

THE VISCOELASTIC BOUSSINESQ'S PROBLEM

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ABSTRACT

We consider an isochemical, isentropic, incompressible fluid half-space and study quasistatic viscoelastic perturbations, induced by two-dimensional surface loads, of a hydrostatic initial state. In view of the regional or local scale required for deformations of planets to be amenable to the half-space approximation, the model is assumed to be externally gravitating. We derive analytic solutions for the displacement and incremental stress components and study several approximations to the expressions. Particular emphasis is placed on discriminating between the material and local incremental stresses. Based on this distinction, new insight is gained into the physical significance of the solution.

1 INTRODUCTION

Detailed investigations into the elastostatic deformation of a plane half-space subject to either displacement or traction boundary conditions have been carried out by Boussinesq and are summarized in a monograph (Boussinesq, 1885). Today, the basic problem is usually referred to as Boussinesq's problem, although in part of his work Boussinesq was preceded by Lamé & Clapeyron (1831) and Cerruti (1882). Since the publication of Boussinesq's monograph, numerous authors have written on particular aspects of Boussinesq's problem (e. g. Lamb, 1902; Terazawa, 1916; Love, 1929; Harding & Sneddon, 1945; Sneddon, 1946; Farrell, 1972).

A common feature of most work on Boussinesq's problem is that the unperturbed half-space is regarded as unstressed. As far as the model is applied to study the regional or local deformation of planets, this assumption clearly cannot be satisfied. Effects due to a planet's *initial stress* are however small for *elastic* perturbations whose lateral wavelength is sufficiently short for the application of the half-space approximation (Cathles, 1975, pp. 35-39).

More significant is the influence of the initial stress for *viscoelastic* perturbations. Such problems were extensively studied by Biot and are reviewed in a monograph (Biot, 1965). Closely related to the present study is Biot's analysis of the quasistatic viscoelastic deformation of an initially hydrostatic half-space (Biot, 1959), in which effects due to the initial stress are accounted for by a separate term included in the incremental equilibrium equation. In general, Biot preferred a formal treatment of the problem, in which the physical significance of the modifications associated with the initial stress was not fully discussed.

Unfortunately, Biot's work has not received much attention from geophysical researchers. This disregard is evident in several subsequent studies of the viscoelastic Boussinesq's problem, in which the initial stress is neglected (e. g. Peltier, 1974; Cathles, 1975, pp. 57-59).

Research on the viscoelastic Boussinesq's problem was resumed by Nakiboglu & Lambeck (1982) and Wolf (1985a, 1985b, 1985c), who studied the response due to surface loading. Whereas Nakiboglu & Lambeck accounted for the influence of the initial stress by an *ad hoc* modification of the incremental boundary condition, Wolf included such effects in the incremental equilibrium equation. The distinction between 'viscoelastic' and 'total' perturbation stresses then allowed Wolf to reduce the incremental field equations formally to those valid in the absence of initial stress, which could be solved using elementary methods.

In retrospect, Wolf's (1985a) method of accounting for the initial stress is seen to be very similar to that used by Biot (1959), although Wolf was not aware of Biot's publication at that time. As in Biot's publication, the significance of the modifications associated with the

initial stress was not fully recognized by Wolf (1985a, 1985b). This, in particular, applies to the physical meaning of the two kinds of incremental stress employed in Wolf's analysis, which was not adequately discussed.

Recently, the theory of *viscoelastodynamics* for fluids in a state of *hydrostatic* initial stress has been reviewed (Wolf, 1991). In particular, rigorous deductions were given of the incremental field equations and continuity conditions and of the asymptotic approximations of the equations for short and long times after the onset of the perturbations. Special emphasis was placed on the distinctions between the *Lagrangian* and *Eulerian* kinematic formulations of the equations and between the *material* and *local* increments of the field quantities. Based on this, it was possible to interpret the short- and long-time asymptotic equations as the incremental field equations and continuity conditions of *elastodynamics* and of *fluid dynamics*, respectively.

In view of the progress achieved in our understanding of the theory of viscoelastodynamics for fluids in a state of hydrostatic initial stress, a re-examination of the viscoelastic Boussinesq's problem within this improved theoretical framework appears to be warranted. In the first place, such a re-examination is intended to clarify the physical interpretation of previous solutions to the problem. However, it should also serve as a guide for the physically correct treatment of more complicated problems.

In agreement with the heuristic character of the present study, the model to be analysed is kept as simple as possible. We therefore consider the viscoelastic Boussinesq's problem using an *isochemical, isentropic, incompressible* half-space deformed by a *two-dimensional, harmonic* surface load. Since effects due to the perturbation of the gravity field are small for local or regional deformations of planets (Cathles, 1975, pp. 72-83), the half-space is assumed to be *externally gravitating*. In Sec. 2, the relevant incremental field equations and boundary conditions and their Laplace transforms are collected. The equations are solved in Sec. 3 by means of Love's strain function and inverse Laplace transformation. Section 4 gives a discussion of the solution. Special care will be taken to discriminate between material and local incremental stresses. This distinction will prove necessary for a physically correct interpretation of the solution and its approximations.

