Photogrammetry and projective geometry—an historical survey

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Abstract

Général Jean-Victor Poncelet published his treatise on projective geometry in 1822. This was the start of an enormous development in geometry in the 19th century. During this period geometry in the plane and in 3-dimensional space was studied in particular detail. The development culminated in the publishing of the Encyclopedia of Mathematics, which appeared in irregular installments from 1900 to 1934.

Photogrammetry—the use of photographic images for surveying, mapping and reconnaissance—began in the second half of the 19th century. By the 1890's substantial theoretical contributions were made by Sebastian Finsterwalder. Finsterwalder reported on his foundational work in a keynote address to the German Mathematical Society in 1897; he also contributed an article on photogrammetry to the Encyclopedia of Mathematics.

Among other things Finsterwalder observed that Rudolf Sturm's analysis of the "homography problem" (1869) can be used to solve the problem of 3D-reconstruction using point matches in two images.

Subsequently, important theoretical advances were made by mathematicians at the Technical University of Vienna. An excellent reference for geometry and its relationship to photogrammetry is a book of Emil Müller on constructive geometry, which appeared in 1923.

Müller's assistent and successor Erwin Kruppa established the "structure-from-motion" theorem in 1913. This theorem was rediscovered by Shimon Ullman in 1977.

1. The role of projective geometry

Projective geometry[40, 45] was introduced by studying the phenomenon of central projection. Usual geometry (so-called *affine geometry*) and projective geometry are equivalent in the sense that affine geometry can be extended to a projective geometry, and the extension is unique up to isomorphism.

The two ways of viewing geometry—projectively and affinely—have coexisted since the inception of the former standpoint. There are good reasons why this is so. First affine geometry has the advantage of being familiar. Also in affine geometry a point has a unique representation in terms of coordinates, which simplifies the investigation of details of geometric phenomena. On the other hand, formulae and theorems most often have a simpler form when they are expressed in the projective context. In particular, there is a considerable reduction of the number of cases in the projective setting. Since geometric phenomena are often very complex, one can profit a great deal from the simplifications which projective geometry entails.

In this historical survey I shall concentrate exclusively on the geometrical theory of photogrammetry, although I have included some early references to engineering, optical and mechanical instruments and experimental results in the bibliography.

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2. The horopter

A stationary camera determines a transformation from 3-space to 2-space. The parameters of this transformation form a 4×3 matrix C. Camera calibration is defined to be the determination of the elements of C using the coordinates of a set of calibration points in 3-space and the coordinates of their respective images.

For camera calibration there exists a *critical set*, which is defined to be the locus of points in 3-space for which calibration is not possible. If the calibration points lie on the critical set, C is not uniquely determined by input data—the coordinates of the calibration points and their respective images, i.e., there exists another matrix C', independent of C, which is also a solution to the problem.

Before photogrammetry got started in the second half of the 19th century, the problem of the critical set for camera calibration was studied, albeit indirectly, in the context of physiological optics.

In physiological optics the eyes of a person can be regarded as two cameras. If we identify the two retinas appropriately, we may ask what the locus of points in 3-space which give corresponding points on the retinas. This locus is called the *horopter*. I shall also use this term when referring to two cameras. The problem of determining the horopter in the plane for a person who has neither raised or lowered his eyes was first solved by G. U. A. Vieth in 1818 and then again in 1826 by Johannes Müller. The general case was analyzed independently by H. Helmholtz and E. Hering in the 1860's. Helmholtz's monograph[14] contains an historical discussion and a list of early references. Actually, Helmholtz and Hering were very much at odds regarding the importance of the horopter, and their papers witness a heated debate.

