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Compact groups of automorphisms of stable planes

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Abstract. It is shown that compact groups of automorphisms of stable planes are either elliptic motion groups, acting in the usual way, or their dimension is bounded by the dimension of a point stabilizer in the elliptic motion group. Examples show that this bound is sharp.

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1. Introduction

The purpose of this paper is to prove:

Main Theorem. *Let* $\mathbb{M} = (M, \mathscr{M})$ *be a stable plane, where* $\dim M = 2l$, *and let* $\mathrm{P}_2\mathbb{F}$ *be the projective plane over the l-dimensional real alternative division algebra (i.e.,* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, *according to l).*

a) *If* Φ *is a compact connected group of automorphisms of* $\mathbb{M}$, *then* Φ *is isomorphic with the elliptic motion group* E *on* $\mathrm{P}_2\mathbb{F}$, *or* $\dim\Phi \leq \dim \mathrm{E} - \dim M$ *(i.e., the dimension of the stabilizer of a point).*

b) *If* Φ *is (locally) isomorphic with* E, *then* $\mathbb{M} \cong \mathrm{P}_2\mathbb{F}$, *and the action of* Φ *on* $\mathbb{M}$ *is equivalent to the usual action of* E *on* $\mathrm{P}_2\mathbb{F}$.

Assertion b) has been proved by R. Löwen [17]. We state it for the sake of completeness.

In this first section, we introduce stable planes, their automorphism groups and certain types of subplanes. Section 2 collects general results on compact groups and their possible actions on stable planes, while sections 3–5 contain the proof of the Main Theorem. Section 6 gives a brief overview of the known examples and indicates some application in the study of stable planes with large (not necessarily compact) groups of automorphisms. The last section, included for the reader's convenience, is an appendix comprising information about compact groups.

(1.1) Definition. A *stable plane* is a linear space $\mathbb{M} = (M, \mathcal{M})$, where the point space M and the line space $\mathcal{M}$ are endowed with locally compact Hausdorff topologies such that

- the geometric operations (joining points, intersecting lines) are continuous,
- the set of pairs of intersecting lines is open in $\mathcal{M} \times \mathcal{M}$ (axiom of stability),
- the point space M has positive and finite covering dimension.

We shall identify each line with the set of points that are incident with it. Thus "$\in$" denotes incidence.

Endowed with the compact-open topology derived from the action on M, the group $\mathrm{Aut}(\mathbb{M})$ of all automorphisms (i.e., continuous collineations) of $\mathbb{M}$ is a locally compact transformation group both on M and $\mathcal{M}$ (see [7: 2.3, 2.9]). By an *action* of a locally compact group Δ on $\mathbb{M}$ we mean a continuous group homomorphism from Δ to $\mathrm{Aut}(\mathbb{M})$. If this homomorphism is injective, we call Δ a *group of automorphisms of* $\mathbb{M}$. Exactly the closed subgroups of $\mathrm{Aut}(\mathbb{M})$ are locally compact (with respect to the induced topology). However, there may be continuous injective group homomorphisms from locally compact groups into $\mathrm{Aut}(\mathbb{M})$ such that the image is not a closed subgroup (e.g., one-parameter subgroups, or Levi complements in Lie groups). Apart from compactness criteria, there do not arise substantial additional problems if one considers actions in this wider sense. By arguments similar to those in [30: 3.2], even results that use compactness criteria may be transferred. Of course, any (continuous) action of a compact group has closed image in $\mathrm{Aut}\,\mathbb{M}$).

A stable plane $\mathbb{M}$ is *projective* (i.e., any two lines meet) if, and only if, the set M of points is compact [7: 1.27]. In this case, the point set M is connected. The compact connected projective planes admitting a point-transitive group Γ of automorphisms have been determined [23], [10]: such a plane is isomorphic with the projective plane $\mathrm{P}_2\mathbb{F}$, where $\mathbb{F}$ is one of the real alternative division algebras (i.e., the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$ or Cayley's octonions $\mathbb{O}$). Moreover, the group Γ contains a so-called *elliptic motion group* E of $\mathrm{P}_2\mathbb{F}$ that acts flag-transitively. This group E is a maximal compact subgroup of the identity component of $\mathrm{Aut}(\mathrm{P}_2\mathbb{F})$. The maximal compact subgroups of a connected locally compact group form a single conjugacy class, so we will talk about *the* elliptic motion group of $\mathrm{P}_2\mathbb{F}$ (by a slight abuse of language). The elliptic motion groups are $\mathrm{PSO}_3\mathbb{R}$, $\mathrm{PSU}_3\mathbb{C}$, $\mathrm{PSU}_3\mathbb{H}$ and the compact real form of the exceptional Lie group F_4, according to $\mathbb{F}$.

(1.2) Remarks. General information about stable planes can be found in the work of R. Löwen (see, e.g., [7], [9], [13]). For our purposes, the following properties are of particular importance (see [13]):

a) The only possible values for $\dim M$ are the integers 2, 4, 8, and 16.

b) For each point $p \in M$, the line pencil $\mathcal{M}_p = \{L \mid p \in L \in \mathcal{M}\}$ is a compact connected homotopy l-sphere, where $l = \dim \mathcal{M}_p = \dim L = \frac{1}{2} \dim \mathcal{M}$.

There are several notions of dimension for topological spaces, see, e.g., [20]. According to results of R. Löwen [13], the spaces M, $\mathcal{M}$ are separable metric spaces for each stable plane $\mathbb{M} = (M, \mathcal{M})$, hence the most commonly used dimension functions (e.g., dim, ind, Ind. ...) coincide on these spaces. The same applies to each line L (considered as a subset of M), each line pencil $\mathcal{M}_x$, and each group Δ of automorphisms of $\mathbb{M}$. If one of these spaces is a topological manifold, i.e., locally homeomorphic with $\mathbb{R}^n$ for some n, then its topological dimension equals n.

The restriction on the dimension of the point space suggests an inductive treatment. For this purpose, we need suitable notions of *subplanes*.

(1.3) Definitions. For any subset D of M, let $\mathcal{D}$ be the set of lines that are incident with at least two points of D. Then $\mathbb{D} = (D, \mathcal{D})$ is called *the geometry induced on D*. If D contains a quadrangle (i.e., four points such that no three of them are collinear), then $\mathbb{D}$ is called a *subplane* of $\mathbb{M}$. A subplane $\mathbb{D} = (D, \mathcal{D})$ is called *full* (*in* $\mathbb{M}$), if D contains each point of M that is the intersection of two lines of $\mathcal{D}$. A subplane is called *open*, *closed*, *d-dimensional* etc., if its point space has the property in question.

There are two important cases where a subplane is a stable plane: if the subplane is open, or if it is a closed full subplane of positive dimension.

The open subplanes of compact connected projective planes form a large class of examples of stable planes. There are, however, stable planes that do not admit open embeddings into projective planes (see [32]).

Note that, for each full subplane $\mathbb{D} = (D, \mathcal{D})$, the geometry induced on the closure of D is a closed full subplane. If a subset X of M contains a quadrangle, then there is a smallest closed full subplane $\langle X \rangle$ of M, called *the subplane generated by X*. There is a description of $\langle X \rangle$ that ensures that $\langle X \rangle$ is invariant under each automorphism α of $\mathbb{M}$ that leaves X invariant, and α acts trivially on $\langle X \rangle$ is invariant under each automorphism α of $\mathbb{M}$ that leaves X invariant, and α acts trivially on $\langle X \rangle$ if the action of α on X is trivial (see [30: 3.1]).

