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ON THE CRITICAL SET FOR PHOTOGRAMMETRIC RECONSTRUCTION USING LINE TOKENS IN $\mathbf{P}_3(\mathbb{C})$

ABSTRACT. We prove that in general the critical set for photogrammetric reconstruction using lines in $\mathbf{P}_3(\mathbb{C})$ is a line congruence Γ of order 3 and class 6; Γ has 10 singular points and no singular planes. The general hyperplane sections of Γ (ruled surfaces formed by intersecting Γ with linear line complexes) have genus 5. Γ can be found in Fano's classification of congruences of order 3, and further properties of Γ can be found in the literature.

1. INTRODUCTION

Traditionally photogrammetry has used point tokens in 2-dimensional images to reconstruct a 3-dimensional scene. (See the references given in [11].) However, in recent years information scientists working in the field of computer vision have begun using line tokens from three images for reconstruction. (See [16]; also [3], [5], [8], [9].)

In this paper we shall assume that the prerequisite correspondence between tokens in the images has been established. Moreover, we assume that a 3-dimensional reconstruction with respect to the given correspondence exists.

For the sake of simplicity we consider only projective planes and spaces over the ground field $\mathbb{C}$ of complex numbers. We shall be concerned exclusively with reconstruction in $\mathbf{P}_3$ up to a collineation, i.e. up to an element of PGL$(3, \mathbb{C})$. Images are isomorphic to $\mathbf{P}_2$. For many of the arguments in this paper, the images are identified with the stars at the respective centers of projection.

The mathematical terms used in this paper can be found in the standard works dealing with line geometry, for example [15] or [19].

I have announced the result of this paper in [2]. The announcement – written for a broader audience – also contains a collection of definitions of concepts from line geometry. In addition, the announcement discusses the relationship between the set which defeats the algorithm described in [8] and the set described here.

2. THE CRITICAL LOCUS FOR RECONSTRUCTION WITH POINTS

The approach to studying the critical set for reconstruction with lines, which is used here, is similar to the determination of the critical locus for reconstruction with points. It is instructional to discuss this critical locus

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before investigating the critical set in the line token case. The proof of the following proposition roughly follows [18].

PROPOSITION 2.1. *Using corresponding point tokens from two images, the reconstruction problem admits more than one noncollinearly related solution if and only if the centers of the points of the scene all lie on a (possibly degenerate) quadric Q.*

Proof. Assume two solutions: one with centers O_1, O_2, the other with centers $\bar{O}_1, \bar{O}_2$. The images induce collineations α_i: star$(O_i) \rightarrow$ star$(\bar{O}_i)$ $(i = 1, 2)$. The plane pencil with axis $\bar{O}_1\bar{O}_2$ gives rise to two homographic plane pencils with axes $(\bar{O}_1\bar{O}_2)^{\alpha_i^{-1}}$, the homography being induced by the restriction of $\alpha_1 \circ \alpha_2^{-1}$.

Consider the locus Q generated by intersecting homographically related planes. If the axes coincide and the homography is the identity map, then $Q = \mathbf{P}_3$. Otherwise, Q is a possibly degenerate quadric, which contains the axes, hence the centers O_1 and O_2.

In the first case, select points P_1, P_2, P_3 so that O_1, O_2, P_1, P_2, P_3 form a frame of reference. Define a collineation $\kappa: \mathbf{P}_3 \rightarrow \mathbf{P}_3$ by setting $O_1^\kappa = \bar{O}_1$, $O_2^\kappa = \bar{O}_2$ and $P_i^\kappa = (O_1P_i)^{\alpha_1} \cap (O_2P_i)^{\alpha_2}$ $(i = 1, 2, 3)$. Then κ induces α_j between star(O_j) and star$(\bar{O}_j)$, as can be seen by comparing the images of the rays O_jP_i and O_jO_k $(j, k = 1, 2)$.