On Felix Klein's suggestion a model of the horopter was included in a collection of plaster and wire models, which were produced commercially by L. Brill.[2]

In the 1930's the horopter curve was again taken up in this context by K. N. Ogle.[38] In Section 10 we shall come across the horopter curve in the context of error analysis. A discussion of the projective form of the horopter in connection with camera calibration was given by E. Waelsch in F. Steiner's textbook.[48]

3. Early photogrammetry

S. Finsterwalder described the historical beginnings of photogrammetry in an article[11] in the Encyclopedia of Mathematics.

> Photogrammetry—the art of making measurements using images—is the task of determining an object or its dimensions using photographs. Preliminary work on this problem was done by Lambert in what he referred to as "inverting the perspective"[29] and by Beautemps-Beaupré (1791-1793).[1] In surveying these methods were first tested by A. Laussedat (1852-59).[31] Starting in 1855 J. Porro[6] began developing instruments for photogrammetry. A. Meydenbauer[35] brought architechtural photogrammetry to high level. W. Jordan[17] and C. Koppe[20, 21] approached the problem from the standpoint of geodesy, and G. Hauck[12, 13] approached it from a theoretical point of view. Photogrammetry was practiced on a large scale in Italy by L. P. Paganini since 1880[39] and in Canada by E. Deville[5] since 1889. S. Finsterwalder has been doing aerial photogrammetry from balloons since 1890. C. Pulfrich has been using stereoscopy since 1890. A. Laussedat has collected material on the history of photographic methods and equipment."

My knowledge of early photogrammetry is primarily from Finsterwalder's articles. Finsterwalder was a mathematician by training—he received his doctorate from my alma mater, the University of Tübingen, under the guidance of the algebraic geometer A. Brill in 1886.[19] Finsterwalder was also an alpinist, who became interested in photogrammetry by studying the flow of glaciers using photographs. His balloon flights launched the era of aerial phtogrammetry.[10]

4. Epipoles

In 1897 Finsterwalder addressed the German Mathematical Society, and he described some of the results of projective geometry he was applying to photogrammetry. [9]

In this talk I shall restrict myself to geometric methods which apply to *derived images*. These are defined to be images which result by applying an arbitrary but fixed collineation mapping the star at the center of projection to the image plane. (A large portion of Finsterwalder's address and photogrammetric literature deal with images where restrictions are placed on this collineation.)

If we have two images of a scene and we hope to reconstruct the scene or at least part of it as well as the positions of the cameras in 3-space, we will be able to tell *a posteriori* where in the image the other camera is located, assuming, of course, the cameras' fields of view are large enough. These points in the images are called *epipoles* (German: Kernpunkte). One approach to 3D-reconstruction is to first determine the epipoles. The lines through the epipoles are called *epipolar lines* (German: Kernstrahlen). The lines arise from planes in 3-space passing through the two camera centers Z_1 and Z_2. After determining the epipoles, we can then determine the scene and the position of the camera up to a projective transformation or in the euclidean case up to a similarty transformation.

Let us assume we have corresponding points $P_1, \ldots, P_n$ and $P'_1, \ldots P'_n$ in the two images $\mathcal{I}$ and $\mathcal{I}'$—the points being images of $X_1, \ldots, X_n$ in 3-space. The determination of the epipoles O and O' in $\mathcal{I}$ and $\mathcal{I}'$ respectively are made possible by the fact that for each ordered subset i, j, k, l of indices the epipolar lines from O to P_i, P_j, P_k, P_l in $\mathcal{I}$ and from O' to P'_i, P'_j, P'_k, P'_l in $\mathcal{I}'$ have the same cross-ratio. This cross-ratio is equal to the cross-ratio of the four planes joining the line Z_1Z_2 to the points X_i, X_j, X_k, X_l in 3-space, because collineations preserve cross-ratios.

