Singularities of Non-Rotationally Symmetric Solutions of Boundary Value Problems for the Lamé Equations in a 3 dimensional Domain with Conical Points

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1 Introduction

It is well known that singularities are present in solutions of boundary value problems for the Lamé equations in conical domains. It follows from the general theory [5, 9] that the solutions consist of singular terms of the form $r^{\alpha}(\ln r)^{q}\,\mathbf{F}(\alpha,\phi,\theta)$ (r is the distance to the vertex of the cone, ϕ and θ are the spherical angles) and a more regular term.

Rotationally symmetric solutions of the Lamé equations under zero boundary displacements or stress free boundary conditions are investigated in [1, 2], where the values of α and q have been computed.Here we are concerned with the more general case, namely that the volume and surface forces of our problems are non rotationally symmetric.That means that the solutions depend not only on r and θ, but on the polar angle ϕ too. Using a monotonicity principle of V.A.Kozlov , V.G.Maz'ja and C.Schwab [6] one can get regularity results for polyhedral domains too.

2 Formulation of the problem

Let Ω be a three-dimensional bounded domain with an only circular conical point on its boundary (see Fig.1). Assume that the displacement field $\mathbf{u}(\mathbf{x})$ of this isotropic elastic body satisfies the linear system of equations

$$\begin{aligned} \mathbf{L}\,\mathbf{u} &= \mathbf{L}(D_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) \\ &= \mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(\mathbf{x})) = -\mathbf{f}(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in\Omega \end{aligned} \tag{1}$$

and the boundary conditions

$$\mathbf{L}_1\,\mathbf{u} = \mathbf{L}_1(D_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) = \mathbf{g}_1(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in\partial\Omega \tag{2}$$

or

$$\mathbf{L}_2\,\mathbf{u} = \mathbf{L}_2(D_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) = \mathbf{S}(\mathbf{u}(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x}) = \mathbf{g}_2(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in\partial\Omega \tag{3}$$

where λ and μ are the Lamé constants, $\mathbf{f}(\mathbf{x})$ is the vector of the volume forces, $\mathbf{g}_1(\mathbf{x})$ is a prescribed displacement (Dirichlet conditions), $\mathbf{g}_2(\mathbf{x})$ is a traction (Neumann conditions). $\mathbf{S}(\mathbf{u}(\mathbf{x}))$ denotes the stress tensor, with Cartesian components

$$S_{ij}(\mathbf{u}(\mathbf{x})) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\nabla\cdot\mathbf{u}(\mathbf{x}) \tag{4}$$

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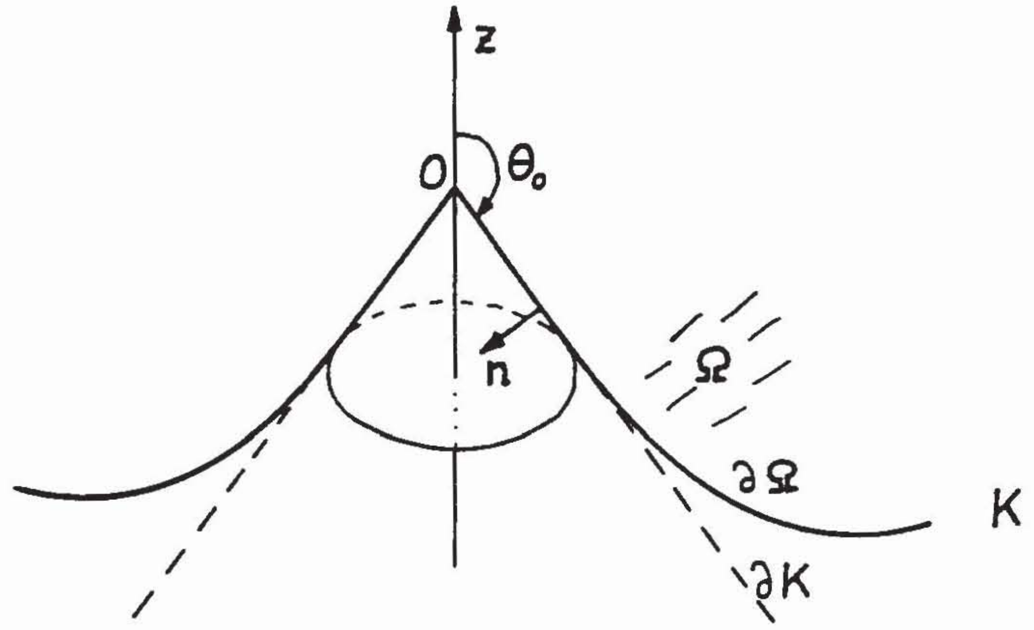

Figure 1: A body with a rotationally symmetric conical vertex

where u_i is the i^{th} component of $\mathbf{u}(\mathbf{x})$ and δ_{ij} is the Kronecker symbol. $\mathbf{n}(\mathbf{x})$ is the unit vector of the outward normal to $\partial\Omega$ in the point $\mathbf{x}$.