In the second case for any point $X \in Q$ we have $(O_iX)^{\alpha_i}$ $(i = 1, 2)$ are coplanar, hence meet. Conversely, if $\bar{r}_i \ni O_i$ $(i = 1, 2)$ meet, they are coplanar. □

REMARK. In the second case in the proof of the proposition, there also exists a corresponding quadric $\bar{Q}$ generated by the plane pencils with axes $(O_1O_2)^{\alpha_1}$ and $(O_1O_2)^{\alpha_2}$.

Using lines, three images are necessary. As in the proof of Proposition 2.1, we shall determine the general critical set by considering two noncollinearly related solutions with centers O_1, O_2, O_3 and $\bar{O}_1$, $\bar{O}_2$, $\bar{O}_3$ and collineations α_i: star$(O_i) \rightarrow$ star$(\bar{O}_i)$ $(i = 1, 2, 3)$.

3. Maximal coaxial relations

A *maximal coaxial relation** $\mathscr{X} \subseteq \mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$ is defined to be a set such that there exists a triple $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ of collineations from $\mathbf{P}_2$ onto plane stars in $\mathbf{P}_3$ with noncollinear centers O_1, O_2, O_3 such that

$$\mathscr{X} = \mathscr{X}(\gamma) = \{(x_1, x_2, x_3) \in \mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2 \mid x_1^{\gamma_1}, x_2^{\gamma_2}, x_3^{\gamma_3} \text{ are coaxial}\}.$$

*A maximal coaxial relation is called 'sectiv trilinear' in the German literature (see [7], [10]).

Discounting the element $(a_1, a_2, a_3) \in \mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$ with $a_i^{\gamma_i} = O_1 O_2 O_3$ $(i = 1, 2, 3)$, we may define an algebraic morphism

$$\chi = \chi(\gamma)\colon \mathscr{X} \backslash (a_1, a_2, a_3) \to \Omega\colon (x_1, x_2, x_3) \mapsto x_1^{\gamma_1} \cap x_2^{\gamma_2} \cap x_3^{\gamma_3}$$

where Ω denotes the set of lines in $\mathbf{P}_3$.

If we restrict χ to the inverse image of

$$\Omega \backslash (\mathrm{star}(O_1) \cup \mathrm{star}(O_2) \cup \mathrm{star}(O_3) \cup \text{ruled plane}(O_1 O_2 O_3)),$$

the restriction is clearly an isomorphism. Hence

PROPOSITION 3.1. *$\mathscr{X}$ is birationally equivalent to Ω.* □

PROPOSITION 3.2. *$\mathscr{X}$ is irreducible, and* $\dim \mathscr{X} = 4$.

Proof. $\mathscr{X}$ is the closure in the Zariski topology of an isomorphic copy of a Zariski open subset of Ω. Furthermore, Ω is irreducible, and $\dim \Omega = 4$. □

REMARK. The fact that $\mathscr{X}$ has dimension 4 was noted in [7, pp. 1–5].

Expressed in terms of maximal coaxial relations the critical sets for reconstruction using lines are $\Gamma = \chi(\gamma)(\mathscr{X}(\gamma) \cap \mathscr{X}(\bar{\gamma}))$ where $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ denotes a triple of collineations $\bar{\gamma}_i\colon \mathbf{P}_2 \to \mathrm{star}(\bar{O}_i)$ in general position with respect to γ. (The assumption on general position of $\bar{\gamma}$ implies $\mathscr{X}(\gamma) \cap \mathscr{X}(\bar{\gamma})$ lies in the domain of definition of χ. See Section 4 below for a detailed discussion.) Set $\alpha_i = \gamma_i \circ \bar{\gamma}_i^{-1}\colon \mathrm{star}(O_i) \to \mathrm{star}(\bar{O}_i)$. The notation is now similar to that of Proposition 2.1.

We shall show that $\mathscr{X}(\gamma)$ and $\mathscr{X}(\bar{\gamma})$ intersect properly, i.e. every irreducible component of $\mathscr{X}(\gamma) \cap \mathscr{X}(\bar{\gamma})$ has dimension 2.

