Seminar Sophus Lie
2 (1992) 69–74

Solvable groups of automorphisms of stable planes

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An interesting problem in the foundations of geometry is the following question: *What is the impact of the interplay of topological assumptions and homogeneity?* One possibility to make this (rather philosophical) question treatable for a mathematician is the classification project for stable planes. We shall briefly outline the necessary definitions and basic (though occasionally deep) results. In section 2, we treat solvable groups of automorphisms of stable planes. This may serve as an example how the project works. Note, however, that the case of (semi-)simple groups of automorphisms is where Lie structure theory shows its full strength, cf. the final Remark.

1. Stable planes

Stable planes are topological incidence structures that share fundamental properties of the "classical compact connected projective planes" (namely, the projective planes over the real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$) and their affine and hyperbolic[1] subplanes. To be precise:

Definition 1.1. A *stable plane* $\mathbb{M} = (M, \mathcal{M})$ is a topological incidence geometry (with point space M and line space $\mathcal{M}$) such that the following holds:

- Any two points $x, y \in M$ are incident with exactly one line $x \vee y \in \mathcal{M}$, but two lines K, L need not meet. If they do, we denote the (unique) common point by $K \wedge L$.
- The spaces M and $\mathcal{M}$ are locally compact Hausdorff spaces of positive and finite covering dimension.
- The mappings $\vee\colon (x, y) \mapsto x \vee y$ and $\wedge\colon (K, L) \mapsto K \wedge L$ are continuous on their respective domain.
- The pairs of meeting lines form an open subset of $\mathcal{M} \times \mathcal{M}$.

For general information, see [2].

If $\mathbb{M} = (M, \mathcal{M})$ is a stable plane, one obtains a stable plane $\mathbb{E} = (E, \mathcal{E})$ for each open subset E of M with $\mathcal{E}$ the set of all lines in M that meet E in at least two points. Since each compact connected projective plane is a stable

[1] A *hyperbolic* subplane of a classical compact connected projective plane is the geometry induced on the interior points of a suitable unital, cf. Klein's model of the real hyperbolic plane.

plane, this provides many examples. Note, however, that there are stable planes that are not embeddable as open subplanes [12: 3.1, 3.2, 3.3, 3.6].
At least topologically, each stable plane is similar to one of the classical planes:

Result 1.2. ([5: Th. 1])

a) *For each stable plane* $(M, \mathcal{M})$*, the covering dimension* $\dim M$ *equals* $\dim \mathcal{M}$ *and is one of the integers* 2*,* 4*,* 8*, or* 16*.*

b) *For each line* $L \in \mathcal{M}$*, we have that*[2] $\dim L = \frac{1}{2} \dim M$*.*

Endowed with the compact-open topology derived from the action on M, the group $\mathrm{Aut}\left(\mathbb{M}\right)$ of all continuous collineations of $\mathbb{M}$ is a locally compact separable transformation group on both M and $\mathcal{M}$ [2: 2.9]. Here is how Lie theory comes in: For each locally compact connected group Γ of finite dimension there is a totally disconnected compact central subgroup N of Γ such that Γ/N is a Lie group. Moreover, the local structure of Γ/N does not depend[3] on the choice of such a N. This implies that important parts of the structure theory of Lie algebras carry over: E.g., *connected* subgroups of Γ commute if, and only if, the corresponding factor groups (mod N) commute. In particular, the notions of simplicity and solvability generalize in a straightforward manner: a locally compact connected group Γ of finite dimension is called *almost simple, semi-simple, solvable,* if, and only if, the Lie group Γ/N has the property in question.

Definition 1.3. An *action* $(\gamma, \Gamma, \mathbb{M})$ of a topological group Γ on a stable plane $\mathbb{M}$ is a continuous homomorphism $\gamma\colon \Gamma \to \mathrm{Aut}\left(\mathbb{M}\right)$. If γ is injective and Γ is locally compact, we shall call Γ *a group of automorphisms of* $\mathbb{M}$.

The **classification project for stable planes** may now be stated as follows: *Find suitable bounds* b_m *for* $m \in \{2, 4, 8, 16\}$ *and determine all actions* $(\gamma, \Gamma, \mathbb{M})$*, such that* $\dim M = m$ *and* $\dim \Gamma \geq b_m$*.*

It turns out that it is convenient to treat the different cases for the structure of Γ (almost simple, semi-simple, solvable, mixed type) separately. Our main line of attack is to proceed inductively: Once a problem has been solved (to some extent) for the lower values of m, then the higher values become tractable if one succeeds in finding invariant subplanes. For such an attempt, two special kinds of actions are of interest:

Definition 1.4. Let Γ be a group of automorphisms of $\mathbb{M}$, and write $\mathrm{Fix}\,(\Gamma)$ for the set of fixed points of Γ.

a) If for each point $x \in M$ the orbit x^Γ is contained in some line F_x then Γ is called *quasi-perspective.* In this case, let $\mathcal{F}_\Gamma := \{F_x \mid x \in M \setminus \mathrm{Fix}\,(\Gamma)\}$.

b) If $\dim \mathrm{Fix}\,(\Gamma) > 0$ and $\mathrm{Fix}\,(\Gamma)$ is not contained in any anti-flag[4], then the geometry induced on $\mathrm{Fix}\,(\Gamma)$ is a stable plane. In this case, we call Γ a *planar* group of automorphisms of $\mathbb{M}$.

