Gravitational Viscoelastodynamics for a Hydrostatic Planet

Habilitationsschrift
zur Erlangung der Venia legendi
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität zu Münster

vorgelegt von
Detlef Karl-Heinz Wolf
Münster, März 1993

Preface

The history of the research reported in this treatise extends back to my period of work on glacial isostasy at the Department of Physics, University of Toronto (1982–1985), when the need of an improved theoretical foundation for modelling the earth's viscoelastic response became apparent to me first. Preliminary attempts of developing a rigorous and complete theory of gravitational viscoelastodynamics for hydrostatic planets were undertaken during my association with the Geological Survey of Canada, Ottawa (1986–1987). These studies were followed up in earnest while being at the Institute of Planetology, University of Münster, since 1988.

Instrumental to the successful completion of the investigation has been the informal and research-oriented atmosphere characterizing the Institute's planetary physics group headed by T. Spohn. At the same time, my progress has markedly benefitted from a number of research visits at the following institutions: (i) Department of Geophysics, University of Uppsala; (ii) Department of Geological Sciences, Queen's University, Kingston; (iii) Department of Geology and Geophysics, University of Calgary; (iv) Geological Survey of Canada, Ottawa; and (v) Pacific Geoscience Centre of Canada, Sidney. The visits were funded by the German Research Society (DFG), the Geological Survey of Canada (GSC), the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Swedish Natural Science Research Council (NFR), whose support is gratefully acknowledged.

During the different stages of the investigation, profound discussions with F. Amelung, E. W. Grafarend, G. Rümpker and P. Wu have been particularly valuable to me. Apart from that, I have received critical comments on special aspects of the study from several colleagues, most notably F. A. Dahlen, P. Gasperini, R. J. Geller, P. Johnston, R. Sabadini, H.-G. Scherneck, C. J. Thomson and D. A. Yuen. The usually difficult access to the older literature was made simple through the steady assistance provided by J. C. Levesque and L. P. Simpson.

Partial results of this study are published as separate articles: (i) Viscoelastodynamics of a stratified, compressible planet: incremental field equations and short- and long-time asymptotes, *Geophys. J. Int.* (1991) **104**, 401–417; (ii) Boussinesq's problem of viscoelasticity, *Terra Nova* (1991) **3**, 401–407; and (iii) Lamé's problem of gravitational viscoelasticity: the isochemical, incompressible planet, *Geophys. J. Int.* (1993) submitted.

Münster, February 1993 Detlef Wolf

Contents

Chapter 1

Introduction

The response of planetary bodies to perturbations with periods of less than several days is conventionally studied assuming elastic constitutive behaviour. The defining feature of elastic solids is the proportionality between stress and strain. The phenomena governed by the associated field theory of elastodynamics can be classified into the propagation of seismic waves, free oscillations, and short-period body and load tides. For the latter two classes, it is necessary to account also for gravity and the presence of initial stress in planetary interiors. This is accomplished by the field theory of *gravitational elastodynamics* (GED). The development of GED has progressed steadily since the pioneering studies of Love (1911) and is reviewed in a number of monographs covering various aspects of the theory and its applications (e. g. Melchior, 1966; Lapwood & Usami, 1981; Lanzano, 1982).

For perturbations of essentially infinite period, planetary bodies can be modelled as hydrostatic fluids; the phenomena explained by the governing field theory of *gravitational hydrostatics* (GHS) are the figures of the planets. The calculation of the level surfaces of gravitating and rotating fluids forms one of the classical branches of geodesy and geophysics and can be traced back to the studies of Clairaut (1743). Modern monographs covering the subject include those by Kopal (1960), Chandrasekhar (1969) and Moritz (1990).

If the perturbation period is in the range of about 1–100 million years, the finite viscosity of the materials forming the planets must normally be taken into account. Conventionally, the stress is taken as proportional to the strain rate; the complete field theory including effects due to gravitation will be called *gravitational viscodynamics* (GVD). The most important planetary phenomena explained by GVD are those known as chemical or thermal convection. A characteristic difference to the phenomena governed by elastic or hydrostatic constitutive behaviour is that convection involves instability, resulting in perturbations of arbitrarily large amplitude. Whereas the onset of instability is amenable to mathematical analysis, the fully developed circulative motion can only be simulated using numerical methods. Stability analyses extend back to the researches of Oberbeck (1879, 1880), Rayleigh (1916) and Jeffreys (1926, 1928); a comprehensive review of a large portion of the work completed has been given by Chandrasekhar (1961). Numerical techniques for modelling convection in planetary interiors have been developed much later. An account of the methods involved is given by Jarvis & McKenzie (1980); for an overview of recent results, Peltier (1989) may be consulted.

For the wide range of perturbation periods between days and millions of years, it is necessary to model planets as viscoelastic fluids. The main classes of phenomena governed by viscoelasticity are long-period body and load tides, the dampening of free nutations and isostatic adjustment processes. An important feature of viscoelastic fluids is their limiting behaviour: if the perturbation period becomes small, their response approaches that of elastic solids while, for sufficiently long periods, they can approximately be described as viscous fluids. In the intervening period range, the viscoelastic constitutive behaviour must be taken into account. This can be formulated mathematically by means of relaxation functionals. Physically, such functionals express the stress at a given position and time in terms of the complete strain history at this position.

Compared with elastodynamics, hydrostatics or viscodynamics, the development of the more general field theory of viscoelastodynamics is recent. A further difference is that this development has been strongly influenced by attempts to construct an axiomatic field theory describing the constitutive behaviour of any type of material and by the necessities of applied materials science for a sound theoretical foundation. Comprehensive monographs of non-linear field theory that encompass viscoelastic constitutive behaviour are those by Eringen (1962), Truesdell & Noll (1965[1], 1992[2]) and Malvern (1969); useful treatises that emphasize viscoelasticity and cover also recent developments include Christensen (1982[2]) and Golden & Graham (1988).

Applications of viscoelastodynamics to the deformation of planetary bodies have been impeded by continuing uncertainities regarding the proper form of the relaxation functional to be adopted. A further difficulty is related to the presence of initial stress in planetary interiors. The modifications to the ordinary theory of viscoelastodynamics necessary to account for initial stress have been carried out by Biot (1965). On the basis of the theory developed by him, Biot also studied a number of geophysical problems but, as a whole, his theory has received little attention from geophysical researchers.

A new development started with the application of viscoelastodynamics to the study of the glacial–isostatic adjustment processes following the retreat of the Pleistocene ice-sheets in Canada and Fennoscandia. The original research on this problem was largely completed by Cathles and Peltier, who stated the theory for the case of hydrostatic initial stress, Maxwell viscoelasticity and gravitation (Peltier, 1974; Cathles, 1975; Peltier, 1982; Wu & Peltier, 1982). Whereas the form of *gravitational viscoelastodynamics* (GVED) proposed by Cathles has a number of deficiencies, the formulation given by Peltier is essentially correct. However, several questions were left unanswered by him, most important among these being (i) the significance of initial stress and (ii) the relation of the field theory of GVED to those of GED and GVD. This has resulted in criticism regarding the consistency of the field equations of GVED as given by Peltier (e. g. Fjeldskaar & Cathles, 1984) and in uncertainty regarding the correct physical interpretation of the field quantities involved (e. g. Nakiboglu & Lambeck, 1982; Wolf, 1985a, b). The principal objective of the present investigation is to clarify these uncertainties by (i) developing the linear field theory of GVED from first principles and stating it in general, complete and rigorous form, (ii) testing its consistency and thoroughly discussing its physical interpretation, and (iii) deducing from it the explicit solutions of the viscoelastic generalizations of two classical problems relevant to surface loading of planets.

We begin with a collection of several basic concepts and conventions in Chapter 2. Most important of these are the development of equivalent kinematic formulations and, based on this, the derivation of different forms of the fundamental perturbation equation used throughout this study. Chapter 3 starts with the rigorous deduction, from the principles of continuum mechanics and potential theory, of the incremental field equations and interface conditions of GVED. The consistency of the incremental equations is confirmed by showing that their short- and long-time asymptotes agree with the corresponding equations of GED and GVD, respectively. After that, sets of incremental field equations and interface conditions valid for incompressible perturbations and subject to additional restrictions are derived. As an elementary application of the specialized theory, the viscoelastic generalization of the Boussinesq–Cerruti problem is considered in Chapter 4. Besides solving the problem in explicit form, we discuss the physical interpretation of the solution functions. The latter serves to clarify the significance of initial stress and to illustrate the relation of viscoelastodynamics to elastodynamics and viscodynamics from a different point of view. Chapter 5 is concerned with the deduction of the solution to the Lamé–Kelvin problem of GVED in complete and explicit form. The solution method used differs from the standard procedure followed for this type of problem. In addition, a complete catalogue of solution functions covering all field quantities of interest in applications is provided.

We note at this point that each of the following chapters is introduced by a detailed review of the topic to be studied followed by a brief overview of the individual sections of the chapter, with the main results summarized in the final section.

Throughout this treatise, we adhere to a uniform notation. A list of the important symbols and the section or paragraph numbers of first reference to them is given in Appendix C. For economy, we continue to employ the abbreviations GVED (gravitational viscoelastodynamics), GED (gravitational elastodynamics), GVD (gravitational viscodynamics) and GHS (gravitational hydrostatics).

Chapter 2

Basic concepts

2.1 Introduction

In this preparatory chapter, we collect the basic concepts used throughout this study. These are the kinematic formulations underlying continuum mechanics (e. g. Malvern, 1969, pp. 138–145) and some essentials of perturbation theory (e. g. Krauss, 1973, pp. 77–88). The concepts have been discussed in a geophysical or geodetic context by Dahlen (1972, 1973, 1974), Grafarend (1982) and others. Here, we adapt and extend their formulations in order to meet our particular requirements. Accordingly, we introduce the *Lagrangian*, *Newtonian* and *Eulerian* kinematic formulations of field quantities (Section 2.2) and specify their *material*, *isopotential* and *local* increments in terms of different forms of the fundamental perturbation equation (Section 2.3). Also, several notational conventions to be used when stating interface conditions are collected (Section 2.4). Finally, the key points are briefly summarized (Section 2.5).

We restrict our study to Cartesian-tensor fields and employ for them the indicial notation and the summation convention: index subscripts i, j, ... run over $\{1,2,3\}$, respectively, and index subscripts repeated in any term imply summation over this range. As usual, δ_{ij} is the Kronecker symbol and ϵ_{ijk} the Levi–Civita symbol.

2.2 Kinematic formulations

Let $\mathcal{E}$ be the unbounded 3-D Euclidian space-domain, $\mathcal{T}$ the time-domain $[0,\infty)$ and consider the mapping

$$r_i = r_i(\mathbf{X},t), \quad X_i \in \mathcal{E}, \quad t \in \mathcal{T}. \tag{2.1}$$

We assume that $r_i(\mathbf{X},t)$ is a one-to-one mapping of $\mathcal{E}$ onto itself with the property $r_i(\mathbf{X},0) = X_i$ and is continuously differentiable with respect to X_i and t as many times as required. Henceforth, t will be called current time, r_i current position, $t=0$ initial time and X_i initial position.

Suppose now that $\mathcal{E}$ is filled by a gravitating material continuum. As a particular mapping of the form of (2.1) satisfying our assumptions, then consider

$$r_i^{(L)} = r_i^{(L)}(\mathbf{X},t) = X_i + u_i(\mathbf{X},t), \quad X_i \in \mathcal{E}, \quad t \in \mathcal{T}. \tag{2.2}$$

This is the kinematic formulation to be used for *material* points (particles). It identifies each material point by its initial position, X_i, and relates to the point its current position, $r_i^{(L)}$, in terms of the material displacement, u_i, from its initial position.

In the course of this study, the concept of *isopotential* points will prove useful. We define such points as follows: An isopotential point is a point which can only move in the direction of the gradient of the gravitational potential currently existing at its position such that the potential at the point remains constant while the latter is being displaced. A second mapping satisfying our assumptions then is

$$r_i^{(N)} = r_i^{(N)}(\mathbf{X},t) = X_i + d_i(\mathbf{X},t), \quad X_i \in \mathcal{E}, \quad t \in \mathcal{T}, \tag{2.3}$$

which is the kinematic formulation to be used for isopotential points. It identifies each isopotential point by its initial position, X_i, and relates to the point its current position, $r_i^{(N)}$, in terms of the isopotential displacement, d_i, from its initial position.

The inverse mappings to (2.2) and (2.3) are

$$\left.\begin{aligned} X_i^{(L)} &= X_i^{(L)}(\mathbf{r},t) = r_i - U_i(\mathbf{r},t) \quad (2.4)\\ X_i^{(N)} &= X_i^{(N)}(\mathbf{r},t) = r_i - D_i(\mathbf{r},t) \quad (2.5)\end{aligned}\right\} \quad r_i \in \mathcal{E}, \quad t \in \mathcal{T}.$$

In contrast to (2.2) and (2.3), equations (2.4) and (2.5), refer to *local* points (places) identified by their position, r_i. In particular, (2.4) relates to each local point the initial position, $X_i^{(L)}$, of the material point currently at r_i by means of the material displacement, U_i. Similarly, (2.5) relates to each local point the initial position, $X_i^{(N)}$, of the isopotential point currently at r_i by means of the isopotential displacement, D_i. Considering the general assumptions to be satisfied by the mappings $r_i^{(L)}(\mathbf{X},t)$ and $r_i^{(N)}(\mathbf{X},t)$, (2.4) and (2.5) define one-to-one mappings that are continuously differentiable with respect to r_i and t as many times as required.

We proceed by specifying the domains of definition of the mappings (2.2)–(2.5) more closely. We begin with (2.2) and decompose $\mathcal{E}$ into two open subdomains: the simply connected *internal* domain, $\mathcal{X}_-^{(L)}$, and the complementary *external* domain, $\mathcal{X}_+^{(L)}$. With $\partial\mathcal{X}^{(L)}$ the 2-D interface between the two domains, it then follows that $\mathcal{E} = \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)} \cup \partial\mathcal{X}^{(L)}$. We now define

$$\mathcal{R}_-^{(L)}(t) = \{r_i^{(L)}(\mathbf{X},t) \,|\, X_i \in \mathcal{X}_-^{(L)}, \, t \in \mathcal{T}\}, \tag{2.6}$$

$$\mathcal{R}_+^{(L)}(t) = \{r_i^{(L)}(\mathbf{X},t) \,|\, X_i \in \mathcal{X}_+^{(L)}, \, t \in \mathcal{T}\}, \tag{2.7}$$

$$\partial\mathcal{R}^{(L)}(t) = \{r_i^{(L)}(\mathbf{X},t) \,|\, X_i \in \partial\mathcal{X}^{(L)}, \, t \in \mathcal{T}\}. \tag{2.8}$$

In view of the physical interpretation of (2.2), $\mathcal{R}_-^{(L)}(t)$, $\mathcal{R}_+^{(L)}(t)$ and $\partial\mathcal{R}^{(L)}(t)$ are the current domains occupied by those material points initially in $\mathcal{X}_-^{(L)}$, in $\mathcal{X}_+^{(L)}$ and on $\partial\mathcal{X}^{(L)}$, respectively. For the present investigation, we suppose that $\mathcal{R}_-^{(L)}(t)$ and $\mathcal{R}_+^{(L)}(t)$ are domains of ***continuity*** for the characteristic parameters of the material and that $\partial\mathcal{R}^{(L)}(t)$ is an interface of ***discontinuity*** for these parameters. We therefore take $\mathcal{R}_-^{(L)}(t) = \mathcal{R}_-^{(N)}(t), \mathcal{R}_+^{(L)}(t) = \mathcal{R}_+^{(N)}(t), \partial\mathcal{R}^{(L)}(t) = \partial\mathcal{R}^{(N)}(t)$ and define

$$\mathcal{X}_-^{(N)}(t) = \{X_i^{(N)}(\mathbf{r},t) \,|\, r_i \in \mathcal{R}_-^{(N)}(t), \, t \in \mathcal{T}\}, \tag{2.9}$$

$$\mathcal{X}_+^{(N)}(t) = \{X_i^{(N)}(\mathbf{r},t) \,|\, r_i \in \mathcal{R}_+^{(N)}(t), \, t \in \mathcal{T}\}, \tag{2.10}$$

$$\partial\mathcal{X}^{(N)}(t) = \{X_i^{(N)}(\mathbf{r},t) \,|\, r_i \in \partial\mathcal{R}^{(N)}(t), \, t \in \mathcal{T}\}. \tag{2.11}$$

In view of the physical interpretation of (2.5), $\mathcal{X}_-^{(N)}(t)$, $\mathcal{X}_+^{(N)}(t)$ and $\partial\mathcal{X}^{(N)}(t)$ are the initial domains occupied by those isopotential points currently in $\mathcal{R}_-^{(N)}(t)$, in $\mathcal{R}_+^{(N)}(t)$ and on $\partial\mathcal{R}^{(N)}(t)$, respectively. Owing to $\mathcal{R}_-^{(L)}(t) = \mathcal{R}_-^{(N)}(t)$, $\mathcal{R}_+^{(L)}(t) = \mathcal{R}_+^{(N)}(t)$ and $\partial\mathcal{R}^{(L)}(t) = \partial\mathcal{R}^{(N)}(t)$, no distinction is required between the domains and the symbols $\mathcal{R}_-(t)$, $\mathcal{R}_+(t)$ and $\partial\mathcal{R}(t)$ can be used.

Next, we give formulations equivalent to (2.2)–(2.5) for arbitrary field quantities. Since we wish to allow for the possibility that the values of such fields or their gradients are discontinuous on $\partial\mathcal{R}(t)$, all material, isopotential and local points currently on this interface are excluded. We thus disregard material points for which $X_i \in \partial\mathcal{X}^{(L)}$, isopotential points for which $X_i \in \partial\mathcal{X}^{(N)}(t)$ and local points for which $r_i \in \partial\mathcal{R}(t)$. With this, the generalizations to (2.2) and (2.3) are

$$f_{ij\ldots}^{(L)} = f_{ij\ldots}^{(L)}(\mathbf{X},t), \quad X_i \in \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)}, \quad t \in \mathcal{T}, \tag{2.12}$$

$$f_{ij\ldots}^{(N)} = f_{ij\ldots}^{(N)}(\mathbf{X},t), \quad X_i \in \mathcal{X}_-^{(N)}(t) \cup \mathcal{X}_+^{(N)}(t), \quad t \in \mathcal{T}. \tag{2.13}$$

The quantity $f_{ij\ldots}^{(L)}$ in (2.12) is the current value of an arbitrary field at the material point whose initial position is X_i. Similarly, $f_{ij\ldots}^{(N)}$ in (2.13) is the current value of that field at the isopotential point whose initial position is X_i. Equation (2.12) is commonly referred to as the *Lagrangian* formulation of the field. Equation (2.13) is non-conventional and will be referred to as the *Newtonian* formulation of that field. The generalization to (2.4) and (2.5) is

$$F_{ij\ldots} = F_{ij\ldots}(\mathbf{r},t), \quad r_i \in \mathcal{R}_-(t) \cup \mathcal{R}_+(t), \quad t \in \mathcal{T}. \tag{2.14}$$

This equation relates to each local point identified by its position, r_i, the current value, $F_{ij\ldots}$, of an arbitrary field at this point. Equation (2.14) is commonly called the *Eulerian* formulation of the field. The mappings defined in (2.12)–(2.14) are assumed to be single-valued and continuously differentiable with respect to X_i, r_i and t as many times as required. Provided the three formulations of the field are defined, they are related by

$$f_{ij\ldots}^{(L)}(\mathbf{X},t) = F_{ij\ldots}[\mathbf{r}^{(L)}(\mathbf{X},t),t], \tag{2.15}$$

$$f_{ij\ldots}^{(N)}(\mathbf{X},t) = F_{ij\ldots}[\mathbf{r}^{(N)}(\mathbf{X},t),t], \tag{2.16}$$

$$F_{ij\ldots}(\mathbf{r},t) = f_{ij\ldots}^{(L)}[\mathbf{X}^{(L)}(\mathbf{r},t),t], \tag{2.17}$$

$$F_{ij\ldots}(\mathbf{r},t) = f_{ij\ldots}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},t),t]. \tag{2.18}$$

As in the preceding equations, we continue to use lower-case symbols for the Lagrangian and Newtonian formulations of fields and upper-case symbols for the Eulerian formulation. The Lagrangian and Newtonian formulations are distinguished by the label superscripts L and N.

The gradients of field quantities in the kinematic formulations considered will be indicated by means of the following abbreviations:

$$f_{ij\ldots,k}^{(L)}(\mathbf{X},t) = \frac{\partial}{\partial X_k} f_{ij\ldots}^{(L)}(\mathbf{X},t), \tag{2.19}$$

$$f_{ij\ldots,k}^{(N)}(\mathbf{X},t) = \frac{\partial}{\partial X_k} f_{ij\ldots}^{(N)}(\mathbf{X},t), \tag{2.20}$$

$$F_{ij\ldots,k}(\mathbf{r},t) = \frac{\partial}{\partial r_k} F_{ij\ldots}(\mathbf{r},t). \tag{2.21}$$

Note that no ambiguity can arise from the notation. This is because a subscript preceded by a comma is to be understood as indicating the gradient with respect to the spatial argument of the field in the formulation considered, i. e. with respect to X_i for $f^{(L)}_{ij...}$ and $f^{(N)}_{ij...}$ and with respect to r_i for $F_{ij...}$. In view of (2.19)–(2.21), differentiation of (2.15)–(2.18) yields

$$f^{(L)}_{ij...,k}(\mathbf{X},t) = F_{ij...,\ell}[\mathbf{r}^{(L)}(\mathbf{X},t),t]\, r^{(L)}_{\ell,k}(\mathbf{X},t), \tag{2.22}$$
$$f^{(N)}_{ij...,k}(\mathbf{X},t) = F_{ij...,\ell}[\mathbf{r}^{(N)}(\mathbf{X},t),t]\, r^{(N)}_{\ell,k}(\mathbf{X},t), \tag{2.23}$$
$$F_{ij...,k}(\mathbf{r},t) = f^{(L)}_{ij...,\ell}[\mathbf{X}^{(L)}(\mathbf{r},t),t]\, X^{(L)}_{\ell,k}(\mathbf{r},t), \tag{2.24}$$
$$F_{ij...,k}(\mathbf{r},t) = f^{(N)}_{ij...,\ell}[\mathbf{X}^{(N)}(\mathbf{r},t),t]\, X^{(N)}_{\ell,k}(\mathbf{r},t). \tag{2.25}$$

2.3 Total, initial and incremental fields

We now assume that the current value of an arbitrary field represents a perturbation of its initial value. Allowing for discontinuities of the field values on $\partial\mathcal{R}(t)$, the Newtonian and Eulerian formulations of the perturbation equation are then straightforward only for isopotential and local points that are initially in $\mathcal{R}_-(0)$ and currently in $\mathcal{R}_-(t)$ or that are initially in $\mathcal{R}_+(0)$ and currently in $\mathcal{R}_+(t)$. We call such points strictly internal or strictly external. For conciseness, we define

$$\mathcal{X}^{(N)}_{\ominus}(t) = \mathcal{X}^{(N)}_-(0) \cap \mathcal{X}^{(N)}_-(t), \tag{2.26}$$
$$\mathcal{X}^{(N)}_{\oplus}(t) = \mathcal{X}^{(N)}_+(0) \cap \mathcal{X}^{(N)}_+(t), \tag{2.27}$$
$$\mathcal{R}_{\ominus}(t) = \mathcal{R}_-(0) \cap \mathcal{R}_-(t), \tag{2.28}$$
$$\mathcal{R}_{\oplus}(t) = \mathcal{R}_+(0) \cap \mathcal{R}_+(t). \tag{2.29}$$

On account of (2.26)–(2.29), the necessary and sufficient conditions for strictly internal or external isopotential points and for strictly internal or external local points are therefore $X_i \in \mathcal{X}^{(N)}_{\ominus}(t) \cup \mathcal{X}^{(N)}_{\oplus}(t)$ and $r_i \in \mathcal{R}_{\ominus}(t) \cup \mathcal{R}_{\oplus}(t)$, respectively.

The Lagrangian, Newtonian and Eulerian formulations of the fundamental perturbation equation can now be stated as follows:

$$\boxed{\begin{aligned}
f^{(L)}_{ij...}(\mathbf{X},t) &= f^{(L)}_{ij...}(\mathbf{X},0) + \delta f^{(L)}_{ij...}(\mathbf{X},t), \quad X_i \in \mathcal{X}^{(L)}_- \cup \mathcal{X}^{(L)}_+, \quad t \in \mathcal{T}, && (2.30)\\
f^{(N)}_{ij...}(\mathbf{X},t) &= f^{(N)}_{ij...}(\mathbf{X},0) + \partial f^{(N)}_{ij...}(\mathbf{X},t), \quad X_i \in \mathcal{X}^{(N)}_{\ominus}(t) \cup \mathcal{X}^{(N)}_{\oplus}(t), \quad t \in \mathcal{T}, && (2.31)\\
F_{ij...}(\mathbf{r},t) &= F_{ij...}(\mathbf{r},0) + \Delta F_{ij...}(\mathbf{r},t), \quad r_i \in \mathcal{R}_{\ominus}(t) \cup \mathcal{R}_{\oplus}(t), \quad t \in \mathcal{T}. && (2.32)
\end{aligned}}$$

We refer to the left-hand sides of the equations as *total* fields, to the first terms on the right-hand sides as the *initial* fields and to the second terms on the right-hand sides as the *incremental* fields. In particular, $\delta f^{(L)}_{ij...}(\mathbf{X},t)$ is called *material* increment, $\partial f^{(N)}_{ij...}(\mathbf{X},t)$ the *isopotential* increment and $\Delta F_{ij...}(\mathbf{r},t)$ the *local* increment. Since $X_i = r^{(L)}_i(\mathbf{X},0) = r^{(N)}_i(\mathbf{X},0)$ and $r_i = X^{(L)}_i(\mathbf{r},0) = X^{(N)}_i(\mathbf{r},0)$, it follows by comparison between (2.2)–(2.5) and (2.30)–(2.32) that

$$\delta r^{(L)}_i(\mathbf{X},t) = u_i(\mathbf{X},t), \tag{2.33}$$
$$\partial r^{(N)}_i(\mathbf{X},t) = d_i(\mathbf{X},t), \tag{2.34}$$

$$\Delta X_i^{(L)}(\mathbf{r},t) = -U_i(\mathbf{r},t), \tag{2.35}$$
$$\Delta X_i^{(N)}(\mathbf{r},t) = -D_i(\mathbf{r},t). \tag{2.36}$$

Equations (2.30)–(2.36) are valid for *finite* perturbations of arbitrarily large amplitude. The remainder of this chapter, however, will be restricted to *infinitesimal* perturbations, for which the increments and their gradients are taken as 'infinitely small'.

In some neighbourhood of $\partial\mathcal{R}(t)$, isopotential and local points are initially in $\mathcal{R}_-(0)$ and currently in $\mathcal{R}_+(t)$ or vice versa. Since the field values are not necessarily continuous on $\partial\mathcal{R}(t)$, such hybrid points require special consideration. In order that this be avoided, we need the Lagrangian forms of (2.31) and (2.32). Beginning with (2.31), we derive, as the first step, its Eulerian form. Putting $X_i = X_i^{(N)}(\mathbf{r},t)$, the equation becomes

$$f_{ij...}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},t),t] = f_{ij...}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},t),0] + \partial f_{ij...}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},t),t],$$
$$r_i \in \mathcal{R}_\ominus(t) \cup \mathcal{R}_\oplus(t), \quad t \in \mathcal{T}, \tag{2.37}$$

where, on account of (2.5) and the assumption of infinitesimal perturbations, the first term on the right-hand side can be expanded into

$$f_{ij...}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},t),0] = f_{ij...}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},0),0] - f_{ij...,k}^{(N)}[\mathbf{X}^{(N)}(\mathbf{r},0),0]\, D_k(\mathbf{r},t). \tag{2.38}$$

Substitution of this into (2.37) and use of (2.18), (2.25) and $X_{i,j}^{(N)}(\mathbf{r},0) = \delta_{ij}$ then gives

$$F_{ij...}(\mathbf{r},t) = F_{ij...}(\mathbf{r},0) + \partial F_{ij...}(\mathbf{r},t) - F_{ij...,k}(\mathbf{r},0)\, D_k(\mathbf{r},t),$$
$$r_i \in \mathcal{R}_\ominus(t) \cup \mathcal{R}_\oplus(t), \quad t \in \mathcal{T}. \tag{2.39}$$

The next step is to find the Lagrangian form of (2.39). Putting $r_i = r_i^{(L)}(\mathbf{X},t)$, the equation becomes

$$F_{ij...}[\mathbf{r}^{(L)}(\mathbf{X},t),t] = F_{ij...}[\mathbf{r}^{(L)}(\mathbf{X},t),0] + \partial F_{ij...}[\mathbf{r}^{(L)}(\mathbf{X},t),t]$$
$$- F_{ij...,k}[\mathbf{r}^{(L)}(\mathbf{X},t),0]\, D_k[\mathbf{r}^{(L)}(\mathbf{X},t),t],$$
$$X_i \in \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)}, \quad t \in \mathcal{T}. \tag{2.40}$$

In view of (2.2), (2.15) and the assumption of infinitesimal perturbations, the first and third terms on the right-hand side of (2.40) can be expanded into

$$F_{ij...}[\mathbf{r}^{(L)}(\mathbf{X},t),0] = F_{ij...}[\mathbf{r}^{(L)}(\mathbf{X},0),0] + F_{ij...,k}[\mathbf{r}^{(L)}(\mathbf{X},0),0]\, u_k(\mathbf{X},t), \tag{2.41}$$

$$F_{ij...,k}[\mathbf{r}^{(L)}(\mathbf{X},t),0]\, D_k[\mathbf{r}^{(L)}(\mathbf{X},t),t] = F_{ij...,k}[\mathbf{r}^{(L)}(\mathbf{X},0),0]\, d_k^{(L)}(\mathbf{X},t). \tag{2.42}$$

Substitution of these equations into (2.40) and use of (2.15), (2.22) and $r_{i,j}^{(L)}(\mathbf{X},0) = \delta_{ij}$ yields

$$f_{ij...}^{(L)}(\mathbf{X},t) = f_{ij...}^{(L)}(\mathbf{X},0) + \partial f_{ij...}^{(L)}(\mathbf{X},t) + f_{ij...,k}^{(L)}(\mathbf{X},0)\,[u_k(\mathbf{X},t) - d_k^{(L)}(\mathbf{X},t)],$$
$$X_i \in \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)}, \quad t \in \mathcal{T}. \tag{2.43}$$

The Lagrangian form of (2.32) follows by taking steps in complete analogy to those leading from (2.39) to (2.43) as

$$f_{ij...}^{(L)}(\mathbf{X},t) = f_{ij...}^{(L)}(\mathbf{X},0) + \Delta f_{ij...}^{(L)}(\mathbf{X},t) + f_{ij...,k}^{(L)}(\mathbf{X},0)\, u_k(\mathbf{X},t),$$
$$X_i \in \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)}, \quad t \in \mathcal{T}. \tag{2.44}$$

We now adopt several notational simplifications to be adhered to throughout this treatise: (i) the arguments X_i, r_i and t are suppressed; (ii) the argument $t = 0$ is indicated by the label superscript 0 appended to the function symbols; (iii) the material, isopotential and local increments are indicated by the label superscripts δ, ∂ and Δ appended to the function symbols. With these conventions, we obtain from the Lagrangian forms of the perturbation equations, (2.30), (2.43) and (2.44), the relations

$$\left.\begin{aligned} f_{ij\ldots}^{(L)} &= f_{ij\ldots}^{(L0)} + f_{ij\ldots}^{(L\delta)} && (2.45)\\ f_{ij\ldots}^{(L)} &= f_{ij\ldots}^{(L0)} + f_{ij\ldots}^{(L\partial)} + f_{ij\ldots,k}^{(L0)}(u_k - d_k^{(L)}) && (2.46)\\ f_{ij\ldots}^{(L)} &= f_{ij\ldots}^{(L0)} + f_{ij\ldots}^{(L\Delta)} + f_{ij\ldots,k}^{(L0)} u_k && (2.47) \end{aligned}\right\} \quad X_i \in \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)}, \quad t \in \mathcal{T},$$

$$\left.\begin{aligned} f_{ij\ldots}^{(L\delta)} &= f_{ij\ldots}^{(L\partial)} + f_{ij\ldots,k}^{(L0)}(u_k - d_k^{(L)}) && (2.48)\\ f_{ij\ldots}^{(L\delta)} &= f_{ij\ldots}^{(L\Delta)} + f_{ij\ldots,k}^{(L0)} u_k && (2.49) \end{aligned}\right\} \quad X_i \in \mathcal{X}_-^{(L)} \cup \mathcal{X}_+^{(L)}, \quad t \in \mathcal{T}.$$

We refer to (2.45), (2.46) and (2.47), respectively, as the material, isopotential and local forms of the (Lagrangian) perturbation equation. The second terms on the right-hand sides of (2.48) and (2.49) are called *advective* increments. They account for the increments resulting from the component of the motion of material or isopotential points parallel to the gradient of the initial field. In the special case of a spatially homogeneous initial field, we have $f_{ij\ldots,k}^{(L0)} = 0$ and, therefore, $f_{ij\ldots}^{(L\delta)} = f_{ij\ldots}^{(L\partial)} = f_{ij\ldots}^{(L\Delta)}$. Henceforth, only the Lagrangian formulation will be employed, which allows us to suppress the label superscript L. Conventionally, the Langrangian formulation is used whenever material points can be identified and the domains initially occupied by them are prescribed but those currently occupied not known *a priori*.

2.4 Interface conditions

We briefly return the behaviour of field values on $\partial\mathcal{R}$. In order to formulate a condition expressing this behaviour, we locally assign to $\partial\mathcal{R}$ (the Lagrangian form of) the unit normal directed outward into $\mathcal{R}_+$. Denoting this vector by n_i and assuming $\epsilon > 0$, we define

$$\left.\begin{aligned} [f_{ij\ldots}]_\pm &= \lim_{\epsilon\to 0+} f_{ij\ldots}(\mathrm{X} \pm \epsilon \mathbf{n}^{(0)}) && (2.50)\\ [f_{ij\ldots}]_-^+ &= [f_{ij\ldots}]_+ - [f_{ij\ldots}]_- && (2.51) \end{aligned}\right\} \quad X_i \in \partial\mathcal{X}, \quad t \in \mathcal{T}.$$

The interface condition for $f_{ij\ldots}$ can then be written as

$$[f_{ij\ldots}]_-^+ = f_{ij\ldots}^{\pm}, \quad X_i \in \partial\mathcal{X}, \quad t \in \mathcal{T}, \tag{2.52}$$

where $f_{ij\ldots}^{\pm}$ is the increase of $f_{ij\ldots}$ in the direction of n_i. For convenience, we may extend the field values onto $\partial\mathcal{R}$ using

$$f_{ij\ldots} = \tfrac{1}{2}\{[f_{ij\ldots}]_- + [f_{ij\ldots}]_+\}, \quad X_i \in \partial\mathcal{X}, \quad t \in \mathcal{T}, \tag{2.53}$$

which is the arithmetic mean of $[f_{ij\ldots}]_-$ and $[f_{ij\ldots}]_+$.

2.5 Summary

The main results of this preparatory chapter can be summarized as follows:

(i) We have defined the Lagrangian, Newtonian and Eulerian kinematic formulations of arbitrary field quantities and have given expressions for the mutual relations of the quantities and their gradients. Whereas the Lagrangian and Eulerian formulations are conventional descriptions used in continuum mechanics, the Newtonian formulation is non-conventional and apt to gravitational continuum mechanics.

(ii) In correspondence with the Lagrangian, Newtonian and Eulerian formulations, we have defined the material, isopotential and local increments of field quantities. Using the mutual relations between the kinematic formulations, this has allowed us to establish the material, isopotential and local forms of the fundamental perturbation equation referred to throughout the following chapters.