5. Chasles' homography problem and Hesse's solution

The problem of finding O and O' using the cross-ratio property of the preceding section is called the *homography problem*. The French geometer M. Chasles posed this problem in 1855 for the case of seven corresponding point pairs.[3] O. Hesse published an answer to Chasles' problem in 1863.[15] Hesse's solution is essentially the same as the first part of the so-called "8-point algorithm" published by H. C. Longuet-Higgins in 1981.[34]

To describe Hesse's solution consider maps from the plane $\mathbb{P}$ which preserve incidence but which reverse the role of the points and lines. In the language of projective geometry such a map κ is a collineation between the projective plane and the dual plane. The map κ is called a *correlation*, and it can be described by a 3×3 matrix M by introducing homogeneous coordinates. M is defined up to a nonzero scalar factor. Hesse's paper focuses on "singular" correlations, specifically, on correlations where the points of $\mathbb{P}$ get mapped to a set of lines all passing through some fixed point. This can be expressed by requiring M to have rank 2. In particular, we have $\det M = 0$.

Besides this equation seven other linear equations for the elements of M are obtained by requiring that for each pair of corresponding points

$$\begin{matrix} P_1, & \ldots, & P_7 \\ P'_1, & \ldots, & P'_7 \end{matrix}$$

P'_i is incident with the line $\kappa(P_i)$.

6. Sturm's analysis

R. Sturm also published a solution to Chasles' problem in 1869.[49, 50] Sturm's solution is particularly interesting, because it also analyzes the solution with four, five and six pairs of points. Sturm makes use of a theorem of Chasles,[47] which states that if X_1, X_2, X_3, X_4 are four points on a conis s, then the cross-ratio of the rays VX_1, VX_2, VX_3, VX_4 is the same for all choices of V on s.

Consider four corresponding pairs of points $P_1, \ldots, P_4$ and $P'_1, \ldots, P'_4$. Suppose we know the epipole O in the first image $\mathcal{I}$. This gives us the cross-ratio θ of $OP_1, \ldots, OP_4$.

Construct a line t through P_1' such that the cross-ratio of the rays $t, P_1'P_2', P_1'P_3', P_1'P_4'$ is equal to θ. Consider the conic s' through $P_1', \ldots, P_4'$ which has t as a tangent. Then by Chasles theorem the cross-ratio of $O'P_1', \ldots, O'P_4'$ is the same for all O' on s'. Let O' and P_1' coalesce; then $O'P_1'$ becomes t, so this cross-ratio is equal to θ. Thus a potential epipole in the first image corresponds to a conic in the second image.

Now let us analyze the situation with five corresponding pairs of points $P_1, \ldots, P_5$ and $P_1', \ldots, P_5'$. Let O be a potential epipole in the first image. We consider subsets of four corresponding pairs of points, omitting P_5, P_5' and P_4, P_4' in succession. For each subset, O coresponds to a conic through P_1', P_2', P_3' and through P_4 or P_5 respectively. Thus we get a configuration as indicated in the figure below.

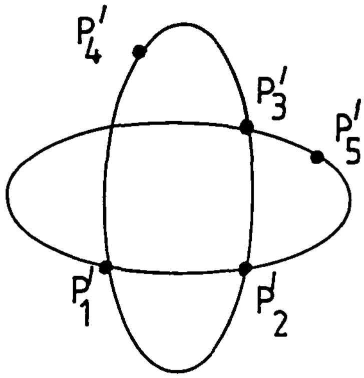

Figure 1

The fourth point O' of intersection of the two conics—the point of intersection distinct from P_1', P_2', P_3'—has the desired property that the cross-ratio of $O'P_1', O'P_2', O'P_3', O'P_4'$ as well as the cross-ratio of $O'P_1', O'P_2', O'P_3', O'P_5'$ are equal to the corresponing cross-ratios in the first image. Hence every potential candidate O' for the epipole in general position in the first image determines a unique candidate for the epipole O' in the second image. (There are some special positions where the correspondence is not defined or not unique.) This correspondence $O \longmapsto O'$ can be described by a polynomial map between the two images. Sturm proves that the degree of the polynomials involved in the correspondence is 5. This means that a general line in first image corresponds to a curve of degree 5 in the second image. A polynomial transformation from $\mathbb{P}$ to $\mathbb{P}$, which is defined everywhere and is bijective except perhaps for points lying on a finite set of curves is called a *Cremona transformation.*

Consider now six pairs of corresponding points. Applying the same approach as above, we consider two subsets of five corresponding point pairs, each of which determines a Cremona transformation. Candidates for the epipoles are points O whose images O_1' and O_2' coincide. Sturm proves that the locus of such points is a curve of degree 3 passing through the points $P_1', \ldots P_6'$ in the second image.

