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DIFFERENTIABLE FIBRATIONS OF THE (2n-1)-SPHERE BY GREAT (n-1)-SPHERES AND THEIR COORDINATIZATION OVER QUASIFIELDS

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1. INTRODUCTION

As a step towards a solution of the Blaschke conjecture, the following theorem has been proved by GLUCK, WARNER and YANG in [7: Theorem B] (see also SATO [13], [15] and SATO - MIZUTANI [14]):

1.1 THEOREM. *A locally trivial differentiable fibre bundle with total space* S^{2n-1} *whose fibres are great* $(n-1)$*-spheres is topologically equivalent to one of the classical Hopf fibrations.*

By a great $(n-1)$-sphere we mean the intersection of S^{2n-1}, considered as the unit sphere in $\mathbb{R}^{2n}$, with an n-dimensional linear subspace of $\mathbb{R}^{2n}$.

We briefly recall the construction of the classical Hopf fibrations. They owe their existence to the classical coordinate structures, namely the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, the skew field $\mathbb{H}$ of quaternions and the alternative field $\mathbb{O}$ of Cayley numbers. Let $\mathbb{K}$ be one of these coordinate domains; $\mathbb{K}$ is a vector space over $\mathbb{R}$ of dimension $n = 1, 2, 4$ or 8. The unit sphere S^{2n-1} in $\mathbb{K}\times\mathbb{K} = \mathbb{R}^{2n}$ is fibered by the great $(n-1)$-spheres which are obtained as the intersections of S^{2n-1}

with the subspaces of the form $\{(x,ax);\ x\in\mathbb{K}\}$ for $0 \neq a\in\mathbb{K}$ and with $\{0\}\times\mathbb{K}$; this fibration is the classical Hopf fibration over $\mathbb{K}$. (It is well known that, quite generally, as a consequence of ADAMS' theorem on maps of Hopf invariant 1, the values $n = 1, 2, 4, 8$ are the only ones for which fibre bundles of the type considered in theorem 1.1 exist.)

Now, the Hopf construction can be carried out over more general coordinate domains. If instead of the classical coordinate domains considered above one uses any (not necessarily associative or alternative) finite dimensional real division algebra, a theorem of BUCHANAN [4: Hilfssatz 2 p. 389] states essentially that the resulting fibration is still topologically equivalent as a fibre bundle to the classical Hopf fibration of the same dimension.

The proof of GLUCK, WARNER and YANG for theorem 1.1 [7: Theorem B] consists in a reduction of the general situation to this particular case. Although not every fibre bundle of the kind considered here can be coordinatized over a division algebra, they show that by a suitable differentiation process any such fibre bundle can be "linearized", without changing its equivalence class, in such a way that the resulting fibration comes from a division algebra (of the general sort mentioned above) via the Hopf construction. Then BUCHANAN's theorem can be applied.

The first purpose of this talk is to describe a variation of this proof which consists in passing to coordinates right from the beginning, not only at the end when division algebras come into play. This approach, while being closely related to the proof by GLUCK, WARNER and YANG, presents some technical advantages. Besides, it is quite natural to anyone who is familiar with coordinate methods in the theory of affine planes. The appropriate algebraic structures, generalizing division algebras, are quasifields (see 2.2), the coordinate domains for translation planes. We shall not make use of the theory of translation planes in the sequel, but a little more on the connections to this topic shall be said at the end of this introduction.

Regarding theorem 1.1 it is natural to try to prove a strengthened version of it in which the differentiability assumptions yield a conclusion which remains within the realm of differentiability, such as in the following

1.2 CONJECTURE. *A locally trivial differentiable fibre bundle with total space* S^{2n-1} *whose fibres are great* $(n-1)$*-spheres is* differentiably *equivalent to one of the classical Hopf fibrations.*

Of course, this is trivially true for $n = 1$ and well-known for $n = 2$.

It is the second objective of this talk to shed some further light on this conjecture for the higher-dimensional cases by the coordinate methods mentioned above. Since a fibre bundle π of this kind is *topologically* equivalent to the corresponding classical Hopf fibration by theorem 1.1, the conjecture would follow by well-known general theorems on the uniqueness of differentiable structures on vector bundles (see e.g. [12: §4 Theorem 3.5 p. 101]) if only one could establish that the base space $B(\pi)$ is diffeomorphic to the base space of the classical Hopf fibration, i.e. S^n with its usual differentiable structure. Now $B(\pi)$ can be described using algebraic operations in coordinatizing quasifields (see 2.9), and again a reduction to the special case of division algebras will be possible by linearization (4.5, 4.6). In this way it will turn out that $B(\pi)$ is a twisted sphere of dimension n (4.7). For $n = 2$ conjecture 1.2 then follows at once since S^2 has essentially only one differentiable structure. For $n = 4$ we may use CERF's result [5: Théorème 1 p.3] to conclude that $B(\pi)$ is diffeomorphic to S^4 and thus to get

1.3 THEOREM. *Conjecture* 1.2 *holds true for* $n = 4$.

One might ask whether it is adequate to use such a deep theorem as CERF's in order to obtain this result. Anyway, this leaves the case $n = 8$ open. In §4, which will furnish the details of what has just been sketched, we shall present several reformulations of the problems regarding conjecture 1.2 in terms of algebraic operations in quasifields and division algebras; they may offer the possibility of a more direct attack upon this conjecture.