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[2] The line L is identified with the set of points that are incident with it: thus L becomes a *closed* subset of M.

[3] Here the finiteness of $\dim \Gamma$ is essential!

[4] An antiflag is a non-incident point-line pair. Of course, this condition implies that $\mathrm{Fix}\,(\Gamma)$ contains a non-degenerate quadrangle. If there is a connected subset of $\mathrm{Fix}\,(\Gamma)$ that is not contained in any line then Γ is planar.

Note that planar groups are just the kernels of restrictions of actions to subplanes.

Result 1.5. ([3: 1.6, 1.5], [13: 5.1, 7.3, 8.21]) *Assume that* Δ *is a connected planar group on a stable plane* $\mathbb{M} = (M, \mathcal{M})$. *Then one of the following holds:*

a) $\dim M \le 4$, *and* $\Delta = \mathbb{1}$.

b) $\dim M = 8$, *and* $\dim \Delta \le 3$.

c) $\dim M = 16$, *and* $\dim \Delta \le 12$, *or* Δ *is a non-solvable group of dimension* 14.

Result 1.6. (cf. [14: 1.5,b)]) *Assume that* Γ *is a group of automorphisms of a stable plane. If a subgroup* Ψ *of* Γ *centralizes a connected, non-trivial quasi-perspective subgroup* Θ *of* Γ, *then there are points* x, y *such that* $\Psi_{x,y}$ *is planar.*

Result 1.7. ([15: 3.3]) *If* Δ *is an abelian group of automorphisms of a stable plane* $\mathbb{M} = (M, \mathcal{M})$, *then* $\dim \Delta \le \dim M$. *If* $\dim \Delta = \dim M$ *then there is a point* $p \in M$ *such that the orbit* p^Δ *is open in* M, *or there is a line* $L \in \mathcal{M}$ *such that* L^Δ *is open in* $\mathcal{M}$. *In either case, the identity component of the centralizer* $\mathrm{C}_{\mathrm{Aut}(\mathbb{M})}(\Delta)$ *of* Δ *is contained in* Δ, *and* Δ *acts freely on the open orbit.*

2. Solvable group of automorphisms

The crucial information about the structure of solvable groups is the following:

Result 2.8. ([1: Ch. III, §9, n° 7, Prop. 22] or [16: Cor. 3.7.5, cf. Ex. 29, p. 252]) *Assume that* Λ *is a solvable locally compact separable group. Then there is a minimal connected abelian normal subgroup* N *of* Λ, *which is either compact or isomorphic with* $\mathbb{R}^n$ *for* $n = \dim \mathrm{N}$. *In the compact case* N *lies in the center of the identity component of* Λ. *If* $\mathrm{N} \cong \mathbb{R}^n$ *then* $\Lambda/\mathrm{C}_\Lambda(\mathrm{N})$ *acts (via conjugation) effectively and irreducibly on* N. *In particular,* $\dim \Lambda/\mathrm{C}_\Lambda(\mathrm{N}) \le n \le 2$.

Lemma 2.9. *Assume that* Γ *is a group of automorphisms of a stable plane* $\mathbb{M} = (M, \mathcal{M})$, *where* $\dim M \le 4$. *Then* $\dim \mathrm{C}_\Gamma(\Theta) \le 2 \dim M$ *for each connected, nontrivial subgroup* Θ *of* Γ.

Proof. We denote the centralizer $\mathrm{C}_\Gamma(\Theta)$ by Ψ. If there is a point $x \in M$ such that the orbit x^Θ is not contained in any line, then x^Θ generates a subplane, and $\dim \Psi_x = 0$ by (1.5). Consequently, $\dim \Psi = \dim x^\Psi \le \dim M$. If Θ is quasi-perspective, then there are points $x, y \in M$ such that $\dim \Psi_{x,y} = 0$ (cf. (1.6), (1.5)). Hence $\dim \Psi \le 2 \dim M$. ■

Theorem 2.10. *Assume that* Λ *is a solvable group of automorphisms of a stable plane* $\mathbb{M} = (M, \mathcal{M})$, *where* $\dim M \le 4$. *Then* $\dim \Lambda \le \frac{5}{2} \dim M$.

