

# CONTINUOUS REPRESENTATION OF UNEVENLY SAMPLED SIGNALS — AN APPLICATION TO THE ANALYSIS OF HEART RATE VARIABILITY —

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## ABSTRACT

*Various methods for the continuous representation of unevenly sampled physiological signals, especially of heart rate and heart rate variability have been published over the past two decades. Due to the mathematical difficulties of analyzing unevenly spaced event series or unevenly sampled signals, all of the previously described techniques are impeded by some severe limitations which result from either representing the signal based on an assumed and often insufficient physiological model of signal generation, or from simplifications in the selected signal processing tools. Though the described methods have their merits and allow sufficient description of heart rate in some special applications, there has been no systematic investigation into a continuous representation of unevenly sampled physiological signals which is free from any limitations or simplifying assumptions and which would allow an objective comparison of the known methods and the determination of the respective errors in order to decide which technique is the best suited for a specific application. In this paper a continuous representation of the point event series describing the temporal sequence of heart beats is presented which does not impose any limiting assumptions on the data analysis. The results obtained were validated through simulations in both time and frequency domains, and compared with the established techniques to allow the determination of errors at the specific times where the heart beats occur, as well as within the intervals between these sample points.*

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## INTRODUCTION

Heart rate (HR) is considered to be one of the most basic and important parameters in medical diagnostics. The generally accepted basis for the clinical application of the HR measurements is the fact that heart rate variability (HRV) reflects the activity of the cardiac control system [10]. Spontaneous variability is related to three major physiological factors: blood-pressure control causing oscillatory fluctuations, thermal regulation appearing as variable frequency oscillations, and respiration resulting in the well known respiratory sinus arrhythmia (RSA). The effects are mainly due to the neural drive of the sinus node by both the sympathetic and the parasympathetic nervous system.

It is a simple task to determine HR either as a mean value by counting the number of heart beats in a given time interval, or as an instantaneous value by measuring and inverting the interbeat intervals, which permits calculation of some statistical parameters such as the various distributions of HR and HRV. Difficulties arise as soon as HR or HRV are to be considered as signals in order to permit the application of various signal processing tools, especially for the purpose of relating them to other physiological measurements such as respiration [1, 6, 13] or blood pressure [4].

Heart beats are generally represented as point events, individual beats being characterized only by their time of occurrence or, equivalently, the interval between successive beats [5]. There are several methodological problems which impede the analysis of point event series. Some statistical methods were developed for the analysis of stochastic series of events but these are not always suited for physiological data [3]. There have been many efforts to interpret HR as a discrete

signal and to find a representation that would accordingly allow the application of the multitude of standard signal processing tools available. Since the intervals between heart beats are irregular, the sequence of HR values as a function of time does not correspond to the desired representation as an evenly sampled signal. In order to get to such a representation, values need to be defined for the HR between the actual heart beats, i.e. some kind of interpolation needs to be performed. Some approaches avoid the problems related to the question whether such interpolation is physiologically valid, by replacing HR with a continuous input function to the heart that would cause the same heart rates as the measured values, based on the IPFM model [10]. In most cases, however, such questions have been of no concern, and the HR values are considered as unevenly spaced samples of a continuous signal. The techniques for interpolation described in the literature are all impeded by some severe limitations, which result from either continuously representing the signal based on an assumed and often insufficient physiological model of signal generation, or from severe simplifications in the selected signal processing tools. The errors caused by interpolation of the HR curve have not been systematically investigated for the various techniques currently being used for continuous representation of HR.

A new technique for the continuous representation of unevenly sampled signals has been developed and applied to the cardiac point event series achieving an exact continuous reconstruction of HR in the sense that the complex Fourier spectra of the measured, unevenly sampled HR and its continuous or evenly sampled representation are equal. This means, that any signal processing applied to the reconstructed HR curve leads to the same results that would be obtained by processing the original HR values if this were possible. The improved continuous representation of HR can serve as the gold standard to investigate the quality of the previous, computationally less expensive techniques.