Finally, let us briefly indicate the connections to the theory of translation planes by explaining how a given fibration of S^{2n-1} by great $(n-1)$-spheres is related to a translation plane: The n-dimensional subspaces of $\mathbb{R}^{2n}$ spanned by the fibres form a spread, i.e. every nonzero vector lies in exactly one of these subspaces, and any two of these subspaces span the whole vector space $\mathbb{R}^{2n}$. Consequently, the images of these subspaces under the vector space translations of $\mathbb{R}^{2n}$ constitute the system of lines of an affine plane with point set $\mathbb{R}^{2n}$, which obviously admits all translations of $\mathbb{R}^{2n}$ as collineations.

Regarding the relationship to translation planes, it should be pointed out that a fibre bundle equivalence of two such fibrations of S^{2n-1} does not tell anything about the question whether the corresponding translation planes are isomorphic as such, since the fibre bundle equivalence need not be induced by a *linear* transformation of $\mathbb{R}^{2n}$ (followed by a retraction of $\mathbb{R}^{2n}-\{0\}$ onto S^{2n-1}). In fact, whole forests of non-isomorphic translation planes with good continuity or even differentiability properties have been systematically explored for all possible dimensions (see BETTEN [3] and further references there as well as GLUCK and WARNER [6] for $n = 2$, [9] and [10] and further references there for $n = 4$, and [11] for $n = 8$).

Most of the translation planes of this kind correspond to topological fibrations of S^{2n-1} which are not differentiable. On the other hand, for all known examples these fibrations are topologically fibre bundle equivalent to the classical Hopf fibrations. Therefore it would be interesting to know if there is a purely topological analogue of theorem 1.1 (without any differentiability assumptions), and it is reasonable to conjecture that there is. This amounts to the validity of BUCHANAN's theorem for general topological quasifields with additive group $\mathbb{R}^n$, not only for division algebras. Up to now, this conjecture has resisted all attempts of proof.

2. THE COORDINATIZATION OF FIBRATIONS BY QUASIFIELDS

2.1 Spreads. Consider a fibering π of S^{2n-1} whose fibres are great $(n-1)$-spheres. By $\mathcal{S}$ we denote the family of n-dimensional linear subspaces of $\mathbb{R}^{2n}$ spanned by the great $(n-1)$-spheres which constitute the fibres of π. The elements of $\mathcal{S}$ cover $\mathbb{R}^{2n}$ (since the fibres of π cover S^{2n-1}) and are pairwise complementary (since two fibres of π are disjoint and since the dimensions are complementary). Thus $\mathcal{S}$ is a *spread* in $\mathbb{R}^{2n}$.

We may assume (by a suitable change of coordinates, if necessary) that the subspaces

$$U_0 = \mathbb{R}^n \times \{0\}, \quad U_\infty = \{0\} \times \mathbb{R}^n$$

of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ belong to $\mathcal{S}$. We arbitrarily fix a "unit" element $e \in \mathbb{R}^n - \{0\}$.

By complementarity, for a fixed subspace $U \in \mathcal{S}$ different from U_∞, and for any $x \in \mathbb{R}^n$ there is a unique $x' \in \mathbb{R}^n$ such that $(x,x') \in U$. In particular, there is a unique $a \in \mathbb{R}^n$ such that $(e,a) \in U$; and conversely, U is uniquely determined by a. We therefore denote U by U_a:

$$(e,a) \in U_a \in \mathcal{S}.$$

Since $U = U_a$ is a linear subspace, the map $x \longmapsto x'$ is a vector space endomorphism which will be denoted by

$$\lambda_a \in \mathrm{End}(\mathbb{R}^n).$$

Thus

$$(x,y) \in U_a \iff y = \lambda_a(x) .$$

From $(e,0) \in U_0$ and $(e,a) \in U_a$ we have

(L0) λ_0 is the zero endomorphism

(L1) $\lambda_a(e) = a$.

For $a,b \in \mathbb{R}^n$ with $a \neq b$ the subspaces $U_a, U_b \in \mathcal{S}$ are complementary; this means that

(L2) $a \neq b \iff \lambda_a - \lambda_b$ is regular .

2.2 <u>Quasifields</u>. We now introduce a multiplication $\circ$ on $\mathbb{R}^n$ by

$$a \circ x = \lambda_a(x) .$$

The properties (L0) - (L2) listed above can be translated as follows:

(Q0) $0 \circ x = 0 = x \circ 0$

(Q1) $x \circ e = x$

(Q2) For $a \neq b \in \mathbb{R}^n$ the map $x \longmapsto a \circ x - b \circ x$ is bijective.

The linearity of λ_a implies the left distributive law

(Q3) $a \circ (x + y) = a \circ x + a \circ y$.

Finally, the fact that $\mathcal{S} = \{U_a ; a \in \mathbb{R}^n \cup \{\infty\}\}$ covers $\mathbb{R}^{2n}$ implies

(Q4) For $b \neq 0$, the map

$$\rho_b : x \longmapsto x \circ b$$

is surjective and therefore bijective

(the bijectivity following from (Q2)).

In current terminology, (Q0) - (Q4) can be expressed by saying that $\mathbb{R}^n$ with the vector addition and the multiplication $\circ$ is a *quasifield*. A quasifield constructed in this way from the fibering π will be called a *quasifield associated with* π.

Let us record explicitly that (Q3) was actually a consequence of the stronger property

(Q3') For fixed a, the map

$$\lambda_a : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \longmapsto a \circ x$$

is linear.