2 FIELD EQUATIONS AND BOUNDARY CONDITIONS

The present study is concerned with Cartesian tensor fields, where we employ the indicial notation and the usual summation and differentiation conventions. In particular, we assume that the current state of a fluid at the time $t \in [0, \infty)$ represents a small increment with respect to a

hydrostatic initial state at the time $t = 0$. Further, we take as the spatial argument the initial particle position, $X_i \in \mathcal{V}^{(0)}$, with $\mathcal{V}^{(0)}$ the region initially occupied by the fluid. This is commonly referred to as the *Lagrangian* kinematic formulation. The perturbation of an arbitrary initial field, $f_{ij\dots}^{(0)}(\mathbf{X})$, can then be alternatively described in terms of the *material* incremental field, $f_{ij\dots}^{(\delta)}(\mathbf{X}, t)$, referring to the particle initially at X_i , or in terms of the *local* incremental field, $f_{ij\dots}^{(\Delta)}(\mathbf{X}, t)$, referring to the initial position, X_i . The material and local incremental fields are related by

$$f_{ij\dots}^{(\delta)}(\mathbf{X}, t) = f_{ij\dots}^{(\Delta)}(\mathbf{X}, t) + f_{ij\dots,k}^{(0)}(\mathbf{X}) u_k(\mathbf{X}, t), \quad (1)$$

where $u_i(\mathbf{X}, t)$ is the particle displacement and $f_{ij\dots,k}^{(0)}(\mathbf{X}) u_k(\mathbf{X}, t)$ the *advective* incremental field (cf. for further details Dahlen, 1974; Grafarend, 1982; Wolf, 1991).

For the present study, we assume that the fluid is isochemical, isentropic, non-rotating, externally gravitating and free of non-gravitational volume forces. On these assumptions, the equations governing the initial state of the fluid (e. g. Wolf, 1991) reduce to

$$-p_{,i}^{(0)} + \rho^{(0)} g_i = 0, \quad (2)$$

$$\rho^{(0)} p_{,i}^{(0)} = k \rho_{,i}^{(0)}, \quad (3)$$

where g_i is the gravitational force per unit mass, k the (isentropic) fluid bulk modulus, $p^{(0)}$ the initial mechanical pressure, $\rho^{(0)}$ the initial mass density and the argument X_i has been suppressed. Equations (2) and (3) are referred to as the initial equilibrium equation and the initial state equation. With g_i and k prescribed parameters, (2) and (3) constitute the system of initial field equations to be satisfied by $p^{(0)}$ and $\rho^{(0)}$ for all $X_i \in \mathcal{V}^{(0)}$.

Assuming now that the fluid undergoes quasistatic viscoelastic perturbations, the following equations apply (e. g. Wolf, 1991):

$$t_{ij,j}^{(\delta)} + (p_{,j}^{(0)} u_j)_{,i} - g_i (\rho^{(0)} u_j)_{,j} = 0, \quad (4)$$

$$t_{ij}^{(\delta)} = \int_0^t [m_1(t-t') - \frac{2}{3} m_2(t-t')] \partial_{t'} u_{k,k}(t') \delta_{ij} dt' + \int_0^t m_2(t-t') \partial_{t'} [u_{i,j}(t') + u_{j,i}(t')] dt', \quad (5)$$

where $m_1(t-t')$ and $m_2(t-t')$ are the bulk and shear relaxation functions, $t_{ij}^{(\delta)}$ is the material incremental Cauchy stress, δ_{ij} the Kronecker symbol, $\partial_{t'}$ the partial derivative operator with respect to t' and the arguments X_i and t have been suppressed. Equations (4) and (5) are referred to as the incremental equation of motion and the incremental constitutive equation of viscoelasticity. With $m_1(t-t')$ and $m_2(t-t')$ prescribed parameters and $p^{(0)}$ and $\rho^{(0)}$ obtained by solving (2) and (3), equations (4) and (5) constitute the system of incremental field equations of viscoelastodynamics to be satisfied by $t_{ij}^{(\delta)}$ and u_i for all $X_i \in \mathcal{V}^{(0)}$ and $t \in [0, \infty)$.

To simplify the problem further, we assume that the fluid is *incompressible*. We obtain for the initial state

$$k \rightarrow \infty, \quad (6)$$

$$\rho_{,i}^{(0)} \rightarrow 0, \quad (7)$$

and for the incremental state

$$m_1(t - t') \rightarrow \infty, \quad (8)$$

$$u_{i,i} \rightarrow 0. \quad (9)$$

We also assume that m_2 is spatially homogeneous. Introducing $m = m_2$ for brevity, we thus require

$$m_{,i}(t - t') = 0. \quad (10)$$

Subject to (6) and (7), the initial field equations, (2) and (3), reduce to the single equation

$$-p_{,i}^{(0)} + \rho g_i = 0, \quad (11)$$

where $\rho \geq 0$ is a prescribed parameter. On bounding surfaces of the fluid, $p^{(0)}$ must satisfy conditions to be specified. In the following, we will assume that the boundary is initially a *free* surface. With $B^{(0)}$ the boundary of $\mathcal{V}^{(0)}$ and $[f_{ij\dots}]_{B^{(0)}}$ the value of $f_{ij\dots}$ for $X_i \in B^{(0)}$, the initial boundary condition then takes the form

$$[p^{(0)}]_{B^{(0)}} = 0. \quad (12)$$

In view of (7)–(9), the incremental field equations, (4) and (5), are replaced by

$$t_{ij,j}^{(\delta)} + (p_{,j}^{(0)} u_j)_{,i} = 0, \quad (13)$$

$$u_{i,i} = 0, \quad (14)$$

$$t_{ij}^{(\delta)} = -p^{(\delta)} \delta_{ij} + \int_0^t m(t - t') \partial_{t'} [u_{i,j}(t') + u_{j,i}(t')] dt', \quad (15)$$

where we have introduced

$$p^{(\delta)} = - \int_0^t \lim_{\substack{u_{i,i} \rightarrow 0 \\ m_1(t-t') \rightarrow \infty}} m_1(t - t') \partial_{t'} u_{i,i}(t') dt'. \quad (16)$$

