SPECKLING IN DIFFRACTION PATTERNS AND OPTICAL IMAGES FORMED WITH THE LASER

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Abstract

Elementary physical considerations are used to show that the waves scattered from an optically rough surface have not only random phases but also random real amplitudes. The same simple physical model serves to establish the main features of speckling in diffraction patterns, namely (i) that the smallest size of detail in any azimuth is determined by the formula $0.5\lambda/\Theta$, where Θ is the angular diameter of the object in that azimuth, (ii) that the scale of the speckling is predominantly of this order of size, and (iii) that speckling is expected to be of high contrast. This last conclusion is easily derived from the randomness of the real amplitudes of the scattered waves. The result (ii) is obtained by finding the autocorrelation function of the diffracted intensity distribution, following Goldfischer (1965).

In the formation of the image of a rough object using coherent light, the entrance pupil is illuminated by a speckled diffraction pattern. This random illumination then appears in the exit pupil, so that the speckling in the image is determined by considering the exit pupil to act as a rough object. The order of size of speckling in the image is thus determined by the formula $0.5\theta_o/\lambda$, where θ_o is now the total convergence angle of the image-forming pencil.

(1) Physical Considerations : The Envelope of the Diffraction Pattern and the Minimum Size of Speckle

We shall consider first the factors which determine the form of the envelope of the intensity in the diffraction pattern. Figure (1) shows schematically a rough object illuminated by a coherent wave converging to a focus in the neighbourhood of the point E. For simplicity in the diagram, transmitted illumination is shown, but the discussion applies equally well to the case of a reflected wave. The roughness is assumed to be localised on the exit surface of the object. A simple physical picture of what occurs may be obtained by considering the irregularities to comprise a random set of positive and negative lenses. The pencils of rays falling on two such lens elements are indicated in the diagram. The rays marked with double arrows pass undeviated to the point E. The convex lens element at O_1 focuses rays at P_1, from which they will diverge as a relatively narrow pencil, giving a wave W_1. The concave lens element at O_2 produces a pencil of rays coming from the virtual image P_2, which, because of the deeper curvature, diverges as a wider-angle pencil, giving the wave W_2. In the diffraction plane the total disturbance will be the resultant of the mutual interference of the coherent waves produced by all such elements of the object. These waves will have random mutual phases because of the random variation of the optical thickness of the object.

At a point such as Q_1 there will be light from all elements of the object, whereas only those elements giving wide-angle scattering will send light to a point such as Q_2 which is further from the centre, E, of the pattern. Since the maximum intensity that can be produced increases with the number of interfering waves, the envelope of the intensity in the diffraction pattern will be expected to have a greater value at Q_1 than at Q_2. Thus, if the sizes of scattering elements have a gaussian type distribution about a mean, the envelope of the diffraction pattern will be expected to have a maximum at the centre, E, and to decrease continuously to the edge. The form of this transition curve will depend on the standard deviations of the sizes and curvatures of the elements. For example, if all elements were of the same size and curvature all the scattered waves would have the same angular spread, and these would be a large constant region for the envelope of the intensity, with a rapid decrease to zero at the edge. The greater the size range of the surface elements, the greater will be the angular range of scattering produced. In this case there will be a smaller constant region, and a more gradual transition to the edge of the pattern.

The smallest size of detail which can occur in the diffraction pattern is determined by the angular size of the diffracting object as seen from the diffraction plane. The diffraction pattern may be regarded as produced by the coherent superposition of the interference fringes of the waves falling on the plane, E,taken in pairs. It can be seen at once that the finest structure in the pattern will correspond to that produced by the waves with the greatest mutual inclination. These would be the waves produced by the elements near to those marked 0^* in the schematic diagram of figure (1). Thus, if the diameter of the object in any azimuth subtends an angle θ at E, the smallest scale of detail in the random pattern will have a half-width of the order of size $0.5\lambda/\theta$ in this azimuth.

The above simple considerations account for the general form of envelope and smallest detail size in the speckling that is found in practice. Thus, if the exposed area of the object is reduced by a field stop, the envelope of the diffraction pattern is little affected, but the scale of the speckling is increased in inverse proportion to the angular subtense θ of the object. This is to be expected provided only that the area of the object employed is large enough to include a representative sample of its random structure.

The above physical considerations serve to establish reasons for some of the general features of the speckled diffraction patterns observed from rough objects. They do not of themselves, however, give information about the expected predominant size and contrast of the speckles.

