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A NOTE ON HILBERT AND BELTRAMI SYSTEMS

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Dedicated to Prof. Benno Artmann, on his 60^{th} birthday

ABSTRACT. Hilbert and Beltrami (line-) systems were introduced by H. Mohrmann, Math. Ann. 85 (1922) p.177–183. These systems give examples of non-desarguesian affine planes, in fact, the earliest known examples are of this type. We describe a construction for "generalized Beltrami systems", and show that every such system defines a topological affine plane with point set $\mathbb{R}^2$. Since our construction uses only the topological structure of $\mathbb{R}^2$-planes, it is possible to iterate this process. As an application, we obtain an embeddability theorem for a class of two-dimensional stable planes, including Strambach's exceptional $SL_2\mathbb{R}$-plane

In this paper, we study various types of topological plane geometries. We shall use these to distort the lines of the real affine plane $\mathbf{A}(\mathbb{R}) = (\mathbb{R}^2, \mathrm{aff}_1(\mathbb{R}^2))$. Our construction uses only the topological properties of $\mathbf{A}(\mathbb{R})$ (in particular, the convexity theory for $\mathbb{R}^2$-planes, cf. [14, p.11] and [2, p.57]). Therefore, it applies to arbitrary affine $\mathbb{R}^2$-planes, as defined below (see [4]; these planes are also called "Salzmann-planes" in the literature).

Let us first consider planes on compact disks:

(1) Definition. An incidence geometry $(D, \mathcal{D})$ is called a *compact disk* (or CD, for short) if the following hold:

(1) $(D, \mathcal{D})$ is a linear space. (I.e., the set $\mathcal{D}$ of "lines" consists of subsets of D such that every line $L \in \mathcal{D}$ contains at least two points, and for any two points $x, y \in D$ there is exactly one line $L \in \mathcal{D}$ such that $x, y \in L$.)
(2) The point set D is homeomorphic to the closed unit disk in $\mathbb{R}^2$.
(3) Each line $L \in \mathcal{D}$ is homeomorphic to the closed unit interval. Moreover, the boundary ∂L is contained in ∂D, and $L \setminus \partial D$ is connected.

Note that we do not exclude the case that $L \setminus \partial D$ is empty for some $L \in \mathcal{D}$. There are, in fact, only two possibilities:

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(2) Lemma. *Let $(D, \mathcal{D})$ be a CD, and let $L \in \mathcal{D}$ be a line. Then either $L \subset \partial D$, or $L \cap \partial D = \partial L$. I.e., the incidence structure that is induced on ∂D has lines that are either connected or consist of exactly two points.*

Proof. The closure H of $L \setminus \partial D$ (in D) is an interval that separates D. Assume that $\partial L \neq L \cap \partial D \neq L$. Then $H \neq L$, and there exists some point $x \in L \setminus H \subset \partial D$. Choosing y in the component of $D \setminus H$ that does not contain x, we infer that the line X that joins x and y meets H, since X is connected. This contradicts the assumption that $(D, \mathcal{D})$ is a linear space. □

(3) Corollary. *For every* CD $\mathbb{D} = (D, \mathcal{D})$, *there exists a "model in* $\mathbf{A}(\mathbb{R})$", *i.e., a homeomorphism $\gamma: D \to E$ onto some compact convex subset of $\mathbf{A}(\mathbb{R})$ such that γ induces an isomorphism of the incidence structures that are induced on ∂D (from $\mathbb{D}$) and on ∂E (from $\mathbf{A}(\mathbb{R})$), respectively.*

Proof. We may assume that D is the unit disk in $\mathbb{R}^2$. If a line $L \in \mathcal{D}$ meets ∂D in more than two points, then $L \subseteq \partial D$ by (2). For every $L \subset \partial D$, we replace L by the line segment that joins the endpoints of L in $\mathbf{A}(\mathbb{R})$. Thus we replace the unit circle ∂D by a Jordan curve $J \subset D$. The projection of ∂D onto J from an interior point q of J yields an isomorphism from the geometry that is induced on ∂D from $\mathbb{D}$ onto the geometry that is induced on J from $\mathbf{A}(\mathbb{R})$. This projection extends to a homeomorphism from D onto the closure E of the interior of J, as is easily seen using polar coordinates (with respect to q) and the fact that the distance function is continuous. □

It seems that the geometries that we call CD were already investigated by E. BELTRAMI, and their spatial analogues by F. KLEIN, see [6, §2], cf. also [7, p. 154]. A main motive for our investigations was the wish to make precise (and, thus, accessible to modern mathematicians) some of the ideas of the second half of the 19^{th} century.