Chapter 3

Linearized field theory

3.1 Introduction

Studies of the response of planetary bodies to perturbing forces of short duration have conventionally been based on the assumption of elastic constitutive behaviour. However, complications arise due to the presence of initial stress in the interiors of the bodies. Rayleigh (1906) discussed this problem and proposed that it be solved if the *total* stress is decomposed into a hydrostatic *initial* stress and a superimposed *incremental* stress. He also took the displacement of a particle with respect to its initial position and suggested to relate the incremental stress to the displacement using equations which formally agree with the ordinary field equations of elastodynamics valid in the absence of initial stress.

Love (1911) elaborated Rayleigh's concepts. In particular, he found it profitable to formulate the incremental field equations in terms of the *material* incremental stress 'experienced' by a displaced particle and conventionally appearing in the incremental constitutive equation of elasticity. However, owing to the initial stress gradient in planetary interiors, the material incremental stress differs from the *local* incremental stress observed at a fixed location and appearing in the Eulerian form of the incremental momentum equation. Love accounted for that difference by introducing the *advective* incremental stress into the latter equation.

A corollary of Love's concept of material incremental stress is that the initial stress associated with a particle at its current position is the hydrostatic stress at its initial position. This has sometimes been described as the particle 'carrying' its hydrostatic initial stress with it while being displaced. In several recent studies, Love's theory has been modified to the extent that a derivation of the incremental field equations of elastodynamics in the Lagrangian formulation has been given preference over his derivation in the Eulerian formulation (e. g. Dahlen, 1974; Grafarend, 1982).

The response of planetary bodies to perturbing forces of long duration has usually been studied assuming fluid constitutive behaviour. In contrast to elastodynamics, no 'advective term' is now required in the incremental momentum equation. This is in consequence of formulating the incremental field equations of fluid dynamics in terms of the local incremental stress appearing in the appropriate incremental constitutive equation. The formal difference between the incremental momentum equations in elastodynamics and fluid dynamics has been discussed by several authors (e. g. Dahlen, 1974; Cathles, 1975, pp. 11–20; Grafarend, 1982).

Particular attention has been devoted to viscous perturbations of an initially hydrostatic state. Studies of this kind had been initiated by Darwin's (1879) investigation into body tides and were continued by several other investigators (e. g. Haskell, 1935, 1936; Parsons, 1972). However, most of these studies are based on the supposition of incompressibility. Notable exceptions include the stability analysis by Jarvis & McKenzie (1980), which allows for pressure-induced density perturbations in terms of a set of approximate incremental field equations, and the study by Li & Yuen (1987), in which effects due to a pressure-induced initial density gradient are considered.

Various types of viscoelastic constitutive behaviour have been suggested over the years in order to explain the response of planets to perturbing forces spanning a wide period range. Comparatively recent is the formal development of the theory of viscoelastodynamics for material continua in a state of initial stress. Pioneering investigations into this problem were published by Biot in a series of papers and are summarized in his monograph (Biot, 1965). Of particular relevance to us is Biot's analysis of viscoelastic perturbations, induced by surface masses, of an initially hydrostatic material occupying a plane half-space (Biot, 1959). Similar to Love, Biot found it necessary to distinguish between local and material incremental stresses. However, in contrast to Love, Biot proceeded by formulating the incremental field equations in terms of the local incremental stress. On the assumption of incompressible perturbations, this allowed their formal reduction to the ordinary field equations valid in the absence of initial stress, whose solutions are well studied.

This 'reduction method' was also found helpful by Wolf (1984, 1985a, b) in his investigations into incompressible, viscoelastic perturbations of initially hydrostatic plane and spherical bodies with two layers. Assuming Maxwell viscoelasticity, he could in particular show that the solution approaches that for a hydrostatic fluid at times long after the onset of forcing (Wolf, 1985b). Since Wolf was not aware of Biot's publications at that time, no reference to them was made. More recently, Wolf's analysis has been extended to include generalized Maxwell viscoelasticity (Müller, 1986; Körnig & Müller, 1989; Rümpker, 1990).

The theory describing compressible, gravitational–viscoelastic perturbations of initially hydrostatic bodies has been largely developed by Peltier and is presented in several papers (e. g. Peltier, 1974; Wu & Peltier, 1982). Peltier's view is to account for elastic compressibility but to ignore bulk relaxation in the incremental constitutive equation. The shear relaxation is assumed to correspond to Maxwell viscoelasticity. A model of such constitutive behaviour that admits a closed-form solution was studied by Wolf (1985c). The simplifying feature of his model is that the incremental gravitational force associated with the perturbations of the initial density and the initial gravitational potential is ignored, which restricts the applicability of the solution to perturbations of sufficiently short wavelength.

Whereas Wolf's simplified model is plane and homogeneous, Peltier's models are spherical and stratified, which allows the initial density distribution to be a prescribed function of the radial distance. This particular feature was criticized by Fjeldskaar & Cathles (1984), who argued that it does not permit the discrimination between chemically induced and pressure-induced initial density gradients. However, no attempt was made by them to elaborate their claim. Instead, Fjeldskaar & Cathles maintained that Peltier's form of the theory of GVED was inconsistent and, therefore, would not apply to the case of a pressure-induced initial density gradient. They supported their view by referring to the necessary existence of an incremental state equation in GVD and to the mentioned difference between the incremental momentum equation in elastodynamics and this equation in fluid dynamics; two features which, in their opinion, were not covered by Peltier's theory. With regard to the second criticism, we incidentally note that, since no assumptions on the constitutive behaviour of the material enter into the derivation of the incremental momentum equation (e. g. Dahlen, 1974), the differences referred to by Fjeldskaar & Cathles can only be formal. Attempts to 'design' a special incremental momentum equation in GVED to provide for a smooth transition to the corresponding equations in GED and GVD (Svensson, 1980) are therefore physically unfounded.

A far-reaching criticism has recently been raised by Geller (1988a), who claimed that the accepted form of the incremental momentum equation describing gravitational–elastic perturbations of an arbitrary static initial state (Dahlen, 1972, 1973) is incorrect. Geller deduced the incremental momentum equation from a variational principle. Furthermore, he maintained that such kind of approach is *necessary*, because the correct form of this equation cannot be derived from the generalization of Newton's second law for continua (Geller, 1988b). Geller proposed modifications to the conventional incremental momentum equation only for a *non-hydrostatic* initial state. The accepted form of the equation for a *hydrostatic* initial state (e. g. Love, 1911; Dahlen, 1974) was confirmed but, since originally derived from the continuum form of Newton's second law, qualified as merely 'fortuitously' correct in Geller's study.

In this chapter, a rigorous exposition of GVED is given with the following objectives in mind:

We wish to demonstrate that the accepted form of the incremental momentum equation for initially hydrostatic planets is a logical consequence of Newton's second law. This conclusion should indeed result from a careful study of Dahlen's criticized papers. However, Dahlen deduced the incremental field equations for a hydrostatic initial state (Dahlen, 1974) from the more complicated incremental field equations for a non-hydrostatic initial state (Dahlen, 1972, 1973). Apart from that, in his Lagrangian formulation for the total fields, the momentum equation contains an error and the gravitational-potential equation is not stated. Since these restrictions impede the verification of Dahlen's results, we will deduce the linearized forms of the incremental field equations and interface conditions of GVED for a hydrostatic initial state directly from the differential forms of the fundamental principles of continuum mechanics and potential theory. The resulting equations describe infinitesimal, gravitational–viscoelastic perturbations of chemically and entropically stratified, compressible, rotating fluids initially in hydrostatic equilibrium and apply to any type of viscoelastic constitutive behaviour characterized by both bulk and shear relaxation (Section 3.2).

After that, the equations derived will be used to probe the justification of the criticism raised by Svensson (1980) and Fjeldskaar & Cathles (1984) regarding the consistency of the incremental field equations and interface conditions of GVED in their conventional form. For that purpose, we will consider perturbations whose short- and long-time limits exist. Based on this supposition, we will deduce two systems of incremental equations which are asymptotically correct for short and long times after the onset of the perturbations. The short-time asymptotic equations are generalizations of the incremental field equations and interface conditions of GED. The long-time asymptotic equations agree with the incremental field equations and interface conditions of GVD, with the incremental thermodynamic pressure satisfying the appropriate incremental state equation of the fluid (Section 3.3).

As an example of how compressibility can be explicitly accounted for in the theory, we will then derive a set of specialized field equations and interface conditions applying to the case of quasistatic, incompressible perturbations of an isochemical, isentropic, compressible initial state (§ 3.4.1). This is followed by a compilation of the corresponding equations for an incompressible material to be studied more closely in Chapters 4 and 5 (§ 3.4.2). The present chapter concludes with a brief summary of the main results obtained (Section 3.5).

We use the Lagrangian formulation developed in Chapter 2; accordingly, field values refer to the current position, r_i, of a material point whose initial position, X_i, is taken as the spatial argument. Also, the field equations and interface conditions are defined for all $X_i \in \mathcal{X}_- \cup \mathcal{X}_+$ and $X_i \in \partial\mathcal{X}$, respectively; the temporal argument is the initial time, $t = 0$, or the current time, $t \in \mathcal{T}$. We assume that, for all $X_i \in \mathcal{X}_- \cup \mathcal{X}_+$ and $t \in \mathcal{T}$, the field quantities are continuously differentiable with respect to X_i and t as many times as required.

3.2 Field equations and interface conditions

In the present section, we deduce the incremental field equations and interface conditions describing infinitesimal, gravitational–viscoelastic perturbations of chemically and entropically stratified, compressible, rotating fluids initially in hydrostatic equilibrium. In deducing the equations, we suppose that the perturbations are isochemical and isentropic. Our assumption is justified if the characteristic times associated with chemical or thermal diffusion are long compared with those associated with viscoelastic relaxation. This condition is met by a wide range of processes in planetary interiors.

We begin by collecting the field equations and interface conditions for the total fields (§ 3.2.1) and the initial fields (§ 3.2.2), from which those for the incremental fields are obtained (§ 3.2.3). After that, the continuity and state equations involving the density and thermodynamic pressure are given (§ 3.2.4). These field quantities are used when studying the large-t asymptotic behaviour of the incremental field equations and interface conditions (§ 3.3.3.2) and when considering the case of local incompressibility (§ 3.4.1).

3.2.1 Equations for the total fields

We recall the relation between the the Cauchy stress, t_{ij}, and the (non-symmetric) Piola–Kirchhoff stress, τ_{ij}:

$$t_{ij}\, da_j = \tau_{ij}\, da_j^{(0)}, \tag{3.1}$$

where da_i is some current surface element at r_i and $da_i^{(0)}$ the associated initial surface element at $r_i^{(0)}$. Using the transformation formula (e. g. Malvern, 1969, pp. 169–170)

$$r_{j,i}\, da_j = j\, da_i^{(0)}, \tag{3.2}$$

with j the Jacobian determinant given by

$$j = \det[r_{i,j}], \tag{3.3}$$

it then follows that

$$r_{i,k}\tau_{jk} = jt_{ij}. \tag{3.4}$$

We suppose in this investigation that the material continuum is without couple stresses, volume couples and spin angular momentum, so that t_{ij} is symmetric (e. g. Malvern, 1969, pp. 217–220).

Consider now a gravitating, rotating material undergoing perturbations of some initial state. On the assumption that the angular velocity, Ω_i, of the material can be regarded as prescribed, the momentum equation relative to a corotating reference frame is (e. g. Malvern, 1969, pp. 220–224)

$$\tau_{ij,j} + \rho^{(0)}(g_i + 2\epsilon_{ijk}\Omega_k\,\partial_t r_j) = \rho^{(0)}\,\partial_t^2 r_i, \tag{3.5}$$

where g_i is the gravity force per unit mass, $2\epsilon_{ijk}\Omega_k\,\partial_t r_j$ the Coriolis force per unit mass and $\rho^{(0)} \geq 0$ the initial volume-mass density. The symbols ∂_t and ∂_t^2 denote the first- and second-order partial-derivative operators with respect to t. The field g_i, henceforth simply referred to as gravity, is given by

$$g_i = (\phi + \chi + \psi)_{,j}X_{j,i}, \tag{3.6}$$

with ϕ the gravitational potential, χ the centrifugal potential and ψ some tidal potential. The gravitational-potential equation can be written as

$$j(\phi_{,ij}X_{i,k}X_{j,k} + \phi_{,i}X_{i,jj}) = -4\pi G\rho^{(0)}, \tag{3.7}$$

where G is Newton's gravitational constant. Using (2.24), equation (3.7) is readily interpreted as the Lagrangian formulation of Poisson's equation (e. g. Ramsey, 1981, pp. 67–69). The rotational-potential equation is

$$2\chi = \Omega_i\Omega_i r_j r_j - \Omega_i\Omega_j r_i r_j, \tag{3.8}$$

which implies an origin of the coordinate system on the spin axis. The constitutive equation is assumed to be of the form

$$t_{ij} = t_{ij}^{(0)} + \mathcal{M}_{ij}[r_{m,k}(t-t')\,r_{m,\ell}(t-t') - \delta_{k\ell}], \tag{3.9}$$

where $\mathcal{M}_{ij}$ is the anisotropic relaxation functional (assumed to be linear for simplicity) transforming the strain *history* given by the term in brackets into the *current* incremental stress and t' is the excitation time. Clearly, $t' \in [0, t]$ must hold as a consequence of the causality principle. With $\mathcal{M}_{ij}$, $t_{ij}^{(0)}$, $\rho^{(0)}$, ψ and Ω_i prescribed parameters and in view of $X_{(i,j)}r_{(j,i)} = 1$ (no summation), equations (3.3)–(3.9) constitute the system of total field equations for g_i, j, r_i, t_{ij}, τ_{ij}, ϕ and χ.

Next, the interface conditions to be satisfied by the fields will be collected. We assume here that $\partial\mathcal{R}$ coincides with a material sheet whose interface-mass density is σ. Considering (2.50)–(2.52) and the direction of n_i agreed upon, the following interface conditions then follow from (3.3)–(3.9):

$$[r_i]_-^+ = 0, \tag{3.10}$$

$$[\phi]_-^+ = 0, \tag{3.11}$$

$$[n_i\phi_{,j}X_{j,i}]_-^+ = -4\pi G\sigma, \tag{3.12}$$

$$[n_j t_{ij}]_-^+ = -g_i\sigma. \tag{3.13}$$

Note that the conditions apply on the supposition that $\partial\mathcal{R}$ is a 'welded' interface not admitting slip or cavitation of the material continuum. The value of g_i on $\partial\mathcal{R}$ is defined according to (2.53) and σ is supposed to be a prescribed function of X_i and t.

3.2.2 Equations for the initial fields

We now assume that (i) the material is a *fluid* and (ii) the initial state applying at $t = 0$ is a *static* equilibrium state. Then, since a fluid at rest cannot maintain deviatoric stresses, the initial state must even be a *hydrostatic* equilibrium state and, with the mechanical pressure defined by $p = -t_{ii}/3$, we have

$$t_{ij}^{(0)} = -\delta_{ij}p^{(0)}. \tag{3.14}$$

Since $r_{i,j}^{(0)} = \delta_{ij}$, equations (3.3) and (3.4) reduce to

$$j^{(0)} = 1, \tag{3.15}$$

$$\tau_{ij}^{(0)} = -\delta_{ij}p^{(0)}. \tag{3.16}$$

Using (3.14)–(3.16), $X_{i,j}^{(0)} = \delta_{ij}$ and $(\partial_t r_i)^{(0)} = (\partial_t^2 r_i)^{(0)} = 0$, equations (3.5)–(3.9) become

$$-p_{,i}^{(0)} + \rho^{(0)}g_i^{(0)} = 0, \tag{3.17}$$

$$g_i^{(0)} = (\phi^{(0)} + \chi^{(0)} + \psi^{(0)})_{,i}, \tag{3.18}$$

$$\phi_{,ii}^{(0)} = -4\pi G\rho^{(0)}, \tag{3.19}$$

$$2\chi^{(0)} = \Omega_i\Omega_i r_j^{(0)} r_j^{(0)} - \Omega_i\Omega_j r_i^{(0)} r_j^{(0)}, \tag{3.20}$$

$$p^{(0)} = \xi(\rho^{(0)}, \lambda^{(0)}, \varphi^{(0)}). \tag{3.21}$$

Equation (3.21) is the form of the state equation assumed for the present investigation, where $\lambda^{(0)}$ is a field quantity representing the initial chemical composition and $\varphi^{(0)}$ the initial entropy density. With the state function, ξ, known and $\lambda^{(0)}$, $\varphi^{(0)}$, $\psi^{(0)}$ and Ω_i prescribed, (3.17)–(3.21) constitute the (non-linear) system of initial field equations of GHS for $g_i^{(0)}$, $p^{(0)}$, $\rho^{(0)}$, $\phi^{(0)}$ and $\chi^{(0)}$. We point out the relation $\epsilon_{ijk}\rho_{,j}^{(0)}g_k^{(0)} = 0$ following from (3.17) and (3.18), whence these equations require that the level surfaces of $p^{(0)}$, $\rho^{(0)}$ and $\phi^{(0)} + \chi^{(0)} + \psi^{(0)}$ coincide. However, since (3.17) and (3.18) represent three scalar equations, respectively, the system (3.17)–(3.21) is overdetermined and solutions for the level surfaces are severely restricted.

Supposing $\sigma^{(0)} = 0$ and using (3.14) and $X_{i,j}^{(0)} = \delta_{ij}$, the following initial interface conditions are obtained from (3.10)–(3.13):

$$[r_i^{(0)}]_-^+ = 0, \tag{3.22}$$

$$[\phi^{(0)}]_-^+ = 0, \tag{3.23}$$

$$[n_i^{(0)}\phi_{,i}^{(0)}]_-^+ = 0, \tag{3.24}$$

$$[p^{(0)}]_-^+ = 0. \tag{3.25}$$

Since solutions to (3.17)–(3.21) admit a jump discontinuity of $\rho^{(0)}$ on $\partial\mathcal{R}^{(0)}$, we also have

$$[\rho^{(0)}]_-^+ = \rho^\pm. \tag{3.26}$$

3.2.3 Equations for the incremental fields

3.2.3.1 Material form

Using (2.33), (2.35) and (2.45), we decompose the total fields in (3.3)–(3.9) into initial and incremental parts. Considering also (3.14)–(3.16), (3.21) and $r_{i,j}^{(0)} = X_{i,j}^{(0)} = \delta_{ij}$, we get

$$(1 + j^{(\delta)}) = \det[\delta_{ij} + u_{i,j}], \tag{3.27}$$

$$(\delta_{ik} + u_{i,k})(-\delta_{jk}p^{(0)} + \tau_{jk}^{(\delta)}) = (1 + j^{(\delta)})(-\delta_{ij}p^{(0)} + t_{ij}^{(\delta)}), \tag{3.28}$$

$$-p_{,i}^{(0)} + \tau_{ij,j}^{(\delta)} + \rho^{(0)}(g_i^{(0)} + g_i^{(\delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_t^2 u_i, \tag{3.29}$$

$$g_i^{(0)} + g_i^{(\delta)} = (\phi^{(0)} + \phi^{(\delta)} + \chi^{(0)} + \chi^{(\delta)} + \psi^{(0)} + \psi^{(\delta)})_{,j}\,(\delta_{ji} - U_{j,i}), \tag{3.30}$$

$$(1 + j^{(\delta)})[(\phi^{(0)} + \phi^{(\delta)})_{,ij}(\delta_{ik} - U_{i,k})(\delta_{jk} - U_{j,k}) - (\phi^{(0)} + \phi^{(\delta)})_{,i}U_{i,jj}] = -4\pi G\rho^{(0)}, \tag{3.31}$$

$$2(\chi^{(0)} + \chi^{(\delta)}) = \Omega_i\Omega_i(r_j^{(0)} + u_j)(r_j^{(0)} + u_j) - \Omega_i\Omega_j(r_i^{(0)} + u_i)(r_j^{(0)} + u_j), \tag{3.32}$$

$$-\delta_{ij}p^{(0)} + t_{ij}^{(\delta)} = -\delta_{ij}\,\xi(\rho^{(0)}, \lambda^{(0)}, \varphi^{(0)}) + \mathcal{M}_{ij}\{[\delta_{mk} + u_{m,k}(t-t')][\delta_{m\ell} + u_{m,\ell}(t-t')] - \delta_{k\ell}\}. \tag{3.33}$$

We note that no restrictions on the magnitude of the perturbations have been imposed so far, i. e. (3.27)–(3.33) are valid for *finite* perturbations. In this study, we only consider *infinitesimal* perturbations, which allows us to *linearize* the field equations. Accordingly, we have

$$U_{i,j} = u_{i,j} \tag{3.34}$$

and (3.27) reduces to

$$j^{(\delta)} = u_{i,i}, \tag{3.35}$$

by which (3.28) can be recast into

$$\tau_{ij}^{(\delta)} = t_{ij}^{(\delta)} + p^{(0)}(u_{j,i} - \delta_{ij}u_{k,k}). \tag{3.36}$$

Considering (3.17)–(3.21) and (3.34)–(3.36), equations (3.29)–(3.33) then become

$$t_{ij,j}^{(\delta)} + p_{,j}^{(0)}u_{j,i} - p_{,i}^{(0)}u_{j,j} + \rho^{(0)}(g_i^{(\delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_t^2 u_i, \tag{3.37}$$

$$g_i^{(\delta)} = (\phi^{(\delta)} + \chi^{(\delta)} + \psi^{(\delta)})_{,i} - (\phi^{(0)} + \chi^{(0)} + \psi^{(0)})_{,j}u_{j,i}, \tag{3.38}$$

$$\phi_{,ii}^{(\delta)} - 2\phi_{,ij}^{(0)}u_{i,j} - \phi_{,i}^{(0)}u_{i,jj} = 4\pi G\rho^{(0)}u_{i,i}, \tag{3.39}$$

$$\chi^{(\delta)} = \chi_{,i}^{(0)}u_i, \tag{3.40}$$

$$t_{ij}^{(\delta)} = \mathcal{M}_{ij}[u_{k,\ell}(t-t') + u_{\ell,k}(t-t')]. \tag{3.41}$$

With $\mathcal{M}_{ij}$, $\psi^{(\delta)}$ and Ω_i prescribed parameters and the initial fields given as special solution to the initial field equations and interface conditions, (3.37)–(3.41) constitute the material form of the incremental field equations of GVED for $g_i^{(\delta)}$, $t_{ij}^{(\delta)}$, u_i, $\phi^{(\delta)}$ and $\chi^{(\delta)}$.

Next, the linearized forms of the associated incremental interface conditions are derived. For that purpose, we decompose the total fields in (3.10)–(3.13) into initial and incremental parts. Using (2.33), (2.35), (2.45), (3.14) and $r_{i,j}^{(0)} = X_{i,j}^{(0)} = \delta_{ij}$, we get

$$[r_i^{(0)} + u_i]_-^+ = 0, \tag{3.42}$$

$$[\phi^{(0)} + \phi^{(\delta)}]_-^+ = 0, \tag{3.43}$$

$$[(n_i^{(0)} + n_i^{(\delta)})(\phi^{(0)} + \phi^{(\delta)})_{,j}(\delta_{ji} - U_{j,i})]_-^+ = -4\pi G\sigma, \tag{3.44}$$

$$[(n_j^{(0)} + n_j^{(\delta)})(-\delta_{ij}p^{(0)} + t_{ij}^{(\delta)})]_-^+ = -(g_i^{(0)} + g_i^{(\delta)})\sigma. \tag{3.45}$$

In view of (3.22)–(3.25) and (3.34) and the assumption of infinitesimal perturbations, the material form of the incremental interface conditions follow as

$$[u_i]_-^+ = 0, \tag{3.46}$$

$$[\phi^{(\delta)}]_-^+ = 0, \tag{3.47}$$

$$[n_i^{(0)}(\phi_{,i}^{(\delta)} - \phi_{,j}^{(0)}u_{j,i})]_-^+ = -4\pi G\sigma, \tag{3.48}$$

$$[n_j^{(0)}t_{ij}^{(\delta)}]_-^+ = -g_i^{(0)}\sigma. \tag{3.49}$$

Since $n_i^{(0)}$ is normal to $\partial\mathcal{R}^{(0)}$, which is a surface of constant $\phi^{(0)} + \chi^{(0)} + \psi^{(0)}$, we may put on this surface

$$g_i^{(0)} = -\gamma n_i^{(0)}, \tag{3.50}$$

which will be used below.

3.2.3.2 Material–local form

The material–local form of the incremental field equations and interface conditions result if we use (2.49) to express $g_i^{(\delta)}$, $\phi^{(\delta)}$ and $\psi^{(\delta)}$ in terms of the respective local increments:

$$g_i^{(\delta)} = g_i^{(\Delta)} + g_{i,j}^{(0)}u_j, \tag{3.51}$$

$$\phi^{(\delta)} = \phi^{(\Delta)} + \phi_{,i}^{(0)}u_i, \tag{3.52}$$

$$\psi^{(\delta)} = \psi^{(\Delta)} + \psi_{,i}^{(0)}u_i. \tag{3.53}$$

In view of (3.17)–(3.19), (3.40) and (3.51)–(3.53), equations (3.37)–(3.41) take the forms

$$t_{ij,j}^{(\delta)} + (p_{,j}^{(0)}u_j)_{,i} - g_i^{(0)}(\rho^{(0)}u_j)_{,j} + \rho^{(0)}(g_i^{(\Delta)} + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_t^2 u_i, \tag{3.54}$$

$$g_i^{(\Delta)} = (\phi^{(\Delta)} + \psi^{(\Delta)})_{,i}, \tag{3.55}$$

$$\phi_{,ii}^{(\Delta)} = 4\pi G(\rho^{(0)}u_i)_{,i}, \tag{3.56}$$

$$t_{ij}^{(\delta)} = \mathcal{M}_{ij}[u_{k,\ell}(t - t') + u_{\ell,k}(t - t')]. \tag{3.57}$$

With $\mathcal{M}_{ij}$, $\psi^{(\Delta)}$ and Ω_i prescribed parameters and the initial fields given as special solution to the initial field equations and interface conditions, (3.54)–(3.57) constitute the material–local form of the incremental field equations of GVED for $g_i^{(\Delta)}$, $t_{ij}^{(\delta)}$, u_i and $\phi^{(\Delta)}$.

Equations (3.54)–(3.56) agree with the incremental momentum equation and gravitational-potential equation given by Love (1911, pp. 89–93) and by Dahlen (1974). Love used the Eulerian formulation, i. e. his incremental equations are functions of the current particle position, r_i. Since the difference between the Lagrangian and Eulerian formulations is of second order in the incremental quantities, it may, however, be ignored in linearized field theory. In contrast to Love, Dahlen used the Lagrangian formulation in terms of the initial particle position, X_i, which has also been adopted here. We note that the expression for the gravitational-force density is given incorrectly in Dahlen's total momentum equation, which therefore does not agree with (3.5) and (3.6). However, this error is not transferred to Dahlen's incremental momentum equation, which is found to be consistent with (3.54) and (3.55). Also, Dahlen does not introduce an equation equivalent to our total gravitational-potential equation, (3.7); accordingly, it is not quite obvious how his equivalent to our incremental gravitational-potential equation, (3.56), has been derived.

The associated incremental interface conditions follow upon substituting (3.52) into (3.47) and (3.48), yielding

$$[\phi^{(\Delta)} + \phi^{(0)}_{,i} u_i]^+_- = 0, \tag{3.58}$$

$$[n^{(0)}_i (\phi^{(\Delta)}_{,i} + \phi^{(0)}_{,ij} u_j)]^+_- = -4\pi G\sigma. \tag{3.59}$$

Observing the constraints imposed by (3.24) on the continuity of the components of $n^{(0)}_j \phi^{(0)}_{,ij}$, (3.59) can be shown to be equivalent to

$$[n^{(0)}_i (\phi^{(\Delta)}_{,i} + \phi^{(0)}_{,jj} u_i)]^+_- = -4\pi G\sigma. \tag{3.60}$$

Upon consideration of (3.19), (3.46), (3.49) and (3.50), the incremental interface conditions are found to be

$$[u_i]^+_- = 0, \tag{3.61}$$

$$[\phi^{(\Delta)}]^+_- = 0, \tag{3.62}$$

$$[n^{(0)}_i (\phi^{(\Delta)}_{,i} - 4\pi G\rho^{(0)} u_i)]^+_- = -4\pi G\sigma, \tag{3.63}$$

$$[n^{(0)}_j t^{(\delta)}_{ij}]^+_- = \gamma n^{(0)}_i \sigma. \tag{3.64}$$

The material-local form of the incremental field equations and interface conditions of GVED is reconsidered below when studying the small-t asymptotes to these equations (§ 3.3.3.1).

3.2.3.3 Local form

We consider (2.49) and (3.14), giving

$$t^{(\delta)}_{ij} = t^{(\Delta)}_{ij} - \delta_{ij} p^{(0)}_{,k} u_k. \tag{3.65}$$

On account of (3.17) and (3.65), the material-local form of the incremental field equations and interface conditions, (3.54)–(3.57) and (3.61)–(3.64), transforms into

$$t^{(\Delta)}_{ij,j} - g^{(0)}_i (\rho^{(0)} u_j)_{,j} + \rho^{(0)} (g^{(\Delta)}_i + 2\epsilon_{ijk} \Omega_k \partial_t u_j) = \rho^{(0)} \partial_t^2 u_i, \tag{3.66}$$

$$g^{(\Delta)}_i = (\phi^{(\Delta)} + \psi^{(\Delta)})_{,i}, \tag{3.67}$$

$$\phi^{(\Delta)}_{,ii} = 4\pi G (\rho^{(0)} u_i)_{,i}, \tag{3.68}$$

$$t^{(\Delta)}_{ij} = \delta_{ij} p^{(0)}_{,k} u_k + \mathcal{M}_{ij}[u_{k,\ell}(t-t') + u_{\ell,k}(t-t')], \tag{3.69}$$

$$[u_i]_-^+ = 0, \tag{3.70}$$

$$[\phi^{(\Delta)}]_-^+ = 0, \tag{3.71}$$

$$[n_i^{(0)}(\phi_{,i}^{(\Delta)} - 4\pi G \rho^{(0)} u_i)]_-^+ = -4\pi G \sigma, \tag{3.72}$$

$$[n_j^{(0)}(t_{ij}^{(\Delta)} - \delta_{ij}\rho^{(0)} g_k^{(0)} u_k)]_-^+ = \gamma n_i^{(0)} \sigma. \tag{3.73}$$

With $\mathcal{M}_{ij}$, $\psi^{(\Delta)}$ and Ω_i prescribed parameters and the initial fields given as special solution to the initial field equations and interface conditions, (3.66)–(3.69) constitute the local form of the incremental field equations of GVED for $g_i^{(\Delta)}$, $t_{ij}^{(\Delta)}$, u_i and $\phi^{(\Delta)}$, whose solution must satisfy the associated incremental interface conditions, (3.70)–(3.73).

The term $[n_i^{(0)}\rho^{(0)}g_j^{(0)}u_j]_-^+$ in (3.73) has sometimes been referred to as 'buoyancy term'. Note that its appearance is solely a consequence of formulating the incremental field equations and interface conditions in terms of the local incremental stress, $t_{ij}^{(\Delta)}$. In the material–local form of the equations, where the material incremental stress, $t_{ij}^{(\delta)}$, is used, no buoyancy term can therefore arise in the corresponding interface condition, (3.64). Conversely, the material-local momentum equation, (3.54), contains the 'advective term' $(p_{,j}^{(0)}u_j)_{,i}$, which is absent from the local momentum equation, (3.66). The local form of the incremental field equations and interface conditions of GVED will be used when deducing the large-t asymptotes to the incremental equations (§ 3.3.3.2) and when considering the case of incompressibility (Section 3.4).

3.2.3.4 Constitutive equation

To obtain an expression for $\mathcal{M}_{ij}$, we use the continuous differentiability of the strain history. On this assumption, $\mathcal{M}_{ij}$ may be written as a convolution integral (e. g. Christensen, 1982[2], pp. 3–9)

$$\mathcal{M}_{ij} = \int_0^t m_{ijk\ell}(t-t')\,\partial_{t'}[u_{k,\ell}(t') + u_{\ell,k}(t')]\,dt', \tag{3.74}$$

with $m_{ijk\ell}(t-t')$ the anisotropic relaxation function. Supposing *isotropic* viscoelasticity from now on and exploiting the symmetry properties of $m_{ijk\ell}$ (e. g. Malvern, 1969, pp. 276–277), this simplifies to

$$\begin{aligned}\mathcal{M}_{ij} = &\int_0^t [m_1(t-t') - \tfrac{2}{3}m_2(t-t')]\delta_{ij}\,\partial_{t'}[u_{k,k}(t')]\,dt' \\ &+ \int_0^t m_2(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt'.\end{aligned} \tag{3.75}$$

We refer to (3.75) as the incremental constitutive equation of viscoelasticity. The independent functions $m_1(t-t')$ and $m_2(t-t')$ are defined for all $t-t' \in [0,\infty)$ and called bulk-relaxation function and the shear-relaxation function, respectively. For notational economy, we frequently use $m_\nu(t-t')$, where $\nu = 1,2$. We assume that the values of $m_\nu(t-t')$ are continuously differentiable in $\mathcal{R}_- \cup \mathcal{R}_+$ but may have a jump discontinuity on $\partial\mathcal{R}$. Furthermore, we take $m_\nu(t-t')$ as continuously differentiable with respect to $t-t'$, where it follows from thermodynamic principles that (e. g. Christensen, 1982[2], pp. 83–87; Golden & Graham, 1988, pp. 12–14)

$$m_\nu(t-t') \geq 0, \tag{3.76}$$

$$\partial_{t-t'} m_\nu(t-t') \leq 0, \tag{3.77}$$

$$\partial^2_{t-t'} m_\nu(t-t') \leq 0. \tag{3.78}$$

To obtain an additional constraint on $m_2(t-t')$, we recall that, by assumption, the material is a fluid. A necessary condition of fluid constitutive behaviour is that deviatoric stresses can relax completely (e. g. Christensen, 1982[2], pp. 9–14; Golden & Graham, 1988, pp. 14–17). In view of (3.75), this is formally expressible as the fluidity condition:

$$\lim_{t-t'\to\infty} m_2(t-t') = 0. \tag{3.79}$$

3.2.4 Continuity and state equations

So far, the incremental density and the incremental thermodynamic pressure have not appeared explicitly in the equations. This is in accordance with the adoption of the Lagrangian formulation, where the displacement, u_i, is preferentially used. However, the incremental density and the incremental thermodynamic pressure are required below in order to interpret the large-t asymptotes to the incremental field equations and interface conditions of GVED (§ 3.3.3.2) and to study the significance of local incompressibility (§ 3.4.1). Here, we collect the equations governing these field quantities.

The current value of the density, ρ, can be related to its initial value, $\rho^{(0)}$, by means of the continuity equation (e. g. Malvern, 1969, pp. 208–210):

$$j\rho = \rho^{(0)}, \tag{3.80}$$

where j is given by (3.3). For a fluid not necessarily in hydrostatic equilibrium, the *thermodynamic* pressure, ϖ, is introduced with the aid of a state equation whose functional relation is identical to that governing the *mechanical* pressure, $p = -t_{ii}/3$, in the case of hydrostatic equilibrium (e. g. Malvern, 1969, p. 296). In view of (3.21), we therefore have

$$\varpi = \xi(\rho, \lambda, \varphi), \tag{3.81}$$

with ϖ in general different from p. However, at $t = 0$, equation (3.81) reduces to

$$\varpi^{(0)} = \xi(\rho^{(0)}, \lambda^{(0)}, \varphi^{(0)}), \tag{3.82}$$

which, by comparison with (3.21), yields

$$\varpi^{(0)} = p^{(0)}. \tag{3.83}$$

A direct consequence of (3.82) and (3.83) is

$$p^{(0)}_{,i} = \left(\frac{\partial\xi}{\partial\rho}\right)^{(0)} \rho^{(0)}_{,i} + \left(\frac{\partial\xi}{\partial\lambda}\right)^{(0)} \lambda^{(0)}_{,i} + \left(\frac{\partial\xi}{\partial\varphi}\right)^{(0)} \varphi^{(0)}_{,i}, \tag{3.84}$$

where the partial derivatives $(\partial\xi/\partial\rho)^{(0)} = [\partial\xi/\partial\rho]_{\rho=\rho^{(0)}}$ etc. are dependent on $X_i \in \mathcal{X}_- \cup \mathcal{X}_+$.