Finally, for seven pairs of corresponding points the candidates for the epipoles are the points of intersection of the two cubic curves distinct from $P_1, \ldots, P_7$. Two cubic curves intersect in nine points. But considering the subsets $P_1, \ldots, P_5, P_6$ and $P_1, \ldots, P_5, P_7$ formed by omitting P_7 and P_6 respectively we see that the cubic curves have $P_1, \ldots, P_5$ in common. Sturm proves that one of the remaining four points of intersection is of a special nature and not a candidate for the epipoles. Hence for seven pairs of corresponding point pairs there exist three corresponding point pairs which are candidates for the epipoles. These correspond to the three solutions of the cubic equation $\det M = 0$ and linear equations in Hesse's solution to Chasles' problem.

7. Euclidean geometry and the absolute conic

Up to now I have discussed the geometry of 3D-reconstruction without taking euclidean geometry into account. Following Laguerre, euclidean transformations are most elegantly described in the plane by introducing circular points. These points are defined to be the points of intersection of an arbitrary circle with equation

$$(x-a)^2+(y-b)^2=r^2 \tag{1}$$

with the line at infinity. Introducing a new variable z to make Eq. 1 homogeneous in the variables x, y, z, we obtain

$$(x-az)^2+(y-bz)^2=r^2z^2.$$

Now setting $z=0$ we obtain

$$x^2+y^2=0$$

which gives solutions $I=(1:i:0)$ and $J=(1:-i:0)$ for $(x:y:z)$. Here i denotes a complex number with $i^2=-1$. The points I and J are called *circular points*. A *similarity transformation* is the composition of a dilation and a euclidean transformation. A similarity transformation in the plane can be characterized as being a collineation of the real plane—which we think of as being embedded in the plane over the complex numbers— which leaves I and J fixed or interchange the two points. In this role if we use a conic instead of circular points, we obtain a *noneuclidean geometry*.

Similarly, 3-dimensional euclidean geometry can be introduced by considering the intersection of an arbitrary sphere with equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \tag{2}$$

with the plane at infinity. The plane at infinity is described by introducing a new variable w to make Eq. 2 homogeneous, then by setting $w=0$. The intersection is then the conic

$$x^2+y^2+z^2=0 \tag{3}$$

in the plane $w=0$. The conic in Eq. 3 is called the *absolute conic* Ω. A conic with no real points is called *virtual*. Ω is a virtual conic.

The approach outlined above is described in the excellent introductory English text by Semple and Kneebone[47] and in Emil Müller's book,[37] where details of applications of the absolute conic to descriptive geometry and photogrammetry are afforded particular prominence.

Consider a series of images from a moving camera. I shall assume that the internal adjustments of the camera like the lens, the ratio of the horizontal height to the vertical height of the image, etc. are not changed while obtaining the images. Then each sphere, in particular the unit sphere about the center of the camera, intersects the plane at infinity in the same curve Ω. Hence the projection of Ω, the "image" of Ω is the same in all of the images. Since Ω is virtual, its image is also virtual. Nonetheless, a computer can perform calulations using the coordinates of imaginary points. When drawing figures, I normally draw Ω to be a circle, since the circle has similar incidence behaviour as Ω.

To find the image of Ω it sufices to calibrate the camera in the laboratory. From calibration we obtain a 3×4 matrix, which we apply to the matrix of coordinates of points of $x^2+y^2+z^2=0$ and $w=0$.

