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A New Estimate for the Ginzburg–Landau Approximation on the Real Axis

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Summary. Modulation equations play an essential role in the understanding of complicated systems near the threshold of instability. For scalar parabolic equations for which instability occurs at nonzero wavelength, we show that the associated Ginzburg–Landau equation dominates the dynamics of the nonlinear problem locally, at least over a long timescale. We develop a method which is simpler than previous ones and allows initial conditions of lower regularity. It involves a careful handling of the critical modes in the Fourier-transformed problem and an estimate of Gronwall's type. As an example, we treat the Kuramoto–Shivashinsky equation. Moreover, the method enables us to handle vector-valued problems [see G. Schneider (1992)].

Key words. modulation equations, Ginzburg–Landau approximation, nonlinear partial differential equations on unbounded domains

1. Introduction

We consider scalar evolutionary problems on the real line. In the parabolic case one is interested in the behavior of systems close to the threshold of instability. If a spatially homogeneous solution becomes unstable, a whole band of wave numbers turns unstable. In this situation the bifurcating solutions can be approximately described by a so-called modulation equation. As an example we study the Kuramoto–Shivashinsky equation

$$\partial_t u = -(1 + \partial_x^2)^2 u + \alpha u + u \partial_x u. \tag{1}$$

The trivial solution $u \equiv 0$ is unstable for $\alpha > 0$ and, linearizing at $u \equiv 0$, we find solutions of the form $u(x, t) = e^{\mu t + ikx}$, where $\mu(k, \alpha) = -(1 - k^2)^2 + \alpha$ is positive for k close to ± 1. Note that center-manifold theory is no longer available for describing bifurcating solutions due to the continuous spectrum. One expects that for small $\alpha > 0$ there are solutions which are small modulations in time and space of the critical modes $e^{\pm ix}$. Using the scalings $\alpha = \epsilon^2$, $T = \epsilon^2 t$, and $X = \epsilon x$ we introduce the formal approximation

$$\psi(x, t, \epsilon) = \epsilon(A(X, T)e^{ix} + \bar{A}(X, T)e^{-ix}). \tag{2}$$

By a formal calculation we find that the amplitude A has to satisfy the Ginzburg–Landau equation

$$\partial_T A = A + 4\partial_X^2 A - (1/9)A|A|^2. \tag{3}$$

One expects that ψ describes the dynamics of solutions which are near the attractor of the system (see [Eck91], [vH92]). This kind of approximation was introduced by Newell and Whitehead (see [NW69]) in 1969 for Bénard's problem. In other hydrodynamic problems, such as the Taylor–Couette problem or Poiseuille flow, such an approximation is also possible, due to the form of the spectrum.

Taking a solution A of (3), the question arises as to how well ψ approximates a solution $u(x, t, \epsilon)$ of the original problem. Of course we have to show that on an $\mathcal{O}(1)$-timescale of (3), ψ is a good approximation. The above question was first treated in [CE90] for the Swift–Hohenberg equation. A simple proof was given in [KSM92] for cases in which the nonlinearity begins with a cubic term. The case of quadratic nonlinearities is more difficult, and until now the question was answered only when the initial data of the Ginzburg–Landau equation (3) was analytic in a strip in the complex plane [vH91]. For the case of the Kuramoto–Shivashinsky equation (1) our theorem specializes to

Theorem 1. *Let $A = A(X, T) \in C([0, T_0], C^4(\mathbb{R}, \mathbb{C}))$ be a solution of the Ginzburg–Landau equation (3). Then there exist $\epsilon_0, C > 0$, such that for all $\epsilon \leq \epsilon_0$ there are solutions $u(x, t, \epsilon)$ of (1) with*

$$\sup_{0 \leq t \leq T_0/\epsilon^2} \|u(x, t, \epsilon) - (\epsilon A(X, T)e^{ix} + c.c.)\|_{C^4(\mathbb{R},\mathbb{R})} < C\epsilon^2.$$

The ideas of the following proof are such that the result can be extended to reaction-diffusion equations and, under some restrictions on the initial conditions, to the Bénard problem or the Taylor–Couette problem (see [Sch92]). This will be described in a forthcoming paper.

In this paper we use the abbreviation $C^n = C^n(\mathbb{R}, \mathbb{C})$ for the space of functions with n bounded and continuous derivatives. Constants are denoted throughout by C.