In the sequel, this will be the relevant property concerning distributivity. To have a short expression, a quasifield whose additive group is $\mathbb{R}^n$ and which satisfies (Q3') will be called a *quasifield on* $\mathbb{R}^n$.

By (Q0) the map λ_0 is the zero endomorphism, whereas by (Q4) we have

$$\lambda_a \in GL_n(\mathbb{R}) \quad \text{for} \quad 0 \neq a \in \mathbb{R}^n.$$

Therefore the map

$$\lambda : \mathbb{R}^n \longrightarrow \mathrm{End}(\mathbb{R}^n) : a \longmapsto \lambda_a$$

restricts to a map

$$\lambda : \mathbb{R}^n - \{0\} \longrightarrow GL_n(\mathbb{R}) .$$

This restriction entirely determines the quasifield and will be called its *characteristic map*.

Note that a (not necessarily associative or alternative) finite dimensional *division algebra* over $\mathbb{R}$ is a quasifield of this type with the extra property that the maps ρ_b of (Q4) are linear as well. In general, this will not be the case.

<u>Some notation</u> which will be used: For $x \neq 0$ we denote by

$$y/x := \rho_x^{-1}(y) \quad \text{and} \quad x\backslash y := \lambda_x^{-1}(y)$$

the unique elements of our quasifield satisfying

$$(y/x)\circ x = y \quad \text{and} \quad x\circ(x\backslash y) = y .$$

2.3 <u>The unit element</u>. In a small detail our axiom (Q1) is a little weaker than customary: we do not ask e to be also a left unit element. This can be remedied easily, however. Changing the coordinates of $\mathbb{R}^n\times\mathbb{R}^n$ by $(x,y)\longmapsto(x,\lambda_e^{-1}(y))$ one can additionally assume that

$$O = \{(x,x);\ x\in\mathbb{R}^n\} \text{ belongs to } S.$$

The quasifield associated with π via these new coordinates satisfies $\lambda_e =$ id since then $(e,e)\in O = U_e$, so that in addition to (Q1) we have

(Q1') $e\circ x = x$.

In algebraic expression this amounts to replacing the original multiplication $\circ$ by $(a,x)\longmapsto a\circ\lambda_e^{-1}(x)$, a well-known process which is called a *prinicipal isotopism*, [2: Theorem 1 p. 510].

2.4 The constructions of 2.1 and 2.2 can be reversed: If we have any quasifield on $\mathbb{R}^n$ as defined in 2.2, we may construct a fibering π of S^{2n-1} by great $(n-1)$-spheres, whose fibres are the intersections of the subspaces $U_a = \{(x,a\circ x);\ x\in\mathbb{R}^n\}$ for $a\in\mathbb{R}^n$ and $U_\infty = \{0\}\times\mathbb{R}^n$ with S^{2n-1}.

We now state how differentiability properties of π translate into differentiability properties of associated quasifields and vice versa:

2.5 PROPOSITION. *The following are equivalent:*

(i) *The fibering π is a locally trivial differentiable fibre bundle.*

(ii) *For every associated quasifield, the map*

$$\gamma : (\mathbb{R}^n-\{0\})\times\mathbb{R}^n \longrightarrow \mathrm{End}(\mathbb{R}^n) \ :\ (x,y)\longmapsto\lambda_{y/x}$$

is differentiable.

(iii) *There is an associated quasifield in which the map γ above and the map*

$$\bar\gamma : \mathbb{R}^n\times(\mathbb{R}^n-\{0\})\longrightarrow \mathrm{End}(\mathbb{R}^n) \ :\ (x,y)\longmapsto\begin{cases}\lambda_{y/x}^{-1} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$$

are differentiable.

This is then true in fact for every associated quasifield.

2.6 DEFINITION. A quasifield on $\mathbb{R}^n$ is called a *differentiable quasifield* if the map γ described in 2.5 is differentiable. It is called *differentiable at infinity* if the map $\bar{\gamma}$ is differentiable.

2.7 Proof of proposition 2.5 (i) $\Longrightarrow$(ii).

A local trivialization of the locally trivial fibre bundle π is a diffeomorphism

$$\Phi : U \times S^{n-1} \longrightarrow W \subseteq S^{2n-1}$$

(where U is an open subset of the base manifold of the fibre bundle π and W an open subset of S^{2n-1} which is a union of fibres) such that for $z \in W$ and

$$\Phi^{-1}(z) = (u,p) \in U \times S^{n-1}$$

the map

$$\Phi_u : S^{n-1} \longrightarrow S^{2n-1} : q \longmapsto \Phi(u,q)$$

is a diffeomorphism onto the fibre F_z of z. Using this trivialization, we prove the following

Assertion. The subspace of $\mathbb{R}^{2n}$ spanned by the fibre F_z has a basis $b_1(z), \ldots, b_n(z)$ depending differentiably on $z \in W$.

Indeed, F_z is a great $(n-1)$-sphere, so for any fixed point $\bar{p} \in S^{n-1}$ the subspace in question is also spanned by the tangent space $T_{\Phi_u(\bar{p})} F_z$ of F_z in $\Phi_u(\bar{p})$ together with $\Phi_u(\bar{p})$ itself. Now

$$T_{\Phi_u(\bar{p})} F_z = T_{\Phi_u(\bar{p})} \Phi_u(S^{n-1}) = d_{\bar{p}}\Phi_u(T_{\bar{p}}S^{n-1})$$

(with the derivative $d_{\bar{p}}\Phi_u$ of Φ_u in $\bar{p}$). Therefore, if $v_1, \ldots, v_{n-1}$ is any basis of the tangent space $T_{\bar{p}}S^{n-1}$, then

$$b_i(z) = d_{\bar{p}}\Phi_u(v_i) \qquad (i = 1,\ldots,n-1)$$

and

$$b_n(z) = \Phi_u(\bar{p})$$

constitute a basis of the span of F_z depending differentiably on u and therefore on z.