The case considered here is that of the Fraunhofer diffraction pattern, since the unscattered wave is assumed to focus at, or near to, E. In other cases, the scattered waves, such as W_1 and W_2, will all be laterally displaced relative to each other, and merely alter the form of envelope of the diffraction pattern.

We shall consider below the size-range to be expected in the speckled diffraction pattern and also the contrast. This analysis is similar to that employed by Goldfischer (1965), but the arguments will continue to be based on the physical model illustrated in figure (1). It will be necessary to consider first the factors determining the amplitudes of the scattered waves.

Referring to figure (2), it will be seen that the angular aperture and the amplitude of the wave scattered by any lens-element will be determined by the area and curvature of the element. For two facets of the same area, the one of greater curvature will scatter a wave of greater angular aperture, and consequently the energy will be distributed through a larger solid angle. For this reason the scattered waves will not be of the same amplitude, even when the incident illumination is uniform. Goldfischer, by contrast, assumes each scattered to radiate a wave of real amplitude equal of the square root of the intensity of the illumination falling on it ; for uniform illumination, the scattered waves would then be wrongly assumed to be all of the same amplitude. That the scattered waves always have random real amplitudes is an important factor in the arguments developed later.

To see the order of magnitude of this effect, consider a circular facet of radius ρ whose surface has radius of curvature r. For light transmitted from glass into air, the focal length of the facet is given by $F = r/(n - 1)$, where n is the refractive index of the glass. The angular radius of the scattered wave is thus.

$$\alpha = \frac{\rho}{F} = \frac{(n-1)\rho}{r} \qquad (1)$$

and the surface area of the scattered wave at E is given by

$$S = \pi(D\alpha)^2 = \pi D^2(n-1)^2\,(\rho/r)^2 \qquad (2)$$

where D is the distance from the object to E. The total energy falling on the facet is proportional to ρ^2, and this energy is distributed over the area S. The energy per unit area of the wave is thus proportional to ρ^2/S, that is proportional to r^2. It follows that the amplitude of the scattered wave is inversely proportional to the curvature, $c = 1/r$, of the facet from which it is scattered. Now the local curvature of a rough surface will be expected to vary appreciably from point to point, and hence it is not admissible to assume the scattered waves to have all the same amplitude. Both the phases and the amplitudes of the scattered waves, superimposing at the diffraction plane will vary randomly about the mean values.

(2) The Intensity in the Diffraction Pattern ; The size Range and Contrast of Speckling.

In figure (3), let (ξ,η) be the coordinates of a point P on the object, and let D be the distance from the object to the diffraction plane. A spherical wave originating from the facet at P, and passing through E will produce at Q, whose coordinates are (X,Y), a disturbance whose phase is in advance of that produced at E by an amount $k(QQ_o)$. Let $PQ_o = R_o$ and $PQ = R$, where both R and R_o are considered to be essentially positive. The lenght $(QQ_o) = (R_o - R)$ is obtained as follows. First, note that

$$(PQ)^2 = R^2 = (X-\xi)^2 + (Y-\eta)^2 + D^2$$

$$(PQ_o)^2 = R_o^2 = \xi^2 + \eta^2 + D^2$$

subtraction of which gives

$$R_o^2 - R^2 = 2(X\xi + Y\eta) - (X^2 + Y^2)$$

Thus, writing $R_o + R = 2R_o - (R_o - R) \simeq 2R_o$, since $|R_o - R| \ll 2R_o$, this last expression becomes

$$R_o - R = \frac{X\xi + Y\eta}{R_o} - \frac{X^2 + Y^2}{2R_o} \tag{3}$$

and the phase difference k (QQ_o) is

$$k(QQ_o) = 2\pi \left\{\frac{X\xi + Y\eta}{\lambda R_o}\right\} - \frac{k(X^2 + Y^2)}{2R_o} \tag{4}$$

The method used here for deriving (3) is much to be preferred to the more conventional way involving the series expansion of square roots, where the validity of the approximation is less easy to consider. It should also be noted that only the first term in (4) involves sigificantly the coordinates (ξ,η) of P on the diffracting object.

Let the phase over the wave EQ_o be $\phi(\xi,\eta)$. The phase of the disturbance produced at Q by the wave from P is then, using (4) given by

$$2\pi\left(\frac{X\xi + Y\eta}{\lambda R_o}\right) + \phi(\xi,\eta) - \varepsilon(X,Y) \tag{5}$$

where $\varepsilon(X,Y)$ is written for the second term of (4). The roughness of the diffracting surface shows itself in randomness of the phases $\phi(\xi,\eta)$.