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## REVIEW OF EXISTING METHODS FOR CONTINUOUS HR REPRESENTATION

The cardiac event series can be represented as a series of Dirac impulses  $\delta(t)$ , located at times  $\{t_i\}$  with  $i=0,1,2,\dots,n$ , at which the peaks of the R waves in the ECG are being observed [12]:

$$p(t) = \sum_i \delta(t - t_i) \quad (1)$$

The HR series is a beat by beat representation of the HR measurements as a function of the sequential number of heart beats. It can be expressed as follows:

$$HR_i = \frac{T}{t_i - t_{i-1}} \quad (2)$$

where  $T$  is a constant and  $t_i$  is the  $i^{\text{th}}$  cardiac event [4].

HR measurements may also be represented as a function of time. The instantaneous heart rate (i.h.r.), based on the representation of the cardiac event series as shown in equation (1), and assigning weights to the Dirac impulses corresponding to the calculated HR values, is defined as [11]:

$$f_{i.h.r.}(t) = \sum_i \frac{T}{t_i - t_{i-1}} \delta(t - t_i) \quad (3)$$

Another often used method to obtain a continuous representation of HR is to perform a constant interpolation. For each time interval between two measurements of HR, the continuous value of HR is defined as the value of HR determined at the beginning or the end of the respective interval. The two possibilities, forward and backward interpolation, result in different representations of HR. For the delayed heart rate (DHR), the measured HR value is assigned to the next interval by forward interpolation:

$$f_{DRH}(t) = \sum_i \frac{T}{t_i - t_{i-1}} [u(t-t_i) - u(t-t_{i+1})] \quad (4)$$

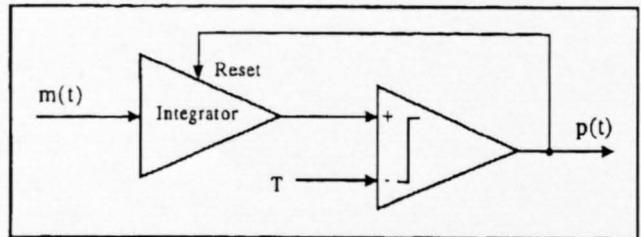
where  $u(t)$  is the unit step function. Instantaneous heart rate (IHR) is the continuous representation of HR

obtained by backward interpolation:

$$f_{IHR}(t) = \sum_i \frac{T}{t_i - t_{i-1}} [u(t-t_{i-1}) - u(t-t_i)] \quad (5)$$

The IHR signal has been considered as an appropriate and practical representation of HR for the analysis of HRV as it is consistent with the accepted physiological models and simple to realize [5].

DHR and IHR are step-wise constant signals with possible discontinuities at each heart beat. These discontinuities in the representations are very controversial as they are not consistent with the general understand-



**Figure 1.** Diagram of the IPFM model. The continuous input function  $m(t)$  is converted into a pulse series  $p(t)$ .  $T$  is the trigger level for the comparator.

ing of physiological systems, nor are they convenient for the theory of signal sampling and digital signal processing techniques.

The low-pass filtered event series (LPFES) results from a completely different approach to arrive at a continuous representation of HR. The LPFES is obtained by simply low-pass filtering the cardiac event series, and is considered consistent with most widely accepted cardiac pacemaker model, the integral pulse frequency modulator (IPFM). Low-pass filtering is implemented as a convolution of the event series with the impulse response of an ideal low-pass filter having a corner frequency  $f_c = 0.5$  Hz [2,5,7,11]:

$$f_{LPFES}(t) = \sum_i \frac{\sin(2\pi f_c(t - t_i))}{\pi(t - t_i)} \quad (6)$$

In contrast to the methods described so far, the derivative cubic spline interpolation (DCSI) achieves a continuous representation of HRV by cubic spline interpolation of an evenly sampled input function to the

IPFM model [10]. This substantial difference in the approach needs to be regarded when comparing the results of DCSI with most other methods of interpolation which attempt a continuous representation of the output of the IPFM.

