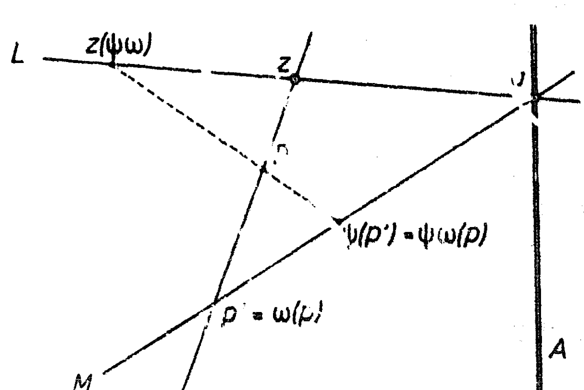

Fig. 2.

through the centres of both ψ and ω, is a fixed line of the homology $\psi\omega$ and therefore contains the centre $z(\psi\omega)$ of $\psi\omega$. Since the image $\psi\omega(p) = \psi(p')$ lies on $z(\psi\omega) \vee p$, it follows that the orbit $\Psi(p') \subseteq M$ is obtained from $\mathfrak{Z}(\Psi\omega) \subseteq L$ by central projection from L into M via p as projection centre. This establishes a homeomorphism between $\mathfrak{Z}(\Psi\omega)$ and the orbit $\Psi(p')$, which is homeomorphic to Ψ by 3.1.

3.4. *For $z \in \mathbb{P}\backslash A$, let Σ be a closed subgroup of the group of homologies with axis A and centre z. Then:*

(i) *The connected component of Σ either is compact or has two ends; in particular, it either has non-trivial compact connected subgroups, or is isomorphic to $\mathbb{R}$, or trivial.*

(ii) *If Σ is not compact, then a compact subset of $\mathbb{P}\backslash A$ can be compressed into any given neighbourhood of the centre z by means of a suitable homology in Σ.*

For the proof, we coordinatize the given plane with coordinates from a locally compact, connected topological ternary field K (see [10, § 7]) in such a way that A is the line at infinity and z is the origin of the coordinate system. It is well known and easy to see that in affine coordinates over K every homology with centre z and axis A is of the form

$$(x, y) \mapsto (xs, ys)$$

with suitable $s \in K\backslash\{0\} = K^{\times}$. The elements $s \in K^{\times}$ corresponding to elements of Σ thus form a closed subgroup N of the multiplicative loop $K^{\times}$ of K, and N is isomorphic to Σ.

Therefore one may study N (and its connected component N^1) instead of Σ. In order to prove that they are either compact or have two ends, we look at their embeddings into the one-point compactification $\hat{K} = K \cup \{\infty\}$ of K.

If N is not compact, then 0 or ∞ must be a cluster point of N, because N is closed in $K^{\times} = \hat{K}\backslash\{0, \infty\}$. In fact, both 0 and ∞ are cluster points, since there are continuous maps $\hat{K} \to \hat{K}$ which leave N invariant and exchange 0 and ∞. For instance, the mapping $K^{\times} \to K^{\times}$ which associates to each $s \in K^{\times}$ its multiplicative left inverse s^{-1} extends continuously to $\hat{K}$ by $0^{-1} = \infty$ and $\infty^{-1} = 0$, see Fig. 3. Thus, $N \cup \{0, \infty\}$ is a two-point compactification of N. The same argument applies to N^1.

Now assume that N^1 has no non-trivial compact connected subgroups. By the Malcev–Iwasawa theorem, N^1 is then homeomorphic to a euclidean space $\mathbb{R}^m$. For $m \geq 2$, however, $\mathbb{R}^m$ does not admit a two-point compactification, as is well-known. [Otherwise, the connected boundary of a large ball containing in its interior the compact complement of two disjoint open neighbourhoods U_1 and U_2 of the two ideal points would be decomposed by U_1 and U_2.] Thus, N^1 is homeomorphic to $\mathbb{R}$ and therefore isomorphic to $\mathbb{R}$, and (i) is proved.