2. The General Situation

The situation of the introduction can appear in more general problems. We consider scalar semilinear problems

$$\partial_t u = \lambda(\partial_x, \epsilon^2)u + f(\partial_x, u) \tag{4}$$

on the real line which are translation invariant with respect to x. The differential operator λ is of degree $2d$ with $\Re\lambda(ik, \epsilon^2) \to -\infty$ for $k \to \pm\infty$ and f is a nonlinear smooth mapping in a neighborhood of zero from C^{2d} to C^1. Therefore, $\lambda(\cdot, \epsilon^2)$ is a polynomial with constant coefficients. If $\mu(k, \epsilon^2) = \lambda(ik, \epsilon^2)$ has the same nature as μ in the above example then it has an expansion around the critical value $k_c \neq 0$ of the form

$$\mu(k_c + \epsilon K, \epsilon^2) = i\omega_0 + \epsilon i \nu_1 K + \epsilon^2(\lambda_0 + i\nu_0) + \epsilon^2 K^2(\lambda_2 + i\nu_2) + \mathcal{O}(\epsilon^3)$$

with $\lambda_0 > 0$ and $\lambda_2 < 0$. The imaginary coefficients ω_0 and ν_1 give rise to fast dynamics. We get rid of ν_1 by the transformation $x \mapsto x - \nu_1 t$. The coefficient ω_0 is taken into account in the following ansatz. The dynamics of the linearly unstable modes $e^{\pm ik_c x}$ are also determined by nonlinear interactions with the linearly damped modes $e^{\pm imk_c x}$ ($m \in Z$). Therefore, these modes are also included in the formal but consistent infinite ansatz

$$u(x - \nu_1 t, t) = \sum_{m \in Z} \sum_{n=0}^{\infty} \epsilon^{\alpha(m)+n}(A_m^n(\epsilon x, \epsilon^2 t)e^{im\hat{\omega}_0 t}e^{imk_c x}) \tag{5}$$

with $\alpha(m) = 1 + \left|1 - |m|\right|$, $\omega_0 - \nu_1 k_c =: \hat{\omega}_0$, and $A_m^n(X) = \bar{A}_{-m}^n(X)$. To compute the equations which the A_m^n have to fulfill, we insert (5) into (4). Then we compute the coefficients in front of $e^{imk_c x}$. Using $\partial_t = \epsilon^2 \partial_T$ and $\partial_x = \epsilon \partial_X$, we expand these coefficients in powers of ϵ, and so obtain an infinite system of equations for the A_m^n (see [vH91]). The Ginzburg–Landau equation for (4) then has the form

$$\partial_T A_1^0 = (\lambda_0 + i\nu_0)A_1^0 - (\lambda_2 + i\nu_2)\partial_X^2 A_1^0 + \gamma A_1^0 |A_1^0|^2, \tag{6}$$

where $\gamma \in \mathbb{C}$ is determined by the above computations. Given the above situation, we can state the following theorem.

Theorem 2. *Let $A_1^0 = A_1^0(X, T) \in C([0, T_0], C^{2d})$ be a solution of the formally derived Ginzburg–Landau equation (6) to problem (4). Then there exist $\epsilon_0, C > 0$ such that for all $\epsilon < \epsilon_0$ there are solutions $u = u(x, t, \epsilon)$ of (4) with*

$$\sup_{0 \le t \le T_0/\epsilon^2} \|u(x - \nu_1 t, t, \epsilon) - \psi(x, t, \epsilon)\|_{C^{2d}} \le C\epsilon^2,$$

where $\psi(x, t, \epsilon) = (\epsilon A_1^0(\epsilon x, \epsilon^2 t)e^{ik_c x}e^{i\hat{\omega}_0 t} + c.c.)$.

3. General Ideas for the Proof

We start by explaining why quadratic terms are the main difficulty in proving Theorem 2. To explain this problem and the ideas for solving it, we write (4) in the form

$$\partial_t u = \lambda u + N_2(u, u) + N_3(u, u, u) + \cdots, \tag{7}$$

where N_j contains the terms of power j and therefore can be written as a symmetric j-linear form. If there is an approximation $\phi = \epsilon\tilde{\phi}$ of order $\mathcal{O}(\epsilon)$, the error should be of order $\mathcal{O}(\epsilon^2)$ on a sufficiently long time interval. Inserting a solution $u = \epsilon\tilde{\phi} + \epsilon^2 R$ in (7), we get

$$\partial_t R = \lambda R + 2\epsilon N_2(\tilde{\phi}, R) + 3\epsilon^2 N_3(\tilde{\phi}, \tilde{\phi}, R) + \epsilon^2 N_2(R, R) + \cdots + \frac{1}{\epsilon^2}\mathrm{Res}(\epsilon\tilde{\phi}) \quad (8)$$

as the differential equation for the error R with initial datum $R|_{t=0} = 0$. The approximation ϕ is chosen such that low ϵ-orders vanish identically, hence the residuum $\mathrm{Res}(\phi) = -\partial_t \phi + \lambda\phi + f(\phi)$ is small. Taking sufficiently many terms of (5) we can construct an approximation ϕ, $\mathcal{O}(\epsilon^2)$-close to the original approximation ψ of the theorem, such that $\mathrm{Res}(\phi)$ will be of order $\mathcal{O}(\epsilon^n)$ for arbitrary $n \geq 1$.

