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**Research note**

# Thick-plate flexure re-examined

Detlef Wolf *Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada*

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**Summary**. The flexure of an incompressible, thick elastic plate floating on an inviscid substratum and subject to an external gravity field is re-analysed. The solution is derived from momentum equations which account for the advection of hydrostatic pre-stress. This is contrasted with a recently published thick-plate solution derived from momentum equations without a pre-stress term. It is demonstrated that neglecting pre-stress advection renders the solution singular when the model degenerates into an inviscid half-space. If pre-stress advection is included, the solution remains correct in this limit. A numerical comparison of both types of thick-plate solution with results based on conventional thin-plate theory further shows that, for geophysically relevant models, the difference in the momentum balance entails discrepancies between the thick-plate solutions which are comparable to the errors introduced by the thin-plate approximation.

## 1 Introduction

Regional deformations of the Earth's lithosphere under superimposed loads have almost exclusively been analysed in terms of thin-plate models. For elastic and viscoelastic materials the theory is well-known and may, for example, be found in Nadai (1963). Until recently, however, this approach has been more motivated by the simplicity of thin-plate theory than it has been based on rigorous theoretical justification. Preliminary comparisons in the wavenumber domain between the response characteristics of thin or thick elastic plates floating on inviscid substrata in fact demonstrated that large differences existed up to wavelengths as large as five times the plate thickness (McKenzie & Bowin 1976). But the significance of this result for specific topographic loads resting on the Earth's lithosphere has not been made clear. Whereas most authors have assumed that the errors introduced by thin-plate theory are generally minor compared to the inaccuracies associated with the observations (e.g. Banks, Parker & Huestis 1977), others (e.g. Turcotte 1979) have pointed out that thin-plate theory may only be marginally valid for the Earth's lithosphere.

The problem of thin-plate versus thick-plate flexure has recently been addressed quantitatively by Comer (1983). Considering realistic axisymmetric loads resting on elastic plates

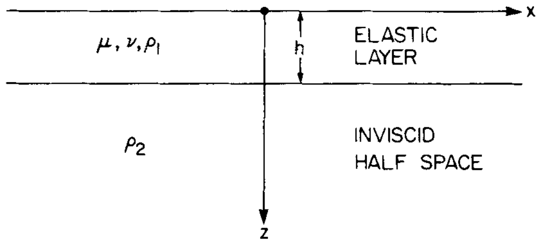


**Figure 1.** Thick elastic plate floating on inviscid substratum. $\nu = \lambda/[2(\lambda + \mu)]$ denotes Poisson's ratio.

characterized by geophysically relevant parameters, he convincingly demonstrated that solutions based on thin-plate theory were usually good approximations. The present note is not so much an attempt to modify this result, which is certainly correct. It is rather intended to discuss some inconsistencies in Comer's thick-plate analysis, which lead to physically unreasonable singularities in the solution in a certain important asymptotic limit.

A cursory inspection of Comer's momentum equations shows that the advection of hydrostatic pre-stress has not been included. In the presence of a gravity field this neglect is, however, inconsistent with the usual elastic boundary conditions. In the plate-flexure problem they require that the normal component of the elastic traction compensates the weight of the superimposed load or the buoyancy of the underlying fluid. This formulation, however, implies that pre-stress advection counterbalances the weight of the material of the elastic plate which is associated with the deflection of the plate from its equilibrium level (e.g. Cathles 1975, pp. 16–23).

The importance of pre-stress advection was first discussed by Love (1911, pp. 89–93) in his study of static deformations of the Earth. Love also incorporated the advection term into the momentum equations governing the Earth's free oscillations. Peltier (1974) extended this analysis and included the effect of pre-stress when stating momentum conservation for a quasi-static Maxwell body. Recently Wu & Peltier (1982) reviewed the relaxation of such continua in detail and noted that the pre-stress term was *required* in order that the correct solution be obtained in the inviscid limit.

In the following, the solution describing the flexure of an incompressible, thick elastic plate will be derived from momentum equations which include the advection of pre-stress. This will permit us to demonstrate explicitly the signficance of the pre-stress term when considering the inviscid limit of the model. Following that, several numerical examples will be presented and compared with results based on Comer's (1983) thick-plate solution and conventional thin-plate theory.