In general, one can consider right hand sides $\mathbf{f} \in [W_p^k(\Omega)]^3$ and $\mathbf{g}_i \in [W_p^{k+3-i-1/p}(\partial\Omega)]^3; i = 1,2; k = 0,1,\ldots; 1 \leq p < \infty$, but we can restrict to boundary forces $\mathbf{g}_i = 0$ subtracting from $\mathbf{u}$ a displacement field $\mathbf{v}$ with $\mathbf{L}_i\mathbf{v} = \mathbf{g}_i$ on $\partial\Omega$.

In order to investigate the behavior of $\mathbf{u}$ near the conical point $\mathbf{0}$ we can use the regularity results of V.A.KONDRATJEV, V.G.MAZ'JA and B.A.PLAMENEVSKIJ in the weighted Sobolev spaces

$$V_p^k(\Omega, \beta) = \left\{ u \in W_p^k(\Omega) : \| u \| = \left(\sum_{|\alpha| \leq k} \int_\Omega r^{p(|\alpha|+\beta-k)} |D^\alpha u| dx \right)^{\frac{1}{p}} < \infty \right\} \tag{5}$$

Thus it is proved in a paper of V.G.MAZ'JA and B.A.PLAMENEVSKIJ [8] (theorem 2.1) that every element-$\mathbf{f}$ from $[W_p^k(\Omega)]^3$ can be written as a sum

$$\mathbf{f} = \mathbf{f}^o + \mathbf{p} \tag{6}$$

where $\mathbf{f}^o$ is from $[V_p^k(\Omega,0)]^3$ and $\mathbf{p}$ is a polynomial vector of degree $\leq k-1$ provided $p > 3$, of degree $\leq k-2$ provided $3/2 < p < 3$ and of degree $\leq k-3$ provided $1 < p < 3/2$.

This decomposition yields an expansion of $\mathbf{u}$ near the conical point $\mathbf{0}$ in two kinds of singular terms; one is coming from $\mathbf{f}^o$ and one from the polynomial vector $\mathbf{p}$ (see [4, 5]). Thus we get for a weak solution $\mathbf{u}$ from $[W_2^1(\Omega)]^3$ of the equations

$$\begin{aligned} \mathbf{L}\mathbf{u} &= -\mathbf{f} \quad \text{in} \quad \Omega \\ \mathbf{L}_i\mathbf{u} &= 0 \qquad \text{on} \quad \partial\Omega \qquad i = 1 \,\text{or}\, 2 \end{aligned} \tag{7}$$

that

$$\mathbf{u} \;=\; \sum_{-1/2<\Re\alpha_i<-3/p+2+k} \eta\, c_i\, r^{\alpha_i} \left(\mathbf{F}_i(\phi,\theta,\alpha_i) + \ln r\, \mathbf{H}_i(\phi,\theta,\alpha_i) + \cdots\right) +$$

$$\sum_{-1/2<\beta_i<-3/p+2+k} \eta r^{\beta_i} (\mathbf{G}_i(\phi,\theta,\beta_i) + \ln r\, \mathbf{J}_i(\phi,\theta,\beta_i) + \cdots) + \mathbf{w} \tag{8}$$

where β_i is an integer, $\mathbf{w}$ is from $[W_p^{k+2}(\Omega)]^3$, $\eta = \eta(r)$ is a cut-off function near $\mathbf{0}$ and α_i are eigenvalues of a parameter depending boundary value problem. Here $\Re$ denotes the real and $\Im$ the imaginary part. In the following we consider only the first type of singularities, that means we assume that $\mathbf{f}$ in eq.(7) is from $[V_p^k(\Omega,0)]^3$. These singular terms are determined by the nontrivial solutions of the following "model" problem in the infinite cone K (see Fig.1).

$$\begin{aligned} \mathbf{Lu} &= \mathbf{0} \quad \text{in} \quad K \\ \mathbf{L}_i\mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial K \qquad i = 1 \,\text{or}\, 2 \end{aligned} \tag{9}$$

3 Solutions of the Lamé equations in an infinite cone

When studying problems over bodies with circular conical points (such as that shown in Fig.1) it is natural to use spherical coordinates (r,θ,ϕ) with origin at the apex $\mathbf{0}$. In these coordinates the local orthonormal basis vectors are

$$\begin{aligned} \mathbf{e}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T \\ \mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)^T \\ \mathbf{e}_\phi &= (-\sin\phi, \cos\phi, 0)^T \end{aligned} \tag{10}$$