Proof. We may assume that Λ is connected. Let N be a minimal connected abelian normal subgroup of Λ, and denote the identity component of $\mathrm{C}_\Lambda(\mathrm{N})$ by Ψ. If N is compact then Ψ equals the identity component of Λ. Hence

$\dim \Lambda = \dim \Psi \le 2 \dim M$ by (2.9). If $\mathrm{N} \cong \mathbb{R}^n$ then $\dim \Lambda/_{\Psi} \le n \le 2$. Now $\dim \Lambda \le \dim \Psi + n$, and we obtain that $\dim \Lambda \le \frac{5}{2} \dim M$, or $\dim M = n = 2$. In the latter case, $\Psi = \mathrm{N}$ by (1.7), and $\dim \Lambda \le 4$. ∎

Example 2.11. The bound in (2.10) is sharp: For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the solvable group

$$\Lambda = \left\{ \begin{pmatrix} a & b & c \\ & d & e \\ & & 1 \end{pmatrix} \,\middle|\, a,b,c,d,e \in \mathbb{K}, ad \ne 0 \right\}$$

acts effectively on the projective plane $(P, \mathcal{P})$ over $\mathbb{K}$, and $\dim \Lambda = \frac{5}{2} \dim P$.

Turning to planes of higher dimension, the bound $\frac{5}{2} \dim M$ is no longer attained in the projective case, cf. [9]. The proof for the projective case rests heavily on the existence of fixed elements for solvable groups and the structure theory for compact connected projective planes. Thus it does not carry over to the general case. We have the following result (which might be improved):

Theorem 2.12. *Assume that Λ is a solvable group of automorphisms of a stable plane.*

a) If $\dim M = 8$, *then* $\dim \Lambda \le 18$ $(< \frac{5}{2} \dim M)$.

b) If $\dim M = 16$, *then* $\dim \Lambda \le 40$ $(= \frac{5}{2} \dim M)$.

Proof. Let N be a minimal connected abelian normal subgroup, and denote the identity component of $\mathrm{C}_\Lambda(\mathrm{N})$ by Ψ.

i) If there is a point $x \in M$ such that the orbit x^{N} is not contained in any line, then Ψ_x acts trivially on the subplane generated by x^{N}. From (1.5,b) and (1.5,c), respectively, we obtain that $\dim \Lambda \le 11$ or $\dim \Lambda \le 28$, according to the dimension of M.

ii) If N is quasi-perspective, we consider the action of Λ on the set $\mathcal{F}_{\mathrm{N}}$ of fixed lines. We denote the kernel of this action by Δ. Then $\Lambda/_{\Delta}$ is a solvable group, and we infer that there is a normal subgroup Φ of Λ such that $\Delta < \Phi$ and $\Phi/_{\Delta}$ is a minimal connected abelian normal subgroup of $\Lambda/_{\Delta}$. Let K denote the kernel of the action of Λ on $\Phi/_{\Delta}$. Then $\dim \Lambda/_{\mathrm{K}} \le \dim \Phi/_{\Delta} \le 2$..

iii) For each point $x \in M \setminus \mathrm{Fix}(\mathrm{N})$, the stabilizer K_{F_x} acts trivially on the orbit F_x^{Φ}. Choose a point $x \in M \setminus \mathrm{Fix}(\mathrm{N})$ such that F_x is moved by Φ, and choose $y \in x^{\Phi} \setminus F_x$. Then K_x fixes F_x and F_y, and $\dim \mathrm{K}_x/_{\mathrm{K}_{x,y}} \le \dim F_y = \frac{1}{2} \dim M$. The centralizer $\Upsilon = \mathrm{C}_{\mathrm{K}_{x,y}}(\mathrm{N})$ acts trivially on $x^{\mathrm{N}} \cup y^{\mathrm{N}}$. Hence Υ is a planar group. We obtain that

$$\begin{aligned} \dim \Lambda &= \Lambda/_{\mathrm{K}} + \dim \mathrm{K}/_{\mathrm{K}_x} + \dim \mathrm{K}_x/_{\mathrm{K}_{x,y}} + \dim \mathrm{K}_{x,y}/_{\Upsilon} + \dim \Upsilon \\ &\le \quad 2 \quad + \quad \dim M \quad + \quad \tfrac{1}{2} \dim M \quad + \quad 2 \quad + \quad b \end{aligned}$$

where $b \in \{3, 12\}$, according to the dimension of M (cf. (1.5)). This implies assertion b). If $\dim M = 8$, we obtain that $\dim \Lambda \le 19$.

