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Viscoelastodynamics of a stratified, compressible planet: incremental field equations and short- and long-time asymptotes

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SUMMARY

We consider a chemically and entropically stratified, compressible, rotating fluid planet and study gravitational–viscoelastic perturbations of a hydrostatic initial state. Using the Lagrangian formulation, we first derive the incremental field equations and continuity conditions governing the perturbations. Following this, we deduce the asymptotes to the equations for short and long times after the onset of the perturbations. The short-time asymptotic equations are referred to as the incremental field equations and continuity conditions of generalized elastodynamics. They include the equations of conventional elastodynamics as zeroth-order approximations. The long-time asymptotic equations agree with the incremental field equations and continuity conditions of Newtonian-viscous fluid dynamics. In particular, the incremental thermodynamic pressure appearing in the long-time asymptote to the incremental constitutive equation satisfies the appropriate incremental state equation. Finally, we introduce the generalized incremental incompressibility condition. Based on it, we derive approximate incremental field equations for gravitational–viscoelastic perturbations of isochemical, isentropic and compressible regions.

Key words: bulk relaxation, compressibility, equation of state, initial stress, thermodynamic pressure, viscoelastodynamics.

1 INTRODUCTION

Studies of the response of a planet to perturbing forces of short duration have been conventionally based on the assumption of elastic constitutive behaviour. However, complications arise due to the presence of initial stress in the interior of the planet. Rayleigh (1906) discussed this problem and proposed that it be solved if the *total* stress is decomposed into a hydrostatic *initial* stress and a superimposed *incremental* stress. He further defined the displacement of a particle with respect to its initial position and suggested to relate the incremental stress to the displacement using equations which formally agree with the ordinary field equations of elastodynamics valid in the absence of initial stress.

Love (1911) elaborated Rayleigh's concepts. In particular, he found it profitable to formulate the incremental field equations in terms of the *material* incremental stress 'experienced' by a displaced particle and conventionally appearing in the incremental constitutive equation of elasticity. Owing to the initial stress gradient in planetary interiors, the material incremental stress however differs from the *local* incremental stress observed at a fixed location and conventionally appearing in the incremental momentum equation. Love accounted for that difference by introducing the *advective* incremental stress into the incremental momentum equation.

A corollary of Love's concept of material incremental stress is that the initial stress associated with a particle at its current position is the hydrostatic stress at its initial position. This has sometimes been described as the particle 'carrying' its hydrostatic initial stress with it while being displaced. In several recent studies, Love's theory has been modified to the extent that a derivation of the incremental field equations of elastodynamics in the Lagrangian formulation has been given preference over his derivation in the Eulerian formulation (e.g. Dahlen 1974; Grafarend 1982).

The response of a planet to perturbing forces of long duration has usually been studied assuming fluid constitutive behaviour. In contrast to elastodynamics, no 'advective term' is now required in the incremental momentum equation. This is in consequence of formulating the incremental field equations of fluid dynamics in terms of the local incremental stress appearing in the incremental constitutive equation of fluids. The differences in the formulation of the incremental momentum equation in elastodynamics and in fluid dynamics have been discussed by several authors (e.g. Dahlen 1974; Cathles 1975; Grafarend 1982).

Particular attention has been devoted to Newtonian-viscous perturbations of an initially hydrostatic planet. Studies of this kind were initiated by Darwin's (1879) investigation into bodily tides and continued by several other investigators (e.g. Haskell 1935; Parsons 1972). However, most of these studies have been based on the supposition of incompressibility. Notable exceptions include the stability analysis by Jarvis & Mackenzie (1980), which allowed for pressure-induced density perturbations in terms of a set of approximate incremental field equations, and the study by Li & Yuen (1987), in which effects due to a pressure-induced initial density gradient were considered.

Various types of viscoelastic constitutive behaviour have been suggested over the years to study the response of a planet to perturbing forces spanning a wide period range. Comparatively recent is the formal development of the theory of viscoelastodynamics for continua in a state of initial stress. Pioneering investigations into this problem have been published by Biot in a series of papers and are summarized in his monograph (Biot 1965).

Relevant to our study is Biot's analysis of load-induced viscoelastic perturbations of an initially hydrostatic half-space (Biot 1959). Similar to Love, Biot found it necessary to distinguish between local and material incremental stresses. However, in contrast to Love, Biot proceeded by formulating the incremental field equations in terms of the local incremental stress. On the assumption of incompressible perturbations, this allowed their formal reduction to the ordinary field equations valid in the absence of initial stress, whose solutions are well studied.

This 'reduction method' was also found useful by Wolf (1984, 1985a,b) in his investigations into incompressible viscoelastic perturbations of initially hydrostatic, two-layer half-spaces and spheres. Assuming Maxwellian viscoelasticity, he could in particular show that the solution approaches that for a hydrostatic fluid at times long after the onset of the loading (Wolf 1985b). Since Wolf was not aware of Biot's publications at that time, no reference to them could be made. More recently, Wolf's studies have been extended to include generalized Maxwellian viscoelasticity (Müller 1986; Körnig & Müller 1989).

The theory describing compressible gravitational–viscoelastic perturbations of an initially hydrostatic planet has been largely developed by Peltier and is presented in several papers (e.g. Peltier 1974; Wu & Peltier 1982). Peltier's view is to account for elastic compressibility but to ignore bulk relaxation in the incremental constitutive equation of viscoelasticity. The shear relaxation is assumed to be Maxwellian viscoelastic. A model with such constitutive behaviour, which however admits a closed-form solution, was also studied by Wolf (1985c). The simplifying feature of his model is that the incremental gravitational force associated with the perturbations of the density and of the gravitational potential is ignored. This restricts the applicability of the solution to perturbations of sufficiently short wavelength.

Whereas Wolf's simplified model is plane and homogeneous, Peltier's models are spherical and stratified. The stratification in particular allows the initial density distribution to be a prescribed function of the radial distance. Although not stated explicitly, it is evident from the field equations used by Peltier that the initial density gradient is assumed to be due to chemical or entropic stratification. Since this view neglects the pressure-induced increase of the initial density with depth in planetary interiors, it effectively takes the 'initial planet' as incompressible while treating its perturbations as compressible.

Peltier's model of an initially incompressible planet was criticized by Fjeldskaar & Cathles (1984), who argued that viscoelastic perturbations are significantly modified by effects due to a pressure-induced initial density gradient. However, no attempt was made by them to generalize Peltier's theory of viscoelastodynamics in order that this feature be included. In fact, Fjeldskaar & Cathles claimed that Peltier's equations were internally inconsistent and could therefore not be adapted to account for initial compressibility. They supported their view by referring to the existence of an incremental state equation in fluid dynamics and to the above-mentioned difference between the incremental momentum equation in elastodynamics and this equation in fluid dynamics; two features which, in their opinion, were not covered by Peltier's theory. With regard to the second criticism, we incidentally note that, since no assumptions on the constitutive behaviour of the continuum enter into the derivation of the incremental momentum equation (e.g. Dahlen 1974), the differences referred to by Fjeldskaar & Cathles can only be formal. Attempts to design a special incremental momentum equation of viscoelastodynamics to provide for a smooth transition between the corresponding equations in elastodynamics and in fluid dynamics (Svensson 1980) are therefore physically unfounded.

A far-reaching criticism has recently been raised by Geller (1988a), who claimed that the accepted form of the incremental momentum equation describing gravitational–elastic perturbations of an arbitrary static initial state (Dahlen 1972, 1973) is incorrect. Geller deduced the incremental momentum equation from a variational principle. Further, he maintained that this approach is *necessary*, because the correct form of the equation *cannot* be obtained by derivation from the continuum form of Newton's second law (Geller 1988b). Geller proposed modifications to the conventional incremental momentum equation only for a *non-hydrostatic* initial state. The accepted form of the equation for a *hydrostatic* initial state (e.g. Love 1911; Dahlen 1974) was confirmed but, since originally derived from the continuum form of Newton's second law, qualified as merely 'fortuitously' correct in Geller's study.

The present study has the following three objectives.

(i) We wish to demonstrate that the accepted form of the incremental momentum equation for an initially hydrostatic planet is not merely *fortuitously* correct but a *logical* consequence of Newton's second law. This conclusion should indeed result from a careful study of Dahlen's criticized papers. However, Dahlen deduced the incremental field equations for a hydrostatic initial state (Dahlen 1974) from the far more complicated incremental field equations for a non-hydrostatic initial state (Dahlen 1972, 1973). Further, in his Lagrangian formulation for the total fields, the momentum equation contains an error and the gravitational-potential equation is not stated. Since these restrictions impede the verification of Dahlen's results, we will deduce the incremental field equations for a hydrostatic initial state directly from the differential forms of the fundamental principles of continuum mechanics (Section 3). In deriving the equations, we will follow Dahlen and use the Lagrangian formulation, i.e. regard the region initially occupied by the continuum as the reference region (Section 2). Of the multitude of kinematic formulations that are equivalent in infinitesimal perturbation theory (e.g. Geller 1988a), the Lagrangian formulation is profitably employed if (a) individual particles can be identified and (b) the region initially occupied by the continuum is prescribed but the region currently occupied not known *a priori*. The resulting equations describe gravitational–viscoelastic perturbations of a chemically and entropically stratified, rotating fluid planet initially in hydrostatic equilibrium. They apply to any type of viscoelastic constitutive behaviour characterized by both bulk and shear relaxation and, in particular, account also for initial compressibility.