On bounding surfaces of the fluid, the solution must satisfy incremental boundary conditions. Here, we are concerned only with perturbations due to surface loads. The incremental boundary condition therefore becomes (e. g. Wolf, 1991)

$$[n_j^{(0)} t_{ij}^{(\delta)}]_{B^{(0)}} = -n_i^{(0)} q, \quad (17)$$

where $n_i^{(0)}$ is the *outward* unit vector normal to $B^{(0)}$ and q is the prescribed incremental load pressure.

The general solution to (13)–(15) can be derived by means of elementary methods if $p^{(\delta)}$ and $t_{ij}^{(\delta)}$ are expressed in terms of the associated local increments. Since $t_{ij}^{(0)} = -p^{(0)}\delta_{ij}$, it follows by (1) that

$$p^{(\delta)} = p^{(\Delta)} + p_{,i}^{(0)}u_i, \quad (18)$$

$$t_{ij}^{(\delta)} = t_{ij}^{(\Delta)} - p_{,k}^{(0)}u_k\delta_{ij}. \quad (19)$$

Substitution of (18) and (19) into (13)–(15) and (17) gives

$$t_{ij,j}^{(\Delta)} = 0, \quad (20)$$

$$u_{i,i} = 0, \quad (21)$$

$$t_{ij}^{(\Delta)} = -p^{(\Delta)}\delta_{ij} + \int_0^t m(t-t') \partial_{t'}[u_{i,j}(t') + u_{j,i}(t')] dt', \quad (22)$$

$$[n_j^{(0)}t_{ij}^{(\Delta)} - n_i^{(0)}p_{,j}^{(0)}u_j]_{B^{(0)}} = -n_i^{(0)}q. \quad (23)$$

Note that the incremental field equations, (20)–(22), no longer depend on $p_{,i}^{(0)}$ and thus in particular agree with the ordinary field equations valid in the absence of initial stress. However, effects due to the initial stress enter through the incremental boundary condition, (23), which explicitly involves $p_{,i}^{(0)}$.

Solutions to (20)–(22) subject to (23) will be obtained by means of the Laplace transforms of the equations. The Laplace transform, $\mathcal{L}[f(t)]$, of a function $f(t)$ is defined by

$$\mathcal{L}[f(t)] = \bar{f}(s) = \int_0^\infty f(t) e^{-st} dt, \quad (24)$$

where the transform variable, s , will be restricted to real values. For the convergence of the integral in (24) for s larger than some value s_0 it is sufficient that $f(t)$ be piecewise continuous for all $t \in [0, \infty)$ and of exponential order as $t \rightarrow \infty$. Elementary consequences of (24) are (e. g. LePage, 1961, pp. 285–328)

$$\mathcal{L}[\partial_t f(t)] = s\bar{f}(s) - f(0+), \quad (25)$$

$$\mathcal{L}\left[\int_0^t f(t-t')g(t')dt'\right] = \bar{f}(s)\bar{g}(s). \quad (26)$$

We will also need

$$\mathcal{L}[H(t)e^{-s't}] = \frac{1}{s+s'}, \quad (27)$$

where $H(t)$ is the unit step function, here formally defined by

$$H(t) = \lim_{\epsilon \rightarrow 0} \begin{cases} \frac{t}{\epsilon}, & 0 \leq t \leq \epsilon \\ 1, & \epsilon < t < \infty \end{cases}. \quad (28)$$

Using (24)–(26) and assuming that $u_{i,j}$ vanishes at $t = 0+$, we obtain from (20)–(23) the Laplace-transformed incremental field equations and boundary condition:

$$\tilde{t}_{ij,j}^{(\Delta)} = 0, \quad (29)$$

$$\tilde{u}_{i,i} = 0, \quad (30)$$

$$\tilde{t}_{ij}^{(\Delta)} = -\tilde{p}^{(\Delta)}\delta_{ij} + s\tilde{m}(\tilde{u}_{i,j} + \tilde{u}_{j,i}), \quad (31)$$

$$[n_j^{(0)}\tilde{t}_{ij}^{(\Delta)} - n_i^{(0)}p_{j,j}^{(0)}\tilde{u}_j]_{B(0)} = -n_i^{(0)}\tilde{q}. \quad (32)$$

As in (29)–(32), the argument s of arbitrary Laplace-transformed incremental fields will be usually suppressed in the following; for brevity, we refer to such fields simply as ‘incremental fields.’

