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Stable planes

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Abstract

Stable planes are a special kind of topological linear spaces. In particular, there is a 'planarity condition' that excludes spaces of geometrical dimension greater than 2. Embeddability problems are posed and answered, and an outline of the classification program is given.

1. Introduction

A topological linear space is a linear space $\mathbb{D}$ whose point and line space are endowed with topologies such that the operations of joining points and intersecting lines are continuous. Dealing with topological linear spaces, one has several possibilities to impose further conditions in order to get satisfying results. One of these possibilities is to strengthen the incidence structure, e.g. to assume that $\mathbb{D}$ is a projective or affine space or a geometric lattice of (geometrical) dimension $\geqslant 3$. The first cases have most recently been studied by Zanella (among others, see e.g. [14, 37, 38, 78–82]) while the last one has been treated by Groh [9–11].

However, in this paper we would like to draw attention to the case where stronger *topological* assumptions make the 'planar' case accessible. Our *planarity condition* (the stability axiom) is actually a combination of topological and incidence properties that leads to a generalization of the well-known topological plane geometries, namely, the real elliptic, euclidean, or hyperbolic plane. The analogues of these geometries over the classical real division algebras (namely, the complex numbers $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$ and Cayley's octonions $\mathbb{O}$) are also covered by this treatment.

Definition. Let $\mathbb{M} = (M, \mathcal{M})$ be a linear space, and assume that there are topologies on the set M of points and on the set $\mathcal{M}$ of lines such that the following hold:

(a) M and $\mathcal{M}$ are locally compact Hausdorff spaces of positive and finite covering dimension;

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(b) the geometric operations of joining points and intersecting lines are continuous;
(c) *stability axiom*, i.e. the set of pairs of intersecting lines is open in $\mathcal{M} \times \mathcal{M}$.
Then $\mathbb{M}$ is called a *stable plane*.

The stability axiom excludes, e.g. topological affine (or projective) spaces of geometric dimension $\geqslant 3$. The restrictions on the covering dimension imply that M and $\mathcal{M}$ share important properties of topological manifolds (see [29]); in fact there is a conjecture that M and $\mathcal{M}$ are always manifolds.

According to deep results of Löwen [29], many of the fundamental properties of compact connected projective planes* of finite covering dimension (which are a special case of stable planes) are common to stable planes in general, e.g.

- the covering dimension of M equals that of $\mathcal{M}$ and is one of the integers 2, 4, 8, 16.
- For each point $p \in M$, the line pencil $\mathcal{M}_p = \{L \mid p \in L \in \mathcal{M}\}$ is a compact connected homotopy l-sphere, where $l = \dim \mathcal{M}_p = \dim L = \frac{1}{2} \dim M$.

2. Embeddings

Let $\mathbb{D} = (D, \mathcal{D})$ and $\mathbb{E} = (E, \mathcal{E})$ be linear spaces. An *embedding* of $\mathbb{D}$ into $\mathbb{E}$ is a pair (π, λ) of injective mappings $\pi : D \to E$ and $\lambda : \mathcal{D} \to \mathcal{E}$ such that $p \in D$ is incident (in $\mathbb{D}$) with $L \in \mathcal{D}$ if and only if p^π is incident (in $\mathbb{E}$) with L^λ.

An embedding (π, λ) is said to preserve pencils if for each point $p \in D$ the pencil $\mathcal{E}_{p^\pi}$ equals $(\mathcal{D}_p)^\lambda$.

Now assume that $\mathbb{D}$ and $\mathbb{E}$ are stable planes. An *open embedding* (π, λ) of $\mathbb{D}$ into $\mathbb{E}$ is an embedding of linear spaces such that π and λ are continuous, and D^π is open in E.

