MARKUS STROPPEL

A CHARACTERIZATION OF QUATERNION PLANES

Dedicated to my teacher, Prof. H. Salzmann, on his 60th birthday

ABSTRACT. The eight-dimensional planes admitting $\mathrm{SL}_2\mathbb{H}$ as a group of automorphisms are determined.

Every open subset of the projective plane over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hamilton quaternions) or $\mathbb{O}$ (Cayley octonions) carries a stable plane in the sense of [4]. There exist stable planes which cannot be embedded into projective planes: Strambach [11] and Löwen [5] have constructed such planes admitting groups of automorphisms isomorphic to $\mathrm{SL}_2\mathbb{R}$ and $\mathrm{SL}_2\mathbb{C}$ respectively. For Strambach's plane, there is a generalization beyond the realm of stable planes [7]. Here we show that there is no quaternion analogue of these examples.

THEOREM. *Let $\mathbb{M} = (M, \mathcal{M})$ be a locally compact stable plane of topological dimension 8 admitting $\Delta = \mathrm{SL}_2\mathbb{H}$ as a group of automorphisms. Then $\mathbb{M}$ contains an open Δ-invariant subplane which is isomorphic to the punctured affine plane over $\mathbb{H}$, and the action of Δ restricted to this subplane is equivalent to the natural (linear) action on $\mathbb{H}^2 \setminus \{(0,0)\}$.*

REMARKS. (a) The subplane above is the geometry induced on the set of points moved by the central involution ζ of Δ. Since ζ cannot be planar, one can show that $\mathbb{M}$ is embedded into the projective plane over $\mathbb{H}$, and that the action of Δ extends to the natural one.

(b) A special case of the stable planes considered here are compact eight-dimensional projective planes. All such planes with automorphism groups of dimension at least 17 have been determined by Salzmann [9]. For semisimple groups, this bound lowers to 16 (see [9, (1)]). Our result extends this classification to the case of the 15-dimensional groups locally isomorphic to $\mathrm{SL}_2\mathbb{H}$. (Since $\mathrm{SO}_5\mathbb{R}$ cannot act on eight-dimensional projective planes, we can exclude $\mathrm{PSL}_2\mathbb{H}$.)

NOTATION. Let $\Delta = \mathrm{SL}_2\mathbb{H}$, and let

$$\Upsilon = \left\{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \middle| \, d\bar{d} = 1 \right\} \cong \mathrm{ASL}_1\mathbb{H}$$

be the stabilizer of the point $(1, 0)$ in the natural linear action on $\mathbb{H}^2$ and

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consider the maximal compact subgroup $\Sigma = \left\{\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix} \middle| d\bar{d} = 1\right\}$ of Υ. Then

$$\Upsilon^* = \left\{\begin{pmatrix}1 & b\\ 0 & d\end{pmatrix} \middle| d\bar{d} = 1\right\}$$

is the stabilizer of the line $\{(1, y) \mid y \in \mathbb{H}\}$. Let

$$\Phi = \mathrm{SU}_2\mathbb{H} = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \middle| \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\bar{a} & \bar{c}\\ \bar{b} & \bar{d}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\} \cong \mathrm{Spin}_5.$$

All subgroups to be considered are closed, and all isomorphisms are isomorphisms of topological groups. The line joining two points x, y will be denoted as xy.

(1) LEMMA. (a) *Every connected subgroup of* Φ *not containing the central involution* $\zeta = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ *of* Φ *is either conjugate to* Σ *or at most one-dimensional.*

(b) *Every subgroup of* Δ *isomorphic to* Υ *is conjugate to* Υ *or to* Υ^*.

(c) *Let* Λ *be a subgroup of* Δ *containing* Σ. *If the centralizer* $\mathrm{C}_\Lambda(\alpha)$ *of the involution* $\alpha = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \Sigma$ *coincides with* Σ, *then either the connected component* Λ^1 *is equal to* Σ, *or* $\Lambda \cong \Upsilon$.

(d) *Any proper subgroup* $\Lambda < \Delta$ *with* $\dim \Delta/\Lambda \leqslant 4$ *is conjugate to the group* $N = \left\{\begin{pmatrix}a & 0\\ c & d\end{pmatrix} \middle| a\bar{a}d\bar{d} = 1\right\}$. *Consequently, the smallest non-trivial orbits of* Δ *are four-dimensional, and the action of* Δ *on such an orbit is equivalent to the natural action on the projective line* $\mathbb{H} \cup \{\infty\} \approx \mathbb{S}_4$.

