Design of Robustly Performing Controllers for a Class of Practical Control Problems

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Abstract

The design of robustly performing controllers for a class of practical control problems is considered, involving both parametric and unstructured uncertainty. The problem is dealt with in the structured singular value (μ)-framework. An iterative approach comprising both μ-analysis of the detailed problem and D-K-iteration for a modified problem is proposed, as no direct solution to the detailed μ-synthesis problem is known to date. With a practical MIMO process control example it is shown that this approach leads to controllers that exhibit robust performance.

Keywords: Robust performance, Structured singular values, μ-Synthesis, Mixed parametric and unstructured uncertainty, Process control.

1 Introduction

Modelling of dynamical systems on the basis of balance equations and conservation laws typically leads to nonlinear state space descriptions with uncertain parameters. Very often nonlinear effects are the prominent characteristic of such systems. To design compensators that take the uncertainties *and* the nonlinearities into account is a very difficult and yet unsolved problem. However in many practical cases the complications imposed by uncertain physical parameters are more restricting than those imposed by nonlinearity. This is e.g. the case when the system is only weakly nonlinear in a wide operating range or when the deviations from the operating point are small (e.g. due to the controller to be designed). In this paper we consider controller design for systems with uncertain physical parameters for which nonlinear effects play an inferior role only. The key issue is that we not only demand stability to be achieved robustly, but require that good performance also be achieved robustly. We restrict our attention to performance objectives that can be formulated in the frequency domain. We will consider this class of control problems within the structured singular value (μ)-framework [1, 2]. An application to a practical process control problem will be used to illustrate the design technique. Simulation results with a nonlinear model of the process show that indeed good performance is achieved robustly.

In Section 2.1 some basic concepts of standard μ-theory are reviewed. The precise problem formulation and the solution in the μ-framework is described in Section 2.2. Finally a robustly performing controller for a continuous stirred tank reactor (CSTR) is designed and evaluated in Section 3.

2 Controller Design for a Specific Class of Control Problems

2.1 Brief Review of μ-Control Theory

The structured singular value (SSV or μ) was introduced to give a quantitative characterisation of the effects of structured uncertainty on stability and performance of linear dynamical systems [1]. Roughly speaking uncertainty

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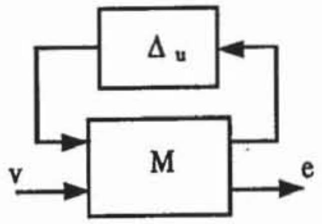

Figure 1: Block structure for robust stability and performance analysis

will be called *structured* if there is more than one "source" for it and these different sources are independently taken into account. Each source is represented by a normalised perturbation Δ_i which represents an arbitrary dynamical system with $\bar{\sigma}(\Delta_i) \leq 1 \; \forall\omega$.

In order to evaluate stability and performance the closed loop loop is represented as so called M-Δ-structure. Control performance as considered in this paper is characterised by the influence that external inputs **v**, like disturbances and sensor noise, have on external outputs **e**, like the tracking error. Fig. 1 represents the extended M-Δ-structure that is a standardised description of the set of all possible closed loop systems [3]. $\mathbf{\Delta_u}$ is a blockdiagonal matrix containing the different perturbations Δ_i and M is an appropriately partitioned matrix ($M = [M_{11}M_{12}; M_{21}M_{22}]$) containing only nominal quantities. For the control problem under consideration the formulation of the M-Δ-structure is elucidated in more detail in Section 2.2.

Conditions for robust stability and also for robust performance can be stated via the structured singular value for systems in this standard M-Δ-structure. In the frequency domain the considered performance is expressed in terms of the H_∞-norm of the transfer matrix $F_u(M, \mathbf{\Delta_u})$ from **v** to **e**:

$$\|F_u(M, \mathbf{\Delta_u})\|_\infty = \sup_\omega \bar{\sigma}(F_u(M, \mathbf{\Delta_u})) < 1. \tag{1}$$

The following theorems hold:

Robust Stability Theorem [1]: *The closed loop system in Fig. 1 is stable for all normalized perturbations* $\mathbf{\Delta_u}$ *if and only if*

$$\mu(M_{11}) < 1 \qquad \forall\omega \quad . \tag{2}$$

Robust Performance Theorem [1]: *The closed loop system in Fig. 1 satisfies the* H_∞*-performance condition* $\|F_u(M, \mathbf{\Delta_u})\|_\infty < 1$ *for all normalized perturbations* $\mathbf{\Delta_u}$ *if and only if*

$$\mu(M) < 1 \qquad \forall\omega \quad . \tag{3}$$

The theorems stated above allow to analyse the closed loop behavior of linear systems with a given controller with respect to the effects of structured uncertainty on stability and performance (*μ-analysis*). Computation of the structured singular value for μ-analysis is possible for a wide range of control problems (e.g. [4]). How to synthesize controllers that guarantee robust stability and robust performance (*μ-synthesis*) is a considerably more difficult question. To date only D-K-iteration [2] and μ-K-iteration [5] are known to solve the μ-synthesis problem for a limited class of uncertainty descriptions. The problem considered here is not suited for D-K-iteration. In Section 2.2 a μ-synthesis approach for this problem is proposed.