Two successive heart beats occurring at  $t_{i-1}$  and  $t_i$  can be related to the IPFM input function  $m(t)$  as follows (see figure 1):

$$\int_{t_{i-1}}^{t_i} m(t)dt = T \quad (7)$$

or, assuming  $t_0 = 0$  as the time origin:

$$\int_0^{t_i} m(t)dt = iT \quad (8)$$

$M(t)$ , the continuous integral of the input function  $m(t)$ , is defined as:

$$M(t) = \int_0^t m(t)dt \quad (9)$$

The necessary sample points for the reconstruction of  $M(t)$  are provided by equation (8):

$$M(t_i) = \int_0^{t_i} m(t)dt = iT \quad (10)$$

From these defined points,  $M(t)$  can be reconstructed by interpolations and  $m(t)$  can be obtained by calculating the derivative of  $M(t)$ . Since any curve passing through all the defined points satisfies equations (7) and (8), the reconstruction of  $m(t)$  is not unique. If  $M(t)$  is obtained by piece-wise linear interpolations, the derivative  $m(t)$  turns out to be the IHR signal. Although  $M(t)$  is continuous at the joints, its derivative  $m(t)$  is not. Higher order interpolation solves this problem. As the method of choice, cubic spline piece-wise polynomial interpolations are chosen providing continuity at joints up to the second derivative [10].

## FOURIER INTEGRAL RECONSTRUCTION

An exact method for the continuous representation of heart rate should not be limited by any unnecessary mathematical simplifications. It should also be free of restrictions on the basis of incomplete physiological

models. The only assumptions about the HR curve made here are that the signal is band limited, and that the representation of cardiac activity by HR as a function of time is physiologically sound. The discussion whether the transition from a discrete series of HR values to a continuous representation of HR is physiologically consistent is irrelevant if the new representation only serves the purpose of allowing digital signal processing techniques to be applied to HR, and if the results of any such operations are the same as they would be if applied to the original data.

In a first approach, the application of Fourier transforms for the interpolation of the HR curve is tested. As mentioned before, it is assumed that the HR signal is band limited and that the sample points provide sufficient density to allow error free reconstruction between these values. The pair of Fourier transforms, forward and inverse, are:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad (11)$$

and:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega \quad (12)$$

The intention now is to calculate the classical Fourier spectrum of HR as a continuous function of frequency. For this purpose, consider the operation for the example of the instantaneous heart rate,  $f_{i,h,r}(t)$ . According to equations (3) and (11), the transform is:

$$F(j\omega) = \int_{-\infty}^{\infty} f_{i,h,r}(t)e^{-j\omega t}dt \quad (13)$$

or

$$F(j\omega) = \int_{-\infty}^{\infty} \sum_i \frac{T}{t_i - t_{i-1}} \delta(t-t_i) e^{-j\omega t}dt \quad (14)$$

Changing the order of integration and summation we obtain:

$$F(j\omega) = \sum_i \int_{-\infty}^{\infty} \frac{T}{t_i - t_{i-1}} \delta(t-t_i) e^{-j\omega t}dt \quad (15)$$

or after integration:

$$F(j\omega) = \sum_i \frac{T}{t_i - t_{i-1}} e^{-j\omega t_i} \quad (16)$$

Using the substitution:

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t) \quad (17)$$

we obtain:

$$F(j\omega) = \sum_i \frac{T\cos(\omega t_i)}{t_i - t_{i-1}} - j\sum_i \frac{T\sin(\omega t_i)}{t_i - t_{i-1}} \quad (18)$$

This simple calculation allows us to obtain the Fourier spectrum of the instantaneous heart rate signal as a continuous function of frequency. The next step is to take the inverse transform according to equation (12), and as the result we will get the continuous approximation of the discrete HR measurements.

An interesting question is, how the Fourier transform method will be affected if the instantaneous HR signal  $f_{i,hr}(t)$  is replaced by the widely used step-wise constant instantaneous HR signal  $f_{iHR}(t)$ . We will use the same calculations as before, only assuming now that each particular value of HR is kept constant until the next heart beat occurs. With equation (5), the spectrum is given by:

$$F(j\omega) = \int_{-\infty}^{\infty} \sum_i \frac{T}{t_i - t_{i-1}} [u(t-t_{i-1}) - u(t-t_i)] e^{-j\omega t} dt \quad (19)$$

Changing the order of integration and summation we obtain:

$$F(j\omega) = \sum_i \int_{-\infty}^{\infty} \frac{T}{t_i - t_{i-1}} [u(t-t_{i-1}) - u(t-t_i)] e^{-j\omega t} dt \quad (20)$$