We may use (31) to eliminate $\tilde{t}_{ij}^{(\Delta)}$ from (29) and (32). Observing (10) and (30), we then arrive at the following incremental field equations and boundary condition:

$$-\tilde{p}_{,i}^{(\Delta)} + s\tilde{m}\tilde{u}_{i,jj} = 0, \quad (33)$$

$$\tilde{u}_{i,i} = 0, \quad (34)$$

$$[n_i^{(0)}(\tilde{p}^{(\Delta)} + p_{j,j}^{(0)}\tilde{u}_j) - n_j^{(0)}s\tilde{m}(\tilde{u}_{i,j} + \tilde{u}_{j,i})]_{B(0)} = n_i^{(0)}\tilde{q}. \quad (35)$$

As an elementary example, we consider perturbations, induced by surface loads, of a plane half-space in a homogeneous gravity field. In applications to planets, this approximation is appropriate only to perturbations whose ‘typical’ lateral wavelength is short compared with the planet’s radius. The symmetry of the half-space suggests to introduce Cartesian coordinates: $X_i = (x, y, z)$. We stipulate that $0 \leq x < \infty$ for all $X_i \in \mathcal{V}^{(0)}$. Then, $g_i = (g, 0, 0)$ must hold, where $g \geq 0$ is a prescribed parameter, and the non-vanishing components of (11) and (12) become

$$-p_{,x}^{(0)} + \rho g = 0, \quad (36)$$

$$[p^{(0)}]_{x=0} = 0. \quad (37)$$

Note that, by (36) and (37), $p^{(0)} = 0$ is equivalent to $\rho g = 0$ for all $X_i \in \mathcal{V}^{(0)}$. The fluid we are concerned with is therefore initially unstressed only if it is non-gravitating.

Supposing in the following that the perturbing load is two-dimensional, we orient the y -axis normal to the strike of the load. It then follows from symmetry considerations that $\tilde{u}_x = 0$, $\tilde{p}_{,x}^{(\Delta)} = 0$, $\tilde{u}_{i,x} = 0$ and thus also $\tilde{t}_{xx}^{(\Delta)} = \tilde{t}_{yx}^{(\Delta)} = 0$. Hence, the non-vanishing scalar components of (31) are

$$\tilde{t}_{xx}^{(\Delta)} = -\tilde{p}^{(\Delta)} + 2s\tilde{m}\tilde{u}_{x,x}, \quad (38)$$

$$\tilde{i}_{xy}^{(\Delta)} = s\bar{m}(\tilde{u}_{x,y} + \tilde{u}_{y,x}), \quad (39)$$

$$\tilde{i}_{yy}^{(\Delta)} = -\tilde{p}^{(\Delta)} + 2s\bar{m}\tilde{u}_{y,y}, \quad (40)$$

$$\tilde{i}_{xx}^{(\Delta)} = -\tilde{p}^{(\Delta)}. \quad (41)$$

Similarly, we find for the non-vanishing scalar components of (33) and (34) the relations

$$-\tilde{p}_{,x}^{(\Delta)} + s\bar{m}(\tilde{u}_{x,xx} + \tilde{u}_{x,yy}) = 0, \quad (42)$$

$$-\tilde{p}_{,y}^{(\Delta)} + s\bar{m}(\tilde{u}_{y,xx} + \tilde{u}_{y,yy}) = 0, \quad (43)$$

$$\tilde{u}_{x,x} + \tilde{u}_{y,y} = 0. \quad (44)$$

With (36) and $n_i^{(0)} = (-1, 0, 0)$, the scalar components of (35) take the forms

$$[\tilde{u}_{x,y} + \tilde{u}_{y,x}]_{x=0} = 0, \quad (45)$$

$$[\tilde{p}^{(\Delta)} + \rho g \tilde{u}_x - 2s\bar{m}\tilde{u}_{x,x}]_{x=0} = \bar{q}. \quad (46)$$

Equations (42)–(44) are three simultaneous second-order partial differential equations for $\tilde{p}^{(\Delta)}$, \tilde{u}_x and \tilde{u}_y , which must be solved subject to (45) and (46). These equations must be completed by suitable conditions requiring that the incremental fields and their spatial derivatives remain finite as $x \rightarrow \infty$.

3 SOLUTION TO THE INCREMENTAL EQUATIONS

We obtain the general solution to the equations by means of Love's strain function, λ , defined by (e. g. Malvern, 1969, pp. 552–554)

$$\tilde{u}_x = \lambda_{,yy}, \quad (47)$$

$$\tilde{u}_y = -\lambda_{,xy}, \quad (48)$$

$$\tilde{p}^{(\Delta)} = -s\bar{m}(\lambda_{,xx} + \lambda_{,yy})_{,x}. \quad (49)$$

Using (47)–(49), equations (38)–(41) can also be expressed in terms of λ :

$$\tilde{i}_{xx}^{(\Delta)} = s\bar{m}(\lambda_{,xx} + 3\lambda_{,yy})_{,x}, \quad (50)$$

$$\tilde{i}_{xy}^{(\Delta)} = -s\bar{m}(\lambda_{,xx} - \lambda_{,yy})_{,y}, \quad (51)$$

$$\tilde{i}_{yy}^{(\Delta)} = s\bar{m}(\lambda_{,xx} - \lambda_{,yy})_{,x}, \quad (52)$$

$$\tilde{i}_{zz}^{(\Delta)} = s\bar{m}(\lambda_{,xx} + \lambda_{,yy})_{,x}. \quad (53)$$

Upon substitution of (47)–(49) into (42)–(44), we find that (42) and (44) are identically satisfied. A necessary condition for (43) to be valid is

$$\lambda_{,xxxx} + 2\lambda_{,xxyy} + \lambda_{,yyyy} = 0, \quad (54)$$

which is the biharmonic equation. The general solution to (54) must satisfy the boundary conditions

$$[(\lambda_{,xx} - \lambda_{,yy}),_y]_{x=0} = 0, \quad (55)$$

$$[\rho g \lambda_{,yy} - s\bar{m}(\lambda_{,xx} + 3\lambda_{,yy}),_x]_{x=0} = \bar{q}, \quad (56)$$

which follow from (45) and (46) upon substitution of (47)–(49). An additional requirement is that λ and its spatial derivatives remain finite as $x \rightarrow \infty$.