Since every compact convex subset $D \subset \mathbb{R}^2$ with non-empty interior is homeomorphic to the closed unit disk, we obtain many examples:

(4) Examples. Let D be a compact convex subset of $\mathbb{R}^2$, and define

$$\mathcal{D} := \left\{ L \cap D; L \in \operatorname{aff}_1(\mathbb{R}^2), |L \cap D| \geq 2 \right\} .$$

Then $(D, \mathcal{D})$ is a CD. In particular, the closure of the BELTRAMI-KLEIN model of the hyperbolic plane is a CD. If D is not strictly convex, we obtain examples with lines $L \in \mathcal{D}$ such that $L \setminus \partial D = \emptyset$.

(5) Proposition. *Assume that $D \subset \mathbb{R}^2$ is a compact convex subset, and that $x \in \mathbb{R}^2 \setminus D$ is a point such that the inversion ι at some circle around x maps D onto a convex set. Then* $\mathbb{D} = (D, \mathcal{D})$ *is a* CD, *where $\mathcal{D}$ consists of the nontrivial intersections with D of circles through x.*

Proof. For any two points $c, d \in D$, there is a unique circle through c, d, x (degenerate, if c, d, x are collinear). Hence $(D, \mathcal{D})$ is a linear space. Since D^ι is convex, we obtain that $(D^\iota, \mathcal{D}^\iota)$ is a CD, cf. (4). □

(6) Definition. Let $(\mathbb{R}^2, \mathcal{L})$ be K. STRAMBACH's $\mathrm{SL}_2\mathbb{R}$-plane, as defined in [15], cf. [12]: The set $\mathcal{L}$ is the union of the set of all ordinary lines through the origin and the set of all images of $H = \left\{ (x, x^{-1}); x > 0 \right\}$ under the usual action of $\mathrm{SL}_2\mathbb{R}$ (i.e., the group of all real 2×2-matrices of determinant 1).

The mapping

$$\rho\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (x,y) \mapsto \frac{(x,y)}{\sqrt{2+x^2+y^2}}$$

maps $\mathbb{R}^2$ onto the open unit disk U. We write D for the closure of U. For each line $L \in \mathcal{L}$, let $\tilde{L}$ denote the closure of L^ρ in D. Finally, let $\mathcal{S} = \{\tilde{L}; L \in \mathcal{L}\}$.

Note that $\tilde{H} = \{(x, 1-x); 0 \le x \le 1\}$, while $\tilde{L}$ is part of a rational curve in general.

(7) Theorem. *The "closed* STRAMBACH*-plane" $(D, \mathcal{S})$ is a* CD.

Proof. Assume that $L \in \mathcal{L}$ is a line of STRAMBACH's plane. Since L is not compact, the image L^ρ is not closed in D. If L passes through the origin, then $L^\rho \subset L$, and $\tilde{L} = L \cap D$. If $L = H^\alpha$ for some $\alpha \in \mathrm{SL}_2\mathbb{R}$, we consider the "asymptotical rays" of H^α; i.e., the half-lines $\{(x,0); x \ge 0\}^\alpha$ and $\{(0,y); y \ge 0\}^\alpha$. The projection $\pi\colon (x,y) \mapsto \{r(x,y); r \ge 0\}$ is a continuous mapping from $\mathbb{R}^2 \setminus \{(0,0)\}$ to the set $\mathcal{R}$ of all rays starting from the origin. Now $H^{\alpha\rho\pi} = H^{\alpha\pi}$ is a proper open interval in $\mathcal{R} \approx \mathbb{S}_1$. Hence $\widetilde{H^\alpha}^\pi \setminus H^{\alpha\pi}$ consists of the two asymptotical rays of H^α. Since $H^{\alpha\rho}$ is closed in U, we conclude that $\widetilde{H^\alpha}$ consists of $H^{\alpha\rho}$ plus the intersection points of the asymptotical rays with ∂D. In particular, we obtain that the point space and each of the lines have the topological properties that are required by Definition (1).