Let $\mathbb{M} = (M, \mathcal{M})$ be a stable plane, and let U be a nonvoid subset of M containing a quadrangle. Define $\mathcal{U}$ to be the set of all lines in $\mathcal{M}$ that are incident with at least two points of U. Then the geometry $\mathbb{U} = (U, \mathcal{U})$ is called the geometry *induced* on U. If U is open in M, then $\mathbb{U}$ is called an *open subplane* of M. In this case $\mathbb{U}$ is a stable plane.

It turns out (see Corrollary 1.2) that, in the case of an open embedding, the mapping π is in fact a homeomorphism of D onto D^π, endowed with the topology induced from E. In particular, the embedding (π, λ) induces a (topological) isomorphism of $\mathbb{D}$ onto the open subplane $(D^\pi, \mathcal{D}^\lambda)$ of $\mathbb{E}$.

Our main interest is the problem whether a given stable plane is isomorphic with an open subplane of a compact connected projective plane, i.e. whether there is an open embedding into a compact projective plane.

Lemma 1.1. *Let $\mathbb{D} = (D, \mathcal{D})$ and $\mathbb{E} = (E, \mathcal{E})$ be stable planes and (π, λ) an embedding of the linear space $\mathbb{D}$ into $\mathbb{E}$. Then (π, λ) is an open embedding if and only if π and λ are continuous and (π, λ) preserves pencils.*

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* For compact connected projective planes, the papers [2, 36, 46, 49, 52] (each of which is, in some way, conclusive) are sources for further references.

Proof. Assume first that (π, λ) is an open embedding. For each point $p \in D$ and each neighbourhood U of p in D, we have that $\mathscr{D}_p = \{p \vee q \mid q \in U \setminus \{p\}\}$. Since U^π is open in E, we conclude that $\mathscr{E}_{p^\pi} = \{p^\pi \vee q^\pi \mid q \in U \setminus \{p\}\} = \mathscr{D}_p^\lambda$. Thus (π, λ) preserves pencils.

Now suppose that π and λ are continuous and that (π, λ) preserves pencils. The restriction of λ to $\mathscr{D}_p$ is a continuous bijection of compact spaces, hence a homeomorphism of $\mathscr{D}_p$ with $\mathscr{E}_{p^\pi}$. For each point $p \in D$ there are lines G, H through p, points $g \in G$ and $h \in H$ and neighbourhoods $\mathscr{X}$ and $\mathscr{Y}$ of G and H in $\mathscr{D}_g$ and $\mathscr{D}_h$, respectively, such that $U = \{X \wedge Y \mid X \in \mathscr{X},\ Y \in \mathscr{Y}\}$ is a neighbourhood of p in D (see [16, 1.4]). Since λ induces homeomorphisms of $\mathscr{D}_g$ onto $\mathscr{E}_{g^\pi}$ and $\mathscr{D}_h$ onto $\mathscr{E}_{h^\pi}$, we obtain that $\mathscr{X}^\lambda$ and $\mathscr{Y}^\lambda$ are neighbourhoods in $\mathscr{E}_{g^\pi}$ and $\mathscr{E}_{h^\pi}$, respectively. Thus, U^π is a neighbourhood of p^π in E, and (π, λ) is an open embedding. □

Since the restriction of an open embedding to the geometry induced on any open nonempty subset U of D is again continuous and preserves pencils, Lemma 1.1 yields the following corollary.

Corollary 1.2. *If (π, λ) is an open embedding of a stable plane $\mathbb{D} = (D, \mathscr{D})$ into a stable plane $\mathbb{E}$, then $\pi: D \to D^\pi$ is a homeomorphism.*

3. Nonembeddable planes

In this section, we give a simple criterion of nonembeddability (in terms of linear spaces). We shall see that this criterion applies to several examples of stable planes.

Let $\mathbb{D} = (D, \mathscr{D})$ be a linear space. For $p \in D$ and $L \in \mathscr{D}$ define $\|_{p,L}$ to be the set of lines through p that do not meet L.