A closed full subplane $\mathbb{B} = (B, \mathcal{B})$ of $\mathbb{M}$ is called a *Baer subplane*, if $\dim B = \frac{1}{2} \dim M$. If α is an automorphism of $\mathbb{M}$ such that the set $\mathrm{Fix}(\alpha)$ of fixed points of α carries a Baer subplane, then α is called a *Baer collineation*.

(1.4) Definitions. Let Δ be a group of automorphisms of a stable plane $\mathbb{M} = (M, \mathcal{M})$.

a) [30] The group Δ is called *planar*, if the set $\mathrm{Fix}(\Delta)$ of fixed points has positive dimension and contains a quadrangle.

b) [31] If each orbit x^Δ is contained in a line L_x, then Δ is called *quasi-perspective*, and its elements are called *quasi-perspectivities*. In this situation, the line L_x is uniquely determined for each $x \in M \setminus \mathrm{Fix}(\Delta)$, and we shall write $\mathscr{L}_\Delta = \{L_x \mid x \in M \setminus \mathrm{Fix}(\Delta)\}$.

c) If Δ acts trivially on $\mathscr{M}_z$ for some point $z \in M$, then z is called the *center* of Δ.

d) If Δ acts trivially on some line $L \in \mathscr{M}$, then L is called an *axis* of Δ.

It is easy to see that the geometry induced on the set of fixed points of a planar group is a full closed subplane $\mathbb{F} = (F, \mathscr{F})$, where $\dim F > 0$. Planar groups occur quite naturally in the following situations:

(1.5) Lemma. *Let Δ be a connected non-trivial group of automorphisms of a stable plane $\mathbb{M} = (M, \mathscr{M})$.*

a) *If there is a point $x \in M$ such that the orbit x^Δ is not contained in any line, then x^Δ generates a subplane $\mathbb{E} = (E, \mathscr{E})$, where $\dim E \geq \dim x^\Delta > 0$. For each subgroup Ψ of $\mathrm{Aut}(\mathbb{M})$ that commutes with Δ, the stabilizer Ψ_x acts trivially on x^Δ and is therefore a planar group.*

b) *If there is no such orbit, then Δ is quasi-perspective. Then for any two points $x \in M \setminus \mathrm{Fix}(\Delta)$ and $y \in M \setminus (\mathrm{Fix}(\Delta) \cup L_x)$, the set $x^\Delta \cup y^\Delta$ generates a subplane $\mathbb{E} = (E, \mathscr{E})$, where $\dim E \geq 2 \dim x^\Delta > 0$. For each subgroup Ψ of $\mathrm{Aut}(\mathbb{M})$ that commutes with Δ, the stabilizer $\Psi_{x,y}$ is a planar group.*

The proof is straightforward.

2. General results

In this section, we collect some basic results that will be needed during the proof of our Main Theorem. The items called "result" are quoted from the literature (adapted to our notation). We omit their proofs but give references.

A. The structure of compact connected groups

Since in any stable plane the stabilizer of a suitable degenerated quintangle has finite dimension [30: 3.4], the full group of automorphisms has finite dimension as well. The following result is well known:

(2.1) Result. (A. Weil [34: § 25], cf. [3: App. I, no. 3, Prop. 2]) *Let Φ be a compact connected group of finite dimension. Then there exist a compact connected abelian group C, almost simple compact Lie groups $S_1, \dots, S_n$ and an epimorphism*

$$\eta : C \times S_1 \times \dots \times S_n \to \Phi \quad \textit{with} \quad \dim \ker \eta = 0.$$

The image C^η is the identity component of the center of Φ, and the commutator group Φ' equals $(S_1 \times \ldots \times S_n)^\eta$.

In particular, a compact connected finite-dimensional group whose center has dimension 0 is a Lie group. More generally:

(2.2) Lemma. *Let Φ be a compact connected group of finite dimension. Each subgroup of Φ that has trivial intersection with the identity component of the center of Φ is a Lie group.*

Proof. Let Z be the identity component of the center of Φ, and assume that Δ is a subgroup of Φ such that $\Delta \cap Z = \mathbb{1}$. Then $\Delta \cong \Delta/\Delta \cap Z \cong \Delta Z/Z \leq \Phi/Z \cong \Phi'/\Phi' \cap Z$. Consequently, the group Δ is isomorphic with a closed subgroup of the Lie group $\Phi'/\Phi' \cap Z$. □

(2.3) Notation. In this paper, a compact connected almost simple Lie group Φ is called *of type* X_n, if the Lie algebra of Φ is the compact real form of the complex simple Lie algebra of type X_n. Besides the exceptional types G_2, F_4, E_6, E_7, E_8, there are the series A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$ and D_n $(n \geq 4)$. More concrete information can be found in the appendix.

(2.4) Result. *Let Φ be a compact connected group acting effectively and transitively on an n-dimensional space X.*

a) (D. Montgomery, L. Zippin [19], cf. [18: Cor. to Th. 2])

The dimension of Φ is at most $\dfrac{n(n+1)}{2}$

b) (L.N. Mann [18: Th. 5]) *If the dimension of Φ falls into one of the ranges*

$$\frac{(n-k)(n-k+1)}{2} + \frac{k(k+1)}{2} < \dim \Phi < \frac{(n-k+1)(n-k+2)}{2}, \quad 1 \leq k < n,$$

then there exist only three possibilities:

i) $n = 4$ *and* $\Phi \cong \mathrm{PSU}_3\mathbb{C}$, *acting on* $\mathrm{P}_2\mathbb{C}$.
ii) $n = 6$ *and* Φ *is of type* G_2, *acting on* $\mathrm{P}_6\mathbb{R}$ *or* $\mathbb{S}_6$.
iii) $n = 10$ *and* $\Phi \cong \mathrm{PSU}_6\mathbb{C}$, *acting on* $\mathrm{P}_5\mathbb{C}$.

For small values of n, we list the excluded dimensions explicitly in the appendix.

(2.5) Result. (H.R. Halder [6]) *Let Δ be a locally compact transformation group on a separable metric space. Then the dimensions of orbits and stabilizers are related in the following way:*

$$\dim x^\Delta = \dim \Delta/\Delta_x = \dim \Delta - \dim \Delta_x .$$

In general, the orbit x^Δ and the coset space Δ/Δ_x are not homeomorphic. If x^Δ is locally compact (in particular, if Δ is compact), the canonical bijection is a homeomorphism.

B. Basis results on compact groups of automorphisms of stable planes

(2.6) Result. (cf. [31: Th. 5]) *Let $\delta \neq \mathbb{1}$ be a quasi-perspectivity of prime order, e.g., an involution. Then one (and only one) of the following holds:*

a) *δ is free, i.e.,* Fix $(\delta) = \emptyset$,
b) *δ is a Baer collineation,*
c) *δ has a center or an axis (or both).*

(2.7) Result. ("Triangle Lemma" [31: Cor. 10]) *Let $\mathbb{M}$ be a stable plane, and let Φ and Δ be subgroups of* $\mathrm{Aut}(\mathbb{M})$ *such that Φ fixes a point x and Δ fixes a triangle pointwise.*

a) *If there are three commuting involutions in Φ, then at least one of them has no axis through x.*

b) *If there are four commuting involutions in Δ, then at least one of them is a Baer collineation.*

c) *If there are three commuting involutions in Φ such that each of them has center and axis, then the centers form a non-degenerate triangle.*

d) *Let α, β, γ be three commuting involutions in Δ. If none of them is a Baer collineation, then $\gamma = \alpha\beta$, and $\langle\alpha, \beta\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

(2.8) Corollary. *Let Λ be a non-trivial compact connected Lie group acting almost effectively on a stable plane in such a way that each involution in Λ has the same axis A. Then Λ has rank 1. In particular,* $\dim \Lambda \leq 3$.