In order to show that $(D, \mathcal{S})$ forms a linear space, let $p, q \in D$ be two points. If both p, q lie in U, then there exists a unique line that contains them, since STRAMBACH's plane is a linear space. So assume that $p \in \partial D$. If $q \in \mathbb{R}p$, then the line $\mathbb{R}p \cap D$ is the unique joining line. If $q \in U \setminus \mathbb{R}p$, then there exists a unique element $\alpha \in \mathrm{SL}_2\mathbb{R}$ that maps $(1,1)$ to $q^{\rho^{-1}}$ and one of the asymptotical rays of H to the ray $\{rp; r \ge 0\}$. Finally, if $q \in \partial D$, then the elements of $\mathrm{SL}_2\mathbb{R}$ that map the asymptotical rays to the rays $\{rp; r \ge 0\}$ and $\{rq; r \ge 0\}$ form a single coset $\Delta\alpha$, where Δ is the stabilizer of H in $\mathrm{SL}_2\mathbb{R}$. In both cases, the line $\widetilde{H^\alpha}$ is the unique joining line. □

If $(D, \mathcal{D})$ is a CD, then the linear space induced on the interior D° is $(D^\circ, \mathcal{D}^\circ)$, where $\mathcal{D}^\circ = \{L \setminus \partial D; L \in \mathcal{D}\} \setminus \{\emptyset\}$. From condition (3) of Definition (1), we infer that $(D^\circ, \mathcal{D}^\circ)$ is an $\mathbb{R}^2$-plane in the sense of [4]:

(8) Definition. An incidence geometry $(U, \mathcal{U})$ is called an $\mathbb{R}^2$*-plane* if the following hold:

(1) $(U, \mathcal{U})$ is a linear space.
(2) The point set U is homeomorphic to $\mathbb{R}^2$.
(3) Each line $L \in \mathcal{U}$ is closed in $\mathbb{R}^2$, and is homeomorphic to $\mathbb{R}$.

If the underlying linear space is an affine plane, we shall speak of an affine $\mathbb{R}^2$-plane.

(9) Examples. The real affine plane $\mathbf{A} = (\mathbb{R}^2, \mathrm{aff}_1(\mathbb{R}^2))$ is an $\mathbb{R}^2$-plane. In analogy with (4), an $\mathbb{R}^2$-plane is induced on every open convex subset $U \subset \mathbb{R}^2$. This includes, in particular, the hyperbolic plane. Numerous other examples have been constructed by H. SALZMANN, K. STRAMBACH, H. GROH, and others, see [14], [4] and the references given there.

There is a convexity theory for $\mathbb{R}^2$-planes, see [14, p.11] and [2, p.57]: Convexity in $(U, \mathcal{U})$ is defined with respect to the lines in $\mathcal{U}$ instead of the ordinary lines of the real affine plane. We shall speak of $\mathcal{U}$-convex sets. Now the fact that the interior of a CD is an $\mathbb{R}^2$-plane has the following immediate consequence (cf. (2)):

(10) Lemma. *Let $(D, \mathcal{D})$ be a CD. Two lines $L, M \in \mathcal{D}$ intersect in an interior point $q \in D^\circ$ if, and only if, none of them is contained in ∂D, and their endpoints separate each other on ∂D (recall that ∂D is homeomorphic to the circle).*

Proof. This follows from the fact that lines in an $\mathbb{R}^2$-plane intersect transversally, see [14, 2.8]. □

Note that (10) implies that, for every CD, the relation "meet in the interior" is just the relation of "separating" for 2-sets in the circle. This fact will be used later on. Our next aim is to show that every CD can be embedded in an affine $\mathbb{R}^2$-plane.

(11) Definition. Let $\mathbf{A} = (\mathbb{R}^2, \mathcal{A})$ be an affine $\mathbb{R}^2$-plane, and let $\mathbb{D} = (D, \mathcal{D})$ be a CD. Moreover, assume that $\gamma: D \to E$ is a homeomorphism onto some $\mathcal{A}$-convex subset $E \subset \mathbb{R}^2$. For every line $L \in \mathcal{D}$, we define $\hat{L} = L^\gamma \cup (K \setminus E)$, where $K \in \mathcal{A}$ joins the endpoints of L^γ. Then the line set $\mathcal{A}_\gamma^{\mathcal{D}} = \{L \in \mathcal{A}; |L \cap E| \le 1\} \cup \{\hat{L}; L \in \mathcal{D}\}$ gives rise to the incidence structure $\mathbf{A}_\gamma^{\mathbb{D}} = (\mathbb{R}^2, \mathcal{A}_\gamma^{\mathcal{D}})$.