Let now π be coordinatized by a quasifield on $\mathbb{R}^n$ as in 2.1 - 2.2. For $(x,y) \in (\mathbb{R}^n - \{0\}) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ consider a local trivialization of π around the fibre containing

$$z = \frac{(x,y)}{\|(x,y)\|}$$

as above. Using the characteristic map λ of the given quasifield the subspace spanned by F_z can be expressed as

$$\langle F_z\rangle = U_{y/x} = \{(c,\lambda_{y/x}(c));\ c\in\mathbb{R}^n\} .$$

We denote by

$$c_1(z),\ \ldots.,\ c_n(z)$$

and

$$d_1(z),\ \ldots.,\ d_n(z)$$

the projections of the basis $b_1(z),\ \ldots.,\ b_n(z)$ of $\langle F_z\rangle$ (constructed according to the assertion above) onto the first and second factor of $\mathbb{R}^n\times\mathbb{R}^n$, so that

$$b_i(z) = (c_i(z),d_i(z)) .$$

Since $b_i(z)\in\langle F_z\rangle = U_{y/x}$, we have

$$(*)\qquad \lambda_{y/x}(c_i(z)) = d_i(z) .$$

Furthermore $U_{y/x}$ is complementary to $U_\infty = \{0\}\times\mathbb{R}^n$ and projects isomorphically onto the first factor of $\mathbb{R}^n\times\mathbb{R}^n$, so that the projection image $c_1(z),\ \ldots,\ c_n(z)$ of the basis $b_1(z),\ \ldots,\ b_n(z)$ of $U_{y/x}$ is a basis of $\mathbb{R}^n$. Thus, the endomorphism $\lambda_{y/x}$ is uniquely determined by (*) and depends differentiably on z and therefore on (x,y), since the basis elements $b_1(z),\ \ldots,\ b_n(z)$ depend differentiably on z according to their construction.

2.8 Proof of proposition 2.5 (ii) $\Longrightarrow$(iii).

Starting from a given quasifield associated with π we can construct another quasifield by reversing the roles of the first and second factor in $\mathbb{R}^{2n} = \mathbb{R}^n\times\mathbb{R}^n$, i.e. of U_0 and U_∞. The map γ for the new quasifield is then precisely the map $\overline{\gamma}$ for the original quasifield.

2.9 Proof of proposition 2.5 (iii) $\Longrightarrow$(i).

This can be proved using a construction of YANG [16: Theorem 2 p. 580]. (He considers only the particular case that the underlying quasifield is a division algebra, but this restriction is not essential to the argument.)

According to this construction, local trivializations for π are obtained in the following way. As base space one takes the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ endowed with the differentiable structure which makes the inclusion of $\mathbb{R}^n$ an embedding and the homeomorphism

$$\psi : (\mathbb{R}^n - \{0\}) \cup \{\infty\} \longrightarrow \mathbb{R}^n : \begin{array}{l} v \longmapsto v \backslash e \quad \text{for} \quad v \in \mathbb{R}^n - \{0\} \\ \infty \longmapsto 0 \end{array}$$

a diffeomorphism. Then one constructs the following two local trivializations for π :

$$g_0 : \mathbb{R}^n \times S^{n-1} \longrightarrow S^{2n-1} - U_\infty : (v,w) \longmapsto \frac{(w, v\circ w)}{\|(w, v\circ w)\|}$$

$$g_1 : ((\mathbb{R}^n - \{0\}) \cup \{\infty\}) \times S^{n-1} \longrightarrow S^{2n-1} - U_0 : \begin{array}{l} (v,u) \longmapsto \dfrac{(v\backslash u, u)}{\|(v\backslash u, u)\|} \quad \text{for} \quad v \neq \infty \\ (\infty, u) \longmapsto (0,u) \; . \end{array}$$

(In comparing these formulas to YANG's, note that unfortunately our notations are in partial disagreement with his : YANG uses the notation λ_u for the map which in our notation is $v \longmapsto v\backslash u$; furthermore, in $S^{2n-1} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ the first and second coordinate are interchanged due to different conventions concerning the coordinatization of π by a quasifield.)

The inverses of these maps can be obtained by straightforward calculation:

$$g_0^{-1} : S^{2n-1} - U_\infty \longrightarrow \mathbb{R}^n \times S^{n-1} : \quad (x,y) \longmapsto (y/x, \frac{x}{\|x\|})$$

$$g_1^{-1} : S^{2n-1} - U_0 \longrightarrow ((\mathbb{R}^n - \{0\}) \cup \{\infty\}) \times S^{n-1} : \begin{array}{l} (x,y) \longmapsto (y/x, \frac{y}{\|y\|}) \quad \text{for } x \neq 0 \\ (0,y) \longmapsto (\infty, y) \; , \end{array}$$