(ii) The equations derived will be used to probe the justification of the criticism raised by Svensson (1980) and by Fjeldskaar & Cathles (1984) regarding the internal consistency of the incremental field equations of viscoelastodynamics. For that purpose, we will consider aperiodic perturbations whose short- and long-time limits exist. Based on this supposition, we will deduce two systems of incremental field equations which are asymptotically correct for short and long times after the onset of the viscoelastic perturbations (Section 4). The short-time asymptotic equations are generalizations of the incremental field equations of elastodynamics. The long-time asymptotic equations agree with the incremental field equations of Newtonian-viscous fluid dynamics, with the incremental thermodynamic pressure satisfying the appropriate incremental state equation of the fluid.

(iii) As an example of how initial compressibility can be included into the incremental field equations, we will derive a set of approximate incremental field equations describing gravitational–viscoelastic perturbations of isochemical, isentropic and compressible regions (Section 5).

2 KINEMATIC FORMULATION

In this preparatory section, we develop the kinematic formulation used in this study. In particular, we introduce the Lagrangian and Eulerian formulations of field quantities and define their material, local and advective increments. We restrict our study to Cartesian tensor fields and employ for the fields the indicial notation with the usual summation convention: the index subscripts $i, j, \ldots$ range over 1, 2, 3 and repeated subscripts imply summation.

Let $V(T)$ denote the region occupied by a continuum at some *reference* time, T, and $V(t)$ the region occupied by this continuum at the *current* time, t. For all $X_i \in V(T)$ and $t \in (-\infty, \infty)$, we then assume the existence of the following unique mapping:

$$r_i(\mathbf{X}, t) = X_i + u_i(\mathbf{X}, t), \tag{1}$$

where X_i is the reference particle position, $r_i(\mathbf{X}, t)$ the Lagrangian formulation of the current particle position and $u_i(\mathbf{X}, t)$ the Lagrangian formulation of the displacement. Supposing also the existence of the inverse mapping of (1), we have, for all $r_i \in V(t)$ and $t \in (-\infty, \infty)$,

$$X_i(\mathbf{r}, t) = r_i - U_i(\mathbf{r}, t), \tag{2}$$

where r_i is the current particle position, $X_i(\mathbf{r}, t)$ the Eulerian formulation of the reference particle position and $U_i(\mathbf{r}, t)$ the Eulerian formulation of the displacement. An obvious consequence of (1) and (2) is

$$u_i(\mathbf{X}, t) = U_i(\mathbf{r}, t). \tag{3}$$

Since $u_i(\mathbf{X}, T) = U_i(\mathbf{r}, T) = 0$, we also have

$$r_i(\mathbf{X}, T) = X_i, \qquad X_i(\mathbf{r}, T) = r_i. \tag{4}$$

More generally, we may consider an arbitrary field describing some aspect of the state of the continuum. Then,

$$f_{ij\ldots}(\mathbf{X}, t) = F_{ij\ldots}(\mathbf{r}, t), \tag{5}$$

where $f_{ij\ldots}(\mathbf{X}, t)$ and $F_{ij\ldots}(\mathbf{r}, t)$ are the Lagrangian and Eulerian formulations of the field. As in (1)–(5), we will henceforth distinguish the Lagrangian and Eulerian formulations of fields by lower-case and upper-case function names. Particle positions appearing explicitly as function arguments will be shown in symbolic notation.

Let the current state of the continuum be also describable as a perturbation of an initial state assumed by the continuum at some initial time, $t^{(0)} \le T$. For all $X_i \in V(T)$, $r_i \in V(t)$ and $t \in [t^{(0)}, \infty)$, we then consider the following decompositions:

$$f_{ij\ldots}(\mathbf{X}, t) = f_{ij\ldots}(\mathbf{X}, t^{(0)}) + \delta f_{ij\ldots}(\mathbf{X}, t), \tag{6}$$

$$F_{ij\ldots}(\mathbf{r}, t) = F_{ij\ldots}(\mathbf{r}, t^{(0)}) + \Delta F_{ij\ldots}(\mathbf{r}, t), \tag{7}$$

where $f_{ij\ldots}(\mathbf{X}, t)$ and $F_{ij\ldots}(\mathbf{r}, t)$ are the Lagrangian and Eulerian formulations of the *total* field, $f_{ij\ldots}(\mathbf{X}, t^{(0)})$ and $F_{ij\ldots}(\mathbf{r}, t^{(0)})$ the Lagrangian and Eulerian formulations of the *initial* field, $\delta f_{ij\ldots}(\mathbf{X}, t)$ is the Lagrangian formulation of the *material incremental* field and $\Delta F_{ij\ldots}(\mathbf{r}, t)$ the Eulerian formulation of the *local incremental* field. Equations (6) and (7) suggest several notational simplifications, which will be adhered to throughout this study: (i) the arguments X_i and r_i of the Lagrangian and Eulerian formulations are suppressed; (ii) the argument t is suppressed; (iii) the arguments $t^{(0)}$ and T are indicated by the label superscripts 0 and T appended to the function names; (iv) the material and local increments are indicated by the label superscripts δ and Δ appended to the function names.

Observing these conventions, (6) and (7) can be abbreviated as

$$f_{ij\ldots} = f_{ij\ldots}^{(0)} + f_{ij\ldots}^{(\delta)}, \tag{8}$$

$$F_{ij\ldots} = F_{ij\ldots}^{(0)} + F_{ij\ldots}^{(\Delta)}. \tag{9}$$

Putting $f_{ij\ldots} = r_i$ and $F_{ij\ldots} = X_i$, we obtain in particular

$$r_i = r_i^{(0)} + r_i^{(\delta)}, \tag{10}$$

$$X_i = X_i^{(0)} + X_i^{(\Delta)}, \tag{11}$$

which are to be distinguished from the abbreviated forms of (1) and (2):

$$r_i = X_i + u_i, \tag{12}$$

$$X_i = r_i - U_i. \tag{13}$$

To obtain simple expressions for $r_i^{(\delta)}$ and $X_i^{(\Delta)}$, we from now on let the initial and reference times coincide: $t^{(0)} = T$. Then, it follows from (4) that

$$r_i^{(0)} = X_i, \qquad X_i^{(0)} = r_i. \tag{14}$$

Equating (10) to (12) and (11) to (13) gives

$$r_i^{(\delta)} = u_i, \qquad X_i^{(\Delta)} = -U_i, \tag{15}$$

hence we have

$$r_i = r_i^{(0)} + u_i, \tag{16}$$

$$X_i = X_i^{(0)} - U_i. \tag{17}$$

The gradient of the Lagrangian field $f_{ij\ldots}$ with respect to X_i is indicated by means of the notation

$$f_{ij\ldots,k} = \frac{\partial f_{ij\ldots}}{\partial X_k}. \tag{18}$$

If $f_{ij\ldots} = r_i$, it follows with (12) that

$$r_{i,j} = \delta_{ij} + u_{i,j}, \tag{19}$$

where δ_{ij} is the Kronecker symbol. The gradient of the Eulerian field $F_{ij\ldots}$ with respect to r_i is denoted by

$$F_{ij\ldots,k} = \frac{\partial F_{ij\ldots}}{\partial r_k}. \tag{20}$$

Note that no ambiguity exists between (18) and (20). This is because a subscript preceded by a comma is understood to indicate the gradient with respect to the spatial argument of the function considered, i.e. with respect to X_i for $f_{ij\ldots}$ and with respect to r_i for $F_{ij\ldots}$. To express $F_{ij\ldots,k}$ in terms of $f_{ij\ldots,k}$, we differentiate (5) with respect to r_i, giving

$$F_{ij\ldots,k} = f_{ij\ldots,\ell} X_{\ell,k}. \tag{21}$$

Differentiating (3) with respect to r_i provides

$$U_{i,j} = u_{i,k} X_{k,j}. \tag{22}$$

In view of (13), we have however

$$X_{i,j} = \delta_{ij} - U_{i,j}, \tag{23}$$

hence, on the assumption of infinitesimal perturbations, it follows from (22) and (23) that

$$U_{i,j} = u_{i,j}. \tag{24}$$

Finally, we give the Lagrangian formulation of (9). Observing that $F^{(0)}_{ij\ldots} = f^{(0)}_{ij\ldots}(\mathbf{X}+\mathbf{U})$, equation (9) can be rewritten as

$$f_{ij\ldots} = f^{(0)}_{ij\ldots}(\mathbf{X}+\mathbf{U}) + f^{(\Delta)}_{ij\ldots}. \tag{25}$$

Using (3) and assuming infinitesimal perturbations, (25) reduces to

$$f_{ij\ldots} = f^{(0)}_{ij\ldots} + f^{(0)}_{ij\ldots,k}u_k + f^{(\Delta)}_{ij\ldots}. \tag{26}$$

Equating (8) to (26) gives

$$f^{(\delta)}_{ij\ldots} = f^{(\Delta)}_{ij\ldots} + f^{(0)}_{ij\ldots,k}u_k. \tag{27}$$

We refer to $f^{(0)}_{ij\ldots,k}u_k$ as the *advective incremental* field. If the initial field is homogeneous, then $f^{(0)}_{ij\ldots,k} = 0$ and, therefore, $f^{(\delta)}_{ij\ldots} = f^{(\Delta)}_{ij\ldots}$, i.e. the distinction between material and local incremental fields is not necessary.

The essential results of this section will be required in Sections 3 and 4 when deriving the incremental field equations and continuity conditions of viscoelastodynamics and their asymptotic approximations.