Next, we use (2.45) to decompose the total fields in (3.80) and (3.81) into initial and incremental parts. Using also (3.15) and (3.35), equation (3.80) becomes

$$(1+u_{i,i})(\rho^{(0)}+\rho^{(\delta)})=\rho^{(0)}. \tag{3.85}$$

Since, by assumption, the perturbations are isochemical and isentropic, we have $\lambda^{(\delta)}=\varphi^{(\delta)}=0$ and the decomposition of (3.81) takes the form

$$\varpi^{(0)}+\varpi^{(\delta)}=\xi(\rho^{(0)},\lambda^{(0)},\varphi^{(0)})+\left(\frac{\partial\xi}{\partial\rho}\right)^{(0)}\rho^{(\delta)}. \tag{3.86}$$

Considering (3.82) and retaining only terms that are linear in the incremental quantities, (3.85) and (3.86) reduce to

$$\boxed{\begin{aligned}\rho^{(\delta)}&=-\rho^{(0)}u_{i,i}, &(3.87)\\ \varpi^{(\delta)}&=\left(\frac{\partial\xi}{\partial\rho}\right)^{(0)}\rho^{(\delta)}, &(3.88)\end{aligned}}$$

which constitute the material forms of the incremental continuity and state equations, respectively. Due to (2.49) and (3.83), we have, however,

$$\rho^{(\delta)}=\rho^{(\Delta)}+\rho^{(0)}_{,i}u_i, \tag{3.89}$$

$$\varpi^{(\delta)}=\varpi^{(\Delta)}+p^{(0)}_{,i}u_i, \tag{3.90}$$

whence the material forms can be replaced by

$$\boxed{\begin{aligned}\rho^{(\Delta)}&=-(\rho^{(0)}u_i)_{,i}, &(3.91)\\ \varpi^{(\Delta)}&=\left(\frac{\partial\xi}{\partial\rho}\right)^{(0)}(\rho^{(\Delta)}+\rho^{(0)}_{,i}u_i)-p^{(0)}_{,i}u_i. &(3.92)\end{aligned}}$$

These are the local forms of the incremental continuity and state equations, respectively. Equation (3.92) takes a more familiar form upon substituting for $p^{(0)}_{,i}$ from (3.84):

$$\varpi^{(\Delta)}=\left(\frac{\partial\xi}{\partial\rho}\right)^{(0)}\rho^{(\Delta)}-\left(\frac{\partial\xi}{\partial\lambda}\right)^{(0)}\lambda^{(0)}_{,i}u_i-\left(\frac{\partial\xi}{\partial\varphi}\right)^{(0)}\varphi^{(0)}_{,i}u_i. \tag{3.93}$$

In §§ 3.3.3.2 and 3.4.1, appropriate expressions for the partial derivatives will be provided.

3.3 Asymptotic behaviour

We proceed by supposing perturbations whose limits exist for both $t\to 0$ and $t\to\infty$. Obviously, these limits correspond to the *initial* and *final* hydrostatic equilibrium states of the fluid. Based on this assumption, we then derive asymptotes, for small and large t, to the incremental field equations and interface conditions of GVED. This problem essentially reduces to finding suitable asymptotic approximations to the Laplace transform of the incremental constitutive equation of viscoelasticity. Upon substituting (3.75) into (3.57) and (3.69), respectively, and observing

(A.2), (A.3), (A.5) and $u_i^{(0)} = 0$, Laplace transformation from the (X_i, t)- to the (X_i, s)-domain yields

$$\widetilde{t}_{ij}^{(\delta)} = (\widetilde{m}_1 - \tfrac{2}{3}\widetilde{m}_2)\delta_{ij} s\widetilde{u}_{k,k} + \widetilde{m}_2 s(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}), \tag{3.94}$$

$$\widetilde{t}_{ij}^{(\Delta)} = \delta_{ij}[p_{,k}^{(0)}\widetilde{u}_k + (\widetilde{m}_1 - \tfrac{2}{3}\widetilde{m}_2)s\widetilde{u}_{k,k}] + \widetilde{m}_2 s(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}), \tag{3.95}$$

where $\widetilde{f}_{ij\ldots}$ denotes the Laplace transform of $f_{ij\ldots}$ and $s \in \mathcal{S}$ the inverse Laplace time (Appendix A). As in these equations, we usually suppress the argument, s, of Laplace-transformed relaxation functions and incremental fields; for brevity, we refer to the latter simply as incremental fields. To make further progress, it is necessary to specify $\widetilde{m}_\nu$. This is achieved by expressing $m_\nu(t-t')$ in terms of the associated relaxation spectrum (§ 3.3.1). Laplace transformation then supplies a formula for $s\widetilde{m}_\nu$, from which asymptotic approximations for large and small s can be derived (§ 3.3.2). Substituting these approximations into (3.94) or (3.95) and applying the generalized initial- and final-value theorems then gives the small- and large-t asymptotes to the incremental constitutive equation of viscoelasticity (§ 3.3.3).

3.3.1 Relaxation functions

For $\nu = 1, 2$, suppose that $m_\nu(t-t')$ can be expressed as (e. g. Christensen, 1982[2], pp. 28–32; Golden & Graham, 1988, pp. 31–32):

$$m_\nu(t-t') = m_{\nu\infty} + \int_0^\infty \overline{m}_\nu(\alpha')\, e^{-\alpha'(t-t')}\, d\alpha', \tag{3.96}$$

where $\overline{m}_\nu(\alpha')$ is the relaxation spectrum, α' is the inverse spectral time and $m_\nu(t-t')$ satisfies the restrictions (3.76)–(3.79). Equation (3.96) implies

$$m_{\nu\infty} = \lim_{t-t'\to\infty} m_\nu(t-t'), \tag{3.97}$$

whence, by (3.79), it follows that

$$m_{2\infty} = 0. \tag{3.98}$$

Defining

$$m_{\nu 0} = m_\nu(0), \tag{3.99}$$

we also get

$$\int_0^\infty \overline{m}_\nu(\alpha')\, d\alpha' = m_{\nu 0} - m_{\nu\infty}. \tag{3.100}$$

A consequence of (3.77), (3.97), (3.99) and (3.100) is that $\int_0^\infty \overline{m}_\nu(\alpha')\, d\alpha' \geq 0$. Here, we impose the more stringent condition that $\overline{m}_\nu(\alpha') \geq 0$ for $\alpha' \in [0, \infty)$; we furthermore require that $\overline{m}_\nu(\alpha')$ vanishes sufficiently rapid as $\alpha' \to 0$ and $\alpha' \to \infty$ that, for $0 < \alpha_1 < \infty$, the integrals $\int_0^{\alpha_1} \overline{m}_\nu(\alpha')/\alpha'\, d\alpha'$ and $\int_{\alpha_1}^\infty \alpha' \overline{m}_\nu(\alpha')\, d\alpha'$ converge. These assumptions are of sufficient generality to include conventional mechanical and molecular models of viscoelasticity.

Next, we apply (A.6) and (A.7) to obtain the s-multiplied Laplace transform of (3.96) with respect to $t-t'$:

$$s\widetilde{m}_\nu = m_{\nu\infty} + \int_0^\infty \frac{s\,\overline{m}_\nu(\alpha')}{s+\alpha'}\, d\alpha'. \tag{3.101}$$

Being interested in asymptotic approximations to $s\tilde{m}_\nu$ for large and small s, we decompose the integral in (3.101) in the following way:

$$\int_0^\infty \frac{s\,\overline{m}_\nu(\alpha')}{s+\alpha'}\,d\alpha' = \int_0^{s-0} \frac{\overline{m}_\nu(\alpha')}{1+\frac{\alpha'}{s}}\,d\alpha' + \int_{s+0}^\infty \frac{s}{\alpha'}\,\frac{\overline{m}_\nu(\alpha')}{1+\frac{s}{\alpha'}}\,d\alpha'. \tag{3.102}$$

Note that, on the right-hand side, $\alpha'/s < 1$ in the first integrand whereas $s/\alpha' < 1$ in the second.

3.3.2 Asymptotic relaxation functions

3.3.2.1 Large-s asymptotes

For sufficiently large s, the second integral on the right-hand side of (3.102) may be neglected and, due to the convergence of $\int_{s+0}^\infty \alpha'\,\overline{m}_\nu(\alpha')\,d\alpha'$, we get the asymptotic approximation

$$\int_0^\infty \frac{s\,\overline{m}_\nu(\alpha')}{s+\alpha'}\,d\alpha' \simeq \int_0^\infty \overline{m}_\nu(\alpha')\Big(1-\frac{\alpha'}{s}\Big)d\alpha'. \tag{3.103}$$

Since $\overline{m}_\nu(\alpha') \geq 0$ for $\alpha' \geq 0$, we can apply the mean-value theorem of integral calculus and obtain the following estimate:

$$\int_0^\infty \alpha'\,\overline{m}_\nu(\alpha')\,d\alpha' = \alpha_{\nu 0}\int_0^\infty \overline{m}_\nu(\alpha')\,d\alpha', \tag{3.104}$$

where $\alpha_{\nu 0} \geq 0$. In view of (3.100), (3.103) and (3.104), equation (3.101) takes the form

$$s\tilde{m}_\nu \simeq m_{\nu 0} - (m_{\nu 0} - m_{\nu\infty})\,\frac{\alpha_{\nu 0}}{s}, \tag{3.105}$$

which is correct to the first order in $\alpha_{\nu 0}/s$. Using the abbreviations

$$\kappa_e = m_{10}, \tag{3.106}$$

$$\kappa_e' = (m_{10} - m_{1\infty})\alpha_{10}, \tag{3.107}$$

$$\mu_e = m_{20}, \tag{3.108}$$

$$\mu_e' = (m_{20} - m_{2\infty})\alpha_{20}, \tag{3.109}$$

we finally obtain

$$s\tilde{m}_1 \simeq \kappa_e - \frac{\kappa_e'}{s}, \tag{3.110}$$

$$s\tilde{m}_2 \simeq \mu_e - \frac{\mu_e'}{s}, \tag{3.111}$$

which are asymptotically correct for large s. From the properties of $m_\nu(t-t')$ specified above it follows that the values of κ_e, κ_e', μ_e and μ_e' are non-negative and continuously differentiable in $\mathcal{R}_- \cup \mathcal{R}_+$, with jump discontinuities admitted on $\partial\mathcal{R}$.

3.3.2.2 Small-s asymptotes

For sufficiently small s, the first integral on the right-hand side of (3.102) may be neglected and, due to the convergence of $\int_0^{s-0} \overline{m}_\nu(\alpha')/\alpha'\, d\alpha'$, we arrive at the asymptotic approximation

$$\int_0^\infty \frac{s\,\overline{m}_\nu(\alpha')}{s+\alpha'}\, d\alpha' \simeq \int_0^\infty \overline{m}_\nu(\alpha')\,\frac{s}{\alpha'}\, d\alpha'. \tag{3.112}$$

Applying the mean-value theorem of integral calculus, we now have

$$\int_0^\infty \frac{\overline{m}_\nu(\alpha')}{\alpha'}\, d\alpha' = \frac{1}{\alpha_{\nu\infty}} \int_0^\infty \overline{m}_\nu(\alpha')\, d\alpha', \tag{3.113}$$

where $\alpha_{\nu\infty} \geq 0$. In view of (3.100), (3.112) and (3.113), equation (3.101) becomes

$$s\tilde{m}_\nu \simeq m_{\nu\infty} + (m_{\nu 0} - m_{\nu\infty})\,\frac{s}{\alpha_{\nu\infty}}, \tag{3.114}$$

which is correct to the first order in $s/\alpha_{\nu 0}$. We simplify this by means of

$$\kappa_h = m_{1\infty}, \tag{3.115}$$

$$\kappa_h' = \frac{m_{10} - m_{1\infty}}{\alpha_{1\infty}}, \tag{3.116}$$

$$\mu_h = m_{2\infty}, \tag{3.117}$$

$$\mu_h' = \frac{m_{20} - m_{2\infty}}{\alpha_{2\infty}}. \tag{3.118}$$

Since $\mu_h = 0$ by (3.98) and (3.117), we obtain

$$\boxed{\begin{aligned} s\tilde{m}_1 &\simeq \kappa_h + \kappa_h' s, &(3.119)\\ s\tilde{m}_2 &\simeq \mu_h' s, &(3.120)\end{aligned}}$$

which are asymptotically correct for small s. From the properties of $m_\nu(t-t')$ specified above, it follows that the values of κ_h, κ_h' and μ_h' are non-negative and continuously differentiable in $\mathcal{R}_- \cup \mathcal{R}_+$, with jump discontinuities admitted on $\partial\mathcal{R}$.

3.3.3 Asymptotic incremental field equations and interface conditions

3.3.3.1 Small-t asymptotes: field theory of GED

By the generalized initial-value theorem for Laplace transforms (Appendix A), the small-t asymptote to (3.57) corresponds to the large-s asymptote to (3.94). However, for s sufficiently large, $s\tilde{m}_1$ and $s\tilde{m}_2$ may be approximated by (3.110) and (3.111), whose substitution into (3.94) provides

$$\tilde{t}_{ij}^{(\delta)} = (\kappa_e - \tfrac{2}{3}\mu_e)\delta_{ij}\tilde{u}_{k,k} + \mu_e(\tilde{u}_{i,j} + \tilde{u}_{j,i}) - (\kappa_e' - \tfrac{2}{3}\mu_e')\delta_{ij}\,\frac{\tilde{u}_{k,k}}{s} - \mu_e'\,\frac{\tilde{u}_{i,j} + \tilde{u}_{j,i}}{s}. \tag{3.121}$$

In view of (A.2), (A.4) and (A.19), inverse Laplace transformation from the (X_i, s)- to the (X_i, t)-domain gives

$$t_{ij}^{(\delta)} = (\kappa_e - \tfrac{2}{3}\mu_e)\delta_{ij}u_{k,k} + \mu_e(u_{i,j} + u_{j,i}) - (\kappa_e' - \tfrac{2}{3}\mu_e')\delta_{ij}\int_0^t u_{k,k}(t')\,dt'$$
$$-\mu_e'\int_0^t [u_{i,j}(t') + u_{j,i}(t')]\,dt', \tag{3.122}$$

with κ_e called elastic bulk modulus, μ_e elastic shear modulus, κ_e' anelastic bulk modulus and μ_e' anelastic shear modulus. Equation (3.122) is to be complemented by the remaining incremental field equations, (3.54)–(3.56), and the associated incremental interface conditions, (3.61)–(3.64). Together, they constitute the material–local form of the small-t asymptotes to the incremental field equations of GVED in terms of $g_i^{(\Delta)}$, $t_{ij}^{(\delta)}$, u_i and $\phi^{(\Delta)}$. We also refer to the equations as the incremental field equations and interface conditions of *generalized* GED.

If the integrals in (3.122) are neglected, it simplifies to the incremental constitutive equation of elasticity. In this case, the small-t asymptotes to the incremental field equations and interface conditions of viscoelastodynamics agree with the incremental field equations and interface conditions of *ordinary* GED (e. g. Love, 1911, pp. 89–93; Dahlen, 1974; Grafarend, 1982).

3.3.3.2 Large-t asymptotes: field theory of GVD

By the generalized final-value theorem for Laplace transforms (Appendix A), the large-t asymptote to (3.69) corresponds to the small-s asymptote to (3.95). However, with s sufficiently large, $s\widetilde{m}_1$ and $s\widetilde{m}_2$ may be replaced by (3.119) and (3.120), whose substitution into (3.95) leads to

$$\widetilde{t}_{ij}^{(\Delta)} = \delta_{ij}(p_{,k}^{(0)}\widetilde{u}_k + \kappa_h\widetilde{u}_{k,k}) + (\kappa_h' - \tfrac{2}{3}\mu_h')\delta_{ij}s\widetilde{u}_{k,k} + \mu_h's(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}). \tag{3.123}$$

Considering (A.2), (A.3), (A.19) and $u_i^{(0)} = 0$, inverse Laplace transformation from the (X_i, s)- to the (X_i, t)-domain provides

$$t_{ij}^{(\Delta)} = \delta_{ij}(p_{,k}^{(0)}u_k + \kappa_h u_{k,k}) + (\kappa_h' - \tfrac{2}{3}\mu_h')\delta_{ij}\,\partial_t u_{k,k} + \mu_h'\,\partial_t(u_{i,j} + u_{j,i}), \tag{3.124}$$

with κ_h referred to as hydrostatic bulk modulus, κ_h' as viscous bulk modulus (bulk viscosity) and μ_h' as viscous shear modulus (shear viscosity). We can reduce the large-t asymptote to an expression for $\varpi^{(\Delta)}$ by recalling that, for a fluid not necessarily in hydrostatic equilibrium, $\varpi^{(\Delta)}$ is related to the other state variables by the same function that relates $p^{(\Delta)} = -t_{ii}^{(\Delta)}/3$ to these variables in the case of hydrostatic equilibrium (e. g. Malvern, 1969, p. 296). Putting $\partial_t = 0$ in (3.124) thus yields

$$\varpi^{(\Delta)} = -p_{,i}^{(0)}u_i - \kappa_h u_{i,i}. \tag{3.125}$$

To replace this by a more familiar form, we compare (3.91), (3.92) and (3.125), giving

$$\left(\frac{\partial\xi}{\partial\rho}\right)^{(0)} = \frac{\kappa_h}{\rho^{(0)}}, \tag{3.126}$$

and set

$$\left(\frac{\partial\xi}{\partial\lambda}\right)^{(0)} = \frac{l}{\lambda^{(0)}}, \tag{3.127}$$

$$\left(\frac{\partial\xi}{\partial\varphi}\right)^{(0)} = \frac{v}{\varphi^{(0)}}, \tag{3.128}$$

where l is the chemical and v the entropic modulus. Upon substitution of (3.126)–(3.128), equation (3.93) takes the form

$$\varpi^{(\Delta)} = \frac{\kappa_h}{\rho^{(0)}}\rho^{(\Delta)} - \frac{l}{\lambda^{(0)}}\lambda^{(0)}_{,i}u_i - \frac{v}{\varphi^{(0)}}\varphi^{(0)}_{,i}u_i, \tag{3.129}$$

which is the incremental state equation of a fluid whose total state equation is given by (3.81).

Considering (3.91), (3.124), (3.125), (3.129) and the assumption of isochemical and isentropic perturbations, $\lambda^{(\delta)} = \varphi^{(\delta)} = 0$, the local form of the incremental field equations of GVED, (3.66)–(3.69) reduces to

$$t^{(\Delta)}_{ij,j} - g^{(0)}_i\rho^{(\Delta)} + \rho^{(0)}(g^{(\Delta)}_i + 2\epsilon_{ijk}\Omega_k\,\partial_t u_j) = \rho^{(0)}\,\partial_t^2 u_i, \tag{3.130}$$

$$g^{(\Delta)}_i = (\phi^{(\Delta)} + \psi^{(\Delta)})_{,i}, \tag{3.131}$$

$$\phi^{(\Delta)}_{,ii} = 4\pi G(\rho^{(0)}u_i)_{,i}, \tag{3.132}$$

$$t^{(\Delta)}_{ij} = -\delta_{ij}\varpi^{(\Delta)} + (\kappa'_h - \tfrac{2}{3}\mu'_h)\delta_{ij}\,\partial_t u_{k,k} + \mu'_h\,\partial_t(u_{i,j} + u_{j,i}), \tag{3.133}$$

$$\varpi^{(\Delta)} = \frac{\kappa_h}{\rho^{(0)}}\rho^{(\Delta)} + \frac{l}{\lambda^{(0)}}\lambda^{(\Delta)} + \frac{v}{\varphi^{(0)}}\varphi^{(\Delta)}, \tag{3.134}$$

$$\rho^{(\Delta)} = -(\rho^{(0)}u_i)_{,i}, \tag{3.135}$$

$$\lambda^{(\Delta)} = -\lambda^{(0)}_{,i}u_i, \tag{3.136}$$

$$\varphi^{(\Delta)} = -\varphi^{(0)}_{,i}u_i. \tag{3.137}$$

These equations are completed by the associated incremental interface conditions, (3.70)–(3.73). Together, they constitute the local form of the large-t asymptotes to the incremental equations of GVED in terms of $g^{(\Delta)}_i$, $t^{(\Delta)}_{ij}$, u_i, $\lambda^{(\Delta)}$, $\varpi^{(\Delta)}$, $\rho^{(\Delta)}$, $\phi^{(\Delta)}$ and $\varphi^{(\Delta)}$; in particular, they agree with the with the incremental field equations and interface conditions of GVD (e. g. Backus, 1967; Cathles, 1975, pp. 15–20; Jarvis & Mackenzie, 1980).

3.4 Specialized field equations and interface conditions

In the present section, we consider simplified field theories by supposing that the fluid is isochemical and isentropic in either of the domains $\mathcal{R}_-$ and $\mathcal{R}_+$:

$$\lambda^{(0)}_{,i} = \varphi^{(0)}_{,i} = 0, \tag{3.138}$$

that rotational and tidal effects are neglegible:

$$\Omega_i = 0, \tag{3.139}$$

$$\chi = \psi = 0, \tag{3.140}$$

that the perturbations are quasistatic:

$$\partial_t^2 u_i = 0, \tag{3.141}$$

and that the bulk relaxation is negligible:

$$m_1(t-t') = \kappa_h. \tag{3.142}$$

We note that, in the absence of tidal forces, the perturbations are now solely due to the incremental interface-mass density, σ. The field theories of GHS and GVED satisfying (3.138)–(3.142) will henceforth be referred to as *specialized* theories. Further below, the perturbations are assumed to be incompressible in addition. The analysis of this feature divides into *local* incompressibility (§ 3.4.1), which accounts for an initial density gradient due to self-compression, and *material* incompressibility (§ 3.4.2), where the initial state is also taken as incompressible.

3.4.1 Local incompressibility

3.4.1.1 Equations for the initial fields

Upon consideration of (3.138)–(3.140) and elimination of $g_i^{(0)}$, the initial field equations of GHS, (3.17)–(3.21), simplify to

$$-p_{,i}^{(0)} + \rho^{(0)}\phi_{,i}^{(0)} = 0, \tag{3.143}$$

$$\phi_{,ii}^{(0)} = -4\pi G\rho^{(0)}, \tag{3.144}$$

$$p^{(0)} = \xi_b(\rho^{(0)}), \tag{3.145}$$

with ξ_b the barotropic state function. The initial interface conditions, (3.22)–(3.26), continue to apply. Solutions to the specialized equations for the initial fields can be shown to exist for the level surfaces of $p^{(0)}$, $\rho^{(0)}$ and $\phi^{(0)}$ being concentric spheres, coaxial cylinders or parallel planes (e. g. Batchelor, 1967, pp. 14–20) To eliminate $p^{(0)}$, consider the gradient of (3.145):

$$p_{,i}^{(0)} = \left(\frac{d\xi_b}{d\rho}\right)^{(0)} \rho_{,i}^{(0)}, \tag{3.146}$$

where $(d\xi_b/d\rho)^{(0)}$ must be constant on the level surfaces. Comparing (3.143) with (3.146) yields

$$\left(\frac{d\xi_b}{d\rho}\right)^{(0)} \rho_{,i}^{(0)} = \rho^{(0)}\phi_{,i}^{(0)}, \tag{3.147}$$

which is the Williamson–Adams equation (Williamson & Adams, 1923; Bullen, 1975, pp. 67–68). With $(d\xi_b/d\rho)^{(0)}$ prescribed, (3.144) and (3.147) are to be solved for $\rho^{(0)}$ and $\phi^{(0)}$.

3.4.1.2 Equations for the incremental fields: local form

Using (3.142) and

$$m_2(t-t') = \mu(t-t'), \tag{3.148}$$

substitution of (3.75) into (3.69) leads to

$$\begin{aligned} t_{ij}^{(\Delta)} &= \delta_{ij}(p_{,k}^{(0)}u_k + \kappa_h u_{k,k}) - \tfrac{2}{3}\int_0^t \mu(t-t')\delta_{ij}\,\partial_{t'}[u_{k,k}(t')]\,dt' \\ &\quad + \int_0^t \mu(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt'. \end{aligned} \tag{3.149}$$

Since, by setting $\partial_t = 0$ in (3.149), we find

$$\varpi^{(\Delta)} = -p^{(0)}_{,i} u_i - \kappa_h u_{i,i} \tag{3.150}$$

and, by comparing (3.91), (3.92), (3.150) and observing $d\xi_b/d\rho = \partial\xi/\partial\rho$, we obtain

$$\left(\frac{d\xi_b}{d\rho}\right)^{(0)} = \frac{\kappa_h}{\rho^{(0)}}, \tag{3.151}$$

it follows from (3.146), (3.150) and (3.151) that

$$\varpi^{(\Delta)} = -\frac{\kappa_h}{\rho^{(0)}} (\rho^{(0)} u_i)_{,i}. \tag{3.152}$$

Note that, with $p^{(\Delta)} = -t^{(\Delta)}_{ii}/3$ *per definitionem*, (3.149) and (3.150) yield

$$\varpi^{(\Delta)} = p^{(\Delta)}, \tag{3.153}$$

which will henceforth be implied.

We now suppose that (3.152) can be replaced by the simultaneous conditions

$$\kappa_h \to \infty, \tag{3.154}$$

$$(\rho^{(0)} u_i)_{,i} \to 0, \tag{3.155}$$

$$p^{(\Delta)} = \text{finite}. \tag{3.156}$$

The significance of (3.155) becomes evident if we notice that, by (3.89) and (3.91), the condition $(\rho^{(0)} u_i)_{,i} = 0$ is equivalent to the condition $\rho^{(\delta)} = \rho^{(0)}_{,i} u_i$ or $\rho^{(\Delta)} = 0$. Equation (3.155) thus states that the compressibility of a displaced particle is constrained to the extent that the material incremental density 'follows' the prescribed initial density gradient such that the local incremental density vanishes. For this reason, we refer to (3.155) as *local* incremental incompressibility condition.

Taking into account (3.139)–(3.141), (3.147), (3.149), (3.151), (3.155) and eliminating $g^{(\Delta)}_i$, the local form of the incremental field equations and interface conditions of GVED, (3.66)–(3.73), reduces to

$$t^{(\Delta)}_{ij,j} + \rho^{(0)} \phi^{(\Delta)}_{,i} = 0, \tag{3.157}$$

$$\phi^{(\Delta)}_{,ii} = 0, \tag{3.158}$$

$$\begin{aligned} t^{(\Delta)}_{ij} &= -\delta_{ij} p^{(\Delta)} + \frac{2\rho^{(0)}\phi^{(0)}_{,k}}{3\kappa_h} \int_0^t \mu(t-t')\delta_{ij}\,\partial_{t'}[u_k(t')]\,dt' \\ &\quad + \int_0^t \mu(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt', \end{aligned} \tag{3.159}$$

$$u_{i,i} = -\frac{\rho^{(0)}\phi^{(0)}_{,i}}{\kappa_h} u_i, \tag{3.160}$$

$$[u_i]^+_- = 0, \tag{3.161}$$

$$[\phi^{(\Delta)}]^+_- = 0, \tag{3.162}$$

$$[n^{(0)}_i (\phi^{(\Delta)}_{,i} - 4\pi G\rho u_i)]^+_- = -4\pi G\sigma, \tag{3.163}$$

$$[n^{(0)}_j (t^{(\Delta)}_{ij} - \delta_{ij}\rho^{(0)}\phi^{(0)}_{,k} u_k)]^+_- = \gamma n^{(0)}_i \sigma. \tag{3.164}$$

The specialized incremental equations are to be solved for $p^{(\Delta)}$, $t_{ij}^{(\Delta)}$, u_i and $\phi^{(\Delta)}$, where $\rho^{(0)}$ and $\phi^{(0)}$ must satisfy the appropriate initial field equations and interface conditions. We observe that the hydrostatic bulk modulus, κ_h, remains finite in (3.159) and (3.160). This is because it enters into these equations in consequence of substituting (3.151) into the *initial* field equation (3.147), for which the approximation (3.154) does not apply. If $\phi^{(0)}$ is prescribed and $\phi^{(\Delta)}$ neglected, the mechanical and gravitational effects decouple. In this case, solutions for the initial state are readily found and the resulting equations for the incremental state become similar to those recently used by Li & Yuen (1987). In contrast to here, Li & Yuen did not point out the approximations implied in their equations and limited their attention to viscous perturbations. The decoupled system has been integrated by Kaufmann (1991), who studied the Boussinesq–Cerruti problem for local incremental incompressibility and Maxwell viscoelasticity.

3.4.2 Material incompressibility

We proceed on the supposition that the *material* is incompressible. Then, also the initial state must be incompressible, whence $\rho^{(0)}$ = constant replaces (3.145) and $\kappa_h \to \infty$ applies also in (3.159) and (3.160), the latter reducing to the conventional *material* incremental incompressibility condition.

3.4.2.1 Equations for the initial fields

With the additional restrictions, the specialized initial field equations of GHS for local incompressibility, (3.143)–(3.145), further simplify to those applying to material incompressibility:

$$-p_{,i}^{(0)} + \rho^{(0)}\phi_{,i}^{(0)} = 0, \tag{3.165}$$

$$\phi_{,ii}^{(0)} = -4\pi G\rho^{(0)}, \tag{3.166}$$

$$\rho^{(0)} = \text{constant}. \tag{3.167}$$

The conditions (3.22)–(3.26) for the initial fields continue to apply.

3.4.2.2 Equations for the incremental fields: local form

Owing to the assumption of material incompressibility, the specialized incremental field equations and interface conditions of GVED for local incompressibility, (3.157)–(3.164), reduce to those for material incompressibility:

$$t_{ij,j}^{(\Delta)} + \rho^{(0)}\phi_{,i}^{(\Delta)} = 0, \tag{3.168}$$

$$\phi_{,ii}^{(\Delta)} = 0, \tag{3.169}$$

$$t_{ij}^{(\Delta)} = -\delta_{ij}p^{(\Delta)} + \int_0^t \mu(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt', \tag{3.170}$$

$$u_{i,i} = 0, \tag{3.171}$$

$$[u_i]_-^+ = 0, \tag{3.172}$$

$$[\phi^{(\Delta)}]_-^+ = 0, \tag{3.173}$$

$$[n_i^{(0)}(\phi_{,i}^{(\Delta)} - 4\pi G\rho^{(0)}u_i)]_-^+ = -4\pi G\sigma, \tag{3.174}$$

$$[n_j^{(0)}(t_{ij}^{(\Delta)} - \delta_{ij}\rho^{(0)}\phi_{,k}^{(0)}u_k)]_-^+ = \gamma n_i^{(0)}\sigma. \tag{3.175}$$

In Chapter 4, a heuristic solution to these equations contingent upon additional simplifying assumptions will be derived. Chapter 5 is concerned with the formal deduction of the complete solution to (3.168)–(3.175).

3.5 Summary

The results of the present chapter can be summarized as follows:

(i) Postulating only the differential forms of the fundamental principles of continuum mechanics and potential theory in the Lagrangian formulation, we have given a concise derivation of the material, material–local and local forms of the incremental field equations and interface conditions of GVED. The equations describe infinitesimal, gravitational–viscoelastic perturbations of chemically and entropically stratified, compressible, rotating fluids initially in hydrostatic equilibrium. In obtaining the equations, deficiencies of previous such derivations for gravitational–elastic perturbations (e. g. Dahlen, 1974) have been avoided.

(ii) The incremental momentum equation deduced agrees with the incremental momentum equation given by Love (1911, pp. 89–93) or by Dahlen (1974). In view of the rigour of our deduction, Geller's (1988a) qualification of this equation as 'fortuitously' correct cannot be supported. Therefore, no restrictions on the derivation of the incremental momentum equation from the generalization of Newton's second law for continua exist for the case of a hydrostatic initial state. Similarly, no such restrictions have been found to exist for the more general case of a non-hydrostatic initial state (Vermeersen & Vlaar, 1991).

(iii) We have obtained, as the short-time asymptotes to the incremental field equations and interface conditions of GVED, a set of equations referred to as the incremental field equations and interface conditions of generalized GED. These equations can be reduced to the equations of ordinary GED. The long-time asymptotes agree with the incremental field equations and interface conditions of GVD. The asymptotic behaviour of the equations is an additional manifestation of their consistency and *ad hoc* modifications to the equations (Svensson, 1980) are therefore not justified.

(iv) We have furthermore shown that the incremental thermodynamic pressure entering into the long-time asymptote to the incremental constitutive equation of viscoelasticity satisfies the correct incremental state equation. This confirms that this asymptote indeed represents the incremental constitutive equation for viscous material response. At the same time, it disproves claims by Svensson (1980) and Fjeldskaar & Cathles (1984), who argued that the conventional formulation of GVED cannot be adapted to describe perturbations of compressible fluids, because it does not include an incremental state equation.

(v) Finally, we have adopted several simplifying assumptions and introduced the concept of local incompressibility. Based on this, we have developed a specialized field theory applying to incompressible perturbations of an isochemical, isentropic, compressible initial state. A solution to the equations governing this type of problem has recently been derived by Kaufmann (1991), who compared it with the corresponding solution for a chemically or entropically stratified, incompressible initial state. The important case of material incompressibility implying incompressible perturbations of an isochemical, isentropic, incompressible initial state will be studied more closely in Chapters 4 and 5.

Chapter 4

Boussinesq–Cerruti problem

4.1 Introduction

Comprehensive investigations into the elastostatic deformation of a plane material half-space subject to prescribed displacement or traction on its surface were carried out by Boussinesq and are summarized in a monograph (Boussinesq, 1885). However, in some aspects of his work, Boussinesq had been preceded by Lamé & Clapeyron (1831) and Cerruti (1882). Since the publication of Boussinesq's monograph, numerous authors have written on particular aspects or extensions of the problem considered by him (e. g. Lamb, 1902, 1917; Terazawa, 1916; Love, 1929; Harding & Sneddon, 1945; Sneddon, 1946; Kuo, 1969; Farrell, 1972). In the present study, we refer to it as the Boussinesq–Cerruti problem.

A common feature of most of the research completed on the Boussinesq–Cerruti problem is that the initial state is regarded as unstressed. As far as the model is applied to study local deformations of planetary bodies, this assumption clearly cannot be satisfied. However, effects due to the initial stress turn out to be small for *elastic* perturbations provided their lateral wavelength is sufficiently short (Cathles, 1975, pp. 35–39).

More significant is the influence of initial stress on *viscoelastic* perturbations. Such problems were extensively studied by Biot and are reviewed in a monograph (Biot, 1965). Closely related to the present chapter is Biot's analysis of viscoelastic perturbations of an initially hydrostatic material half-space (Biot, 1959), where the initial stress is accounted for by a separate term included in the incremental momentum equation. In general, Biot favoured a formal treatment of the problem and did not fully discuss the physical significance of the modifications associated with initial stress. Unfortunately, Biot's work has not received much attention from geophysical researchers. This disregard is evident in several later studies of the viscoelastic Boussinesq–Cerruti problem, in which the initial stress was neglected (e. g. Peltier, 1974; Cathles, 1975, pp. 57–59).

