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EMBEDDING A NON-EMBEDDABLE STABLE PLANE

ABSTRACT. In [4], K. Strambach describes a 2-dimensional stable plane $\mathscr{S}_{\mathbb{R}}$ admitting $\Sigma = \mathrm{SL}_2\mathbb{R}$ as a group of automorphisms such that there exists no Σ-equivarient embedding into a 2-dimensional projective plane. R. Löwen [3] has given a 4-dimensional analogue $\mathscr{S}_{\mathbb{C}}$, admitting $\Delta = \mathrm{SL}_2\mathbb{C}$. He posed the question whether there are embeddings of Strambach's plane $\mathscr{S}_{\mathbb{R}}$ into $\mathscr{S}_{\mathbb{C}}$. We show that such embeddings exist, in fact we determine all Σ-equivariant embeddings of 2-dimensional stable planes admitting Σ as a *transitive* group of automorphisms.

1. THE PLANES

In the original definitions, the point space is taken to be $\mathbb{R}^2$ or $\mathbb{C}^2$, respectively, and the lines are described as subsets of the point space. We wish to describe the resulting geometries by the method given in [6]. Since this method applies only to point homogeneous geometries, we have to delete the origin.

(1.1) NOTATION. We write

$$\Sigma = \mathrm{SL}_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R}; ad - bc = 1 \right\}$$

$$\Delta = \mathrm{SL}_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{C}; ad - bc = 1 \right\}.$$

The following stabilizers (with respect to the linear actions on $\mathbb{R}^2$ and $\mathbb{C}^2$) shall be useful:

$$\Pi = \Sigma_{(1,1)} = \left\{ \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \middle| t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}^{\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)}$$

$$\Phi = \Delta_{(1,i)} = \left\{ \begin{pmatrix} 1-iz & z \\ z & 1+iz \end{pmatrix} \middle| z \in \mathbb{C} \right\} = \left\{ \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} \middle| z \in \mathbb{C} \right\}^{\left(\begin{smallmatrix} 1 & i \\ & 1 \end{smallmatrix}\right)}$$

For the description of the planes that are of interest in this paper, we consider the following sets of points:

$$R_o = \{(x, x) \mid x \in \mathbb{R}\},\ R_e = \{(1, y) \mid y \in \mathbb{R}\},\ R_s = \{(x, 1/x) \mid x > 0\}$$

$$S_o = \{(u, iu) \mid u \in \mathbb{C}\},\ S_e = \{(1, v) \mid v \in \mathbb{C}\},$$

$$S_s = \{(x, i/x + y) \mid x, y \in \mathbb{R}, x > 0\}$$

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and the corresponding stabilizers:

$$\Lambda_o = \Sigma_{R_o} = \left\{ \begin{pmatrix} s-t & s-1/s-t \\ t & 1/s+t \end{pmatrix} \middle| s, t \in \mathbb{R};\ s \neq 0 \right\}$$

$$= \left\{ \begin{pmatrix} s & \\ t & 1/s \end{pmatrix} \middle| s, t \in \mathbb{R};\ s \neq 0 \right\}^{\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)},$$

$$\Lambda_e = \Sigma_{R_e} = \left\{ \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\},\ \Lambda_s = \Sigma_{R_s} = \left\{ \begin{pmatrix} a & \\ & 1/a \end{pmatrix} \middle| a > 0 \right\},$$

$$\Omega_o = \Delta_{S_o} = \left\{ \begin{pmatrix} u-iz & i(u-1/u-iz) \\ z & 1/u+iz \end{pmatrix} \middle| u, z \in \mathbb{C};\ u \neq 0 \right\},$$

$$\Omega_e = \Delta_{S_e} = \left\{ \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

$$\Omega_s = \Delta_{S_s} = \left\{ \begin{pmatrix} a & b \\ & 1/a \end{pmatrix} \middle| a, b \in \mathbb{R};\ a > 0 \right\}.$$

Note that, for each $k \in \{o, e, s\}$, we have that $R_k = (1,1)\Lambda_k$, and $S_k = (1, i)\Omega_k$.