3 FIELD EQUATIONS AND CONTINUITY CONDITIONS

In this section, we deduce the incremental field equations and continuity conditions describing infinitesimal gravitational–viscoelastic perturbations of a chemically and entropically stratified, compressible, rotating fluid planet initially in hydrostatic equilibrium. In deducing the equations, we suppose the perturbations are *isochemical* and *isentropic*. Our assumption is justified if the characteristic times associated with chemical and thermal diffusion are long compared with those associated with viscoelastic relaxation. This condition is met by a wide range of processes in planetary interiors.

The derivation is given in the Lagrangian formulation, i.e. the reference particle position, $X_i \in V^{(T)}$, is used as the spatial argument. We assume without loss of generality that $t^{(0)} = 0$, i.e. we have $t \in [0, \infty)$ for the temporal argument. All field quantities are taken as continuously differentiable with respect to t as many times as desired.

In Sections 3.1 and 3.2, we collect the field equations and continuity conditions for the total and initial fields, from which, in Section 3.3, those for the incremental fields are obtained. In Section 3.4, we derive several field equations involving the incremental density and the incremental thermodynamic pressure. These field quantities are less frequently used in combination with the Lagrangian formulation but will be required in Sections 4.3 and 5.

3.1 Equations for the total fields

We begin by recalling the relation between the Piola stress, τ_{ij}, (also referred to as the non-symmetric Piola–Kirchhoff stress) and the Cauchy stress, t_{ij}:

$$\tau_{ij}N_j\,dA = t_{ij}n_j\,da, \tag{28}$$

where N_i is the unit vector normal to an arbitrarily oriented reference surface element, dA, at X_i and n_i is the unit vector normal to the associated current surface element, da, at r_i. Note that, in view of $r^{(0)}_i = X_i$, we also have $n^{(0)}_i = N_i$ and $da^{(0)} = dA$. With the transformation formula (e.g. Malvern 1969, pp. 169–170)

$$jN_i\,dA = r_{j,i}n_j\,da, \tag{29}$$

where j is the Jacobian determinant given by

$$j = \det(r_{i,j}), \tag{30}$$

it then follows that

$$r_{i,k}\tau_{jk} = jt_{ij}. \tag{31}$$

Consider now a self-gravitating, rotating continuous planet undergoing perturbations of an initial state. On the assumption that the angular velocity, Ω_i, of the planet can be regarded as prescribed, the momentum equation relative to a corotating reference frame is (e.g. Malvern 1969, pp. 220–224)

$$\tau_{ij,j} + \rho^{(0)}[(\phi+\psi)_{,j}X_{j,i} + f_i + 2\epsilon_{ijk}\Omega_k\,\partial_t r_j] = \rho^{(0)}\,\partial_{tt}r_i, \tag{32}$$

where f_i is the force per unit mass, ϵ_{ijk} the permutation symbol, $\rho^{(0)}$ the initial mass density (assumed to be non-negative), ϕ the gravitational potential and ψ the centrifugal potential. In view of (21), the term $(\phi+\psi)_{,j}X_{j,i}$ is the Lagrangian formulation of the gravity force per unit mass; the term $2\epsilon_{ijk}\Omega_k\,\partial_t r_j$ is the Coriolis force per unit mass. Note that, by (18) and (20), we have $X_{(i),(j)}r_{(j),(i)} = 1$ (no summation). The gravitational-potential equation can be written as

$$j(\phi_{,ij}X_{i,k}X_{j,k} + \phi_{,i}X_{i,jj}) = -4\pi G\rho^{(0)}, \tag{33}$$

where G is Newton's gravitational constant. Using (21), equation (33) is readily interpreted as the Lagrangian formulation of Poisson's equation. The equation for ψ is

$$2\psi = \Omega_i\Omega_i r_j r_j - \Omega_i\Omega_j r_i r_j, \tag{34}$$

which implies an origin of the coordinate system on the spin axis. The constitutive equation is assumed to be of the form

$$t_{ij} = t_{ij}^{(0)} + \mathcal{M}_{ijk\ell}[r_{m,k}(t-t')r_{m,\ell}(t-t') - \delta_{k\ell}], \tag{35}$$

where the constitutive functional, $\mathcal{M}_{ijk\ell}$, (assumed to be linear for simplicity) transforms the strain history given by the term in brackets into the incremental stress and where $t' \leq t$ must hold as a consequence of the causality principle. With f_i, $\mathcal{M}_{ijk\ell}$, $t_{ij}^{(0)}$, $\rho^{(0)}$ and Ω_i prescribed, (30)–(35) constitute the system of total field equations for j, r_i, t_{ij}, τ_{ij}, ϕ and ψ.

To write down the continuity conditions for the total fields in the direction of N_i normal to dA, we introduce the abbreviation

$$[f_{ij\ldots}]_-^+ = \lim_{\epsilon\to 0}[f_{ij\ldots}(\mathbf{X}+\epsilon\mathbf{N}) - f_{ij\ldots}(\mathbf{X}-\epsilon\mathbf{N})]. \tag{36}$$

With f_i, $\mathcal{M}_{ijk\ell}$, $n_j^{(0)}t_{ij}^{(0)}$ and $\rho^{(0)}$ continuously differentiable with respect to $X_i \in V^{(T)}$ as many times as desired (jump discontinuities of $\mathcal{M}_{ijk\ell}$ and $\rho^{(0)}$ are permitted, *cf.* below), the following continuity conditions apply:

$$[r_i]_-^+ = 0, \tag{37}$$

$$[\phi]_-^+ = 0, \tag{38}$$

$$[n_i\phi_{,j}X_{j,i}]_-^+ = 0, \tag{39}$$

$$[n_j t_{ij}]_-^+ = 0. \tag{40}$$

We note that the continuity of r_i expressed by (37) excludes cavitation and slip in the fluid.

3.2 Equations for the initial fields

We now assume that (i) the continuum is a *fluid* and (ii) the initial state is a static equilibrium state. Since a fluid at rest cannot maintain shear stresses, the initial state must even be a *hydrostatic* equilibrium state and, with the mechanical pressure defined by $p = -t_{ii}/3$, we must have

$$t_{ij}^{(0)} = -p^{(0)}\delta_{ij}. \tag{41}$$

Since, by (14), $r_{i,j}^{(0)} = \delta_{ij}$, equations (30) and (31) reduce to

$$j^{(0)} = 1, \tag{42}$$

$$\tau_{ij}^{(0)} = -p^{(0)}\delta_{ij}. \tag{43}$$

Using (41)–(43) and observing $X_{i,j}^{(0)} = \delta_{ij}$, which follows from (14), and $(\partial_t r_i)^{(0)} = (\partial_{tt} r_i)^{(0)} = 0$, which holds by assumption, (32)–(35) become

$$-p_{,i}^{(0)} + \rho^{(0)}(\phi_{,i}^{(0)} + \psi_{,i}^{(0)} + f_i^{(0)}) = 0, \tag{44}$$

$$\phi_{,ii}^{(0)} = -4\pi G\rho^{(0)}, \tag{45}$$

$$2\psi^{(0)} = \Omega_i\Omega_i r_j^{(0)} r_j^{(0)} - \Omega_i\Omega_j r_i^{(0)} r_j^{(0)}, \tag{46}$$

$$p^{(0)} = \gamma(\rho^{(0)}, \lambda^{(0)}, \sigma^{(0)}). \tag{47}$$

Equation (47) is the form of the state equation assumed in this study, with $\lambda^{(0)}$ a field quantity representing the initial chemical composition and $\sigma^{(0)}$ the initial entropy density. An immediate consequence of (44) is

$$\epsilon_{ijk}\rho_{,j}^{(0)}(\phi_{,k}^{(0)} + \psi_{,k}^{(0)} + f_k^{(0)}) + \rho^{(0)}\epsilon_{ijk}f_{k,j}^{(0)} = 0. \tag{48}$$

If the state function, γ, is known and $f_i^{(0)}$, $\lambda^{(0)}$, $\sigma^{(0)}$ and Ω_i are prescribed, (44)–(47) constitute the system of initial field equations for $p^{(0)}$, $\rho^{(0)}$, $\phi^{(0)}$ and $\psi^{(0)}$.

With (44) representing three scalar equations, this system is however overdetermined and solutions are severely restricted. If $f_i^{(0)} = 0$, it follows from (44) and (48) that the level surfaces of $p^{(0)}$, $\rho^{(0)}$ and $(\phi^{(0)} + \psi^{(0)})$ coincide, where solutions can be shown to exist for these surfaces being oblate spheroids with respect to the spin axis. If $f_i^{(0)} \neq 0$ but $\epsilon_{ijk}f_{k,j}^{(0)} = 0$, the force field $f_i^{(0)}$ is conservative. Then, we may put $f_i^{(0)} = \chi_{,i}^{(0)}$ and it follows from (44) and (48) that the level surfaces of $p^{(0)}$, $\rho^{(0)}$ and $(\phi^{(0)} + \psi^{(0)} + \chi^{(0)})$ coincide.