REMARK. One could expect that the cycles determined by $\mathscr{X}(\gamma)$ and $\mathscr{X}(\bar{\gamma})$ are rationally equivalent in the Chow ring A of $\mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$. (See the appendix on intersection theory in [6, p. 426] for definitions.) If this is so, the cycle in the Chow ring of Ω corresponding to the critical set Γ is the image under χ_* of the self-intersection of $\mathscr{X} = \mathscr{X}(\gamma)$. This can be expressed in terms of the second Chern class of the normal sheaf $\mathscr{N}$ of $\mathscr{X}$ in $\mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$, namely

$$\chi_*(\mathscr{X} . \mathscr{X}) = \chi_*(i_*(c_2(\mathscr{N})))$$

where $i\colon \mathscr{X} \to \mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$ is the inclusion map.

PROPOSITION 3.3. *Each component of Γ has dimension* $\geqslant 2$.

Proof. The dimension of Γ is the same as that of $\mathscr{X}(\gamma) \cap \mathscr{X}(\bar{\gamma})$. Since the dimension of a component is a local property, it suffices to consider intersections in affine subspaces of the type $\mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_2$ covering $\mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$. The result then follows from the affine dimension theorem [6, p. 48, Prop. 7.1]. □

4. Assumptions on general position

We shall deal only with the general configuration of centres and general collineations α_i. Explicitly, we make the following assumptions:

(i) Each of the sets $\{O_1, O_2, O_3\}$ and $\{\bar{O}_1, \bar{O}_2, \bar{O}_3\}$ is assumed to span a plane.

(ii) The three planes $(O_1O_2O_3)^{\alpha_i}$ $(i = 1, 2, 3)$ are assumed to intersect in a single point $\bar{P}$. Similarly, the three planes $(\bar{O}_1\bar{O}_2\bar{O}_3)^{\alpha_i^{-1}}$ are assumed to intersect in a single point P. Moreover, we assume $\bar{O}_1 \notin (O_1O_2O_3)^{\alpha_1} \cap (O_1O_2O_3)^{\alpha_2}$ and a similar relation holds for centers $\bar{O}_2$ and $\bar{O}_3$.

(iii) The rays $(O_iP)^{\alpha_i}$ $(i = 1, 2, 3)$ are assumed not to be concurrent, where P denotes the intersection of the $(\bar{O}_1\bar{O}_2\bar{O}_3)^{\alpha_i^{-1}}$ as in assumption (ii) above.

(iv) Consider the three critical loci Q_{ij} $(\bar{Q}_{ij})$ with respect to the pairs of centers O_i and O_j $(\bar{O}_i$ and $\bar{O}_j)$ $(i, j = 1, 2, 3)$. Each Q_{ij} $(\bar{Q}_{ij})$ is assumed to be a proper quadric. (This implies $(O_iO_j)^{\alpha_i}$ and $(O_jO_i)^{\alpha_j}$ are skew.) Each intersection $Q_{ij} \cap Q_{ik}$ $(\bar{Q}_{ij} \cap \bar{Q}_{ik})$ is assumed to be an irreducible quartic.

(v) For any fixed $j \in \{1, 2, 3\}$ the two lines $(O_iO_j)^{\alpha_i}$ for $i = 1, 2, 3$; $i \neq j$ are assumed to be skew.

(vi) The set $Q_{23} \cap Q_{13} \cap Q_{12}$ is assumed to consist of eight associated points. Moreover, $\{O_1, O_2, O_3\} \cap Q_{23} \cap Q_{13} \cap Q_{12} = \varnothing$. (Cf. [15, pp. 327–347] for a general discussion of the intersections of quadrics.)