The approximation $\epsilon\tilde{\phi}$ is a good approximation of a solution u if a solution R of (8) is $\mathcal{O}(1)$-bounded on a sufficiently large time interval. In our case λ is the generator of a slowly increasing semigroup, whose norm can be estimated by $C\exp(C\epsilon^m t)$ with $m = 2$. Moreover, suppose that all N_j vanish identically for $j \leq n$; this means that ϵ^n is the lowest ϵ-order on the right-hand side of (8). Using the transformation $T = \epsilon^{\min(m,n)}t$ and Gronwall's inequality, one can show that R is $\mathcal{O}(1)$-bounded on a time interval of length $\mathcal{O}(1/\epsilon^{\min(m,n)})$. As stated, the Ginzburg–Landau approximation has to be a good approximation on a time interval of length $\mathcal{O}(1/\epsilon^2)$. That means $n = 2$ and the quadratic terms $N_2(u, u)$ should disappear. This approach was used in [KSM92].

If quadratic terms are present, the estimation of the error over time intervals of length $\mathcal{O}(1/\epsilon^2)$ is more difficult. The main tool for attaining the required estimate on this long time interval is Fourier transform. The approximation $\epsilon\tilde{\phi}$ that we take will be of order $\mathcal{O}(\epsilon)$ for linearly unstable (critical) Fourier modes and of order $\mathcal{O}(\epsilon^2)$ for the linearly damped Fourier modes. Therefore, we suppose that the error $\epsilon^2 R$ is of order $\mathcal{O}(\epsilon^2)$ for linearly unstable Fourier modes and of order $\mathcal{O}(\epsilon^3)$ for the linearly damped Fourier modes. To make this formal, let P be a λ-invariant operator which extracts Fourier modes corresponding to an $\mathcal{O}(1)$-neighborhood of wave numbers around the critical wave numbers $\pm k_c$. We set $\phi_c = P\tilde{\phi}$, $\epsilon\phi_s = (1-P)\tilde{\phi}$, $R_c = PR$, and $\epsilon R_s = (1-P)R$. Inserting a solution

$$u = \epsilon\phi_c + \epsilon^2\phi_s + \epsilon^2 R_c + \epsilon^3 R_s$$

in (7) and applying the operator P, we obtain

$$\begin{aligned} \partial_t R_c &= \lambda R_c + 2\epsilon P N_2(R_c, \phi_c) + \mathcal{O}(\epsilon^2), \\ \partial_t R_s &= \lambda R_s + 2(1-P) N_2(R_c, \phi_c) + \mathcal{O}(\epsilon) \end{aligned} \quad (9)$$

as equations for R_c and R_s. The restriction $\lambda_2 = \lambda|_{\mathrm{range}\,(1-P)}$ is the generator of an exponentially damped semigroup. Hence R_s will be of order $\mathcal{O}(1)$ over any timescale. The norm of the semigroup, generated by $\lambda_1 = \lambda|_{\mathrm{range}\,P}$, increases as $Ce^{C\epsilon^2 t}$. As outlined above, the error R_c can be shown to be of order $\mathcal{O}(1)$ over the $\mathcal{O}(1/\epsilon^2)$ timescale if the quadratic term

$$P N_2(R_c, \phi_c) = 0 \quad (10)$$

disappears. But this is true if a suitable P is chosen! The wave numbers of the critical Fourier modes are contained in the set $[-5k_c/4, -3k_c/4] \cup [3k_c/4, 5k_c/4]$. Therefore, the wave numbers of the convolution of such modes are contained in

$[-5k_c/2, -3k_c/2] \cup [-k_c/2, k_c/2] \cup [3k_c/2, 5k_c/2]$. Since the intersection of these two sets is empty, the application of the operator P on this quadratic interaction yields zero. Therefore, (10) is fulfilled and the proof of the theorem will be possible.

Up to now, all arguments have been formal. In the next section we will provide a mathematically rigorous framework. We choose the space of n-times differentiable functions C^n because it contains many interesting solutions; for example, fronts, quasi-periodic, or space-periodic functions. Therefore, we have to translate the above ideas from Fourier space into the physical space C^n. This will be done in the next section, where we define the mode-filters, which extract the critical modes. In the fifth section we estimate the approximation error $u - \psi$.

4. The Mode-Filters

Before we start we have to say a few words about the functional analytic devices. The main tool here is Fourier transform. It is well known that the space of n-times differentiable functions with bounded derivatives C^n is a subset of the space of tempered distributions $\mathcal{S}'$ (see [W182]). Therefore, it is possible to define the Fourier transform $\mathcal{F}$ of functions $u \in C^n$, and thus we can define operators by their action in Fourier space. As stated, we want to separate linearly unstable from linearly damped modes in Fourier space.