(12) Theorem. *Let $\mathbf{A} = (\mathbb{R}^2, \mathcal{A})$ be an affine $\mathbb{R}^2$-plane. Assume that $\mathbb{D} = (D, \mathcal{D})$ is a CD, and that $\gamma: D \to E$ is a homeomorphism of D onto some $\mathcal{A}$-convex subset $E \subset \mathbb{R}^2$ such that γ induces an isomorphism of the incidence structures that are induced on ∂D (from $\mathbb{D}$) and ∂E (from $\mathbf{A}$), respectively. Then $\mathbf{A}_\gamma^{\mathbb{D}}$ is an affine $\mathbb{R}^2$-plane.*

Proof. Let $p, q \in \mathbb{R}^2$ be two points. Existence and uniqueness of a line $L \in \mathcal{A}_\gamma^{\mathcal{D}}$ that joins p and q are obvious if either both $p, q \in E$ or both $p, q \in \mathbb{R}^2 \setminus E^\circ$. So assume that $p \in \mathbb{R}^2 \setminus E$, and $q \in E^\circ$. There exists a line $H \in \mathcal{A}$ such that $H \cap (\{p\} \cup E) = \emptyset$. We fix an ordering on H. For each line L that meets H, we have the notion of the lower and the upper half plane with respect to this ordering on L. We consider the set $q^\downarrow$ of all points $x \in H$ with the property that q does not belong to the lower half plane with respect to the line that joins p and x in $\mathcal{A}_\gamma^{\mathcal{D}}$, and the set $q^\uparrow$ of all points $y \in H$ with the property that q does not belong to the upper half plane with respect to the line that joins p and y in $\mathcal{A}_\gamma^{\mathcal{D}}$. Obviously, $\inf q^\uparrow = \sup q^\downarrow$, and q belongs to the line that joins p and $\inf q^\downarrow$ in $\mathcal{A}_\gamma^{\mathcal{D}}$. According to (10), two lines that meet in q never meet outside E. This implies uniqueness of joining lines between points of $\mathbb{R}^2 \setminus E$ and points in E°. The validity of the parallel axiom for $\mathbf{A}_\gamma^{\mathbb{D}}$ also follows easily from (10). □

Note that the assumption about the isomorphism between ∂D and ∂E holds trivially if $\mathbb{D}$ is "strictly convex", i.e., if $L \cap \partial D = \partial L$ for every $L \in \mathcal{D}$. Moreover, we know from (3) that there always exists a model in $\mathbf{A}(\mathbb{R})$. Hence every CD can be embedded in some affine $\mathbb{R}^2$-plane. From L.A. Skornjakov's result [13], see [14, p.7], we infer:

(13) Corollary. *Each CD is embeddable in a projective plane with the property that the point space is a compact surface (in fact, homeomorphic to the point space of the real projective plane).*

The planes $\mathbf{A}(\mathbb{R})_\gamma^{\mathbb{D}}$ are slight generalizations of Hilbert and Beltrami systems in the sense of [10]: Actually, H. Mohrmann's definitions imply that $\mathbf{A}(\mathbb{R})_\gamma^{\mathbb{D}}$ is a Hilbert system if $\mathbb{D}$ is locally desarguesian, and that $\mathbf{A}(\mathbb{R})_\gamma^{\mathbb{D}}$ is a Beltrami system if there exists no desarguesian neighbourhood for any point of $\mathbb{D}$. Note that, even if one starts with the real affine plane, iteration of the process described in (11) requires the general setting.

(14) Example. Using a CD of the type described in (5), D. HILBERT [5, Kap. V, §23] obtains his *first example of a non-desarguesian plane*[1]. The existence of a point x such that some inversion with center x maps the convex set D onto a convex set is a restriction to the shape of D: if the boundary ∂D is a (piecewise) smooth curve, this is equivalent to the assertion that the curvature of ∂D is bounded below (in this case, choose x such that the distance from x to D is bigger than the inverted infimum, i.e., the supremum over the radii of curvature). In particular, there is no such point x for any polygon D. For any ellipse, however, there exists a suitable x. For his example, D. HILBERT explicitly gives an ellipse and a center of inversion.

