Arch. Math., Vol. 58, 621–624 (1992)

0003-889X/92/5806-0621 $ 2.30/0

Reconstruction of incidence geometries from groups of automorphisms

By

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In [4], Freudenthal describes a method to construct an incidence geometry from a group such that the given group acts transitively on the set of flags (incident point-line pairs) of the constructed geometry. This method can be found in [5], too. Here we give a useful generalization to geometries that are not flag-homogeneous. Such geometries occur quite naturally in the study of stable planes (see e.g. [16]). We consider geometries $G = (P, L, F)$ with a group Δ of automorphisms of G acting transitively on the point set P of G such that the orbit decomposition L_p/Δ_p of the pencil $L_p = \{l \in L \mid (p, l) \in F\}$ via the stabilizer $\Delta_p = \{\delta \in \Delta \mid p^\delta = p\}$ coincides with the induced decomposition $L_p \cap L/\Delta = \{l^\Delta \cap L_p \mid l \in L_p\}$ (cp. (R 2)). Additional assumptions are made in (R 3) to distinguish the different orbits.

(1) Definition. Let $G = (P, L, F)$ be an incidence geometry in the sense of Dembowski [2].

a) a subgroup $\Delta \leqq \mathrm{Aut}(G)$ is said to *represent* G, if the following conditions hold:

(R 1) Δ acts transitively on P.

(R 2) There is a point $p \in P$ and a subset $R \subseteqq L_p$ that forms a cross section for L/Δ and L_p/Δ_p, simultaneously.

(R 3) For any two different elements $r, s \in R$, the stabilizers Δ_r, Δ_s are different.

In this situation, $(\Delta, \Delta_p, (\Delta_r)_{r\in R})$ is called a *representing triplet* for G.

b) Let $(\Delta, \Delta_p, (\Delta_r)_{r\in R})$ be a representing triplet for G. The geometry $\tilde{G} = (\tilde{P}, \tilde{L}, \tilde{F})$ defined by

$$\tilde{P} = \Delta/\Delta_p, \quad \tilde{L} = \bigcup_{r\in R} \Delta/\Delta_r,$$

$$\tilde{F} = \{(\Delta_p \alpha, \Delta_r \beta) \in \tilde{P} \times \tilde{F} \mid \Delta_p \alpha \cap \Delta_r \beta \neq \emptyset\}$$

is called *the geometry represented by* $(\Delta, \Delta_p, (\Delta_r)_{r\in R})$.

(2) Proposition. *Let* $(\Delta, \Delta_p, (\Delta_r)_{r\in R})$ *be a representing triplet for* $G = (P, L, F)$. *Then* G *and the geometry* $\tilde{G}$ *represented by* $(\Delta, \Delta_p, (\Delta_r)_{r\in R})$ *are isomorphic.*

Proof. i) The mapping $\pi: \tilde{P} = \Delta/\Delta_p \to P: \Delta_p \alpha \mapsto p^\alpha$ is a bijection.

ii) For $r, s \in R$ and $\alpha, \beta \in \Delta$ we have

$$\Delta_r \alpha = \Delta_s \beta \Leftrightarrow \Delta_r = \Delta_s \beta \alpha^{-1}.$$

Since Δ_r is a group, the second assertion is equivalent to $\beta\alpha^{-1} \in \Delta_r = \Delta_s$. By condition (R 3), this is equivalent to $r = s$ and $r^\alpha = s^\beta$. Thus the mapping

$$\lambda: \tilde{L} = \bigcup_{r \in R} \Delta/\Delta_r \to L: \Delta_r \alpha \mapsto r^\alpha$$

is well-defined and injective. Since R is a cross section for L/Δ, it is surjective, too.

iii) For $r \in R$ and $\alpha, \beta \in \Delta$ we have

$$(p^\alpha, r^\beta) \in F \Leftrightarrow (p, r^{\beta\alpha^{-1}}) \in F$$

since Δ consists of automorphisms. By (R 2), the second assertion is equivalent to the existence of $\gamma \in \Delta_p$ with $r^{\beta\alpha^{-1}} = r^\gamma$. This is equivalent to $\Delta_p \alpha \cap \Delta_r \beta \neq \emptyset$, which means $(\Delta_p \alpha, \Delta_r \beta) \in \tilde{F}$. Thus we have shown that the pair (π, λ) is an isomorphism. □

(3) Remark. Via π and λ, the given action of Δ on G is equivalent to the action of Δ on G via $\Delta_x \gamma \mapsto \Delta_x \gamma\delta$.