(1.2) PROPOSITION. *For cosets* $\mathrm{A}\alpha$, $\mathrm{B}\beta$ *(where* $\mathrm{A} \in \{\Pi, \Phi\}$, $\mathrm{B} \in \{\Lambda_o, \Lambda_e, \Lambda_s, \Omega_o, \Omega_e, \Omega_s\}$*) define the relation*

$$\mathrm{A}\alpha\, I\, \mathrm{B}\beta \Leftrightarrow \mathrm{A}\alpha \cap \mathrm{B}\beta \neq \varnothing.$$

Then the incidence geometries

$$(\Sigma/\Pi,\ \Sigma/\Lambda_o \cup \Sigma/\Lambda_e,\ I) \qquad (\Sigma/\Pi,\ \Sigma/\Lambda_o \cup \Sigma/\Lambda_s,\ I)$$
$$(\Delta/\Phi,\ \Delta/\Omega_o \cup \Delta/\Omega_e,\ I) \qquad (\Delta/\Phi,\ \Delta/\Omega_o \cup \Delta/\Omega_s,\ I)$$

are isomorphic with the real affine plane $\mathscr{A}_{\mathbb{R}}$, *Strambach's plane* $\mathscr{S}_{\mathbb{R}}$, *the complex affine plane* $\mathscr{A}_{\mathbb{C}}$, *and Löwen's plane* $\mathscr{S}_{\mathbb{C}}$, *respectively (each minus the origin).*

Proof. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the group $\mathrm{SL}_2\mathbb{K}$ acts transitively on the points of $\mathscr{A}_{\mathbb{K}}$ and $\mathscr{S}_{\mathbb{K}}$ (minus the origin). It is easy to see that in each of the considered cases there are exactly two line orbits: namely, the line pencil in the origin and the complement thereof (for $\mathscr{S}_{\mathbb{K}}$ this holds by definition), and that the stabilizer of any point p acts with two orbits on the line pencil in p. Thus the orbit decomposition of the pencil (under the action of the stabilizer) coincides with the decomposition that is induced from the action of $\mathrm{SL}_2\mathbb{K}$ on the line space. Obviously, the stabilizers of representatives for the different orbits are different. Thus [6: 4] yields the assertion. □

(1.3) PROPOSITION. *Assume that $\mathcal{S}$ is a linear space (resp., a stable plane) admitting a group Γ of automorphisms such that the reconstruction method of* [6] *applies; i.e. the group Γ acts transitively on the point set, and there is a cross-section $\mathcal{R}$ for the line orbits that simultaneously forms a cross-section for the orbits in the line pencil through some point p under the stabilizer Γ_p.*

If Γ acts point-transitively on a linear space (resp., a stable plane) $\mathcal{S}'$ such that

- *for each $R \in \mathcal{R}$ there is a line R' of $\mathcal{S}'$ such that $\Gamma_R = \Gamma_{R'}$,*
- *there is a point p' of $\mathcal{S}'$ that is incident with each of the R', and $\Gamma_p = \Gamma_{p'}$*

then $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, and the actions of Γ are equivalent.

Proof. According to [6], the linear space $\mathcal{S}$ is isomorphic with the incidence geometry whose point space is the orbit p'^{Γ} and whose line space is $\bigcup_{R \in \mathcal{R}} R'^{\Gamma}$. Therefore we just have to show that the line space of $\mathcal{S}'$ equals the union of the orbits R'^{Γ}, where $R \in \mathcal{R}$. Because joining lines are unique in $\mathcal{S}'$, this follows immediately from the fact that two points of $\mathcal{S}$ are joined by a line in $\bigcup_{R \in \mathcal{R}} R^{\Gamma}$. □

To recognize Strambach's plane, we shall need the following:

