Research note

On Boussinesq's problem for Maxwell continua subject to an external gravity field

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Summary. Boussinesq's problem is solved for a uniform and incompressible Maxwell half-space subject to an external gravity field. The solution is based on momentum equations which account for stress advection in the hydrostatically pre-stressed continuum during its deformation. The analysis shows that disregarding the pre-stress term renders the theoretical stress distribution incorrect and the deformation singular in the inviscid limit of the Maxwell continuum. Our solution is contrasted with a recently published alternative solution of the same problem, where regularity in the inviscid limit was forced by modified boundary conditions.

1 Introduction

In the previous note, the load-induced flexure of an incompressible, thick elastic plate subject to an external gravity field was analysed (Wolf 1985). It was shown that the validity of the solution in the inviscid limit is contingent upon the inclusion of the pre-stress term in the momentum balance. In the present note this result is extended to viscoelastic models, and Boussinesq's problem for an externally gravitating Maxwell continuum is solved.

The related non-gravitating problem was discussed by Peltier (1974), who also pointed out the errors incurred by neglecting gravitational restoring forces. Hence Peltier included pre-stress advection in his general analysis of the relaxation of self-gravitating Maxwell spheres [see his equation (44) on p. 656]. Recently, Wu & Peltier (1982) have again called attention to stress advection in a hydrostatically pre-stressed Maxwell continuum by noting that the appropriate term is *required* in the momentum balance in order that the correct solution be obtained at large times after the onset of loading, i.e. in the inviscid limit. Their claim is, however, at variance with Nakiboglu & Lambeck's (1982) treatment of the same problem. These authors do not include the advective term in their momentum balance but modify the boundary conditions by adding a 'buoyancy term'.

Although both methods yield the correct surface deflection in the inviscid limit, their equivalence is not clear. Here we will attempt to illuminate this problem and derive the formal solution for the deformation of the uniform, incompressible and gravitationally pre-stressed Maxwell half-space. This will allow us to discuss the stress distribution in the

continuum during its relaxation. As will be shown, the role of pre-stress is crucial, since it ensures that the imposed boundary conditions be satisfied at any time.

2 Theory

In the previous note I derived the solution for the deformation, under a two-dimensional load, of an incompressible thick elastic plate overlying an inviscid half-space and subject to an external gravity field (Wolf 1985). For very large plate thicknesses the solution reduces to that appropriate to a uniform elastic half-space. Imposing the condition that all field quantities remain regular as $z \to \infty$, we then obtain, from Wolf's equations (2.28) and (2.29), for the horizontal and vertical displacement components, respectively,

$$2\mu\hat{u}(k, z) = ik[kA - (1 - kz)B] \exp(-kz), \qquad (2.1)$$

$$2\mu\hat{w}(k, z) = -k^2(A + Bz) \exp(-kz). \qquad (2.2)$$

Parameter μ denotes Lamé's second constant and A and B are new integration constants appropriate to the half-space model. A carat denotes the Fourier transformation of the pertinent quantity with respect to the horizontal coordinate x, k is the associated transform variable (wavenumber) and z the vertical coordinate in the direction of the gravity field. Similarly, we obtain for the stress components from Wolf's (1985) equations (2.30)–(2.32)

$$\hat{\sigma}_{xx}(k, z) = -k^2[kA - (2 - kz)B] \exp(-kz), \qquad (2.3)$$

$$\hat{\sigma}_{zz}(k, z) = k^3(A + Bz) \exp(-kz), \qquad (2.4)$$

$$\hat{\sigma}_{xz}(k, z) = -ik^2[kA - (1 - kz)B] \exp(-kz), \qquad (2.5)$$

$\hat{\sigma}_{ij}$ with the Fourier transform of the total perturbation stress defined by

$$\sigma_{ij} = \sigma_{ij}^{(e)} + \rho g w \delta_{ij}. \qquad (2.6)$$

Here $\sigma_{ij}^{(e)}$ denotes the elastic perturbation stress, and $\rho g w \delta_{ij}$ is the pre-stress term from the momentum balance (see Wolf 1985 for a more complete discussion); ρ is the density and g the acceleration due to gravity.