(ii) is equivalent to the statement that if N is not compact, then for any given compact subset C of the ternary field K and for any neighbourhood U of 0 in K there is an element $s \in N$ such that $Cs \subseteq U$. Such an element may be found by choosing a neighbourhood W of 0 with $CW \subseteq U$ (this is possible since $C \cdot 0 = \{0\} \subseteq U$ and since

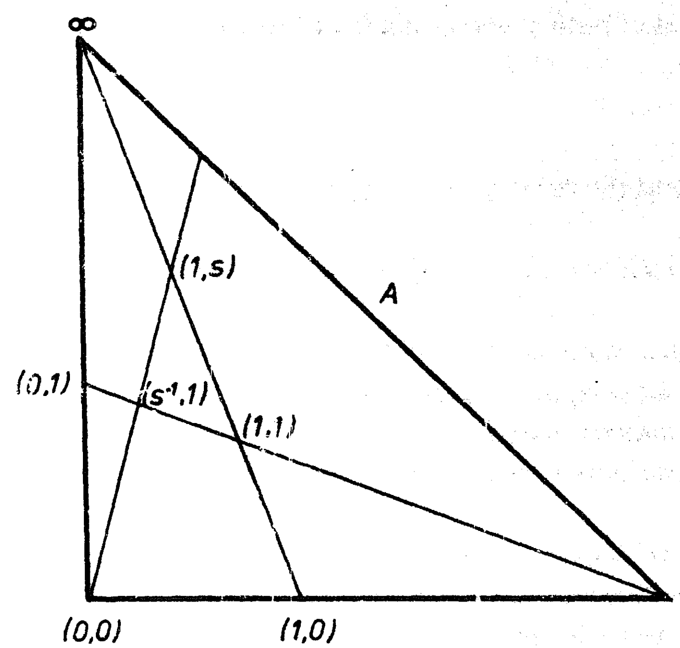

Fig. 3.

C is compact), and by taking $s \in N \cap W$; this intersection is nonvoid if N is not compact, since then 0 is a cluster point of N.

An immediate corollary of 3.4(i) is

3.5. *For $z \in \mathbb{P}\setminus A$, a non-trivial closed connected Lie subgroup of the group of all homologies with axis A and centre z either contains involutions, or is isomorphic to $\mathbb{R}$.*

Restricting the compressions of 3.4(ii) to certain submanifolds of $\mathbb{P}\setminus A$, we obtain:

3.6. *Suppose that a locally euclidean subspace M of $\mathbb{P}\setminus A$ is invariant, for some $z \in M$, under a closed non-compact subgroup Σ of the group of all homologies with axis A and centre z. Then M has trivial homotopy groups.*

Supplementary remark. Using a well-known theorem of M. Brown [3], we show moreover that M is in fact a euclidean space $\mathbb{R}^k$. However, in what follows we may do without this extra information.

Proof. Every element of Σ induces a homeomorphism of the Σ-invariant subspace M. Therefore by 3.4(ii) every compact subset of M may be compressed homeomorphically into a given neighbourhood of z in M. By choosing this neighbourhood to be homeomorphic to a euclidean ball $\mathbb{D}^k$ and by considering its inverse image under the above compressions, one sees that every compact subset of M has a neighbourhood homeomorphic to $\mathbb{D}^k$. The triviality of the homotopy groups of M now follows at once.

Furthermore, since M is the countable union of compact subsets, it follows that we may represent M as the union of a monotone sequence of open k-balls. By Brown's theorem, M itself is an open k-ball.

Concerning the homeomorphism type of elation groups, we now can prove:

3.7. *Let Φ be a closed connected Lie subgroup of the group of elations with axis A. Then Φ is homeomorphic to a euclidean space $\mathbb{R}^k$ provided one of the following conditions holds:*

(i) *For some centre $z \in A$, the normalizer of Φ in the group of homologies with centre z and axis A is not compact.*

(ii) *The point space $\mathbb{P}$ is of finite topological dimension.*

Proof. To deal with case (i) we shall use 3.6 above. Let Σ denote the normalizer of Φ in the group of homologies with centre z and axis A. The orbit $\Phi(z)$ is Σ-invariant; it is homeomorphic to Φ (3.1) and therefore is a manifold. We now may conclude directly by applying the supplementary remark of 3.6. If one wants to avoid this strong version of 3.6, one may proceed as follows: 3.6 then tells only that Φ has trivial homotopy groups. By the Malcev–Iwasawa theorem, Φ is homeomorphic to the product of a euclidean space $\mathbb{R}^k$ and a maximal compact subgroup C. But C must be trivial since non-trivial compact Lie groups always have some non-vanishing homotopy group.