The continuity conditions (37)–(40) reduce to

$$[r_i^{(0)}]_-^+ = 0, \tag{49}$$

$$[\phi^{(0)}]_-^+ = 0, \tag{50}$$

$$[n_i^{(0)}\phi_{,i}^{(0)}]_-^+ = 0, \tag{51}$$

$$[n_i^{(0)}p^{(0)}]_-^+ = 0. \tag{52}$$

We note that solutions to (44)–(47) admit jump discontinuities of $\rho^{(0)}$ normal its level surfaces. With $X_i \in \mathcal{A}^{(T)}$, where $\mathcal{A}^{(T)} \subset \mathcal{V}^{(T)}$ is any of the discontinuity surfaces of $\rho^{(0)}$, we then have for N_i normal to $\mathcal{A}^{(T)}$

$$[\rho^{(0)}]_-^+ \neq 0. \tag{53}$$

3.3 Equations for the incremental fields

Using (8), (16) and (17), we decompose the total fields f_i, j, r_i, t_{ij}, X_i, τ_{ij}, ϕ and ψ in (30)–(35) into initial and incremental parts. Observing (19), (23), (41)–(43) and (47), we get

$$(1 + j^{(\delta)}) = \det(\delta_{ij} + u_{i,j}), \tag{54}$$

$$(\delta_{ik} + u_{i,k})(-p^{(0)}\delta_{jk} + \tau_{jk}^{(\delta)}) = (1 + j^{(\delta)})(-p^{(0)}\delta_{ij} + t_{ij}^{(\delta)}), \tag{55}$$

$$-p_{,i}^{(0)} + \tau_{ij,j}^{(\delta)} + \rho^{(0)}[(\phi^{(0)} + \phi^{(\delta)} + \psi^{(0)} + \psi^{(\delta)})_{,j}(\delta_{ji} - U_{j,i}) + f_i^{(0)} + f_i^{(\delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j)] = \rho^{(0)}\,\partial_{tt}u_i, \tag{56}$$

$$(1 + j^{(\delta)})[(\phi^{(0)} + \phi^{(\delta)})_{,ij}(\delta_{ik} - U_{i,k})(\delta_{jk} - U_{j,k}) - (\phi^{(0)} + \phi^{(\delta)})_{,i}U_{i,jj} = -4\pi G\rho^{(0)}, \tag{57}$$

$$2(\psi^{(0)} + \psi^{(\delta)}) = \Omega_i\Omega_i(r_j^{(0)} + u_j)(r_j^{(0)} + u_j) - \Omega_i\Omega_j(r_i^{(0)} + u_i)(r_j^{(0)} + u_j), \tag{58}$$

$$-p^{(0)}\delta_{ij} + t_{ij}^{(\delta)} = -\gamma(\rho^{(0)}, \lambda^{(0)}, \sigma^{(0)})\delta_{ij} + \mathcal{M}_{ijk\ell}\{[\delta_{mk} + u_{m,k}(t-t')][\delta_{m\ell} + u_{m,\ell}(t-t')] - \delta_{k\ell}\}. \tag{59}$$

We note that no restriction on the magnitude of the perturbations has been imposed so far, i.e. (54)–(59) are valid even for *finite* perturbations.

To make further progress, we from now on assume that the perturbations are *infinitesimal*. This supposition allows us to neglect products of incremental quantities and, in particular, to use (24) and (27). It thus follows from (54) that

$$j^{(\delta)} = u_{i,i}, \tag{60}$$

by which (55) can be reduced to

$$\tau_{ij}^{(\delta)} = t_{ij}^{(\delta)} + p^{(0)}(u_{j,i} - u_{k,k}\delta_{ij}). \tag{61}$$

With (24), (44)–(47), (60) and (61), equations (56)–(59) then become

$$t_{ij,j}^{(\delta)} + p_{,j}^{(0)}u_{j,i} - p_{,i}^{(0)}u_{j,j} + \rho^{(0)}[(\phi^{(\delta)} + \psi^{(\delta)})_{,i} - (\phi^{(0)} + \psi^{(0)})_{,j}u_{j,i} + f_i^{(\delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j] = \rho^{(0)}\,\partial_{tt}u_i, \tag{62}$$

$$\phi_{,ii}^{(\delta)} - 2\phi_{,ij}^{(0)}u_{i,j} - \phi_{,i}^{(0)}u_{i,jj} = 4\pi G\rho^{(0)}u_{i,i}, \tag{63}$$

$$\psi^{(\delta)} = \psi_{,i}^{(0)}u_i, \tag{64}$$

$$t_{ij}^{(\delta)} = \mathcal{M}_{ijk\ell}[u_{k,\ell}(t-t') + u_{\ell,k}(t-t')]. \tag{65}$$

An equivalent system of field equations results if we use (27) to express $f_i^{(\delta)}$ and $\phi^{(\delta)}$ in terms of the associated local increments:

$$f_i^{(\delta)} = f_i^{(\Delta)} + f_{i,j}^{(0)}u_j, \tag{66}$$

$$\phi^{(\delta)} = \phi^{(\Delta)} + \phi_{,i}^{(0)}u_i. \tag{67}$$

In view of (64), (66) and (67), equation (62) takes the form

$$t_{ij,j}^{(\delta)} + p_{,j}^{(0)}u_{j,i} - p_{,i}^{(0)}u_{j,j} + (\phi_{,ij}^{(0)} + \psi_{,ij}^{(0)} + f_{i,j}^{(0)})\rho^{(0)}u_j + \rho^{(0)}(\phi_{,i}^{(\Delta)} + f_i^{(\Delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_{tt}u_i. \tag{68}$$

Using (44), this can be rewritten as

$$\boxed{t_{ij,j}^{(\delta)} + (p_{,j}^{(0)}u_j)_{,i} - (\phi_{,i}^{(0)} + \psi_{,i}^{(0)} + f_i^{(0)})(\rho^{(0)}u_j)_{,j} + \rho^{(0)}(\phi_{,i}^{(\Delta)} + f_i^{(\Delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_{tt}u_i.} \tag{69}$$

Upon substitution of (67) and use of (45), equation (63) reduces to

$$\boxed{\phi_{,ii}^{(\Delta)} = 4\pi G(\rho^{(0)}u_i)_{,i}.} \tag{70}$$

Equations (69) and (70) agree with the incremental momentum equation and gravitational-potential equation given by Love (1911, pp. 89–93) and by Dahlen (1974). Love used the Eulerian formulation, i.e. his incremental equations are functions of the current particle position, r_i. Since the difference between the Lagrangian and Eulerian formulations is of second order in the incremental quantities, it may however be ignored in infinitesimal perturbation theory. In contrast to Love, Dahlen used the Lagrangian formulation in terms of the reference particle position, X_i, which is adopted here. We note that the expression for the gravitational-force density is given incorrectly in Dahlen's total momentum equation, which therefore does not agree with (32). However, this error is not transferred to Dahlen's incremental momentum equation, which is found to be consistent with (69). Also, Dahlen does not introduce an equation equivalent to our total gravitational-potential equation (33). Hence, it is not quite obvious how his equivalent to our incremental gravitational-potential equation (70) has been derived.

To obtain an expression for $\mathcal{M}_{ijk\ell}$, we use the continuous differentiability of the strain history. On this assumption, $\mathcal{M}_{ijk\ell}$ may be written as a convolution integral of the form $\int_0^t m_{ijk\ell}(t-t')\,\partial_{t'}[u_{k,\ell}(t')+u_{\ell,k}(t')]\,dt'$ (e.g. Christensen 1982, pp. 3–9). Supposing further *isotropic* viscoelasticity and exploiting the symmetry properties of $m_{ijk\ell}$ (e.g. Malvern 1969, pp. 276–277), equation (65) takes the form

$$\boxed{t_{ij}^{(\delta)} = \int_0^t [m_1(t-t') - \tfrac{2}{3}m_2(t-t')]\,\partial_{t'}[u_{k,k}(t')]\delta_{ij}\,dt' + \int_0^t m_2(t-t')\,\partial_{t'}[u_{i,j}(t')+u_{j,i}(t')]\,dt'.} \tag{71}$$

We refer to (71) as the incremental constitutive equation of viscoelasticity. The independent functions $m_1(t-t')$ and $m_2(t-t')$, which are defined only for $t-t'\geq 0$, are the bulk relaxation function and the shear relaxation function. For notational economy, we frequently use $m_\nu(t-t')$, where $\nu=1, 2$. We assume that $m_\nu(t-t')$ is continuously differentiable with respect to X_i, where possible jump discontinuities on $\mathcal{A}^{(T)}$ are excepted. We further take $m_\nu(t-t')$ as continuously differentiable with respect to $t-t'$, where it follows from thermodynamic principles that (e.g. Christensen 1982, pp. 83–87; Golden & Graham 1988, pp. 12–14)

$$m_\nu(t-t')\geq 0, \tag{72}$$

$$\partial_{t-t'}m_\nu(t-t')\leq 0. \tag{73}$$

To obtain an additional constraint on $m_2(t-t')$, we recall that, by assumption, the continuum is a fluid. A necessary condition of fluid constitutive behaviour is that deviatoric stresses cannot be maintained indefinitely (e.g. Christensen 1982, pp. 9–14; Golden & Graham, 1988, pp. 14–17). In view of (71), this can be formally expressed by means of the fluidity condition:

$$\lim_{t-t'\to\infty} m_2(t-t')=0. \tag{74}$$

With $f_i^{(0)}$, m_ν and Ω_i prescribed and $p^{(0)}$, $\rho^{(0)}$, $\phi^{(0)}$ and $\psi^{(0)}$ given as a solution to (44)–(47), equations (69)–(71) constitute the system of incremental field equations of viscoelastodynamics for $t_{ij}^{(\delta)}$, u_i and $\phi^{(\Delta)}$.