(2.9) Result. (R. Löwen [14]) *Let $\mathbb{M} = (M, \mathcal{M})$ be a stable plane, and let Γ be a group of automorphisms of $\mathbb{M}$. If there are two points $x_1, x_2 \in M$ such that the stabilizer Γ_{x_i} acts transitively on the line pencil $\mathcal{M}_{x_i}$ (for $i \in \{1, 2\}$), then $\mathbb{M}$ contains a flag homogeneous open subplane $\mathbb{E}$. This subplane is isomorphic with the elliptic, hyperbolic or euclidean plane (of the adequate dimension). In particular, $\mathbb{M}$ is isomorphic with an open subplane of* $\mathrm{P}_2\mathbb{F}$ *(where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, according to* $\dim M$*), and Γ is isomorphic with a subgroup of* $\mathrm{Aut}(\mathrm{P}_2\mathbb{F})$.

(2.10) Lemma. *Let Δ be a compact connected group of automorphisms of a stable plane $\mathbb{M} = (M, \mathcal{M})$.*

a) *If there is a point x such that* $\dim x^\Delta = \dim M$*, then $\mathbb{M}$ is isomorphic with the corresponding compact projective Moufang plane (i.e., the plane over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, according to* $\dim M$*), and Δ is isomorphic with the corresponding elliptic motion group* E *(i.e.,* $\mathrm{PSO}_3\mathbb{R}$, $\mathrm{PSU}_3\mathbb{C}$, $\mathrm{PSU}_3\mathbb{H}$*, or F_4, respectively). Moreover, the action of Δ on $\mathbb{M}$ is equivalent to the usual action of* E.

b) *If Δ fixes a point $x \in M$ and there is a line L through x such that $l = \dim \mathcal{M}_x = \dim L^\Delta$, then $\mathcal{M}_x$ is homeomorphic with the sphere $\mathbb{S}_l$, and Δ is a two-fold covering group of* $\mathrm{SO}_{l+1}\mathbb{R}$ *(i.e.* $\mathrm{SO}_2\mathbb{R}$ *for $l = 1$, and* Spin_{l+1} *for $l > 1$). Moreover, the action of Δ on $\mathcal{M}_x$ is equivalent to the usual (linear, almost effective) action on $\mathbb{S}_l$. In particular, the central involution of Δ has center x.*

c) *If Δ fixes a line L and there is a point $x \in L$ such that $l = \dim L = \dim x^\Delta$, then L is homeomorphic with the sphere $\mathbb{S}_l$, and Δ is a two-fold covering group of* $\mathrm{SO}_{l+1}\mathbb{R}$. *Moreover, the action of Δ on L is equivalent to the usual action on $\mathbb{S}_l$. In particular, the line L is a projective line (i.e., it meets each other line), and the central involution of Δ has axis L.*

Proof. According to [13: Th. 11c)], compact orbits of full dimension are open in M, $\mathcal{M}_x$, L, respectively.

i) Assume that $\dim x^\Delta = \dim M$. For each point $y \in M \setminus \{x\}$, there is some $\delta \in \Delta$ such that x^δ is not incident with the line xy. Moreover, the intersection with x^Δ is open in xy and $x^\delta y$, respectively. Hence both lines are lines of the geometry induced on x^Δ. But this geometry is a projective plane by [7: 1.27], and $y = xy \wedge x^\delta y \in x^\Delta$. Therefore Δ acts transitively on M, and [23] or [10] completes the proof of a).

ii) In the situation of b), the orbit L^Δ is a compact open of the connected line pencil $\mathcal{M}_x$. We conclude that acts transitively on $\mathcal{M}_x$, and b) follows from [14: 3.11].

iii) Assume the situation of c). The set $X = L \setminus x^\Delta$ is closed in L, hence closed in M. Thus $\mathbb{M} = (M \setminus X, \mathcal{M})$ is a stable plane, and x^Δ is a compact line of $\tilde{\mathbb{M}}$. According to [7: 1.15], each line $H \in \mathcal{M}$ meets x^Δ. This implies that each line through $y \in X$ has at least two points with L in common. Consequently, the set X is empty, and Δ acts transitively on L. Applying b) to the opposite plane $\mathbb{M}^*$ (cf. [9: 1.1]), we infer that c) holds. □

(2.11) Corollary.

a) *If a compact group $\Psi \leq \mathrm{Aut}(\mathbb{M})$ centralizes a quasi-perspective group Θ, then there are points x and y such that the set $x^\Theta \cup y^\Theta$ generates a subplane of positive dimension and* $\dim \Psi/\Psi_{x,y} \leq (2l-1) + (l-1) + (l-1) = 4l - 3$.

b) *Each non-simple compact group on a stable plane has only orbits of dimension less than the dimension of the point set.*

Proof. The bound for the dimension follows from the fact that the compact orbits x^Ψ, $(xy)^{\Psi_x}$ and $y^{\Psi_{xy}}$ are not open in M, $\mathcal{M}_x$ and xy, respectively. Lemma (1.5) yields the rest of assertion a).

For each $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the elliptic motion group E of $\mathrm{P}_2\mathbb{F}$ is a simple group. Using the fact that each automorphism of $\mathrm{P}_2\mathbb{F}$ fixes a point (cf. [13: Cor. 6]), one infers that the centralizer of E in Aut $(\mathrm{P}_2\mathbb{F})$ is trivial. Together with (2.10, a), this yields assertion b). □

(2.12) Results. *Let* Δ *be a compact connected planar group on* $\mathbb{M}$.

a) (R. Löwen) [8: 1.6]) *If* $\dim M \leq 4$, *then* $\Delta = \mathbb{1}$.

b) ([30: 6.13, a)]) *If* $\dim M = 8$, *then* $\Delta \cong \mathrm{SO}_3\mathbb{R}$, $\Delta \cong \mathrm{SO}_2\mathbb{R}$, $\Delta = \mathbb{1}$, *or* Δ *is an abelian non-Lie group. In any case,* $\dim \Delta \leq 3$.

c) ([30: 6.13, b)]) *If* $\dim M = 16$, *then* Δ *is of type* G_2, *or* Δ *is isomorphic with one of the groups* $\mathrm{SU}_3\mathbb{C}$, $\mathrm{SO}_4\mathbb{R}$, $\mathrm{U}_2\mathbb{C}$, $\mathrm{SU}_2\mathbb{C}$, $\mathrm{SO}_3\mathbb{R}$, $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$, $\mathrm{SO}_2\mathbb{R}$, $\mathbb{1}$, *or* Δ *is a non-Lie group of dimension* ≤ 7.

d) ([30: 6.3, 6.8]) *If the set of fixed points of* Δ *carries a Baer subplane of* $\mathbb{M}$, *then* $\dim M = 8$ *and* $\dim \Delta \leq 1$, *or* $\dim M = 16$ *and* $\dim \Delta \leq 3$, *or* $\dim M = 16$ *and* Δ *is a non-Lie group (and* $\dim \Delta \leq 7$*).*

e) ([30: 6.10]) *If* Δ *is abelian, then* $\dim \Delta < \frac{1}{2} \dim M$.

(2.13) Corollary. *If, in particular,* $\dim \Delta > 8$, *then* Δ *is of type* G_2 *and* $\dim \mathrm{Fix}(\Delta) = 2$.