We define the mode-filter for the critical modes

$$E_c u := \int G_c(x - \xi) u(\xi) d\xi,$$

where G_c is defined by $G_c(x) = (1/2\pi) \int e^{ikx} \chi_c(k) dk$ and χ_c is a positive even C_0^∞-function with

$$\chi_c(k) = \begin{cases} 1 & \text{for } k \in I_c = [-9k_c/8, -7k_c/8] \cup [7k_c/8, 9k_c/8], \\ 0 & \text{for } k \in \mathbb{R} \backslash U_{kc/8}(I_c), \end{cases}$$

where $U_r(M) = \{k | \, |k - s| < r \text{ for an } s \in M\}$. Moreover, we define the operator $E_s = 1 - E_c$ for the uncritical modes. The Fourier transform of $E_c u$ is a distribution with support $[-5k_c/4, -3k_c/4] \cup [3k_c/4, 5k_c/4]$. This set contains the wave numbers of the critical modes. In the same way, we define an operator E_0 with the same properties as E_c but with the set $I_0 = [-k_c/8, k_c/8]$. For the complement we define $E_0^c = 1 - E_0$. Since E_c and E_s are not projections we define additional operators E_c^h and E_s^h with $E_c^h E_c = E_c$ and $E_s^h E_s = E_s$. The C^∞-function χ_c^h defining E_c^h vanishes outside $U_{k_c/4}(I_c)$, and χ_s^h defining E_s^h vanishes in $[-17k_c/16, -15k_c/16] \cup [15k_c/16, 17k_c/16]$. For example, the Fourier transform of $(E_0 u(x))e^{ijx}$ is a distribution with support $[j - (k_c/4), j + (k_c/4)]$.

The kernels G_c, G_0 lie in $\mathcal{S}$, which is the space of rapidly decaying C^∞ functions. This means that the function as well as all derivatives decay faster than $1/|x|^n$ for $|x| \to \infty$ and all $n \in \mathbb{N}$ (see [W182]).

In the following we give three properties of the mode-filters. They are basic for the estimates of the next section. First we show that mode-filters corresponding to bounded intervals of wave numbers have smoothing properties.

Lemma 3. *The operators E_c and E_0 are linear and continuous mappings from C^0 to C^m. For every $m \geq 0$ there exists $C_m > 0$ with $\|E_0 u\|_{C^m} + \|E_c u\|_{C^m} \leq C_m \|u\|_{C^0}$.*

Proof. The differentiability follows from $G_c \in C^\infty$. The norm can be estimated by

$$\|E_c u\|_{C^m} \leq \sum_{r=0}^{m} \sup_x \left| \int \partial_x^r G_c(x-\xi) u(\xi) d\xi \right| \leq \left(\sum_{r=0}^{m} \|\partial_x^r G_c\|_{L^1} \right) \|u\|_{C^0}.$$

Since $G_c \in \mathcal{S}$ the result follows. □

The application of E_0 on a scaled $A_m^n = A_m^n(\epsilon x)$ smoothes A_m^n. The scaling $X = \epsilon x$ concentrates the Fourier modes $\mathcal{O}(\epsilon)$-close to the wave number zero in Fourier space. Therefore the application of E_0 on a scaled function cannot change the scaled function A_m^n very much; that is, $E_0 A_m^n(\epsilon x) - A_m^n(\epsilon x)$ is small.

Lemma 4. *For $n \in \mathbb{N}$ there is a $C_n > 0$ such that $\|(E_0^c A(\epsilon \cdot))\|_{C^n} = \|(E_0 A(\epsilon \cdot)) - A(\epsilon \cdot)\|_{C^n} \leq C_n \epsilon^n \|A\|_{C^n}$.*

Proof. We use the well-known relation $\int G_0(x) x^n dx = (-i)^n \chi_0^{(n)}(0)$ which yields 1 for $n = 0$ and 0 for $n \geq 1$ to obtain with $\theta \in [\min(x, \xi), \max(x, \xi)]$

$$\begin{aligned}
\|E_0^c A(\epsilon x)\|_{C^n} &= \sup_{r=0,\dots n;x} |\partial_x^r [A(\epsilon x) - \int G_0(x-\xi) A(\epsilon \xi) d\xi]| \\
&= \sup_{r=0,\dots n;x} |\epsilon^r \int G_0(x-\xi)[\partial_X^r A(\epsilon x) - \partial_X^r A(\epsilon \xi)] d\xi| \\
&= \sup_{r=0,\dots n;x} |\epsilon^r \int G_0(x-\xi)[(\sum_{j=r+1}^{n-1} \partial_X^j A(\epsilon x) \epsilon^{j-r} (x-\xi)^{j-r}/(j-r)!) \\
&\qquad + \partial_X^n A(\theta)(x-\xi)^{n-r} \epsilon^{n-r}/(n-r)!] d\xi| \\
&\leq C\epsilon^n \|\partial_X^n A\|_{L^\infty} \sup_{r=0,\dots,n} \int |G(x)|\, |x|^r dx \leq C\epsilon^n \|A\|_{C^n}.
\end{aligned}$$

□

The third property is equivalent to (10) and makes rigorous what has already been stated in an informal way. The application of E_c to quadratic terms, whose factors are filtered by E_c, vanishes.