The point of this discussion is that the images contain a conic (with no real points) which was introduced in a rather natural way. This virtual conic endows each image with an *elliptic geometry*.

8. Kruppa's "structure-from-motion" theorem

In 1913 E. Kruppa, who was Müller's assistent and who subsequently became Müller's successor at the Technical University of Vienna, published a theorem now referred to as the "structure-from-motion" theorem.[28] The theorem assumes five corresponding point pairs and a camera which was calibrated in the laboratory. To find the epipoles under these assumptions, Kruppa derived an analytical expression for Sturm's Cremona transformation. He then considered the two nonreal tangent lines from candidates for the epipoles to the absolute conic, thereby obtaining a set of seven rays, which then sufficed to determine the epipoles up to finite ambiguity. The upper bound given by Kruppa on the number of solutions, was sharpened to 10 by Faugeras and Maybank in 1990,[8] when they analyzed Kruppa's work using a computer symbolic math package.

The "structure-from-motion" theorem was rediscovered by S. Ullman in 1977.[51]

9. The critical set for 3D-reconstruction

After World War I a number of mathematicians at the Technical University of Vienna published papers on photogrammetry. I learned of these from the bibliography in the monograph of Rinner and Burkhardt.[41]

During World War II the critical set for 3D-reconstruction was studied from both the projective and euclidean point of view. The German term for the critical set is the "gefährlicher Ort", which literally means "dangerous locus".

The *critical set C for 3D-reconstruction* is the locus of points in 3-space which admits an ambiguous reconstruction when images of points of C are given in two views. C depends on the positions of the cameras in space. With respect to reconstruction up to a general projective space collineation, C is an arbitrary quadric surface passing through the centers of projection.[52]

In 1940 J. Krames[22] published an analysis of the critical set for 3D-reconstruction in the euclidean setting, which was based on a monograph of H. Schröter[46] from 1880 on quadrics.

To describe Krames' result, first consider an *orthogonal cone*, which is defined to be a quadric cone whose base curve is a circle having the additional propertv that a line lying on the cone is perpendicular to the plane of the circle.

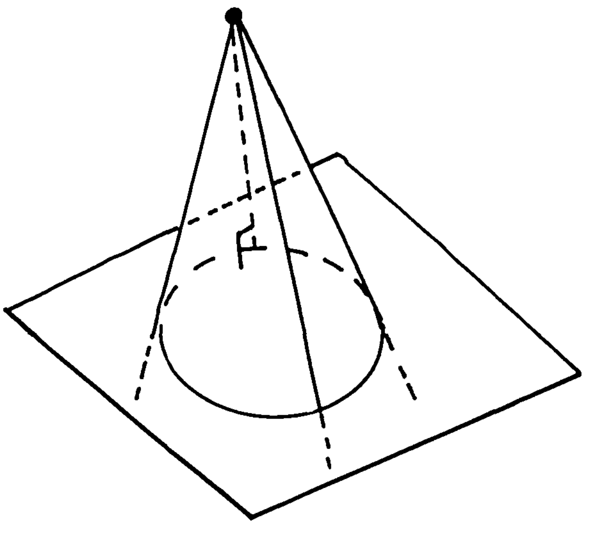

Figure 2

Now consider a hyperboloid of one sheet. Such a hyperboloid has an asymptotic cone lying in the interior region bounded by the surface. Krames proves that the general critical set C for 3D-reconstruction in the euclidean setting is a hyperboloid whose asymptotic cone is an orthogonal cone. These hyperboloids are also called *orthogonal.*

Let us introduce a coordinate system in 3-space so that the coordinate axes correspond to the principal axes of C. Then C has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

with $a > b$. Schröter shows that the orthogonality of C is equivalent to

$$\frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2} = 0.$$

There is a restriction as to where the centers Z_1 and Z_2 of projection can lie on C. Two lines l_1, l_2 on C are said to be *adjoint* if, when we rotate C about the x-axis (the line $y = 0 = z$) by 180°, l_1 and l_2 are interchanged. Krames proves that we must have $Z_1 \in l_1$ and $Z_2 \in l_2$ where l_1 and l_2 are adjoint.