## 2 Theoretical analysis

The problem to be solved is the static deformation of an elastic layer or thick plate (characterized by its thickness $h$, Lamé parameters $\lambda$ and $\mu$, and density $\rho_1$) floating on an inviscid substratum (characterized by its density $\rho_2$) and subject to a normal surface traction $-q(x)$ acting at $z = 0$. Here all parameters are assumed to be spatially constant. A cross-section of the general configuration in the $(x, z)$-plane is shown in Fig. 1.

Within the range of validity of flat earth models it is reasonable to neglect perturbations of the geopotential by the load or by internal mass redistributions. The gravity field is therefore taken to be externally applied. He will further assume that it is constant and acts parallel to the direction of the $z$-axis. If the analysis is restricted to an incompressible plate, its density will remain unperturbed, and the momentum balance reduces to

$$\frac{\partial}{\partial x}\sigma_{xx}^{(e)} + \frac{\partial}{\partial z}\sigma_{xz}^{(e)} = -\frac{\partial}{\partial x}\left(\frac{\partial p_0}{\partial z} w\right), \tag{2.1}$$

$$\frac{\partial}{\partial x}\sigma_{xz}^{(e)} + \frac{\partial}{\partial z}\sigma_{zz}^{(e)} = -\frac{\partial}{\partial z}\left(\frac{\partial p_0}{\partial z} w\right). \tag{2.2}$$

Here $\sigma_{xx}^{(e)}$ etc. denote the Cartesian components of the elastic stress tensor $\sigma_{ij}^{(e)}$. The term in brackets on the right sides of (2.1) and (2.2) accounts for stress advection in a hydrostatically pre-stressed elastic continuum of initial pressure $p_0$. In the elastic plate

$$\frac{\partial p_0}{\partial z} w = \rho_1 g w, \tag{2.3}$$

where $g$ denotes gravity, and (2.1) and (2.2) become

$$\frac{\partial}{\partial x}(\sigma_{xx}^{(e)} + \rho_1 g w) + \frac{\partial}{\partial z}\sigma_{xz}^{(e)} = 0, \tag{2.4}$$

$$\frac{\partial}{\partial z}(\sigma_{zz}^{(e)} + \rho_1 g w) + \frac{\partial}{\partial x}\sigma_{xz}^{(e)} = 0. \tag{2.5}$$

Since the elastic plate is incompressible, its constitutive equations are

$$\sigma_{xx}^{(e)} = -p^{(e)} + 2\mu\frac{\partial u}{\partial x}, \tag{2.6}$$

$$\sigma_{zz}^{(e)} = -p^{(e)} + 2\mu\frac{\partial w}{\partial z}, \tag{2.7}$$

$$\sigma_{xz}^{(e)} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{2.8}$$

$$\sigma_{yy}^{(e)} = (\sigma_{xx}^{(e)} + \sigma_{zz}^{(e)})/2. \tag{2.9}$$

Here $u$ and $w$ denote displacement components in the $x$- and $z$-directions, respectively. $p^{(e)}$ is the elastic perturbation pressure. For an incompressible elastic solid it is defined by

$$p^{(e)} = \lim_{\substack{\lambda\to\infty \\ \Delta\to 0}} (\lambda\Delta), \tag{2.10}$$

where $\Delta = \partial u/\partial x + \partial w/\partial z$ is the dilatation (Love 1927, pp. 255–257). If the total perturbation stress

$$\sigma_{ij} = \sigma_{ij}^{(e)} + \rho_1 g w \delta_{ij} \tag{2.11}$$

is introduced, the momentum balance becomes

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial z}\sigma_{xz} = 0, \tag{2.12}$$

$$\frac{\partial}{\partial x}\sigma_{xz}+\frac{\partial}{\partial z}\sigma_{zz}=0. \tag{2.13}$$

Similarly, the constitutive equations reduce to

$$\sigma_{xx}=-p+2\mu\frac{\partial u}{\partial x}, \tag{2.14}$$

$$\sigma_{zz}=-p+2\mu\frac{\partial w}{\partial z}, \tag{2.15}$$

$$\sigma_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \tag{2.16}$$

$$\sigma_{yy}=(\sigma_{xx}+\sigma_{zz})/2, \tag{2.17}$$

where $p$ denotes the total perturbation pressure. Since a two-dimensional model has been adopted, the compatability equations reduce to one condition, viz.