2.2 Solution of the Control Problem in the μ-Framework

In this paper we consider dynamic systems that are modelled on the basis of balance equations and first principles. This typically results in nonlinear dynamical equations with uncertain physical parameters. The controllers are designed on basis of a linear MIMO state space model $[A, B, C, D]$ that is obtained by linearising the nonlinear equations about the operating point.

The particular characteristic of the systems studied is that the elements of A, B, C and D depend on physical parameters $\mathbf{p}$ and the stationary values of the states $\mathbf{x_S}$ and inputs $\mathbf{u_S}$. This means for the elements of A:

$$a_{ij} = a_{ij}(\mathbf{p}, \mathbf{x_S}, \mathbf{u_S}) \tag{4}$$

and the analogous is true for the elements of B, C and D. Very often, the values of the physical parameters are only known to lie within an interval, giving rise to a structured uncertainty description.

In the following it is shown how the uncertainties can be modelled and specifically how this leads to the M-Δ-structure required for μ-synthesis. The basic idea of the proposed uncertainty description is to write out the parameter dependence of the elements of the system matrices. The functional relations of the coefficients of the state-space matrices in which the uncertain parameters $\mathbf{p}$ appear are kept, e.g.

$$a_{ij} = a_{ij}(\mathbf{p}) \, , \tag{5}$$

instead of just calculating bounds for each element of the state-space matrices, e.g.

$$\underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij} \, . \tag{6}$$

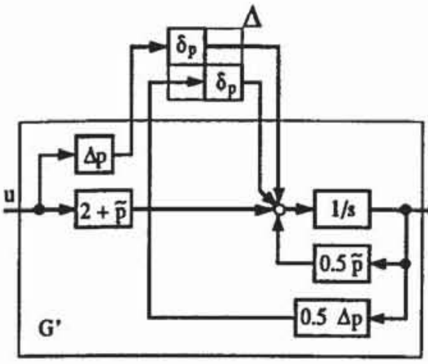

Figure 2: Block diagram of uncertainty description of example system

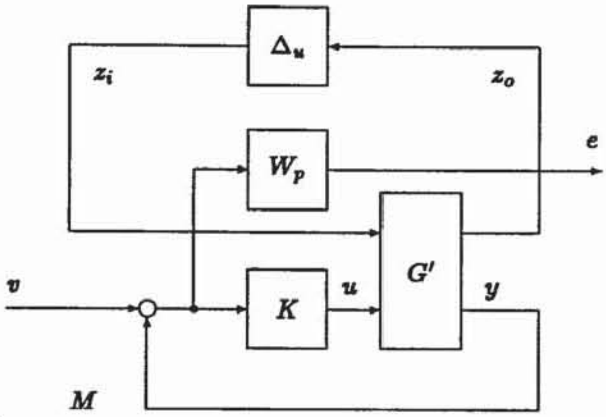

Figure 3: M-Δ-structure for robust performance containing a plant G' with uncertainty description and a performance specification

Consider for instance the following simple SISO-example:

$$\begin{aligned} \dot{x} &= ax + bu \quad \text{with} \quad a = \tfrac{1}{2}p,\ b = 2 + p \\ y &= cx \qquad\quad \text{with} \quad c = 1 \end{aligned}$$

p is a physical parameter whose value is just known to lie in an interval $[\tilde{p} - \Delta p, \tilde{p} + \Delta p]$ around its nominal value $\tilde{p}$. The plant with uncertainties is described with a normalised perturbation δ_p and the uncertainty "weight" Δp as follows:

$$\begin{aligned} \dot{x} &= [\tfrac{1}{2}\tilde{p}]x + [2 + \tilde{p}]u + [\tfrac{1}{2}\Delta p]\delta_p x + \\ &\quad + [\Delta p]\delta_p u\,, \ \text{with } |\delta_p| \leq 1, \delta_p \in \mathbb{R} \\ y &= x\,. \end{aligned}$$

Pulling out the perturbations δ_p leads to the block diagram shown in Fig. 2. This results in an M-Δ-structure with a diagonal Δ_u-matrix, containing a repeated, real, scalar perturbation δ_p.