Finally, we derive the incremental continuity conditions (e.g. Dahlen 1974; Grafarend 1982). For that purpose, we use (8), (16), (17), (19) and (23) to decompose the total fields in (37)–(40) into initial and incremental parts. It follows directly from (37) that

$$[r_i^{(0)}+u_i]_-^+=0, \tag{75}$$

which, by (49), reduces to

$$\boxed{[u_i]_-^+=0.} \tag{76}$$

Decomposition of (38) and use of (67) gives

$$[\phi^{(0)}+\phi^{(\Delta)}+\phi_{,i}^{(0)}u_i]_-^+=0. \tag{77}$$

In view of (50), (51), (76) and the continuity of $n_i^{(0)}$, this is equivalent to

$$\boxed{[\phi^{(\Delta)}]_-^+=0.} \tag{78}$$

To obtain the incremental continuity condition for $n_i^{(0)}\phi_{,i}^{(\Delta)}$, we decompose (39). Using $n_i=n_i^{(0)}+n_i^{(\delta)}$, substituting (67) and neglecting products of incremental quantities, we obtain

$$[n_i^{(0)}(\phi_{,i}^{(0)}+\phi_{,i}^{(\Delta)}+\phi_{,ij}^{(0)}u_j)+\phi_{,i}^{(0)}n_i^{(\delta)}]_-^+=0. \tag{79}$$

In view of (51) and the continuity of $n_i^{(0)}$ and $n_i^{(\delta)}$, this reduces to

$$[n_i^{(0)}(\phi_{,i}^{(\Delta)}+\phi_{,ij}^{(0)}u_j)]_-^+=0. \tag{80}$$

Observing the constraints imposed by (51) on the continuity of the components of $n_i^{(0)}\phi_{,ij}^{(0)}$, equation (80) can be shown to be equivalent to

$$[n_i^{(0)}(\phi_{,i}^{(\Delta)} + \phi_{,jj}^{(0)}u_i)]_-^+ = 0. \tag{81}$$

Substituting for $\phi_{,ii}^{(0)}$ from (45), this takes the form

$$\boxed{[n_i^{(0)}(\phi_{,i}^{(\Delta)} - 4\pi G\rho^{(0)}u_i)]_-^+ = 0.} \tag{82}$$

If, in particular, $X_i \in \mathcal{A}^{(T)}$ and N_i normal to $\mathcal{A}^{(T)}$, then $[\rho^{(0)}]_-^+ \neq 0$ according to (53) whence $n_i^{(0)}\phi_{,i}^{(\Delta)}$ is discontinuous in this case. The incremental continuity condition for $n_j^{(0)}t_{ij}^{(\delta)}$ can be obtained by decomposing (40). Putting again $n_i = n_i^{(0)} + n_i^{(\delta)}$ and neglecting products of incremental quantities, we obtain

$$[n_j^{(0)}(p^{(0)}\delta_{ij} + t_{ij}^{(\delta)}) + p^{(0)}n_i^{(\delta)}]_-^+ = 0. \tag{83}$$

With (52) and the continuity of $n_i^{(0)}$ and $n_i^{(\delta)}$, it follows that

$$\boxed{[n_j^{(0)}t_{ij}^{(\delta)}]_-^+ = 0.} \tag{84}$$

3.4 Continuity and state equations

So far, the incremental density and the incremental thermodynamic pressure have not appeared explicitly in the equations. This is in accordance with the adoption of the Lagrangian formulation, where the displacement, u_i, is preferentially used. However, the incremental density and the incremental thermodynamic pressure will be required to interpret the long-time asymptotes to the incremental field equations and continuity conditions of viscoelastodynamics in Section 4.3 and to study the special case of generalized incompressibility in Section 5. Here, we collect the equations governing these field quantities.

The density, ρ, is related to $\rho^{(0)}$ by the continuity equation (e.g. Malvern 1969, pp. 208–210):

$$j\rho = \rho^{(0)}, \tag{85}$$

where j is given by (30). For a fluid not necessarily in hydrostatic equilibrium, the *thermodynamic* pressure, π, is introduced with the aid of a state equation whose functional relation is identical to that governing the *mechanical* pressure, $p = -t_{ii}/3$, in the case of hydrostatic equilibrium (e.g. Malvern 1969, p. 296). Hence,

$$\pi = \gamma(\rho, \lambda, \sigma), \tag{86}$$

where the state functions γ in (47) and (86) are identical. Note that, with this definition, π is in general different from p. However, at $t = 0$ equation (86) reduces to

$$\pi^{(0)} = \gamma(\rho^{(0)}, \lambda^{(0)}, \sigma^{(0)}). \tag{87}$$

Equating (47) to (87) gives

$$\pi^{(0)} = p^{(0)}. \tag{88}$$

A direct consequence of (87) and (88) is

$$p_{,i}^{(0)} = \left(\frac{\partial\gamma}{\partial\rho}\right)^{(0)}\rho_{,i}^{(0)} + \left(\frac{\partial\gamma}{\partial\lambda}\right)^{(0)}\lambda_{,i}^{(0)} + \left(\frac{\partial\gamma}{\partial\sigma}\right)^{(0)}\sigma_{,i}^{(0)}, \tag{89}$$

where $(\partial\gamma/\partial\rho)^{(0)} = [\partial\gamma/\partial\rho]_{\rho=\rho^{(0)}}$ etc. and the partial derivatives are functions of $X_i \in \mathcal{V}^{(T)}$.

Next, we use (8) to decompose the total field equations (85) and (86) into initial and incremental parts. Using also (42) and (60), equation (85) becomes

$$(1 + u_{i,i})(\rho^{(0)} + \rho^{(\delta)}) = \rho^{(0)}. \tag{90}$$

Since, by assumption, the perturbations are isochemical and isentropic, we have $\lambda^{(\delta)} = \sigma^{(\delta)} = 0$ and the decomposition of (86) takes the form

$$\pi^{(0)} + \pi^{(\delta)} = \gamma(\rho^{(0)}, \lambda^{(0)}, \sigma^{(0)}) + \left(\frac{\partial\gamma}{\partial\rho}\right)^{(0)}\rho^{(\delta)}. \tag{91}$$

Considering (87) and retaining only terms that are linear in the incremental quantities, (90) and (91) reduce to

$$\rho^{(\delta)} = -\rho^{(0)}u_{i,i}, \tag{92}$$

$$\pi^{(\delta)} = \left(\frac{\partial\gamma}{\partial\rho}\right)^{(0)}\rho^{(\delta)}. \tag{93}$$

Due to (27) and (88), we have however

$$\rho^{(\delta)} = \rho^{(\Delta)} + \rho^{(0)}_{,i}u_i, \tag{94}$$

$$\pi^{(\delta)} = \pi^{(\Delta)} + p^{(0)}_{,i}u_i, \tag{95}$$

which allow us to replace (92) and (93) by

$$\rho^{(\Delta)} = -(\rho^{(0)}u_i)_{,i}, \tag{96}$$

$$\pi^{(\Delta)} = \left(\frac{\partial\gamma}{\partial\rho}\right)^{(0)}(\rho^{(\Delta)} + \rho^{(0)}_{,i}u_i) - p^{(0)}_{,i}u_i. \tag{97}$$

Equations (92) and (96) are different forms of the incremental continuity equation; similarly, (93) and (97) are alternative forms of the incremental state equation. Equation (97) takes a more familiar form upon substituting for $p^{(0)}_{,i}$ from (89):

$$\pi^{(\Delta)} = \left(\frac{\partial\gamma}{\partial\rho}\right)^{(0)}\rho^{(\Delta)} - \left(\frac{\partial\gamma}{\partial\lambda}\right)^{(0)}\lambda^{(0)}_{,i}u_i - \left(\frac{\partial\gamma}{\partial\sigma}\right)^{(0)}\sigma^{(0)}_{,i}u_i. \tag{98}$$

In Sections 4.3 and 5, suitable expressions for the partial derivatives appearing in (98) will be provided.

4 ASYMPTOTIC BEHAVIOUR

In this section, we derive the asymptotes, for small and large times, to the incremental field equations and continuity conditions of viscoelastodynamics. Central to our derivation is to find suitable asymptotic approximations to the Laplace transform of the incremental constitutive equation of viscoelasticity, (71).

The Laplace transform, $L[f(t)]$, of a scalar function $f(t)$ (e.g. LePage 1961, pp. 285–328) is defined by

$$L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt, \tag{99}$$

where the transform variable, s, will be restricted to real values. For the convergence of the integral in (99) for s larger than some value s_0 it is sufficient that $f(t)$ be piecewise continuous for all $t \in [0, \infty)$ and of exponential order as $t \to \infty$. Here, we are concerned with the more restrictive case that $f(t)$ is continuously differentiable for all $t \in [0, \infty)$, where $\lim_{t\to 0} f(t)$ and $\lim_{t\to\infty} f(t)$ are supposed to exist, thereby ensuring also the existence of asymptotic approximations to $f(t)$ for small and large t. This assumption is of sufficient generality to apply to the temporal dependences of the relaxation functions and of the incremental fields associated with *aperiodic* perturbations. Introducing $\tilde{f}(s) = L[f(t)]$ for brevity, the following formulae can be established:

$$L\left[\int_0^t f(t')\,dt'\right] = \frac{\tilde{f}(s)}{s}, \tag{100}$$

$$L[\partial_t f(t)] = s\tilde{f}(s) - f(0), \tag{101}$$

$$L\left[\int_0^t f(t-t')g(t')\,dt'\right] = \tilde{f}(s)\tilde{g}(s), \tag{102}$$

$$L[e^{-s't}] = \frac{1}{s+s'}. \tag{103}$$

Further consequences are the generalized initial- and final-value theorems. According to the first theorem, an asymptotic approximation $p(t)$ to $f(t)$ for small t corresponds to an asymptotic approximation $\tilde{p}(s)$ to $\tilde{f}(s)$ for large s. Similarly, the second theorem states that an asymptotic approximation $q(t)$ to $f(t)$ for large t corresponds to an asymptotic approximation $\tilde{q}(s)$ to $\tilde{f}(s)$ for small s.