$$\nabla^2(\sigma_{xx}+\sigma_{zz})=0. \tag{2.18}$$

Following Malvern (1969, pp. 552–554), we may introduce a displacement potential $\chi$, such that

$$2\mu u=-\frac{\partial^2\chi}{\partial x\partial z}, \tag{2.19}$$

$$2\mu w=\frac{\partial^2\chi}{\partial x^2}, \tag{2.20}$$

$$2p=-\nabla^2\frac{\partial\chi}{\partial z}. \tag{2.21}$$

Substituting for the displacements in (2.14)–(2.16) yields

$$2\sigma_{xx}=\left(\nabla^2-2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\chi}{\partial z}, \tag{2.22}$$

$$2\sigma_{zz}=\left(\nabla^2+2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\chi}{\partial z}, \tag{2.23}$$

$$2\sigma_{xz}=\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)\frac{\partial\chi}{\partial x}. \tag{2.24}$$

It may be shown by direct substitution that (2.12) is identically satisfied by (2.22) and (2.24). Substituting for $\sigma_{xx}$ and $\sigma_{zz}$ in (2.18), under consideration of (2.13), yields the biharmonic equation

$$\nabla^4\chi(x,z)=0. \tag{2.25}$$

Using the transform pair

$$\hat{f}(k)=\int_{-\infty}^{\infty}f(x)\exp(-ikx)\,dx,$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) \exp(ikx)\, dk,$$

with $k$ the wavenumber, the Fourier transform of (2.25) is

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 \hat{\chi}(k, z) = 0 \tag{2.26}$$

with the general solution

$$\hat{\chi}(k, z) = (A + Bz)\cosh(kz) + (C + Dz)\sinh(kz). \tag{2.27}$$

Fourier transformation of (2.19), (2.20), (2.22)–(2.24) and substitution for $\hat{\chi}$ yields

$$2\mu\hat{u}(k, z) = -ik [\![ \sinh(kz)kA + [kz\sinh(kz) + \cosh(kz)]B + \cosh(kz)kC + [kz\cosh(kz) + \sinh(kz)]D ]\!], \tag{2.28}$$

$$2\mu\hat{w}(k, z) = -k[\cosh(kz)kA + kz\cosh(kz)B + \sinh(kz)kC + kz\sinh(kz)D], \tag{2.29}$$

$$\hat{\sigma}_{xx}(k, z) = k^2 [\![ \sinh(kz)kA + [kz\sinh(kz) + 2\cosh(kz)]B + \cosh(kz)kC + [kz\cosh(kz) + 2\sinh(kz)]D ]\!], \tag{2.30}$$

$$\hat{\sigma}_{zz}(k, z) = -k^2[\sinh(kz)kA + kz\sinh(kz)B + \cosh(kz)kC + kz\cosh(kz)D], \tag{2.31}$$

$$\hat{\sigma}_{xz}(k, z) = -ik^2 [\![ \cosh(kz)kA + [kz\cosh(kz) + \sinh(kz)]B + \sinh(kz)kC + [kz\sinh(kz) + \cosh(kz)]D ]\!], \tag{2.32}$$

where $i \equiv (-1)^{1/2}$. In the transform domain the solution must satisfy the boundary conditions

$$\hat{\sigma}_{zz}^{(e)}(k, 0) = -\hat{q}(k), \tag{2.33}$$

$$\hat{\sigma}_{xz}^{(e)}(k, 0) = 0, \tag{2.34}$$

$$\hat{\sigma}_{zz}^{(e)}(k, h) = -\rho_2 g\hat{w}(k, h), \tag{2.35}$$

$$\hat{\sigma}_{xz}^{(e)}(k, h) = 0. \tag{2.36}$$

With (2.11), this becomes

$$\hat{\sigma}_{zz}(k, 0) = \rho_1 g\hat{w}(k, 0) - \hat{q}(k), \tag{2.37}$$

$$\hat{\sigma}_{xz}(k, 0) = 0, \tag{2.38}$$

$$\hat{\sigma}_{zz}(k, h) = (\rho_1 - \rho_2)g\hat{w}(k, h), \tag{2.39}$$

$$\hat{\sigma}_{xz}(k, h) = 0. \tag{2.40}$$

Equations (2.37) and (2.39) simply state that the *total* perturbation pressure balances the external load or buoyancy force *and* the weight of the material of the elastic plate displaced from its undisturbed level surface.