This scheme can easily and non-conservatively be generalised for more complicated systems, including MIMO-systems. It leads to M-Δ-structures where Δ_u contains several diagonal blocks $\delta_i I$ with repeated, real, scalar uncertainties, one block for each uncertain parameter.

Up to now, the part of the M-Δ-structure that is related to Δ_u was treated. The remaining part of the M-Δ-structure to be specified is the performance specification, related to Δ_p (see also Fig. 1 and 3).

The performance is specified by the H_∞-Norm of some closed-loop transfer function, e.g. the weighted sensitivity function S:

$$\|W_p S\|_\infty < 1\,. \tag{7}$$

Now, the M-Δ-structure is fully specified. Theoretically, one could execute a μ-synthesis now. However, as mentioned in Section 2.1, it is not possible to perform a D-K-iteration for systems containing real, repeated uncertainties. In D-K-iteration μ-analysis, leading to so called D-matrices, and H_∞-synthesis, making use of the D-matrices and leading to controllers K, alternate with each other. μ-analysis for systems with real, repeated uncertainties does not lead to D-matrices that are suitable for H_∞-synthesis.

As alternative approach, the following iterative scheme on basis of D-K-iteration is proposed. If the set of permissible perturbations in Δ_u is enlarged D-K-iteration is possible. Each real, repeated scalar $\delta_i I_{k\times k}$ is replaced by a diagonal block consisting of several independent complex scalars:

$$\mathrm{diag}(\delta_{i1}, \delta_{i2}, \ldots, \delta_{ik})\,, \quad \text{with } \delta_{ij} \in \mathbf{C}\,. \tag{8}$$

We suggest a D-K-iteration with the modified perturbations, followed by a μ-analysis for the original problem. μ-analysis for problems with repeated real scalar perturbation blocks can be performed (e.g. [6, 7]). In this first step temporarily conservatism is introduced by modifying the uncertainty description.

Fig. 4 shows a flowchart of the proposed approach to μ-synthesis. The subsequent μ-analysis determines exactly whether robust performance has been achieved for the original problem. One cannot expect that the computed controller meets the desired robust performance condition at once. This comes from the potentially significant difference between the modified and the original problem. A robustly performing controller for the original problem can be synthesized in an iterative process: The outcome of the μ-synthesis is crucially dependent on the performance

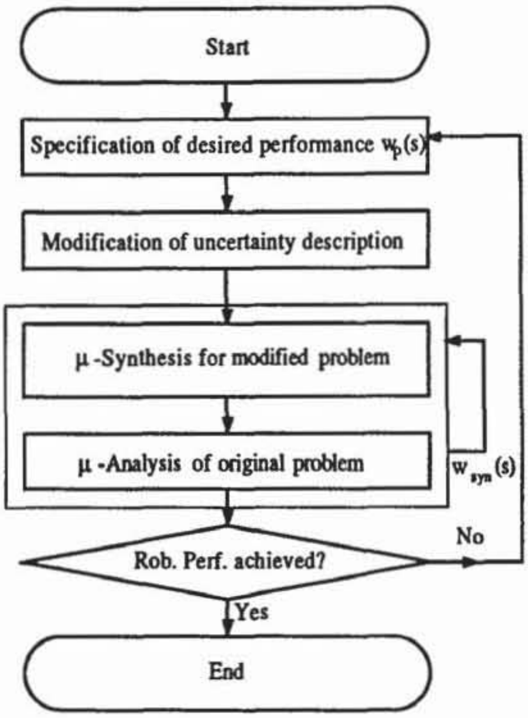

Figure 4: Diagram of proposed iterative μ-synthesis

weight used. By using a different performance weight $W_{syn}(s)$ from the actually wanted $W_p(s)$ it is in many cases possible to find a controller that satisfies the robust performance objective $W_p(s)$ for the original problem. The μ-analysis step gives information about the necessary modifications of $W_{syn}(s)$. One might e.g. choose a performance weight $W_{syn}(s)$ that demands higher performance in a certain frequency range in order to stress achievement of performance over stability.

There is no guarantee to find a controller that satisfies the robust performance criterion even when iterating over $W_{syn}(s)$. If such a controller cannot be found, either the desired performance weight $W_p(s)$ has to be relaxed or the uncertainty description might have to be changed. It has to be emphasized that if the robust performance test fails, it does not necessarily signify that a μ-optimal controller for the original problem does not exist. This just means that it is not possible to calculate it with the approach suggested.