We now turn to stable planes in the sense of [10], i.e. we assume that P and L are endowed with locally compact Hausdorff topologies such that their covering dimension is positive and finite; joining of points is continuous and intersecting of lines is stable (i.e. continuous with open domain of definition). Of special interest are the geometries induced on open orbits in the point space of a compact connected projective plane. Without additional assumptions, Proposition 2 carries over to the topological case:

(4) Proposition. *Let $G = (P, L, F)$ be a stable plane, and assume that a closed subgroup Δ of* Aut(G) *(the group of continuous collineations, endowed with the compact-open topology) represents G. If the point set $\tilde{P} = \Delta/\Delta_p$ is endowed with the quotient topology, then there is exactly one topology on $\tilde{L}$ such that $\tilde{G}$ becomes a stable plane. With these topologies, the stable planes G and $\tilde{G}$ are isomorphic.*

Proof. The underlying incidence geometries are isomorphic by Proposition 2. The group Aut(G) is locally compact and separable metric by [7 : 2.9], a result of Freudenthal [3] says that π is a homeomorphism. By [7 : 1.4] there is exactly one topology on L such that G is a stable plane. This topology is carried over to $\tilde{L}$ via λ^{-1}. □

(5) Corollary. *Let $(\Delta, \Delta_p, (\Delta_r)_{r \in R})$ be a representing triplet for both the incidence geometries (stable planes) G and G'. Then G and G' are isomorphic, and the actions of Δ are (topologically) equivalent.*

(6) Remarks.

a) The validity of incidence properties can be characterized in terms of the representing group analogously to [4: 6.3] and [5]. E.g. a geometry represented by $(\Delta, \Delta_p, (\Delta_r)_{r \in R})$ is a linear space if, and only if, the following conditions hold:

i) $\Delta = \bigcup_{r \in R} \Delta_p \Delta_r \Delta_p$ (existence of joining lines)

ii) For all $r \in R$: $\Delta_p \Delta_r \cap \Delta_r \Delta_p = \Delta_p \cup \Delta_r$ (uniqueness of joining lines).

b) The conditions for (P, L, F) to form a projective plane seem to be of a rather complicated nature. There are, however, two cases where our conditions yield flag homogeneity (i.e. the situation studied in [4] and [5]): For *finite* projective planes, the orbit theorem [6: Th. 13.4] says that any point transitive automorphism group is transitive on the set of lines, too. Condition (R 2) implies flag homogeneity. Point homogeneous *compact connected* projective planes are always flag homogeneous [15], [9].

c) In the case of affine translation planes, our result is known already (though stated differently): cp. [1], [14: 8.1, Satz 2, p. 201], [12: I.1, p. 1–7].

(7) Example. Let h be a hermitian or symmetric form on a (left) vector space V of dimension 3 over a (skew) field F of characteristic $\neq 2$. The corresponding unitary group induces a group Δ of collineations of the projective plane $(P, L, \leqq)$ described by V. According to Witt's theorem [13], a line $l_1 \in L$ (i.e. a two-dimensional subspace of V) can be moved to a line l_2 by an element of Δ if, and only if, the induced forms $H|_{l_i \times l_i}$ are equivalent. In this case, there are $x_1, x_2, y_1, y_2 \in V$ such that $l_i = Fx_i + Fy_i$ and $h(v_1, w_1) = h(v_2, w_2)$ for $i \in \{1, 2\}$ and $v, w \in \{x, y\}$. Moreover, one can choose $x_1 \in l_1 \cap l_2$ and $x_2 = x_1$. This means that there is $\delta \in \Delta$ such that $x_1^\delta = x_1$ and $y_1^\delta = y_2$. Therefore, conditions (R 1) and (R 2) hold for the geometry $(p^\Delta, L', \leqq)$ induced on any orbit p^Δ, where $p = Fx \in P$. Using Witt's theorem, one may find a cross section R for L'/Δ and compute the corresponding stabilizers. Now it is checked easily whether condition (R 3) is valid or not. These geometries are generalizations of the familiar euclidean, elliptic and hyperbolic planes. Topological examples have been described by Löwen [8], [11].

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Eingegangen am 13. 2. 1991

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