Using the substitution in equation(17), we obtain:

$$F(j\omega) = \sum_i \int_{t_{i-1}}^{t_i} \frac{T}{t_i - t_{i-1}} (\cos(\omega t) - j\sin(\omega t)) dt \quad (21)$$

The last equation can be split into two parts, resulting in:

$$F(j\omega) = \sum_i \int_{t_{i-1}}^{t_i} \frac{T\cos(\omega t) dt}{t_i - t_{i-1}} - j\sum_i \int_{t_{i-1}}^{t_i} \frac{T\sin(\omega t) dt}{t_i - t_{i-1}} \quad (22)$$

or:

$$F(j\omega) = \sum_i \frac{T[\sin(\omega t_i) - \sin(\omega t_{i-1})]}{\omega(t_i - t_{i-1})} + j\sum_i \frac{T[\cos(\omega t_i) - \cos(\omega t_{i-1})]}{\omega(t_i - t_{i-1})} \quad (23)$$

Interestingly, the spectrum contains the term  $1/\omega$ . This means that the transform behaves as a low pass filter, attenuating the high frequency components that

were to be expected as a consequence of the discontinuities at the single steps of the IHR curve. Inverse Fourier transformation results in a smooth, continuous representation of the discrete HR measurements.

## CONTINUOUS REPRESENTATION OF HR BASED ON SPECTRAL DECONVOLUTION

In the previous section, the effects of signal sampling have been disregarded. As long as we are dealing with evenly sampled signals, we do not need to care about these effects, except, of course, for windowing problems and the sampling theorem's request for high enough sampling rate since the spectrum of the sampling function, the comb function is a pulse train in both time and frequency domain. The situation changes completely, if an unevenly spaced sampling function, similar to the window function, causes distortions of the Fourier spectrum which result from its convolution with the signal spectrum.

If we consider the measured values of HR as unevenly spaced samples of a continuous HR signal, these samples are obtained by multiplication of the continuous HR signal with the train of Dirac impulses  $\delta(t-t_i)$  which are located at the times where the heart beats occur (see figure 2). The unevenly sampled values can thus be expressed mathematically as:

$$p(t) = HR(t)\delta(t - t_i) \quad (24)$$

The Fourier spectrum of  $p(t)$  is determined as:

$$P(j\omega) = \int_{-\infty}^{\infty} p(t)e^{-j\omega t} dt \quad (25)$$

Substituting  $p(t)$  as expressed by equation (24), we can rewrite the Fourier spectrum as:

$$P(j\omega) = \int_{-\infty}^{\infty} HR(t)\delta(t - t_i)e^{-j\omega t} dt \quad (26)$$

At this point we use the well known properties of the Dirac impulse  $\delta(t)$  to obtain:

$$P(j\omega) = \sum_i HR(t_i)e^{-j\omega t_i} \tag{27}$$

Using the substitution from equation (17), the final expression for the complex Fourier spectrum is:

$$P(j\omega) = \sum_i HR(t_i)(\cos(\omega t_i) - j\sin(\omega t_i)) \tag{28}$$

We can also easily calculate the spectrum of the train of  $\delta(t)$  functions:

$$D(j\omega) = \int_{-\infty}^{\infty} \delta(t-t_i)e^{-j\omega t} dt = \sum_i (\cos(\omega t_i) - j\sin(\omega t_i)) \tag{29}$$

At this stage we need to express our unknown continuous HR function in the frequency domain as:

$$HR(j\omega) = \int_{-\infty}^{\infty} HR(t)e^{-j\omega t} dt \tag{30}$$

We further use the correspondence of multiplication in the time domain and convolution in frequency domain:

$$f_1(t) \cdot f_2(t) \Leftrightarrow F_1(j\omega) * F_2(j\omega) \tag{31}$$

Thus, we can express the spectrum of the discrete

HR measurement as the convolution of the spectra of the unknown continuous HR function and the train of  $\delta(t)$  impulses:

$$P(j\omega) = HR(j\omega) * D(j\omega) \tag{32}$$

The last equation can be expressed in terms of convolution integral as:

$$P(j\omega) = \int_{-\infty}^{\infty} HR(u)D(j\omega-u)du \tag{33}$$

where  $u$  is an auxiliary variable.