Proof. (a) Let Ξ be a non-trivial connected subgroup of Φ not containing ζ. There is no pair of commuting involutions in Ξ since their product would be ζ. Therefore Ξ is a compact Lie group of rank 1, and $\dim \Xi = 1$ or $\Xi \cong \mathrm{Spin}_3$ (cf. [1, 22 §3, no. 6, Prop. 6]). In the second case, we may assume that $\alpha = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \Xi$. Then Ξ is a subgroup of the centralizer $\mathrm{C}_\Phi(\alpha) = \left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \middle| a\bar{a} = d\bar{d} = 1\right\}$. The projections of Ξ to the quasi-simple factors Σ and $\Theta = \left\{\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} \middle| a\bar{a} = 1\right\}$ must be bijective or trivial. Since the kernel of the projection to Θ contains α, we have $\Xi = \Sigma$.

(b) Let $\Upsilon \cong \Lambda \leqslant \Delta$ and write $\Lambda = \Xi\Omega$ with $\Xi \cong \mathrm{Spin}_3$, $\Omega \cong \mathbb{R}^4$ and consider the natural linear action on $\mathbb{H}^2 = \mathbb{C}^4$ as a complex representation. Since Ω is abelian, there is a vector $v \in \mathbb{C}^4$ with $v^\Omega \subseteq \mathbb{C}v$ by Lie's theorem. We may assume that $v = (1, 0)$. Then $\Omega < \mathrm{N} = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \middle| \, a\bar{a}d\bar{d} = 1 \right\}$. From $v^\Omega \subseteq \mathbb{C}v$ we deduce $\dim \Omega_v \geqslant 2$. The stabilizer Ω_v consists of axial collineations of the affine quaternion plane. Since Ξ acts effectively on Ω, all elements of Ω are axial with axes through the origin. The group Δ leaves invariant the line at infinity W, therefore all centers of elements of Ω lie on W. The homologies in Δ are contained in compact subgroups. Thus Ω consists of elations. Commutativity yields $\Omega = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \middle| \, c \in \mathbb{H} \right\}$. The normalizer of Ω is N, and we may assume that the compact group Ξ is contained in $\Psi = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, a\bar{a}d\bar{d} = 1 \right\}$. Acting effectively on Ω, the group Ξ cannot contain ζ. By (a) we have $\Xi = \Sigma$ or $\Xi = \Sigma^\iota$, where $\iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \Phi$. In the second case, $\Lambda^\iota = \Upsilon^*$.

(c) Assume that $\Sigma \neq \Lambda^1$. By (a) the subgroup Σ of Λ^1 is maximal compact. Therefore $\Sigma = \Phi \cap \Lambda^1$, and $\dim \Delta/\Lambda = \dim \Delta/\Lambda^1 \geqslant \dim \Phi/\Sigma = 7$. Since Δ is 15-dimensional, we have $\dim \Lambda \leqslant 8$. Any semisimple linear group with a maximal compact subgroup isomorphic to Σ contains a central involution which lies in each maximal compact subgroup (cf. [12]). Thus $\mathrm{C}_\Lambda(\alpha) = \Sigma$ implies that Λ is not semisimple. Now the Levi decomposition shows that there is a connected solvable group Ω such that $\Lambda^1 = \Sigma\Omega$. Let N be a minimal abelian normal subgroup of Λ^1. Being reductive, the group Σ acts completely reducibly on the Lie algebra of Ω. Therefore there is an invariant complement K of N in Ω. Since Σ acts effectively on N, we have $\mathrm{N} \cong \mathbb{R}^4$ and $\dim \mathrm{K} \leqslant 1$. Thus Σ acts trivially on K, and from the centralizer condition we get $\mathrm{K} = \mathbb{1}$. By (b) the connected component Λ^1 is conjugate to Υ or Υ^*. The normalizer of Υ in Δ being the product of Υ and the centralizer $\mathrm{C}_\Delta(\Sigma)$, we obtain $\Lambda = \Lambda^1$.