The four integration constants $A$, $B$, $C$ and $D$ are determined to satisfy the four boundary conditions in the usual way. After some algebraic manipulation we obtain

$$k^2 A = -kD, \tag{2.41}$$

$$k^2B = \frac{-\hat{\sigma}_{zz}(k,0)\sinh(kh)\,[2\mu k\sinh(kh) + (\rho_1-\rho_2)g\cosh(kh)]}{2\mu k(kh)^2 - (\rho_1-\rho_2)g(kh) - \sinh(kh)\,[2\mu k\sinh(kh) + (\rho_1-\rho_2)g\cosh(kh)]}, \tag{2.42}$$

$$k^2C = -\hat{\sigma}_{zz}(k,0), \tag{2.43}$$

$$k^2D = \frac{\hat{\sigma}_{zz}(k,0)\,[\![2\mu k(kh) + \sinh(kh)\,[(\rho_1-\rho_2)g\sinh(kh) + 2\mu k\cosh(kh)]]\!]}{2\mu k(kh)^2 - (\rho_1-\rho_2)g(kh) - \sinh(kh)\,[2\mu k\sinh(kh) + (\rho_1-\rho_2)g\cosh(kh)]}. \tag{2.44}$$

Equation (2.29) together with (2.41) yields

$$2\mu\hat{w}(k,0) = kD. \tag{2.45}$$

Using (2.37) and (2.44) this may be transformed to

$$\hat{w}(k,0) = \hat{q}(k)/[\rho_1 g + 2\mu k P(k)], \tag{2.46}$$

where

$$P(k) = \frac{2\mu k(kh)^2 - (\rho_1-\rho_2)g(kh) - \sinh(kh)\,[(\rho_1-\rho_2)g\cosh(kh) + 2\mu k\sinh(kh)]}{2\mu k(kh) + \sinh(kh)\,[(\rho_1-\rho_2)g\sinh(kh) + 2\mu k\cosh(kh)]}. \tag{2.47}$$

Only for $\rho_1 = 0$, i.e. for an elastic plate without mass, is (2.46), with $P$ given by (2.47), identical to Comer's (1983) result. One might argue that Comer effectively disregarded gravity in his analysis, since its effect can be shown to be small in most situations. If $g = 0$ in (2.46) and (2.47), a purely elastic solution is obtained, however, i.e. the buoyancy effects of the inviscid substratum are also neglected. More illuminating is a discussion of the limit $\mu k \to 0$ for $\rho_1 = \rho_2$. Then the model degenerates into an inviscid half-space. We find $P = 0$ and

$$\hat{w}(k,0) = \hat{q}(k)/(\rho_1 g). \tag{2.48}$$

This is the correct solution. At $\mu k = 0$ Comer's solution has a singularity, however, which is a consequence of his neglect of pre-stress in (2.1) and (2.2).

If axisymmetric loads are modelled, all previous equations in the Fourier transform domain can be interpreted as equations in the Hankel transform domain. Then the horizontal directions $x$ and $y$ must be taken as being radial and azimuthal directions $r$ and $\phi$, respectively. Furthermore, the Fourier transforms of $\chi$, $w$, $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{zz}$ must be interpreted as zeroth-order Hankel transforms, whereas those of $iu$ and $i\sigma_{xz}$ as first-order Hankel transforms of the corresponding field quantities in cylindrical coordinates.

## 3 Numerical examples

For a demonstration of the numerical differences between the improved thick-plate theory outlined in the last section and the solution previously obtained by Comer (1983), it will be sufficient to restrict the discussion to vertical surface displacements.

Fig. 2 compares results obtained in the wavenumber domain for plate thicknesses of 125 and 250 km. The values of the model parameters are given in the caption. The quantity plotted is the response function $\phi(\lambda)$, which normalize $\hat{w}(\lambda, 0)$ with respect to the static response of the underlying inviscid half-space alone (Walcott 1976). The response according to our improved theory exceeds Comer's response curves significantly for $\lambda = 2\pi/k > 2000$ km and $h > 200$ km.