Proof. According to (2.12, c), the group Δ is of type G_2. Assume that fim $\mathrm{Fix}(\Delta) \geq 4$. For each point $x \in M \backslash \mathrm{Fix}(\Delta)$, the subplane generated by $\mathrm{Fix}(\Delta) \cup \{x\}$ has dimension at least 8. Since Δ is a Lie group, we infer from (2.12, d) that $\dim \Delta_x \leq 3$, and obtain that $\dim \Delta < \dim \mathsf{G}_2$, a contradiction.

(2.14) Lemma. *Let* Θ *be a compact connected quasi-perspective group on a stable plane* $\mathbb{M} = (M, \mathcal{M})$.

a) *Each point orbit has dimension strictly less than* $\frac{1}{2} \dim M$.

b) *If* $\dim M = 2$, *then* $\Theta = \mathbb{1}$.

c) *If* $\dim M = 4$, *then* Θ *is abelian, and* $\dim \Theta \leq 2$.

d) *If* Θ *is almost simple, then* $\dim M = 8$ *and* $\dim \Theta = 3$, *or* $\dim M = 16$, *and* $\dim \Theta \leq 28$.

Proof. i) Assume that there is a point x such that $\dim x^\Theta = \frac{1}{2} \dim M$. Then $\dim x^\Theta = \dim L_x$, and $L_x = x^\Theta$ meets each of the lines L_y by (2.10, c). But the intersection of two lines in $\mathcal{L}_\Theta$ is a fixed point of Θ and cannot belong to x^Θ. This proves a).

ii) Assertion b) is an immediate consequence of a).

iii) If Θ contains an almost simple group Δ, then each non-trivial orbit x^Δ is at least 2-dimensional (2.4). If $\dim M = 4$, we infer from a) that Θ is abelian (cf. (2.1)). According to (2.12, a) and (1.5), there are points x, y such that the stabilizer $\Theta_{x,y}$ is trivial. This implies that $\dim \Theta \leq 2$.

iv) Assertion d) follows from a) and (2.4). □

(2.15) Corollary. *Let Δ be a compact connected almost simple group on a stable plane $\mathbb{M} = (M, \mathscr{M})$. If Δ fixes a point x, then $2l = \dim M \geq 4$, and $\dim \Delta \leq \frac{(l-1)l}{2}$, or $\Delta \cong \mathrm{Spin}_{l+1}$ and Δ acts transitively on $\mathscr{M}_x$.*

Proof. If Δ acts trivially on $\mathscr{M}_x$, then Δ is quasi-perspective and the assertion follows from (2.14, a) and (2.4). If Δ acts non-trivially on $\mathscr{M}_x$ then it acts almost effectively on a non-trivial orbit in $\mathscr{M}_x$, and (2.4) and (2.10, b) give the result. □

(2.16) Lemma. *Let Δ be a compact abelian group on a stable plane $\mathbb{M} = (M, \mathscr{M})$. Then $\dim \Delta < \dim M$. If $\dim \Delta = \dim M - 1$, then Δ acts freely on some point orbit.*

Proof. It suffices to consider the case where Δ is connected. Assume first that there is a point $x \in M$ such that x^Δ is not contained in any line. In this case, the orbit x^Δ generates a non-degenerates subplane $\mathbb{E}$. The stabilizer Δ_x acts trivially on $\mathbb{E}$ and is therefore a planar group. If $\mathbb{M} = \mathbb{E}$, then $\Delta_x = \mathbb{1}$ and $\dim \Delta \leq \dim M - 1$. If $\mathbb{E}$ is a proper subplane, we have that $\dim \Delta/\Delta_x < \frac{1}{2}\dim M$, and $\dim \Delta_x < \frac{1}{2}\dim M$ by (2.12, e). Hence $\dim \Delta \leq \dim M - 2$.

Now assume that Δ is quasi-perspective. Choose points $x \in M \setminus \mathrm{Fix}(\Delta)$ and $y \in M \setminus (\mathrm{Fix}(\Delta) \cup L_x)$. Then the set $x^\Delta \cup y^\Delta$ generates a subplane $\mathbb{E}$, and $\Delta_{x,y}$ is a planar group. If $\mathbb{E} = \mathbb{M}$, then $\Delta_{x,y}$ is trivial, and $\dim \Delta \leq \dim M - 2$ by (2.14). If $\mathbb{E}$ is a proper subplane, then $\dim \Delta/\Delta_z < \frac{1}{4}\dim M$ for $z \in \{x, y\}$ and $\dim \Delta_{x,y} < \frac{1}{2}\dim M$ by (2.12, e). Hence $\dim \Delta \leq \dim M - 3$, and the lemma is proved. □

3. Stable planes of low dimension

(3.1) Theorem. *Let Φ be a compact connected group of automorphisms of a 2-dimensional stable plane $\mathbb{M}$. Then $\Phi \cong \mathrm{SO}_2\mathbb{R}$, or $\Phi \cong \mathrm{SO}_3\mathbb{R}$. In the latter case, the plane $\mathbb{M}$ is isomorphic with the projective plane over $\mathbb{R}$, and the action of Φ is equivalent to the usual one.*

Proof. According to (2.16), each abelian compact subgroup of $\mathrm{Aut}(\mathbb{M})$ has dimension at most 1. Since each almost simple Lie group of rank r contains a group that is isomorphic with $\mathbb{T}^r$, we conclude that either Φ is abelian, or Φ is almost simple of rank 1. In the latter case, each non-trivial orbit has dimension 2, and the assertion follows from (2.10). □

(3.2) Theorem. *Let Φ be a compact connected group of automorphisms of a 4-dimensional stable plane $\mathbb{M}$. Then $\dim \Phi \leq 4$, or $\Phi \cong \mathrm{PSU}_3\mathbb{C}$. In the latter case, the plane $\mathbb{M}$ is isomorphic with the projective plane over $\mathbb{C}$, and the action of Φ is equivalent to the usual one.*

Proof. Semi-simple groups of automorphisms of 4-dimensional stable planes have exhaustively been studied by R. Löwen [8]: such groups are always almost simple, and the only possible compact groups are $SO_3\mathbb{R}$, $SU_2\mathbb{C}$ and $PSU_3\mathbb{C}$. The first two groups have dimension 3, the latter is the elliptic motion group.

We may therefore assume that the identity component Z of the center of Φ is not trivial. If there is a point x such that x^{Z} is not contained in any line, then the stabilizer Φ_x has dimension 0 by (2.12, a) and $\dim\Phi \leq 3$ by (2.11). So assume that Z is quasi-perspective. If there is a point x such that $\dim Z_x = 0$, then $\dim Z = 1$. Since Φ' cannot be the elliptic motion group, we know that $\dim\Phi' \leq 3$ and $\dim\Phi = \dim\Phi' + \dim Z \leq 4$.

If $\dim Z_x > 0$ for each point x, then x^{Φ} is contained in L_x for each x (otherwise, Z_x would be planar, in contradiction to (2.12, a)). Hence Φ is quasi-perspective, and $\Phi = Z$ by (2.14). Now $\dim\Phi \leq 2$ by (2.14).

(3.3) Remark. More details on the actions of $SO_3\mathbb{R}$ and Spin_3 on 4-dimensional stable planes (depending on the fact that no connected group of automorphisms of a 4-dimensional stable plane contains a planar involution) can be found in the work of R. Löwen, see [15], [16].

4. Stable planes of dimension 8

The result of this section has been proved in [29: Th. A], using Richardson's classification of actions of compact groups on the 4-sphere. The study of the case of 16-dimensional planes required different methods, since there is no such classification for the 8-sphere. Now the methods that were developped to treat the 16-dimensional case apply to the lower dimensions, yielding simpler proofs. We give this alternate proof in the sequel.

We distinguish three cases, according to the structure of the group.