Lemma 2.1. *Assume that a linear space $\mathbb{D} = (D, \mathscr{D})$ has points x, y and a line L such that the sets $\|_{x,L}$ and $\|_{y,L}$ have different cardinality. Then there is no pencil-preserving embedding of $\mathbb{D}$ into a projective (or an affine) plane.*

Proof. Assume that (π, λ) is a pencil-preserving embedding into a projective plane $\mathbb{P} = (P, \mathscr{P})$. Then the bijection $\alpha: \mathscr{D}_x \to \mathscr{D}_y: G \mapsto ((G^\lambda \wedge L^\lambda) \vee y^\pi)^{\lambda^{-1}}$ restricts to a bijection of $\|_{x,L}$ onto $\|_{y,L}$. □

In the special case where $\mathbb{D}$ is a stable plane, the bijection α is a homeomorphism of $\mathscr{D}_x$ onto $\mathscr{D}_y$. Thus, we obtain the following Corollary from Corollary 1.2 and from Lemma 2.1:

Corollary 2.2. *If a stable plane $\mathbb{D} = (D, \mathscr{D})$ has points x, y and a line L such that there is no homeomorphism of $\mathscr{D}_x$ onto $\mathscr{D}_y$ that maps $\|_{x,L}$ onto $\|_{y,L}$ (in particular, if $\|_{x,L}$ is not homeomorphic with $\|_{y,L}$) then there is no open embedding of $\mathbb{D}$ into a compact projective plane.*

4. Examples

The construction of examples of stables planes is particularly convenient in the case where the point space is $\mathbb{R}^2$ (i.e. the point set of the real affine plane, endowed with the usual topology): For any linear space $\mathbb{M}=(M,\mathcal{M})$ such that $M=\mathbb{R}^2$ and $\mathcal{M}$ consists of *closed connected* subsets of M, there is a unique topology on $\mathcal{M}$ such that $\mathbb{M}$ is a stable plane [46, 2.12] (cf. [59, 60]). Therefore, we just describe the set $\mathcal{M}$.

Let $M=\mathbb{R}^2$. The set $\mathcal{M}$ of all ordinary lines of nonnegative slope, together with the set of all translates of some suitable arc A forms the line space of a stable plane $\mathbb{M}=(M,\mathcal{M})$ (cf. [7]). Two special cases are of particular importance.

Example 4.1 (*Salzmann* [47]). *Let* $A=\{(\xi,\xi^{-d})\,|\,\xi>0\}$, where $d\geqslant 1$.

Example 4.2 (*Betten* [1]; *Strambach* [66, (*vi*)]). Choose $A=\{(\xi,e^{-\xi})\,|\,\xi\in\mathbb{R}\}$.

In both cases, we obtain stable planes that do not allow any open embeddings into projective planes: choosing a line L of slope 0 and points x and y that lie on different sides with respect to L, one verifies easily that Corollary 2.2 applies.

Example 4.3. In $E=\mathbb{R}^2$, let X and Y denote the horizontal and the vertical axis, respectively. Let $\mathcal{E}$ be the set of all ordinary lines of nonnegative slope, all lines of negative slope and positive intercept and all 'lines' of the form

$$L_{\sigma,\tau}=\{(\xi,-\sigma\xi-\tau)\,|\,\xi<0\}\cup\{(\xi,-\sigma(1+\tau)\xi-\tau)\,|\,\xi\geqslant 0\},$$

where $\sigma,\tau>0$. Then $\mathbb{E}=(E,\mathcal{E})$ is a stable plane.

The construction in Example 4.3 was inspired by Moulton's example of a nondesarguesian affine plane. Yet $\mathbb{E}$ has no open embedding into any projective plane, as can be seen by applying Lemma 2.2 to a line L that has negative slope and passes through the origin, and two points that lie on different sides with respect to L.