In case (ii), we use Smith fixed point theory. The affine point space $\mathbb{P}\backslash A$ is contractible [10, 7.11]. Therefore, and since $\mathbb{P}\backslash A$ is finite dimensional by assumption, any involutory homeomorphism of $\mathbb{P}\backslash A$ has a fixed point [14]. Consequently, the elation group Φ, which acts freely on $\mathbb{P}\backslash A$, cannot contain a compact connected Lie group. Using again the Malcev–Iwasawa theorem, we conclude that Φ is homeomorphic to some $\mathbb{R}^k$.

4. Proof of the theorem

We distinguish several cases, of which the simpler ones are treated first. Frequent use will be made of the fact that the connected component Ξ of Ω, as well as the elation group $\Omega_{[A]}$ and its connected component Θ, are closed normal subgroups of Ω.

4.1. *If $\Theta = \Xi \neq \{\mathrm{id}\}$, then either Ω consists entirely of elations (so that $\mathfrak{Z}\backslash A = \emptyset$), or we are in one of the situations* (iii) *or* (iv) *of the theorem.*

Proof. Since Ω has a countable basis, the index of the (open) connected component Ξ in Ω is at most countable so by our assumption Ω is the union of at most countably many cosets of $\Theta = \Xi$. If such a coset does not entirely consist of elations, then by 3.3

the centres of its elements form an orbit under Θ. Therefore $\mathfrak{Z}\backslash A$ is the union of at most countably many Θ-orbits. By 3.1, these orbits are closed in $\mathsf{P}\backslash A$ and homeomorphic to Θ.

4.2. *If* $\Xi = \{\mathrm{id}\}$, *then* $\mathfrak{Z}$ *is at most countable*,

since in this case Ω is discrete and therefore is at most countable (having a countable basis).

4.3. *If* $\dim \Xi = 1$ *and* $\Theta \neq \Xi$, *then* $\mathfrak{Z}$ *is a point.*

Proof. In this case, the connected Lie group Ξ is a one-parameter group, that is, a homomorphic image of $\mathbf{R}$. Since $\mathbf{R}$ contains the locally cyclic group $\mathbf{Q}$ as a dense subgroup, all elements of Ξ have the same centre z, which is not contained in A if $\Theta \neq \Xi$. Since z is the only fixed point of Ξ in $\mathsf{P}\backslash A$, and since Ξ is a normal subgroup of Ω, z is a fixed point (and therefore the centre) of each element in Ω. Thus $\mathfrak{Z} = \{z\}$.

In the special cases considered above, the theorem was nearly obvious. The remaining generic case is more involved; at first sight it even seems astonishing that it may be covered by such a uniform result:

4.4. Proposition. *If* $\dim \Xi \geq 2$ *and* $\Theta \neq \Xi$, *then* $\mathfrak{Z}$ *is compact and is the union of an orbit under* Θ *and of the set* $\mathfrak{Z}(\Theta) \subseteq A$ *of all centres of elements in* $\Theta\backslash\{\mathrm{id}\}$.

For the proof of 4.4, we study the one-parameter subgroups of the connected Lie group Ξ. They correspond to the one-dimensional subspaces of the Lie algebra $\mathscr{L}(\Xi)$ of Ξ; therefore we consider the projective space $P\mathscr{L}(\Xi)$ of all these one-dimensional subspaces. As we have already noted, all elements in a one-parameter group of perspectivities have the same centre. Therefore one may define the *centre map*

$$\zeta : P\mathscr{L}(\Xi) \to \mathsf{P}$$

mapping each one-dimensional subspace of $\mathscr{L}(\Xi)$ onto the common centre of the perspectivities in the corresponding one-parameter subgroup. This map is continuous, since the exponential function and the map assigning the centre to each perspectivity are continuous. The image $\mathfrak{Z}^1$ of ζ, i.e. the set of all centres belonging to non-identical perspectivities in one-parameter subgroups of Ω, is therefore compact. From the conjugation formula (1) in the introduction and the fact that the set of one-parameter subgroups is conjugation-invariant, it follows that $\mathfrak{Z}^1$ is invariant under Ω.