A. Almost simple groups

(4.1) Lemma. *Let Φ be a compact connected almost simple group of type A_3. Then each subgroup $\Delta \leq \Phi$ with $\dim\Delta \geq 8$ contains a group of type A_2 or B_2.*

Proof. According to [2], a group of type A_n has no proper semi-simple subgroup of maximal rank. Therefore the rank of Δ' is at most 2. If this rank is less than 2, then $\dim\Delta \leq 5$, in contradiction to the hypothesis. So Δ' has rank 2, and $\dim\Delta' \geq \dim\Delta - 1 \geq 7$. This yields that Δ' is almost simple (cf. (7.3)), hence Δ' is of type A_2, B_2 or G_2. The last case is excluded by (2.4) since $\dim\mathsf{G}_2 = 14$.

(4.2) Lemma. *Let $\mathbb{M} = (M, \mathscr{M})$ be a stable plane. If* $\mathrm{Aut}(\mathbb{M})$ *has a compact connected almost simple subgroup of type G_2 or A_3, then* $\dim M = 16$.

Proof. According to (3.1) and (3.2), we may assume that $\dim M = 8$. Let $\Phi \leq \mathrm{Aut}(\mathbb{M})$ be a compact connected almost simple group, and choose a point x that is not fixed by Φ.

i) If Φ is of type G_2, then $\dim \Phi_x \geq 7$. Therefore the identity component Δ of Φ_x has rank 2 and is almost simple (cf. (7.3)). Since there is no proper subgroup of dimension greater than 9 (cf. (2.4)), we conclude that Δ is of type A_2 in contradiction to (2.15).

ii) If Φ is of type A_3, then $\dim \Phi_x \geq 8$. According to (4.1), the stabilizer Φ_x contains a subgroup Δ of type A_2 or B_2. From (2.15) we infer that $\Delta \cong \mathrm{Spin}_5$. Now Φ_y acts transitively on $\mathcal{M}_y$ for each point $y \in x^{\Phi}$, and by (2.9) the group Φ is a subgroup of the elliptic motion group $\mathrm{E} = \mathrm{PSU}_3\mathbb{H}$. But E is of type C_3 and has no subgroups of type A_3 (cf. [2]).

(4.3) Proposition. *If* Φ *is a compact connected almost simple group of automorphisms of an 8-dimensional stable plane* $\mathbb{M}$, *then* $\dim \Phi \leq 10$, *or* Φ *is isomorphic with the elliptic motion group* $\mathrm{PSU}_3\mathbb{H}$.

Proof. If Φ has rank greater than 2, then Φ contains a subgroup of type A_3, B_3, or C_3. The first two cases are excluded by (4.2), note that B_3 contains G_2. The last case implies that Φ is the elliptic motion group. Almost simple groups of rank ≤ 2 are of type G_2, or their dimension is at most 10. Since the first case is excluded by (4.2), the assertion follows. □

B. Semi-simple groups

(4.4) Proposition. *Let* Δ *be a compact connected group of automorphisms of an 8-dimensional stable plane* $\mathbb{M}$, *and assume that* Δ *is semi-simple, but not almost simple. Then* $\dim \Delta \leq 13$. *If* $\dim \Delta = 13$, *then* Δ *is a product of a factor of type* A_1 *(acting quasi-perspectively) and a factor of type* B_2.

Proof. Let Θ be an almost simple factor of Δ such that $\dim \Theta$ is minimal. Write Ψ for the identity component of $\mathrm{C}_\Delta(\Theta)$. Then Ψ is the product of all almost simple factors except Θ. In particular, $\dim \Theta \leq \dim \Psi$.

Assume first that there is a point x such that the orbit x^Θ is not contained in any line. The stabilizer Ψ_x acts trivially on the subplane $\mathbb{E}$ generated by x^Θ. There are the following cases:

i) $1 < \dim \Psi_x \leq 3$
According to (2.12, d), the subplane $\mathbb{E}$ is 2-dimensional. By (3.1), the group Θ induces the elliptic motion group on $\mathbb{E} \cong \mathrm{P}_2\mathbb{R}$. In particular, $\dim \Theta = 3$ and the stabilizer Θ_x is not trivial. Therefore $\dim x^\Psi \leq 4$, and $\dim \Delta = \dim \Theta + \dim x^\Psi + \dim \Psi_x \leq 3 + 4 + 3 = 10$.

ii) $\dim \Psi_x = 1$
In this case, the group Ψ has dimension at most 8 by (2.11, b). If $\dim \Theta = 8$, then Θ induces the elliptic motion group on $\mathbb{E} \cong P_2\mathbb{C}$. Again, the stabilizer Θ_x is not trivial, and $\dim \Psi \leq 5$, a contradiction. Therefore $\dim \Theta = 3$ and $\dim \Delta \leq 11$.

iii) $\dim \Psi_x = 0$
Since there are no semi-simple groups of dimension 7, this yields that $\dim \Psi \leq 6$. Consequently, $\dim \Theta = 3$ and $\dim \Delta \leq 9$.

Now assume that Θ is quasi-perspective. From (2.14, d) we obtain that $\dim \Theta = 3$. Let x be a point that is moved by Θ. If an involution $\alpha \in \Psi_x$ is planar, then Θ acts as a quasi-perspective almost simple group on the subplane induced on $\mathrm{Fix}(\alpha)$. This is impossible by (2.14, d). Consequently, each involution in Ψ_x has axis L_x. By (2.7) the identity component of Ψ_x has rank 1. Hence $\dim \Psi_x \leq 3$, and $\dim \Delta \leq 13$. Equality holds only if $\dim \Psi = 10$. The only semi-simple groups of dimension 10 are the almost simple groups of type B_2. This completes the proof of the proposition. □

C. Groups that are not semi-simple

Recall from (2.1) that a connected compact group is not semi-simple if, and only if, the identity component of its center is not trivial.

(4.5) Lemma. *If the identity component* Z *of the center of a compact group* Φ *of automorphisms of an 8-dimensional stable plane* $\mathbb{M}$ *is not quasi-perspective, then* $\dim \Phi \leq 10$.

Proof. Let x be a point such that x^{Z} is not contained in any line. Then Φ_x is a planar group, hence $\dim \Phi_x \leq 3$, and $\dim \Phi \leq 10$. □

(4.6) Lemma. *Let* Φ *be a compact group of automorphisms of an 8-dimensional stable plane* $\mathbb{M}$*, and assume that the identity component* Z *of the center of* Φ *has dimension at least* 2. *Then* $\dim \Phi \leq 13$.

Proof. By (4.5) we may assume that Z is quasi-perspective. Choosing points $x \in M \setminus \mathrm{Fix}(\mathrm{Z})$ and $y \in M \setminus (\mathrm{Fix}(\mathrm{Z}) \cup L_x)$, we obtain that the set $x^{\mathrm{Z}} \cup y^{\mathrm{Z}}$ generates a subplane $\mathbb{E} = (E, \mathscr{E})$. From (2.14, b) we infer that $\dim E \geq 4$, and $\dim \Phi_{x,y} \leq 1$ by (2.12, d). If there is a point $x \in M \setminus \mathrm{Fix}(\mathrm{Z})$ such that $\mathrm{Z}_x \neq \mathbb{1}$, then the orbit x^Φ cannot generate $\mathbb{M}$. Hence $\dim x^\Phi \leq 4$, and $\dim \Phi = \dim x^\Phi + \dim y^{\Phi_x} + \dim \Phi_{x,y} \leq 4 + 6 + 1 = 11$. So assume that $\mathrm{Z}_x = \mathbb{1}$. In this case, $\dim E > 2 \dim x^{\mathrm{Z}} = 2 \dim \mathrm{Z} \geq 4$ by (2.14). Thus $\mathbb{E} = \mathbb{M}$, and $\Phi_{x,y} = \mathbb{1}$. This yields that $\dim \Phi = \dim x^\Phi + \dim (xy)^{\Phi_x} + \dim y^{\Phi_{x,xy}} \leq 7 + 3 + 3 = 13$. □

(4.7) Proposition. *Let* Φ *be a compact group of automorphisms of an 8-dimensional stable plane* $\mathbb{M}$*, and assume that the identity component* Z *of the center of* Φ *is not trivial. Then* $\dim \Phi \leq 13$.