At this point it is instructive to compare Comer's figs 3(a) and 3(c). Although the ratios between wavelength of deformation and thickness of lithosphere are similar for both models,

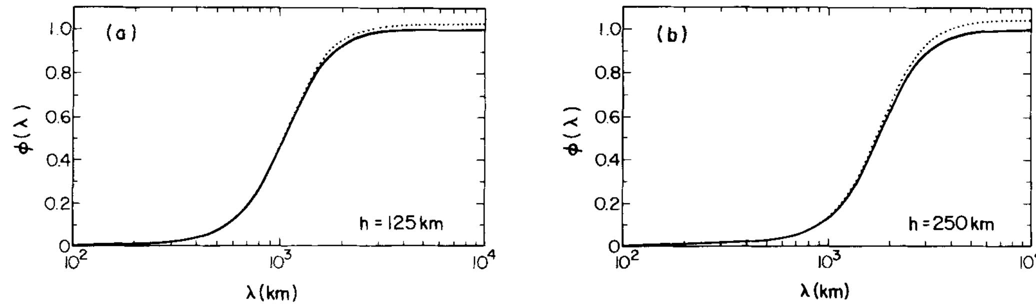


**Figure 2.** Response function $\phi$ for elastic-plate thicknesses $h$ of 125 km (a) and 250 km (b) versus wavelength $\lambda$ of the deformation. Improved (solid) and simplified (dotted) thick-plate solutions for $\mu = 0.4 \times 10^{11}$ N m$^{-2}$, $\nu = 0.5$, $\rho_1 = \rho_2 = 3000$ kg m$^{-3}$.

the discrepancies between thin-plate and simplified thick-plate solutions are not. For the 200 km thick (Martian) lithosphere they exceed the differences obtained for the 25 km thick (oceanic) lithosphere by almost a factor of 3. Since similarity should hold closely, this result is unexpected. Here we suggest that the difference is related to Comer's neglect of pre-stress advection in the momentum balance. This hypothesis will be tested by calculating the deformation caused by strip loads of different widths according to thin-plate theory. Comer's simplified thick-plate theory and our improved thick-plate theory.

The Fourier transform of a unit strip or box-car load $q(x) = [H(x + l) - H(x - l)]/(2l)$ is $\hat{q}(k) = \sin(kl)/(kl)$, where $l$ is the half-width of the strip and $H$ denotes the Heaviside step function. Substituting for $\hat{q}(k)$ in (2.37) and numerical inverse Fourier transformation then yields $w(x, 0)$.

Fig. 3 presents results obtained for a 250 km thick elastic plate. This is very close to the thickness chosen by Comer in his model of the Martian lithosphere. Similar values have recently been suggested for the continental lithosphere from the interpretation of deglaciation-induced relative sea-level rise along the east coast of North America and from polar-wander information (Peltier 1984). For a line load (a) the thin-plate approximation results in an underestimation of the central displacement by almost 40 per cent. Compared to this the differences between the two thick-plate solutions remain insignificant. Line loads acting on thin plates have been considered by several authors, although for thinner lithospheres (e.g. Walcott 1970). For a 250 km wide load (d), however, one-third of the 15 per cent discrepancy between the thin- and thick-plate solutions, as implied from Comer's simplified thick-plate theory, is spurious. For a width of 500 km (e), the actual difference between thin- and thick-plate solutions is overestimated by 50 per cent and the simplified thick-plate solution is no longer superior to conventional thin-plate theory.

## 4 Conclusions

This study has attempted to illuminate the physical significance of the pre-stress term in the momentum equations appropriate to static deformations of incompressible elastic continua in an external gravity fields, by demonstrating that its neglect results in physically unreasonable singularities in the solution. The theoretical analysis of the correctly posed problem involves no additional complications and leads to solutions in terms of total perturbation stresses instead of elastic perturbation stresses. In view of this, we feel that simplifications of the kind introduced by Comer (1983) are unnecessary. If the continuum is compressible,