The equations are solved on the supposition that the incremental fields can be expanded into a Fourier integral. For an arbitrary incremental field component, \bar{f} , and the load pressure, \bar{q} , we thus require that

$$\bar{f}(x, y, s) = \int_{-\infty}^{\infty} \bar{F}(x, \xi, s) e^{i\xi y} d\xi, \quad (57)$$

$$\bar{q}(y, s) = \int_{-\infty}^{\infty} \bar{Q}(\xi, s) e^{i\xi y} d\xi, \quad (58)$$

where \bar{F} and \bar{Q} are the (complex) spectral densities associated with \bar{f} and \bar{q} and ξ is the wave number. Due to their linearity, (47)–(53) then become

$$\bar{U}_x = -\xi^2 \Lambda, \quad (59)$$

$$\bar{U}_y = -i\xi \Lambda_{,x}, \quad (60)$$

$$\bar{P}^{(\Delta)} = -s\bar{m}(\Lambda_{,xx} - \xi^2 \Lambda)_{,x}, \quad (61)$$

$$\bar{T}_{xx}^{(\Delta)} = s\bar{m}(\Lambda_{,xx} - 3\xi^2 \Lambda)_{,x}, \quad (62)$$

$$\bar{T}_{xy}^{(\Delta)} = -i\xi s\bar{m}(\Lambda_{,xx} + \xi^2 \Lambda), \quad (63)$$

$$\bar{T}_{yy}^{(\Delta)} = s\bar{m}(\Lambda_{,xx} + \xi^2 \Lambda)_{,x}, \quad (64)$$

$$\bar{T}_{zz}^{(\Delta)} = s\bar{m}(\Lambda_{,xx} - \xi^2 \Lambda)_{,x}, \quad (65)$$

where the arguments x , ξ and s have been suppressed. Similarly, (54)–(56) reduce to

$$\Lambda_{,xxxx} - 2\xi^2 \Lambda_{,xx} + \xi^4 \Lambda = 0, \quad (66)$$

$$[\Lambda_{,xx} + \xi^2 \Lambda]_{x=0} = 0, \quad (67)$$

$$-[\xi^2 \rho g \Lambda + s\bar{m}(\Lambda_{,xx} - 3\xi^2 \Lambda)_{,x}]_{x=0} = \bar{Q}. \quad (68)$$

In addition to (68), we require that Λ and its spatial derivatives remain finite as $x \rightarrow \infty$.

For the purposes of the following discussion, it will be sufficient to assume $\xi \geq 0$. The solution to (66) satisfying the condition at infinity can then be written in the form

$$\Lambda = \frac{1}{\xi^2} (A + B\xi x)e^{-\xi x}, \quad (69)$$

where A and B are integration constants. Using (69) to substitute for Λ and its derivatives in (59)–(65), we obtain

$$\bar{U}_x = -(A + B\xi x)e^{-\xi x}, \quad (70)$$

$$\bar{U}_y = i[A - (1 - \xi x)B]e^{-\xi x}, \quad (71)$$

$$\bar{P}^{(\Delta)} = -2\xi s\bar{m}Be^{-\xi x}, \quad (72)$$

$$\bar{T}_{xx}^{(\Delta)} = 2\xi s\bar{m}(A + B\xi x)e^{-\xi x}, \quad (73)$$

$$\bar{T}_{xy}^{(\Delta)} = -2i\xi s\bar{m}[A - (1 - \xi x)B]e^{-\xi x}, \quad (74)$$

$$\bar{T}_{yy}^{(\Delta)} = -2\xi s\bar{m}[A - (2 - \xi x)B]e^{-\xi x}, \quad (75)$$

$$\bar{T}_{zz}^{(\Delta)} = 2\xi s\bar{m}Be^{-\xi x}. \quad (76)$$

The constants A and B can be determined by substitution of (69) and its derivatives into (67) and (68). We get

$$A = B = -\frac{\bar{Q}}{2\xi s\bar{m} + \rho g}, \quad (77)$$

whence (70)–(76) take the forms

$$\bar{U}_x = \frac{\bar{Q}}{2\xi s\bar{m} + \rho g} (1 + \xi x)e^{-\xi x}, \quad (78)$$

$$\bar{U}_y = -i \frac{\bar{Q}}{2\xi s\bar{m} + \rho g} \xi x e^{-\xi x}, \quad (79)$$

$$\bar{P}^{(\Delta)} = \frac{2\xi s\bar{m}\bar{Q}}{2\xi s\bar{m} + \rho g} e^{-\xi x}, \quad (80)$$

$$\bar{T}_{xx}^{(\Delta)} = -\frac{2\xi s\bar{m}\bar{Q}}{2\xi s\bar{m} + \rho g} (1 + \xi x)e^{-\xi x}, \quad (81)$$

$$\bar{T}_{xy}^{(\Delta)} = i \frac{2\xi s\bar{m}\bar{Q}}{2\xi s\bar{m} + \rho g} \xi x e^{-\xi x}, \quad (82)$$

$$\bar{T}_{yy}^{(\Delta)} = -\frac{2\xi s\bar{m}\bar{Q}}{2\xi s\bar{m} + \rho g} (1 - \xi x)e^{-\xi x}, \quad (83)$$

$$\bar{T}_{zz}^{(\Delta)} = -\frac{2\xi s\bar{m}\bar{Q}}{2\xi s\bar{m} + \rho g} e^{-\xi x}. \quad (84)$$