(15) Proposition. *If one chooses the closed unit disk for D in (5), and for x any point outside D, then the resulting affine plane $\mathbb{A}(\mathbb{R})^{\mathbb{D}}_{\mathrm{id}}$ is isomorphic to the real affine plane.*

Proof. Let K be the circle with center x that is orthogonal to ∂D. If ι denotes the inversion at K, and σ denotes the (hyperbolic) reflection with center x and axis through the intersection points of K and ∂D, then $\iota\sigma$ fixes ∂D pointwise. Defining φ by $\varphi|_D = \iota\sigma|_D$ and $\varphi|_{\mathbb{R}^2\setminus D} = \mathrm{id}$, we obtain that φ is an isomorphism. □

(16) Examples. In [10], H. MOHRMANN uses rectangles for $D = E$, and gives the system $\mathcal{D}$ explicitly (using parabolas, or exponential curves).

(17) Example. Combining (7), (12), (13), we obtain an embedding of K. STRAMBACH's exceptional $\mathrm{SL}_2\mathbb{R}$-plane in a projective plane whose point space is a surface. This is of particular interest since K. STRAMBACH showed that the $\mathrm{SL}_2\mathbb{R}$-plane does not admit any $\mathrm{SL}_2\mathbb{R}$-*equivariant* embedding in a projective plane of the type mentioned above. The homeomorphism ρ in (6) is choosen in such a way that the action of the subgroup $\mathrm{SO}_2\mathbb{R}$ extends to $\mathbb{A}(\mathbb{R})^{\mathbb{D}}_{\rho}$, i.e., we obtain an $\mathrm{SO}_2\mathbb{R}$-equivariant embedding. Note that $\mathrm{SO}_2\mathbb{R}$ is a maximal subgroup of $\mathrm{SL}_2\mathbb{R}$.

(18) Remark. Corollary (13) gives a criterion for embeddability as an open subplane of an affine or projective plane. There are some criteria that imply *non-embeddability*, see [8, Sect. 5], [16, Sect. 2 and 3].

Recall that a Beltrami system is of the form $\mathbb{A}(\mathbb{R})^{\mathbb{D}}_{\mathrm{id}}$, where $\mathbb{D}$ is a CD with the property that no point has a desarguesian neighbourhood. Since every collineation of an $\mathbb{R}^2$-plane is continuous [14, 3.5], we obtain:

(19) Theorem. *The group of collineations of a Beltrami system $\mathbb{A}(\mathbb{R})^{\mathbb{D}}_{\mathrm{id}}$ leaves D invariant. The same assertion holds for the projective closure of the system.*

(20) Remark. The situation becomes more difficult for Hilbert systems, as is indicated by (15). In particular, a hard part in the determination of the automorphism group of D. HILBERT's first example (cf. (14)) will be to show the invariance of D. Once this has been established, R. LÖWEN's "local Fundamental Theorem" [9] can be used to determine the restriction to the complement of D (and to the interior D°, if one has a Hilbert system). Since the boundary ∂D contains one-dimensional orbits, results of H. GROH [3] may be useful as well.

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[1] Actually, there was an earlier example by D. HILBERT, see [17, p.158]. However, this example never appeared in print. Note the interesting fact that, as early as 1873, F. KLEIN [6, p.135f] seems to use the construction described in (11) to give an example of a non-desarguesian plane.

(21) Conjecture. It is tempting to conjecture that, among the affine $\mathbb{R}^2$-planes, the planes $\mathbb{A}(\mathbb{R})^{\mathbf{D}}_{\gamma}$ can be characterized by the fact that "the distortion is bounded": I.e., the set of points that do not have desarguesian neighbourhoods is contained in some compact set. However, we face the problem that it is not known whether the complement of a compact set in an affine $\mathbb{R}^2$-plane is isomorphic to a subgeometry of $\mathbb{A}(\mathbb{R})$, if it is desarguesian. The results of H. BUSEMANN [2, 11.2, 13.1] and C. POLLEY [11] use the assumption that each line is connected. For locally desarguesian planes, this assumption is not superfluous, as is shown by examples like F.R. MOULTON's famous plane (see D. BETTEN's description [1]), or the example in [16, 3.3].

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