Research on the viscoelastic Boussinesq–Cerruti problem was resumed by Nakiboglu & Lambeck (1982) and Wolf (1985a–d), who studied the response to surface loading. Whereas Nakiboglu & Lambeck accounted for the initial stress by an *ad hoc* modification of the loading condition, Wolf included such effects in the incremental momentum equation. The distinction between 'viscoelastic' and 'total' perturbation stresses then allowed the latter to reduce the incremental field equations formally to those valid in the absence of initial stress, which can be integrated using elementary methods.

In retrospect, Wolf's (1985a, b) method of accounting for the initial stress is seen to be equivalent to that used by Biot (1959), although Wolf was not aware of Biot's publication at that time. As in Biot's study, the significance of the modifications associated with the initial stress was not fully recognized by Wolf. This, in particular, applies to the physical interpretation of the two types of incremental stress employed in Wolf's analysis, which was not adequately discussed.

In Chapter 3, a detailed exposition has been given of the field theory of GVED for fluids in a state of hydrostatic initial stress. In particular, rigorous deductions of the incremental field equations and interface conditions and of the asymptotic approximations to the equations for short and long times after the onset of the perturbations have been presented. Emphasis has been placed on the distinction between the *material* and *local* increments of field quantities, which has allowed us to interpret the short- and long-time asymptotic equations as the incremental field equations and interface conditions of GED and GVD, respectively. The progress achieved suggests a new analysis of the viscoelastic Boussinesq–Cerruti problem on the basis of the improved theoretical foundation. In the first place, such an analysis serves to clarify the physical interpretation of the formal solution to the problem. At the same time, it prepares for the rigorous treatment of more complicated problems, such as the viscoelastic Lamé–Kelvin problem (Chapter 5).

In agreement with the heuristic character of the present chapter, the model to be analysed will be kept as simple as possible. We therefore study the viscoelastic Boussinesq–Cerruti problem using an isochemical, incompressible, hydrostatic material half-space perturbed by 2-D harmonic surface masses. Since effects due to perturbations of the gravity field are small for planetary deformations amenable to the half-space approximation (Cathles, 1975, pp. 72–83), the gravity field is taken as prescribed. We begin with a compilation of the pertinent incremental field equations and interface conditions and their Laplace transforms (Section 4.2). The equations are solved by means of Love's strain function and inverse Laplace transformation (Section 4.3). This is followed by a discussion of the solution functions of main interest in studies of planetary deformations (Section 4.4). The chapter concludes with a brief summary of the results obtained (Section 4.5).

4.2 Specialized field equations and interface conditions

We start from the specialized field theories of GHS and GVED for material incompressibility applying to infinitesimal, quasistatic, gravitational–viscoelastic perturbations of isochemical, incompressible, non-rotating fluids initially in hydrostatic equilibrium (§ 3.4.2). In addition, we suppose that the gravity field is *fixed* and *spatially homogeneous*. This assumption allows us to regard $\phi^{(0)}$ as prescribed and to neglect $\phi^{(\Delta)}$. For brevity, the parameters $g_i = \phi^{(0)}_{,i}$ and $\rho = \rho^{(0)}$ will be used in the following. We begin by listing the tensor forms of the equations for the initial fields, the incremental fields and the Laplace transforms of the latter (§ 4.2.1), from which the scalar forms appropriate to plane symmetry are obtained (§ 4.2.2).

4.2.1 Tensor equations

4.2.1.1 Equations for the initial fields

On account of the parameterization of the gravity and density, the initial field equations, (3.165)–(3.167), and the initial interface conditions, (3.22)–(3.26), further reduce to

$$-p_{,i}^{(0)} + \rho g_i = 0, \tag{4.1}$$

$$[r_i^{(0)}]_-^+ = 0, \tag{4.2}$$

$$[p^{(0)}]_-^+ = 0, \tag{4.3}$$

$$[\rho]_-^+ = \rho^\pm. \tag{4.4}$$

4.2.1.2 Equations for the incremental fields: local form

In a similar way, the incremental field equations and interface conditions, (3.168)–(3.175), take the simplified forms

$$t_{ij,j}^{(\Delta)} = 0, \tag{4.5}$$

$$t_{ij}^{(\Delta)} = -\delta_{ij} p^{(\Delta)} + \int_0^t \mu(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt', \tag{4.6}$$

$$u_{i,i} = 0, \tag{4.7}$$

$$[u_i]_-^+ = 0, \tag{4.8}$$

$$[n_j^{(0)}(t_{ij}^{(\Delta)} - \rho\delta_{ij} g_k u_k)]_-^+ = \gamma n_i^{(0)} \sigma. \tag{4.9}$$

Solutions to these equations will be obtained upon Laplace transformation from the (X_i, t)- to the (X_i, s)-domain. Using (A.2), (A.3), (A.5) and $u_i^{(0)} = 0$, we obtain the following system of simultaneous differential equations and associated interface conditions for $\widetilde{p}^{(\Delta)}$, $\widetilde{t}_{ij}^{(\Delta)}$ and $\widetilde{u}_i$:

$$\widetilde{t}_{ij,j}^{(\Delta)} = 0, \tag{4.10}$$

$$\widetilde{t}_{ij}^{(\Delta)} = -\delta_{ij}\widetilde{p}^{(\Delta)} + s\widetilde{\mu}(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}), \tag{4.11}$$

$$\widetilde{u}_{i,i} = 0, \tag{4.12}$$

$$[\widetilde{u}_i]_-^+ = 0, \tag{4.13}$$

$$[n_j^{(0)}(\widetilde{t}_{ij}^{(\Delta)} - \rho\delta_{ij} g_k \widetilde{u}_k)]_-^+ = \gamma n_i^{(0)} \widetilde{\sigma}. \tag{4.14}$$

We may use (4.11) to eliminate $\widetilde{t}_{ij}^{(\Delta)}$ from (4.10) and (4.14). Observing (4.12), we arrive at a set of equations in terms $\widetilde{p}^{(\Delta)}$ and $\widetilde{u}_i$:

$$-\widetilde{p}_{,i}^{(\Delta)} + s\widetilde{\mu}\widetilde{u}_{i,jj} = 0, \tag{4.15}$$

$$\widetilde{u}_{i,i} = 0, \tag{4.16}$$

$$[\widetilde{u}_i]_-^+ = 0, \tag{4.17}$$

$$[n_i^{(0)}\widetilde{p}^{(\Delta)} - s\widetilde{\mu} n_j^{(0)}(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}) + \rho n_i^{(0)} g_j \widetilde{u}_j]_-^+ = -\gamma n_i^{(0)} \widetilde{\sigma}. \tag{4.18}$$

Note that these equations imply the spatial homogeneity of the values of $\widetilde{\mu}$ in each of the domains $\mathcal{R}_-$ and $\mathcal{R}_+$, which conforms with the assumed spatial homogeneity of ρ in these domains.

4.2.2 Scalar equations in Cartesian coordinates

4.2.2.1 Geometrical considerations

We now suppose that the material continuum is confined to the internal domain, $\mathcal{R}_-$, so that $\mu(t-t') = 0$, $p = 0$ and $\rho = 0$ in the external domain, $\mathcal{R}_+$. It then follows from the incremental field equations that u_i = continuous on $\partial\mathcal{R}$, u_i = indeterminate in $\mathcal{R}_+$ and the values of the remaining incremental quantities vanish identically in $\mathcal{R}_+$. Furthermore, we consider the case that the level surfaces of $p^{(0)}$ and $\phi^{(0)}$ are *parallel planes*. In applications to planets, this approximation is appropriate only to perturbations whose 'typical' lateral wavelength is short compared with the radius of the planet. The plane symmetry then suggests to introduce Cartesian coordinates: $X_i = (x,y,z)$. We stipulate $\mathcal{X}_- = \{X_i \,|\, x \in (0,-\infty)\}$ whence $\partial\mathcal{X} = \{X_i \,|\, x = 0\}$. The following field equations and interface conditions therefore apply to $x \in (0,-\infty)$ and $x = 0$, respectively, with $y, z \in (-\infty,\infty)$. The material continuum occupying $\mathcal{R}_-$ is referred to as the *half-space*, the material sheet occupying $\partial\mathcal{R}$ as the *load* and the coordinate triple (x,y,z) as the *observation point*. The coordinate x is called vertical distance. We use label subscripts x, y, z appended to tensor symbols to denote their appropriate scalar components in Cartesian coordinates; the summation convention no longer applies. Note that we have $(g_x, g_y, g_z) = (-\gamma, 0, 0)$ and $(n_x^{(0)}, n_y^{(0)}, n_z^{(0)}) = (1,0,0)$ applies on $\partial\mathcal{R}^{(0)}$.

4.2.2.2 Equations for the initial fields

In view of the symmetry of the problem, the relevant scalar components of (4.1)–(4.4) reduce to

$$p_{,x}^{(0)} + \gamma\rho = 0, \tag{4.19}$$

$$[p^{(0)}]_- = 0. \tag{4.20}$$

4.2.2.3 Equations for the incremental fields: (x,y,s)-domain

Consider in the following a 2-D load, orient the y-axis at right angle to its strike and refer to y and z as transversal and longitudinal distances, respectively. It then follows from symmetry considerations that $\widetilde{u}_z = 0$, $\partial_z = 0$ and $\widetilde{t}_{xz}^{(\Delta)} = \widetilde{t}_{yz}^{(\Delta)} = 0$, whence the relevant scalar components of (4.11) take the forms

$$\widetilde{t}_{xx}^{(\Delta)} = -\widetilde{p}^{(\Delta)} + 2s\widetilde{\mu}\widetilde{u}_{x,x}, \tag{4.21}$$

$$\widetilde{t}_{xy}^{(\Delta)} = s\widetilde{\mu}(\widetilde{u}_{x,y} + \widetilde{u}_{y,x}), \tag{4.22}$$

$$\widetilde{t}_{yy}^{(\Delta)} = -\widetilde{p}^{(\Delta)} + 2s\widetilde{\mu}\widetilde{u}_{y,y}, \tag{4.23}$$

$$\widetilde{t}_{zz}^{(\Delta)} = -\widetilde{p}^{(\Delta)}. \tag{4.24}$$

Similarly, we find for the relevant scalar components of (4.15) and (4.16) the relations

$$-\widetilde{p}_{,x}^{(\Delta)} + s\widetilde{\mu}(\widetilde{u}_{x,xx} + \widetilde{u}_{x,yy}) = 0, \tag{4.25}$$

$$-\widetilde{p}_{,y}^{(\Delta)} + s\widetilde{\mu}(\widetilde{u}_{y,xx} + \widetilde{u}_{y,yy}) = 0, \tag{4.26}$$

$$\widetilde{u}_{x,x} + \widetilde{u}_{y,y} = 0. \tag{4.27}$$

Observing the symmetry of the problem, the relevant scalar components of (4.17) and (4.18) become

$$[\widetilde{u}_{x,y} + \widetilde{u}_{y,x}]_- = 0, \tag{4.28}$$

$$[\widetilde{p}^{(\Delta)} - 2s\widetilde{\mu}\widetilde{u}_{x,x} - \gamma\rho\widetilde{u}_x]_- = \gamma\widetilde{\sigma}. \tag{4.29}$$

Equations (4.25)–(4.27) are three simultaneous second-order partial differential equations for $\widetilde{p}^{(\Delta)}$, $\widetilde{u}_x$ and $\widetilde{u}_y$, which must be solved subject to (4.28) and (4.29). These equations are to be completed by conditions requiring that the incremental fields and their spatial derivatives remain bounded as $x \to -\infty$.

4.3 Integration of the incremental equations

We proceed by integrating the scalar incremental equations listed in Section 4.2. Assuming fundamental solutions with respect to y in terms of Fourier coefficients, we deduce the general solution (§ 4.3.1) and the special solution (§ 4.3.2) to the equations. Upon specifying the shear-relaxation and loading functions (§ 4.3.3), the inverse Laplace transformation of the solution functions from the (x, k, s)- to the (x, k, t)-domain is implemented (§ 4.3.4).

4.3.1 General solution: (x, k, s)-domain

We obtain the general solution to the incremental field equations by means of Love's strain function, ζ, defined by (e. g. Malvern, 1969, pp. 552–554)

$$\widetilde{u}_x = \zeta_{,yy}, \tag{4.30}$$

$$\widetilde{u}_y = -\zeta_{,xy}, \tag{4.31}$$

$$\widetilde{p}^{(\Delta)} = -s\widetilde{\mu}(\zeta_{,xx} + \zeta_{,yy})_{,x}. \tag{4.32}$$

Using this, (4.21)–(4.24) can be expressed in terms of ζ:

$$\widetilde{t}^{(\Delta)}_{xx} = s\widetilde{\mu}(\zeta_{,xx} + 3\zeta_{,yy})_{,x}, \tag{4.33}$$

$$\widetilde{t}^{(\Delta)}_{xy} = -s\widetilde{\mu}(\zeta_{,xx} - \zeta_{,yy})_{,y}, \tag{4.34}$$

$$\widetilde{t}^{(\Delta)}_{yy} = s\widetilde{\mu}(\zeta_{,xx} - \zeta_{,yy})_{,x}, \tag{4.35}$$

$$\widetilde{t}^{(\Delta)}_{zz} = s\widetilde{\mu}(\zeta_{,xx} + \zeta_{,yy})_{,x}. \tag{4.36}$$

Upon substitution of these equations into (4.25)–(4.27), equations (4.25) and (4.27) are found to be identically satisfied. The necessary condition for (4.26) to be valid is

$$\zeta_{,xxxx} + 2\zeta_{,xxyy} + \zeta_{,yyyy} = 0, \tag{4.37}$$

which is the biharmonic equation. The general solution to (4.37) must satisfy the interface conditions

$$[(\zeta_{,xx} - \zeta_{,yy})_{,y}]_- = 0, \tag{4.38}$$

$$[s\widetilde{\mu}(\zeta_{,xx} + 3\zeta_{,yy})_{,x} + \gamma\rho\zeta_{,yy}]_- = -\gamma\widetilde{\sigma} \tag{4.39}$$

following from (4.28) and (4.29) upon substitution of (4.30)–(4.32). An additional constraint is that ζ and its spatial derivatives remain bounded as $x \to -\infty$.

To solve the equations, we introduce fundamental solutions with respect to y. In view of the limited objectives of this chapter, expressions of sufficient generality are

$$\widetilde{f}(x,y,s) = \widetilde{F}(x,k,s)\, e^{iky}, \tag{4.40}$$

$$\widetilde{\sigma}(y,s) = \widetilde{S}(k,s)\, e^{iky}, \tag{4.41}$$

with $\widetilde{F}$ and $\widetilde{S}$ the (complex) Fourier coefficients of the arbitrary incremental field component $\widetilde{f}$ and the interface-mass density $\widetilde{\sigma}$, respectively, and $k \in [0,\infty)$ the Fourier wave-number. Equations (4.30)–(4.36) then become

$$\widetilde{U}_x = -k^2 Z, \tag{4.42}$$

$$\widetilde{U}_y = -ikZ_{,x}, \tag{4.43}$$

$$\widetilde{P}^{(\Delta)} = -s\widetilde{\mu}(Z_{,xx} - k^2 Z)_{,x}, \tag{4.44}$$

$$\widetilde{T}^{(\Delta)}_{xx} = s\widetilde{\mu}(Z_{,xx} - 3k^2 Z)_{,x}, \tag{4.45}$$

$$\widetilde{T}^{(\Delta)}_{xy} = -iks\widetilde{\mu}(Z_{,xx} + k^2 Z), \tag{4.46}$$

$$\widetilde{T}^{(\Delta)}_{yy} = s\widetilde{\mu}(Z_{,xx} + k^2 Z)_{,x}, \tag{4.47}$$

$$\widetilde{T}^{(\Delta)}_{zz} = s\widetilde{\mu}(Z_{,xx} - k^2 Z)_{,x}, \tag{4.48}$$

where the arguments of the coefficients have been suppressed. Similarly, (4.37)–(4.39) reduce to

$$Z_{,xxxx} - 2k^2 Z_{,xx} + k^4 Z = 0, \tag{4.49}$$

$$[Z_{,xx} + k^2 Z]_- = 0, \tag{4.50}$$

$$[s\widetilde{\mu}(Z_{,xx} - 3k^2 Z)_{,x} - k^2\gamma\rho Z]_- = -\gamma\widetilde{S}, \tag{4.51}$$

with Z and its spatial derivatives remaining bounded as $x \to -\infty$.

Considering the signs of the values of k and x, the appropriate bounded solution to (4.49) can be written in the form

$$Z = \frac{1}{k^2}(A + Bkx)e^{kx}, \tag{4.52}$$

where A and B are arbitrary integration constants. Substituting this into (4.42)–(4.48), we obtain

$$\widetilde{U}_x = -(A + Bkx)e^{kx}, \tag{4.53}$$

$$\widetilde{U}_y = -i[A + B(1 + kx)]e^{kx}, \tag{4.54}$$

$$\widetilde{P}^{(\Delta)} = -2ks\widetilde{\mu}Be^{kx}, \tag{4.55}$$

$$\widetilde{T}^{(\Delta)}_{xx} = -2ks\widetilde{\mu}(A + Bkx)e^{kx}, \tag{4.56}$$

$$\widetilde{T}^{(\Delta)}_{xy} = -2iks\widetilde{\mu}[A + B(1 + kx)]e^{kx}, \tag{4.57}$$

$$\widetilde{T}^{(\Delta)}_{yy} = 2ks\widetilde{\mu}[A + B(2 + kx)]e^{kx}, \tag{4.58}$$

$$\widetilde{T}^{(\Delta)}_{zz} = 2ks\widetilde{\mu}Be^{kx}. \tag{4.59}$$

4.3.2 Special solution: (x, k, s)-domain

The integration constants A and B are determined by substitution of (4.52) into (4.50) and (4.51). We get

$$A = -B = \frac{\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}, \tag{4.60}$$

whence (4.53)–(4.59) take the forms

$$\widetilde{U}_x = -\frac{\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}(1 - kx)e^{kx}, \tag{4.61}$$

$$\widetilde{U}_y = i\frac{\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}kxe^{kx}, \tag{4.62}$$

$$\widetilde{P}^{(\Delta)} = \frac{2ks\widetilde{\mu}\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}e^{kx}, \tag{4.63}$$

$$\widetilde{T}_{xx}^{(\Delta)} = -\frac{2ks\widetilde{\mu}\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}(1 - kx)e^{kx}, \tag{4.64}$$

$$\widetilde{T}_{xy}^{(\Delta)} = i\frac{2ks\widetilde{\mu}\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}kxe^{kx}, \tag{4.65}$$

$$\widetilde{T}_{yy}^{(\Delta)} = -\frac{2ks\widetilde{\mu}\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}(1 + kx)e^{kx}, \tag{4.66}$$

$$\widetilde{T}_{zz}^{(\Delta)} = -\frac{2ks\widetilde{\mu}\gamma\widetilde{S}}{2ks\widetilde{\mu} + \gamma\rho}e^{kx}. \tag{4.67}$$

Further insight is gained by considering the maximum shear stress in the half-space, which is related to the difference between the largest and smallest principal stresses. Since the initial state is hydrostatic, it is sufficient to consider principal incremental stresses. With $\widetilde{t}_{zz}^{(\Delta)} = -\widetilde{p}^{(\Delta)}$ one of the principal incremental stresses, the remaining two are obtained from the characteristic equation

$$\det\begin{bmatrix} \widetilde{t}_{xx}^{(\Delta)} - \widetilde{t}^{(\Delta)} & \widetilde{t}_{xy}^{(\Delta)} \\ \widetilde{t}_{xy}^{(\Delta)} & \widetilde{t}_{yy}^{(\Delta)} - \widetilde{t}^{(\Delta)} \end{bmatrix} = 0. \tag{4.68}$$

Upon expansion of the determinant and use of (4.32), (4.33) and (4.35), we find

$$\widetilde{t}_{1,2}^{(\Delta)} = -\widetilde{p}^{(\Delta)} \pm \tfrac{1}{2}\sqrt{(\widetilde{t}_{xx}^{(\Delta)} - \widetilde{t}_{yy}^{(\Delta)})^2 + 4(\widetilde{t}_{xy}^{(\Delta)})^2}. \tag{4.69}$$

Obviously, $\widetilde{t}_1^{(\Delta)}$ and $\widetilde{t}_2^{(\Delta)}$ are the largest and smallest principal incremental stresses, respectively; hence, the maximum shear stress, $\widetilde{t}_s$, is given by

$$\widetilde{t}_s = \tfrac{1}{2}(\widetilde{t}_1^{(\Delta)} - \widetilde{t}_2^{(\Delta)}). \tag{4.70}$$

Note that we have not added a superscript to the symbol $\widetilde{t}_s$. This is because, for a hydrostatic initial state, no difference exists between material and local incremental shear stress components. Substitution of (4.69) into (4.70) yields

$$\widetilde{t}_s = \tfrac{1}{2}\sqrt{(\widetilde{t}_{xx}^{(\Delta)} - \widetilde{t}_{yy}^{(\Delta)})^2 + 4(\widetilde{t}_{xy}^{(\Delta)})^2}. \tag{4.71}$$

Evaluating the square root separately for the real and imaginary parts of the stress components, we obtain with (4.40) and (4.64)–(4.66) upon superposition of the results

$$\tilde{t}_s = |\tilde{T}_s|(1+i), \tag{4.72}$$

where

$$\boxed{\tilde{T}_s = -\frac{2ks\tilde{\mu}\gamma\tilde{S}}{2ks\tilde{\mu}+\gamma\rho}\,kxe^{kx}. \qquad (4.73)}$$

We note that, according to these equations, the maximum shear stress is independent of y.

4.3.3 Maxwell viscoelasticity and Heaviside loading

In order that (4.61)–(4.67) and (4.73) can be transformed from the (x,k,s)- to the (x,k,t)-domain, the functional forms of $\tilde{\mu}$ and $\tilde{S}$ must be specified. In this chapter, we only consider the cases of Maxwell viscoelasticity and Heaviside loading; the general cases being studied in Chapter 5.

The form of the shear-relaxation function for Maxwell viscoelasticity can be derived from (3.96) expressing the bulk- and shear-relaxation functions in terms of the associated spectra. For $\mu(t-t')$, the appropriate formula is

$$\mu(t-t') = \int_0^\infty \overline{\mu}(\alpha')\,e^{-\alpha'(t-t')}d\alpha', \tag{4.74}$$

where $\overline{\mu}(\alpha')$ is the shear-relaxation spectrum. A particularly simple form of $\overline{\mu}(\alpha')$ is

$$\overline{\mu}(\alpha') = \mu_e\,\delta(\alpha'-\alpha), \tag{4.75}$$

with $\alpha > 0$ the inverse Maxwell time and $\delta(\alpha'-\alpha)$ the Dirac delta-function. Upon substitution of (4.75) into (4.74) and use of the properties of the Dirac delta-function, we get

$$\mu(t-t') = \mu_e e^{-\alpha(t-t')}, \tag{4.76}$$

which is the shear-relaxation function for Maxwell viscoelasticity (e. g. Christensen, 1982[2], pp. 16–20). Note that $\mu(t-t')$ is determined by two parameters: the inverse Maxwell time, α, and the shear modulus, μ_e. The functional form expressing Heaviside loading is

$$S = S^*(k)\,H(t-\tau), \tag{4.77}$$

where $\tau > 0$ is the placement time and $H(t-\tau)$ the *symmetric* Heaviside step-function defined in (A.13).

In view of (A.7) and (A.17), the appropriate expressions for the Laplace transforms of (4.76) and (4.77) are

$$\tilde{\mu} = \frac{\mu_e}{s+\alpha}, \tag{4.78}$$

$$\tilde{S} = S^*\frac{e^{-\tau s}}{s}. \tag{4.79}$$

4.3.4 Special solution: (x, k, t)-domain

Using (4.78) and (4.79), we get the following expressions:

$$\frac{\gamma \tilde{S}}{2ks\tilde{\mu} + \gamma\rho} = \frac{\gamma S^*}{2k\mu_e + \gamma\rho} e^{-\tau s}\left[\frac{1}{s} + \frac{2k\mu_e}{\gamma\rho}\left(\frac{1}{s} - \frac{1}{s+\beta}\right)\right], \tag{4.80}$$

$$\frac{2ks\tilde{\mu}\gamma\tilde{S}}{2ks\tilde{\mu} + \gamma\rho} = \frac{2k\mu_e\gamma S^*}{2k\mu_e + \gamma\rho}\frac{e^{-\tau s}}{s+\beta}, \tag{4.81}$$

where

$$\beta = \frac{\gamma\rho}{2k\mu_e + \gamma\rho}\alpha. \tag{4.82}$$

With (A.2), (A.6), (A.7), (A.18) and (A.19), the inverse Laplace transforms of (4.80) and (4.81) are

$$\mathcal{L}^{-1}\left[\frac{\gamma\tilde{S}}{2ks\tilde{\mu} + \gamma\rho}\right] = \frac{\gamma S^* H(t-\tau)}{2k\mu_e + \gamma\rho}\left\{1 + \frac{2k\mu_e}{\gamma\rho}[1 - e^{-\beta(t-\tau)}]\right\}, \tag{4.83}$$

$$\mathcal{L}^{-1}\left[\frac{2ks\tilde{\mu}\gamma\tilde{S}}{2ks\tilde{\mu} + \gamma\rho}\right] = \frac{2k\mu_e\gamma S^* H(t-\tau)}{2k\mu_e + \gamma\rho} e^{-\beta(t-\tau)}, \tag{4.84}$$

where $\mathcal{L}^{-1}$ is the inverse of the Laplace transformation functional $\mathcal{L}$ (Appendix A). Defining the *right-hand* Heaviside step-function by $H_+[t - (\tau - 0)] = H(t - \tau)$ and considering, in particular, placement of the load at $\tau = 0+$, inverse Laplace transformation of (4.61)–(4.67) and (4.73) thus yields

$$U_x = -\frac{\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho}\left[1 + \frac{2k\mu_e}{\gamma\rho}(1 - e^{-\beta t})\right](1 - kx)e^{kx}, \tag{4.85}$$

$$U_y = i\frac{\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho}\left[1 + \frac{2k\mu_e}{\gamma\rho}(1 - e^{-\beta t})\right]kxe^{kx}, \tag{4.86}$$

$$P^{(\Delta)} = \frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho} e^{-\beta t}e^{kx}, \tag{4.87}$$

$$T_{xx}^{(\Delta)} = -\frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho} e^{-\beta t}(1 - kx)e^{kx}, \tag{4.88}$$

$$T_{xy}^{(\Delta)} = i\frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho} e^{-\beta t}kxe^{kx}, \tag{4.89}$$

$$T_{yy}^{(\Delta)} = -\frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho} e^{-\beta t}(1 + kx)e^{kx}, \tag{4.90}$$

$$T_{zz}^{(\Delta)} = -\frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho} e^{-\beta t}e^{kx}, \tag{4.91}$$

$$T_s = -\frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho} e^{-\beta t}kxe^{kx}. \tag{4.92}$$

The parameter β controls the time-dependence of the solution; henceforth, it will be referred to as inverse relaxation time.

4.4 Selected solution functions for the incremental fields: (x, k, t)-domain

We focus on the behaviour of the solution functions immediately below the interface. Setting $x = 0-$, it follows from (4.85)–(4.92) that $U_y = 0$, $T_{xx}^{(\Delta)} = T_{yy}^{(\Delta)} = T_{zz}^{(\Delta)} = -P^{(\Delta)}$ and $T_{xy}^{(\Delta)} = T_s = 0$, whence the expressions for the nonvanishing independent components are

$$\boxed{\left.\begin{aligned} U_x &= -\tfrac{\gamma S^* H_+(t)}{2k\mu_e+\gamma\rho}\left[1 + \tfrac{2k\mu_e}{\gamma\rho}(1 - e^{-\beta t})\right] \qquad (4.93)\\ P^{(\Delta)} &= \tfrac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e+\gamma\rho}\, e^{-\beta t} \qquad (4.94)\end{aligned}\right\} \quad x = 0- \,.}$$

In addition, we consider the vertical distance, x_m, where T_s reaches its maximum. Since

$$\frac{d}{d(kx)}(kxe^{kx}) = (1 + kx)e^{kx}, \tag{4.95}$$

we obtain

$$x_m = -\frac{1}{k}, \tag{4.96}$$

whose substitution into (4.92) gives

$$\boxed{T_s = \frac{1}{e}\,\frac{2k\mu_e\gamma S^* H_+(t)}{2k\mu_e + \gamma\rho}\, e^{-\beta t}, \quad x = x_m. \qquad (4.97)}$$

For the remainder of this section, we are concerned with the discussion of the small- and large-t limits of (4.93), (4.94) and (4.97) (§§ 4.4.1 and 4.4.2), their small- and large-k asymptotes (§§ 4.4.3 and 4.4.4) and the influence of initial stress on viscoelastic perturbations (§ 4.4.5).

4.4.1 Small-t limits

The limits of (4.93), (4.94) and (4.97) for $t \to 0+$ are

$$\left.\begin{aligned} U_x &= -\tfrac{\gamma S^*}{2k\mu_e+\gamma\rho} \qquad (4.98)\\ P^{(\Delta)} &= \tfrac{2k\mu_e\gamma S^*}{2k\mu_e+\gamma\rho} \qquad (4.99)\end{aligned}\right\} \quad x = 0-,$$

$$T_s = \tfrac{1}{e}\,\tfrac{2k\mu_e\gamma S^*}{2k\mu_e+\gamma\rho}, \quad x = x_m. \tag{4.100}$$

These equations apply to *elastostatic* equilibrium in a half-space specified by its elastic shear modulus, μ_e, and its hydrostatic initial stress gradient, $\gamma\rho$ (e. g. Wolf, 1985a, b).

4.4.2 Large-t limits

Using (4.82), the limits of (4.93), (4.94) and (4.97) for $t \to \infty$ are

$$\left.\begin{aligned} U_x &= -\tfrac{S^*}{\rho} \qquad (4.101)\\ P^{(\Delta)} &= 0 \qquad (4.102)\end{aligned}\right\} \quad x = 0-,$$

$$T_s = 0, \quad x = x_m, \tag{4.103}$$

which describe *hydrostatic* equilibrium in a half-space whose hydrostatic initial stress gradient is $\gamma\rho$. The transition from the instantaneous elastostatic to the final hydrostatic equilibrium state is seen from (4.93), (4.94) and (4.97) to be exponential in time and controlled by the inverse relaxation time, β.

4.4.3 Small-k asymptotes

If $2k\mu_e/(\gamma\rho) \ll 1$, equations (4.82), (4.93), (4.94) and (4.97) become, correct to the first order in the small quantity,

$$\beta \simeq \left(1 - \tfrac{2k\mu_e}{\gamma\rho}\right)\alpha, \tag{4.104}$$

$$\left.\begin{aligned} U_x &\simeq -\tfrac{S^* H_+(t)}{\rho}\left(1 - \tfrac{2k\mu_e}{\gamma\rho}\, e^{-\beta t}\right) & (4.105)\\ P^{(\Delta)} &\simeq \tfrac{2k\mu_e S^* H_+(t)}{\rho}\, e^{-\beta t} & (4.106)\end{aligned}\right\} \quad x = 0-,$$

$$T_s \simeq \tfrac{1}{e}\, \tfrac{2k\mu_e S^* H_+(t)}{\rho}\, e^{-\beta t}, \quad x = x_m. \tag{4.107}$$

Inspection of (4.82) and (4.104) shows that β is now close to its largest value, α, i. e. the relaxation proceeds rapidly. According to (4.105)–(4.107), the relaxation is, however, insignificant in amplitude and vanishes for $k\mu_e = 0$. For perturbations of sufficiently long wavelength, the difference between the elastostatic and hydrostatic equilibrium states may therefore be ignored.

4.4.4 Large-k asymptotes

If $\gamma\rho/(2k\mu_e) \ll 1$ and $\beta t \ll 1$, equations (4.82), (4.93), (4.94) and (4.97) become, correct to the first order in the small quantities,

$$\beta \simeq \tfrac{\gamma\rho}{2k\mu_e}\, \alpha, \tag{4.108}$$

$$\left.\begin{aligned} U_x &\simeq -\tfrac{\gamma S^* H_+(t)}{2k\mu_e}\left(1 - \tfrac{\gamma\rho}{2k\mu_e} + \alpha t\right) & (4.109)\\ P^{(\Delta)} &\simeq \gamma S^* H_+(t)\left(1 - \tfrac{\gamma\rho}{2k\mu_e} - \beta t\right) & (4.110)\end{aligned}\right\} \quad x = 0-,$$

$$T_s \simeq \tfrac{1}{e}\gamma S^* H_+(t)\left(1 - \tfrac{\gamma\rho}{2k\mu_e} - \beta t\right), \quad x = x_m. \tag{4.111}$$

Equation (4.108) is the expression of the inverse relaxation time for *viscous* perturbations in a half-space specified by its shear viscosity, μ_e/α, and its hydrostatic initial stress gradient, $\gamma\rho$ (Haskell, 1935, 1936; Ranalli, 1987, pp. 192–199).

4.4.5 Significance of initial stress

If $\gamma\rho/(2k\mu_e) = 0$, equations (4.108)–(4.111) further simplify to

$$\beta = 0, \tag{4.112}$$

$$\left.\begin{aligned} U_x &= -\tfrac{\gamma S^* H_+(t)}{2k\mu_e}\,(1 + \alpha t) & (4.113)\\ P^{(\Delta)} &= \gamma S^* H_+(t) & (4.114)\end{aligned}\right\} \quad x = 0-,$$

$$T_s = \tfrac{1}{e}\gamma S^* H_+(t), \quad x = x_m. \tag{4.115}$$

For $t \to 0+$, equations (4.113)–(4.115) apply to elastostatic equilibrium in an *initially unstressed* half-space (e. g. Jeffreys, 1976[6], pp. 265–267). Equation (4.112) shows that the subsequent relaxation proceeds infinitely slow. On the other hand, it follows from (4.113) for $t \to \infty$ that

$$U_x \to -\infty, \quad x = 0-, \tag{4.116}$$

where the instability is due to the absence of the gravitational force necessary to balance the load in the final hydrostatic state.

More practically, we may use the condition $\gamma\rho/(2k\mu_e) \leq 10^{-2}$ to estimate the minimum value of k in order that effects due to the hydrostatic initial stress may be neglected. Considering the earth as an example and taking $\gamma \simeq 10$ m s^{-2}, $\rho \simeq 10^3$ kg m^{-3} and $\mu_e \simeq 10^{11}$ Pa as representative values, we obtain $k \simeq 10^{-5}$ m^{-1} as minimum wave-number. This approximately corresponds to a maximum wavelength of 10^6 m. Within the limit of the half-space approximation, the influence of the initial stress is therefore negligible as far as elastic perturbations are concerned. However, if $\mu_e \ll 10^{11}$ Pa, the initial stress becomes noticeable at much shorter wavelengths. This condition applies to viscoelastic perturbations *in general*, because the 'effective' shear modulus, $s\tilde{\mu}$, may become arbitrarily small. Claims that the initial stress can be ignored for viscoelastic perturbations amenable to the half-space approximation (e. g. Cathles, 1975, pp. 57–59) are therefore unfounded.

Additional insight is gained by discussing also the material incremental pressure, $P^{(\delta)}$. In view of (2.49), (4.19) and (4.40), we have

$$P^{(\delta)} = P^{(\Delta)} - \gamma\rho U_x, \tag{4.117}$$

which, upon taking $x = 0-$ and substituting (4.93) and (4.94), becomes

$$P^{(\delta)} = \gamma S^* H_+(t), \quad x = 0- \, . \tag{4.118}$$

Since it is readily shown that $T_{xx}^{(\delta)} = -P^{(\delta)}$ applies for $x = 0-$, equation (4.118) expresses the balance required by (3.64) between the material incremental stress component normal to the interface and the incremental load pressure.