A useful incremental field quantity to consider is the maximum shear stress occurring in the half-space. Since the maximum shear stress can be related to the difference between the largest and smallest principal stresses, we must first determine the principal stresses. With $\bar{i}_{xx}^{(\Delta)} = -\bar{p}^{(\Delta)}$ one of the principal stresses, the other two are obtained from the characteristic equation

$$\det \begin{bmatrix} \bar{i}_{xx}^{(\Delta)} - \bar{i}^{(\Delta)} & \bar{i}_{xy}^{(\Delta)} \\ \bar{i}_{xy}^{(\Delta)} & \bar{i}_{yy}^{(\Delta)} - \bar{i}^{(\Delta)} \end{bmatrix} = 0. \quad (85)$$

Upon expansion of the determinant and use of (49), (50) and (52), the principal stresses are found to be

$$\bar{i}_{1,2}^{(\Delta)} = -\bar{p}^{(\Delta)} \pm \left[\frac{(\bar{i}_{xx}^{(\Delta)} - \bar{i}_{yy}^{(\Delta)})^2}{4} + (\bar{i}_{xy}^{(\Delta)})^2 \right]^{\frac{1}{2}}. \quad (86)$$

Since $\bar{i}_1^{(\Delta)}$ and $\bar{i}_2^{(\Delta)}$ are the largest and smallest principal stresses, respectively, the maximum shear stress, \bar{i}_s , is given by

$$\bar{i}_s = \frac{1}{2}(\bar{i}_1^{(\Delta)} - \bar{i}_2^{(\Delta)}). \quad (87)$$

Note that we have not added a superscript to the symbol \bar{i}_s . This is in accordance with (19), which shows that no distinction between material and local increments is required if shear stress components or differences between normal stress components are considered. Substitution of (86) into (87) gives

$$\bar{i}_s = \frac{1}{2}[(\bar{i}_{xx}^{(\Delta)} - \bar{i}_{yy}^{(\Delta)})^2 + 4(\bar{i}_{xy}^{(\Delta)})^2]^{\frac{1}{2}}. \quad (88)$$

Using (81)–(83) and

$$\bar{i}_s = \int_{-\infty}^{\infty} \bar{T}_s d\xi, \quad (89)$$

we find

$$\bar{T}_s = (1 + i) \frac{2\xi s \bar{m} \bar{Q}}{2\xi s \bar{m} + \rho g} \xi x e^{-\xi x}. \quad (90)$$

Note that, according to (89) and (90), \bar{i}_s is independent of y .

To proceed beyond (78)–(84) and (90), we must specify \bar{m} and \bar{Q} . As a simple example, we consider the shear relaxation function for *Maxwellian* viscoelasticity (e. g. Christensen, 1982, pp. 16–20):

$$m = \mu H(t) e^{-\alpha t}. \quad (91)$$

Note that m is determined by two parameters: the inverse Maxwell time, α , and the shear modulus, μ . We further consider an ‘instantaneous’ loading event:

$$Q = Q' H(t). \quad (92)$$

In view of (27), the Laplace transforms of (91) and (92) are

$$\bar{m} = \frac{\mu}{s + \alpha}, \quad (93)$$

$$\bar{Q} = \frac{Q'}{s}. \quad (94)$$

From (93) and (94), we get

$$\frac{\bar{Q}}{2\xi s \bar{m} + \rho g} = \frac{Q'}{2\xi\mu + \rho g} \left[\frac{1}{s} + \frac{2\xi\mu}{\rho g} \left(\frac{1}{s} - \frac{1}{s + \beta} \right) \right], \quad (95)$$

$$\frac{2\xi s \bar{m} \bar{Q}}{2\xi s \bar{m} + \rho g} = \frac{2\xi\mu Q'}{2\xi\mu + \rho g} \frac{1}{s + \beta}, \quad (96)$$

where

$$\beta = \frac{\rho g \alpha}{2\xi\mu + \rho g}. \quad (97)$$

Upon substitution of (95) and (96) and subsequent inverse Laplace transformation using (27), equations (78)–(84) and (90) become

$$U_x = \frac{Q'H(t)}{2\xi\mu + \rho g} \left[1 + \frac{2\xi\mu}{\rho g} (1 - e^{-\beta t}) \right] (1 + \xi x) e^{-\xi x}, \quad (98)$$

$$U_y = -i \frac{Q'H(t)}{2\xi\mu + \rho g} \left[1 + \frac{2\xi\mu}{\rho g} (1 - e^{-\beta t}) \right] \xi x e^{-\xi x}, \quad (99)$$

$$P(\Delta) = \frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t} e^{-\xi x}, \quad (100)$$

$$T_{xx}^{(\Delta)} = -\frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t} (1 + \xi x) e^{-\xi x}, \quad (101)$$

$$T_{xy}^{(\Delta)} = i \frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t} \xi x e^{-\xi x}, \quad (102)$$

$$T_{yy}^{(\Delta)} = -\frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t} (1 - \xi x) e^{-\xi x}, \quad (103)$$

$$T_{zz}^{(\Delta)} = -\frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t} e^{-\xi x}, \quad (104)$$

$$T_s = (1 + i) \frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t} \xi x e^{-\xi x}. \quad (105)$$

In view of (98)–(105), β will be referred to as the inverse relaxation time.