C can be characterized by the relation of its intersection with the plane at infinity ι. Let $s = C \cap \iota$. Then s and the absolute conic Ω intersect in four distinct points. For two points of intersection A, B the *pole* of the line AB with respect to Ω, i.e., the point of intersection of the tangents at A and B, lies on s.

Krames also made a series of models of the critical set from plaster and rods and used these models for experiments.[23] The theoretical observations of W. Wunderlich regarding the critical surface deserve particlar mention.[52]

10. The critical region

If we assume that the coordinates of the points of an image and the results of camera calibration are simply estimates with a certain known tolerance, the critical surface described in the last section must be expanded to become a critical region in space. This region is mentioned by Wunderlich.[52] It was analyzed by Krames in detail starting in 1948, who also compared his theoretical findings with experimental data.[26, 24, 25]

In general, the critical region is bounded by two orthogonal hyperboloids as indicated in Fig. 3 below.

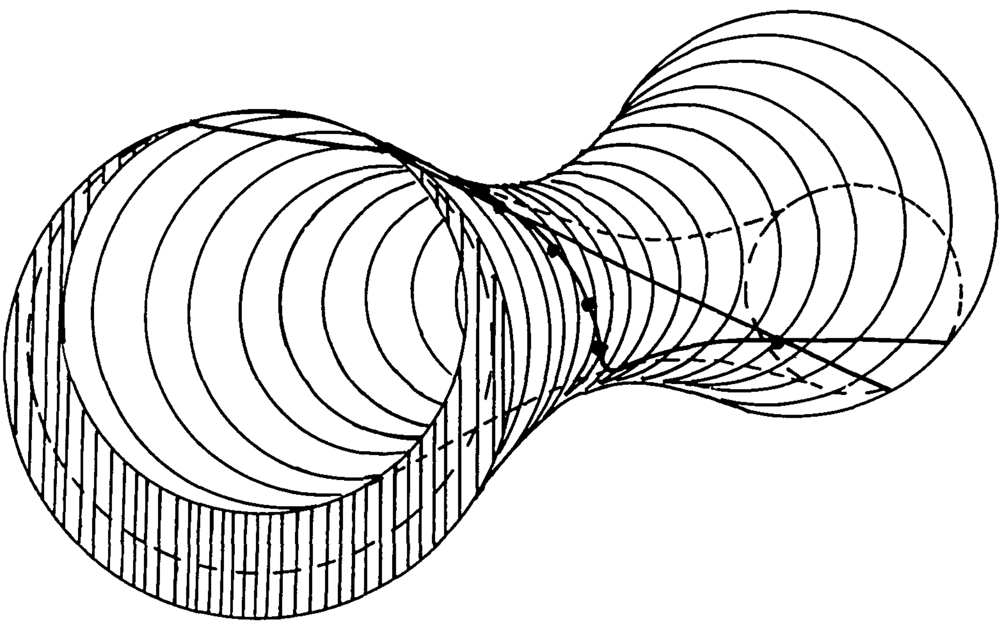

Figure 3

Actually the region between the two hyperboloids divides up into four parts, bounded by the curve of intersection of the two surfaces.

This curve of intersection is of particular interest. Krames proves that it breaks up into a cubic space curve c and a chord f of c. The points indicated in Fig. 3 are the points of intersection of c and f, the two centers of projection

Z_1 and Z_2, and the center of symmetry of c.

The curve c is the euclidean form of the horopter discussed in Section 2. The euclidean form of the horopter intersects the plane at infinity in one real point P and two nonreal points which lie on the Ω. Moreover, the tangents to Ω at these points intersect in P.

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