In view of (4.102), it follows from (4.117) for $t \to \infty$ that

$$P^{(\delta)} = -\gamma\rho U_x, \quad x = 0- \, . \tag{4.119}$$

In the final hydrostatic equilibrium state, the material incremental pressure is thus completely maintained by the advective incremental pressure associated with the displacement component in the direction of the initial pressure gradient. Assuming that (4.102) remains valid for $\gamma\rho \to 0$, equation (4.119) also applies in this limit. On account of (4.118), we therefore have for $t \to \infty$ the relation

$$\lim_{\substack{\gamma\rho \to 0 \\ U_x \to \infty}} -\gamma\rho U_x = \gamma S^*, \quad x = 0- \, . \tag{4.120}$$

Together, (4.119) and (4.120) ensure that the incremental interface condition given by (4.118) is satisfied in the final hydrostatic state even in the absence of initial stress. This illustrates the physical significance of the singularity in the solution function for the displacement noted above from a different point of view.

4.5 Summary

The results of the present chapter can be summarized as follows:

(i) We have deduced an elementary solution to the Boussinesq–Cerruti problem for the case of viscoelastic perturbations. The analysis presented strictly adheres to the distinction between material and local incremental fields and, in this way, avoids the misinterpretations prevailing in previous treatments of the problem.

(ii) Furthermore, we have discussed the solution functions of principal interest in deformation studies. We have found in particular that the limiting responses for short and long times after the commencement of loading correspond to the elastostatic and hydrostatic responses, respectively. This agrees with the results obtained in Section 3.3 more directly from a study of the asymptotic behaviour of the incremental field equations and interface conditions of GVED and, thus, further supports the consistency of the theory.

(iii) Finally, we have analysed the significance of initial stress for elastic and viscoelastic perturbations. Whereas it may be neglected for elastic perturbations of sufficiently short wavelength, it is *always* required when considering viscoelastic perturbations. The significance of initial stress has recently been studied also by Wu (1992b), whose conclusions essentially agree with ours.

Chapter 5

Lamé–Kelvin problem

5.1 Introduction

The problem of the elastostatic deformation of a spherical body was first investigated by Lamé (1854), who considered a spherical shell subject to given volume forces and prescribed conditions on its inner and outer surfaces. Lamé formulated the field equations in spherical coordinates and derived the solution for the displacement in terms of surface harmonics. The same problem was independently solved by Thomson (1864). In difference to Lamé, he employed Cartesian coordinates and expanded the solution into solid harmonics. Here, we refer to the type of spherical problem first considered by Lamé and Thomson (later Lord Kelvin) as the Lamé–Kelvin problem.

Applications of the Lamé–Kelvin problem to deformation studies of planetary bodies were closely connected to the problem of correctly accounting for gravitation. The modifications introduced by gravitation were discussed by Love (1908), who pointed out two basic effects. The first is related to the presence of *initial stress* in planetary interiors and requires a modification of the ordinary momentum equation valid in the absence of initial stress. Love (1911, pp. 89–93) implemented the necessary adjustments to the theory and derived the incremental momentum equation for a *hydrostatic* initial state. The second effect only arises if the planet is taken as compressible. In this case, the incremental gravitational force associated with perturbations of the initial density introduces a tendency toward *instability*. Normally, this tendency is, however, compensated by the opposing force resulting from the compressibility of the material. The stability of planetary bodies was studied in detail by Love (1911, pp. 89–104, 111–125).

A simplification of the investigations by Lamé, Thomson and Love as far as applications to planets are concerned is the assumption of homogeneous distributions of density, bulk modulus and shear modulus in the initial state. This constraint was removed by Herglotz (1905) and Hoskins (1910, 1920), who gave analytic solutions for elastostatic and gravitational–elastostatic deformations of a sphere, due to tidal volume forces, for simple types of variation of density and elasticity with radial distance. Later, Takeuchi (1950) extended these studies in order to allow for more realistic radial variations. This generalization required the use of numerical integration techniques. Takeuchi's approach was also employed in studies of gravitational–elastostatic perturbations of planetary bodies due to arbitrary surface loads. The latter problem was first investigated by Slichter & Caputo (1960), Caputo (1961, 1962) and Longman (1962, 1963), who calculated the Green's functions for displacement and incremental gravity. Longman's theory and numerical results were reviewed and extended by Farrell (1972).

More recently, a number of studies were concerned with the proper treatment of gravitational–elastostatic perturbations of a compressible planet with a fluid core. The solution to this problem proved to be not straightforward and occupied the investigators involved for several years. The competing approaches were reviewed by Longman (1975) and could be essentially reconciled by Dahlen & Fels (1978); in both works, comprehensive bibliographies of the relevant publications may be found.

The development of the theory governing gravitational–viscoelastic perturbations of initially hydrostatic planetary bodies has been strongly inspired by attempts to understand the earth's response to glacial surface loads and is reviewed in Peltier (1974), Sabadini, Yuen & Boschi (1982) and Wu & Peltier (1982). The virtue of the theory presented in these publications is that it admits the construction of realistic models of spherically symmetric planets. On the other hand, explicit analytic solutions for elementary models are indispensable for a *physical* interpretation of the solution (Chapter 4) or for tests of the accuracy of numerical solution methods (e. g. Gasperini & Sabadini, 1989; Wu, 1992a). So far, only a limited number of explicit solutions have been obtained mostly on the assumption of incompressible *Maxwell* viscoelasticity. Thus, Wu & Peltier (1982) derived an expression for the surface displacement, due to surface loading, of a homogeneous viscoelastic sphere. Dragoni, Yuen & Boschi (1983) gave the analytic solution for the displacement, caused by volume forces, in a sphere consisting of a homogeneous elastic lithosphere overlying a homogeneous viscoelastic mantle. Explicit solutions were also given by Wolf (1984), Wu (1990) and Amelung (1991), who analysed load-induced gravitational–viscoelastic perturbations of two-layer spheres with contrasts in density, shear modulus or viscosity across the interface.

In this chapter, infinitesimal, quasistatic, gravitational–viscoelastic perturbations, due to surface loads, of a spherical, isochemical, incompressible, non-rotating, fluid planet initially in hydrostatic equilibrium are reconsidered. The distinctive features of our analysis are the following:

We show that, for an isochemical, incompressible fluid, the incremental field equations can be recast into a form in which the equation for the (mechanical) momentum is *decoupled* from the equation for the (gravitational) potential. The *coupling* between the mechanical and gravitational aspects of the problem is restricted to the interface between the domains and expressed by appropriate incremental interface conditions. Instrumental to the decoupling of the incremental field equations is the use of a field quantity referred to as *isopotential* incremental pressure measuring the increment of the hydrostatic initial pressure with respect to a (perturbed) level surface of the gravitational potential (Section 5.2).

Using the appropriate ansatz for the decoupled incremental field equations, we then establish two independent (4×4) and (2×2) first-order ordinary differential systems for the mechanical and gravitational aspects of the problem, respectively. The deduction of the general solutions to these systems is algebraically much simpler than the deduction of the general solution to the conventional (6×6) system; the special solution is obtained in the conventional way by means of the incremental interface conditions (Section 5.3).

Following that, we list the special solution functions. In difference to previous studies, a comprehensive catalogue of formulae covering all field quantities of interest in deformation studies is provided. Furthermore, *transfer* functions, *impulse-response* functions and *Green's* functions for the incremental fields in the appropriate solution domains are collected. The solution functions given involve explicit expressions for the Legendre degrees $n = 0$, $n = 1$ and $n \geq 2$, apply to any location in the interior or exterior of planetary bodies and are valid for any type of *generalized Maxwell* viscoelasticity and for arbitrary surface loads (Section 5.4).

We conclude the chapter with an assessment of the results obtained and a brief outlook on possible consequences (Section 5.5).

5.2 Specialized field equations and interface conditions

We employ the specialized field theories of GHS and GVED for material incompressibility governing infinitesimal, quasistatic gravitational–viscoelastic perturbations of isochemical, incompressible, non-rotating fluids initially in hydrostatic equilibrium (§ 3.4.2). We begin with the tensor forms of the equations for the initial fields, the incremental fields and the Laplace-transformed incremental fields (§ 5.2.1), from which the scalar forms of the equations for spherical symmetry are derived (§ 5.2.2). The incremental field equations and interface conditions are given in isopotential–local form, which results in the decoupling of the field equations. Since $\rho^{(0)}$ = constant, it will be replaced by the parameter ρ.

5.2.1 Tensor equations

5.2.1.1 Equations for the initial fields

In view of the parameterization of the density, the initial field equations, (3.165)–(3.168), and the initial interface conditions, (3.22)–(3.26), take the forms

$$-p_{,i}^{(0)} + \rho\phi_{,i}^{(0)} = 0, \tag{5.1}$$
$$\phi_{,ii}^{(0)} = -4\pi G\rho, \tag{5.2}$$
$$[r_i^{(0)}]_-^+ = 0, \tag{5.3}$$
$$[\phi^{(0)}]_-^+ = 0, \tag{5.4}$$
$$[n_i^{(0)}\phi_{,i}^{(0)}]_-^+ = 0, \tag{5.5}$$
$$[p^{(0)}]_-^+ = 0, \tag{5.6}$$
$$[\rho]_-^+ = \rho^\pm. \tag{5.7}$$

5.2.1.2 Equations for the incremental fields: isopotential–local form

The *isopotential–local* form of the incremental field equations and interface conditions governing the problem is obtained upon using (2.48), (2.49) and (3.14) to express the local increments $p^{(\Delta)}$ and $t_{ij}^{(\Delta)}$ in terms of the corresponding isopotential increments:

$$p^{(\Delta)} = p^{(\partial)} - p_{,i}^{(0)} d_i, \tag{5.8}$$
$$t_{ij}^{(\Delta)} = t_{ij}^{(\partial)} + \delta_{ij} p_{,k}^{(0)} d_k. \tag{5.9}$$

In order that d_i be eliminated, we note that, by the definition of isopotential increments, $\phi^{(\partial)} = 0$. Consideration of (2.48) to (2.49) thus yields $\phi^{(0)}_{,i} d_i = -\phi^{(\Delta)}$ in this particular case, which, in view of (5.1), is equivalent to $p^{(0)}_{,i} d_i = -\rho^{(0)}\phi^{(\Delta)}$. Equations (5.8) and (5.9) can therefore be rewritten as

$$p^{(\Delta)} = p^{(\partial)} + \rho^{(0)}\phi^{(\Delta)}, \tag{5.10}$$

$$t^{(\Delta)}_{ij} = t^{(\partial)}_{ij} - \delta_{ij}\rho^{(0)}\phi^{(\Delta)}. \tag{5.11}$$

On account of this, the isopotential–local form of the incremental field equations and interface conditions corresponding to (3.168)–(3.175) is found to be

$$t^{(\partial)}_{ij,j} = 0, \tag{5.12}$$

$$\phi^{(\Delta)}_{,ii} = 0, \tag{5.13}$$

$$t^{(\partial)}_{ij} = -\delta_{ij} p^{(\partial)} + \int_0^t \mu(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt', \tag{5.14}$$

$$u_{i,i} = 0, \tag{5.15}$$

$$[u_i]^+_- = 0, \tag{5.16}$$

$$[\phi^{(\Delta)}]^+_- = 0, \tag{5.17}$$

$$[n^{(0)}_i(\phi^{(\Delta)}_{,i} - 4\pi G\rho u_i)]^+_- = -4\pi G\sigma, \tag{5.18}$$

$$[n^{(0)}_j t^{(\partial)}_{ij} - \rho n^{(0)}_i(\phi^{(0)}_{,j} u_j + \phi^{(\Delta)})]^+_- = \gamma n^{(0)}_i \sigma. \tag{5.19}$$

Note that the incremental momentum equation, (5.12), involves only the isopotential incremental stress and thus *formally* agrees with the ordinary momentum equation valid in the absence of initial stress and gravitation. However, effects due to the initial stress and gravitation are introduced by (5.19), which explicitly involves $\rho\phi^{(0)}_{,i}u_i$ and $\rho\phi^{(\Delta)}$.

The first step toward integrating the incremental field equations and interface conditions is their Laplace-transformation from the (X_i,t)- to the (X_i,s)-domain. On account of (A.2), (A.3), (A.5) and $u^{(0)}_i = 0$, we get the following system of simultaneous differential equations and associated interface conditions for $\tilde{p}^{(\partial)}$, $\tilde{t}^{(\partial)}_{ij}$, $\tilde{u}_i$ and $\tilde{\phi}^{(\Delta)}$:

$$\tilde{t}^{(\partial)}_{ij,j} = 0, \tag{5.20}$$

$$\tilde{\phi}^{(\Delta)}_{,ii} = 0, \tag{5.21}$$

$$\tilde{t}^{(\partial)}_{ij} = -\tilde{p}^{(\partial)}\delta_{ij} + s\tilde{\mu}(\tilde{u}_{i,j} + \tilde{u}_{j,i}), \tag{5.22}$$

$$\tilde{u}_{i,i} = 0, \tag{5.23}$$

$$[u_i]^+_- = 0, \tag{5.24}$$

$$[\tilde{\phi}^{(\Delta)}]^+_- = 0, \tag{5.25}$$

$$[n^{(0)}_i(\tilde{\phi}^{(\Delta)}_{,i} - 4\pi G\rho\tilde{u}_i)]^+_- = -4\pi G\tilde{\sigma}, \tag{5.26}$$

$$[n^{(0)}_j \tilde{t}^{(\partial)}_{ij} - \rho n^{(0)}_i(\phi^{(0)}_{,j}\tilde{u}_j + \tilde{\phi}^{(\Delta)})]^+_- = \gamma n^{(0)}_i\tilde{\sigma}. \tag{5.27}$$

We proceed by eliminating $\widetilde{t}_{ij}^{(\partial)}$ from (5.20). Using (5.22) and (5.23) and assuming the spatial homogeneity of $\widetilde{\mu}$, we get

$$-\widetilde{p}_{,i}^{(\partial)} + s\widetilde{\mu}\widetilde{u}_{i,jj} = 0. \tag{5.28}$$

This can be rewritten by means of the rotation, $\widetilde{\omega}_i$, defined by

$$\widetilde{\omega}_i = \tfrac{1}{2}\epsilon_{ijk}\widetilde{u}_{k,j}. \tag{5.29}$$

With (5.23) and the identity $\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$, it follows that $\widetilde{u}_{i,jj} = -2\epsilon_{ijk}\widetilde{\omega}_{k,j}$, whose substitution into (5.28) gives

$$\widetilde{p}_{,i}^{(\partial)} + 2s\widetilde{\mu}\epsilon_{ijk}\widetilde{\omega}_{k,j} = 0. \tag{5.30}$$

Equations (5.21), (5.23), (5.29) and (5.30) constitute an alternative set of incremental field equations in terms of $\widetilde{p}^{(\partial)}$, $\widetilde{u}_i$, $\widetilde{\phi}^{(\Delta)}$ and $\widetilde{\omega}_i$. The incremental interface condition for $\widetilde{p}^{(\partial)}$ results upon substitution of (5.22) into (5.27):

$$[n_i^{(0)}\widetilde{p}^{(\partial)} - s\widetilde{\mu}n_j^{(0)}(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}) + \rho n_i^{(0)}(\phi_{,j}^{(0)}\widetilde{u}_j + \widetilde{\phi}^{(\Delta)})]_-^+ = -\gamma n_i^{(0)}\widetilde{\sigma}. \tag{5.31}$$

Equations (5.24)–(5.26) and (5.31) are the incremental interface conditions associated with the alternative set of incremental field equations.

5.2.2 Scalar equations in spherical coordinates

5.2.2.1 Geometrical considerations

Next, we consider the scalar forms in spherical coordinates of the field equations and interface conditions. As in Chapter 4, the material continuum will be confined to $\mathcal{R}_-$. Then, $\mu(t-t') = 0$, $p = 0$ and $\rho = 0$ in $\mathcal{R}_+$, so that u_i = continuous on $\partial\mathcal{R}$, u_i = indeterminate in $\mathcal{R}_+$ and the remaining incremental mechanical quantities vanish identically in $\mathcal{R}_+$. However, in difference to Chapter 4, we now suppose that the level surfaces of $p^{(0)}$ and $\phi^{(0)}$ are *concentric spheres*. This suggests to take the common centre of these surfaces as the origin, O, of the Cartesian coordinate system $OX_1X_2X_3$. The spherical coordinates, r, θ and λ, are related to the Cartesian coordinates, X_1, X_2 and X_3, by

$$\left.\begin{aligned} r &= \sqrt{X_1^2 + X_2^2 + X_3^2} \qquad (5.32)\\ \theta &= \tan^{-1}\tfrac{\sqrt{X_1^2+X_2^2}}{X_3} \qquad (5.33)\\ \lambda &= \tan^{-1}\tfrac{X_2}{X_3} \qquad (5.34)\end{aligned}\right\}\quad X_1 \in (0,\infty),\quad X_2 \in (0,\infty),\quad X_3 \in [0,\infty),$$

where $r \in (0,\infty)$ is the radial distance, $\theta \in (0,\pi)$ the colatitude and $\lambda \in [0,2\pi)$ the longitude of the *observation point*. For brevity, we refer to the material continuum occupying $\mathcal{R}_-$ as the *sphere* and to the material sheet occupying $\partial\mathcal{R}$ as the *load*. With a the radius of the sphere, we then have $\mathcal{X}_- = \{X_i \,|\, r \in (0,a)\}$, $\mathcal{X}_+ = \{X_i \,|\, r \in (a,\infty)\}$ and $\partial\mathcal{X} = \{X_i \,|\, r = a\}$. The label subscripts r, θ and λ appended to tensor symbols denote their appropriate scalar components in spherical coordinates, where the summation convention is no longer effective. Note that $(n_r^{(0)}, n_\theta^{(0)}, n_\lambda^{(0)}) = (1,0,0)$ applies on $\partial\mathcal{R}^{(0)}$.

5.2.2.2 Equations for the initial fields

On account of the spherical symmetry, we have $\partial_\theta = \partial_\lambda = 0$ and the relevant components of the initial field equations and interface conditions, (5.1)–(5.7), reduce to

$$p^{(0)}_{,r} - \rho\phi^{(0)}_{,r} = 0, \quad r < a, \tag{5.35}$$

$$(r^2\phi^{(0)}_{,r})_{,r} = \begin{cases} -4\pi r^2 G\rho, & r < a \\ 0, & r > a \end{cases}, \tag{5.36}$$

$$\left.\begin{aligned} [\phi^{(0)}]^+_- &= 0 \quad (5.37) \\ [\phi^{(0)}_{,r}]^+_- &= 0 \quad (5.38) \\ [p^{(0)}]_- &= 0 \quad (5.39) \end{aligned}\right\} \quad r = a.$$

These equations are to be supplemented by conditions ensuring that the initial fields remain bounded as $r \to 0$ and $r \to \infty$.

5.2.2.3 Equations for the incremental fields: (r, θ, s)-domain

Since the solution functions in the (r, θ, λ, t)-domain are to be expressed in terms of Green's functions representing the contributions from point loads (§ 5.4.4.2), it is sufficient to restrict the following derivations to *axisymmetric* perturbations. For convenience, we let the X_3-axis coincide with the symmetry axis. Then, $\partial_\lambda = 0$ and, in particular, $\widetilde{u}_\lambda = 0$, and the relevant components of (5.21), (5.23), (5.29) and (5.30) simplify to

$$\left.\begin{aligned} &\widetilde{u}_{r,\theta} - (r\widetilde{u}_\theta)_{,r} + 2r\widetilde{\omega}_\lambda = 0 \quad (5.40) \\ &r\sin\theta\, \widetilde{p}^{(\partial)}_{,r} + 2s\widetilde{\mu}(\sin\theta\,\widetilde{\omega}_\lambda)_{,\theta} = 0 \quad (5.41) \\ &\widetilde{p}^{(\partial)}_{,\theta} - 2s\widetilde{\mu}(r\widetilde{\omega}_\lambda)_{,r} = 0 \quad (5.42) \\ &\sin\theta\,(r^2\widetilde{u}_r)_{,r} + r(\sin\theta\,\widetilde{u}_\theta)_{,\theta} = 0 \quad (5.43) \end{aligned}\right\} \quad r < a,$$

$$\sin\theta\,(r^2\widetilde{\phi}^{(\Delta)}_{,r})_{,r} + (\sin\theta\,\widetilde{\phi}^{(\Delta)}_{,\theta})_{,\theta} = 0, \quad r \neq a. \tag{5.44}$$

Equations (5.40)–(5.43) are four simultaneous partial differential equations of first order for the *mechanical* quantities $\widetilde{p}^{(\partial)}$, $\widetilde{u}_r$, $\widetilde{u}_\theta$ and $\widetilde{\omega}_\lambda$; note that they are decoupled from (5.44), which is a second-order partial differential equation in terms of the *gravitational* quantity $\widetilde{\phi}^{(\Delta)}$.

The solutions to the equations must satisfy the appropriate incremental interface conditions. In view of the symmetry of the problem, we find for the relevant components of (5.24)–(5.26) and (5.31) the expressions

$$\left.\begin{aligned} &[\widetilde{p}^{(\partial)} - 2s\widetilde{\mu}\widetilde{u}_{r,r} + \rho(\widetilde{\phi}^{(\Delta)} + \phi^{(0)}_{,r}\widetilde{u}_r)]_- = \gamma\widetilde{\sigma} \quad (5.45) \\ &[\widetilde{u}_{r,\theta} + r\widetilde{u}_{\theta,r} - \widetilde{u}_\theta]_- = 0 \quad (5.46) \\ &[\widetilde{\phi}^{(\Delta)}]^+_- = 0 \quad (5.47) \\ &[\widetilde{\phi}^{(\Delta)}_{,r}]^+_- + [4\pi G\rho\widetilde{u}_r]_- = -4\pi G\widetilde{\sigma} \quad (5.48) \end{aligned}\right\} \quad r = a.$$

These equations are to be supplemented by conditions ensuring the boundedness of the incremental fields as $r \to 0$ and $r \to \infty$.

5.3 Integration of the equations

In this section, the scalar equations compiled above are solved. We begin with the special solution to the initial field equations and interface conditions (§ 5.3.1). Following this, we seek fundamental solutions to the incremental field equations and interface conditions in terms of Legendre polynomials (§ 5.3.2). Based on this assumption, the special solution with respect to r is deduced in three steps. First, we establish the general solution to the system of four simultaneous ordinary differential equations of first order governing the mechanical quantities (§ 5.3.2.1). After that, we derive the general solution for the gravitational quantities (§ 5.3.2.2). Finally, we determine the integration constants using the incremental interface conditions (§ 5.3.2.3).

5.3.1 Solution for the initial fields

The special solution to (5.35) and (5.36) which satisfies (5.37)–(5.39) and remains bounded as $r \to 0$ and $r \to \infty$ is well known (e. g. Ramsey, 1981, pp. 45–51). In order that this solution be expressed concisely, we introduce the non-dimensional radial distance, $R = r/a$. Using also $\partial_R = a\partial_r$ and

$$\gamma = \tfrac{4}{3}\pi G a \rho, \tag{5.49}$$

the following formulae for $\phi^{(0)}$ and $g_r^{(0)} = \phi^{(0)}_{,r}$ are obtained:

$$\phi^{(0)} = \begin{cases} \tfrac{1}{2}a\gamma(3 - R^2), & R < 1 \\ a\gamma R^{-1}, & R > 1 \end{cases}, \tag{5.50}$$

$$g_r^{(0)} = \begin{cases} -\gamma R, & R < 1 \\ -\gamma R^{-2}, & R > 1 \end{cases}. \tag{5.51}$$

In (5.50), the additive constant in the potential function has been chosen such that $\lim_{r\to\infty} \phi^{(0)} = 0$. Also, (5.51) shows that, in accordance with (3.50), γ equals the magnitude of the initial gravity on the interface. The special solution for the initial pressure takes the form

$$p^{(0)} = \tfrac{1}{2}a\gamma\rho(1 - R^2), \quad R < 1. \tag{5.52}$$

Equations (5.49), (5.50) and (5.52) show that the initial state is completely determined if any two of the parameters a, γ and ρ are given.

5.3.2 Solution for the incremental fields: (r, n, s)-domain

We seek solutions to (5.40)–(5.44) subject to (5.45)–(5.48) in terms of Legendre polynomials of the first kind, $P_n(\cos\theta)$, where $n \in \{0, 1, \ldots\}$ is the degree of the polynomial (e. g. Lebedev, 1972, pp. 44–51). We recall that $P_n(\cos\theta)$ satisfies Legendre's equation:

$$P_{n,\theta\theta}(\cos\theta) + \cot\theta\, P_{n,\theta}(\cos\theta) + n(n+1)\, P_n(\cos\theta) = 0. \tag{5.53}$$

5.3.2.1 General solution for the mechanical quantities

We suppose fundamental solutions with respect to θ of the form

$$\tilde{u}_r(r,\theta,s) = \tilde{U}_{rn}(r,s)\,\tilde{\varsigma}_n(s)\,P_n(\cos\theta), \tag{5.54}$$

$$\tilde{u}_\theta(r,\theta,s) = -\tilde{U}_{\theta n}(r,s)\,\tilde{\varsigma}_n(s)\,P_{n,\theta}(\cos\theta), \tag{5.55}$$

$$\tilde{p}^{(\partial)}(r,\theta,s) = \tilde{P}_n^{(\partial)}(r,s)\,\tilde{\varsigma}_n(s)\,P_n(\cos\theta), \tag{5.56}$$

$$\tilde{\omega}_\lambda(r,\theta,s) = -\tilde{\Omega}_{\lambda n}(r,s)\,\tilde{\varsigma}_n(s)\,P_{n,\theta}(\cos\theta), \tag{5.57}$$

where $\tilde{\varsigma}_n(s)$ is the *non-dimensional* Legendre coefficient of $\tilde{\sigma}(\theta,s)$ (§ 5.3.2.3) and $\tilde{F}_n(r,s)$ the *normalized* Legendre coefficient of $\tilde{f}(r,\theta,s)$. Note that $\tilde{F}_n(r,s)$ is *assumed* to be a Laplace transform, which is confirmed below (§ 5.4.3). To proceed further, we distinguish the degrees $n = 0$ and $n \geq 1$. For brevity, the arguments of the functions will usually be suppressed.

Degree $n = 0$:
With $P_0' = 0$, equations (5.55) and (5.57) show that we may put

$$\tilde{U}_{\theta 0} = \tilde{\Omega}_{\lambda 0} = 0. \tag{5.58}$$

Therefore, (5.40)–(5.43) reduce to

$$\left.\begin{aligned} &r\tilde{P}_{0,r}^{(\partial)} = 0 &(5.59)\\ &r\tilde{U}_{r0,r} + 2\tilde{U}_{r0} = 0 &(5.60)\end{aligned}\right\}\quad r < a,$$

whose general solutions are

$$\left.\begin{aligned} &\tilde{U}_{r0} = A^{(2)}R^{-2} &(5.61)\\ &\tilde{P}_0^{(\partial)} = A^{(1)} &(5.62)\end{aligned}\right\}\quad R < 1,$$

with $A^{(1)}$ and $A^{(2)}$ arbitrary constants.

Degrees $n \geq 1$:
Upon substitution of (5.54)–(5.57) and use of (5.53), equations (5.40)–(5.43) take the forms

$$\left.\begin{aligned} &\tilde{U}_{rn} + r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n} - 2r\tilde{\Omega}_{\lambda n} = 0 &(5.63)\\ &r\tilde{P}_{n,r}^{(\partial)} + 2n(n+1)s\tilde{\mu}\tilde{\Omega}_{\lambda n} = 0 &(5.64)\\ &\tilde{P}_n^{(\partial)} + 2s\tilde{\mu}(r\tilde{\Omega}_{\lambda n,r} + \tilde{\Omega}_{\lambda n}) = 0 &(5.65)\\ &r\tilde{U}_{rn,r} + 2\tilde{U}_{rn} + n(n+1)\tilde{U}_{\theta n} = 0 &(5.66)\end{aligned}\right\}\quad r < a.$$

We now introduce non-dimensional quantities, $Y_1, \ldots, Y_4$, by

$$\begin{bmatrix} \tilde{U}_{rn} \\ \tilde{U}_{\theta n} \\ \tilde{P}_n^{(\partial)} \\ \tilde{\Omega}_{\lambda n} \end{bmatrix} = \begin{bmatrix} a & & & \\ & a & & \\ & & 2s\tilde{\mu}R^{-1} & \\ & & & R^{-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \tag{5.67}$$

and consider fundamental solutions with respect to R of the form

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = R^{\kappa^{(k)}} \begin{bmatrix} Y_1^{(k)} \\ Y_2^{(k)} \\ Y_3^{(k)} \\ Y_4^{(k)} \end{bmatrix}. \tag{5.68}$$

Following successive substitution of (5.67) and (5.68), equations (5.63)–(5.66) can be recast into the following matrix equation:

$$\begin{bmatrix} \kappa^{(k)}+2 & n(n+1) & 0 & 0 \\ 1 & \kappa^{(k)}+1 & 0 & -2 \\ 0 & 0 & \kappa^{(k)}-1 & n(n+1) \\ 0 & 0 & 1 & \kappa^{(k)} \end{bmatrix} \begin{bmatrix} Y_1^{(k)} \\ Y_2^{(k)} \\ Y_3^{(k)} \\ Y_4^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.69}$$

Denoting the (4×4)-matrix by $[M_{ij}^{(k)}]$ and the non-vanishing (4×1)-matrix by $[Y_i^{(k)}]$, where $i,j \in \{1,2,3,4\}$, this equation can be abbreviated as $[M_{ij}^{(k)}][Y_j^{(k)}] = 0$, which has non-trivial solutions only if $\det[M_{ij}^{(k)}] = 0$. Expansion of the determinant gives the quartic equation

$$[(\kappa^{(k)}+1)(\kappa^{(k)}+2) - n(n+1)][\kappa^{(k)}(\kappa^{(k)}-1) - n(n+1)] = 0, \tag{5.70}$$

whose roots are

$$\boxed{\begin{aligned} \kappa^{(1)} &= n-1, & (5.71)\\ \kappa^{(2)} &= n+1, & (5.72)\\ \kappa^{(3)} &= -(n+2), & (5.73)\\ \kappa^{(4)} &= -n. & (5.74) \end{aligned}}$$

The determination of the eigenvectors, $[Y_i^{(k)}]$, associated with the eigenvalues, $\kappa^{(k)}$, follows standard procedures (Appendix B). The resulting expressions, (B.8), (B.10), (B.12) and (B.14), can be simplified by putting

$$\begin{aligned} Y_1^{(1)} &= n, & (5.75)\\ Y_3^{(2)} &= (n+1)(2n+3), & (5.76)\\ Y_1^{(3)} &= n+1, & (5.77)\\ Y_3^{(4)} &= n(2n-1). & (5.78) \end{aligned}$$

With this, the following eigenvectors are obtained:

$$\boxed{\begin{aligned} [Y_i^{(1)}] &= [n,-1,0,0]^T, & (5.79)\\ [Y_i^{(2)}] &= [n(n+1),-(n+3),(n+1)(2n+3),-(2n+3)]^T, & (5.80)\\ [Y_i^{(3)}] &= [n+1,1,0,0]^T, & (5.81)\\ [Y_i^{(4)}] &= [n(n+1),n-2,n(2n-1),2n-1]^T. & (5.82) \end{aligned}}$$

Since we have four fundamental solutions, the general solution with respect to R for the mechanical quantities can be written as

$$\boxed{[Y_i] = \sum_{k=1}^{4} A^{(k)} R^{\kappa^{(k)}} [Y_i^{(k)}], \quad R < 1,} \tag{5.83}$$

with the eigenvalues, $\kappa^{(k)}$, given by (5.71)–(5.74), the eigenvectors, $[Y_i^{(k)}]$, by (5.79)–(5.82) and the constants, $A^{(k)}$, arbitrary.

5.3.2.2 General solution for the gravitational quantities

We introduce $g_i^{(\Delta)} = \phi_{,i}^{(\Delta)}$, whose Laplace-transformed radial component is

$$\tilde{g}_r^{(\Delta)} = \tilde{\phi}_{,r}^{(\Delta)}. \tag{5.84}$$

Subsituting this into (5.44), we obtain

$$\sin\theta\,(r^2\tilde{g}_r^{(\Delta)})_{,r} + (\sin\theta\,\tilde{\phi}_{,\theta}^{(\Delta)})_{,\theta} = 0, \quad r \neq a. \tag{5.85}$$

As for the mechanical quantities, we consider fundamental solutions with respect to θ of the form

$$\tilde{\phi}^{(\Delta)}(r,\theta,s) = \tilde{\Phi}_n^{(\Delta)}(r,s)\,\tilde{\varsigma}_n(s)\,P_n(\cos\theta), \tag{5.86}$$

$$\tilde{g}_r^{(\Delta)}(r,\theta,s) = \tilde{G}_{rn}^{(\Delta)}(r,s)\,\tilde{\varsigma}_n(s)\,P_n(\cos\theta). \tag{5.87}$$

Upon substitution of these expressions and use of (5.53), equations (5.84) and (5.85) reduce to

$$\left.\begin{array}{l} \tilde{\Phi}_{n,r}^{(\Delta)} - \tilde{G}_{rn}^{(\Delta)} = 0 \quad (5.88) \\ n(n+1)\tilde{\Phi}_n^{(\Delta)} - r^2\tilde{G}_{rn,r}^{(\Delta)} - 2r\tilde{G}_{rn}^{(\Delta)} = 0 \quad (5.89) \end{array}\right\} \quad r \neq a.$$

Next, we introduce non-dimensional quantities, Z_1 and Z_2, by

$$\begin{bmatrix} \tilde{\Phi}_n^{(\Delta)} \\ \tilde{G}_{rn}^{(\Delta)} \end{bmatrix} = \begin{bmatrix} 3a\gamma & \\ & 3\gamma R^{-1} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \tag{5.90}$$

and consider fundamental solutions with respect to R of the form

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = R^{\lambda^{(\ell)}} \begin{bmatrix} Z_1^{(\ell)} \\ Z_2^{(\ell)} \end{bmatrix}. \tag{5.91}$$

Upon successive substitution of (5.90) and (5.91), equations (5.88) and (5.89) can be recast into the following matrix equation:

$$\begin{bmatrix} \lambda^{(\ell)} & -1 \\ n(n+1) & -(\lambda^{(\ell)}+1) \end{bmatrix} \begin{bmatrix} Z_1^{(\ell)} \\ Z_2^{(\ell)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.92}$$

Denoting the (2×2)-matrix by $[N_{ij}^{(\ell)}]$ and the non-vanishing (2×1)-matrix by $[Z_i^{(\ell)}]$, where $i,j \in \{1,2\}$, this equation can be abbreviated as $[N_{ij}^{(\ell)}][Z_j^{(\ell)}] = 0$, which has non-trivial solutions only if $\det[N_{ij}^{(\ell)}] = 0$. Expansion of the determinant gives the quadratic equation

$$\lambda^{(\ell)}(\lambda^{(\ell)}+1) - n(n+1) = 0, \tag{5.93}$$

whose roots are

$$\boxed{\begin{aligned} \lambda^{(1)} &= n, & (5.94)\\ \lambda^{(2)} &= -(n+1). & (5.95) \end{aligned}}$$

The elements of the eigenvectors, $[Z_i^{(\ell)}]$, associated with the eigenvalues, $\lambda^{(\ell)}$, are found to satisfy

$$Z_2^{(1)} = nZ_1^{(1)}, \tag{5.96}$$

$$Z_2^{(2)} = -(n+1)Z_1^{(2)}, \tag{5.97}$$

which, upon putting $Z_1^{(1)} = Z_1^{(2)} = 1$, yield the following eigenvectors:

$$\boxed{\begin{aligned} [Z_i^{(1)}] &= [1,n]^T, & (5.98)\\ [Z_i^{(2)}] &= [1,-(n+1)]^T. & (5.99) \end{aligned}}$$

Since we have two fundamental solutions, the general solution with respect to R for the gravitational quantities takes the form

$$\boxed{[Z_i] = \sum_{k=1}^{2} B^{(\ell)} R^{\lambda^{(\ell)}} [Z_i^{(\ell)}], \quad R \neq 1,} \tag{5.100}$$

with the eigenvalues, $\lambda^{(\ell)}$, given by (5.94) and (5.95), the eigenvectors, $[Z_i^{(\ell)}]$, by (5.98) and (5.99) and the constants, $B^{(\ell)}$, arbitrary.