(1.4) LEMMA. *Let Σ, Π and Λ_o be as in* (1.1), *and write*

$$\Lambda(t) = \left\{ \begin{pmatrix} a & t(a - 1/a) \\ & 1/a \end{pmatrix} \middle| \, a > 0 \right\} \quad \textit{for each } t \in \mathbb{R}.$$

With the incidence relation I *as in* (1.2), *the incidence structure* $(\Sigma/\Pi, \Sigma/\Lambda_o \cup \Sigma/\Lambda(t), \mathrm{I})$ *is isomorphic with $\mathcal{S}_{\mathbb{R}}$ for each choice of $t \in \mathbb{R} \setminus \{1\}$.*

Proof. Conjugation with $\begin{pmatrix} 1 & t \\ & 1 - t \end{pmatrix} \in \mathrm{GL}_2\mathbb{R}$ leaves both Π and Λ_o invariant, but carries Λ_s to $\Lambda(t)$. In view of (1.2) and [6], this yields the assertion. □

(1.5) REMARKS. (a) The 2-dimensional stable planes admitting $\mathrm{SL}_2\mathbb{R}$ as a group of automorphisms have been determined by R. Löwen [2]. Among these, the punctured real affine and the punctured Strambach plane are the only ones where $\mathrm{SL}_2\mathbb{R}$ acts point-transitively.

(b) There is no analogue to $\mathcal{S}_{\mathbb{R}}$ and $\mathcal{S}_{\mathbb{C}}$ over Hamilton's quaternions [5].

We shall not use this classification of $\mathrm{SL}_2\mathbb{R}$-planes, since our special situation allows to identify the planes that are isomorphic with the punctured real affine plane or the punctured Strambach plane by means of (1.3) and (1.4).

2. Orbits

As a subgroup of Δ, the group Σ has a natural action on $\mathbb{C}^2$. We characterize the orbits:

(2.1) LEMMA. *For $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, one of the following holds:*

(a) *There is some $w \in \mathbb{C}^*$ such that $w(u, v) \in \mathbb{R}^2$. Then the orbit $(u, v)\Sigma$ has dimension 2, and the action of Σ is equivalent to the natural action on $\mathbb{R}^2 \setminus \{(0, 0)\} = (1, 1)\Sigma$. Moreover, the orbit $(u, v)\Sigma$ is closed in $\mathbb{C}^2 \setminus \{(0, 0)\}$, hence locally compact.*

(b) *For each $w \in \mathbb{C}^*$, the vector $w(u, v)$ does not belong to $\mathbb{R}^2$. Then the orbit $(u, v)\Sigma$ has dimension 3, and Σ acts freely on that orbit.*

Proof. We have to determine the stabilizer $\Sigma_{(u,v)}$. Since Σ acts $\mathbb{C}$-linearly, we have that $\Sigma_{(u,v)} = \Sigma_{w(u,v)}$ for each $w \in \mathbb{C}^*$, and scalar multiplication with w is a homeomorphism from $\mathbb{C}^2 \setminus \{(0, 0)\}$ onto itself that carries $(u, v)\Sigma$ to $w(u, v)\Sigma$. This implies assertion (a) (recall that Σ acts transitively on $\mathbb{R}^2 \setminus \{(0, 0)\}$). In the situation of (b), we may assume that $(u, v) = (1, v)$ for some $v \in \mathbb{C} \setminus \mathbb{R}$. Then 1 and v form an $\mathbb{R}$-basis for $\mathbb{C}$. The stabilizer condition $(1, v) = (a + vc, b + vd)$ yields that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, since $a, b, c, d \in \mathbb{R}$. This proves assertion (b).