(d) Let K be a maximal compact subgroup of the connected component Λ^1. By [6] we have $\dim \mathrm{K} = 6$. Thus K is locally isomorphic to Spin_4 because Φ has rank 2. Since Φ does not contain a quadruple of commuting involutions, the group K is isomorphic to Spin_4 and centralizes two involutions. We can assume that α is one of them, and obtain $\mathrm{K} = \mathrm{C}_\Phi(\alpha)$. Now Λ has dimension 11. Any semisimple connected linear Lie group containing Spin_4 as a maximal compact subgroup centralizes all its involutions (cf. [12]). The Levi decomposition shows that there is a solvable connected invariant subgroup Ω of Λ such that $\Lambda^1 = \mathrm{K}\Omega$. Considering the usual linear action of Δ on $\mathbb{H}^2 = \mathbb{C}^4$ as a complex representation, we find $v \in \mathbb{H}^2$ such that $v^\Omega \subseteq \mathbb{C}v$ (Lie's theorem). If Ω had two linearly independent eigenvectors, we could assume that $\Omega \leqslant \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a\bar{a}d\bar{d}=1 \right\}$, a contradiction to the fact that Ω is homeomorphic to $\mathbb{R}^5$ (by the Malcev–Iwasawa decomposition). Normalizing Ω, the group Λ is therefore conjugate to a subgroup of N. Equality of dimensions yields that $\Lambda = \mathrm{N}$ since N is connected.

In order to determine the stabilizer of a line we need the following

(2) LEMMA. *Let $\mathbb{M}$ be a locally compact stable plane of finite positive dimension and assume that the lines are (topological) manifolds. Let α be an axial involution and x any point on the axis A. Then α fixes precisely one line $Z \neq A$ through x.*

Proof. We consider the action of α on the line pencil through x. By [3, (1.19)] the pencil $\mathcal{M}_x$ is homeomorphic to a sphere $\mathbb{S}_l$. For the restricted action on $\mathcal{M}_x \setminus \{A\} \approx \mathbb{R}^l$, there is at least one fixed element Z by [10]. Let X be a further fixed line through x. Choosing $y \in A$ near x, one finds a fixed line Y through y intersecting at least one of the two lines X, Z in a point z outside A. Having center z, the automorphism α has two axes and is trivial.

PROOF OF THE THEOREM. Without loss of generality we may assume that M is connected and that $\zeta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has no fixed points; in the general case, the fixed points of ζ form a Δ-invariant closed set which can be removed.

(i) *The action of Δ on M is equivalent to the natural (linear) action of Δ on $\mathbb{H}^2 \setminus \{0\}$.*

Proof. We show that $\Delta_x \cong \Upsilon$ for any point $x \in M$. Then all orbits are eight-dimensional and thus open. Connectedness yields transitivity, and (1.b) completes the proof of (i).

Since $x \neq x^\zeta$, a maximal compact subgroup of the connected component of $\Lambda = \Delta_x$ can be assumed to be Σ (cf. (1.a)). The centralizer of $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in Δ is the direct product of the groups

$$\Sigma,\ \Theta = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a\bar{a} = 1 \right\} \quad \text{and} \quad \mathrm{P} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \middle| r > 0 \right\}.$$

Let $G = xx^\zeta \in \mathcal{F}_\zeta = \{yy^\zeta \mid y \in M\}$. The set $\mathcal{F}_\zeta$ is connected and locally homeomorphic to a line (cf. [4, (1.1)]). Applying (1.d) to the action of Φ on $\mathcal{F}_\zeta$ we obtain transitivity on $\mathcal{F}_\zeta$ and see that α fixes precisely two lines G, H of $\mathcal{F}_\zeta$. Moreover, the set $\{G, H\}$ is the set of fixed lines of $\Sigma\Theta$ in $\mathcal{F}_\zeta$.

Next, we determine the geometric type of α. If α is a central involution, its center has to be fixed by ζ. Analogously, no line $L \neq G$ through x can be an axis of α. Planarity of α would yield a two-dimensional set of fixed lines in $\mathscr{F}_\zeta$. In the only remaining case, the involution α has axis G and acts freely on $M \backslash G$. Via conjugation by $\iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \Phi$ we obtain the same result for $-\alpha$ and H.