Lemma 5. *For $u_1, u_2 \in C^n$ and $r_1, r_2 \in \mathbb{N}$ it is true that*

$$E_c(\partial_x^{r_1} E_c u_1 \cdot \partial_x^{r_2} E_c u_2) = 0.$$

Proof. The Fourier transforms $\mathcal{F}(\partial_x^{r_1} E_c u_1)$ and $\mathcal{F}(\partial_x^{r_2} E_c u_2)$ are tempered distributions with support $[-5k_c/4, -3k_c/4] \cup [3k_c/4, 5k_c/4]$. The convolution of distributions with compact support is well defined (see [W182]). The convolution $\mathcal{F}(\partial_x^{r_1} E_c u_1) * \mathcal{F}(\partial_x^{r_2} E_c u_2)$ has support $[-5k_c/2, -3k_c/2] \cup [-k_c/2, k_c/2] \cup [3k_c/2, 5k_c/2]$ and so the application of E_c on the convolution yields zero. □

5. Proof of Theorem 2

First we seek a good approximation ϕ near ψ for which the residuum $\mathrm{Res}(\phi)$ will be $\mathcal{O}(\epsilon^n)$ with n sufficiently large. Next we estimate the error R, as explained in the third section.

5.1. The Approximation and the Residuum

Because of notational complexity we restrict this part to the case of the Kuramoto–Shivashinsky equation. But the ideas for the general case will be clear. We write (1) in the form

$$\partial_t u = \lambda(\partial_x, \epsilon^2)u + \rho(\partial_x)u^2 \tag{11}$$

with $\lambda = -(1+\partial_x^2)^2 + \epsilon^2$ and $\rho = \partial_x/2$. When we insert the ansatz (5) in (11) we get in lowest order the modulation equations

$$\begin{aligned} \partial_T A_1^0 &= (1 + 4\partial_x^2)A_1^0 + iA_{-1}^0 A_2^0, \\ 0 &= -9A_2^0 + iA_1^0 A_1^0, \\ 0 &= -A_0^0. \end{aligned} \tag{12}$$

Upon elimination of $A_{\pm 2}^0$ we arrive at the Ginzburg–Landau equation (3). Formally, in the coefficient in front of e^{ix} all terms of order $\mathcal{O}(\epsilon^3)$, and for 1 and e^{2ix} all terms of order $\mathcal{O}(\epsilon^2)$, have vanished.

We take a finite sum of (5) and modify this formal approximation by applying appropriate mode-filters and so let

$$\begin{aligned} \phi = {} & \epsilon(E_0 A_1^0(\epsilon x, \epsilon^2 t))e^{ix} + \epsilon(E_0 A_{-1}^0(\epsilon x, \epsilon^2 t))e^{-ix} \\ & + \epsilon^2(E_0 A_2^0(\epsilon x, \epsilon^2 t))e^{2ix} + \epsilon^2(E_0 A_{-2}^0(\epsilon x, \epsilon^2 t))e^{-2ix}. \end{aligned}$$

The approximation ψ used in the theorem and the above approximation ϕ are close. Using Lemma 4 we have

$$\sup_{t \in [0, T_0/\epsilon^2]} \|\psi - \phi\|_{C^4} \leq \mathcal{O}(\epsilon^2).$$

Moreover, ϕ is in C^∞ because of Lemma 3. Next we have to compute the residuum

$$\mathrm{Res}(\phi) = -\partial_t \phi + \lambda\phi + \rho\phi^2.$$

If we had not applied E_0 in the definition of ϕ, $\mathrm{Res}(\phi)$ would be only in C^0, since the initial conditions of the Ginzburg–Landau equation are assumed to be in C^4 only. But to solve the equations for the error, $\mathrm{Res}(\phi)$ should be at least in C^1. We denote by $\tilde{\delta}_j$ the term in front of e^{ijx}; that is, $\mathrm{Res}(\phi) = \sum_{j=-4}^{4} \tilde{\delta}_j e^{ijx}$. By using the formulas

$$\begin{aligned}\lambda(\partial_x, 0)B(\epsilon x)e^{inx} &= -e^{inx}[(1-n^2)^2 B + 4\epsilon in(1-n^2)\partial_X B \\ &\quad + \epsilon^2(2-6n^2)\partial_X^2 B + \epsilon^3 4in\partial_X^3 B + \epsilon^4 \partial_X^4 B], \\ 2\rho(\partial_x)B(\epsilon x)e^{inx} &= e^{inx}[inB + \epsilon\partial_X B]\end{aligned}$$

and the fact that the A_j^0 are solutions of the modulation equations (12) we get