Now it is easy to see that these maps are homeomorphisms which indeed constitute a complete system of local trivializations for π . In order to see that they are even *diffeomorphisms* one has to take a closer look; for the proof of the differentiability of g_1 and g_1^{-1} one still has to transform the elements of $(\mathbb{R}^n - \{0\}) \cup \{\infty\}$ by the chart ψ given above which determines the differentiable structure of the base space, i.e. to study the maps $g_1 \circ (\psi^{-1} \times \mathrm{id})$ and $(\psi \times \mathrm{id}) \circ g_1^{-1}$. They can be calculated directly:

$$g_1 \circ (\psi^{-1} \times \mathrm{id}) : \mathbb{R}^n \times S^{n-1} \longrightarrow S^{2n-1} - U_0$$

$$\begin{array}{ll} (v,u) \longmapsto \dfrac{((e/v)\backslash u, u)}{\|((e/v)\backslash u, u)\|} & \text{for} \quad v \neq 0 \\ (0,u) \longmapsto (0,u) & \end{array}$$

$$\begin{aligned}(\psi\times id)\circ g_1^{-1} : S^{2n-1} - U_0 &\longrightarrow \mathbb{R}^n \times S^{n-1}\\ (x,y) &\longmapsto (((y/x)\backslash e, \tfrac{y}{\|y\|}) \quad \text{for} \quad x \neq 0\\ (0,y) &\longmapsto (0,y)\ .\end{aligned}$$

The crucial expressions appearing in all these formulas are

$$y/x = \lambda_{y/x}(e)$$
$$(y/x)\backslash e = \lambda_{y/x}^{-1}(e)$$
$$(e/v)\backslash u = \lambda_{e/v}^{-1}(u)\ ;$$

thus by our differentiability conditions 2.5(iii) all these maps are indeed differentiable, and 2.5(i) follows.

Remark. These local trivializations suggest themselves by a geometrical construction in the translation plane corresponding to the given fibration π; by this construction one would be lead directly to the maps g_0^{-1} and $(\psi\times id)\circ g_1^{-1}$. One considers the "line" passing through the point (x,y) and the origin. For $x \neq 0$, this is the line $U_{y/x}$ of "slope" y/x, and the slope can be determined geometrically, the intersection point of $U_{y/x}$ with the "vertical unit line" $\{e\}\times\mathbb{R}^n$ being just $(e,y/x)$. Similarly, for $y \neq 0$ the intersection point of $U_{y/x}$ with the "horizontal unit line" $\mathbb{R}^n\times\{e\}$ is $(\lambda_{y/x}^{-1}(e),e) = ((y/x)\backslash e\ , e)$.

2.10 REMARK. In [16], loc. cit. the maps $g_1\circ(\psi^{-1}\times id)$ and $(\psi\times id)\circ g_1^{-1}$ were not studied explicitly, so the fact that the differentiability condition at infinity in 2.5(iii) is really necessary went unnoticed, and it was concluded that for an arbitrary real division algebra the corresponding fibration of S^{2n-1} is always differentiable. Now in fact this is only true for the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, as will be shown in [8].

2.11 ADDENDUM: The characteristic map of π. According to the local trivializations given in 2.9 the base space of the fibre bundle π is homeomorphic to S^n, so that π is determined up to fibre bundle equivalence by the homotopy class of its characteristic map, which can be read off the maps g_0^{-1} and g_1^{-1}. We consider the characteristic map to be defined on the whole intersection $\mathbb{R}^n-\{0\}$ of the two pieces of the base space over which the two local trivializations g_0 and g_1 are given; it maps a fixed element $a \in \mathbb{R}^n-\{0\}$ to the transformation $\chi_a \in \mathrm{Diff}(S^{n-1})$ with $\chi_a(\frac{x}{\|x\|}) = \frac{y}{\|y\|}$ if $y/x = a$,

i.e.
$$x_a(z) = \frac{a \circ z}{\|a \circ z\|} = \frac{\lambda_a(z)}{\|\lambda_a(z)\|} .$$
The homotopy class of the characteristic map of η is therefore completely determined by the homotopy class of the map
$$\lambda : \mathbb{R}^n - \{0\} \longrightarrow GL_n(\mathbb{R}) : a \longmapsto \lambda_a ,$$
i.e. the characteristic map of the coordinatizing quasifield.

2.12 REMARK. The introduction of a two-sided unit element as explained in 2.3 does not affect the topological properties of a differentiable quasifield on $\mathbb{R}^n$. One then gets an H-space structure on the sphere $S \cong S^{n-1}$ in $\mathbb{R}^n$ through e with center 0 (by radially projecting products in the quasifield back to the sphere). Therefore according to ADAMS' theorem [1], such quasifields can exist only for $n = 1, 2, 4$ and 8.

3. INFINITESIMAL DIVISION ALGEBRAS OF DIFFERENTIABLE QUASIFIELDS

In this section it will be shown that every differentiable quasifield on $\mathbb{R}^n$ can be transformed into a division algebra without changing the homotopy class of the characteristic map.

For a given differentiable quasifield on $\mathbb{R}^n$, we consider the map
$$\lambda : \mathbb{R}^n \longrightarrow \mathrm{End}(\mathbb{R}^n) : a \longmapsto \lambda_a .$$
Appropriate division algebras will be obtained by differentiating λ, thus linearizing the multiplication in the first argument, too. This linearization procedure is equivalent on the coordinate level to the linearization procedure for fibrations described in [7: Section 6 p.1057] ("from fibrations to algebras"). The following lemma ensures that this process will not produce zero divisors. It corresponds to lemmas 4.5 and 4.8 in [7: p. 1051 ff.], but is easier.