REMARK. There is a restriction regarding the generality of the configuration of the quadrics Q_{ij} and points O_i. Let Σ_{ij} denote the quadratic envelope consisting of tangent planes of Q_{ij}. Then there exists an element $\pi \in \Sigma_{ij} \cap \Sigma_{ik}$ with P and O_i in π, where $P \in Q_{23} \cap Q_{13} \cap Q_{12}$ denotes the point of intersection of $(\bar{O}_1\bar{O}_2\bar{O}_3)^{\alpha_k^{-1}}$ $(k = 1, 2, 3)$ as in assumption (ii). Indeed, π is $(\bar{O}_1\bar{O}_2\bar{O}_3)^{\alpha_i^{-1}}$. This type of configuration is mentioned in [1, p. 12] in connection with a line congruence like the one described in the main theorem in the next section.

5. Main result

The main result of this paper is the following:

THEOREM. *Under the assumption of general position given above, the critical set for reconstruction using corresponding line tokens from three images is an irreducible line congruence Γ of order 3 and class 6. Γ has precisely 10 singular points: the centers O_1, O_2, O_3 and seven points in $Q_{23} \cap Q_{13} \cap Q_{12}$. The singular* *points are vertices of cubic cones contained in Γ. Γ has no singular planes.*

The proof uses properties of sets generated by three collinear plane stars and sets generated by three trilinearly related plane pencils in $\mathbf{P}_3$ as expounded in [17]. Part of the proof consists of analyzing various special cases to ascertain the singular points and planes.

We can identify Γ in Fano's classification [4] of congruences on the basis of Γ's singular points and planes. From Fano's classification we see that the general hyperplane sections of Γ considered as a subset of Ω (ruled surfaces formed by intersecting Γ with linear complexes) have genus 5.

According to [4, pp. 70–72], the irreducible (3, 6)-congruence with 'sectional' genus 5 has a plane representation consisting of curves of order 7 having double points in 10 base points. The cones of Γ correspond to elliptic curves of order 3 passing through 9 of the 10 base points. (See also [14, pp. 319–321]. In [12, pp. 383–408] varieties related to the general configuration as well as some of the special configurations of the base points are also discussed.) According to [1, pp. 12–13] the base points correspond to ruled surfaces of order 2 contained in Γ. Γ has rank 5. It has a focal surface of order 14 and class 20. (In [1, p. 11] it is reported that focal surface has order 16.)

The proof of the theorem is broken up into a series of propositions. It is convenient to first discuss the centers, then the class and singular planes, and lastly the order and singular points.

6. THE CENTERS

PROPOSITION 6.1. *The centers O_i $(i = 1, 2, 3)$ are vertices of cubic cones contained in Γ. Each plane π through O_i contains a finite number of lines of Γ.*

Proof. We prove the proposition for $i = 1$. Take an arbitrary plane $\pi \ni O_1$.

CASE 1: $\pi \cap \{O_2, O_3\} = \varnothing$. We show that π contains at most three rays of Γ passing through O_1.

Now π determines a perspectivity between star(O_2) and star(O_3). This induces a homography between the plane pencils $(O_2O_1)^{\alpha_2}$ and $(O_3O_1)^{\alpha_3}$. These axes are skew by assumption (v).

The homography generates a ruled surface $\bar{\Sigma}$, which is proper since the axes are skew. Moreover, $\bar{\Sigma}$ is homographically related to the pencil $(\bar{O}_1, \pi^{\alpha_1})$. By abuse of notation we identify $\bar{\Sigma}$ with the quadric locus supporting it.