□

3. Equivariant embeddings

(3.1) DEFINITION. Let $\mathscr{S}$ be a 2-dimensional stable plane, and assume that $\Sigma = \mathrm{SL}_2\mathbb{R}$ acts effectively and transitively on $\mathscr{S}$. A Σ-equivariant embedding of $\mathscr{S}$ into $\mathscr{S}_{\mathbb{C}}$ is a triplet (π, λ, γ) with the property that

- π and λ are embeddings of the point and line space of $\mathscr{S}$ into the point and line space of $\mathscr{S}_{\mathbb{C}}$, respectively;
- (π, λ) is a collineation onto $\mathscr{S}^{(\pi,\lambda)}$;
- γ is a continuous monomorphism of Σ into Δ such that the diagram

$$\begin{array}{ccc} \mathscr{S} \times \Sigma & \xrightarrow{(\pi, \lambda, \gamma)} & \mathscr{S}_{\mathbb{C}} \times \Delta \\ \alpha \downarrow & & \downarrow \beta \\ \mathscr{S} & \xrightarrow[(\pi, \lambda)]{} & \mathscr{S}_{\mathbb{C}} \end{array}$$

commutes, where α and β denote the action of Σ and Δ, respectively.

(3.2) In the sequel, we shall consider the case where $\gamma = \mathbb{1}$. According to (2.1) we may assume that the image of π equals $w(s, t)\Sigma$ for some $w \in \mathbb{C}^*$ and $(s, t) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Adapting (s, t), one obtains that $w\bar{w} = 1$. Now there is $\sigma \in \Sigma$ such that $(s, t)\sigma = (1, 1)$, and we may assume that the image of π equals $(w, w)\Sigma$ for some $w = \mathbb{C}^*$ with $w\bar{w} = 1$.

(3.3) We write

$$\varphi = \begin{pmatrix} w & w - i\bar{w} \\ & \bar{w} \end{pmatrix} \in \Delta, \quad \text{hence } (1, i)\varphi = (w, w).$$

To determine the incidence structure on $\mathcal{S}^{(\pi,\lambda)}$ with the help of (1.3) we put $p = (1, 1)$, $\mathcal{R} = \{L_o, L_e\}$ resp. $\mathcal{R} = \{L_o, L_s\}$ and $p' = (1, i)\varphi = (w, w)$, $L'_o = S_o\varphi$, $L'_k = S_s\varphi$ for $k = e$ or $k = s$, respectively. Now we are able to compute the stabilizers Σ_x (for $x \in \{p', S_o\varphi, S_s\varphi\}$) in the following way:

$$\Sigma_x = \Sigma \cap \Delta_x = \Sigma \cap \Delta_{x\phi}^{\phi^{-1}}.$$

Observe that

$$\begin{pmatrix} \bar{w} & i\bar{w} - w \\ & w \end{pmatrix} \begin{pmatrix} w & w - i\bar{w} \\ & \bar{w} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

(3.4) For the action (via conjugation) of φ on Δ we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\varphi} = \begin{pmatrix} a - w(w - i\bar{w})c & \bar{w}(w - i\bar{w})a - (w - i\bar{w})^2 c + \bar{w}^2 b - \bar{w}(w - i\bar{w})d \\ w^2 c & w(w - i\bar{w})c + d \end{pmatrix}.$$

(3.5) Applying (3.4) to elements of Φ we get

$$\begin{pmatrix} 1 - iz & z \\ z & 1 + iz \end{pmatrix}^{\varphi} = \begin{pmatrix} 1 - w^2 z & -w^2 z \\ w^2 z & 1 + w^2 z \end{pmatrix}.$$

Therefore,

$$\Sigma_{(w,w)} = \Phi^{\varphi} \cap \Sigma = \left\{ \begin{pmatrix} 1 - t & -t \\ t & 1 + t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} = \Pi.$$

(3.6) For the elements of Ω_o we obtain:

$$\begin{pmatrix} u - iz & i(u - 1/u - iz) \\ z & 1/u + iz \end{pmatrix}^{\varphi} = \begin{pmatrix} u - w^2 z & u - 1/u - w^2 z \\ w^2 z & 1/u + w^2 z \end{pmatrix}.$$

Therefore $\Sigma_{S_o\varphi} = \Omega_o^{\varphi} \cap \Sigma = \Lambda_o$.