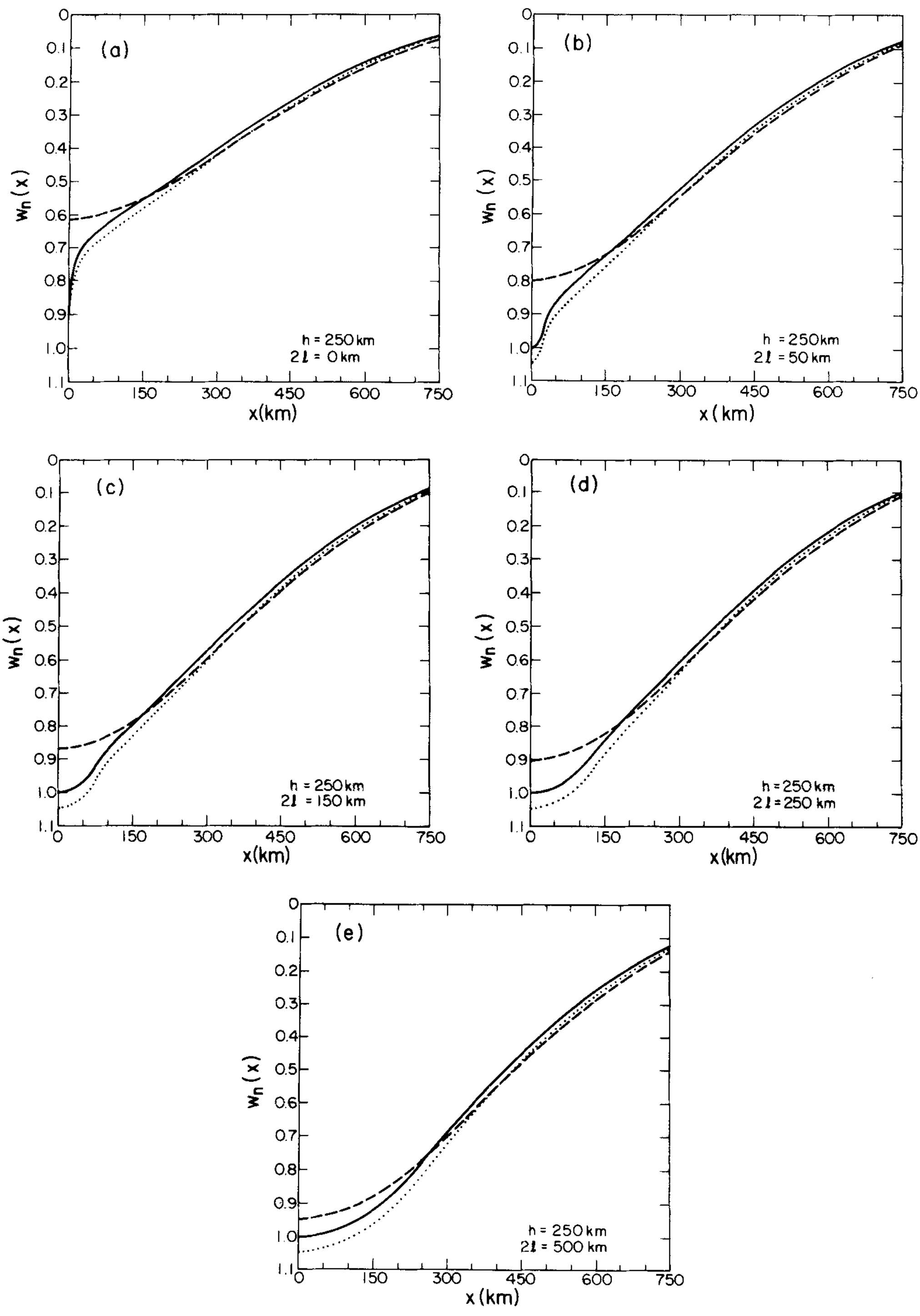


**Figure 3.** Normalized vertical surface displacements $w$ for strip-load widths $2l$ of 0 km (a), 50 km (b), 150 km (c), 250 km (d) and 500 km (e) versus horizontal distance $x$ from load axis. Improved (solid) and simplified (dotted) thick-plate solutions, together with associated thin-plate solutions (dashed). Normalization with respect to central displacement according to improved thick-plate theory. Parameters as for Fig. 2.

the modifications introduced by pre-stress advection are much more substantial. This is because a formulation in terms of the total perturbation stress no longer simplifies the analysis. The compressible problem is at present being followed up in a separate investigation.

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