The integration constants are determined from the boundary conditions

$$\hat{\sigma}_{zz}^{(e)}(k, 0) = -\hat{q}(k), \qquad (2.7)$$

$$\hat{\sigma}_{xz}^{(e)}(k, 0) = 0. \qquad (2.8)$$

Substituting for the field quantities in (2.7) and (2.8) from (2.2), (2.4) and under consideration of (2.5), (2.6), we obtain

$$k^2A = kB = -2\mu\hat{q}(k)/(2\mu k + \rho g). \qquad (2.9)$$

In the Fourier transform domain the solution of Boussinesq's problem is therefore

$$\begin{bmatrix} \hat{u}(k, z) \\ \hat{w}(k, z) \end{bmatrix} = \frac{\hat{q}}{2\mu k + \rho g} \begin{bmatrix} -ikz \\ 1 + kz \end{bmatrix} \exp(-kz), \qquad (2.10)$$

$$\begin{bmatrix} \hat{\sigma}_{xx}(k, z) \\ \hat{\sigma}_{zz}(k, z) \\ \hat{\sigma}_{xz}(k, z) \end{bmatrix} = -\frac{2\mu k \hat{q}}{2\mu k + \rho g} \begin{bmatrix} 1 - kz \\ 1 + kz \\ -ikz \end{bmatrix} \exp(-kz). \qquad (2.11)$$

The correspondence principle allows us to interpret (2.10) and (2.11) as Laplace transforms of an associated linear viscoelastic problem. This method is well-known and has been discussed in a geophysical context by Peltier (1974) and others. We therefore only recall that (2.10) and (2.11) constitute the Laplace-transformed solution for a Maxwell half-space subject to an impulsive load $\hat{q}(k)\,\delta(t)$, provided that $\mu \to \mu s/(s + \tau^{-1})$. Here s is the Laplace transform variable associated with the time t and $\tau = \eta/\mu$ the Maxwell time; η denotes the dynamic viscosity. Then (2.10) and (2.11) take the form

$$\begin{bmatrix} \tilde{\tilde{u}}(k, z) \\ \tilde{w}(k, z) \end{bmatrix} = \frac{\hat{q}}{2\mu k + \rho g} \left(1 + \frac{\tau^{-1} - s^{(1)}}{s + s^{(1)}}\right) \begin{bmatrix} -ikz \\ 1 + kz \end{bmatrix} \exp(-kz), \tag{2.12}$$

$$\begin{bmatrix} \tilde{\hat{\sigma}}_{xx}(k, z) \\ \tilde{\sigma}_{zz}(k, z) \\ \tilde{\hat{\sigma}}_{xz}(k, z) \end{bmatrix} = -\frac{2\mu k \hat{q}}{2\mu k + \rho g} \left(1 - \frac{s^{(1)}}{s + s^{(1)}}\right) \begin{bmatrix} 1 - kz \\ 1 + kz \\ -ikz \end{bmatrix} \exp(-kz), \tag{2.13}$$

where the tilde denotes Laplace transformation. $s^{(1)}$ will later be interpreted as inverse relaxation time and is given by

$$s^{(1)} = \rho g \tau^{-1}/(2\mu k + \rho g). \tag{2.14}$$

If we introduce viscoelastic transfer functions by

$$\tilde{T}^{(\text{ve})}(s) = \frac{1}{2\mu k + \rho g} \left(1 + \frac{\tau^{-1} - s^{(1)}}{s + s^{(1)}}\right), \tag{2.15}$$

$$\tilde{S}^{(\text{ve})}(s) = -\frac{2\mu k}{2\mu k + \rho g} \left(1 - \frac{s^{(1)}}{s + s^{(1)}}\right), \tag{2.16}$$

the impulse reponse is obtained by taking the inverse Laplace transforms of (2.15) and (2.16). We obtain

$$T^{(\text{ve})}(t) = \frac{1}{2\mu k + \rho g} [\delta(t) + (\tau^{-1} - s^{(1)}) \exp(-s^{(1)}t)], \tag{2.17}$$

$$S^{(\text{ve})}(t) = -\frac{2\mu k}{2\mu k + \rho g} [\delta(t) - s^{(1)} \exp(-s^{(1)}t)]. \tag{2.18}$$

For our purposes it is necessary to know the response due to a Heaviside load $\hat{q}(k)H(t)$. From (2.17) and (2.18), by convolution and for $t > 0$,

$$T^{(\text{ve})}(t) = \frac{1}{2\mu k + \rho g} \left[\!\left[1 - \frac{\tau^{-1} - s^{(1)}}{s^{(1)}} [\exp(-s^{(1)}t) - 1]\right]\!\right], \tag{2.19}$$

$$S^{(\text{ve})}(t) = -\frac{2\mu k}{2\mu k + \rho g} [\![1 + [\exp(-s^{(1)}t) - 1]]\!]. \tag{2.20}$$

In (2.19) and (2.20) the first term in the square brackets is the instantaneous elastic response to the load, whereas the second term describes the time-dependent viscous part of the response.