3.1 LEMMA. *For $c \in \mathbb{R}^n$, consider the derivative of λ in c:*
$$d_c\lambda : \mathbb{R}^n \longrightarrow \mathrm{End}(\mathbb{R}^n) .$$
Then
$$d_c\lambda(u) \text{ is non-singular for } 0 \neq u \in \mathbb{R}^n .$$

Proof: In the given differentiable quasifield, the map

$$\rho_u : x \longmapsto x \circ u$$

is a diffeomorphism for $u \neq 0$, since $\rho_u(x) = \lambda_x(u)$ and $\rho_u^{-1}(y) = y/u = \lambda_{y/u}(e)$. For $u, v \in \mathbb{R}^n - \{0\}$ we have

$$(d_e\lambda)(u)(v) = \frac{d}{dt}\lambda_{e+tu}(v)\Big|_{t=0} = \frac{d}{dt}((e+tu)\circ v)\Big|_{t=0} = \frac{d}{dt}\rho_v(e+tu)\Big|_{t=0} = (d_e\rho_v)(u) ;$$

and this is $\neq 0$, since the derivative $d_e\rho_v$ of the diffeomorphism ρ_v is non-singular.

3.1 CONSTRUCTION. Starting from a given differentiable quasifield on $\mathbb{R}^n$, we define, for fixed $e \in \mathbb{R}^n$, a new multiplication $\cdot$ on $\mathbb{R}^n$ by

$$u \cdot v := (d_e\lambda)(u)(v) .$$

Obviously, this multiplication is linear in both arguments, and by lemma 3.1 it has no zero divisors. By definition, e remains a right multiplicative unit element also for the new multiplication. Thus $\cdot$ is the multiplication of a real division algebra, which we call the infinitesimal division algebra at e of the given differentiable quasifield.

In what follows we concentrate on the infinitesimal division algebra at e.

3.2 <u>The characteristic map of the infinitesimal division algebra at</u> e is

$$d_e\lambda : \mathbb{R}^n - \{0\} \to GL_n(\mathbb{R}) : a \longmapsto (d_e\lambda)(a) ;$$

it is homotopic to the characteristic map

$$\lambda : \mathbb{R}^n - \{0\} \to GL_n(\mathbb{R}) : a \longmapsto \lambda_a$$

of the original quasifield:

A homotopy between the two maps is

$$(a,t) \longmapsto \begin{cases} \frac{1}{t}\lambda_{ta} & \text{for } t \neq 0 \\ d_e\lambda(a) = \lim_{t\to 0}\frac{1}{t}\lambda_{ta} & \text{for } t = 0 \end{cases}$$

3.3 Note that the infinitesimal division algebra at e does not necessarily admit a two-sided unit element, even if the original quasifield does. In this case, however, the introduction of a unit element by the method of 2.3 does not affect the homotopy class of the characteristic map:

Consider the map

$$\theta = d_0\lambda(e) : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \longmapsto e * x .$$

The new multiplication

$$u \times v := u * \theta^{-1}(v)$$

defines a division algebra again, which now admits e as *two-sided* unit element. The characteristic map of this division algebra is $u \longmapsto d_0\lambda(u) \circ \theta^{-1}$. It is homotopic to $d_0\lambda|\mathbb{R}^n - \{0\}$ in $GL_n(\mathbb{R})$ since by the homotopy 3.3(*) $\theta = d_0\lambda(e)$ can be differentiably deformed in $GL_n(\mathbb{R})$ into λ_e, which is the identity as we have assumed that the original quasifield has a two-sided unit element.

This is summarized by

3.5 *For a differentiable quasifield on* $\mathbb{R}^n$ *with characteristic map* $\lambda|\mathbb{R}^n - \{0\}$ *admitting a two-sided unit element* e *there is a real division algebra whose characteristic map is homotopic in* $GL_n(\mathbb{R})$ *to* $\lambda|\mathbb{R}^n - \{0\}$ *and which admits* e *as two-sided unit element, too.*

As a direct corollary to BUCHANAN's theorem [4: Hilfssatz 2 p. 389] on the characteristic map of real division algebras we now obtain:

3.6 PROPOSITION. *The characteristic map of a differentiable quasifield on* $\mathbb{R}^n$ *with two-sided unit element is homotopic to the characteristic map of one of the classical division algebras* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ *or the opposite algebras* $\mathbb{H}^{op}$, $\mathbb{O}^{op}$.

Proof: By ADAMS' theorem the only dimensions to be considered are $n = 1, 2, 4, 8$ (see 2.12). By 3.5 the general case of an arbitrary differentiable quasifield is reduced to the particular case of a real (not necessarily classical) division algebra. In this case, for $n = 4$ or 8, the claim of the proposition is nothing else than BUCHANAN's theorem. In dimension 1 resp. 2 there is nothing left to prove, since in these dimensions the only division algebras with unit (without associativity preassumed!) are $\mathbb{R}$ and $\mathbb{C}$, as is well-known.

[For $n = 1$, this is obvious. For a division algebra of dimension 2 with two-sided unit e, one can argue as follows: Let x be an element which is not a real multiple of e, then the whole algebra is spanned by e and x. An elementary calculation shows that there is a linear combination $z = \varepsilon\cdot e + \zeta\cdot x$ with $\varepsilon,\zeta\in\mathbb{R}$; $\zeta \neq 0$ such that $z^2 = \alpha\cdot e$ for suitable $\alpha\in\mathbb{R}$. Then $\alpha < 0$, since otherwise $(z - \sqrt{\alpha}\cdot e)(z + \sqrt{\alpha}\cdot e) = 0$ and $z = \pm\sqrt{\alpha}\cdot e$ which would contradict $\zeta \neq 0$. Putting $i = \frac{1}{\sqrt{-\alpha}}\cdot z$ one has $i^2 = -e$.]