Proof. Assume that $\dim \Phi \geq 14$. From (4.6) we infer that $\dim \mathsf{Z} = 1$. Hence $\dim \Phi' \geq 13$, and (4.4) yields that Φ is a product of Z with a factor Θ of type A_1 and a factor Ψ of type B_2. Since $\dim(\mathsf{Z}\Psi)_p \geq 4$ for each point p, the group $\mathsf{Z}\Theta$ acts quasi-perspectively. Now $\dim p^{\mathsf{Z}\Theta} \leq 3$ for each point p by (2.14, a), and $\dim(\mathsf{Z}\Theta)_p \geq 1$. This implies that $\dim p^{\Psi} \leq 4$. Consequently, $\dim \Psi_{p,q} \geq 2$ for any two points p, q. On the other hand, there are points x, y such that the set $x^{\mathsf{Z}\Theta} \cup y^{\mathsf{Z}\Theta}$ generates a subplane $\mathbb{E}$, and the almost simple group Θ acts non-trivially and quasi-perspectively on $\mathbb{E}$. Consequently, $\mathbb{E} = \mathbb{M}$, and $\dim \Psi_{x,y} = \mathbb{1}$. We have reached a contradiction. □

Combining (4.3), (4.4) and (4.7), we obtain:

(4.8) Theorem. *Let Φ be a compact connected group of automorphisms of an 8-dimensional stable plane $\mathbb{M}$. Then* $\dim \Phi \leq 13$, *or* $\Phi \cong \mathrm{PSU}_3\mathbb{H}$. *In the latter case, the plane $\mathbb{M}$ is isomorphic with the projective plane over $\mathbb{H}$, and the action of Φ is equivalent to the usual one.*

5. Stable planes of dimension 16

Again, we distinguish the three possibilities for the structure of the group.

A. Semi-simple groups

Assume that Δ is semi-simple, but not almost simple, and choose a factor Θ of minimal dimension. Let Ψ be the identity component of the centralizer of Θ in Δ (i.e., the product of all factors except Θ).

(5.1) Lemma. *If there is a point $x \in M$ such that the orbit x^{Θ} is not contained in any line, then* $\dim \Delta \leq 33$.

Proof. The connected orbit x^{Θ} generates a subplane $\mathbb{E} = (E, \mathscr{E})$ of $\mathbb{M}$, and Θ acts almost effectively on $\mathbb{E}$. The stabilizer Ψ_x acts trivially on x^{Θ} and hence trivially on $\mathbb{E}$. We distinguish the following cases:

i) $\dim \Psi_x > 8$
According to (2.13), the stabilizer Ψ_x is of type G_2, and $\dim E = 2$. From (3.1) we infer that $\dim \Theta = 3$, and $\dim \Delta = \dim \Theta + \dim \Psi_x + \dim x^{\Psi} \leq 3 + 14 + 15 = 32$.

ii) $8 \geq \dim \Psi_x > 3$
By (2.12, d), the subplane $\mathbb{E}$ is not a Baer subplane of $\mathbb{M}$. From (3.1) and (3.2) we infer that $\dim \Theta \leq 8$. Now $\dim \Delta \leq 8 + 8 + 15 = 31$.

iii) $\dim \Psi_x \leq 3$
In this case, $\dim \Theta \leq \dim \Psi \leq 18$. There are no almost simple compact groups of dimension 16, 17, or 18. Hence $\dim \Theta \leq 15$, and $\dim \Delta \leq 15 + 18 = 33$. □

Now assume that Θ acts quasi-perspectively. Recall that each non-trivial point orbit x^Θ is contained in exactly one line called L_x. Since Θ has minimal dimension, we have that $\dim\Delta \le 2\dim\Psi$. If $\dim\Psi_x = 0$ for some point $x \in M$, then $\dim\Psi \le 15$ and $\dim\Delta \le 30$. For the remainder of this section, we will therefore assume that $\dim\Psi_x > 0$. In particular, there is an involution $\alpha \in \Psi_x$. This involution acts trivially on the non-trivial orbit x^Θ. Therefore x is not a center of α, and by (2.6), the involution α is planar or has axis L_x.

(5.2) Lemma. *If α is planar, then* $\dim\Delta \le 35$.

Proof. Since Θ centralizes α, the Baer subplane $\mathbb{F}$ induced on $\mathrm{Fix}(\alpha)$ is Θ-invariant, and Θ acts almost effectively as a quasi-perspective group on $\mathbb{F}$. From (2.14) we infer that $\dim\Theta = 3$, and that there is no Θ-invariant proper full subplane of $\mathbb{F}$ containing x^Θ. Consequently, there is some point y such that the stabilizer $\Psi_{x,y}$ acts trivially on the Baer subplane, and $\dim\Psi_{x,y} \le 3$ by (2.12, d). This yields that $\dim\Delta = \dim\Theta + \dim\Psi_{x,y} + \dim x^\Psi + \dim y^{\Psi_x} \le 3 + 3 + 15 + 14 = 35$. □

If there is no planar involution in Ψ_x, we conclude that each involution in Ψ_x has axis L_x. Consequently, the stabilizer Ψ_x has rank 1 by (2.7), and $\dim\Psi_x \le 3$. This yields that $\dim\Psi \le 18$, and we infer that $\dim\Delta \le 33$ as in step iii) of the proof of (5.1).

Thus we have shown:

(5.3) Proposition. *If a compact group Δ of automorphisms of a 16-dimensional stable plane is semi-simple, but not almost simple, then* $\dim\Delta \le 35$.

B. Groups that are not semi-simple

(5.4) Lemma. *If the identity component Z of the center of a compact group Φ of automorphisms of a 16-dimensional stable plane $\mathbb{M}$ is not quasi-perspective, then* $\dim\Phi \le 29$.

Proof. Let x be a point such that x^{Z} is not contained in any line. Then Φ_x is a planar group, hence $\dim\Phi_x \le 14$, and $\dim\Phi \le 29$. □

(5.5) Lemma. *Let Φ be a compact group of automorphisms of a 16-dimensional stable plane $\mathbb{M}$, and assume that the identity component Z of the center of Φ has dimension at least* 2. *Then* $\dim\Phi \le 32$.