In the case where $\pi^{\alpha_1} \cap \bar{\Sigma}$ is a proper conic, coordinatize π^{α_1} so that $\pi^{\alpha_1} \cap \bar{\Sigma}$ is given by

$$\theta \mapsto (1, \theta, \theta^2) \quad (\theta \in \mathbf{P}_1)$$

and pencil$(\bar{O}_1, \pi^{\alpha_1})$ is given by

$$u_1 + \theta u_2 = 0$$

where u_1, u_2 are homogeneous linear forms in the three coordinate variables. Then each solution for θ in the cubic equation

$$(1) \qquad u_1(1, \theta, \theta^2) + \theta u_2(1, \theta, \theta^2) = 0$$

determines an element $\bar{a} \in \bar{\Sigma}$ and thus homographically related planes $\bar{\eta}_j \supseteq (O_j O_1)^{\alpha_j}$ $(j = 2, 3)$ intersecting in $\bar{a}$. Set $\bar{\eta}_1 = \bar{O}_1 \bar{a}$. Then for each solution θ, the planes $\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3$ are coaxial. Also η_1, η_2, η_3 are coaxial, because $\bar{\eta}_1$ meets π^{α_1} precisely in the line where η_2 and η_3 (and π) meet.

Equation (1) cannot be satisfied by all θ, for otherwise it is easy to see $\pi^{\alpha_1} \cap \bar{\Sigma} \subseteq \bar{Q}_{12} \cap \bar{Q}_{13}$ contradicting generality assumption (iii). Hence there are at most three values of θ satisfying (1).

Consider the special case where $\pi^{\alpha_1} \cap \bar{\Sigma}$ consists of two lines: $\bar{u}$ and a transversal $\bar{v}$ of $(O_i O_1)^{\alpha_i}$ $(i = 2, 3)$.

If $\bar{u} \ni \bar{O}_1$, there can only be one element in $\Gamma \cap$ pencil(O_1, π) namely $l = \bar{u}^{\alpha_1^{-1}}$.

If $\bar{u} \not\ni \bar{O}_1$, then in general we have a $(1, 1)$ correspondence between pencil$(\bar{O}_1, \pi^{\alpha_1})$ and $\bar{u}$. It follows that in general $\bar{u}$ has two united points $\bar{M}_j$, and only $(\bar{O}_1 \bar{M}_j)^{\alpha_1^{-1}} \in \Gamma \cap$ pencil(O_1, π), $(j = 1, 2)$.

If $\bar{u}$ and pencil$(\bar{O}_1, \pi^{\alpha_1})$ were perspectively related, then it is easy to see that $\bar{u} \in \bar{Q}_{12} \cap \bar{Q}_{13}$; a contradiction.

In general, the collineation between star$(\bar{O}_2)$ and star$(\bar{O}_3)$ generates a $(1, 3)$ congruence. Besides the lines through O_1, π contains at most only three other lines l such that π^{α_1}, $(O_2 l)^{\alpha_2}$ and $(O_3 l)^{\alpha_3}$ meet.

In cases where the collineation generates a congruence with a singular plane, this plane contains the centers $\bar{O}_2$ and $\bar{O}_3$; moreover, the plane is self-corresponding [17, III, pp. 182–183]. Because of the generality assumptions π^{α_1} cannot be such a plane for then $\pi^{\alpha_1} = \bar{O}_1 \bar{O}_2 \bar{O}_3$, and $(\bar{O}_1 \bar{O}_2 \bar{O}_3)^{\alpha_i^{-1}}$ $(i = 1, 2, 3)$ would meet in a line; a contradiction.

CASE 2: Now consider $\pi \supseteq O_1 O_2$ with $\pi \not\ni O_3$. Which $l \in$ pencil(O_1, π) is in Γ? This occurs whenever $\pi^{\alpha_2} \cap (O_3 l)^{\alpha_3}$ meets l^{α_1}. We shall first consider the general case and then consider possible anomalies.

Coordinatize π^{α_2} so that $\pi^{\alpha_2} \cap \pi^{\alpha_1}$ has points $(1, \theta, 0)$ $(\theta \in \mathbf{P}_1)$ corresponding to l^{α_1} for l varying in pencil(O_1, π). The corresponding $\pi^{\alpha_2} \cap (O_3 l)^{\alpha_3}$ can then be given by $u_1 + \theta u_2 = 0$, where u_1, u_2 denote homogeneous linear forms in the coordinate variables. Then for each l corresponding to a θ satisfying the quadratic equation

$$u_1(1, \theta, 0) + \theta u_2(1, \theta, 0) = 0$$

we have $\eta_1 = (l^{\alpha_1}\bar{O}_3)^{\alpha_1^{-1}}$, $\eta_2 = \pi$ and $\eta_3 = lO_3$ meet in l, and $\eta_i^{\alpha_i}$ are coaxial.