The following technical lemma will be useful in the sequel:

3.7 LEMMA. *For a differentiable quasifield on* $\mathbb{R}^n$ *with multiplication* $\circ$ *and characteristic map* λ *and its infinitesimal division algebra at* 0 *with multiplication* $*$ *consider the element*

$$y/x \quad \textit{with} \quad (y/x)\circ x = y$$

and analogously the element

$$y/\!/x \quad \textit{with} \quad (y/\!/x) * x = y \ .$$

Then the following formulas hold:

(i) $$y/\!/x = \frac{d}{dt}\,(ty)/x\Big|_{t=0}$$

(ii) $$d_0\lambda(y/\!/x) = \frac{d}{dt}\,\lambda_{(ty)/x}\Big|_{t=0} .$$

Proof: By the chain rule one has

(1) $$d_0\lambda\Big(\frac{d}{dt}\,(ty)/x\Big|_{t=0}\Big) = \frac{d}{dt}\,\lambda_{(ty)/x}\Big|_{t=0} ,$$

in particular, by the definition of $*$ (see 3.2)

$$\frac{d}{dt}\,(ty)/x\Big|_{t=0} * x = \frac{d}{dt}\,\lambda_{(ty)/x}(x)\Big|_{t=0} = \frac{d}{dt}\,((ty)/x)\circ x\Big|_{t=0} = \frac{d}{dt}\,ty\Big|_{t=0} = y \ .$$

This shows (i), and (ii) follows by (1).

4. THE TOPOLOGICAL AND DIFFERENTIABLE EQUIVALENCE CLASS OF FIBRATIONS OVER DIFFERENTIABLE QUASIFIELDS

In this section we shall prove theorems 1.1 and 1.3 and discuss conjecture 1.2.

Let π be a locally trivial differentiable fibre bundle with total space S^{2n-1} whose fibres are great $(n-1)$-spheres. Consider a quasifield associated with π (see 2.2), and let

$$\lambda : \mathbb{R}^n - \{0\} \longrightarrow GL_n(\mathbb{R})$$

be its characteristic map; by 2.3 we may assume that the quasifield has a two-sided unit e. According to 2.5, the quasifield is differentiable and differentiable at infinity.

4.1 <u>Proof of theorem 1.1</u>. The topological equivalence class of the fibre bundle π depends only on the homotopy class of the characteristic map $\lambda|\mathbb{R}^n-\{0\}$ (see 2.11). Now by 3.6, the characteristic map is homotopic to the characteristic map of one of the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{H}^{op}$, $\mathbb{O}$ or $\mathbb{O}^{op}$. Therefore, π is fibre bundle equivalent to the corresponding Hopf fibration.

[Note that the fibrations corresponding to $\mathbb{H}$ and $\mathbb{H}^{op}$ are equivalent, a fibre bundle equivalence being given by the map

$$S^{2n-1} \longrightarrow S^{2n-1} : (x,y) \longmapsto (\bar{x},\bar{y}) \qquad (x,y \in \mathbb{R}^n = \mathbb{H}),$$

and similarly for $\mathbb{O}$ and $\mathbb{O}^{op}$.]

4.2 π *is differentiably equivalent to the classical Hopf fibration of* S^{2n-1} *if and only if the base space* $B(\pi)$ *of* π *is diffeomorphic to the base space of the classical Hopf fibration (namely* S^n *with the usual differentiable structure).*

Proof: π can be considered as the sphere bundle of a vector bundle; by 2.11, the characteristic map of this vector bundle is just λ (which indeed maps into the appropriate structure group $GL_n(\mathbb{R})$). The topological fibre bundle equivalence established in the proof of theorem 1.1 carries over to the corresponding vector bundles, since the homotopy between characteristic maps (3.6) takes place in $GL_n(\mathbb{R})$. Now it is well known that two vector bundles over the same differentiable manifold are differentiably equivalent if only they are topologically equivalent. (This is obtained by differentiable approximation of a homotopy between the classifying maps, see [12: §4 Theorem 2.5 p. 101].)

We now recall the description of $B(n)$ from 2.9:

4.3 $B(n)$ is obtained by gluing together two copies of $\mathbb{R}^n$ along $\mathbb{R}^n \setminus \{0\}$ via the "inversion function" of the associated quasifield

$$\psi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\} : a \longmapsto a \backslash e = \lambda_a^{-1}(e).$$

Note that ψ is a diffeomorphism, the inverse being the map

$$b \longmapsto e/b = \lambda_{e/b}(e).$$

4.4 CONJECTURE. ψ is isotopic to the inversion map of the corresponding classical division algebra, i.e. the map

$$\mathbb{K} \setminus \{0\} \longrightarrow \mathbb{K} \setminus \{0\} : x \longmapsto x^{-1}$$

where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ if $n = 1, 2, 4, 8$, respectively.

Whenever this is the case, $B(n)$ is diffeomorphic to the base space S^n of the classical Hopf fibration over $\mathbb{K}$ (as can be deduced easily from the isotopy extension theorem by using for instance [12: §8 Theorem 1.9 p. 182]), and then conjecture 1.2 holds true by 4.2.