Solving the integral Volterra equation of the first kind (equation 33), and taking the inverse Fourier transform of the calculated  $HR(j\omega)$ , we obtain a continuous function representing our discrete HR measurements [9].

There are several methods for solving integral Volterra equations of the first kind. Almost all of them are burdensome and time consuming. In this study, a simplified solution which gives HR samples with an arbitrary density and accuracy will be presented below. For this purpose we can rewrite equation (33) into the following form:

$$P(\omega) = \sum_{m=0}^N HR(m)D(\omega-m) \tag{34}$$

Instead of the ideally infinite number of sample points, the real situation is always limited to a finite time or finite number  $N+1$  of HR values. The computational procedure is as follows:

$$P(\omega_1) = \sum_{m=0}^N HR(m)D(\omega_1-m) = HR(0)D(\omega_1) \tag{35}$$

Thus we can calculate  $HR(0)$  as:

$$HR(0) = \frac{P(\omega_1)}{D(\omega_1)} \tag{36}$$

In the next step we calculate:

$$\begin{aligned} P(\omega_2) &= \sum_{m=0}^N HR(m)D(\omega_2-m) \\ &= HR(0)D(\omega_2) + HR(1)D(\omega_1) \end{aligned} \tag{37}$$

and obtain  $HR(1)$  as:

$$HR(1) = \frac{P(\omega_2) - HR(0)D(\omega_2)}{D(\omega_1)} \tag{38}$$

The general formula for all values of the HR

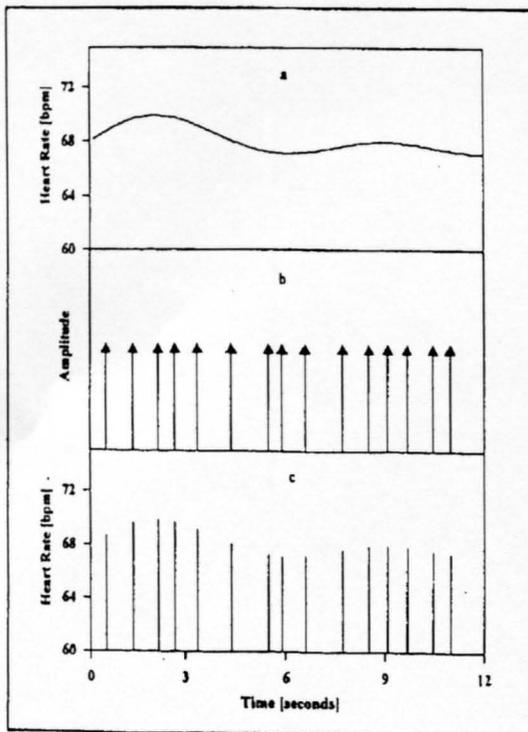


Figure 2. Generation of unevenly sampled heart rate signals. The signal representing a continuous heart rate curve (a) is sampled by multiplication with a series of Dirac impulses (b) which are located at the times of the individual heart beats. The product (c) is the measured, discrete HR.

signal is:

$$P(\omega_N) = \sum_{m=0}^N HR(m)D(\omega_N-m) \quad (39)$$

$$HR(N) = \frac{P(\omega_N) - \sum_{m=0}^{N-1} HR(m)D(\omega_N-m)}{D(\omega_1)} \quad (40)$$

It is important to remember that all the mathematical operations concern complex numbers, so care needs to be taken that the suitable software is written for all of these operations.

## RESULTS

The various methods for continuous representation of heart rate were tested on artificial signals as well as on real patient data. The limitation for the tests on real HR data is, that no error for the HR representation can be calculated for the values between measurements since these are unknown.

The first test was performed on a simple sine wave representing the interbeat intervals:

$$f_{IBI}(t) = 70 \cdot \sin(0.1\pi t) + 700 \quad (41)$$

Figure 3 shows the original values for the HR as an unevenly sampled representation of the artificial test signal, and the results of the various methods for reconstruction. The abbreviations are: HR for the original discrete HR signal, IHR for the instantaneous heart rate as defined in equation (5), LPFES for the low-pass filtered event series, DCSI for the derivative cubic spline interpolation, EFOU for the Fourier transform technique, and DECON for the deconvolution technique.