Proof. By (5.4) we may assume that Z is quasi-perspective. Choosing points $x \in M\setminus\mathrm{Fix}(\mathrm{Z})$ and $y \in M\setminus(\mathrm{Fix}(\mathrm{Z}) \cup L_x)$, we obtain that the set $x^{\mathrm{Z}} \cup y^{\mathrm{Z}}$ generates a subplane $\mathbb{E} = (E, \mathscr{E})$. From (2.14) we infer that $\dim E \ge 4$, and $\dim\Phi_{x,y} \le 8$ by (2.13). If there is a point $x \in M\setminus\mathrm{Fix}(\mathrm{Z})$ such that $\mathrm{Z}_x \ne \mathbb{1}$, then the orbit x^Φ cannot generate $\mathbb{M}$. Hence $\dim x^\Phi \le 8$, and $\dim\Phi = \dim x^\Phi + \dim y^{\Phi_x} + \dim\Phi_{x,y}$

$\leq 8+14+8=30$. So assume that $Z_x = \mathbb{1}$. In this case, $\dim E > 2\dim x^Z = 2\dim Z \geq 4$ by (2.14). Thus $\mathbb{E}$ is a Baer subplane (or $\mathbb{E} = \mathbb{M}$). On the other hand, the stabilizer Φ_x is a Lie group by (2.2), and $\dim \Phi_{x,y} \leq 3$ by (2.12, d). This yields that $\dim \Phi = \dim x^\Phi + \dim y^{\Phi_x} + \dim \Phi_{x,y} \leq 15 + 14 + 3 = 32$. □

(5.6) Lemma. *Let Δ be an almost simple compact group of automorphisms of a 16-dimensional stable plane* $\mathbb{M}$. *If there is a compact subgroup Θ of* Aut$(\mathbb{M})$ *such that Δ and Θ commute, then* $\dim \Delta \leq 28$, *or* $\dim \Theta = 0$.

Proof. Assume that $\dim \Theta > 0$. According to (5.4), we may assume that Θ is quasi-perspective. We consider the action of Δ on $\mathscr{L}_\Theta$. If this action is trivial, then Δ is quasi-perspective, and $\dim x^\Delta \leq 7$ for each point x by (2.14). Since the almost simple group Δ acts almost effectively on each non-trivial orbit, we infer from (2.4) that $\dim \Delta \leq 22$.

Now assume that Δ acts almost effectively on $\mathscr{L}_\Theta$. From (2.4) we known that $\dim \Delta = 36$ or $\dim \Delta \leq 29$. Since there are no almost simple compact groups of dimension 29, we just have to exclude the case where $\dim \Delta = 36$. In this case, Δ is of type B_4 or C_4. Let $L_x \in \mathscr{L}_\Theta$ be a line that is moved by Δ. Since Δ has no non-trivial orbit of dimension less than 8, we know that $\dim \Delta_{L_x} = 28$. This implies that the identity component Υ of Δ_{L_x} has maximal rank (cf. (7.3)). Now [2] yields hat Υ is almost simple. The group Υ acts almost effectively on L_x by (2.7). Choosing a one-dimensional connected subgroup T of Θ such that $\mathrm{T} \neq \mathrm{T}_x$, we infer that there is an almost effective action of the 29-dimensional group $\mathrm{T}\Upsilon$ on the orbit $x^{\mathrm{T}\Upsilon}$, which has dimension at most 7. This contradicts (2.4). □

(5.7) Proposition. *Let Φ be a compact group of automorphisms of a 16-dimensional stable plane* $\mathbb{M}$, *and assume that the identity component* Z *of the center of* Φ *is not trivial. Then* $\dim \Phi \leq 36$.

Proof. Let Φ be a group satisfying the assumptions. If $\dim \Phi > 36$, we conclude from (5.5) that $\dim \mathrm{Z} = 1$. Consequently, the commutator group Φ' has dimension greater than 35, and Φ' is almost simple by (5.3). Now we have reached a contradiction to (5.6). □

C. Almost simple groups

We consider a special case first.

(5.8) Lemma. *Assume that* $\Delta = \mathrm{PSO}_{10}\mathbb{R}$ *acts on a stable plane, and let*

$$\alpha = \pm\,\mathrm{diag}(-1,-1,1,1,1,1,1,1,1,1),$$
$$\beta = \pm\,\mathrm{diag}(1,1,-1,-1,1,1,1,1,1,1).$$

If these involutions have centers, then the centers are different.

Proof. A common center z of α and β would be fixed by both $C_\Delta(\alpha)$ and $C_\Delta(\beta)$. The Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ that correspond to these two groups are both isomorphic with $\mathfrak{so}_2\mathbb{R} \times \mathfrak{so}_8\mathbb{R}$, and their intersection is isomorphic with $\mathfrak{so}_2\mathbb{R} \times \mathfrak{so}_2\mathbb{R} \times \mathfrak{so}_6\mathbb{R}$. For the algebra $\mathfrak{c}$ that is generated by $\mathfrak{a}$ and $\mathfrak{b}$ we have that

$$\dim \mathfrak{c} \geq \dim(\mathfrak{a} + \mathfrak{b}) = \dim \mathfrak{a} + \dim \mathfrak{b} - \dim(\mathfrak{a} \cap \mathfrak{b}) = 2 \cdot 29 - 17 = 41 .$$

According to (2.4), there are no proper subgroups of Δ with dimension greater than 36. Therefore we obtain that the centralizers $C_\Delta(\alpha)$ and $C_\Delta(\beta)$ generate Δ, and Δ fixes the point z. By (2.4), the group Δ cannot act non-trivially on $\mathcal{M}_z$. On the other hand, a trivial action on $\mathcal{M}_z$ implies that Δ is quasi-perspective, which is impossible by (2.14). □

(5.9) Lemma. *No compact group of type* D_5 *can act on a stable plane.*

Proof. Each non-simple group Δ of type D_5 contains a central involution ζ. According to (2.4), the group Δ acts trivially on $\mathcal{L}_\zeta$ and is therefore quasi-perspective. But this contradicts (2.14). Hence we may assume that $\Delta = \mathrm{PSO}_{10}\mathbb{R}$. Choose a point $x \in M \setminus \mathrm{Fix}(\Delta)$. The identity component Υ of Δ_x has dimension at least 30. Therefore the rank of Υ is at least 4 (cf. (7.3)). The groups of rank 4 and dimension at least 30 are of type B_4, C_4, or F_4. According to (2.15), the only possibility is that $\Upsilon \cong \mathrm{Spin}_9$. In this case, the stabilizer Δ_y acts transitively on the line pencil $\mathcal{M}_y$ for each $y \in x^\Delta$, and Δ is a subgroup of the elliptic motion group by (2.9). But this is impossible since the rank of Δ is greater than the rank of F_4.

There remains the case that Υ is of rank 5. Consequently, the group Υ contains a maximal torus $\Theta \cong \mathbb{T}^5$ of Δ. We may assume that

$$\Theta = \left\{ \pm \begin{pmatrix} T_1 & & & & \\ & T_2 & & & \\ & & T_3 & & \\ & & & T_4 & \\ & & & & T_5 \end{pmatrix} \middle| \; T_i \in \mathrm{SO}_2\mathbb{R} \right\}.$$

The involutions

$$\begin{aligned}
\alpha &= \pm\,\mathrm{diag}(-1,\ -1,1,1,1,1,1,1,1,1),\\
\beta &= \pm\,\mathrm{diag}(1,1,\ -1,-1,1,1,1,1,1,1),\\
\gamma &= \pm\,\mathrm{diag}(1,1,1,1,\ -1,-1,1,1,1,1),\\
\delta &= \pm\,\mathrm{diag}(1,1,1,1,1,1,\ -1,-1,1,1)
\end{aligned}$$

are all in the same conjugacy class and commute with each other. If x is the center of α, then each of them has a center. According to (5.8), the point x is not center of β. Consequently, the involution β and each of its conjugates has center and axis. Now the centers of α, β and γ form a triangle by (2.7, c) and we obtain a contradiction to (2.7, d). There remains the case that α is planar. The centralizer $C_\Delta(\alpha)$ contains a group that is isomorphic with $\mathrm{SO}_8\mathbb{R}$. This group cannot act trivially on the subplane $\mathbb{F}$ induced on $\mathrm{Fix}(\alpha)$ by (2.12, c). Therefore it acts almost effectively on $\mathbb{F}$. But this contradicts (4.8). □