5.3.2.3 Special solution

We now adjust the general solutions with respect to R to the incremental interface conditions. This requires that the interface-mass density is of the form

$$\tilde{\sigma}(\theta, s) = \tilde{\sigma}_n(s)\, P_n(\cos\theta), \tag{5.101}$$

where $\tilde{\sigma}_n(s)$ is the *ordinary* Legendre coefficient of $\tilde{\sigma}(\theta, s)$. Its relation to the *non-dimensional* Legendre coefficient, $\tilde{\varsigma}_n(s)$, used above is

$$\tilde{\sigma}_n(s) = a\rho(2n+1)\,\tilde{\varsigma}_n(s). \tag{5.102}$$

In view of

$$\int_0^\pi P_n(\cos\theta)\sin\theta\, d\theta = \begin{cases} 2, & n = 0 \\ 0, & n \geq 1 \end{cases}, \tag{5.103}$$

we notice that $\tilde{\sigma}_0(s)$ corresponds to an *accreted* load and $\tilde{\sigma}_n(s)$ for $n \geq 1$ to a *redistributed* load.

Degree $n = 0$:
As for the initial fields, we seek solutions that remain bounded as $r \to 0$ and $r \to \infty$. According to (5.61), this requires $A^{(2)} = 0$. Considering also (5.58) and (5.62), we thus have

$$\left.\begin{aligned} \tilde{U}_{r0} = \tilde{U}_{\theta 0} = \tilde{\Omega}_{\lambda 0} = 0 \qquad (5.104)\\ \tilde{P}_0^{(\vartheta)} = A^{(1)} \qquad (5.105) \end{aligned}\right\} \quad R < 1.$$

Imposing the constraint $\lim\limits_{r\to\infty} \tilde{\Phi}^{(\delta)} = 0$, it follows from (5.86), (5.87), (5.90), (5.94), (5.95) and (5.98)–(5.100) that

$$\tilde{\Phi}_0^{(\Delta)} = \begin{cases} 3a\gamma B^{(1)}, & R < 1 \\ 3a\gamma B^{(2)} R^{-1}, & R > 1 \end{cases}, \tag{5.106}$$

$$\tilde{G}_{r0}^{(\Delta)} = \begin{cases} 0, & R < 1 \\ -3\gamma B^{(2)} R^{-2}, & R > 1 \end{cases}. \tag{5.107}$$

The three constants are determined using (5.45)–(5.48). Expressing the incremental fields in these equations in terms of (5.54)–(5.56), (5.86), (5.87) and (5.101), we obtain with (5.49)–(5.51), (5.84), (5.102) and (5.104) the conditions

$$\left.\begin{aligned} [\tilde{P}_0^{(\vartheta)} + \rho\tilde{\Phi}_0^{(\Delta)}]_- = a\gamma\rho \qquad (5.108)\\ [\tilde{\Phi}_0^{(\Delta)}]_-^+ = 0 \qquad (5.109)\\ [a\rho\tilde{G}_{r0}^{(\Delta)}]_-^+ = -3a\gamma\rho \qquad (5.110) \end{aligned}\right\} \quad R = 1.$$

Substituting (5.105)–(5.107), this becomes

$$A^{(1)} = -2a\gamma\rho, \tag{5.111}$$

$$B^{(1)} = B^{(2)} = 1, \tag{5.112}$$

which completely determines the special solution with respect to R for $n = 0$.

Degrees $n \geq 1$:
Again, we require a bounded solution for $r \to 0$ and $r \to \infty$. Observing the signs in (5.71)–(5.74), (5.94) and (5.95), equations (5.83) and (5.100) reduce to

$$\boxed{\begin{aligned} &[Y_i] = A^{(1)} R^{n-1} [Y_i^{(1)}] + A^{(2)} R^{n+1} [Y_i^{(2)}], \quad R < 1, && (5.113)\\ &[Z_i] = \begin{cases} B^{(1)} R^n [Z_i^{(1)}], & R < 1 \\ B^{(2)} R^{-(n+1)} [Z_i^{(2)}], & R > 1 \end{cases}. && (5.114) \end{aligned}}$$

The four constants in (5.113) and (5.114) are determined following steps similar to those taken for degree $n = 0$. We obtain the conditions

$$\left.\begin{aligned} [\tilde{P}_n^{(\vartheta)} - 2s\tilde{\mu}\tilde{U}_{rn,r} + \rho(\tilde{\Phi}_n^{(\Delta)} - \gamma\tilde{U}_{rn})]_- = a\gamma\rho(2n+1) \qquad (5.115)\\ [\tilde{U}_{rn} - r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n}]_- = 0 \qquad (5.116)\\ [\tilde{\Phi}_n^{(\Delta)}]_-^+ = 0 \qquad (5.117)\\ [a\rho\tilde{G}_{rn}^{(\Delta)}]_-^+ + [3\gamma\rho\tilde{U}_{rn}]_- = -3a\gamma\rho(2n+1) \qquad (5.118) \end{aligned}\right\} \quad R = 1,$$

which, using (5.67) and (5.90), can be rewritten as

$$[-2s\tilde{\mu}Y_{1,R} - a\gamma\rho Y_1 + 2s\tilde{\mu}Y_3 + 3a\gamma\rho Z_1]_- = a\gamma\rho(2n+1), \tag{5.119}$$

$$[Y_1 - Y_{2,R} + Y_2]_- = 0, \tag{5.120}$$

$$[Z_1]_-^+ = 0, \tag{5.121}$$

$$[Y_1]_- + [Z_2]_-^+ = -(2n+1). \tag{5.122}$$

In view of (5.113) and (5.114), this further transforms into

$$\begin{aligned}
&\{-2s\tilde{\mu}[(n-1)Y_1^{(1)} - Y_3^{(1)}] - a\gamma\rho Y_1^{(1)}\}A^{(1)} \\
&+ \{-2s\tilde{\mu}[(n+1)Y_1^{(2)} - Y_3^{(2)}] - a\gamma\rho Y_1^{(2)}\}A^{(2)} \\
&\qquad + 3a\gamma\rho Z_1^{(1)}B^{(1)} = a\gamma\rho(2n+1),
\end{aligned} \tag{5.123}$$

$$[Y_1^{(1)} - (n-2)Y_2^{(1)}]A^{(1)} + (Y_1^{(2)} - nY_2^{(2)})A^{(2)} = 0, \tag{5.124}$$

$$Z_1^{(1)}B^{(1)} - Z_1^{(2)}B^{(2)} = 0, \tag{5.125}$$

$$Y_1^{(1)}A^{(1)} + Y_1^{(2)}A^{(2)} - Z_2^{(1)}B^{(1)} + Z_2^{(2)}B^{(2)} = -(2n+1). \tag{5.126}$$

Substituting (5.79), (5.80), (5.98) and (5.99) and eliminating $B^{(2)}$ by means of (5.125), we arrive at the following matrix equation:

$$\begin{bmatrix} -n\left[\frac{2(n-1)}{a\gamma\rho}s\tilde{\mu}+1\right] & -(n+1)\left[\frac{2(n^2-n-3)}{a\gamma\rho}s\tilde{\mu}+n\right] & 3 \\ n-1 & n(n+2) & 0 \\ -n & -n(n+1) & 2n+1 \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ B^{(1)} \end{bmatrix} = \begin{bmatrix} 2n+1 \\ 0 \\ 2n+1 \end{bmatrix}. \tag{5.127}$$

If $n \geq 2$, its solution is given by

$$A^{(1)} = -(n+2)\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho}, \tag{5.128}$$

$$A^{(2)} = \frac{n-1}{n}\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho}, \tag{5.129}$$

$$B^{(1)} = B^{(2)} = \frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, \tag{5.130}$$

$$k_n = \frac{2n^2+4n+3}{an}, \tag{5.131}$$

with k_n the Legendre wave-number.

If $n = 1$, equation (5.127) reduces to

$$\begin{bmatrix} -1 & \frac{2(6s\tilde{\mu}-a\gamma\rho)}{a\gamma\rho} & 3 \\ 0 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ B^{(1)} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}. \tag{5.132}$$

Since the determinant of the square matrix vanishes, the system is underdetermined and no unique solution exists. Considering the 'reduced' system

$$\begin{bmatrix} 0 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 3(1-B^{(1)}) \end{bmatrix}, \tag{5.133}$$

we may put

$$B^{(1)} = B^{(2)} = C, \tag{5.134}$$

whence we find

$$A^{(1)} = 3(C-1), \tag{5.135}$$

$$A^{(2)} = 0. \tag{5.136}$$

Constant C can be determined by bearing in mind that, for redistributed loads, sphere and load constitute a closed system whose centre of mass remains unperturbed. Since $P_1 = \cos\theta$, this is satisfied for $n = 1$ only if the interface-mass density associated with the load is anulled by the effective interface-mass density resulting from the perturbation of the surface of the sphere. The mathematical expression of this condition is $[\rho\tilde{u}_r]_- = -\tilde{\sigma}$, which, by (5.54), (5.101) and (5.102) for $n = 1$, is equivalent to $[\tilde{U}_{r\,1}]_- = -3a$. However, using (5.67), (5.113) and (5.135), we alternatively obtain $[\tilde{U}_{r\,1}]_- = -3a(1-C)$. Hence, $C = 0$ and (5.134)–(5.136) simplify to

$$A^{(1)} = -3, \tag{5.137}$$

$$A^{(2)} = B^{(1)} = B^{(2)} = 0. \tag{5.138}$$

With (5.128)–(5.131), (5.137) and (5.138), the special solution with respect to R is also completely determined for $n \geq 1$.

5.4 Solution functions for the incremental fields

We begin by compiling the solution functions for the individual fields in the (r, n, s)-domain, where *transfer* functions are introduced (§ 5.4.1). This is followed by the specification of the shear-relaxation function for *generalized Maxwell* viscoelasticity (§ 5.4.2). After that, the solution functions are transformed to the (r, n, t)-domain, where *impulse-response* functions are established (§ 5.4.3). Finally, we consider the transformation to the (r, θ, λ, t)-domain and provide the appropriate *Green's* functions (§ 5.4.4).

5.4.1 Functions in the (r, n, s)-domain

We first give closed-form solution functions for the following fields: material displacement, isopotential incremental pressure and local incremental potential (§ 5.4.1.1); isopotential height (§ 5.4.1.2); strain (§ 5.4.1.3); rotation (§ 5.4.1.4); material, isopotential and local incremental stress (§ 5.4.1.5); and material, isopotential and local incremental gravity (§ 5.4.1.6). After that, the relation between the Lamé–Kelvin and the Boussinesq–Cerruti problems is discussed (§ 5.4.1.7). Finally, the general form of the individual solution functions is established and transfer functions are defined (§ 5.4.1.8). All formulae are written in terms of the non-dimensional radial distance, $R = r/a$, with the degrees $n = 0$, $n = 1$ and $n \geq 2$ considered explicitly for each field. We recall that, for redistributed loads, $\tilde{\sigma}_0 = 0$, so that $n = 0$ is without relevance in this case.

5.4.1.1 Material displacement, isopotential incremental pressure and local incremental potential

The incremental fields $\tilde{u}_r$, $\tilde{u}_\theta$, $\tilde{p}^{(\partial)}$ and $\tilde{\phi}^{(\Delta)}$ are *basic* in the sense that all other incremental fields can be expressed in terms of them or their spatial derivatives. For convenient reference, we recollect the pertinent fundamental solutions with respect to θ listed as (5.54)–(5.56) and (5.86) above:

$$\tilde{u}_r = \tilde{U}_{rn}\tilde{\varsigma}_n P_n, \tag{5.139}$$

$$\tilde{u}_\theta = -\tilde{U}_{\theta n}\tilde{\varsigma}_n P_{n,\theta}, \tag{5.140}$$

$$\tilde{p}^{(\partial)} = \tilde{P}_n^{(\partial)}\tilde{\varsigma}_n P_n, \tag{5.141}$$

$$\tilde{\phi}^{(\Delta)} = \tilde{\Phi}_n^{(\Delta)}\tilde{\varsigma}_n P_n. \tag{5.142}$$

Degree $n = 0$:
In view of (5.111) and (5.112), equations (5.104)–(5.106) reduce to

$$\left.\begin{aligned} \tilde{U}_{r0} &= 0 \quad (5.143)\\ \tilde{U}_{\theta 0} &= 0 \quad (5.144)\\ \tilde{P}_0^{(\partial)} &= -2a\gamma\rho \quad (5.145)\end{aligned}\right\} \quad R < 1,$$

$$\tilde{\Phi}_0^{(\Delta)} = \begin{cases} 3a\gamma, & R < 1 \\ 3a\gamma R^{-1}, & R > 1 \end{cases}. \tag{5.146}$$

Degree $n = 1$:
Using (5.67), (5.90), (5.113) and (5.114) and substituting for the eigenvectors and integration constants from (5.79), (5.80), (5.98), (5.99), (5.137) and (5.138), we obtain the formulae

$$\left.\begin{aligned} \tilde{U}_{r1} &= -3a \quad (5.147)\\ \tilde{U}_{\theta 1} &= -3a \quad (5.148)\\ \tilde{P}_1^{(\partial)} &= 0 \quad (5.149)\end{aligned}\right\} \quad R < 1,$$

$$\tilde{\Phi}_1^{(\Delta)} = 0, \quad R \neq 0. \tag{5.150}$$

Degrees $n \geq 2$:
From (5.67), (5.90), (5.113) and (5.114) and upon substitution of the eigenvectors and integration constants using (5.79), (5.80), (5.98), (5.99) and (5.128)–(5.130), we find

$$\boxed{\begin{aligned} &\left.\begin{aligned} \tilde{U}_{rn} &= -a\left[n(n+2)R^{n-1} - (n^2-1)R^{n+1}\right]\tfrac{\gamma\rho}{k_{ns}\bar{\mu}+\gamma\rho} \quad (5.151)\\ \tilde{U}_{\theta n} &= \tfrac{a}{n}\left[n(n+2)R^{n-1} - (n-1)(n+3)R^{n+1}\right]\tfrac{\gamma\rho}{k_{ns}\bar{\mu}+\gamma\rho} \quad (5.152)\\ \tilde{P}_n^{(\partial)} &= 2a\gamma\rho\,\tfrac{(2n+3)(n^2-1)}{2n^2+4n+3}\,R^n\,\tfrac{k_{ns}\bar{\mu}}{k_{ns}\bar{\mu}+\gamma\rho} \quad (5.153)\end{aligned}\right\} R < 1,\\ &\tilde{\Phi}_n^{(\Delta)} = \begin{cases} 3a\gamma R^n\,\tfrac{k_{ns}\bar{\mu}}{k_{ns}\bar{\mu}+\gamma\rho} & R < 1 \\ 3a\gamma R^{-(n+1)}\,\tfrac{k_{ns}\bar{\mu}}{k_{ns}\bar{\mu}+\gamma\rho} & R > 1 \end{cases}. \quad (5.154)\end{aligned}}$$

With formulae for the basic incremental fields, $\tilde{u}_r$, $\tilde{u}_\theta$, $\tilde{p}^{(\partial)}$ and $\tilde{\phi}^{(\Delta)}$, for the degrees $n = 0$, $n = 1$ and $n \geq 2$ established, we can now derive corresponding formulae for any other incremental field quantity of interest. In order that isopotential increments can be related to local or material increments, we first need the solution for the isopotential height, $\tilde{h}$.

5.4.1.2 Isopotential height

Since $\widetilde{\phi}^{(\partial)} = 0$ *per definitionem*, (2.48) and (2.49) lead to

$$\phi_{,i}^{(0)}\widetilde{d}_i = -\widetilde{\phi}^{(\Delta)}, \quad X_i \in \mathcal{X}_- \cup \mathcal{X}_+, \tag{5.155}$$

which, upon assigning to any level surface of ϕ locally the outward unit normal ν_i, can be replaced by

$$\nu_i^{(0)}\nu_j^{(0)}\phi_{,i}^{(0)}\widetilde{d}_j = -\widetilde{\phi}^{(\Delta)}, \quad X_i \in \mathcal{X}_- \cup \mathcal{X}_+. \tag{5.156}$$

An elementary consideration shows that, on the assumption of infinitesimal perturbations, $\nu_i^{(0)}\widetilde{d}_i$ equals the height of the current level surface $\phi = \phi_0$ passing through $X_i + d_i$ with respect to the associated initial level surface $\phi^{(0)} = \phi_0$ passing through X_i, as measured at X_i in the direction of $\nu_i^{(0)}$. We call $\nu_i^{(0)}\widetilde{d}_i$ the isopotential height. Introducing

$$\widetilde{h} = \nu_i^{(0)}\widetilde{d}_i, \tag{5.157}$$

it then follows from (5.156) with $g_r^{(0)} = \phi_{,r}^{(0)}$ in spherical coordinates

$$\widetilde{h} = -\frac{1}{g_r^{(0)}}\widetilde{\phi}^{(\Delta)}, \quad r \neq a. \tag{5.158}$$

Using (5.142) and

$$\widetilde{h} = \widetilde{H}_n\widetilde{\varsigma}_n P_n, \tag{5.159}$$

equation (5.158) takes the form

$$\widetilde{H}_n = -\frac{1}{g_r^{(0)}}\widetilde{\Phi}_n^{(\Delta)}, \quad r \neq a. \tag{5.160}$$

Degree $n = 0$:
Substituting (5.51) and (5.146), equation (5.160) reduces to

$$\widetilde{H}_0 = \begin{cases} 3aR^{-1}, & R < 1 \\ 3aR, & R > 1 \end{cases}. \tag{5.161}$$

We note that, with $g_r^{(0)}$ vanishing for $R = 0$ and $R \to \infty$, $\widetilde{H}_0$ becomes singular at these points.

Degree $n = 1$:
In view of (5.51) and (5.150), it follows from (5.160) that

$$\widetilde{H}_1 = 0. \tag{5.162}$$

Degrees $n \geq 2$:
Following substitution of (5.51) and (5.154), equation (5.160) becomes

$$\boxed{\widetilde{H}_n = \begin{cases} 3aR^{n-1}\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho}, & R < 1 \\ 3aR^{-(n-1)}\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho}, & R > 1 \end{cases}. \quad (5.163)}$$

5.4.1.3 Strain

In Cartesian-tensor notation, the strain, $\tilde{e}_{ij}$, is defined by

$$\tilde{e}_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i}), \quad X_i \in \mathcal{X}_-. \tag{5.164}$$

The non-vanishing components of $\tilde{e}_{ij}$ in spherical coordinates are given by

$$\left.\begin{aligned}
\tilde{e}_{rr} &= \tilde{u}_{r,r} && (5.165)\\
\tilde{e}_{r\theta} &= \tfrac{1}{2r}(\tilde{u}_{r,\theta} + r\tilde{u}_{\theta,r} - \tilde{u}_\theta) && (5.166)\\
\tilde{e}_{\theta\theta} &= \tfrac{1}{r}(\tilde{u}_r + \tilde{u}_{\theta,\theta}) && (5.167)\\
\tilde{e}_{\lambda\lambda} &= \tfrac{1}{r}(\tilde{u}_r + \cot\theta\,\tilde{u}_\theta) && (5.168)
\end{aligned}\right\} \quad r < a.$$

If we introduce

$$\tilde{e}_{rr} = \tilde{E}_{rrn}\tilde{\varsigma}_n P_n, \tag{5.169}$$

$$\tilde{e}_{r\theta} = -\tilde{E}_{r\theta n}\tilde{\varsigma}_n P_{n,\theta}, \tag{5.170}$$

$$\tilde{e}_{\theta\theta} = \tilde{E}^{(1)}_{\theta\theta n}\tilde{\varsigma}_n P_n - \tilde{E}^{(2)}_{\theta\theta n}\tilde{\varsigma}_n \cot\theta\, P_{n,\theta}, \tag{5.171}$$

$$\tilde{e}_{\lambda\lambda} = \tilde{E}^{(1)}_{\lambda\lambda n}\tilde{\varsigma}_n P_n - \tilde{E}^{(2)}_{\lambda\lambda n}\tilde{\varsigma}_n \cot\theta\, P_{n,\theta} \tag{5.172}$$

and observe (5.53), (5.139) and (5.140), we obtain from (5.165)–(5.168) the following formulae:

$$\left.\begin{aligned}
\tilde{E}_{rrn} &= \tilde{U}_{rn,r} && (5.173)\\
\tilde{E}_{r\theta n} &= -\tfrac{1}{2r}(\tilde{U}_{rn} - r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n}) && (5.174)\\
\tilde{E}^{(1)}_{\theta\theta n} &= \tfrac{1}{r}[\tilde{U}_{rn} + n(n+1)\tilde{U}_{\theta n}] && (5.175)\\
\tilde{E}^{(2)}_{\theta\theta n} &= -\tfrac{1}{r}\tilde{U}_{\theta n} && (5.176)\\
\tilde{E}^{(1)}_{\lambda\lambda n} &= \tfrac{1}{r}\tilde{U}_{rn} && (5.177)\\
\tilde{E}^{(2)}_{\lambda\lambda n} &= \tfrac{1}{r}\tilde{U}_{\theta n} && (5.178)
\end{aligned}\right\} \quad r < a.$$

Degree $n = 0$:
In view of (5.143) and (5.144), it follows from (5.173)–(5.178) that

$$\tilde{E}_{rr0} = \tilde{E}_{r\theta 0} = \tilde{E}^{(1)}_{\theta\theta 0} = \tilde{E}^{(2)}_{\theta\theta 0} = \tilde{E}^{(1)}_{\lambda\lambda 0} = \tilde{E}^{(2)}_{\lambda\lambda 0} = 0, \quad R < 1. \tag{5.179}$$

Degree $n = 1$:
Substituting (5.147) and (5.148) into (5.173)–(5.178), we arrive at

$$\left.\begin{aligned}
\tilde{E}_{rr1} &= 0 && (5.180)\\
\tilde{E}_{r\theta 1} &= 3R^{-1} && (5.181)\\
\tilde{E}^{(1)}_{\theta\theta 1} &= -9R^{-1} && (5.182)\\
\tilde{E}^{(2)}_{\theta\theta 1} &= 3R^{-1} && (5.183)\\
\tilde{E}^{(1)}_{\lambda\lambda 1} &= -3R^{-1} && (5.184)\\
\tilde{E}^{(2)}_{\lambda\lambda 1} &= -3R^{-1} && (5.185)
\end{aligned}\right\} \quad R < 1.$$

Degrees $n \geq 2$:
Using (5.151) and (5.152), we obtain from (5.173)–(5.178) the following formulae:

$$\left.\begin{array}{ll}\widetilde{E}_{rrn} = -(n-1)[n(n+2)R^{n-2} - (n+1)^2R^n]\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho} & (5.186)\\ \widetilde{E}_{r\theta n} = (n-1)(n+2)(R^{n-2} - R^n)\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho} & (5.187)\\ \widetilde{E}^{(1)}_{\theta\theta n} = (n+2)[n^2R^{n-2} - (n^2-1)R^n]\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho} & (5.188)\\ \widetilde{E}^{(2)}_{\theta\theta n} = -\frac{1}{n}[n(n+2)R^{n-2} - (n-1)(n+3)R^n]\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho} & (5.189)\\ \widetilde{E}^{(1)}_{\lambda\lambda n} = -[n(n+2)R^{n-2} - (n^2-1)R^n]\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho} & (5.190)\\ \widetilde{E}^{(2)}_{\lambda\lambda n} = \frac{1}{n}[n(n+2)R^{n-2} - (n-1)(n+3)R^n]\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho} & (5.191)\end{array}\right\} R < 1.$$

5.4.1.4 Rotation

The non-vanishing component of (5.29) in spherical coordinates is

$$\widetilde{\omega}_\lambda = -\frac{1}{2r}(\widetilde{u}_{r,\theta} - r\widetilde{u}_{\theta,r} - \widetilde{u}_\theta), \quad r < a. \tag{5.192}$$

Using (5.139), (5.140) and

$$\widetilde{\omega}_\lambda = -\widetilde{\Omega}_{\lambda n}\widetilde{\varsigma}_n P_{n,\theta}, \tag{5.193}$$

equation (5.192) takes the form

$$\widetilde{\Omega}_{\lambda n} = \frac{1}{2r}(\widetilde{U}_{rn} + r\widetilde{U}_{\theta n,r} + \widetilde{U}_{\theta n}), \quad r < a. \tag{5.194}$$

Degree $n = 0$:
By (5.143) and (5.144), equation (5.194) reduces to

$$\widetilde{\Omega}_{\lambda 0} = 0, \quad R < 1. \tag{5.195}$$

Degree $n = 1$:
With (5.147) and (5.148), equation (5.194) becomes

$$\widetilde{\Omega}_{\lambda 1} = -3R^{-1}, \quad R < 1. \tag{5.196}$$

Degrees $n \geq 2$:
Using (5.151) and (5.152), equation (5.194) can be written as

$$\boxed{\widetilde{\Omega}_{\lambda n} = -\frac{(n-1)(2n+3)}{n}R^n\frac{\gamma\rho}{k_n s\bar{\mu}+\gamma\rho}, \quad R < 1.} \tag{5.197}$$

5.4.1.5 Material, isopotential and local incremental stresses

The main reason for introducing the isopotential incremental stress, $\widetilde{t}^{(\vartheta)}_{ij}$, has been to decouple the incremental field equations for the mechanical quantities from those for the gravitational quantities (§ 5.2.1.2). Apart from that, $\widetilde{t}^{(\vartheta)}_{ij}$ may serve as a measure of the deviation from the hydrostatic equilibrium state. In this respect, it resembles the local incremental stress, $\widetilde{t}^{(\Delta)}_{ij}$, which provides such a measure in the absence of gravitational perturbations. On the other hand, observations of the incremental stress in planetary interiors usually refer to material particles and therefore yield values for the material incremental stress, $\tilde{t}_{ij}^{(\delta)}$.

The relations between $\tilde{t}_{ij}^{(\delta)}$, $\tilde{t}_{ij}^{(\partial)}$ and $\tilde{t}_{ij}^{(\Delta)}$ are conveniently expressed upon decomposition of the stresses according to

$$\left.\begin{aligned} \tilde{t}_{ij}^{(\delta)} &= \tfrac{1}{3}\tilde{t}_{kk}^{(\delta)}\delta_{ij} + \tilde{s}_{ij}^{(\delta)} \\ \tilde{t}_{ij}^{(\partial)} &= \tfrac{1}{3}\tilde{t}_{kk}^{(\partial)}\delta_{ij} + \tilde{s}_{ij}^{(\partial)} \\ \tilde{t}_{ij}^{(\Delta)} &= \tfrac{1}{3}\tilde{t}_{kk}^{(\Delta)}\delta_{ij} + \tilde{s}_{ij}^{(\Delta)} \end{aligned}\right\} \quad X_i \in \mathcal{X}_-, \qquad \begin{aligned}(5.198)\\(5.199)\\(5.200)\end{aligned}$$

where the first and second terms on the right-hand sides are the *spherical* and *deviatoric* increments, respectively. Since $\tilde{p}^{(\delta)} = -\tilde{t}_{ii}^{(\delta)}/3$, $\tilde{p}^{(\partial)} = -\tilde{t}_{ii}^{(\partial)}/3$, $\tilde{p}^{(\Delta)} = -\tilde{t}_{ii}^{(\Delta)}/3$ and $\tilde{s}_{ij}^{(\delta)} = \tilde{s}_{ij}^{(\partial)} = \tilde{s}_{ij}^{(\Delta)} = \tilde{s}_{ij}$, equations (5.198)–(5.200) may be recast into

$$\left.\begin{aligned} \tilde{t}_{ij}^{(\delta)} &= -\tilde{p}^{(\delta)}\delta_{ij} + \tilde{s}_{ij} \\ \tilde{t}_{ij}^{(\partial)} &= -\tilde{p}^{(\partial)}\delta_{ij} + \tilde{s}_{ij} \\ \tilde{t}_{ij}^{(\Delta)} &= -\tilde{p}^{(\Delta)}\delta_{ij} + \tilde{s}_{ij} \end{aligned}\right\} \quad X_i \in \mathcal{X}_-. \qquad \begin{aligned}(5.201)\\(5.202)\\(5.203)\end{aligned}$$

Spherical increments. With the spherical incremental stress equal to the negative of the incremental pressure, we may proceed by considering the incremental pressure. In view of (2.48) and (2.49), it follows for the three measures of increment in spherical coordinates

$$\left.\begin{aligned} \tilde{p}^{(\delta)} &= \tilde{p}^{(\partial)} - p_{,r}^{(0)}(\tilde{h} - \tilde{u}_r) \\ \tilde{p}^{(\Delta)} &= \tilde{p}^{(\partial)} - p_{,r}^{(0)}\tilde{h} \end{aligned}\right\} \quad r < a. \qquad \begin{aligned}(5.204)\\(5.205)\end{aligned}$$

With (5.139), (5.141), (5.159) and

$$\tilde{p}^{(\delta)} = (\tilde{P}_n^{(\Delta)} + \tilde{P}_n^{(\alpha)})\tilde{\varsigma}_n P_n, \tag{5.206}$$

$$\tilde{p}^{(\Delta)} = \tilde{P}_n^{(\Delta)}\tilde{\varsigma}_n P_n, \tag{5.207}$$

we find from (5.204) and (5.205) the following equations:

$$\left.\begin{aligned} \tilde{P}_n^{(\Delta)} &= \tilde{P}_n^{(\partial)} - p_{,r}^{(0)}\tilde{H}_n \\ \tilde{P}_n^{(\alpha)} &= p_{,r}^{(0)}\tilde{U}_{rn} \end{aligned}\right\} \quad r < a. \qquad \begin{aligned}(5.208)\\(5.209)\end{aligned}$$

Degree $n = 0$:
Upon substitution of (5.52), (5.143), (5.145) and (5.161), equations (5.208) and (5.209) reduce to

$$\left.\begin{aligned} \tilde{P}_0^{(\Delta)} &= a\gamma\rho \\ \tilde{P}_0^{(\alpha)} &= 0 \end{aligned}\right\} \quad R < 1. \qquad \begin{aligned}(5.210)\\(5.211)\end{aligned}$$

Degree $n = 1$:
In view of (5.52), (5.147), (5.149) and (5.162), equations (5.208) and (5.209) become

$$\left.\begin{aligned} \tilde{P}_1^{(\Delta)} &= 0 \\ \tilde{P}_1^{(\alpha)} &= 3a\gamma\rho R \end{aligned}\right\} \quad R < 1. \qquad \begin{aligned}(5.212)\\(5.213)\end{aligned}$$

Degrees $n \geq 2$:
Using (5.52), (5.151), (5.153) and (5.163), we get from (5.208) and (5.209) the formulae

$$\left.\begin{aligned} \widetilde{P}_n^{(\Delta)} &= 3a\gamma\rho\left[1+\tfrac{2}{3}\tfrac{(2n+3)(n^2-1)}{2n^2+4n+3}\right]R^n\,\tfrac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho} \qquad (5.214)\\ \widetilde{P}_n^{(\alpha)} &= a\gamma\rho\left[n(n+2)R^n-(n^2-1)R^{n+2}\right]\tfrac{\gamma\rho}{k_n s\widetilde{\mu}+\gamma\rho} \qquad (5.215)\end{aligned}\right\} \quad R<1.$$

Deviatoric increment. In Cartesian-tensor notation, comparison of (5.22), (5.164) and (5.202) yields

$$\widetilde{s}_{ij} = 2s\widetilde{\mu}\widetilde{e}_{ij}, \quad X_i \in \mathcal{X}_-, \tag{5.216}$$

whose non-vanishing components of $\widetilde{s}_{ij}$ in spherical coordinates are given by

$$\left.\begin{aligned} \widetilde{s}_{rr} &= 2s\widetilde{\mu}\widetilde{e}_{rr} \qquad (5.217)\\ \widetilde{s}_{r\theta} &= 2s\widetilde{\mu}\widetilde{e}_{r\theta} \qquad (5.218)\\ \widetilde{s}_{\theta\theta} &= 2s\widetilde{\mu}\widetilde{e}_{\theta\theta} \qquad (5.219)\\ \widetilde{s}_{\lambda\lambda} &= 2s\widetilde{\mu}\widetilde{e}_{\lambda\lambda} \qquad (5.220)\end{aligned}\right\} \quad r<a.$$

Introducing

$$\widetilde{s}_{rr} = \widetilde{S}_{rrn}\widetilde{\varsigma}_n P_n, \tag{5.221}$$

$$\widetilde{s}_{r\theta} = -\widetilde{S}_{r\theta n}\widetilde{\varsigma}_n P_{n,\theta}, \tag{5.222}$$

$$\widetilde{s}_{\theta\theta} = \widetilde{S}^{(1)}_{\theta\theta n}\widetilde{\varsigma}_n P_n - \widetilde{S}^{(2)}_{\theta\theta n}\widetilde{\varsigma}_n \cot\theta\, P_{n,\theta}, \tag{5.223}$$

$$\widetilde{s}_{\lambda\lambda} = \widetilde{S}^{(1)}_{\lambda\lambda n}\widetilde{\varsigma}_n P_n - \widetilde{S}^{(2)}_{\lambda\lambda n}\widetilde{\varsigma}_n \cot\theta\, P_{n,\theta} \tag{5.224}$$

and observing (5.169)–(5.172), we obtain from (5.217)–(5.220) the formulae

$$\left.\begin{aligned} \widetilde{S}_{rrn} &= 2s\widetilde{\mu}\widetilde{E}_{rrn} \qquad (5.225)\\ \widetilde{S}_{r\theta n} &= 2s\widetilde{\mu}\widetilde{E}_{r\theta n} \qquad (5.226)\\ \widetilde{S}^{(1)}_{\theta\theta n} &= 2s\widetilde{\mu}\widetilde{E}^{(1)}_{\theta\theta n} \qquad (5.227)\\ \widetilde{S}^{(2)}_{\theta\theta n} &= 2s\widetilde{\mu}\widetilde{E}^{(2)}_{\theta\theta n} \qquad (5.228)\\ \widetilde{S}^{(1)}_{\lambda\lambda n} &= 2s\widetilde{\mu}\widetilde{E}^{(1)}_{\lambda\lambda n} \qquad (5.229)\\ \widetilde{S}^{(2)}_{\lambda\lambda n} &= 2s\widetilde{\mu}\widetilde{E}^{(2)}_{\lambda\lambda n} \qquad (5.230)\end{aligned}\right\} \quad r<a.$$

Degree $n = 0$:
In view of (5.179), it follows from (5.225)–(5.230) that

$$\widetilde{S}_{rr0} = \widetilde{S}_{r\theta 0} = \widetilde{S}^{(1)}_{\theta\theta 0} = \widetilde{S}^{(2)}_{\theta\theta 0} = \widetilde{S}^{(1)}_{\lambda\lambda 0} = \widetilde{S}^{(2)}_{\lambda\lambda 0} = 0, \quad R<1. \tag{5.231}$$

Degree $n = 1$:
Upon substitution of (5.180)–(5.185) into (5.225)–(5.230) and definition of $k_1 = 9/a$, which is consistent with (5.131) for $n = 1$, the following formulae result:

$$\left.\begin{array}{rl}\widetilde{S}_{rr1} = & 0 \\ \widetilde{S}_{r\theta 1} = & \frac{2}{3}aR^{-1}k_1 s\widetilde{\mu} \\ \widetilde{S}^{(1)}_{\theta\theta 1} = & -2\,aR^{-1}k_1 s\widetilde{\mu} \\ \widetilde{S}^{(2)}_{\theta\theta 1} = & \frac{4}{3}aR^{-1}k_1 s\widetilde{\mu} \\ \widetilde{S}^{(1)}_{\lambda\lambda 1} = & -\frac{2}{3}aR^{-1}k_1 s\widetilde{\mu} \\ \widetilde{S}^{(2)}_{\lambda\lambda 1} = & -\frac{2}{3}aR^{-1}k_1 s\widetilde{\mu}\end{array}\right\} \quad R < 1. \qquad \begin{array}{r}(5.232)\\(5.233)\\(5.234)\\(5.235)\\(5.236)\\(5.237)\end{array}$$

Degrees $n \geq 2$:
Using (5.131) and substituting (5.186)-(5.191) into (5.225)-(5.230), we arrive at

$$\boxed{\left.\begin{array}{rl}\widetilde{S}_{rrn} = & -2a\gamma\rho\,\frac{n(n-1)}{2n^2+4n+3}\left[n(n+2)R^{n-2}-(n+1)^2R^n\right]\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho} \\ \widetilde{S}_{r\theta n} = & 2a\gamma\rho\,\frac{n(n-1)(n+2)}{2n^2+4n+3}\left(R^{n-2}-R^n\right)\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho} \\ \widetilde{S}^{(1)}_{\theta\theta n} = & 2a\gamma\rho\,\frac{n(n+2)}{2n^2+4n+3}\left[n^2R^{n-2}-(n^2-1)R^n\right]\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho} \\ \widetilde{S}^{(2)}_{\theta\theta n} = & -2a\gamma\rho\,\frac{1}{2n^2+4n+3}\left[n(n+2)R^{n-2}-(n-1)(n+3)R^n\right]\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho} \\ \widetilde{S}^{(1)}_{\lambda\lambda n} = & -2a\gamma\rho\,\frac{n}{2n^2+4n+3}\left[n(n+2)R^{n-2}-(n^2-1)R^n\right]\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho} \\ \widetilde{S}^{(2)}_{\lambda\lambda n} = & 2a\gamma\rho\,\frac{1}{2n^2+4n+3}\left[n(n+2)R^{n-2}-(n-1)(n+3)R^n\right]\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho}\end{array}\right\} R < 1. \quad \begin{array}{r}(5.238)\\(5.239)\\(5.240)\\(5.241)\\(5.242)\\(5.243)\end{array}}$$

5.4.1.6 Material, isopotential and local incremental gravity

With the isopotential height, $\widetilde{h}$, the material displacement, $\widetilde{u}_i$, and the local incremental potential, $\widetilde{\phi}^{(\Delta)}$, given, any of the increments of gravity, $\widetilde{g}_i^{(\delta)}$, $\widetilde{g}_i^{(\partial)}$ and $\widetilde{g}_i^{(\Delta)}$, can be calculated.