4 DISCUSSION OF THE SOLUTION

For the discussion of the solution, we consider the surface of the half-space, $x = 0$. It then follows from (98)–(105) that $U_y = 0$, $T_{xx}^{(\Delta)} = T_{yy}^{(\Delta)} = T_{zz}^{(\Delta)} = -P^{(\Delta)}$ and $T_{xy}^{(\Delta)} = T_s = 0$. Hence, the non-vanishing independent surface components are

$$[U_x]_{x=0} = \frac{Q'H(t)}{2\xi\mu + \rho g} \left[1 + \frac{2\xi\mu}{\rho g} (1 - e^{-\beta t}) \right], \quad (106)$$

$$[P^{(\Delta)}]_{x=0} = \frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t}. \quad (107)$$

We also consider the depth, x_m , where T_s assumes a maximum. Since

$$\frac{d}{d(\xi x)} (\xi x e^{-\xi x}) = (1 - \xi x) e^{-\xi x}, \quad (108)$$

we obtain

$$x_m = \frac{1}{\xi}. \quad (109)$$

Substitution of (109) into (105) then gives

$$[T_s]_{x=x_m} = \frac{1+i}{e} \frac{2\xi\mu Q'H(t)}{2\xi\mu + \rho g} e^{-\beta t}. \quad (110)$$

The short-time limits of (106), (107) and (110) are

$$[U_x]_{x=0, t=0+} = \frac{Q'}{2\xi\mu + \rho g}, \quad (111)$$

$$[P^{(\Delta)}]_{x=0, t=0+} = \frac{2\xi\mu Q'}{2\xi\mu + \rho g}, \quad (112)$$

$$[T_s]_{x=x_m, t=0+} = \frac{1+i}{e} \frac{2\xi\mu Q'}{2\xi\mu + \rho g}. \quad (113)$$

Equations (111)–(113) apply to *elastostatic* equilibrium, governed by the shear modulus μ , in a half-space subject to the hydrostatic initial stress gradient ρg (e. g. Wolf, 1985a, 1985b). Using (97), the long-time limits of (106), (107) and (110) are found to be

$$[U_x]_{x=0, t \rightarrow \infty} = \frac{Q'}{\rho g}, \quad (114)$$

$$[P^{(\Delta)}]_{x=0, t \rightarrow \infty} = 0, \quad (115)$$

$$[T_s]_{x=x_m, t \rightarrow \infty} = 0. \quad (116)$$

Equations (114)–(116) describe *hydrostatic* equilibrium in a half-space subject to the hydrostatic initial stress gradient ρg . The transition from the instantaneous elastostatic to the final hydrostatic equilibrium state is seen from (106), (107) and (110) to be exponential in time, where the inverse relaxation time, β , is related to the parameters of the half-space by (97).

If $2\xi\mu/(\rho g) \ll 1$, equations (97), (106), (107) and (110) become, correct to the first order in the small quantity,

$$\beta \simeq \alpha \left(1 - \frac{2\xi\mu}{\rho g}\right), \quad (117)$$

$$[U_x]_{x=0} \simeq \frac{Q'H(t)}{\rho g} \left(1 - \frac{2\xi\mu}{\rho g} e^{-\beta t}\right), \quad (118)$$

$$[P^{(\Delta)}]_{x=0} \simeq \frac{2\xi\mu Q'H(t)}{\rho g} e^{-\beta t}, \quad (119)$$

$$[T_s]_{x=x_m} \simeq \frac{1+i}{e} \frac{2\xi\mu Q'H(t)}{\rho g} e^{-\beta t}. \quad (120)$$

A comparison between (97) and (117) shows that β is now close to its largest value, α , i. e. the relaxation proceeds rapidly. According to (118)–(120), the relaxation is however insignificant in amplitude and vanishes for $\xi\mu = 0$. For perturbations of sufficiently long wavelength, the differences between the elastostatic and hydrostatic equilibrium states may therefore be ignored.

If $\rho g/(2\xi\mu) \ll 1$ and $\beta t \ll 1$, equations (97), (106), (107) and (110) become, correct to the first order in the small quantities,

$$\beta \simeq \frac{\rho g \alpha}{2\xi\mu}, \quad (121)$$

$$[U_x]_{x=0} \simeq \frac{Q'H(t)}{2\xi\mu} \left(1 - \frac{\rho g}{2\xi\mu} + \alpha t\right), \quad (122)$$

$$[P^{(\Delta)}]_{x=0} \simeq Q'H(t) \left(1 - \frac{\rho g}{2\xi\mu} - \beta t\right), \quad (123)$$

$$[T_s]_{x=x_m} \simeq \frac{1+i}{e} Q'H(t) \left(1 - \frac{\rho g}{2\xi\mu} - \beta t\right). \quad (124)$$

Equation (121) is the expression for the inverse relaxation time for *Newtonian-viscous* perturbations, controlled by the shear viscosity modulus μ/α , of a half-space subject to the hydrostatic initial stress gradient ρg (Haskell, 1935, 1936; Ranalli, 1987, pp. 192–199).