Many of the large almost simple groups can be excluded because they contain large subgroups that are not almost simple:

(5.10) Proposition. *Let Δ be a compact almost simple group of automorphisms of a stable plane If $\dim \Delta > 28$, then Δ is locally isomorphic with $\mathrm{SU}_6\mathbb{C}$, $\mathrm{SO}_9\mathbb{R}$, $\mathrm{SU}_4\mathbb{H}$ or the elliptic motion group F_4.*

Proof. The groups listed in the assertion are of type A_5, B_4, C_4, and F_4. Any other compact almost simple group Δ with $\dim \Delta > 28$ contains a subgroup of type A_6, D_5 or C_5. It is therefore sufficient to exclude these groups. The groups of type D_5 cannot occur by (5.9). Each group of type A_6 contains a subgroup that is locally isomorphic with $\mathrm{U}_6\mathbb{C}$, which is impossible by (5.6). Each group of type C_5 has a subgroup that is locally isomorphic with $\mathrm{U}_4\mathbb{H} \times \mathrm{U}_1\mathbb{H}$, and cannot act by (5.3). □

Combining (5.3), (5.7) and (5.10), we obtain:

(5.11) Theorem. *Let Φ be a compact connected group of automorphisms of a 16-dimensional stable plane $\mathbb{M}$. Then $\dim \Phi \leq 36$, or Φ is of type F_4. In the latter case, the plane $\mathbb{M}$ is isomorphic with the projective plane over $\mathbb{O}$, and the action of Φ is equivalent to the usual one.*

Since each stable plane has dimension 2, 4, 8, or 16, the proof of the Main Theorem is accomplished by (3.1), (3.2), (4.8), and (5.11).

6. About examples

(6.1) The following procedure yields examples that show that our Main Theorem gives sharp dimension bounds: Let $\mathrm{P}_2\mathbb{F} = (P, \mathscr{P})$ be any of the classical compact connected projective planes, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, and let E be the corresponding elliptic motion group. For every point $p \in P$, the stabilizer E_p leaves invariant the point set $D := P \setminus C$ if, and only if, C is a union of orbits under E_p. If, moreover, C is closed and $\emptyset \neq C \neq P$ then the subplane $(D, \mathscr{D})$ is a stable plane (cf. (1.3)), and E_p is a maximal compact subgroup of $\mathrm{Aut}((D, \mathscr{D}))$, cf. [11]. As a special case, we obtain the classical affine planes.

In a way, these examples are not too satisfactory: One would like to see examples that do not admit any open embedding in a plane that admits a larger compact group. For the low dimensional cases, such examples are known:

(6.2) Examples. In the realm of two-dimensional planes, the group $\mathrm{SO}_2\mathbb{R}$ tends to lie inside a (not necessarily compact) non-solvable group, cf. H. Groh's results [5].

An overview of two-dimensional stable planes that admit non-solvable groups is given in [12]: Apart from examples obtained by the procedure described in (6.1), there are the modified hyperbolic planes [22], the Moulton-planes (cf. [1]), and an exceptional plane found by K. Strambach [28]. In each of these cases, the identity component of a maximal compact subgroup of the automorphism group is isomorphic with $\mathrm{SO}_2\mathbb{R}$. Projective planes that admit a two-dimensional abelian group containing $\mathrm{SO}_2\mathbb{R}$ were studied by I. Schellhammer in her thesis [26].

(6.3) Examples. For the case of four-dimensional planes, the possibilities for compact non-abelian groups are strongly restricted. Apart from the elliptic group and the groups $\mathrm{SO}_3\mathbb{R}$ and $\mathrm{SU}_2\mathbb{C}$, there remain the groups that are almost split extensions of $\mathrm{SU}_2\mathbb{C}$ by a one-dimensional group, see [15] and [16]. The examples for planes that admit groups of the latter type include the complex hyperbolic planes and a class of affine planes constructed by P. Sperner [27].

(6.4) Turning to planes of dimension 8 or 16, respectively, one has to face the fact that all the known examples are open subplanes of compact connected projective planes. The classification of compact connected projective planes with large automorphism groups shows that such planes are mostly translation planes. In these cases, a large compact group already implies that the plane is classical. However, our bound for the dimension of compact groups serves as a first step towards a classification of sufficiently homogeneous planes, cf. [24], [29]. In particular, this proved to be a good tool for the study of almost simple groups. Note also that the group E_p plays a prominent role in R. Löwen's classification of stable planes with isotropic points [14].

7. Appendix

(7.1) We list the results of É. Cartan's classification of the compact simple real Lie algebras ([4], cf. [21] or [33]) in a table:

type	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
dimension	n^2+2n	$2n^2+n$	$2n^2+n$	$2n^2-n$	14	52	78	133	248
matrix realization	$\mathfrak{su}_{n+1}\mathbb{C}$	$\mathfrak{so}_{2n+1}\mathbb{R}$	$\mathfrak{su}_n\mathbb{H}$	$\mathfrak{so}_{2n}\mathbb{R}$					

Table A. Types of compact simple Lie algebras.

(7.2) The next table attempts to visualize the distribution of compact simple Lie algebras in the range that is relevant for this paper. The parentheses signalize the exceptional isomorphisms $\mathsf{A}_1 \cong \mathsf{B}_1 \cong \mathsf{C}_1$, $\mathsf{B}_2 \cong \mathsf{C}_2$, $\mathsf{A}_3 \cong \mathsf{D}_3$. Note also that $\mathsf{D}_2 \cong \mathsf{A}_1 \times \mathsf{A}_1$ is not simple.

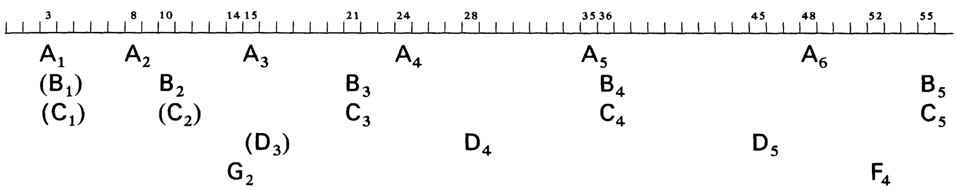

Table B. Dimension of compact simple Lie algebras.

(7.3) If Φ is a compact connected Lie group of rank r, then Φ is almost simple or Φ is a product of commuting subgroups Σ and Θ of rank s and t, respectively, such that $s + t = r$ and $\dim \Phi = \dim \Sigma + \dim \Theta$.

Let a_r be the maximal dimension of a compact almost simple Lie group of rank r. If the maximal dimension m_s of compact Lie groups of rank s is determined for all $s < r$, one obtains the value of m_r in the following way: Determine $m_s + m_t$ for all choices s, t with $s + t = r$, and let n_r be the greatest of these numbers. Then m_r is the maximum of a_r and n_r. The following table lists the results for $t \leq 5$:

r	1	2	3	4	5
n_r	–	6	17	28	55
a_r	3	14	21	52	55

Table C. Dimensions of compact Lie groups of low rank.

(7.4) If a compact connected group of dimension d acts effectively and transitively on an n-dimensional space X, then certain values of d are excluded ([18: Th. 5], cf. (2.4)). For $n \leq 10$, these values are:

n	2	3	4	5	6	7	8	9
maximal dimension	3	6	10	15	21	28	36	45
excluded dimensions		5	9	12–14	17–20	19–20 23–27	25–27 30–35	32–35 38–44

Table D. Gaps in the dimensions of compact transformation groups.

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