Local increment. We first consider the local incremental gravity, $\widetilde{g}_i^{(\Delta)} = \widetilde{\phi}^{(\Delta)}_{,i}$, whose non-vanishing components in spherical coordinates are

$$\left.\begin{array}{l}\widetilde{g}_r^{(\Delta)} = \widetilde{\phi}^{(\Delta)}_{,r} \\ \widetilde{g}_\theta^{(\Delta)} = \frac{1}{r}\widetilde{\phi}^{(\Delta)}_{,\theta}\end{array}\right\} \quad r \neq a. \qquad \begin{array}{r}(5.244)\\(5.245)\end{array}$$

If we define

$$\widetilde{g}_r^{(\Delta)} = \widetilde{G}^{(\Delta)}_{rn}\widetilde{\varsigma}_n P_n, \tag{5.246}$$

$$\widetilde{g}_\theta^{(\Delta)} = -\widetilde{G}^{(\Delta)}_{\theta n}\widetilde{\varsigma}_n P_{n,\theta} \tag{5.247}$$

and observe (5.142), we obtain from (5.244) and (5.245) the following equations:

$$\left.\begin{array}{rl}\widetilde{G}^{(\Delta)}_{rn} = & \widetilde{\Phi}^{(\Delta)}_{n,r} \\ \widetilde{G}^{(\Delta)}_{\theta n} = & -\frac{1}{r}\widetilde{\Phi}^{(\Delta)}_{n}\end{array}\right\} \quad r \neq a. \qquad \begin{array}{r}(5.248)\\(5.249)\end{array}$$

Degree $n = 0$:
In view of (5.146), it follows from (5.248) that

$$\widetilde{G}^{(\Delta)}_{r0} = \begin{cases} 0, & R < 1 \\ -3\gamma R^{-2}, & R > 1 \end{cases}, \tag{5.250}$$

whereas, since $P_0' = 0$, we may put

$$\tilde{G}_{\theta 0}^{(\Delta)} = 0, \quad R \neq 1. \tag{5.251}$$

Degree $n = 1$:
With (5.150), equations (5.248) and (5.249) give

$$\tilde{G}_{r1}^{(\Delta)} = \tilde{G}_{\theta 1}^{(\Delta)} = 0, \quad R \neq 1. \tag{5.252}$$

Degrees $n \geq 2$:
Following substitution of (5.154), equations (5.248) and (5.249) take the forms

$$\boxed{\begin{aligned} \tilde{G}_{rn}^{(\Delta)} &= \begin{cases} 3\gamma n R^{n-1} \frac{k_n s\bar{\mu}}{k_n s\bar{\mu}+\gamma\rho}, & R < 1 \\ -3\gamma(n+1)R^{-(n+2)} \frac{k_n s\bar{\mu}}{k_n s\bar{\mu}+\gamma\rho}, & R > 1 \end{cases}, & (5.253) \\ \tilde{G}_{\theta n}^{(\Delta)} &= \begin{cases} -3\gamma R^{n-1} \frac{k_n s\bar{\mu}}{k_n s\bar{\mu}+\gamma\rho}, & R < 1 \\ -3\gamma R^{-(n+2)} \frac{k_n s\bar{\mu}}{k_n s\bar{\mu}+\gamma\rho}, & R > 1 \end{cases}. & (5.254) \end{aligned}}$$

Material and isopotential increments. Observations of gravity changes on planetary surfaces usually refer to material points and are frequently reduced to the geoid. Therefore, it is necessary to relate $\tilde{g}_i^{(\Delta)}$ to the material incremental gravity, $\tilde{g}_i^{(\delta)}$, and the isopotential incremental gravity, $\tilde{g}_i^{(\partial)}$. In spherical coordinates and with $g_\theta^{(0)} = 0$, it follows from (2.48) and (2.49) that

$$\left.\begin{aligned} \tilde{g}_r^{(\delta)} &= \tilde{g}_r^{(\Delta)} + g_{r,r}^{(0)}\tilde{u}_r & (5.255) \\ \tilde{g}_\theta^{(\delta)} &= \tilde{g}_\theta^{(\Delta)} & (5.256) \end{aligned}\right\} \quad r < a,$$

$$\left.\begin{aligned} \tilde{g}_r^{(\partial)} &= \tilde{g}_r^{(\Delta)} + g_{r,r}^{(0)}\tilde{h} & (5.257) \\ \tilde{g}_\theta^{(\partial)} &= \tilde{g}_\theta^{(\Delta)} & (5.258) \end{aligned}\right\} \quad r \neq a.$$

Note that, by (5.256) and (5.258), the colatitudinal components of the three measures of incremental gravity are identical in the joint spatial domains. The following analysis is therefore limited to the radial components. If we introduce

$$\tilde{g}_r^{(\delta)} = (\tilde{G}_{rn}^{(\Delta)} + \tilde{G}_{rn}^{(\alpha)})\tilde{\varsigma}_n P_n, \tag{5.259}$$

$$\tilde{g}_r^{(\partial)} = \tilde{G}_{rn}^{(\partial)}\tilde{\varsigma}_n P_n \tag{5.260}$$

and observe (5.139), (5.159) and (5.246), we get from (5.255) and (5.257) the formulae

$$\left.\begin{aligned} \tilde{G}_{rn}^{(\alpha)} &= g_{r,r}^{(0)}\tilde{U}_{rn} & (5.261) \\ \tilde{G}_{rn}^{(\partial)} &= \tilde{G}_{rn}^{(\Delta)} + g_{r,r}^{(0)}\tilde{H}_n & (5.262) \end{aligned}\right\} \quad r \neq a.$$

Expressions for $\tilde{G}_{rn}^{(\Delta)}$ have been provided above as (5.250), (5.252) and (5.253).

Degree $n = 0$:
Upon substitution of (5.51), (5.143), (5.161) and (5.250), equations (5.261) and (5.262) become

$$\tilde{G}_{r0}^{(\alpha)} = 0, \quad R < 1, \tag{5.263}$$

$$\widetilde{G}_{r0}^{(\vartheta)} = \begin{cases} -3\gamma R^{-1}, & R < 1 \\ 3\gamma R^{-2}, & R > 1 \end{cases} . \tag{5.264}$$

Degree $n = 1$:

In view of (5.51), (5.147), (5.162) and (5.252), equations (5.261) and (5.262) take the forms

$$\widetilde{G}_{r1}^{(\alpha)} = 3\gamma, \quad R < 1, \tag{5.265}$$

$$\widetilde{G}_{r1}^{(\vartheta)} = 0, \quad R \neq 1. \tag{5.266}$$

Degrees $n \geq 2$:

Considering (5.51), (5.151), (5.163) and (5.253), we obtain from (5.261) and (5.262) the expressions

$$\boxed{\begin{aligned} &\widetilde{G}_{rn}^{(\alpha)} = \gamma\,[n(n+2)R^{n-1} - (n^2-1)R^{n+1}]\,\tfrac{\gamma\rho}{k_n s\widetilde{\mu}+\gamma\rho}, \quad R < 1, && (5.267) \\ &\widetilde{G}_{rn}^{(\vartheta)} = \begin{cases} 3\gamma(n-1)R^{n-1}\,\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho}, & R < 1 \\ -3\gamma(n-1)R^{-(n+2)}\,\frac{k_n s\widetilde{\mu}}{k_n s\widetilde{\mu}+\gamma\rho}, & R > 1 \end{cases} . && (5.268) \end{aligned}}$$

5.4.1.7 Half-space limit

If the wavelength of planetary deformations is sufficiently short compared with the radius of the planet, the sphericity becomes irrelevant and gravitational perturbations may be neglected. Then, the half-space theory developed in Chapter 4 is profitably employed and yields the desired results more easily than the spherical theory considered in the present chapter. The accuracy of the half-space approximation has been tested computationally for a number of earth models (Wolf, 1984; Amelung, 1991); here, we show how the solution functions derived for the sphere can be formally reduced to those for the half-space.

Since the problem is primarily of theoretical interest, we restrict our analysis to the radial displacement. Using (5.102), (5.139), (5.151), $R = 1 + x/a$ and $\widetilde{u}_r = \widetilde{u}_x$, it can be expressed as

$$\widetilde{u}_x = -\frac{(1+\frac{2}{n})(1+\frac{x}{a})^{n-1} - (1-\frac{1}{n^2})(1+\frac{x}{a})^{n+1}}{\frac{1}{n}(2+\frac{1}{n})[(2+\frac{4}{n}+\frac{3}{n^2})\frac{n}{a}s\widetilde{\mu} + \gamma\rho]}\,\gamma\widetilde{\sigma}_n P_n. \tag{5.269}$$

If we put $n/a = k$, this can be recast into

$$\widetilde{u}_x = -\Gamma_n(1 + \tfrac{1}{n}kx)^n\,\gamma\widetilde{\sigma}_n P_n, \tag{5.270}$$

where

$$\Gamma_n = \frac{(1+\frac{2}{n})(1+\frac{1}{n}kx)^{-1} - (1-\frac{1}{n^2})(1+\frac{1}{n}kx)}{\frac{1}{n}(2+\frac{1}{n})[(2+\frac{4}{n}+\frac{3}{n^2})ks\widetilde{\mu} + \gamma\rho]} \tag{5.271}$$

and, by the binomial theorem,

$$(1+\tfrac{1}{n}kx)^n = 1 + kx + (1-\tfrac{1}{n})\,\frac{(kx)^2}{2!} + (1-\tfrac{1}{n})(1-\tfrac{2}{n})\,\frac{(kx)^3}{3!} + \ldots\,. \tag{5.272}$$

We now suppose that $x/a \to 0$ and $1/n \to 0$ such that $k =$ finite. Following some manipulation, (5.271) then yields

$$\lim_{n\to\infty} \Gamma_n = \frac{1-kx}{2ks\widetilde{\mu}+\gamma\rho}, \tag{5.273}$$

whereas (5.272) takes the form

$$\lim_{n\to\infty}(1+\tfrac{1}{n}kx)^n = e^{kx}. \tag{5.274}$$

According to Hobson (1931, p. 299), $\tilde{\sigma}_n$ can always be chosen such that, correct to the order $1/\sqrt{n}$, the relation $\tilde{\sigma}_n P_n = \tilde{S}\sin[(n+\frac{1}{2})\theta + \frac{\pi}{4}]$ applies in some neighbourhood of θ. Selecting $\theta = \pi/2$ and putting $n\theta = ky$, this simplifies to

$$\tilde{\sigma}_n P_n = \tilde{S}\cos(ky). \tag{5.275}$$

In view of (5.270) and (5.273)–(5.275), we obtain for $x/a \to 0$ and $1/n \to 0$ the limit

$$\boxed{\tilde{u}_x = -\frac{\gamma\tilde{S}}{2ks\tilde{\mu}+\gamma\rho}(1-kx)e^{kx}\cos(ky).} \tag{5.276}$$

Equations (5.275) and (5.276) are fully consistent with (4.40), (4.41) and (4.61) and, accordingly, give the vertical displacement in a half-space subject to a fixed and homogeneous gravity field and a sinusoidal load of wavelength $2\pi/k$ and amplitude $\tilde{S}$.

5.4.1.8 Transfer functions

Inspection of the solution functions listed in §§ 5.4.1.1–5.4.1.6 shows that the ordinary Legendre coefficients, $\tilde{f}_n(r,s) = \tilde{F}_n(r,s)\,\tilde{\varsigma}_n(s)$, of the field quantities analysed can be decomposed according to

$$\boxed{\tilde{f}_n(r,s) = \overline{F}_n(r)\,\tilde{T}_n(s)\,\tilde{\varsigma}_n(s),} \tag{5.277}$$

which is the general form of the solution functions in the (r,n,s)-domain. Function $\overline{F}_n(r) = \tilde{F}_n(r,s)/\tilde{T}_n(s)$ specifies the radial dependence of $\tilde{f}_n(r,s)$ and can be directly obtained from the individual solution functions. Function $\tilde{T}_n(s)$ is referred to as transfer function and found to be of either of two types:

$$\tilde{T}_n^{(A)}(s) = \begin{cases} 1, & n=1 \\ \frac{\gamma\rho}{k_n s\tilde{\mu}(s)+\gamma\rho}, & n\ge 2 \end{cases}, \tag{5.278}$$

$$\tilde{T}_n^{(B)}(s) = \begin{cases} 1, & n=0 \\ k_1 s\tilde{\mu}(s), & n=1 \\ \frac{k_n s\tilde{\mu}(s)}{k_n s\tilde{\mu}(s)+\gamma\rho}, & n\ge 2 \end{cases}. \tag{5.279}$$

As in the foregoing equations, the arguments of all functions considered will henceforth be displayed.

5.4.2 Generalized Maxwell viscoelasticity

For the inversion of $\tilde{T}_n^{(A)}(s)$ for $n\ge 2$ and $\tilde{T}_n^{(B)}(s)$ for $n\ge 1$, it is necessary to specify $\tilde{\mu}(s)$. As in § 4.3.3, we start from the general formula

$$\mu(t-t') = \int_0^\infty \overline{\mu}(\alpha')\,e^{-\alpha'(t-t')}d\alpha', \tag{5.280}$$

which expresses $\mu(t-t')$ in terms of its spectrum, $\overline{\mu}(\alpha')$. We can approximate the latter to any degree of accuracy required by

$$\overline{\mu}(\alpha') = \sum_{q=1}^{Q} \mu^{(q)}\, \delta(\alpha' - \alpha^{(q)}), \tag{5.281}$$

where $\alpha^{(q)} > 0$ is the q-th inverse elemental Maxwell time and $\mu^{(q)} > 0$ the q-th elemental elastic shear modulus, both prescribed for $q \in \{1, 2, \ldots, Q\}$; as before, $\delta(\alpha' - \alpha^{(q)})$ denotes the Dirac delta-function. Upon substitution of (5.281) and use of the properties of the Dirac delta-function, (5.280) reduces to

$$\mu(t-t') = \sum_{q=1}^{Q} \mu^{(q)}\, e^{-\alpha^{(q)}(t-t')}, \tag{5.282}$$

which is the shear-relaxation function for *generalized Maxwell* viscoelasticity (e. g. Christensen, 1982[2], pp. 16–20; Müller, 1986; Wang, 1986). Recalling that $\mu_e = \lim_{t-t' \to 0} \mu(t-t')$, we obtain in particular

$$\mu_e = \sum_{q=1}^{Q} \mu^{(q)}. \tag{5.283}$$

By (A.2) and (A.7), the Laplace transform of (5.282) with respect to $t - t'$ is

$$\widetilde{\mu}(s) = \sum_{q=1}^{Q} \frac{\mu^{(q)}}{s + \alpha^{(q)}}. \tag{5.284}$$

5.4.3 Functions in the (r, n, t)-domain

5.4.3.1 Impulse-response functions

We proceed with the transformation of the solution functions specified in § 5.4.1 from the (r, n, s)- to the (r, n, t)-domain. This requires inverse Laplace transformation of (5.277). Using (A.5) and (A.19), we obtain

$$\boxed{f_n(r,t) = \overline{F}_n(r) \int_0^t T_n(t-t')\, \varsigma_n(t')\, dt'} \tag{5.285}$$

as general form of the solution functions in the (r, n, t)-domain. Function $T_n(t-t')$ is the impulse-response function associated with the transfer function $\widetilde{T}_n(s)$, which is of type $\widetilde{T}_n^{(A)}(s)$ or $\widetilde{T}_n^{(B)}(s)$.

With $\widetilde{\mu}(s)$ specified and the general functional form of $f_n(r,t)$ established, (5.278) and (5.279) can now be inverted. We consider the degrees $n = 0$, $n = 1$ and $n \geq 2$ individually.

Degree $n = 0$:
By (A.12) and (A.19), the shifted inverse Laplace transform of (5.279) is

$$T_0^{(B)}(t-t') = \delta(t-t'). \tag{5.286}$$

Degree $n = 1$:
As for $n = 0$, equation (5.278) leads to

$$T_1^{(A)}(t-t') = \delta(t-t'). \tag{5.287}$$

Upon substitution of (5.284) into (5.279), it follows that

$$\tilde{T}_1^{(B)}(s) = k_1 \sum_{q=1}^{Q} \mu^{(q)} \left(1 - \frac{\alpha^{(q)}}{s + \alpha^{(q)}}\right), \tag{5.288}$$

whence, in view of (A.2), (A.7), (A.12) and (A.19), the shifted inverse Laplace transform takes the form

$$T_1^{(B)}(t-t') = k_1 \sum_{q=1}^{Q} \mu^{(q)} [\delta(t-t') - \alpha^{(q)} e^{-\alpha^{(q)}(t-t')}]. \tag{5.289}$$

Degrees $n \geq 2$:
Substituting (5.284), we obtain from (5.278) the equation

$$\tilde{T}_n^{(A)}(s) = \gamma\rho \left(k_n \sum_{q=1}^{Q} \frac{\mu^{(q)} s}{s + \alpha^{(q)}} + \gamma\rho\right)^{-1}. \tag{5.290}$$

After some algebraic manipulation, this can be recast into

$$\tilde{T}_n^{(A)}(s) = \frac{\gamma\rho}{k_n \mu_e + \gamma\rho} [1 + \tilde{W}_n(s)], \tag{5.291}$$

where

$$\tilde{W}_n(s) = \frac{U_n(s)}{V_n(s)}, \tag{5.292}$$

$$U_n(s) = k_n \sum_{q=1}^{Q} \frac{\mu^{(q)} \alpha^{(q)}}{s + \alpha^{(q)}}, \tag{5.293}$$

$$V_n(s) = k_n \left(\mu_e - \sum_{q=1}^{Q} \frac{\mu^{(q)} \alpha^{(q)}}{s + \alpha^{(q)}}\right) + \gamma\rho. \tag{5.294}$$

Using (A.2), (A.12) and (A.19), the shifted inverse Laplace transform of (5.291) can formally be written as

$$\boxed{T_n^{(A)}(t-t') = \frac{\gamma\rho}{k_n \mu_e + \gamma\rho} [\delta(t-t') + W_n(t-t')].} \tag{5.295}$$

Comparing (5.278) and (5.279), we get

$$\tilde{T}_n^{(B)}(s) = 1 - \tilde{T}_n^{(A)}(s), \tag{5.296}$$

so that, with (5.295), (A.2), (A.12) and (A.19), the shifted inverse Laplace transform is found to be

$$\boxed{T_n^{(B)}(t-t') = \frac{\gamma\rho}{k_n \mu_e + \gamma\rho} \left[\frac{k_n \mu_e}{\gamma\rho} \delta(t-t') - W_n(t-t')\right].} \tag{5.297}$$

5.4.3.2 Stability analysis

It remains to establish the functional form of $W_n(t-t')$. Inspecting (5.292)–(5.294), we note that $\widetilde{W}_n(s)$ can be rewritten as the quotient of two polynomials in s (without common roots) of degrees $L = Q - 1$ in the enumerator and $M = Q$ in the denominator. Accordingly, the inverse Laplace transform of $\widetilde{W}(s)$ exists and can be specified upon determination of the roots of $V_n(s)$ (Appendix A). To prove that all roots are simple and negative, we assume that the M poles of $V_n(s)$ have been ordered such that $0 > -\alpha^{(1)} > -\alpha^{(2)} > \ldots > \alpha^{(M)}$. Considering the interval $\mathcal{I}^{(1)} = (-\alpha^{(1)}, 0)$ first, we note that $\lim_{s\to-\alpha^{(1)}+0} V_n(s) = -\infty$ and $V_n(0) = 1$, whence one root must lie in $\mathcal{I}^{(1)}$. The remaining roots are found by considering the interval $\mathcal{I}^{(m)} = (-\alpha^{(m)}, -\alpha^{(m-1)})$, where $m \in \{2, 3, \ldots, M\}$. Since $\lim_{s\to-\alpha^{(m)}+0} V_n(s) = -\infty$ and $\lim_{s\to-\alpha^{(m-1)}-0} V_n(s) = \infty$, one root must also lie in each of $\mathcal{I}^{(2)}, \mathcal{I}^{(3)}, \ldots, \mathcal{I}^{(M)}$. However, $V_n(s)$ can have either M roots if all are simple or less than M roots if at least one is multiple. Taking this into account, it then follows that there is *exactly* one root in the intervals $\mathcal{I}^{(1)}, \mathcal{I}^{(2)}, \ldots, \mathcal{I}^{(M)}$, respectively.

Having established that $\widetilde{V}_n(s)$ has M simple and negative roots, the functional form of $W_n(t-t')$ can now be given. Denoting the pole in $\mathcal{I}^{(m)}$ by $-\beta_n^{(m)}$, (A.21) yields

$$W_n(t-t') = \sum_{m=1}^{M} \frac{U_n^{(m)}}{V_n^{(m)}} e^{-\beta_n^{(m)}(t-t')}, \quad M = Q, \tag{5.298}$$

$$U_n^{(m)} = U_n(-\beta_n^{(m)}) = k_n \sum_{q=1}^{Q} \frac{\mu^{(q)}\alpha^{(q)}}{\alpha^{(q)} - \beta_n^{(m)}}, \tag{5.299}$$

$$V_n^{(m)} = d_s V_n(-\beta_n^{(m)}) = k_n \sum_{q=1}^{Q} \frac{\mu^{(q)}\alpha^{(q)}}{(\alpha^{(q)} - \beta_n^{(m)})^2}, \tag{5.300}$$

where $\alpha^{(q)}$ and $\mu^{(q)}$ are prescribed by (5.282). The m-th term of the sum in (5.298) is called the m-th relaxation mode, with $U_n^{(m)}/V_n^{(m)}$ the modal amplitude, $\beta_n^{(m)}$ the inverse modal relaxation time and M the total number of relaxation modes. With (5.131) and (5.283), these quantities completely determine $T_n^{(A)}(t-t')$ and $T_n^{(B)}(t-t')$ for $n \geq 2$ and, by (5.285), the corresponding solution functions in the (r, n, t)-domain.

5.4.3.3 Maxwell and Burgers viscoelasticity

General methods of obtaining closed-form expressions of the roots, $-\beta_n^{(m)}$, of $V_n(s)$ exist only for $M \leq 4$. In all other cases, numerical methods must normally be applied. In practice, such methods are even used for $M = 3$ or $M = 4$. Here, we evaluate $W_n(t-t')$ exactly for $M = 1$ and $M = 2$; in view of $M = Q$, this is equivalent to $Q = 1$ and $Q = 2$ corresponding to Maxwell viscoelasticity and Burgers viscoelasticity, respectively.

Case $M = 1$:
Equation (5.294) takes the simplified form

$$V_n(s) = \frac{(k_n\mu^{(1)} + \gamma\rho)s + \gamma\rho\alpha^{(1)}}{s + \alpha^{(1)}} \tag{5.301}$$

with the root

$$-\beta_n^{(1)} = -\frac{\gamma\rho}{k_n\mu^{(1)}+\gamma\rho}\,\alpha^{(1)}, \tag{5.302}$$

whereas (5.298)–(5.300) lead to reduces to

$$W_n(t-t') = \frac{k_n\mu^{(1)}\alpha^{(1)}}{k_n\mu^{(1)}+\gamma\rho}\,e^{-\beta_n^{(1)}(t-t')}. \tag{5.303}$$

Case $M = 2$:
Equation (5.294) reduces to

$$V_n(s) = \frac{(k_n\mu_e+\gamma\rho)s^2 + [(k_n\mu^{(2)}+\gamma\rho)\alpha^{(1)} + (k_n\mu^{(1)}+\gamma\rho)\alpha^{(2)}]s + \gamma\rho\alpha^{(1)}\alpha^{(2)}}{(s+\alpha^{(1)})(s+\alpha^{(2)})}, \tag{5.304}$$

which has the following roots:

$$\left.\begin{array}{r}-\beta_n^{(1)}\\ -\beta_n^{(2)}\end{array}\right\} = -\frac{(k_n\mu^{(2)}+\gamma\rho)\alpha^{(1)} + (k_n\mu^{(1)}+\gamma\rho)\alpha^{(2)}}{2(k_n\mu_e+\gamma\rho)}$$
$$\pm\frac{\sqrt{[(k_n\mu^{(2)}+\gamma\rho)\alpha^{(1)} - (k_n\mu^{(1)}+\gamma\rho)\alpha^{(2)}]^2 + 4k_n^2\mu^{(1)}\mu^{(2)}\alpha^{(1)}\alpha^{(2)}}}{2(k_n\mu_e+\gamma\rho)}. \tag{5.305}$$

Following substitution of (5.299) and (5.300), equation (5.298) becomes

$$W_n(t-t') = \sum_{m=1}^{2}\frac{(k_n\mu^{(1)}\alpha^{(1)} + k_n\mu^{(2)}\alpha^{(2)})\beta_n^{(m)} - k_n\mu_e\alpha^{(1)}\alpha^{(2)}}{2(k_n\mu_e+\gamma\rho)\beta_n^{(m)} + (k_n\mu^{(2)}+\gamma\rho)\alpha^{(1)} + (k_n\mu^{(1)}+\gamma\rho)\alpha^{(2)}}\,e^{-\beta_n^{(m)}(t-t')}. \tag{5.306}$$

5.4.4 Functions in the (r,θ,λ,t)-domain

The final step is the transformation of the solution function (5.285) from the (r,n,t)- to the (r,θ,λ,t)-domain. In the following, we will distinguish *axisymmetric* and *non-symmetric* loads and calculate the respective Green's functions.

5.4.4.1 Axisymmetric Green's functions

We first consider the Green's function for axisymmetric loads whose distribution is given by $\sigma(\theta',t')$, where θ' is the colatitude of the excitation point. On the assumption that $\sigma(\theta',t')$ is twice continuously differentiable with respect to θ' in $(0,\pi)$ and that $\int_0^\pi[\sigma(\theta',t')]^2\sin\theta'\,d\theta' =$ finite, the distribution can be expanded into a convergent Legendre series (e. g. Lebedev, 1972, pp. 53–60):

$$\sigma(\theta',t') = \sum_{n=n_0}^{\infty}\sigma_n(t')\,P_n(\cos\theta'), \tag{5.307}$$

with

$$\sigma_n(t') = (n+\tfrac{1}{2})\int_0^\pi\sigma(\theta',t')\sin\theta'\,P_n(\cos\theta')\,d\theta'. \tag{5.308}$$

We recall that, in the series (5.307), only the term $\sigma_0(t')$ corresponds to a net mass (§ 5.3.2.3); accordingly, $n_0 = 0$ applies if $\sigma(\theta',t')$ specifies an *accreted* load and $n_0 = 1$ if it specifies a *redistributed* load. In view of the linearity of the problem, the solution for the load given by (5.307) can be expressed as

$$f(r,\theta,t)=\sum_{n=n_0}^{\infty} f_n(r,t)\begin{cases} P_n(\cos\theta) \\ -P_{n,\theta}(\cos\theta) \\ -\cot\theta\, P_{n,\theta}(\cos\theta)\end{cases}, \tag{5.309}$$

which, upon use of (5.102) and (5.285), takes the form

$$f(r,\theta,t)=\frac{1}{a\rho}\sum_{n=n_0}^{\infty}\frac{\overline{F}_n(r)}{2n+1}\left\{\begin{array}{l} P_n(\cos\theta) \\ -P_{n,\theta}(\cos\theta) \\ -\cot\theta\, P_{n,\theta}(\cos\theta)\end{array}\right\}\int_0^t T_n(t-t')\,\sigma_n(t')\,dt'. \tag{5.310}$$

Substituting (5.308) and changing the sequence of summation and integrations, this becomes

$$f(r,\theta,t)=\int_0^{\pi}\int_0^t f^{(as)}(r,\theta,\theta',t-t')\,\sigma(\theta',t')\sin\theta'\,d\theta'\,dt', \tag{5.311}$$

$$f^{(as)}(r,\theta,\theta',t-t')=\frac{1}{2a\rho}\sum_{n=n_0}^{\infty}\overline{F}_n(r)\left\{\begin{array}{l} P_n(\cos\theta) \\ -P_{n,\theta}(\cos\theta) \\ -\cot\theta\, P_{n,\theta}(\cos\theta)\end{array}\right\}P_n(\cos\theta')\,T_n(t-t'), \tag{5.312}$$

where $f^{(as)}(r,\theta,\theta',t-t')$ denotes the axisymmetric Green's function for the (r,θ,t)-domain.

5.4.4.2 Non-symmetric Green's functions

It is now straightforward to deduce the Green's function for non-symmetric loads described by the distribution $\sigma(\theta',\lambda',t')$, where θ' and λ' are the colatitude and longitude of the excitation point, respectively. For this, we take into account that $f^{(as)}(r,\theta,\theta',t-t')$ gives the normalized contribution to $f(r,\theta,t)$ from an *annular* load at colatitude θ'. The contribution $f(r,\theta,t)$ from a *point* load on the symmetry axis thus follows from

$$f^{(ns)}(r,\theta,t-t')=\lim_{\theta'\to 0} f^{(as)}(r,\theta,\theta',t-t'). \tag{5.313}$$

Noting that $f(r,\theta,\lambda)$ for a non-symmetric load, $\sigma(\theta',\lambda',t')$, can be obtained by superposing the contributions from the appropriate distribution of point loads, the generalizations of (5.311) and (5.312) are

$$f(r,\theta,\lambda,t)=\int_0^{\pi}\int_0^{2\pi}\int_0^t f^{(ns)}(r,\vartheta,t-t')\,\sigma(\theta',\lambda',t')\sin\theta'\,d\theta'\,d\lambda'\,dt', \tag{5.314}$$

$$f^{(ns)}(r,\vartheta,t-t')=\frac{1}{2a\rho}\sum_{n=n_0}^{\infty}\overline{F}_n(r)\left\{\begin{array}{l} P_n(\cos\vartheta) \\ -P_{n,\vartheta}(\cos\vartheta) \\ -\cot\vartheta\, P_{n,\vartheta}(\cos\vartheta)\end{array}\right\}T_n(t-t'), \tag{5.315}$$

$$\cos\vartheta=\cos(\theta-\theta')\cos(\lambda-\lambda'), \tag{5.316}$$

where ϑ is the angle between the observation point and the excitation point and $f^{(ns)}(r, \vartheta, t-t')$ the non-symmetric Green's function for the (r, θ, λ, t)-domain. With $\overline{F}_n(r) = \tilde{F}_n(r,s)/\tilde{T}_n(s)$ implied by the solution functions listed in § 5.4.1, $T_n(t-t')$ of the types $T_n^{(A)}(t-t')$ or $T_n^{(B)}(t-t')$ given by (5.286), (5.287), (5.289), (5.295) and (5.297), and $\sigma(\theta', \lambda', t')$ prescribed, (5.314)–(5.316) completely specify the solution to the Lamé–Kelvin problem of gravitational viscoelasticity.

5.5 Summary

The results of the final chapter can be summarized as follows:

(i) We have deduced the complete solution describing infinitesimal, quasistatic, gravitational–viscoelastic perturbations, induced by surface loads, of a spherical, isochemical, incompressible, non-rotating, fluid planet initially in hydrostatic equilibrium. The kinematic formulation used is uniformly Lagrangian in the internal and external domains. The resulting solution functions are specified for the (r, n, s)-, (r, n, t)- and (r, θ, λ, t)-domains. They involve explicit expressions for the Legendre degrees $n = 0$, $n = 1$ and $n \geq 2$ and comprise all incremental field quantities of interest in studies of the deformation of planetary bodies.

(ii) A notable aspect of the solution is that it applies to arbitrary types of *generalized Maxwell* viscoelasticity. The inverse modal relaxation times characterizing a particular type are given as the poles of the quotient of two polynomials in s. Since all poles are simple and negative, the planet is always *stable*, with its impulse response involving a series of *exponential* decay modes.

(iii) The solution method used differs from the methods conventionally employed for the type of problem studied. Its main characteristic is that the incremental field equations are recast into two *decoupled* (4×4) and (2×2) first-order differential systems in terms of the mechanical and gravitational quantities of the problem, respectively; the coupling is restricted to the incremental interface conditions. Interesting features of our solution method are the following: (a) The complexity of the algebraic manipulations necessary to solve the decoupled (4×4)- and (2×2)-systems is distinctly less than that for the (6×6)-system commonly used. It is expected that this simplification is also of significance when modelling perturbations of layered planets. (b) The (4×4)-system obtained formally agrees with the system governing the corresponding problem without gravitation. Since the numerical simulation of viscoelastic deformations of planets has, so far, been based on techniques developed for non-gravitating continua, our solution method opens a way of consistently accounting for gravitation when using these techniques.

(iv) The decoupling of the incremental field equations is contingent upon the use of a quantity referred to as *isopotential* incremental pressure measuring the increment of the hydrostatic pressure with respect to a perturbed level surface of the potential. Isopotential increments can be defined for any field quantity but are of special interest to gravimetry, where the free-air gravity anomaly is usually calculated such that it agrees with the isopotential incremental gravity considered here.

Bibliography

Amelung, F., 1991. *Viskoelastische Perturbationen sphärischer Erdmodelle: Bedeutung der Selbstgravitation beim Modellieren glazial-induzierter Relaxationsvorgänge*, Diploma thesis, University of Münster, Münster.

Backus, G. E., 1967. Converting vector and tensor equations to scalar equations in spherical coordinates, *Geophys. J. R. Astron. Soc.*, **13**, 71–101.

Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge.

Biot, M. A., 1959. The influence of gravity on the folding of a layered viscoelastic medium under compression, *J. Franklin Inst.*, **267**, 211–228.

Biot, M. A., 1965. *Mechanics of Incremental Deformations*, J. Wiley, New York.

Boussinesq, J., 1885. *Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques, principalement au calcul des déformations et des pressions que produisent, dans ces solides, des efforts quelconques exercés sur une petite partie de leur surface ou de leur intérieur*, Gauthier-Villars, Paris.

Bullen, K. E., 1975. *The Earth's Density*, Chapman & Hall, London.

Caputo, M., 1961. Deformation of a layered earth by an axially symmetric surface mass distribution, *J. Geophys. Res.*, **66**, 1479–1483.

Caputo, M., 1962. Tables for the deformation of an earth model by surface mass distributions, *J. Geophys. Res.*, **67**, 1611–1616.

Cathles, L. M., 1975. *The Viscosity of the Earth's Mantle*, Princeton University Press, Princeton.

Cerruti, V., 1882. Ricerche intorno all' equilibrio de' corpi elastici isotropi, *Reale Accad. Linc. Roma, Ser. 3A*, **13**, 81–122.

Chandrasekhar, S., 1961. *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford.

Chandrasekhar, S., 1969. *Ellipsoidal Figures of Equilibrium*, Yale University Press, New Haven.

Christensen, R. M., 1982^2. *Theory of Viscoelasticity*, Academic Press, New York.

Clairaut, A. C., 1743. *Théorie de la figure de la terre, tirée des principes de l'hydrostatique*, Durand, Paris.

Dahlen, F. A., 1972. Elastic dislocation theory for a self-gravitating elastic configuration with an initial static stress field, *Geophys. J. R. Astron. Soc.*, **28**, 357–383.

Dahlen, F. A., 1973. Elastic dislocation theory for a self-gravitating elastic configuration with an initial static stress field II: energy release, *Geophys. J. R. Astron. Soc.*, **31**, 469–484.

Dahlen, F. A., 1974. On the static deformation of an earth model with a fluid core, *Geophys. J. R. Astron. Soc.*, **36**, 461–485.

Dahlen, F. A. & Fels, S. B., 1978. A physical explanation of the static core paradox, *Geophys. J. R. Astron. Soc.*, **55**, 317–331.

Darwin, G. H., 1879. On the bodily tides of viscous and semi-elastic spheroids, and on the ocean tides upon a yielding nucleus, *Phil. Trans. R. Soc. London, Part 1*, **170**, 1–35.

Dragoni, M., Yuen, D. A. & Boschi, E., 1983. Global postseismic deformation in a stratified viscoelastic earth: effects on Chandler wobble excitation, *J. Geophys. Res.*, **88**, 2240–2250.

Eringen, A. C., 1962. *Non-linear Theory of Continuous Media*, McGraw-Hill, New York.

Farrell, W. E., 1972. Deformation of the earth by surface loads, *Rev. Geophys. Space Phys.*, **10**, 761–797.

Fjeldskaar, W. & Cathles, L. M., 1984. Measurement requirements for glacial uplift detection of nonadiabatic density gradients in the mantle, *J. Geophys. Res.*, **89**, 10115–10124.

Gasperini, P. & Sabadini, R., 1989. Lateral heterogeneities in mantle viscosity and post-glacial rebound, *Geophys. J. Int.*, **98**, 413–428.

Geller, R. J., 1988a. Elastodynamics in a laterally heterogeneous, self-gravitating body, *Geophys. J. Int.*, **94**, 271–283.

Geller, R. J., 1988b. On the derivation of the elastic equation of motion, *J. Phys. Earth*, **36**, 201–228.

Golden, J. M. & Graham, G. A. C., 1988. *Boundary Value Problems in Linear Viscoelasticity*, Springer-Verlag, Berlin.

Grafarend, E. W., 1982. Six lectures on geodesy and global geodynamics, *Mitt. Geodät. Inst. Techn. Univ. Graz*, **41**, 531–685.

Harding, J. W. & Sneddon, I. N., 1945. The elastic stresses produced by the indentation of the plane surface of a semi-infinite elastic solid by a rigid punch, *Proc. Cambridge Phil. Soc.*, **41**, 16–26.

Haskell, N. A., 1935. The motion of a viscous fluid under a surface load, *Physics*, **6**, 265–269.

Haskell, N. A., 1936. The motion of a viscous fluid under a surface load, 2, *Physics*, **7**, 56–61.

Herglotz, G., 1905. Über die Elastizität der Erde bei Berücksichtigung ihrer variablen Dichte, *Z. Math. Phys.*, **52**, 275–299.

Hobson, E. W., 1931. *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, Cambridge.

Hoskins, L. M., 1910. The strain of a non-gravitating sphere of variable density, *Trans. Am. Math. Soc.*, **11**, 494–504.

Hoskins, L. M., 1920. The strain of a gravitating sphere of variable density and elasticity, *Trans. Am. Math. Soc.*, **21**, 1–43.

Jarvis, G. T. & McKenzie, D. P., 1980. Convection in a compressible fluid with infinite Prandtl number, *J. Fluid Mech.*, **96**, 515–583.

Jeffreys, H., 1926. The stability of a layer of fluid heated below, *Phil. Mag.*, **2**, 833–844.

Jeffreys, H., 1928. Some cases of instability in fluid motion, *Proc. R. Soc. London, Ser. A*, **118**, 195–208.

Jeffreys, H., 1976[6]. *The Earth: Its Origin, History and Physical Constitution*, Cambridge University Press, Cambridge.

Kaufmann, G., 1991. *Auswirkungen adiabatischer und chemischer Dichtegradienten auf das Relaxationsverhalten eines Maxwell-Halbraums*, Diploma thesis, University of Münster, Münster.

Kopal, Z., 1960. *Figures of Equilibrium of Celestial Bodies, with Emphasis on Problems of Motion of Artificial Satellites*, University of Wisconsin Press, Madison.

Körnig, M. & Müller, G., 1989. Rheological models and interpretation of postglacial uplift, *Geophys. J. Int.*, **98**, 243–253.

Krauss, W., 1973. *Methods and Results of Theoretical Oceanography, Vol. 1: Dynamics of the Homogeneous and the Quasihomogeneous Ocean*, Bornträger, Berlin.

Kuo, J. T., 1969. Static response of a multilayered medium under inclined surface loads, *J. Geophys. Res.*, **74**, 3195–3207.

Lamb, H., 1917. On the deflection of the vertical by tidal loading of the earth's surface, *Proc. R. Soc. London, Ser. A*, **93**, 293–312.

Lamé, G., 1854. Mémoire sur l'équilibre d'élasticité des enveloppes sphériques, *Liouville's J. Math. Pures Appl.*, **19**, 51–87.

Lamé, G. & Clapeyron, E., 1831. Mémoire sur l'équilibre intérieur des corps solides homogènes, *Crelle's J. Reine Angew. Math.*, **7**, 381–413.

Lanzano, P., 1982. *Deformations of an Elastic Earth*, Academic Press, New York.

Lapwood, E. R. & Usami, T., 1981. *Free Oscillations of the Earth*, Cambridge University Press, Cambridge.

Lebedev, N. N., 1972. *Special Functions and their Applications*, Dover, New York.

LePage, W. R., 1980. *Complex Variables and the Laplace Transform for Engineers*, Dover, New York.

Li, G. & Yuen, D. A., 1987. Viscous relaxation of a compressible spherical shell, *Geophys. Res. Lett.*, **14**, 1227–1230.

Longman, I. M., 1962. A Green's function for determining the deformation of the earth under surface mass loads I: theory, *J. Geophys. Res.*, **67**, 845–850.

Longman, I. M., 1963. A Green's function for determining the deformation of the earth under surface mass loads, 2: computations and numerical results, *J. Geophys. Res.*, **68**, 485–496.

Longman, I. M., 1975. On the stability of a spherical gravitating compressible liquid planet without spin, *Geophys. J. R. Astron. Soc.*, **42**, 621–635.

Love, A. E. H., 1908. The gravitational stability of the earth, *Phil. Trans. R. Soc. London, Ser. A*, **207**, 171–241.

Love, A. E. H., 1911. *Some Problems of Geodynamics*, Cambridge University Press, Cambridge.

Love, A. E. H., 1929. The stress produced in a semi-infinite solid by pressure on part of the boundary, *Phil. Trans. R. Soc. London, Ser. A*, **228**, 377–420.

Malvern, L. E., 1969. *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs.

Melchior, P., 1966. *The Earth Tides*, Pergamon Press, New York.

Moritz, H., 1990. *The Figure of the Earth*, H. Wichmann, Karlsruhe.

Müller, G., 1986. Generalized Maxwell bodies and estimates of mantle viscosity, *Geophys. J. R. Astron. Soc.*, **87**, 1113–1141.

Nakiboglu, S. M. & Lambeck, K., 1982. A study of the earth's response to surface loading with application to Lake Bonneville, *Geophys. J. R. Astron. Soc.*, **70**, 577–620.

Oberbeck, A., 1879. Ueber die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen, *Ann. Phys. Chem.*, **7**, 271–292.

Oberbeck, A., 1880. Strömungen von Flüssigkeiten infolge ungleicher Temperatur innerhalb derselben, *Ann. Phys. Chem.*, **11**, 489–495.

Parsons, B. E., 1972. *Changes in the Earth's Shape*, Ph. D. thesis, Cambridge University, Cambridge.

Peltier, W. R., 1974. The impulse response of a Maxwell earth, *Rev. Geophys. Space Phys.*, **12**, 649–669.

Peltier, R., 1982. Dynamics of the ice age earth, *Adv. Geophys.*, **24**, 1–146.

Peltier, W. R., ed., 1989. *Mantle Convection, Plate Tectonics and Geodynamics*, Gordon and Breach, New York.

Ramsey, A. S., 1981. *Newtonian Attraction*, Cambridge University Press, Cambridge.

Ranalli, G., 1987. *Rheology of the Earth: Deformation and Flow Processes in Geophysics and Geodynamics,* Allen & Unwin, Boston.

Rayleigh, Lord, 1906. On the dilatational stability of the earth, *Proc. R. Soc. London, Ser. A,* **77**, 486–499.

Rayleigh, Lord, 1916. On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side, *Phil. Mag.,* **32**, 529–546.

Rümpker, G., 1990. *Viskoelastische Relaxation eines Burgers-Halbraums: Theorie und Anwendung,* Diploma thesis, University of Münster, Münster.

Sabadini, R., Yuen, D. A., Boschi, E., 1982. Polar wandering and the forced responses of a rotating, multilayered, viscoelastic planet, *J. Geophys. Res.,* **87**, 2885–2903.

Slichter, L. B. & Caputo, M., 1960. Deformation of an earth model by surface pressures, *J. Geophys. Res.,* **65**, 4151–4156.

Sneddon, I. N., 1946. The elastic stresses produced in a thick plate by the application of pressure to its free surfaces, *Proc. Cambridge Phil. Soc.,* **42**, 260–271.

Svensson, S. L., 1980. A model for glacial uplift, in *Earth Rheology, Isostasy, and Eustasy,* pp. 65–72, ed. Mörner, N.-A., J. Wiley, Chichester.

Takeuchi, H., 1950. On the earth tide of the compressible earth of variable density and elasticity, *Trans. Am. Geophys. Union,* **31**, 651–689.

Terazawa, K., 1916. On the elastic equilibrium of a semi-infinite solid under given boundary conditions, with some applications, *J. College Sci. Tokyo,* **37**, 7, 1–64.

Thomson, W., 1864. Dynamical problems regarding elastic spheroidal shells and spheroids of incompressible liquid, *Phil. Trans. R. Soc. London,* **153**, 583–616.

Truesdell, C. & Noll, W., 1965[1], 1992[2]. *The Nonlinear Field Theories of Mechanics,* Springer-Verlag, Berlin.

Vermeersen, L. L. A. & Vlaar, N. J., 1991. The gravito-elastodynamics of a pre-stressed elastic earth, *Geophys. J. Int.,* **104**, 555–536.

Wang, R., 1986. *Das viskoelastische Verhalten der Erde auf langfristige Gezeitenterme,* Diploma thesis, University of Kiel, Kiel.

Williamson, E. D. & Adams, L. H., 1923. Density distribution in the earth, *J. Washington Acad. Sci.,* **13**, 413–428.

Wolf, D., 1984. The relaxation of spherical and flat Maxwell earth models and effects due to the presence of the lithosphere, *J. Geophys.,* **56**, 24–33.

Wolf, D., 1985a. Thick-plate flexure re-examined, *Geophys. J. R. Astron. Soc.,* **80**, 265–273.

Wolf, D., 1985b. On Boussinesq's problem for Maxwell continua subject to an external gravity field, *Geophys. J. R. Astron. Soc.,* **80**, 275–279.

Wolf, D., 1985c. The normal modes of a uniform, compressible Maxwell half-space, *J. Geophys.,* **56**, 100–105.

Wolf, D., 1985d. The normal modes of a layered, incompressible Maxwell half-space, *J. Geophys.,* **57**, 106–117.

Wu, P., 1990. Deformation of internal boundaries in a viscoelastic earth and topographic coupling between the mantle and the core, *Geophys. J. Int.,* **101**, 213–231.

Wu, P., 1992a. Deformation of an incompressible viscoelastic flat earth with power-law creep: a finite element approach, *Geophys. J. Int.,* **108**, 35–51.

Wu, P., 1992b. Viscoelastic versus viscous deformation and the advection of prestress, *Geophys. J. Int.,* **108**, 136–142.

Wu, P. & Peltier, W. R., 1982. Viscous gravitational relaxation, *Geophys. J. R. Astron. Soc.,* **70**, 435–485.

Appendix A

Laplace transforms

A.1 Forward transforms

The Laplace transform, $\mathcal{L}[f(t)]$, of a function, $f(t)$, is defined by (e. g. LePage, 1980, pp. 285–318)

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st} dt, \quad s \in \mathcal{S}, \tag{A.1}$$

where s is the inverse Laplace time and $\mathcal{S}$ the complex domain. We suppose here that $f(t)$ is continuous for all $t \in \mathcal{T}$ and of exponential order as $t \to \infty$, which are sufficient conditions for the convergence of the Laplace integral in (A.1) for Re s larger than some value, s_R. Defining $\mathcal{L}[f(t)] = \tilde{f}(s)$ and assuming the same properties for $g(t)$, elementary consequences are

$$\mathcal{L}[a\, f(t) + b\, g(t)] = a\, \tilde{f}(s) + b\, \tilde{g}(s), \quad a, b = \text{constant}, \tag{A.2}$$

$$\mathcal{L}[\partial_t f(t)] = s\, \tilde{f}(s) - f(0), \tag{A.3}$$

$$\mathcal{L}\left[\int_0^t f(t')\, dt'\right] = \frac{\tilde{f}(s)}{s}, \tag{A.4}$$

$$\mathcal{L}\left[\int_0^t f(t-t')\, g(t')\, dt'\right] = \tilde{f}(s)\, \tilde{g}(s), \tag{A.5}$$

$$\mathcal{L}[1] = \frac{1}{s}, \tag{A.6}$$

$$\mathcal{L}[e^{-s_0 t}] = \frac{1}{s+s_0}, \quad s_0 = \text{constant}. \tag{A.7}$$

To find the Laplace transform of the symmetric Dirac delta-function, $\delta(t-t_0)$, we regard it as the limit of a series of continuous functions:

$$\delta(t-t_0) = \lim_{\epsilon \to 0} \delta^{(\epsilon)}(t-t_0), \quad \epsilon \in [0, \infty), \tag{A.8}$$

where the approximation function has the following properties:

$$\delta^{(\epsilon)}(t-t_0) = \begin{cases} 0, & t \in (-\infty, t_0) \\ \text{positive}, & t \in [t_0, t_0+\epsilon] \\ 0, & t \in (t_0+\epsilon, \infty) \end{cases}, \tag{A.9}$$

$$\int_{t_0}^{t_0+\epsilon} \delta^{(\epsilon)}(t'-t_0)\,dt' = 1. \tag{A.10}$$

The Laplace transform of $\delta^{(\epsilon)}(t-t_0)$ is given by $\mathcal{L}[\delta^{(\epsilon)}(t-t_0)] = \int_{t_0}^{t_0+\epsilon} \delta^{(\epsilon)}(t'-t_0)\exp(-st')\,dt'$. Using the mean-value theorem of integral calculus and (A.10), we obtain $\mathcal{L}[\delta^{(\epsilon)}(t-t_0)] = \exp[-s(t_0+\epsilon_0)]$ where $\epsilon_0 \in [0,\epsilon]$. If ϵ is sufficiently small that $s\epsilon \ll 1$, this can be replaced by

$$\mathcal{L}[\delta^{(\epsilon)}(t-t_0)] = e^{-t_0 s}(1-\epsilon s), \tag{A.11}$$

whence, defining $\mathcal{L}[\delta(t-t_0)] = \lim_{\epsilon\to 0}\mathcal{L}[\delta^{(\epsilon)}(t-t_0)]$, it follows that

$$\boxed{\mathcal{L}[\delta(t-t_0)] = e^{-t_0 s}, \quad t_0 = \text{constant.}} \tag{A.12}$$

Next, we consider the symmetric Heaviside step-function, $H(t-t_0)$, which we regard as the limit of a series of continuous functions:

$$H(t-t_0) = \lim_{\epsilon\to 0} H^{(\epsilon)}(t-t_0), \quad \epsilon \in [0,\infty), \tag{A.13}$$

with the approximation function defined by

$$H^{(\epsilon)}(t-t_0) = \int_{-\infty}^{t} \delta^{(\epsilon)}(t'-t_0)\,dt'. \tag{A.14}$$

In view of the properties of $\delta^{(\epsilon)}(t-t_0)$, we find

$$H^{(\epsilon)}(t-t_0) = \begin{cases} 0, & t \in (-\infty, t_0) \\ \text{monotonic}, & t \in [t_0, t_0+\epsilon] \\ 1, & t \in (t_0+\epsilon, \infty) \end{cases}. \tag{A.15}$$

For ϵ sufficiently small, (A.4), (A.11) and (A.14) yield

$$\mathcal{L}[H^{(\epsilon)}(t-t_0)] = e^{-t_0 s}\left(\frac{1}{s} - \epsilon\right), \tag{A.16}$$

from which, with $\mathcal{L}[H(t-t_0)] = \lim_{\epsilon\to 0}\mathcal{L}[H^{(\epsilon)}(t-t_0)]$, it follows that

$$\mathcal{L}[H(t-t_0)] = \frac{e^{-t_0 s}}{s}, \quad t_0 = \text{constant.} \tag{A.17}$$

A generalization of (A.17) is

$$\boxed{\mathcal{L}[H(t-t_0)\,f(t-t_0)] = e^{-t_0 s}\tilde{f}(s), \quad t_0 = \text{constant.}} \tag{A.18}$$

A.2 Inverse transforms

If $\mathcal{L}[f(t)]$ is the forward Laplace transform of $f(t)$, then $f(t)$ is called the inverse Laplace transform of $\mathcal{L}[f(t)]$, which is written $\mathcal{L}^{-1}\{\mathcal{L}[f(t)]\} = f(t)$. Since $\mathcal{L}[f(t)] = \tilde{f}(s)$, it follows that

$$\mathcal{L}^{-1}[\tilde{f}(s)] = f(t), \quad t \in \mathcal{T}, \tag{A.19}$$

which admits the immediate inversion of the forward transforms listed above.

Useful consequences of (A.1) and (A.19) are the generalized initial- and final-value theorems. Assuming that the appropriate limits exist, the first theorem states that an asymptotic approximation, $p(t)$, to $f(t)$ for small t corresponds to an asymptotic approximation, $\tilde{p}(s)$, to $\tilde{f}(s)$ for large s; similarly, according to the second theorem, an asymptotic approximation, $q(t)$, to $f(t)$ for large t corresponds to an asymptotic approximation, $\tilde{q}(s)$, to $\tilde{f}(s)$ for small s.

The inverse Laplace transform of $\tilde{f}(s)$ can always be expressed by (e. g. LePage, 1980, pp. 318–328)

$$\mathcal{L}^{-1}[\tilde{f}(s)] = \frac{1}{2\pi i}\int_{s_R-i\infty}^{s_R+i\infty} \tilde{f}(s)\, e^{ts} ds, \quad t \in \mathcal{T}, \tag{A.20}$$

where s_R is located 'to the right' of all singularities of $\tilde{f}(s)$ in $\mathcal{S}$. Frequently, a function, $F(s)$, not known to be a Laplace transform *a priori* is given. Of interest to us is the case that $F(s)$ can be expressed as the quotient of two polynomials in s, $F(s) = U(s)/V(s)$, where the degree, L, of $U(s)$ is lower than the degree, M, of $V(s)$. Then, $F(s) = \tilde{f}(s)$ and (A.20) exists. Supposing, in particular, that $V(s)$ has only simple roots, $s^{(m)}$, the evaluation of the inversion integral by means of the residue theorem yields

$$\boxed{\mathcal{L}^{-1}\left[\frac{U(s)}{V(s)}\right] = \sum_{m=1}^{M} \frac{U(s^{(m)})}{d_s V(s^{(m)})}\, e^{s^{(m)}t}, \quad t \in \mathcal{T},} \tag{A.21}$$

where $d_s V(s^{(m)}) = [dV/ds]_{s=s^{(m)}}$ applies.

Appendix B

Eigenvectors of (4×4)-system

To determine the eigenvectors, $[Y_i^{(k)}]$, associated with the eigenvalues, $\kappa^{(k)}$, of the differential system (5.69), we consider the (3×3)-submatrix $[S_{ij}^{(k)}]$ resulting upon erasing the first row and column of the (4×4)-matrix $[M_{ij}^{(k)}]$ of this system:

$$[S_{ij}^{(k)}] = \begin{bmatrix} \kappa^{(k)}+1 & 0 & -2 \\ 0 & \kappa^{(k)}-1 & n(n+1) \\ 0 & 1 & \kappa^{(k)} \end{bmatrix}. \tag{B.1}$$

We also introduce the (3×3)-submatrix $[Q_{ij}^{(k)}]$ obtained by erasing the third row and column of $[M_{ij}^{(k)}]$:

$$[Q_{ij}^{(k)}] = \begin{bmatrix} \kappa^{(k)}+2 & n(n+1) & 0 \\ 1 & \kappa^{(k)}+1 & -2 \\ 0 & 0 & \kappa^{(k)} \end{bmatrix}. \tag{B.2}$$

If we substitute for the eigenvalues $\kappa^{(k)}$ and expand $\det[S_{ij}^{(k)}]$ and $\det[Q_{ij}^{(k)}]$, either determinant is found to be non-vanishing for each value of k:

$$\det[S_{ij}^{(1)}] = -2n(2n-1), \tag{B.3}$$

$$\det[Q_{ij}^{(2)}] = 2(n+1)(2n+3), \tag{B.4}$$

$$\det[S_{ij}^{(3)}] = -2(n+1)(2n+3), \tag{B.5}$$

$$\det[Q_{ij}^{(4)}] = 2n(2n-1). \tag{B.6}$$

Accordingly, $[M_{ij}^{(k)}]$ is of rank 3 for each value of k, i. e. each eigenvalue $\kappa^{(k)}$ is associated with exactly one eigenvector $[Y_i^{(k)}]$. We determine $[Y_i^{(k)}]$ by considering the differential system (5.69) for $k = 1, \ldots, 4$, respectively.

Case $k = 1$:
Since $\det[M_{ij}^{(1)}] = 0$ and $\det[S_{ij}^{(1)}] \neq 0$, it follows that the first row and column of $[M_{ij}^{(1)}]$ are linearly dependent on its remaining rows and columns. Using $\kappa^{(1)} = n-1$, equation (5.69) reduces to

$$\begin{bmatrix} n & 0 & -2 \\ 0 & n-2 & n(n+1) \\ 0 & 1 & n-1 \end{bmatrix} \begin{bmatrix} Y_2^{(1)} \\ Y_3^{(1)} \\ Y_4^{(1)} \end{bmatrix} = \begin{bmatrix} -Y_1^{(1)} \\ 0 \\ 0 \end{bmatrix}, \tag{B.7}$$

whose solution is

$$\begin{bmatrix} Y_2^{(1)} \\ Y_3^{(1)} \\ Y_4^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{n}Y_1^{(1)} \\ 0 \\ 0 \end{bmatrix}. \tag{B.8}$$

Case $k = 2$:

Since $\det[M_{ij}^{(2)}] = 0$ and $\det[Q_{ij}^{(2)}] \neq 0$, it follows that the third row and column of $[M_{ij}^{(2)}]$ are linearly dependent on its remaining rows and columns. Using $\kappa^{(2)} = n + 1$, equation (5.69) reduces to

$$\begin{bmatrix} n+3 & n(n+1) & 0 \\ 1 & n+2 & -2 \\ 0 & 0 & n+1 \end{bmatrix} \begin{bmatrix} Y_1^{(2)} \\ Y_2^{(2)} \\ Y_4^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -Y_3^{(2)} \end{bmatrix}, \tag{B.9}$$

whose solution is

$$\begin{bmatrix} Y_1^{(2)} \\ Y_2^{(2)} \\ Y_4^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{n}{2n+3}Y_3^{(2)} \\ -\frac{n+3}{(n+1)(2n+3)}Y_3^{(2)} \\ -\frac{1}{n+1}Y_3^{(2)} \end{bmatrix}. \tag{B.10}$$

Case $k = 3$:

As for $k = 1$, it follows that the first row and column of $[M_{ij}^{(3)}]$ are linearly dependent on its remaining rows and columns. Using $\kappa^{(3)} = -(n+2)$, equation (5.69) reduces to

$$\begin{bmatrix} -(n+1) & 0 & -2 \\ 0 & -(n+3) & n(n+1) \\ 0 & 1 & -(n+2) \end{bmatrix} \begin{bmatrix} Y_2^{(3)} \\ Y_3^{(3)} \\ Y_4^{(3)} \end{bmatrix} = \begin{bmatrix} -Y_1^{(3)} \\ 0 \\ 0 \end{bmatrix}, \tag{B.11}$$

whose solution is

$$\begin{bmatrix} Y_2^{(3)} \\ Y_3^{(3)} \\ Y_4^{(3)} \end{bmatrix} = \begin{bmatrix} \frac{1}{n+1}Y_1^{(3)} \\ 0 \\ 0 \end{bmatrix}. \tag{B.12}$$

Case $k = 4$:

As for $k = 2$, it follows that the third row and column of $[M_{ij}^{(4)}]$ are linearly dependent on its remaining rows and columns. Using $\kappa^{(4)} = -n$, equation (5.69) reduces to

$$\begin{bmatrix} -(n-2) & n(n+1) & 0 \\ 1 & -(n-1) & -2 \\ 0 & 0 & -n \end{bmatrix} \begin{bmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_4^{(4)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -Y_3^{(4)} \end{bmatrix}, \tag{B.13}$$

whose solution is

$$\begin{bmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_4^{(4)} \end{bmatrix} = \begin{bmatrix} \frac{n+1}{2n-1}Y_3^{(4)} \\ \frac{n-2}{n(2n-1)}Y_3^{(4)} \\ \frac{1}{n}Y_3^{(4)} \end{bmatrix}. \tag{B.14}$$

Appendix C

List of important symbols

The entries in the third column denote the section or paragraph numbers of first reference to the corresponding symbols; no entries are given for symbols in general use.

C.1 Latin symbols

Symbol	*Name*	*Reference*
A	integration constant	4.3.1
$A^{(k)}$	integration coefficient of fundamental solution to (4×4)-system	5.3.2.1
a	radius of sphere	5.2.2.1
B	integration constant	4.3.1
$B^{(\ell)}$	integration coefficient of fundamental solution to (2×2)-system	5.3.2.2
d_i	isopotential displacement	2.2
da	surface element	3.2.1
e	2.71828...	—
e_{ij}	strain	5.4.1.3
F	Fourier coefficient of f	4.3.1
F_n	normalized Legendre coefficient of f	5.3.2.1
$\overline{F}_n$	r-dependent part of F_n	5.4.1.8
f	scalar field or function	—
$\tilde{f}$	Laplace transform of f	3.3
f_n	ordinary Legendre coefficient of f	5.3.2.3
$f^{(as)}$	axisymmetric Green's function for f	5.4.4.1
$f^{(ns)}$	non-symmetric Green's function for f	5.4.4.2
$f_{ij...}$, $f_{ij...}^{(L)}$	Cartesian tensor field of arbitrary rank	2.2
$f_{ij...,k}$	partial derivative of $f_{ij...}$ with respect to X_k	2.2
$f_{ij...}^{(\Delta)}$	local increment of $f_{ij...}^{(0)}$	2.3
$f_{ij...}^{(\delta)}$	material increment of $f_{ij...}^{(0)}$	2.3
$f_{ij...}^{(\vartheta)}$	isopotential increment of $f_{ij...}^{(0)}$	2.3
$f_{ij...}^{(0)}$	initial value of $f_{ij...}$	2.3

Symbol	*Name*	*Reference*
$f^{\pm}_{ij\ldots}$	increase of $f_{ij\ldots}$ across $\partial\mathcal{R}$ in direction of n_i	2.4
G	Newton's gravitational constant	3.2.1
g_i	gravity force per unit mass	3.2.1
H	symmetric Heaviside step-function	4.3.3
H_+	right-hand Heaviside step-function	4.3.4
h	isopotential height	5.4.1.2
i	imaginary unit	—
$i, j, \ldots$	index subscripts of Cartesian tensor	2.1
j	Jacobian determinant	3.2.1
k	Fourier wave-number	4.3.1
	sequential number of fundamental solution to (4×4)-system	5.3.2.1
k_n	Legendre wave-number for $n \geq 2$	5.3.2.3
l	chemical modulus	3.3.3.2
ℓ	sequential number of fundamental solution to (2×2)-system	5.3.2.2
M	total number of relaxation modes	5.4.3.2
$[M^{(k)}_{ij}]$	matrix of (4×4)-system	5.3.2.1
m	sequential number of relaxation mode	5.4.3.2
m_ν	relaxation function	3.2.3.4
$\overline{m}_\nu$	relaxation spectrum	3.3.1
m_1	bulk-relaxation function	3.2.3.4
m_2	shear-relaxation function	3.2.3.4
$m_{\nu 0}$	small-t limit of relaxation function	3.3.1
$m_{\nu\infty}$	large-t limit of relaxation function	3.3.1
$m_{ijk\ell}$	anisotropic relaxation function	3.2.3.4
$[N^{(\ell)}_{ij}]$	matrix of (2×2)-system	5.3.2.2
n	degree of Legendre coefficient or Legendre polynomial	5.3.2
n_i	outward unit normal with respect to $\partial\mathcal{R}$	2.4
O	origin of coordinate system	5.2.2.1
P_n	Legendre polynomial of the first kind	5.3.2
p	mechanical pressure	3.2.2
Q	total number of Maxwell elements	5.4.2
q	sequential number of Maxwell element	5.4.2
R	non-dimensional radial distance of observation point	5.3.1
r	radial distance of observation point	5.2.2.1
r_i, $r_i^{(L)}$	current position of material point	2.2
s	inverse Laplace time	3.3
s_{ij}	deviatoric incremental stress	5.4.1.5
T_n	impulse-response function	5.4.3.1
$\widetilde{T}_n$	transfer function	5.4.1.8

Symbol	*Name*	*Reference*
t	current time	2.2
t'	excitation time	3.2.1
t_s	maximum shear stress	4.3.2
t_1	largest principal incremental stress	4.3.2
t_2	smallest principal incremental stress	4.3.2
t_{ij}	Cauchy stress	3.2.1
u_i	material displacement	2.2
v	entropic modulus	3.3.3.2
X_i	initial position of material point	2.2
x	vertical distance of observation point	4.2.2.1
x_m	value of x for maximum of t_s	4.4
$[Y_i]$	solution vector of (4×4)-system	5.3.2.1
$[Y_i^{(k)}]$	eigenvector of (4×4)-system	5.3.2.1
y	transversal distance of observation point	4.2.2.1
$[Z_i]$	solution vector of (2×2)-system	5.3.2.2
$[Z_i^{(\ell)}]$	eigenvector of (2×2)-system	5.3.2.2
z	longitudinal distance of observation point	4.2.2.1

C.2 Greek symbols

Symbol	*Name*	*Reference*
α	inverse Maxwell time	4.3.3
α'	inverse spectral time	3.3.1
$\alpha^{(q)}$	inverse elemental Maxwell time	5.4.2
β	inverse relaxation time	4.3.4
$\beta_n^{(m)}$	inverse modal relaxation time	5.4.3.2
γ	magnitude of $g_i^{(0)}$ on $\partial\mathcal{R}^{(0)}$	3.2.3.1
δ	symmetric Dirac delta-function	4.3.3
δ_{ij}	Kronecker symbol	2.1
∂_t, ∂_t^2	first- and second-order partial-derivative operators with respect to t	3.2.1
ϵ_{ijk}	Levi–Civita symbol	2.1
ζ	Love's strain function	4.3.1
θ	colatitude of observation point	5.2.2.1
θ'	colatitude of excitation point	5.4.4.1
ϑ	angle between observation point and excitation point	5.4.4.2
κ_e	elastic bulk modulus	3.3.2.1
κ_e'	anelastic bulk modulus	3.3.2.1
κ_h	hydrostatic bulk modulus	3.3.2.2
κ_h'	viscous bulk modulus	3.3.2.2

Symbol	*Name*	*Reference*
$\kappa^{(k)}$	eigenvalue of (4×4)-system	5.3.2.1
λ	chemical composition	3.2.2
	longitude of observation point	5.2.2.1
λ'	longitude of excitation point	5.4.4.2
$\lambda^{(\ell)}$	eigenvalue of (2×2)-system	5.3.2.2
μ	shear-relaxation function	3.4.1.2
$\overline{\mu}$	shear-relaxation spectrum	4.3.3
μ_e	elastic shear modulus	3.3.2.1
μ_e'	anelastic shear modulus	3.3.2.1
μ_h'	viscous shear modulus	3.3.2.2
$\mu^{(q)}$	elemental elastic shear modulus	5.4.2
ν_i	outward unit normal with respect to level surface of ϕ	5.4.1.2
ξ	state function	3.2.2
ξ_b	barotropic state function	3.4.1.1
π	3.14159...	—
ϖ	thermodynamic pressure	3.2.4
ρ	volume-mass density	3.2.1
σ	interface-mass density	3.2.1
ς_n	non-dimensional Legendre coefficient of σ	5.3.2.1
τ	placement time	4.3.3
τ_{ij}	Piola–Kirchhoff stress	3.2.1
ϕ	gravitational potential	3.2.1
φ	entropy density	3.2.2
χ	centrifugal potential	3.2.1
ψ	tidal potential	3.2.1
Ω_i	angular velocity	3.2.1
ω	rotation	5.2.1.2

C.3 Calligraphic symbols

Symbol	*Name*	*Reference*
$\mathcal{E}$	Euclidian space-domain	2.2
$\mathcal{L}$	Laplace-transformation functional	4.3.4
$\mathcal{L}^{-1}$	inverse Laplace-transformation functional	4.3.4
$\mathcal{M}_{ij}$	anisotropic relaxation functional	3.2.1
$\mathcal{R}_-$	internal domain of r_i	2.2
$\mathcal{R}_+$	external domain of r_i	2.2
$\mathcal{S}$	domain of s	3.3

Symbol	*Name*	*Reference*
$\mathcal{T}$	domain of t	2.2
$\mathcal{X}_-$	internal domain of X_i	2.2
$\mathcal{X}_+$	external domain of X_i	2.2
$\partial\mathcal{R}$	interface between $\mathcal{R}_-$ and $\mathcal{R}_+$	2.2
$\partial\mathcal{X}$	interface between $\mathcal{X}_-$ and $\mathcal{X}_+$	2.2