The real amplitude produced at Q by the wave from P will be proportional to the square root of the intensity, B (ξ,η); of the wave scattered at P. This intensity will, in turn, be proportional to both the intensity of the incident intensity at P and, as has been seen above, to the square of the curvature of the elementary facet at P. Because of the randomness of the curvatures of the different facets, the values of $B(\xi,\eta)$ will also be random.

The complex amplitude produced at Q by the wave from P may now be written as

$$\exp\{-i\varepsilon(X,Y)\}\sqrt{B(\xi,\eta)}\ \exp\left\{i2\pi\left(\frac{X\xi + Y\eta}{\lambda R_o}\right) + \phi(\xi,\eta)\right\}$$

the phase being given by (5). If the facet at P is denoted by the subscript n, the total amplitude at Q will be given by

$$U_Q = \exp\{-i\varepsilon(X,Y)\}\sum_{n=1}^{N}\sqrt{B(\xi_n,\eta_n)}\ \exp\left\{i2\pi\left(\frac{X\xi_n + Y\eta_n}{\lambda R_o}\right) + \phi(\xi_n,\eta_n)\right\} \tag{6}$$

N being the total number of facets. The intensity produced at Q is given by the squared modulus of (5), that is, with an obvious notation, by

$$I(X,Y) = \sum_{n}\sum_{m}\sqrt{B_n B_m}\ \exp\{i(\phi_n - \phi_m)\}\exp\left\{i2\pi\left(\frac{X(\xi_n - \xi_m) + Y(\eta_n - \eta_m)}{\lambda R_o}\right)\right\} \tag{7}$$

This double sum may be separated into those parts locally independent of (X,Y), and those terms which do depend on the position (X,Y) of Q. This gives, grouping the terms m = n,

$$I(X,Y) = \sum_n B_n + \sum_n \sum_{\substack{m \\ m\neq n}} \sqrt{B_n B_m} \; \exp\{i(\phi_n-\phi_m)\} \; \exp\{i2\pi \left[\frac{X(\xi_n-\xi_m)+Y(\eta_n-\eta_m)}{\lambda R_o}\right]\} \quad (8)$$

The sum $\sum_n B_n$ will only be locally independent of (X,Y), because for widely differing positions for Q the number of waves to be summed will be different. Indeed, at sufficiently large distances from the centre of the pattern, E, nearly all of the intensities B_n will be zero, since only a few facets will scatter light trough very large angles.

In any local region of the pattern, the term $\sum_n B_n$ will be constant, but even small changes in (X,Y) will affect the terms in the double summation. The first sum in (8) thus represents the mean intensity over a small region surrounding the point Q, and the double sum accounts for the interference effects responsible for the speckling. A typical term (n,m) of the double sum represents the interference pattern produced by the waves from the two (different) facets, n and m, of the object. It should be noted that the terms (m,n) and (n,m) are complex conjugates, and this permits (8) to the written

$$I(X,Y) = \sum_n B_n + 2 \sum_n \sum_{\substack{m \\ n\neq m}} \sqrt{B_n B_m} \; \cos\{(\phi_n-\phi_m)+2\pi \left[\frac{(\xi_n-\xi_m)X+(\eta_n-\eta_m)Y}{\lambda R_o}\right]\} \quad (9)$$

provided the second summation is restricted to avoid each term being counted twice.

To find the distribution of sizes among the speckles the autocorrelation function of I(X,Y) may be found. To simplify the notation, this will be carried out using one variable only. Thus, writing (9) in the form

$$I(X) = \bar{I} + \tilde{I}(X) \quad (10)$$

where $\bar{I}$ and $\tilde{I}(X)$ are the mean and variable parts of I(X), the autocorrelation function of I(X) is defined by the mean value

$$C(x) = \langle I(X)I^*(X+x)\rangle_X \quad (11)$$

Substituting from (10) in (11), this leaves

$$C(x) = \bar{I}^2 + \langle\tilde{I}(X)\tilde{I}^*(X+x)\rangle_X \quad (12)$$

since the local mean value of $\tilde{I}(X)$ will be zero. The presence of structure in I(X) is related only to the second term in (12), namely

$$\tilde{C}(x) = \langle\tilde{I}(X)\tilde{I}^*(X+x)\rangle_X \quad (13)$$

which is the autocorrelation of the variable part of I(X), that is the variation of this function about its mean value.