If $u_1 + \theta u_2 = 0$ and $(1, \theta, 0)$ were perspectively related, then $\pi^{\alpha_2} \cap \pi^{\alpha_1}$ would belong to $\bar{Q}_{13} \cap \bar{Q}_{12}$ contradicting the generality assumption.

Since $\bar{Q}_{12}$ is proper, we have $\pi^{\alpha_1} \neq \pi^{\alpha_2}$. Besides lines through O_1 or O_2 only one other line of π can be in Γ, namely $\pi \cap (\bar{O}_3(\pi^{\alpha_1} \cap \pi^{\alpha_2}))^{\alpha_3^{-1}}$. In the case where $(O_3O_1)^{\alpha_3} \subseteq \pi^{\alpha_2}$, there are again only two cases of interest: an $l \in \text{pencil}(O_1, \pi)$ such that $(O_3l)^{\alpha_3} = \pi^{\alpha_2}$, and an l such that $l^{\alpha_1} \cap \pi^{\alpha_2} \subseteq (O_3O_1)^{\alpha_3}$.

CASE 3: $\pi = O_1O_2O_3$. Consider first lines in π not containing O_2. Note that

$$(\bar{O}_1((O_1O_2O_3)^{\alpha_2} \cap (O_1O_2O_3)^{\alpha_3}))^{\alpha_1^{-1}}$$

meets $O_1O_2O_3$ in a line of Γ.

Furthermore, $O_1O_2 \in \Gamma$. In fact, let $\bar{l}$ join $(O_1O_2)^{\alpha_1} \cap (O_1O_2O_3)^{\alpha_3}$ and $(O_1O_2)^{\alpha_2} \cap (O_1O_2O_3)^{\alpha_3}$. Then the planes $\bar{\eta}_i$ spanned by $\bar{l}$ and $(O_1O_2)^{\alpha_i}$ $(i = 1, 2)$ have the property that $\bar{\eta}_i^{\alpha_i^{-1}}$ and $O_1O_2O_3$ meet in O_1O_2.

The same holds for O_1O_3. □

7. Γ'S CLASS AND ORDER

PROPOSITION 7.1. *Γ has class* 6.

Proof. Take a plane π with $\pi \cap \{O_1, O_2, O_3\} = \varnothing$. Then π determines a perspectivity between star(O_i) and star(O_j) $(i, j = 1, 2, 3)$ via

$$r \mapsto (r \cap \pi)O_j.$$

The α_i induce collineations between the plane stars at $\bar{O}_i$. In general, there exist precisely six corresponding planes $\bar{\eta}_i \in \text{plane star}(\bar{O}_i)$ which are coaxial [17, III, p. 185]. Then the $\bar{\eta}_i^{\alpha_i^{-1}}$ are coaxial, and their axes lie in π. □

PROPOSITION 7.2. *Γ has no singular planes.*

Proof. In general, three collinear plane stars generate a cubic surface F. Special effects can occur only when

(i) three corresponding rays meet;
(ii) two or more centers coincide;
(iii) there exists a line in every point of which three corresponding rays meet;
(iv) the centers lie all on one line;
(v) the plane containing the centers is self-corresponding.

See [17, III, pp. 197–202].

Regarding case (i) the point of intersection of three corresponding rays is a node on the surface F. This may result in a double line but not an infinite number of lines (cf. [13, II, p. 170]).

Cases (ii) and (iv) cannot occur because of the assumption that O_1, O_2 and O_3 span a plane.

Case (iii) cannot occur, because we have assumed $Q_{23} \cap Q_{13} \cap Q_{12}$ is finite.