Using (101), (102) and $u_i^{(0)} = 0$, the Laplace transform of (71) is found to be

$$\tilde{t}_{ij}^{(\delta)} = (\tilde{m}_1 - \tfrac{2}{3}\tilde{m}_2)s\tilde{u}_{k,k}\delta_{ij} + \tilde{m}_2 s(\tilde{u}_{i,j} + \tilde{u}_{j,i}). \tag{104}$$

As in (104), we will in the following suppress the argument s of the Laplace-transformed incremental fields and relaxation functions. To make further progress with (104), it is necessary to specify $\bar{m}_\nu$ for $\nu = 1, 2$. This is achieved by expressing $m_\nu(t - t')$ in terms of its relaxation spectrum, $\hat{m}_\nu(\alpha)$. Laplace transformation then supplies a formula for $\bar{m}_\nu$, from which asymptotic approximations to $s\bar{m}_\nu$ for large and small s can be derived (Section 4.1). Substituting these approximations into (104) and applying the generalized initial- and final-value theorems, finally gives the asymptotes to the incremental constitutive equation of viscoelasticity, (71), for small and large t (Sections 4.2 and 4.3).

4.1 Asymptotic relaxation functions

For $\nu = 1, 2$, suppose that $m_\nu(t - t')$ can be expressed in terms of its relaxation spectrum, $\hat{m}_\nu(\alpha)$, (e.g. Christensen 1982, pp. 28–32; Golden & Graham 1988, pp. 31–32):

$$m_\nu(t - t') = m_\nu^{(f)} + \int_0^\infty \hat{m}_\nu(\alpha) e^{-\alpha(t-t')} \, d\alpha, \tag{105}$$

where α is the inverse constitutive relaxation time. Equation (105) implies

$$m_\nu^{(f)} = \lim_{t-t' \to \infty} m_\nu(t - t'), \tag{106}$$

hence, by (74), it follows that

$$m_2^{(f)} = 0. \tag{107}$$

We further define

$$m_\nu^{(e)} = m_\nu(0), \tag{108}$$

so that we get

$$\int_0^\infty \hat{m}_\nu(\alpha) \, d\alpha = m_\nu^{(e)} - m_\nu^{(f)}. \tag{109}$$

It follows from (73), (106), (108) and (109) that $\int_0^\infty \hat{m}_\nu(\alpha) \, d\alpha \geq 0$. Here, we impose the more stringent condition that $\hat{m}_\nu(\alpha) \geq 0$ for $\alpha \geq 0$; we further require that $\hat{m}_\nu(\alpha)$ vanishes sufficiently fast as $\alpha \to 0$ and $\alpha \to \infty$ that, for $0 < \alpha' < \infty$, the integrals $\int_0^{\alpha'} \hat{m}_\nu(\alpha)/\alpha \, d\alpha$ and $\int_{\alpha'}^\infty \alpha \hat{m}_\nu(\alpha) \, d\alpha$ converge. These assumptions are of sufficient generality to include conventional mechanical and molecular models of viscoelasticity.

Next, we apply (103) and obtain the s-multiplied Laplace transform of (105) with respect to $t - t'$:

$$s\bar{m}_\nu = m_\nu^{(f)} + \int_0^\infty \frac{s\hat{m}_\nu(\alpha)}{s + \alpha} \, d\alpha. \tag{110}$$

Being interested in asymptotic approximations to $s\bar{m}_\nu$ for large and small s, we decompose the integral in (110) in the following way:

$$\int_0^\infty \frac{s\hat{m}_\nu(\alpha)}{s + \alpha} \, d\alpha = \int_0^{s-0} \frac{\hat{m}_\nu(\alpha)}{1 + \alpha/s} \, d\alpha + \int_{s+0}^\infty \frac{s}{\alpha} \frac{\hat{m}_\nu(\alpha)}{1 + s/\alpha} \, d\alpha. \tag{111}$$

Note that, on the right-hand side, $\alpha/s < 1$ in the first integrand whereas $s/\alpha < 1$ in the second.

For sufficiently large s, the second integral on the right side of (111) may be neglected and, due to the convergence of $\int_{s+0}^\infty \alpha \hat{m}_\nu(\alpha) \, d\alpha$, we get the asymptotic approximation

$$\int_0^\infty \frac{s\hat{m}_\nu(\alpha)}{s + \alpha} \, d\alpha \simeq \int_0^\infty \hat{m}_\nu(\alpha) \left(1 - \frac{\alpha}{s}\right) d\alpha. \tag{112}$$

Since $\hat{m}_\nu(\alpha) \geq 0$ for $\alpha \geq 0$, we can apply the mean-value theorem of integral calculus to obtain the following estimate:

$$\int_0^\infty \alpha \hat{m}_\nu(\alpha) \, d\alpha = \alpha_\nu^{(e)} \int_0^\infty \hat{m}_\nu(\alpha) \, d\alpha, \tag{113}$$

where $\alpha_\nu^{(e)} \geq 0$. In view of (109), (112) and (113), equation (110) takes the form

$$s\bar{m}_\nu \simeq m_\nu^{(e)} - (m_\nu^{(e)} - m_\nu^{(f)}) \frac{\alpha_\nu^{(e)}}{s}, \tag{114}$$

which is correct to the first order in $\alpha_\nu^{(e)}/s$.

For sufficiently small s, the first integral on the right side of (111) may be neglected and, due to the convergence of

$\int_0^{s-0} \hat{m}_\nu(\alpha)/\alpha\, d\alpha$, we arrive at the asymptotic approximation

$$\int_0^\infty \frac{s\hat{m}_\nu(\alpha)}{s+\alpha}\, d\alpha \simeq \int_0^\infty \hat{m}_\nu(\alpha)\frac{s}{\alpha}\, d\alpha. \tag{115}$$

Applying the mean-value theorem of integral calculus again, we now have

$$\int_0^\infty \frac{\hat{m}_\nu(\alpha)}{\alpha}\, d\alpha = \frac{1}{\alpha_\nu^{(f)}}\int_0^\infty \hat{m}_\nu(\alpha)\, d\alpha, \tag{116}$$

where $\alpha_\nu^{(f)} \geq 0$. In view of (109), (115) and (116), equation (110) becomes

$$s\tilde{m}_\nu \simeq m_\nu^{(f)} + (m_\nu^{(e)} - m_\nu^{(f)})\frac{s}{\alpha_\nu^{(f)}}, \tag{117}$$

which is correct to the first order in $s/\alpha_\nu^{(e)}$.

In Sections 4.2 and 4.3, it will be helpful to have available compact forms of (114) and (117). To simplify (114) for $\nu = 1$, 2, we put

$$\kappa = m_1^{(e)}, \tag{118}$$

$$\kappa' = (m_1^{(e)} - m_1^{(f)})\alpha_1^{(e)}, \tag{119}$$

$$\mu = m_2^{(e)}, \tag{120}$$

$$\mu' = (m_2^{(e)} - m_2^{(f)})\alpha_2^{(e)}. \tag{121}$$

We obtain

$$\boxed{s\tilde{m}_1 \simeq \kappa - \frac{\kappa'}{s},} \tag{122}$$

$$\boxed{s\tilde{m}_2 \simeq \mu - \frac{\mu'}{s},} \tag{123}$$

which are asymptotically correct for large s. Similarly, we simplify (117) for $\nu = 1, 2$ using

$$k = m_1^{(f)}, \tag{124}$$

$$k' = \frac{m_1^{(e)} - m_1^{(f)}}{\alpha_1^{(f)}}, \tag{125}$$

$$\eta = m_2^{(f)}, \tag{126}$$

$$\eta' = \frac{m_2^{(e)} - m_2^{(f)}}{\alpha_2^{(f)}}. \tag{127}$$

Since $\eta = 0$ by (107), we now obtain

$$\boxed{s\tilde{m}_1 \simeq k + k's,} \tag{128}$$

$$\boxed{s\tilde{m}_2 \simeq \eta's,} \tag{129}$$

which are asymptotically correct for small s. Note that, from the properties of $m_\nu(t-t')$ specified in Section 3.3, it follows that k, k', η', κ, κ', μ and μ' are continuously differentiable, non-negative functions of $X_i \in V^{(T)}$ with possible jump discontinuities on $\mathcal{A}^{(T)}$.

4.2 Short-time asymptotic incremental field equations and continuity conditions

If s is sufficiently large, $s\tilde{m}_1$ and $s\tilde{m}_2$ may be approximated by (122) and (123). Substitution of these equations into (104) provides its asymptote for large s:

$$\tilde{t}_{ij}^{(\delta)} = (\kappa - \tfrac{2}{3}\mu)\hat{u}_{k,k}\delta_{ij} + \mu(\tilde{u}_{i,j} + \tilde{u}_{j,i}) - (\kappa' - \tfrac{2}{3}\mu')\frac{\tilde{u}_{k,k}}{s}\delta_{ij} - \mu'\frac{\tilde{u}_{i,j} + \tilde{u}_{j,i}}{s}. \tag{130}$$

In view of (100), inverse Laplace transformation of (130) gives

$$t_{ij}^{(\delta)} = (\kappa - \tfrac{2}{3}\mu)u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - (\kappa' - \tfrac{2}{3}\mu')\int_0^t u_{k,k}(t')\delta_{ij}\,dt' - \mu'\int_0^t [u_{i,j}(t') + u_{j,i}(t')]\,dt', \tag{131}$$

which, by the generalized initial-value theorem, is the asymptote to (71) for small t. Together, (69), (70) and (131) constitute the short-time asymptotes to the incremental field equations of viscoelastodynamics for $t_{ij}^{(\delta)}$, u_i and $\phi^{(\Delta)}$. These equations are complemented by the incremental continuity conditions (76), (78), (82) and (84) for $n_j^{(0)}t_{ij}^{(\delta)}$, u_i, $\phi^{(\Delta)}$ and $n_i^{(0)}\phi_{,i}^{(\Delta)}$. We refer to these asymptotic equations as the incremental field equations and continuity conditions of *generalized* elastodynamics.