Removing the points on the lower half of Y, we obtain the open subplane $\mathbb{D}=(D,\mathcal{D})$, where $\mathcal{D}=\mathcal{M}$ and $D=E\setminus\{(0,\eta)\,|\,\eta\leqslant 0\}$. Note that D is again homeomorphic with $\mathbb{R}^2$, but there are some disconnected lines now. The plane $\mathbb{D}$ has the following remarkable property: each point $x\in D$ has a neighbourhood U_x in D such that the geometry induced on U_x is desarguesian (i.e. Desargues' theorem holds for each configuration such that all required intersections exist in U_x). In fact, D is the union of the three desarguesian open half planes $E_1=\{(\xi,\eta)\,|\,\xi<0\}$, $E_2=\{(\xi,\eta)\,|\,\eta>0\}$, $E_3=\{(\xi,\eta)\,|\,\xi>0\}$. However, the geometry $\mathbb{D}$ is not desarguesian. Thus, this example shows that the assumption of connected lines in Polley's work on locally desarguesian geometries (see [39–41]) is not superfluous.

Example 4.4. The so-called $\mathrm{SL}_2\mathbb{R}$-plane [64] is obtained in the following way: Let the group $\Sigma = \mathrm{SL}_2\mathbb{R}$ act in the usual (linear) way on $M = \mathbb{R}^2$. If $\mathcal{M}$ denotes the set of all ordinary lines through the origin, together with all Σ-images of a branch of some hyperbola, then $(M, \mathcal{M})$ is a stable plane.

According to [64], there is no Σ-equivariant open embedding (i.e. an open embedding such that the action of Σ extends) of this plane into a projective plane. In fact, this plane does not admit any proper Σ-invariant open embedding into a stable plane (see [28]). However, our criterion does not apply to this plane, so the question remains open whether there is an open embedding such that the action of Σ does not extend.

Example 4.5. Löwen [34, pp. 9–12] constructed an analogue to Strambach's $\mathrm{SL}_2\mathbb{R}$-plane: the point space is $\mathbb{C}^2$, and the group Σ is replaced by $\mathrm{SL}_2\mathbb{C}$. Löwen's plane shares the embeddability properties of Strambach's plane. For the quaternion case, there is no such analogue [69].

Example 4.6. In [22], Löwen describes a stable plane $E_\alpha = (\mathbb{R}^2, \mathcal{L}_\alpha)$ for each twice differentiable, convex, increasing function α. We will not repeat the construction but just note that Corollary 2.2 applies: e.g. choose the points $(0,0)$ and $(0,2)$ and the line $L_{1,0}$ (in the notation of [22, 5.4]) (cf. [22, Remarks, p. 314]). Therefore, there is no open embedding of E_α into a projective plane (for the extension $\bar{E}_\alpha$ defined in [22, 2.3], this property has been established in [22]).

5. Classification of stable planes (a short report)

Following (yet extending) the ideas of F. Klein, we consider pairs $(\Gamma, \mathbb{M})$, where $\mathbb{M} = (M, \mathcal{M})$ is a stable plane and Γ is a closed subgroup of $\mathrm{Aut}(\mathbb{M})$ (i.e. the group of all continuous collineations, endowed with the compact-open topology derived from the action on M).

The general classification problem can be stated as follows:

- find suitable homogeneity conditions,
- list all pairs $(\Gamma, \mathbb{M})$ satisfying these homogeneity conditions.

Several special cases have been treated successfully.

Case 1: The 'classical' homogeneity conditions are *transitivity* of Γ on the set of *points*, the set of *lines*, or on the set of *flags* (incident point–line pairs).

The last condition yields that $\mathbb{M}$ is isomorphic with one of the 'classical' stable planes (in particular, the group Γ contains an elliptic, hyperbolic or euclidean motion group (see [35] for details)).

In the special case of compact connected projective planes, transitivity on the set of points characterizes the compact connected Moufang planes (i.e. the projective planes over the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$ or Cayley's octonions $\mathbb{O}$). Moreover, the group Γ contains an elliptic motion group (see [50] or [25]).