iv) Assume now that $\dim M = 8$. If N is compact then $\dim \Lambda/_{\Psi} = 0$, and the set $x^{\mathrm{N}} \cup y^{\mathrm{N}}$ generates a 4-dimensional plane. This yields that $\dim \Upsilon \le 1$ and $\dim \Lambda \le 15$.

v) If $\mathrm{N}_x \neq \mathbb{1}$ then x^Ψ cannot generate $\mathbb{M}$. Consequently $\dim \Psi/_{\Psi_x} \leq 4$, hence $6 \geq \dim \Lambda/_{\Lambda_x} \geq \dim \mathrm{K}/_{\mathrm{K}_x}$. We conclude that $\dim \Lambda \leq 17$.

vi) If $\mathrm{N} \cong \mathbb{R}^2$ and $\mathrm{N}_x = \mathbb{1}$ then $x^\mathrm{N} \cup y^\mathrm{N}$ generates a 4-dimensional subplane, hence $\dim \Upsilon \leq 1$ and $\dim \Lambda \leq 17$. If $\mathrm{N} \cong \mathbb{R}$ then $\dim \Lambda/_{\Psi} \leq 1$, which implies that $\dim \Lambda \leq 18$. ∎

Remark 2.13. Parts iv)-vi) of the proof show that the bound in (2.12) can only be attained if each minimal connected abelian normal subgroup of Λ is isomorphic with $\mathbb{R}$, not central and acts quasi-perspectively.

Example 2.14. The solvable group

$$\Lambda = \left\{ \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} \middle| \; b, c, e \in \mathbb{H};\ a, d, f \in \mathbb{C};\ adf \neq 0;\ f\bar{f} = 1 \right\}$$

acts almost effectively on the projective plane $(P, \mathcal{P})$ over $\mathbb{H}$, and $\dim \Lambda = 17$. According to [9], this is the maximal possible dimension for a solvable group of automorphisms of an eight-dimensional compact projective plane. For the case of 16-dimensional compact projective planes, M. LÜNEBURG has proved that a solvable group of automorphisms can have dimension 30 at most [9]. He also gives examples that show that this bound is sharp.

Remark 2.15. The results of this paper are of rather a negative kind. However, they contribute to the classification program for stable planes (cf. [12]) since they exclude solvable groups of "large" dimension, thus concentrating further efforts on groups that contain non-trivial semi-simple subgroups. Both the structure theory of semi-simple groups and the theory of representations (on minimal abelian normal subgroups) provide convenient tools for the non-solvable case. Moreover, the existence of involutions (which is granted in most semi-simple groups) imposes strong restrictions to the geometries. See [4], [6], [7], [8], [10], [14] for results in that direction.

References

[1] Bourbaki, N., "Groupes et algèbres de Lie", Hermann, 1972.

[2] Löwen, R., *Vierdimensionale stabile Ebenen*, Geom. Ded., **5** (1976), 239-294.

[3] —, *Halbeinfache Automorphismengruppen von vierdimensionalen stabilen Ebenen sind quasi-einfach*, Math. Ann., **236** (1978), 15-28.

[4] —, *Zweidimensionale stabile Ebenen mit nichtauflösbarer Automorphismengruppe*, Arch. Math., **41** (1983), 286-288.

[5] —, *Topology and dimension of stable planes: On a conjecture by H. Freudenthal*, J. Reine Angew. Math., **343** (1983), 108-122.

[6] —, *Actions of* SO(3) *on 4-dimensional stable planes*, Aequ. Math., **30** (1986), 212-222.

[7] —, *Actions of* Spin$_3$ *on 4-dimensional stable planes*, Geom. Ded., **21** (1986), 1-12.

[8] —, *Stable planes admitting a classical motion group*, Res. Math., **9** (1986), 119-130.

[9] Lüneburg, M., "Involutionen, auflösbare Gruppen und die Klassifikation topologischer Ebenen", Dissertation, Tübingen, 1992.

[11] Stroppel, M., *A characterization of quaternion planes*, Geom. Ded., **36** (1990), 405-410.

[10] —, "Achtdimensionale stabile Ebenen mit quasieinfacher Automorphismengruppe", Dissertation, Tübingen, 1990.

[12] —, *Stable Planes*, Discrete Math., to appear (1992).

[13] —, "Planar groups of automorphisms of stable planes", Manuscript, Tübingen, 1991.

[14] —, "Compact groups of automorphisms of stable planes", Manuscript, Tübingen, 1991.

[15] —, "Quaternion Hermitian Planes", Manuscript, Darmstadt, 1991.

[16] Varadarajan, V. S., "Lie groups, Lie algebras, and their representations", Prentice Hall, 1974 = Springer, 1984.

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Received February 10, 1992