Using our knowledge about the structure of homology groups, we now distinguish two possible cases:

Case 4.4(a). Ξ *contains a reflection* (i.e. a homology of order two);

if this does not hold, then by 3.5 we have

Case 4.4(b). *For all centres $z \notin A$, either the homology group $\Omega_{[z]}$ is discrete, or its connected component is isomorphic to* $\mathbb{R}$.

In the second case, which we take up first since it involves the core of the argument, we know in particular that all the homology groups $\Omega_{[z]}$ are at most one-dimensional. This implies that the centre map ζ is injective on $P\mathscr{L}(\Xi)\backslash P\mathscr{L}(\Theta)$. It therefore induces a continuous bijection $P\mathscr{L}(\Xi)\backslash P\mathscr{L}(\Theta) \to \mathfrak{Z}^1\backslash A$, which, moreover, is open, since $P\mathscr{L}(\Xi)$ is compact; so we have a homeomorphism

$$P\mathscr{L}(\Xi)\backslash P\mathscr{L}(\Theta) \cong \mathfrak{Z}^1\backslash A.$$

In particular $\mathfrak{Z}^1\backslash A$ is a manifold of dimension k, where

$$k = \dim P\mathscr{L}(\Xi) = \dim \Xi - 1.$$

Now consider a fixed centre $z \in \mathfrak{Z}^1\backslash A$. From the definition of $\mathfrak{Z}^1$ we know that $\Omega_{[z]}$ contains a one-parameter subgroup and so is not discrete; therefore by the assumption of case 4.4(b) the connected component of $\Omega_{[z]}$ is isomorphic to $\mathbb{R}$. In particular $\Omega_{[z]}$ is not compact. By 3.6, the $\Omega_{[z]}$-invariant manifold $\mathfrak{Z}^1\backslash A$ is simply connected. Therefore the complement of the unit sphere of $\mathscr{L}(\Theta)$ in the unit sphere of $\mathscr{L}(\Xi)$ is disconnected, since it is a two-fold covering space of $P\mathscr{L}(\Xi)\backslash P\mathscr{L}(\Theta) \cong \mathfrak{Z}^1\backslash A$. But this means

$$\dim \Theta = \dim \Xi - 1 = k.$$

Since $\mathfrak{Z}^1\backslash A$ is Θ-invariant, it contains a Θ-orbit. By 3.1, this orbit is closed in $\mathfrak{Z}^1\backslash A$ on the one hand; on the other hand, it is homeomorphic to Θ and therefore open in the connected k-manifold $\mathfrak{Z}^1\backslash A$ by the Brouwer invariance theorem, because $k = \dim \Theta$. Consequently, this orbit fills the whole of $\mathfrak{Z}^1\backslash A$; in particular, it is an Ω-invariant Θ-orbit. This situation is dealt with in the following lemma (which will be re-used for the handling of the remaining case 4.4(a) and therefore starts from less information):

4.5. Lemma. *Suppose that for some $z \in \mathbb{P}\backslash A$ we have $\Xi(z) = \Theta(z)$ and $\Xi \cap \Omega_{[z]} \neq \{\mathrm{id}\}$. Then $\mathfrak{Z} = \Theta(z) \cup \mathfrak{Z}(\Theta) = \mathfrak{Z}^1$; in particular, $\mathfrak{Z}$ is compact.*

Proof. For $\xi \in \Xi$ let $\vartheta \in \Theta$ be the elation with $\iota(z) = \xi(z)$. Since $z = \vartheta^{-1}\xi(z)$ we then have $\vartheta^{-1}\xi \in \Omega_{[z]}$; thus $\Xi = \Theta \cdot (\Xi \cap \Omega_{[z]})$. By 3.3 we get $\mathfrak{Z}(\Xi\backslash\Theta) = \Theta(z)$, from which it follows that the whole of Ω leaves $\Theta(z)$ invariant, since Ξ is normal in Ω. From this, one may deduce as above that