Transitivity on the set of lines is equivalent to transitivity on the set of points in the projective case (by duality). In the case of locally compact connected *affine* planes, however, transitivity on the set of lines yields the Moufang planes [51], while there are many affine planes admitting a group that acts transitively on the set of points (e.g. translation planes).

For stable planes in general, transitivity on the set of points is rather a weak condition. In fact, examples are obtained whenever $\Gamma \leqslant \mathrm{Aut}(\mathbb{M})$ has some open orbit $U \subseteq M$: the restriction of the action of Γ to the subplane induced on U is effective. Open orbits do occur in the non-Moufang planes, e.g. the translation planes, Hughes planes, Moulton planes, or each of the nonembeddable examples 3.1–3.4. Under suitable additional conditions (concerning line orbits) it is possible to reconstruct the geometry induced on the open orbit from the action of Γ (see [70, 6.3] or [71] for a method and [69]; [70, Ch. 7, pp. 36–47]; [74, 76] for concrete applications).

Case 2: Each closed subgroup Γ of $\mathrm{Aut}(\mathbb{M})$ is locally compact and separable [16, 2.9], and its identity component is a projective limit of Lie groups (cf. [70, 2.3]). In particular, the different notions of topological dimension (covering dimension, inductive dimension etc.) coincide for Γ, and $\dim \Gamma$ equals the ($\mathbb{R}$-)dimension of the corresponding Lie algebra (in the sense of Lashof [15]). This dimension serves as a measure for the size of Γ and hence as a measure for the homogeneity of $\mathbb{M}$. Moreover, the theory of Lie algebras supplies a description of the structure of Γ.

In the late 1950s, H. Salzmann initiated the project of determining all pairs $(\Gamma, \mathbb{M})$, where $\dim \Gamma$ is 'sufficiently' large. In the projective case, quite satisfying results have been obtained.

For each value $m = \dim M$, there is a 'critical' dimension g_m such that $\dim \Gamma > g_m$ implies that the plane is isomorphic with the Moufang plane $P_2\mathbb{F}$ (where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, according to m). The geometries with $\dim \Gamma \geqslant g_m - 1$ have been determined (mostly translation planes, Moulton planes or Hughes planes). It seems worth noting that g_m lies in the vicinity of $\frac{1}{2}\dim \Sigma$, where $\Sigma = \mathrm{Aut}(P_2\mathbb{F})$. The main results (and further references) can be found in [2, 36, 46, 49, 52, 53].

A similar classification has been obtained for stable planes under the special assumption that the point space is homeomorphic with $\mathbb{R}^2$ and that each line is connected (the so-called 'Salzmann-planes') (see [12] and the references given there).

In the case of stable planes in general, special assumptions about the structure of Γ have led to satisfying partial results. We attempt to describe the state of the art by Table 1.

Table 1

Structure of Γ	$\dim M = 2$	$\dim M = 4$	$\dim M = 8$	$\dim M = 16$
Almost simple	$(\Gamma, \mathbb{M})$ known [28]	$\dim \Gamma > 3 \Rightarrow$ $(\Gamma, \mathbb{M})$ known [32, 34, 35]	$\dim \Gamma > 16 \Rightarrow$ $(\Gamma, \mathbb{M})$ known [70]	?
Semisimple	$\Uparrow$ [28]	$\Uparrow$ [18]	*	?
Solvable	$\dim \Gamma \leqslant \frac{5}{2} \dim M$ [77]		*	?
Abelian	$\dim \Gamma \leqslant \dim M$ [74, 3.4]			
Compact	$(\Gamma, \mathbb{M}) \cong (\mathrm{E}, P_2\mathbb{F})$ or $\dim \Gamma \leqslant \dim \mathrm{E} - \dim M$ [75]			

Note: E = Elliptic motion group on $P_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (according to $\dim M$),
* Bounds for $\dim \Gamma$ have been proved, but probably these bounds are not sharp.

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