Case (v) cannot occur, because of assumption (ii).

Hence all planes not meeting $\{O_1, O_2, O_3\}$ are nonsingular. By Proposition 2.1 no planes through the centers are singular. $\square$

PROPOSITION 7.3. Γ *has order* 3.

Proof. Let $X \notin (Q_{23} \cap Q_{13} \cap Q_{12}) \cup \{O_1, O_2, O_3\}$. Then the relation

$$T = \{(\eta_1, \eta_2, \eta_3) \mid XO_i \subseteq \eta_i \text{ and } \eta_i \text{ coaxial } (i = 1, 2, 3)\}$$

is a trilinearity between the plane pencils with axes XO_i. If X does not lie on a critical locus, each pair of lines $(XO_i)^{\alpha_i}$ is skew. T induces a trilinearity $\bar{T}$ between the plane pencils with axes $(XO_i)^{\alpha_i}$. In general, this will not be a perspectivity, i.e. there will not exist a plane π with the property that $(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \in \bar{T}$ iff $\bar{\eta}_1 \cap \pi$, $\bar{\eta}_1 \cap \pi$, $\bar{\eta}_1 \cap \pi$ are concurrent. By [17, II, p. 325] there exist precisely three triples $(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \in \bar{T}$ such that $\bar{\eta}_1$, $\bar{\eta}_2, \bar{\eta}_3$ are coaxial. By definition of $\bar{T}$, the $\bar{\eta}_i^{\gamma_i^{-1}}$ are coaxial. Thus precisely those axes, which by construction pass through X, belong to Γ. $\square$

PROPOSITION 7.4. Γ *has precisely* 10 *singular points. The singular cones have order* 3.

Proof. By Proposition 6.1, O_1, O_2, O_3 are three singular points which are vertices of cubic cones. Also points $X \in Q_{23} \cap Q_{13} \cap Q_{12}$, where $\bar{r}_i = (O_i X)^{\alpha_i}$ $(i = 1, 2, 3)$ are concurrent, are singular and vertices of cubic cones [17, II, pp. 326–327].

Note that $P = \bigcap_{i=1,2,3} (\bar{O}_1 \bar{O}_2 \bar{O}_3)^{\alpha_i^{-1}}$ lies on $Q_{23} \cup Q_{13} \cup Q_{12}$. But by assumption (iii) P does not have the property that $(O_i P)^{\alpha_i}$ are concurrent, whereas the other points of $Q_{23} \cup Q_{13} \cup Q_{12}$ do. Thus $Q_{23} \cup Q_{13} \cup Q_{12}$ contributes seven singular points.

For $X \in Q_{23} \cap Q_{13} \cap Q_{12}$ some of the axes $\bar{r}_i$ may meet. If only two axes meet, $\bar{T}$ generates a cubic surface with a node [17, II, pp. 326–327].

In the perspective case described in the proof of Proposition 6.1, every common transversal of $(XO_i)^{\alpha_i}$ $(i = 1, 2, 3)$ is the axis of a trilinearly related triple $(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)$. In this case the lines $\bar{\eta}_1^{\alpha_1^{-1}} \cap \bar{\eta}_2^{\alpha_2^{-1}}$ would be a plane pencil with vertex X. Thus Γ would have a singular plane, contradicting Proposition 7.2.

$\square$

PROPOSITION 7.5. Γ *is irreducible.*

Proof. By Proposition 3.3 each component of Γ has dimension $\geqslant 2$. Since Γ does not contain any subsets of dimension $\geqslant 3$, each component must be a congruence. Apparently the orders and classes of the components add up to 3and 6 respectively. Hence if Γ were irreducible, there must be components of order 1 and possibly of order 0.

A congruence of order 0 is a ruled plane. But by Proposition 7.2, Γ has no singular planes.

Any congruence of order 1 has a curve consisting of singular points [19, p. 1184]. But such singularities are not compatible with those we have established in Proposition 7.4. □

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