$$\begin{aligned}\tilde{\delta}_0 &= \epsilon^3\partial_X((E_0A_1^0)(E_0A_{-1}^0)) + \epsilon^5\partial_X((E_0A_2^0)(E_0A_{-2}^0)), \\ \tilde{\delta}_1 &= (-\epsilon^4 4i\partial_X^3 - \epsilon^5\partial_X^4)E_0A_1^0 + \epsilon^4\partial_X((E_0A_2^0)(E_0A_{-1}^0)) + \epsilon^3\partial_T E_0^c A_1^0 \\ &\quad - (\epsilon^3 + 4\epsilon^3\partial_X^2)E_0^c A_1^0 - \epsilon^3 i((E_0A_2^0)(E_0^cA_{-1}^0) + (E_0^cA_2^0)(E_0A_{-1}^0) \\ &\quad + (E_0^cA_2^0)(E_0^cA_{-1}^0)), \\ \tilde{\delta}_2 &= -\epsilon^4\partial_T E_0A_2^0 - \epsilon^2(-12i\epsilon\partial_X - 22\epsilon^2\partial_X^2 + 8i\epsilon^3\partial_x^3 + \epsilon^4\partial_X^4)E_0A_2^0 + \epsilon^4 E_0A_2^0 \\ &\quad + \tfrac{1}{2}\epsilon^3\partial_X((E_0A_1^0)(E_0A_1^0)) + 9\epsilon^2E_0^cA_2^0 - \epsilon^2 i(2(E_0A_1^0)(E_0^cA_1^0) + (E_0^cA_1^0)(E_0^cA_1^0)), \\ \tilde{\delta}_3 &= \epsilon^3(3i + \epsilon\partial_X)((E_0A_1^0)(E_0A_2^0)), \\ \tilde{\delta}_4 &= \tfrac{1}{2}\epsilon^4(4i + \epsilon\partial_X)((E_0A_2^0)(E_0A_2^0)), \\ \tilde{\delta}_{-n} &= \bar{\tilde{\delta}}_n.\end{aligned}$$

Since we have a semilinear problem and we can use smoothing properties of the semigroup generated by λ, we have to estimate $\tilde{\delta}_j$ only in C^1. The solution A_1^0 of the Ginzburg–Landau equation is bounded in C^4 on the considered time interval. And since E_0 commutes with λ and ρ we easily get $\|\tilde{\delta}_0\|_{C^1} = \mathcal{O}(\epsilon^3)$, $\|\tilde{\delta}_3\|_{C^1} = \mathcal{O}(\epsilon^3)$, and $\|\tilde{\delta}_4\|_{C^1} = \mathcal{O}(\epsilon^4)$ uniformly on $[0, T_0/\epsilon^2]$.

To estimate $\|\tilde{\delta}_2\|_{C^1}$ we note that all terms without time derivatives are of order $\mathcal{O}(\epsilon^3)$ due to Lemma 4 and $A_j^0 \in C^4$. Since E_0 also commutes with ∂_T, and $\partial_T A_2^0$ can be expressed with the help of (12), we get

$$\|\partial_T A_2^0\|_{C^1} \le C\|\partial_T A_1^0\|_{C^1}\|A_1^0\|_{C^1} \le C(\|A_1^0\|_{C^3} + \|A_1^0\|_{C^1}^3)\|A_1^0\|_{C^1}$$

and so $\sup_{t\in[0,T_0/\epsilon^2]}\|\tilde{\delta}_2\|_{C^1} = \mathcal{O}(\epsilon^3)$. In the same way we obtain $\sup_{t\in[0,T_0/\epsilon^2]}\|\tilde{\delta}_1\|_{C^1} = \mathcal{O}(\epsilon^4)$.

The $\tilde{\delta}_j$ always have compact support $[-1/2, 1/2]$ in Fourier space since ϕ is chosen such that this is true. Therefore, we obtain $E_c(\tilde{\delta}_j e^{ijx}) = 0$ for $|j| \neq 1$. It is clear that the above calculations are also valid for the more general case, and so we can state the following lemma.

Lemma 6. *Let* $A_1^0 \in C([0, T_0], C^{2d})$ *be a solution of the Ginzburg–Landau equation (6) and let* ψ *be as in Theorem 2. Then there exists an approximation* ϕ *of the form*

$$\phi = \epsilon(E_0 A_1^0) e^{ik_c x} e^{i\hat{\omega}_0 t} + \epsilon(E_0 A_{-1}^0) e^{-ik_c x} e^{-i\hat{\omega}_0 t} + \epsilon^2 \phi_s$$

with

$$\sup_{t\in[0,T_0/\epsilon^2]} \|\phi - \psi\|_{C^{2d}} = \mathbb{O}(\epsilon^2), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|\phi_s\|_{C^{2d}} = \mathbb{O}(1),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|E_s(\mathrm{Res}(\phi))\|_{C^1} = \mathbb{O}(\epsilon^3), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|E_c(\mathrm{Res}(\phi))\|_{C^1} = \mathbb{O}(\epsilon^4).$$