It can clearly be seen that only two of the methods for continuous representation of HR actually retain the original HR values: IHR and DECON. The mean and absolute errors for the match of test signal and continuous representation at the times of the original HR sample values is shown in figure 4, both for the time and the frequency domain. Figure 5 shows the

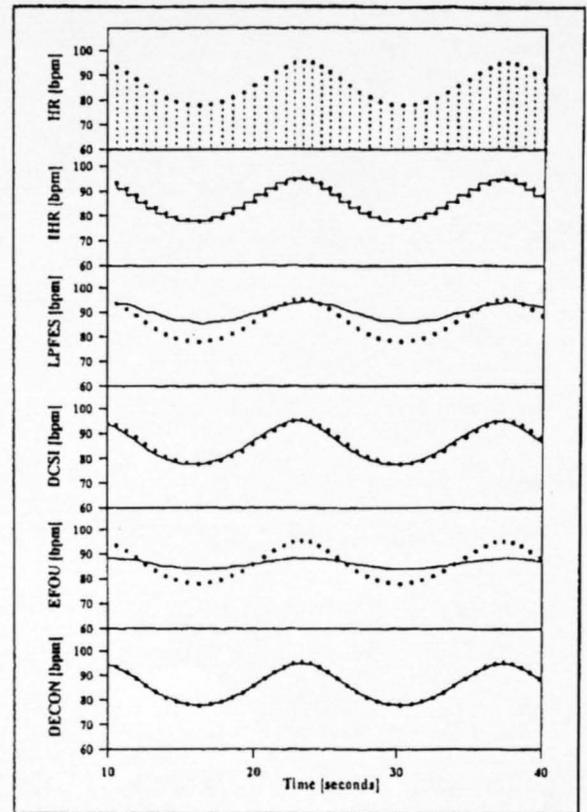


Figure 3. Five continuous representations of the discrete heart rate shown in the uppermost row. Closest representation of the artificial test signal (1/sin) is achieved by the spectral deconvolution technique.

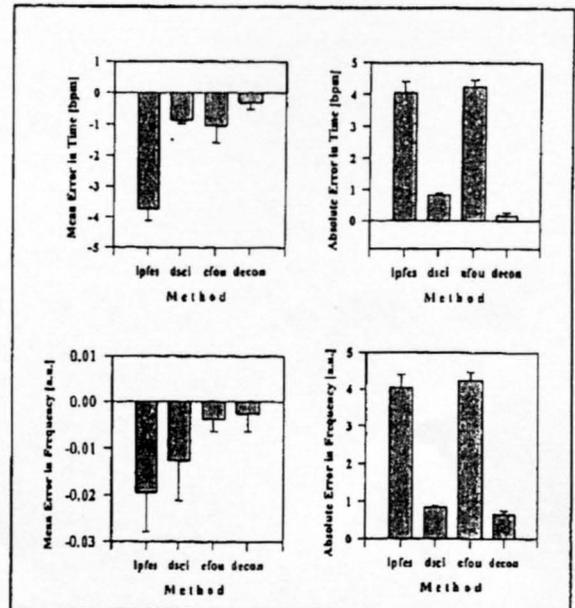


Figure 4. Mean and absolute errors in time and frequency domains for the continuous representations of HR shown in figure 3. Calculation is limited to the original HR samples.

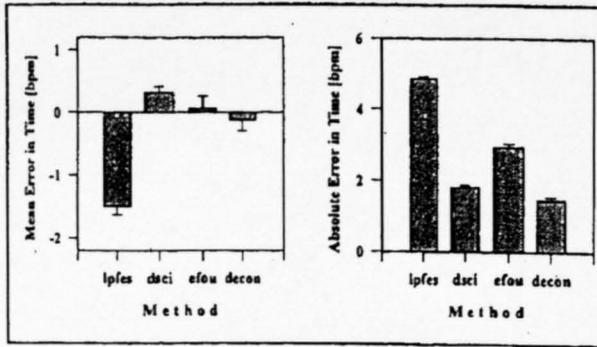


Figure 6. Mean and absolute errors for the continuous HR representations, calculated for the interpolated data points between the original sample points.