If the integrals in (131) are neglected, it reduces to the incremental constitutive equation of elasticity with κ the *elastic* bulk modulus and μ the shear modulus. In this case, the short-time asymptotes to the incremental field equations and continuity conditions of viscoelastodynamics agree with the incremental field equations and continuity conditions of conventional elastodynamics (e.g. Love 1911, pp. 89–93; Dahlen 1974; Grafarend 1982).

4.3 Long-time asymptotic incremental field equations and continuity conditions

If s is sufficiently small, $s\tilde{m}_1$ and $s\tilde{m}_2$ may be approximated by (128) and (129). Upon substitution of these equations into (104), its asymptote for small s results:

$$\tilde{t}_{ij}^{(\delta)} = k\tilde{u}_{k,k}\delta_{ij} + (k' - \tfrac{2}{3}\eta')s\tilde{u}_{k,k}\delta_{ij} + \eta' s(\tilde{u}_{i,j} + \tilde{u}_{j,i}). \tag{132}$$

Considering (101) and $u_i^{(0)} = 0$, inverse Laplace transformation of (132) provides

$$t_{ij}^{(\delta)} = ku_{k,k}\delta_{ij} + (k' - \tfrac{2}{3}\eta')\,\partial_t u_{k,k}\delta_{ij} + \eta'\,\partial_t(u_{i,j} + u_{j,i}), \tag{133}$$

which, by the generalized final-value theorem, is the asymptote to (71) for large t. We can rewrite (133) by introducing $\pi^{(\delta)}$ explicitly into the equation. For that purpose, we recall that, in a fluid not necessarily in hydrostatic equilibrium, $\pi^{(\delta)}$ is related to the other state variables by the same function that relates $p^{(\delta)} = -t_{ii}^{(\delta)}/3$ to these variables in the case of hydrostatic equilibrium (e.g. Malvern 1969, p. 296). In view of this, (133) takes the form

$$t_{ij}^{(\delta)} = -\pi^{(\delta)}\delta_{ij} + (k' - \tfrac{2}{3}\eta')\,\partial_t u_{k,k}\delta_{ij} + \eta'\,\partial_t(u_{i,j} + u_{j,i}), \tag{134}$$

where

$$\pi^{(\delta)} = -ku_{i,i}. \tag{135}$$

Together, (69), (70), (134) and (135) constitute the long-time asymptotes to the incremental field equations of viscoelastodynamics for $t_{ij}^{(\delta)}$, u_i, $\pi^{(\delta)}$ and $\phi^{(\Delta)}$.

Equations (69), (70) and (134) will be identified with the incremental field equations of fluid dynamics if $t_{ij}^{(\delta)}$ and $\pi^{(\delta)}$ are expressed by the associated local increments and $\rho^{(\Delta)}$ is introduced explicitly into the equations. Using therefore (95), (96) and

$$t_{ij}^{(\delta)} = t_{ij}^{(\Delta)} - p_{,k}^{(0)}u_k\delta_{ij}, \tag{136}$$

which is a consequence of (27) and (41), the following equations result:

$$t_{ij,j}^{(\Delta)} + (\phi_{,i}^{(0)} + \psi_{,i}^{(0)} + f_i^{(0)})\rho^{(\Delta)} + \rho^{(0)}(\phi_{,i}^{(\Delta)} + f_i^{(\Delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_{tt}u_i, \tag{137}$$

$$\phi_{,ii}^{(\Delta)} = -4\pi G\rho^{(\Delta)}, \tag{138}$$

$$\rho^{(\Delta)} = -(\rho^{(0)}u_i)_{,i}, \tag{139}$$

$$t_{ij}^{(\Delta)} = -\pi^{(\Delta)}\delta_{ij} + (k' - \tfrac{2}{3}\eta')\,\partial_t u_{k,k}\delta_{ij} + \eta'\,\partial_t(u_{i,j} + u_{j,i}). \tag{140}$$

Equations (137)–(140) have the desired forms; in particular, (140) agrees with the incremental constitutive equation of Newtonian viscosity with k' the bulk viscosity modulus and η' the shear viscosity modulus.

To replace (135) by the associated incremental state equation in terms of $\pi^{(\Delta)}$, we need expressions for the partial derivatives in (98). By comparison between (92), (93) and (135), we find

$$\left(\frac{\partial\gamma}{\partial\rho}\right)^{(0)} = \frac{k}{\rho^{(0)}}, \tag{141}$$

whence k is referred to as the *fluid* bulk modulus. For the remaining partial derivatives, we put

$$\left(\frac{\partial \gamma}{\partial \lambda}\right)^{(0)} = \frac{\ell}{\lambda^{(0)}}, \tag{142}$$

$$\left(\frac{\partial \gamma}{\partial \sigma}\right)^{(0)} = \frac{c}{\sigma^{(0)}}, \tag{143}$$

where ℓ and c are the chemical and entropic moduli corresponding to k. Upon substitution of (141)–(143), equation (98) becomes

$$\boxed{\pi^{(\Delta)} = \frac{k}{\rho^{(0)}} \rho^{(\Delta)} - \frac{\ell}{\lambda^{(0)}} \lambda_{,i}^{(0)} u_i - \frac{c}{\sigma^{(0)}} \sigma_{,i}^{(0)} u_i.} \tag{144}$$

Equation (144) can be written in a more 'symmetric' form. To achieve that, we recall the supposition of isochemical and isentropic perturbations: $\lambda^{(\delta)} = \sigma^{(\delta)} = 0$ or, by (27),

$$\lambda^{(\Delta)} = -\lambda_{,i}^{(0)} u_i, \tag{145}$$

$$\sigma^{(\Delta)} = -\sigma_{,i}^{(0)} u_i. \tag{146}$$

Upon substitution of (145) and (146) into (144), it takes the form

$$\pi^{(\Delta)} = \frac{k}{\rho^{(0)}} \rho^{(\Delta)} + \frac{\ell}{\lambda^{(0)}} \lambda^{(\Delta)} + \frac{c}{\sigma^{(0)}} \sigma^{(\Delta)}. \tag{147}$$

This is the general form of the incremental state equation of a fluid whose total state equation is of the form of (86).

The incremental continuity conditions for u_i, $\phi^{(\Delta)}$ and $n_i^{(0)} \phi_{,i}^{(\Delta)}$ are given by (76), (78) and (82). The condition for $n_j^{(0)} t_{ij}^{(\Delta)}$ follows from (84) and (136):

$$\boxed{[n_j^{(0)} (t_{ij}^{(\Delta)} - p_{,k}^{(0)} u_k \delta_{ij})]_-^+ = 0.} \tag{148}$$

Using (44), we may express $[p_{,i}^{(0)}]_-^+$ in terms of $[\rho^{(0)}]_-^+$. In view of (53), it is then obvious that $n_j^{(0)} t_{ij}^{(\Delta)}$ is discontinuous normal to $\mathcal{A}^{(T)}$. The term $[n_i^{(0)} p_{,j}^{(0)} u_j]_-^+$ has sometimes been referred to as the 'buoyancy term.' Note that its appearance in (148) is solely a consequence of formulating the long-time asymptotic field equations in terms of the local incremental stress, $t_{ij}^{(\Delta)}$. In the short-time asymptotic field equations, where the material incremental stress, $t_{ij}^{(\delta)}$, is conventionally used, no buoyancy term therefore arises and the continuity condition is given by (84). Conversely, the short-time asymptotic momentum equation (69) contains the 'advective term' $(p_{,j}^{(0)} u_j)_{,i}$, which is absent from the long-time asymptotic momentum equation (137).

Equations (137)–(140) and (144) are the incremental field equations for a Newtonian-viscous fluid in terms of $t_{ij}^{(\Delta)}$, u_i, $\pi^{(\Delta)}$, $\rho^{(\Delta)}$ and $\phi^{(\Delta)}$; equations (76), (78), (82) and (148) are the associated incremental continuity conditions for $n_j^{(0)} t_{ij}^{(\Delta)}$, u_i, $\phi^{(\Delta)}$ and $n_i^{(0)} \phi_{,i}^{(\Delta)}$, (e.g. Backus 1967; Cathles 1975, pp. 15–20; Jarvis & Mackenzie 1980). For a chemically and entropically stratified, compressible fluid, the long-time asymptotes to the incremental field equations and continuity conditions of viscoelastodynamics therefore agree with the incremental field equations and continuity conditions of Newtonian-viscous fluid dynamics.

5 GENERALIZED INCOMPRESSIBILITY

In this section, we show how a pressure-induced initial density gradient can be practically accounted for in the incremental field equations of viscoelastodynamics. For that purpose, we introduce several simplifying assumptions.