Substituting $\tilde{I}(X)$ from (7) in (12) gives

$$\tilde{C}(x) = \left\langle \sum_{n}\sum_{\substack{m \\ m\neq n}} \sqrt{B_n B_m}\ \exp\{i(\phi_n-\phi_m)\}\exp\{i2\pi \frac{(\xi_n-\xi_m)X}{\lambda R_o}\}\right.$$

$$\left.\sum_{p}\sum_{\substack{q \\ q\neq p}} \sqrt{B_p B_q}\ \exp\{-i(\phi_p-\phi_q)\}\exp\{-i2\pi\frac{(\xi_p-\xi_q)(X+x)}{\lambda R_o}\}\right\rangle X \qquad (14)$$

In this average with respect to X, it is only in the exponential factors which group to give

$$\exp\{i2\pi \frac{(\xi_n-\xi_m)X-(\xi_p-\xi_q)X}{\lambda\ R_o}\}$$

that X occurs, and each such term will average to zero except when $\xi_n-\xi_m=\xi_p-\xi_q$. Thus, writing $\xi_m=\xi_n+\alpha$ and $\xi_q = \xi_p + \alpha$ and omitting those terms which average to zero, (14) becomes

$$\tilde{C}(X)=\sum_{n}\sum_{p}\sum_{\alpha\neq 0}\sqrt{B(\xi_n)B(\xi_n+\alpha)}\ \sqrt{B(\xi_p)B\ (\xi_p+\alpha)}\ \exp\{i\ [\phi(\xi_n)\ -\phi(\xi_n+\alpha)]\}$$

$$\exp\{-i[\phi(\xi_p)-\ \phi(\xi_p+\alpha)]\}\ \exp\{-i2\pi\frac{\alpha x}{\lambda R_o}\} \qquad (15)$$

In (15) the phase differences $\phi(\xi_n)-\phi\ (\xi_n+\alpha)$ and $\phi(\xi_p)-\phi(\xi_p+\alpha)$, with $\alpha\neq 0$, will be randomly related except for the terms $p = n$. The terms $p \neq n$ will thus sum to zero in (15), leaving

$$\tilde{C}(x) = \sum_{n}\sum_{\alpha}B(\xi_n)B\ (\xi_n+\alpha)\exp\{-i2\pi\ \frac{\alpha x}{\lambda R_o}\} \qquad (16)$$

for the autocorrelation of the variable part of I(X).

If the roughness of the object is of very small scale, the discrete sums in (16) may be replaced by integrals, giving

$$\tilde{C}(x) = \iint_{\xi\alpha} B(\xi)\ B(\xi+\alpha)\ \exp\{-i2\pi\frac{\alpha x}{\lambda R_o}\}\ d\xi d\alpha$$

This may be written

$$\tilde{C}(x) = \int_{\xi} B(\xi)\ \exp\{+i2\pi(\frac{x}{\lambda R_o})\xi\}d\xi\int_{\alpha}B(\xi+\alpha)\ \exp\{-i2\pi(\frac{x}{\lambda R_o})\ (\xi+\alpha)\}\ d\alpha \qquad (17)$$

Definie now the (inverse) Fourier transform of $B(\xi)$ to be

$$b(\sigma) = \int_{\xi} B(\xi) \exp\{-2\pi\sigma\xi\}\, d\xi \tag{18}$$

The first factor in (17) is then $b^*(\frac{x}{\lambda R_o})$, and the second factor is $b(\frac{x}{\lambda R_o})$. Thus, (17) merely reduces to

$$\tilde{C}(x) = |b(\frac{x}{\lambda R_o})|^2 \tag{19}$$

Reverting to two variables, the autocorrelation function of the variable part of $I(X,Y)$ is given by

$$\tilde{C}(x,y) = |b\,(\frac{x}{\lambda R_o}, \frac{y}{\lambda R_o})|^2 \tag{20}$$

where $b(\sigma,\tau)$ is the (inverse) Fourier transform of $B(\xi,\eta)$. Essentially this result was obtained by Goldfischer.