3 Application to a CSTR

3.1 Description of Control Problem

The plant considered is a continuous stirred tank reactor (CSTR) in which cyclopentenol (B) is produced from cyclopentadiene (A). The by-products cyclopentanediol (C) and dicyclopentadiene (D) are produced in unwanted chain- and secondary reactions.

Only the reactant cyclopentadiene is fed into the reactor with concentration c_{A0} and flow rate $\dot{V}$. The exchange of heat between the reactor and its surroundings is $\dot{q}_H$. The dynamics of the system are described by a system of three nonlinear equations which can be derived from mass and energy balances. Certain parameters are only known within bounds, with a relative uncertainty ranging from 1.3 % to 56 %.

The concentration of cyclopentenol c_B and the temperature ϑ are the controlled variables and can directly be measured. For performance mainly c_B is interesting. The manipulated variables are $\dot{V}$ and $\dot{q}_H$. Thus we have a TITO control problem. Specific bounds are also given for the range of $\dot{V}$. The input concentration c_{A0} of the plant varies, since it depends on some upstream unit and is regarded as disturbance. The control objective is to regulate changes in the disturbance c_{A0} by ± 12 % and to allow for set point changes for c_B by -23 % and $+6$ %.

This problem was proposed by Klatt and Engell [8] as a benchmark problem for controller design.

3.2 Design of a μ-optimal Controller for the CSTR

The control problem posed in Section 3.1 is of the class considered in this paper. The CSTR introduced there is modelled by a third order system of nonlinear differential equations. It has two inputs and two outputs. The reactor is only weakly nonlinear in the operating range considered. The system exhibits one zero in the right half plane of which the location varies significantly with the operating point. Parametric uncertainty are brought in by seven uncertain chemo-physical constants. We want to use the framework proposed in Section 2.2 to design a robustly performing controller.

The μ-framework requires a linear model plus uncertainty description that can be rearranged into an M-Δ-structure as in Fig. 1.

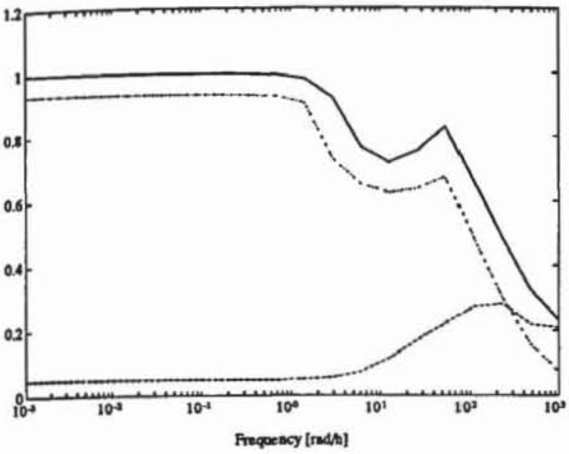

Figure 5: μ for robust performance (solid line), robust stability (dashdotted) and nominal performance (dashed)

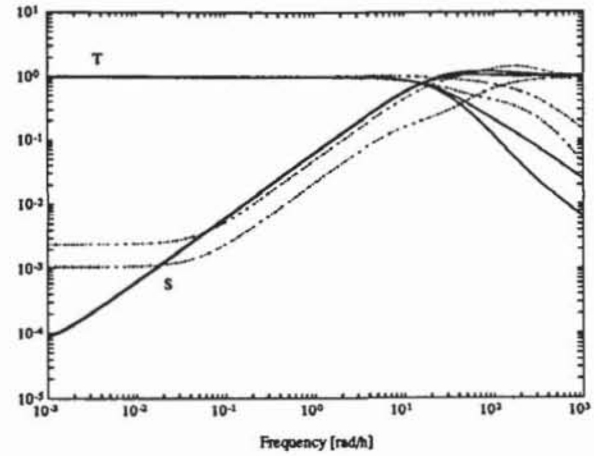

Figure 6: Singular values of S and T for the μ-controller (dashed) and the H_∞-controller (solid)

The construction of the uncertainty description is based on the equations of the linearized system and is performed similar to the example in Section 2.2.