5.2. The Estimates for the Error

We already know that the quadratic terms generate the main difficulties. Since higher-order terms are not problematic, we suppose, for notational reasons, that only quadratic terms appear, and hence (4) is of the form

$$\partial_t u = \lambda u + B(u, u), \tag{13}$$

where $B(u, v) = \sum_{i,j=0}^{2d-1} a_{ij} (\partial_x^i u)(\partial_x^j v)$ with $a_{ij} = a_{ji}$. Moreover, we suppose that the transformation $x \mapsto x - \nu_1 t$ was already done. We now make an approximation ϕ with the properties of Lemma 6. As explained, we suppose the error to be the sum

$$R(x, t, \epsilon) = \epsilon^2 R_c(x, t) + \epsilon^3 R_s(x, t) \tag{14}$$

with $R_c = \epsilon^{-2} E_c R \in C^{2d}$ and $R_s = \epsilon^{-3} E_s R \in C^{2d}$. By later calculations we show that such an ansatz is possible. We abbreviate

$$\phi_c := (E_0 A_1^0(\epsilon x, \epsilon^2 t)) e^{ik_c x} e^{i\hat{\omega}_0 t} + c.c.$$

and insert

$$u = \phi + R = \epsilon\phi_c + \epsilon^2\phi_s + \epsilon^2 R_c + \epsilon^3 R_s \tag{15}$$

into equation (13). After dividing by ϵ^2 and after selecting linear and nonlinear terms, and terms for which the application of the operator E_c vanishes, we get the equation

$$\begin{aligned} \partial_t R_c + \epsilon\partial_t R_s =& \lambda R_c + \epsilon\lambda R_s + \epsilon\tilde{L}_2(R_c) + \epsilon^2\tilde{L}_1(R) \\ &+ \epsilon^2\tilde{N}_2(R_c) + \epsilon^3\tilde{N}_1(R) + \frac{1}{\epsilon^2}\mathrm{Res}(\phi), \end{aligned} \tag{16}$$

where the abbreviations stand for

$$\tilde{L}_2(R_c) = 2B(R_c, \phi_c), \qquad \tilde{L}_1(R) = 2B(R_c, \phi_s) + 2B(R_s, \phi_c),$$

$$\tilde{N}_2(R_c) = B(R_c, R_c), \qquad \tilde{N}_1(R) = B(R_s, 2R_c + 2\phi_s + \epsilon R_s).$$

If all R_s, R_c, ϕ_c, and ϕ_s are of order $\mathcal{O}(1) \in C^{2d}$, then $\tilde{L}_1, \tilde{L}_2, \tilde{N}_1$, and $\tilde{N}_2$ are also of order $\mathcal{O}(1) \in C^{2d}$. To get equations for R_c and R_s we apply the mode–filters E_c and E_s. By Lemma 5 we have $E_c\tilde{L}_2(R_c) = E_c\tilde{N}_2(R_c) = 0$. We now separate (16) into two parts and define R_c and R_s to be the solutions of the system

$$\begin{aligned} \partial_t R_c &= \lambda R_c + \epsilon^2 L_c(R) + \epsilon^3 N_c(R) + \epsilon^2 \delta_c, \\ \partial_t R_s &= \lambda R_s + L_s(R_c) + \epsilon N_s(R) + \delta_s \end{aligned} \tag{17}$$

with the abbreviations

$$\begin{aligned} \delta_c &= (1/\epsilon^4)E_c(\mathrm{Res}(\phi)), & \delta_s &= (1/\epsilon^3)E_s(\mathrm{Res}(\phi)), \\ L_c(R) &= E_c(\tilde{L}_1(R)), & L_s(R_c) &= E_s(\tilde{L}_2(R_c)), \\ N_c(R) &= E_c(\tilde{N}_1(R)), & N_s(R) &= E_s(\tilde{L}_1(R) + \tilde{N}_2(R_c) + \epsilon\tilde{N}_1(R)), \end{aligned}$$

and the initial data $(R_1(0), R_2(0)) = (0, 0)$. Adding the two equations (17) we get (16). We remark that the equation for R_c has only nonvanishing Fourier modes for wave numbers $k \in U_{k_c/8}(I_c)$, and the Fourier modes in the equation for R_s vanish for $k \in I_c$. We solve this system in the space

$$\mathcal{B}_n = C([0, T_0/\epsilon^2], C^n)^2$$

with the norm

$$\|(R_1, R_2)\|_{\mathcal{B}_n} = \sum_{i=1}^{2} \sup_{0 \le t \le T_0/\epsilon^2} \|R_i(\cdot, t)\|_{C^n}$$

for $n = 2d$. We have to show that the solution is of order $\mathcal{O}(1)$ in $\mathcal{B}_{2d}$. We do that by inverting the linear part of (17) and by applying a contraction principle.