Consider a fluid planet that is non-rotating and initially without volume forces:

$$\Omega_i = \psi_{,i}^{(0)} = f_i^{(0)} = 0. \tag{149}$$

Suppose, in addition, that the planet consists of isochemical and isentropic regions:

$$\lambda_{,i}^{(0)} = \sigma_{,i}^{(0)} = 0. \tag{150}$$

On these assumptions, the initial field equations (44)–(47) reduce to

$$-p_{,i}^{(0)} + \rho^{(0)} \phi_{,i}^{(0)} = 0, \tag{151}$$

$$\phi_{,ii}^{(0)} = -4\pi G \rho^{(0)}, \tag{152}$$

$$p^{(0)} = \gamma'(\rho^{(0)}). \tag{153}$$

Solutions to (151)–(153) can be shown to exist for the level surfaces of $p^{(0)}$, $\rho^{(0)}$ and $\phi^{(0)}$ being concentric spheres. To eliminate $p^{(0)}$, consider the gradient of (153):

$$p^{(0)}_{,i} = \left(\frac{d\gamma'}{d\rho}\right)^{(0)} \rho^{(0)}_{,i}, \tag{154}$$

where $(d\gamma'/d\rho)^{(0)}$ must be constant on the level surfaces. Comparing (151) with (154) yields

$$\left(\frac{d\gamma'}{d\rho}\right)^{(0)} \rho^{(0)}_{,i} = \rho^{(0)}\phi^{(0)}_{,i}, \tag{155}$$

which is the Williamson–Adams equation (Williamson & Adams 1923; Bullen 1975, pp. 67–68). With $(d\gamma'/d\rho)^{(0)}$ prescribed, (152) and (155) are to be solved for $\rho^{(0)}$ and $\phi^{(0)}$.

The incremental field equations (69)–(71) simplify if the perturbations are quasi-static:

$$\partial_{tt}u_i = 0, \tag{156}$$

and the bulk relaxation can be neglected:

$$m_1(t-t') = k, \tag{157}$$

where k is time independent. Using (149), (156) and (157) and introducing

$$\pi^{(\delta)} = -ku_{i,i}, \tag{158}$$

equations (69) and (71) reduce to

$$t^{(\delta)}_{ij,j} + (p^{(0)}_{,j}u_j)_{,i} - \phi^{(0)}_{,i}(\rho^{(0)}u_j)_{,j} + \rho^{(0)}(\phi^{(\Delta)}_{,i} + f^{(\Delta)}_i) = 0, \tag{159}$$

$$t^{(\delta)}_{ij} = -\pi^{(\delta)}\delta_{ij} - \frac{2}{3}\int_0^t m_2(t-t')\,\partial_{t'}[u_{k,k}(t')]\delta_{ij}\,dt' + \int_0^t m_2(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt'. \tag{160}$$

By means of (95) and (136), equations (159) and (160) can be rewritten in terms of $t^{(\Delta)}_{ij}$ and $\pi^{(\Delta)}$. Listing (70) again for easier reference, we thus have

$$t^{(\Delta)}_{ij,j} - \phi^{(0)}_{,i}(\rho^{(0)}u_j)_{,j} + \rho^{(0)}(\phi^{(\Delta)}_{,i} + f^{(\Delta)}_i) = 0, \tag{161}$$

$$\phi^{(\Delta)}_{,ii} = 4\pi G(\rho^{(0)}u_i)_{,i}, \tag{162}$$

$$t^{(\Delta)}_{ij} = -\pi^{(\Delta)}\delta_{ij} - \frac{2}{3}\int_0^t m_2(t-t')\,\partial_{t'}[u_{k,k}(t')]\delta_{ij}\,dt' + \int_0^t m_2(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt'. \tag{163}$$

To find the incremental state equation in terms of $\pi^{(\Delta)}$, we combine (92) and (93). Since $d\gamma'/d\rho = \partial\gamma/\partial\rho$, we obtain

$$\pi^{(\delta)} = -\left(\frac{d\gamma'}{d\rho}\right)^{(0)} \rho^{(0)}u_{i,i}. \tag{164}$$

Equating (158) to (164) provides

$$\left(\frac{d\gamma'}{d\rho}\right)^{(0)} = \frac{k}{\rho^{(0)}}. \tag{165}$$

From (95), (154), (164) and (165) it then follows that

$$\pi^{(\Delta)} = -\frac{k}{\rho^{(0)}}(\rho^{(0)}u_i)_{,i}. \tag{166}$$

Recalling that $p^{(\Delta)} = -t^{(\Delta)}_{ii}/3$ by definition, it follows from (163) that

$$\pi^{(\Delta)} = p^{(\Delta)}. \tag{167}$$

This identity will be implied in the remainder of this section.

An interesting approximation to the incremental field equations (161)–(163) and (166) results if (166) is replaced by the simultaneous conditions

$$k \to \infty, \tag{168}$$

$$(\rho^{(0)}u_i)_{,i} \to 0, \tag{169}$$

$$p^{(\Delta)} = \text{finite}. \tag{170}$$

The significance of (169) becomes obvious if we note that, by (94) and (96), the condition $(\rho^{(0)}u_i)_{,i}=0$ is equivalent to the conditions $\rho^{(\delta)}=\rho^{(0)}_{,i}u_i$ and $\rho^{(\Delta)}=0$. Equation (169) thus states that the compressibility of a displaced particle is restricted to the extent that the material incremental density 'follows' the prescribed initial density gradient such that the local incremental density vanishes. Equation (169) is referred to as the *generalized* incremental incompressibility condition.

With (155), (165) and (169), the incremental field equations for isochemical and isentropic regions can now be written as

$$t^{(\Delta)}_{ij,j}+\rho^{(0)}(\phi^{(\Delta)}_{,i}+f^{(\Delta)}_i)=0, \tag{171}$$

$$\phi^{(\Delta)}_{,ii}=0, \tag{172}$$

$$t^{(\Delta)}_{ij}=-p^{(\Delta)}\delta_{ij}+\frac{2\rho^{(0)}\phi^{(0)}_{,k}}{3k}\int_0^t m_2(t-t')\,\partial_{t'}[u_k(t')]\delta_{ij}\,dt'+\int_0^t m_2(t-t')\,\partial_{t'}[u_{i,j}(t')+u_{j,i}(t')]\,dt', \tag{173}$$

$$u_{i,i}=-\frac{\rho^{(0)}\phi^{(0)}_{,i}}{k}u_i. \tag{174}$$

Equations (171)–(174) are to be solved for $p^{(\Delta)}$, $t^{(\Delta)}_{ij}$, u_i and $\phi^{(\Delta)}$, where $\rho^{(0)}$ and $\phi^{(0)}$ must satisfy (152) and (155) subject to (165).

We note that k remains finite in (173) and (174). This is because k enters into these equations in consequence of substituting (165) into the *initial* field equation (155), for which the approximation (168) does not apply. If the initial state is also taken as incompressible, then $k\rightarrow\infty$ also in (173) and (174). In this case, $\rho^{(0)}$ must be homogeneous and (171)–(174) reduce to the well-known incremental field equations describing incompressible gravitational–viscoelastic perturbations of homogeneous regions.

If, on the other hand, $\phi^{(0)}$ is prescribed and the term $\rho^{(0)}\phi^{(\Delta)}_{,i}$ in (171) neglected, the mechanical and gravitational effects decouple. Then, solutions for the initial state are readily found and the resulting equations for the incremental state become similar to those recently used by Li & Yuen (1987). In contrast to here, Li & Yuen however restricted their attention to viscous perturbations; further, they did not discuss the approximations involved in the equations used in their study. As shown here, their theory can be extended to include any type of viscoelastic constitutive behaviour.

6 CONCLUSIONS

We summarize the results of our study in the following five statements.

(i) Postulating only the fundamental principles of continuum mechanics in differential form, we have given a concise derivation, in the Lagrangian formulation, of the incremental field equations and continuity conditions describing gravitational–viscoelastic perturbations of a chemically and entropically stratified, compressible, rotating fluid planet initially in hydrostatic equilibrium. In obtaining the equations, minor deficiencies of previous such derivations for gravitational–elastic perturbations (e.g. Dahlen 1974) have been corrected.

(ii) The incremental momentum equation deduced agrees with the incremental momentum equation given by Love (1911, pp. 89–93) or that given by Dahlen (1974). In view of the rigour of our deduction, Geller's (1988a) qualification of this equation as 'fortuitously' correct cannot be supported. Therefore, no restrictions on the derivation of the incremental momentum equation from the continuum form of Newton's second law exist for the case of a hydrostatic initial state considered. Similarly, no such restrictions can be reasonably conceived for the more general case of a non-hydrostatic initial state.

(iii) We have obtained, as the short-time asymptotes to the incremental field equations and continuity conditions of viscoelastodynamics, a set of equations referred to as the incremental field equations and continuity conditions of generalized elastodynamics. These equations can be reduced to the equations of conventional elastodynamics. The long-time asymptotes agree with the incremental field equations and continuity conditions of Newtonian-viscous fluid dynamics. The asymptotic behaviour of the equations of viscoelastodynamics is an additional manifestation of their internal consistency and the modifications to the equations suggested by Svensson (1980) are therefore not justified.

(iv) We have further shown that the incremental thermodynamic pressure appearing in the long-time asymptote to the incremental constitutive equation of viscoelasticity satisfies the correct incremental state equation. This confirms that this asymptote indeed represents the incremental constitutive equation of Newtonian viscosity. At the same time, it disproves claims by Svensson (1980) and Fjeldskaar & Cathles (1984), who argued that the formulation of the incremental field theory of viscoelastodynamics developed by Peltier (1974) and others cannot be adapted to describe perturbations of a compressible planet, because it does not provide for an incremental state equation.

(v) Finally, we have introduced the generalized incremental incompressibility condition. Based on it, we have derived approximate incremental field equations describing gravitational–viscoelastic perturbations of isochemical, isentropic and compressible regions. Solutions to these equations are currently studied and will be compared with corresponding solutions for chemically and entropically stratified, incompressible regions.

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