For any point $y \in G$ we have $\Phi_y \leqslant \Phi_G$, and the connected component of Φ_y is Σ or Θ by (1.a). Since $-\alpha$ acts freely on $M \backslash H$, we obtain that Σ is contained in the group $\Phi_{[G]} = \{\varphi \in \Phi_G \mid \varphi \text{ acts trivially on } G\}$. Again, conjugation by ι shows that $\Theta \leqslant \Phi_{[H]}$. Free action outside the axes enforces equality in both cases.

For any point $y \in G$ we consider the stabilizer in $\Sigma\Theta\mathrm{P} = \mathrm{C}_\Delta(\alpha)$. Since ι induces inversion on P, we have $\mathrm{P}_y = \mathrm{P}_y^{-1} = \mathrm{P}_y^\iota = \mathrm{P}_{y^\iota}$. The stabilizer P_y therefore acts trivially on the three-dimensional orbits y^Θ and $y^{\iota\Sigma}$. Since these two orbits generate the whole plane, one gets $\mathrm{P}_y = \mathbb{1}$. Now $(\Sigma\Theta\mathrm{P})_x = \Sigma(\Theta\mathrm{P})_x$. The kernel of the projection of $(\Theta\mathrm{P})_x$ to Θ lies in $\mathrm{P}_x = \mathbb{1}$, therefore $(\Theta\mathrm{P})_x \leqslant \Theta_x = \mathbb{1}$. Finally, (1.c) shows that $\Delta_x \cong \Upsilon$.

(ii) *Let $x \in M$ and $\Delta_x = \Upsilon = \Sigma\Omega$ with $\Omega = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \middle| c \in \mathbb{H} \right\}$. Then Ω is sharply transitive on $\mathscr{M}_x \backslash \{xx^\zeta\}$.*

Proof. Let $L \in \mathscr{M}_x \backslash \{xx^\zeta\}$. In a positive-dimensional stabilizer Ω_L one finds a one-parameter group $\Xi = \left\{ \begin{pmatrix} 1 & 0 \\ rc & 1 \end{pmatrix} \middle| r \in \mathbb{R} \right\}$. This group is invariant under $\mathrm{P} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \middle| r > 0 \right\}$ and therefore acts trivially on the orbit $L^{\Omega\mathrm{P}}$. If L^Ω is not trivial, there are $\rho \in \mathrm{P}$ and $\omega \in \Omega$ such that L^ρ and L^ω meet outside $xx^\zeta = G$. This contradicts the fact that Ω acts freely outside G. Thus $\Omega = \Omega_L$. For any further line L' fixed by Ω there is again $\rho \in \mathrm{P}$ such that L' and L^ρ meet outside G. Since all four-dimensional orbits in $\mathscr{M}_x$ are open, this yields a (sharply) transitive action of $\Omega \cong \mathbb{R}^4$ on $\mathscr{M}_x \backslash \{G, L\}$, which contradicts $\mathscr{M}_x \approx \mathbb{S}_4$.

(iii) *The stabilizer of any line $L \neq L^\zeta$ is isomorphic to Υ.*

Proof. By the preceding results, one can assume that $L = Z$ (compare (2)). Thus $\Sigma = \Delta_{x,L}$. By (1.a) Σ is a maximal compact subgroup of Δ_L^1. For any $\rho \in \Delta_L$ commuting with $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \Sigma$ we have $x^\rho = x^{\rho\alpha}$. Since α acts freely outside xx^ζ, we obtain $\rho \in \Delta_{x,L} = \Sigma$ and $\Delta_L \cong \Upsilon$ by (1.c).

Since $\Delta_{x,L} = \Sigma \neq \Delta_L$, we have $\Delta_x \neq \Delta_L$. The maximal compact subgroup Σ of

Δ_L is also maximal compact in Υ and in Υ^*. Therefore there is $\delta \in N_\Delta(\Sigma)$ such that $\Delta_L = \Upsilon^\delta$ or $\Delta_L = \Upsilon^{*\delta}$. But the normalizer of Σ in Δ is precisely the centralizer of α, which leaves both Υ and Υ^* invariant. Therefore $\Delta_L = \Upsilon^*$. Transitivity of Δ on points and lines moved by ζ allows the reconstruction of the geometry analogously to [2]. This concludes the proof.

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Author's address:

Markus Stroppel,
Mathematisches Institut der Universität Tübingen,
Auf der Morgenstelle 10,
D-7400 Tübingen 1,
F.R.G.

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