If $\rho g/(2\xi\mu) = 0$, equations (121)–(124) further reduce to

$$\beta = 0, \quad (125)$$

$$[U_x]_{x=0} = \frac{Q'H(t)}{2\xi\mu} (1 + \alpha t), \quad (126)$$

$$[P^{(\Delta)}]_{x=0} = Q'H(t), \quad (127)$$

$$[T_s]_{x=x_m} = \frac{1+i}{e} Q'H(t). \quad (128)$$

For $t = 0+$, equations (126)–(128) apply to elastostatic equilibrium in an *initially unstressed* half-space (e. g. Jeffreys, 1976, pp. 265–267). Equation (125) shows that the subsequent relaxation proceeds infinitely slow. On the other hand, it follows with (114) that

$$[U_x]_{x=0, t \rightarrow \infty} \rightarrow \infty, \quad (129)$$

where the instability is due to the absence of the gravitational force necessary to balance the load in the final hydrostatic state.

More practically, we may use the condition $\rho g/(2\xi\mu) \leq 10^{-2}$ to estimate the minimum value of ξ in order that effects due to the hydrostatic initial stress may be neglected. Considering the earth as an example and taking $\rho \simeq 10^3 \text{ kg m}^{-3}$, $g \simeq 10 \text{ m s}^{-2}$ and $\mu \simeq 10^{11} \text{ Pa}$ as values typical of the earth, we obtain $\xi \simeq 10^{-5} \text{ m}^{-1}$ as the minimum wave number. This approximately corresponds to a maximum wavelength of 10^6 m . Within the limits of the half-space approximation, the influence of the initial stress is therefore negligible as far as elastic perturbations are concerned. However, if $\mu \ll 10^{11} \text{ Pa}$, the initial stress becomes noticeable at much shorter wavelengths. This condition in particular applies to viscoelastic perturbations, where the 'effective' shear modulus, $s\bar{m}$, may become arbitrarily small.

Additional insight is gained by considering also $P^{(\delta)}$. In view of (18), (36) and (57), we have in particular

$$[P^{(\delta)}]_{x=0} = [P^{(\Delta)}]_{x=0} + \rho g[U_x]_{x=0}. \quad (130)$$

Upon substitution of (106) and (107), this becomes

$$[P^{(\delta)}]_{x=0} = Q'H(t). \quad (131)$$

Since it is readily shown that $[T_{xx}^{(\delta)}]_{x=0} = -[P^{(\delta)}]_{x=0}$, equation (131) simply expresses the balance required by (17) between the material incremental stress component normal to the surface of the half-space and the incremental load pressure.

In view of (115), it follows from (130) that

$$[P^{(\delta)}]_{x=0, t \rightarrow \infty} = \rho g[U_x]_{x=0, t \rightarrow \infty}. \quad (132)$$

In the final hydrostatic equilibrium state, the material incremental pressure is thus completely maintained by the advective incremental pressure associated with the displacement component in the direction of the initial pressure gradient. Assuming that (115) remains valid for $\rho g \rightarrow 0$, equation (132) also applies in this case and we require

$$\lim_{\substack{\rho g \rightarrow 0 \\ [U_x]_{x=0, t \rightarrow \infty} \rightarrow \infty}} \rho g[U_x]_{x=0, t \rightarrow \infty} = Q'. \quad (133)$$

Together, (132) and (133) ensure that the incremental boundary condition given by (131) is satisfied in the final hydrostatic state even in the absence of initial stress. This illustrates the physical significance of the singularity in the solution for the displacement noted in (129) from a different point of view.

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LIST OF SYMBOLS

Latin symbols

e	2.71828 ...
f	scalar field
$F_{ij\dots}$	spectral density of $f_{ij\dots}$
$f_{ij\dots}$	tensor field
$f_{ij\dots,k}$	partial derivative of $f_{ij\dots}$ with respect to X_k
$\bar{f}_{ij\dots}$	Laplace transform of $f_{ij\dots}$
$f_{ij\dots}^{(0)}$	initial value of $f_{ij\dots}$
$f_{ij\dots}^{(\Delta)}$	local increment of $f_{ij\dots}^{(0)}$
$f_{ij\dots}^{(\delta)}$	material increment of $f_{ij\dots}^{(0)}$
g	magnitude of g_i
g_i	gravitational force per unit mass
H	unit step function
i	imaginary unit
k	fluid bulk modulus
m_1	bulk relaxation function
m, m_2	shear relaxation function
n_i	unit vector normal to \mathcal{B}
p	mechanical pressure
q	incremental load pressure
s	Laplace-transform variable
t	current time
t'	previous time
t_s	maximum shear stress
t_1	largest principal stress
t_2	smallest principal stress
t_{ij}	Cauchy stress
u_i	particle displacement
X_i	initial particle position
(x, y, z)	Cartesian components of X_i
z_m	depth of maximum value of t_s

Greek symbols

α	inverse Maxwell time
β	inverse relaxation time
δ_{ij}	Kronecker symbol
∂_t	partial derivative operator with respect to t
λ	Love's strain function
μ	shear modulus
ξ	wave number
ρ	mass density

Calligraphic symbols

B	2-D boundary
\mathcal{L}	Laplace transformation
\mathcal{V}	3-D region

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