3 Discussion

Except for spatially harmonic load distributions, the inverse Fourier transformation of our solution can, in general, be implemented only numerically. Here it is sufficient, however, to analyse the behaviour in the wavenumber domain. We will show in particular that the boundary conditions are satisfied at any time and discuss the role of pre-stress in this context.

At $z = 0$ we obtain from (2.12) and (2.13)

$$\begin{bmatrix} \hat{u}(k, 0) \\ \hat{w}(k, 0) \end{bmatrix} = T^{(ve)} \hat{q} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.1}$$

$$\begin{bmatrix} \hat{\sigma}_{xx}(k, 0) \\ \hat{\sigma}_{zz}(k, 0) \\ \hat{\sigma}_{xz}(k, 0) \end{bmatrix} = S^{(ve)} \hat{q} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tag{3.2}$$

where $T^{(ve)}$ and $S^{(ve)}$ are given by (2.19) and (2.20), respectively. At the surface the tangential displacement component therefore vanishes. The total perturbation stress is non-deviatoric, such that (2.8) is satisfied. If we denote the viscoelastic portion of the perturbation stress by $\sigma_{ij}^{(ve)}$, we have, in correspondence to (2.6),

$$\hat{\sigma}_{zz}^{(ve)} = \hat{\sigma}_{zz} - \rho g \hat{w}. \tag{3.3}$$

Substituting for $\hat{\sigma}_{zz}$ and $\hat{w}$ from (3.1), we obtain after a short calculation

$$\hat{\sigma}_{zz}^{(ve)}(k, 0) = -\hat{q}(k). \tag{3.4}$$

The advected portion of the perturbation stress is thus given by

$$\rho g \hat{w}(k, 0) = \left[1 - \frac{2\mu k}{2\mu k + \rho g} \exp(-s^{(1)}t) \right] \hat{q}(k), \tag{3.5}$$

which, in the inviscid limit, reduces to

$$\rho g \hat{w}(k, 0) = \hat{q}(k). \tag{3.6}$$

Equation (3.4) simply states that the viscoelastic portion of the perturbation stress balances the applied load $\hat{q}(k)H(t)$ for $t > 0$. The advected portion, on the other hand, increases in magnitude and finally becomes equal and opposite to the viscoelastic portion, such that both cancel each other in the inviscid limit. This, however, expresses the familiar result that, in a perfect fluid, the initial state of hydrostatic pressure cannot be perturbed.

As mentioned previously, Nakiboglu & Lambeck (1982) did not introduce pre-stress into their analysis of the Maxwell continuum. According to (2.6), this is equivalent to equating the total perturbation stress with the elastic perturbation stress. In order to ensure regular surface displacements $\hat{w} = \hat{q}(\rho g)^{-1}$ for Heaviside loads in the inviscid limit, they were, however, forced to modify boundary condition (2.7) and introduced

$$\hat{\sigma}_{zz}^{(e)}(k, 0) = \rho g \hat{w}(k, 0) - \hat{q}(k). \tag{3.7}$$

Nakiboglu & Lambeck (1982) regarded (3.7) as the appropriate modification of the usual elastic boundary condition in order that the associated viscoelastic problem be solved. They justified their reasoning by the erroneous assumption that the correspondence principle does not apply at large times. This is certainly not true, and singularities in the inviscid limit are in fact a consequence of neglecting pre-stress in the momentum balance.

Although (3.7) leads to correct solutions for the displacement field, it is nevertheless an incorrect boundary condition and consequently renders the theoretical stress distribution incorrect also. Obviously, a formal substitution of $\hat{\sigma}_{zz}^{(e)}$ by $\hat{\sigma}_{zz}$ in (3.7) remedies this situation. This replacement can, however, only be understood if the concept of pre-stress has been introduced before.

Acknowledgment

References

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