(3.7) Finally, we turn the elements of Ω_s:

$$\begin{pmatrix} a & b \\ & 1/a \end{pmatrix}^{\varphi} = \begin{pmatrix} a & \bar{w}^2 b + (1 - i\bar{w}^2)(a - 1/a) \\ & 1/a \end{pmatrix}.$$

If $\bar{w}^2 \in \mathbb{R}$ (i.e. if $w \in \{1, -1, i, -i\}$) then the imaginary part of $\bar{w}^2 b + (1 - i\bar{w}^2)(a - 1/a)$ equals $-i\bar{w}^2(a - 1/a)$ (recall that $a, b \in \mathbb{R}$). If $\begin{pmatrix} a & b \\ & 1/a \end{pmatrix} \in \Sigma$, we infer that $a = 1$. Hence $\Sigma_{S_s\varphi} = \Omega_s^\varphi \cap \Sigma = \Lambda_e$.

If $\bar{w}^2 \in \mathbb{C}\backslash\mathbb{R}$, we write $\bar{w}^2 = x + iy$, where $x, y \in \mathbb{R}$ and $y \neq 0$. Then

$$\bar{w}^2 b + (1 - i\bar{w}^2)\left(a - \frac{1}{a}\right) \in \mathbb{R}$$

if and only if

$$\bar{w}^2 b - i\bar{w}^2\left(a - \frac{1}{a}\right) \in \mathbb{R}.$$

This implies that $yb - x(a - 1/a) = 0$. Hence $b = x/y(a - 1/a)$, and we obtain that

$$\Omega_s^\varphi \cap \Sigma = \left\{ \begin{pmatrix} a & (1 + 1/y)(a + 1/a) \\ & 1/a \end{pmatrix} \middle| a > 0 \right\}$$

(recall that $x^2 + y^2 = (\bar{w}w)^2 = 1$). Using the notation of (1.4), this means that $\Sigma_{S_s\varphi} = \Omega_s^\varphi \cap \Sigma = \Lambda(1 + 1/y)$.

Applying (1.3) and (1.4), we infer from (3.5), (3.6) and (3.7):

(3.8) THEOREM. *Assume that* $\Sigma = \mathrm{SL}_2\mathbb{R}$ *acts effectively and transitively on a 2-dimensional stable plane* $\mathcal{S}$. *If there is a* Σ*-equivariant embedding* $(\pi, \lambda, \mathbb{1})$ *of* $\mathcal{S}$ *into* $\mathcal{S}_{\mathbb{C}}$, *then there is* $w \in \mathbb{C}$ *such that* $w\bar{w} = 1$ *and* (w, w) *belongs to the image of* π. *If* $w^2 \in \mathbb{R}$, *then* $\mathcal{S}$ *is isomorphic with the punctured affine plane over* $\mathbb{R}$; *if* $w^2 \in \mathbb{C}\backslash\mathbb{R}$, *then* $\mathcal{S}$ *is isomorphic with the punctured Strambach plane.*

(3.9) FINAL REMARKS. (a) Since we fix the embedding (and thus the action) of Σ, we obtain the same subgeometry for choices $w \in \{u, v\}$ if, and only if, the point (u, u) is contained in the orbit $(v, v)\Sigma$. Using the fact that Σ acts transitively on $\mathbb{R}^2\backslash\{(0, 0)\}$, one infers that this condition is equivalent to the condition that $u^{-1}v \in \mathbb{R}$. Under the additional assumption that $u\bar{u} = 1 = v\bar{v}$ this amounts to $u = \pm v$.

(b) One can show (see [1]) that for each continuous monomorphism $\iota: \Sigma = \mathrm{SL}_2\mathbb{R} \to \Delta = \mathrm{SL}_2\mathbb{C}$, the image Σ' is a conjugate of Σ in Δ. Thus in (3.2) the assumption that $\gamma = \mathbb{1}$ is not an essential restriction.

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