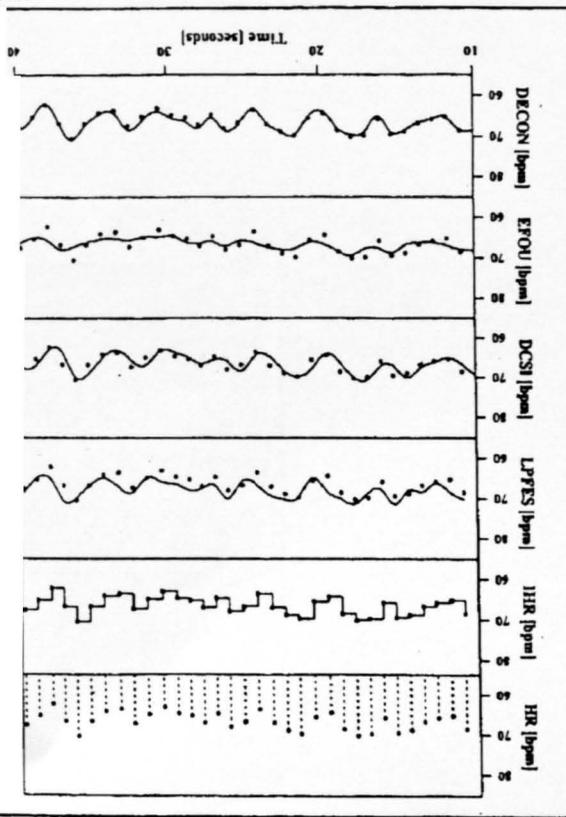


Figure 5. Continuous representation of a real HR curve.

errors between the original sample points for the time domain. It should be noted that this test does some injustice to the DCSI method which is actually supposed to continuously represent the input signal to the IPFM, not the output signal, the heart rate. Similar test results were obtained for more sophisticated artificial test signals simulating the different frequency bands contained in real HR signals. Overall, the spectral deconvolution technique showed the closest signal

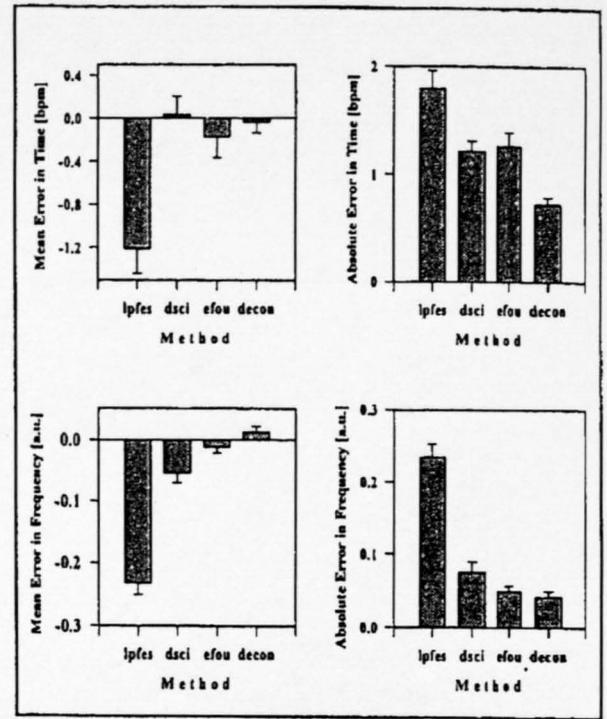


Figure 7. Mean and absolute errors of continuous representations in both time and frequency domains for real heart rate data.

reconstruction and the smallest errors.

The test results obtained from a real HR curve are shown in figures 6 and 7. Again, the newly developed deconvolution technique for continuous representation of unevenly sampled signals shows the best results.

By improvements of the algorithm used for the deconvolution method in terms of higher computational precision, the errors obtained with this technique can be further reduced. But even in its current state, DECON is clearly superior to all the other techniques and can serve to judge the quality of all simplified techniques for the continuous representation of heart rate.

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