If the object occupies the rectangle inside the lines $X = \pm a$ and $I = \pm b$, and the average local value of $B(\xi,\eta)$ is constant, the form of $\tilde{C}(x,y)$ may be found using

$$b(\sigma,\tau) = \int_{-a}^{+a}\int_{-b}^{+b} \exp\{-i2\pi(\sigma\xi+\tau\eta)\}\, d\xi d\eta$$

$$= \frac{\sin(2\pi a\sigma)}{\pi\sigma} \quad \frac{\sin(2\pi b\tau)}{\pi\tau}$$

so that

$$\tilde{C}(x,y) = \left[\frac{\sin(\frac{2\pi a}{\lambda R_o}x)}{(\frac{\pi}{\lambda R_o})x} \quad \frac{\sin(\frac{2\pi b}{\lambda R_o}y)}{(\frac{\pi}{\lambda R_o})y}\right]^2 \tag{21}$$

From (21), the value $\tilde{C}(x,y)$ is first zero when

$$x = x_o = \frac{0.5\lambda}{\theta_x};\ \theta_x = 2a/R_o \tag{22}$$

and when

$$y = y_o = \frac{0.5\lambda}{\theta_y};\ \theta_y = 2b/R_o \tag{23}$$

Here θ_x and θ_y are the angular substenses of the width and height of the object as seen from the diffraction plane. When the autocorrelation of the variable part of $I(X,Y)$falls to zero for $x = x_o$, it implies that $\bar{I}(X,Y)$ on the average changes appreciably over a distance of the order of x_o. Since, as shown above, the smallest size of detail in any azimuth is given by a formula identical to (22), it follows that the size of speckle tends to be uniform, and predominantly of a size in any azimuth equal to $0.5\lambda/\theta$, where θ is the angular subtense of that azimuth of the object as seen from the diffraction plane.

It remains to consider the expectation value of the contrast in the speckle pattern, an important aspect of the phenomenon not considered in Goldfischer's analysis. Since the autocorrelation $\tilde{C}(x)$ refers only to the varying part of the intensity distribution, it says nothing regarding the ratio of the modulation of this varying intensity to the magnitude of the mean intensity.

Again using only one variable, the mean value of the intensity is given by

$$\bar{I} = \sum_{n=1}^{N} B_n = N\bar{B}_n \qquad (24)$$

where N is the number of scattering elements, and $\bar{B}$ is the mean value of the intensities of the scattered waves on arrival at the diffraction plane. The values of B_n, it has been shown, will be expected to show a random variation. The expected depth of modulation of $\tilde{I}(X)$ will be given by the expected modulus of this term in (7), namely of

$$\tilde{I}(X) = \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m \neq n}}^{N} \sqrt{B_n B_m} \quad \exp\{i(\phi_n - \phi_m)\} \exp\{i\frac{2\pi(\xi_n - \xi_m)X}{\lambda R_o}\} \qquad (25)$$

This comprises the sum of $(N^2 - N)$ 2-dimensional vectors, of modulus $\sqrt{B_n B_m}$ and with phases which are random because of the presence of $(\phi_n - \phi_m)$ with $m \neq n$. This is the random walk problem, so that the expected value of the modulus of $\tilde{I}(X)$ is given by the root mean square

$$\left[\sum_{\substack{n=1 \\ n \neq m}}^{N} \sum_{m=1}^{N} B_n B_m\right]^{\frac{1}{2}} = \{(N^2 - N)\ \overline{B_n B_m}\}^{\frac{1}{2}} \qquad (26)$$

the bar denoting the mean value of the product. Since N is large, $(N^2 - N)^{\frac{1}{2}} \simeq N$ and since the values of B_n are uncorrelated,

$$\overline{B_n B_m} = \overline{B_n}\,\overline{B_m} = (\overline{B_n})^2$$

The expected value of the modulus of (25) is thus, from (26) equal to $N\bar{B}_n$. This is precisely the same as the value of $\bar{I}$, and demonstrates that the contrast, or visibility, of the speckles will have a value near to unity. The variation of intensity is then from maxima to zero minima of intensity.

(3) Speckling in optical Images

Image formation using coherent light is a two-stage diffraction process. In figure (4), O is an object illuminated coherently. This gives a speckle diffraction pattern in the entrance pupil E of the optical system, which is indicated only schematically in the diagram. The speckle pattern at E appears at E', the exit pupil of the system, and this illuminates the image plane at O'. It needs only to be remembered that the size of speckling seen at E is determined by the angle θ_0, and that the exit pupil itself behaves excatly like a rough object with an effective roughness of a lateral size equal to $0.5\lambda/\theta_0$. This is very small compared with the size of the exit pupil. It follows that the image has a speckle pattern whose size is determined solely by the aperture angle, θ, subtented by the exit pupil at the image plane O'.

(4) Summary

It has been shown that the known characteristics of speckling in both diffraction patterns and optical images can be explained on the basis of a simple physical model. This explains not only the occurence of a predominant size but also the high contrast in speckle patterns.

Reference

Goldfischer, L.I. 1966, J.O.S.A., 55, 247.