$$\Omega = \Theta \cdot \Omega_{[z]} \tag{3}$$

and therefore by 3.3

$$\mathfrak{Z}\backslash A = \Theta(z).$$

Since $\mathfrak{Z}^1\setminus A$ is a Θ-invariant subset, we get even

$$\mathfrak{Z}^1\setminus A = \Theta(z).$$

By 3.2, either all elations in Θ have the same centre $a \in A$, or Θ is a vector group and therefore the union of its one-parameter subgroups, so that $\mathfrak{Z}(\Theta) \subseteq \mathfrak{Z}^1$. From (3) we infer $\Omega_{[A]} = \Theta$. We now have obtained

$$\mathfrak{Z}^1 \subseteq \mathfrak{Z} = (\mathfrak{Z}\setminus A) \cup \mathfrak{Z}(\Omega_{[A]}) = (\mathfrak{Z}\setminus A) \cup \mathfrak{Z}(\Theta) = \Theta(z) \cup \mathfrak{Z}(\Theta) \subseteq \mathfrak{Z}^1;$$

so the lemma is proved.

Thus, in the case 4.4(b) our proposition 4.4 is established by the conclusion of the lemma above.

The remaining case 4.4(a), which is characterized by the presence of reflections in Ξ, may be settled quite comfortably by means of well-known information how to generate elations from reflections. For instance, the following lemma tells us that in this case, too, we may apply Lemma 4.5:

4.6. Lemma. *Let Δ be a closed connected Lie subgroup of the group of all continuous collineations leaving the line A invariant, and consider the connected component Θ of the group of elations in Δ with axis A. If Δ contains a reflection with axis A and centre $z \notin A$, then $\Delta(z) = \Theta(z)$.*

Proof. If $\Delta(z) = z$, the lemma is trivial. Otherwise, the codimension d of the isotropy subgroup Δ_z in Δ is positive. By considering a local cross section of the fibering $\pi: \Delta \to \Delta/\Delta_z$ one obtains a d-dimensional compact connected subset $D \subseteq \Delta$ with $\mathrm{id} \in D$ such that the map $D \to \mathrm{P}\setminus A: \delta \mapsto \delta(z)$ is injective.

Now let ι be a reflection with axis A and centre z, and $\delta \in D$. Then $\delta\iota\delta^{-1}$ is a reflection with the same axis and with centre $\delta(z)$; therefore the map $D \to \Delta$: $\delta \to \delta\iota\delta^{-1}$, too, is injective. The image J of this map is connected and has dimension $d = \dim D$, since D is compact. It is well known that $J \cdot \iota = \{\delta\iota\delta^{-1}\iota;\ \delta \in D\}$ consists of elations (see [2, p. 103]); so $J \cdot \iota$ is contained in Θ, since it is connected and contains the identity by definition of D. In particular, we have $d \leq \dim \Theta$. On the other hand, since $\Theta \cap \Delta_z = \{\mathrm{id}\}$, the canonical projection $\pi: \Delta \to \Delta/\Delta_z$ is injective on Θ so that $\dim \Theta \leq \dim \Delta/\Delta_z = d$. All in all, we have $\dim \Theta = d = \dim \Delta/\Delta_z$; therefore, the Brouwer invariance theorem implies that $\pi(\Theta)$ is open in Δ/Δ_z. But $\pi(\Theta)$ is also closed in Δ/Δ_z because it is the inverse image of the orbit $\Theta(z)$ under the continuous map $\Delta/\Delta_z \to \mathrm{P}\setminus A: \delta\Delta_z \mapsto \delta(z)$, and because $\Theta(z)$ is closed in $\mathrm{P}\setminus A$ (3.1). By the connectedness of Δ/Δ_z we conclude that $\pi(\Theta) = \Delta/\Delta_z$, or, equivalently, $\Theta(z) = \Delta(z)$.

In the case 4.4(a), we now apply this lemma putting $\Delta = \Xi$. By hypothesis, Ξ contains a reflection with axis A. For the centre $z \notin A$ of this reflection, Lemma 4.6 tells us that $\Xi(z) = \Theta(z)$. It follows from Lemma 4.5 that $\mathfrak{Z} = \Theta(z) \cup \mathfrak{Z}(\Theta)$ and that $\mathfrak{Z}$ is compact. This proves proposition 4.4. Thus the main Theorem 1.1, which is a condensation of 4.1–4.4, is established.

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