Because of Lemma 6 the inhomogeneity $\delta = (\delta_c, \delta_s)$ is obviously in $\mathcal{B}_1$ and of order $\mathcal{O}(1)$. The nonlinearity $N = (N_c(R), N_s(R))$ is a sum of linear and bilinear terms, and, since C^n is an algebra, N is a local Lipschitz-continuous mapping from $\mathcal{B}_{2d}$ to $\mathcal{B}_1$, which maps bounded sets of $\mathcal{B}_{2d}$ in bounded sets in $\mathcal{B}_1$. What remains is the estimation of the solutions of the system

$$\begin{aligned} \partial_t R_c &= \lambda R_c + \epsilon^2 L_c(R) + \epsilon^2 f_c \\ \partial_t R_s &= \lambda R_s + L_s(R_c) + f_s \end{aligned} \tag{18}$$

for $f_c = E_c g_c$ and $f_s = E_s g_s$ with $g = (g_c, g_s) \in \mathcal{B}_1$. Since $L = (L_c(R), L_s(R_c))$ is a bounded linear mapping from $\mathcal{B}_{2d}$ into $\mathcal{B}_1$, the local existence in time of solutions of this system is clear. More interesting is the question of the $\mathcal{O}(1)$-boundedness of these solutions on the time interval $[0, T_0/\epsilon^2]$. To show this, we remark that the operator λ is the generator of a semigroup $e^{\lambda t}$. Its action on a function $u(x)$ can be expressed by

$$e^{\lambda t}u(x) = \int H(x - \xi, t)u(\xi)d\xi$$

where H is defined by $H(x,t) = (1/2\pi)\int e^{ikx} e^{\lambda(ik,\epsilon^2)t} dk$. We have

$$\|e^{\lambda t} E_c^h\|_{\mathcal{L}(C^1,C^{2d})} \leq C \max_{r=0\ldots2d-1} \int \left| \int e^{\lambda(ik,\epsilon^2)t} \chi_c^h(k) e^{ikl} k^r \, dk \right| dl \leq C e^{\epsilon^2 t}$$

due to the compact support of χ_c^h. Moreover,

$$\|e^{\lambda t} E_s^h\|_{\mathcal{L}(C^1,C^{2d})} \leq C e^{-\sigma t} \max(1, t^{-(2d-1)/(2d)})$$

with an ϵ-independent constant $\sigma > 0$ due to the fact that only damped modes appear. Remember that $E_c^h E_c = E_c$ and $E_s^h E_s = E_s$. For the second equation in (18) we obtain

$$R_s(t) = \int_0^t \int H(x-\xi, t-\tau)[[(L_s(R_c) + f_s)](\xi,\tau)] d\xi d\tau.$$

With the abbreviations $S_i(s) := \sup_{t\leq s} \|R_i(t)\|_{C^{2d}}$, $(i = s, c)$, we find

$$\begin{aligned} S_s(t) &\leq \left(\int_0^t C \max(1, \tau^{-(2d-1)/(2d)}) e^{-\sigma\tau} d\tau \right) (CS_c(t) + \|f\|_{\mathcal{B}_1}) \\ &\leq CS_c(t) + C\|f\|_{\mathcal{B}_1}. \end{aligned}$$

Similarly, we can estimate the first equation

$$\begin{aligned} S_c(t) &\leq \epsilon^2 \int_0^t Ce^{C\epsilon^2(t-\tau)} (S_c(\tau) + S_s(\tau) + \|f\|_{\mathcal{B}_1}) d\tau \\ &\leq \epsilon^2 C \int_0^t S_c(\tau) d\tau + C\|f\|_{\mathcal{B}_1}. \end{aligned}$$

With the help of Gronwall's inequality we see that

$$S_c(t) \leq C\|f\|_{\mathcal{B}_1} e^{CT_0} = \mathcal{O}(1) \quad \text{and} \quad S_s(t) \leq C\|f\|_{\mathcal{B}_1} = \mathcal{O}(1). \tag{19}$$

Now we define the inverse J of the linear part by $R = Jf$ if R is a solution of (18). Therefore, $J \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_{2d})$ is a linear operator, bounded independently of ϵ. When we apply J on (17) we obtain

$$R = \epsilon JN(R) + J\delta =: F(R). \tag{20}$$

For $\epsilon \leq \epsilon_0$ the function $F : \mathcal{B}_{2d} \to \mathcal{B}_{2d}$ is a contraction on a ball with center $J\delta$ in $\mathcal{B}_{2d}$, because of the Lipschitz-continuity of N and the ϵ in front of N. Therefore, there exists a unique fixed point of (20) which is a solution of order $\mathcal{O}(1)$ of (17). By (14) and (15) we have constructed a solution of the original problem (13). The estimate of Theorem 2 follows since

$$\|u - \psi\|_{\mathcal{B}_{2d}} \leq \|u - \phi\|_{\mathcal{B}_{2d}} + \|\phi - \psi\|_{\mathcal{B}_{2d}} = \|(\epsilon^2 R_c, \epsilon^3 R_s)\|_{\mathcal{B}_{2d}} + \mathcal{O}(\epsilon^2) = \mathcal{O}(\epsilon^2).$$

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