Again, as for the characteristic map, the problem posed by conjecture 4.4 may be reduced to the infinitesimal division algebra:

4.5 The inversion map ψ of a differentiable quasifield is isotopic to the inversion map ψ of the infinitesimal division algebra at 0.

Proof: Since the characteristic map of the infinitesimal division algebra at 0 is $d_0\lambda$, its inversion map is

$$\psi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\} : a \longmapsto d_0\lambda(a)^{-1}(e) .$$

Now for $t \in [0,1]$, the maps

$$\psi_t : a \longmapsto \begin{cases} (\frac{1}{t}\cdot\lambda_{ta})^{-1}(e) & \text{for } t \neq 0 \\ d_0\lambda(a)^{-1}(e) & \text{for } t = 0 \end{cases}$$

are diffeomorphisms, with inverses $b \longmapsto \frac{1}{t}\cdot(te)/b$ for $t \neq 0$ and

$b \longrightarrow e/\!/b$ for $t = 0$ in the notation of lemma 3.7, which shows that moreover the map $(b,t) \longmapsto \psi_t^{-1}(b)$ is differentiable in both variables also at $t = 0$; for the map $(a,t) \longmapsto \psi_t(a)$ this is obvious. Thus the ψ_t constitute an isotopy between $\psi_0 = \Psi$ and $\psi_1 = \psi$.

Introducing a unit element as in 3.4 will alter the inversion map only by a linear map from the connected component of the identity; thus we obtain from 4.5:

4.6 *The inversion map* ψ *of a differentiable quasifield with two-sided unit is isotopic to the inversion map of a division algebra with two-sided unit.*

Since $\mathbb{C}$ is the only 2-dimensional division algebra with unit, this shows that conjecture 4.4 and therefore conjecture 1.2 are true for $n = 2$. Of course this is already well-known and follows directly from fundamental results of differential topology in dimension 2, but the present argument may offer a pleasant alternative. See also [6: Remark 9.8 p. 131].

4.7 PROPOSITION. $B(\pi)$ *is diffeomorphic to the twisted sphere obtained by gluing together two copies of the unit ball* $\mathbb{D}^n$ *along their boundaries via the diffeomorphism*

$$S^{n-1} \longrightarrow S^{n-1} : z \longmapsto \frac{\Psi(z)}{\|\Psi(z)\|}$$

as gluing map (Ψ *being the inversion map of the infinitesimal division algebra at* 0 *of a quasifield associated with* π *).*

Proof: For $a \in \mathbb{R}^n - \{0\}$, one has by definition $a * \Psi(a) = e$, where $*$ is the multiplication of the infinitesimal division algebra. Since this multiplication is linear in both arguments, it follows that for $t \in \mathbb{R} - \{0\}$

$$\Psi(ta) = \frac{1}{t} \cdot \Psi(a) .$$

Therefore Ψ maps every ray of $\mathbb{R}^n$ diffeomorphically onto a ray of $\mathbb{R}^n$, and it is easy to deform Ψ isotopically into a diffeomorphism of $\mathbb{R}^n - \{0\}$ which maps S^{n-1} onto itself. Explicitly, the diffeomorphisms

$$\Psi_t : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\} : a \longmapsto \frac{\Psi(a)}{(1-t) + t \cdot \left\|\Psi\left(\frac{a}{\|a\|}\right)\right\|}$$

with inverses

$$b \longmapsto \frac{\psi^{-1}(b)}{(1-t) + t\cdot\|\psi^{-1}(\frac{b}{\|b\|})\|}$$

constitute an isotopy between $\psi_0 = \psi$ and

$$\psi_1 : \mathbb{R}^n-\{0\} \longrightarrow \mathbb{R}^n-\{0\} : a \longmapsto \frac{\psi(a)}{\|\psi(\frac{a}{\|a\|})\|} .$$

Combining this with 4.5, we see that ψ_1 is isotopic to the inversion map ψ of the original quasifield. Therefore $B(\pi)$, obtained by gluing together two copies of $\mathbb{R}^n$ along $\mathbb{R}^n-\{0\}$ via ψ (4.3), is diffeomorphic to the space which is obtained by using ψ_1 as gluing map instead of ψ (again by the isotopy extension theorem). Since $\psi_1(S^{n-1}) = S^{n-1}$, the latter space can also be described as stated in the proposition.

4.8 Proof of theorem 1.3. We now show for $n = 4$ that $B(\pi)$ is always diffeomorphic to S^n (theorem 1.3 will then follow by 4.2).

In view of 4.7, this is an immediate consequence of CERF's result [5: Théorème 1 p. 3] according to which every diffeomorphism of S^3 is isotopic to the identity or to a reflection: By 4.7 and [12: §8 Theorem 2.3 p. 185] it follows that $B(\pi)$ is diffeomorphic to the space obtained by gluing together two copies of D^4 along their boundaries via the identity or a reflection; in both cases the result is S^4 with its usual differentiable structure.

4.9 REMARK. In general it might well be that the reduction to an infinitesimal division algebra achieved in 4.5 results in a loss of information since the infinitesimal division algebra at 0 of a quasifield associated with π need not be differentiable *at infinity* any more. In fact, according to [8], this is only true if the infinitesimal division algebra is isomorphic to the classical algebra of the corresponding dimension (after the introduction of a two-sided unit element, if necessary); but in this case our problem is solved anyhow by 4.5. One can construct examples of fibrations for which none of the infinitesimal division algebras of associated quasifields is classical in this sense; on the other hand, for all examples known to the author conjecture 4.4 can be verified individually.

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