The performance objective chosen is expressed in terms of the weighted output sensitivity function:

$$\|W_p S\|_\infty < 1. \tag{9}$$

The desired performance is described by $W_p = w_{p1} I$:

$$w_{P1}(s) = \frac{s+10}{2(s+0.1)}. \tag{10}$$

The weight $w_{P1}(s)$ is also chosen as first synthesis weight $W_{syn}(s)$ and a μ-synthesis step is performed with it. This step consists of D-K-iteration for the modified problem and μ-analysis of the original problem, according to the approach explained in Section 2.2. The controller found does not even achieve the desired performance nominally. A more demanding synthesis weight W_{syn} is chosen. μ-analysis of the original problem suggests a stronger weighting of the tracking error and an increase in the bandwidth. The maximally allowed tracking error is lowered from 0.02 to 0.005 and the loop bandwidth is increased by a factor of two. With this synthesis weight, the D-K-iteration converges after six steps to a controller that fails only marginally to meet the μ-test for the robust performance objective w_{P1}. Therefore the μ-synthesis is not repeated but instead a modified performance objective is chosen so that robust performance with the μ-controller obtained above can be guaranteed:

$$w_{P2}(s) = \frac{s+5}{5(s+0.05)}. \tag{11}$$

Fig. 5 shows the graph of μ for robust performance, robust stability and nominal performance for this performance objective. It can be seen, that the condition for robust performance ($\mu(M) < 1\ \forall\omega$) is satisfied, guaranteeing performance not worse than $1/|w_{P2}(j\omega)|$ not only for the nominal plant but also for the perturbed plant.

An H_∞-controller, designed with an (S, T, KS)-criterion is compared to the μ-controller. Its singular values of the sensitivity S and complementary sensitivity T are shown in Fig. 6, together with the respective singular values of the μ-controller. The loop shapes of the H_∞-controller suggest superior dynamic behavior compared to the μ-controller. However $\sigma(S)$ and $\sigma(T)$ only describe nominal performance and do not give any information about robust performance. The crucial test is a μ-analysis. The H_∞-controller has a peak value of $\mu = 1.106 > 1$. Hence the H_∞-controller does not satisfy the robust performance objective and therefore no guarantee for achievement of performance in the presence of uncertainty can be given. It should be noted that the H_∞-design is by loop-shaping, disregarding all information about the uncertainty.

The performance objective was posed in the frequency domain. Another question is whether the μ-controller meets time domain

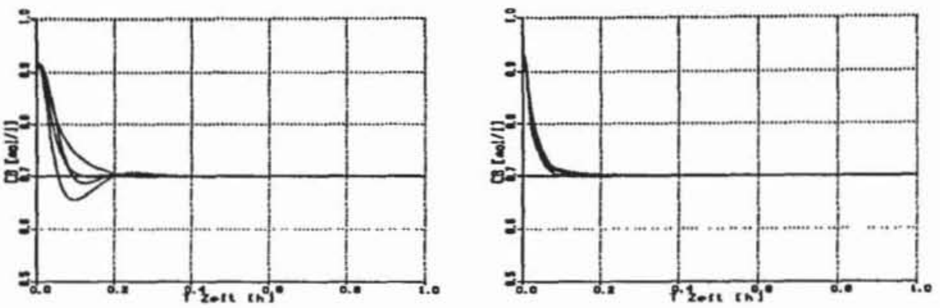

Figure 7: Step responses of $c_B(t)$ for H_∞-controller (left) and μ-controller (right). The disturbance c_{A0} is changed by +12 %. Simultaneously the setpoint $c_{B,ref}$ is changed by −23 %.

demands. To examine this, nonlinear simulations are performed. Fig. 7 shows typical step responses of c_B for the μ- and the H_∞-controller for simultaneous disturbance and setpoint changes. The full line is the step response for the nominal plant while the other lines are responses for plants with physical parameters that are changed within their uncertainty limits. It is clearly visible that the μ-controller does not degrade its performance significantly despite perturbed parameters while the H_∞-controller does alter its behavior considerably. The nonlinear simulations confirm the results of the μ-analysis and show that the μ-controller achieves a truly robust performance.

4 Conclusions

Design of robustly performing controllers for a class of practical control problems is considered. This class appears naturally when the design is based on models obtained by linearizing nonlinear balance equations with uncertain physical parameters.

This controller synthesis problem can be treated in the structured singular value (μ)-framework. However standard D-K-iteration for μ-synthesis cannot be applied. Instead we propose a technique involving μ-analysis of the problem and D-K-iteration of a modified problem. By this a μ-suboptimal controller can be found in an iterative procedure. It should however be stressed that there are cases for which the proposed scheme does not lead to satisfying results. An alternative solution to this problem could possibly be obtained by μ-K-iteration. Experience suggests however that an increased computational effort will be necessary in general.

The method is applied to the control of a CSTR for cyclopentenol synthesis. Without unrealistic effort it is possible to design a controller satisfying the robust performance requirement. Time domain simulations with a detailed nonlinear process model confirm the guaranteed performance. The increased computational effort necessary (e.g. compared to H_∞-design) pays out in a decreased conservatism and a guarantee for robust performance.

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