Then any vector $\mathbf{u}$ can be written as $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_\phi\mathbf{e}_\phi$ or shortly

$$\mathbf{u} = \mathbf{u}(r,\theta,\phi) = \begin{pmatrix} u_r(r,\theta,\phi) \\ u_\theta(r,\theta,\phi) \\ u_\phi(r,\theta,\phi) \end{pmatrix} \tag{11}$$

Now we have to look for solutions of $\mathbf{Lu} = \mathbf{0}$ of the shape $r^\alpha\,\mathbf{F}(\phi,\theta,\alpha)$ (see section 2). Since the domain K is rotationally symmetric we are only interested in solutions with $\mathbf{F}(\phi,\theta,\alpha) = \mathbf{F}(\phi + 2\pi,\theta,\alpha)$. Therefore we consider the expansion of $\mathbf{u}$ in a Fourier series

$$\begin{aligned} \mathbf{u} &= r^\alpha\mathbf{F} \\ &= r^\alpha\{\mathbf{A}_0(\theta,\alpha) + \mathbf{A}_1(\theta,\alpha)\cos\phi + \mathbf{A}_2(\theta,\alpha)\cos 2\phi \\ &\quad +\mathbf{B}_1(\theta,\alpha)\sin\phi + \mathbf{B}_2(\theta,\alpha)\sin 2\phi + \cdots\} \end{aligned} \tag{12}$$

In order to construct the unknown coefficients $\mathbf{A}_i(\theta,\alpha)$ and $\mathbf{B}_i(\theta,\alpha)$ we use the Papkovich - Neuber representation of the displacement fields through harmonic functions. It holds that

$$\mathbf{u} = (3-4\nu)\begin{pmatrix} \mathbf{B}\cdot\mathbf{e}_r \\ \mathbf{B}\cdot\mathbf{e}_\theta \\ \mathbf{B}\cdot\mathbf{e}_\phi \end{pmatrix} - \begin{pmatrix} r\mathbf{e}_r\cdot\partial\mathbf{B}/\partial r + \partial B_4/\partial r \\ \mathbf{e}_r\cdot\partial\mathbf{B}/\partial\theta + r^{-1}\partial B_4/\partial\theta \\ \sin^{-1}\theta\,\mathbf{e}_r\cdot\partial\mathbf{B}/\partial\phi + (r\sin\theta)^{-1}\partial B_4/\partial\phi \end{pmatrix} \tag{13}$$

where $\mathbf{B} = (B_1,B_2,B_3)$, $\Delta B_i = 0$ in K, $i = 1...4$ and $\nu = \lambda/2(\lambda+\mu)$ is the Poisson ratio. The general form of the harmonic functions B_i, which yield displacement fields of the type eq.(12), is known [3], namely

$$\begin{aligned} B_1 &= r^\alpha\sum_{k=0}^{\infty} P_\alpha^{-k}(\cos\theta)\,(C_{k1}\cos k\phi + C_{k2}\sin k\phi) \\ B_2 &= r^\alpha\sum_{k=0}^{\infty} P_\alpha^{-k}(\cos\theta)\,(C_{k3}\cos k\phi + C_{k4}\sin k\phi) \end{aligned}$$

$$
\begin{aligned}
B_3 &= r^{\alpha}\sum_{k=0}^{\infty} P_{\alpha}^{-k}(\cos\theta)\,(C_{k5}\cos k\phi + C_{k6}\sin k\phi)\\
B_4 &= r^{\alpha+1}\sum_{k=0}^{\infty} P_{\alpha+1}^{-k}(\cos\theta)\,(C_{k7}\cos k\phi + C_{k8}\sin k\phi)
\end{aligned}
\tag{14}
$$

Here $P_\alpha^{-k} = P_\alpha^{-k}(\cos\theta)$ are the associated Legendre functions of first kind [7] with the special notation $P_\alpha = P_\alpha^0(\cos\theta)$.

Inserting eq.(14) into eq.(13) and using recurrence formulae for the Legendre functions [7] we get for $i = 0$

$$
\begin{aligned}
\mathbf{A}_0(\theta,\alpha) &= c_{10}(\alpha)\begin{pmatrix} (3-4\nu-\alpha)\,P_\alpha\cos\theta \\ -(3-4\nu)\,P_\alpha\sin\theta + (P_\alpha)'\sin\theta\cos\theta \\ 0 \end{pmatrix} + \\
&\quad c_{20}(\alpha)\begin{pmatrix} (\alpha+1)P_{\alpha+1} \\ -(P_{\alpha+1})'\sin\theta \\ 0 \end{pmatrix} + \\
&\quad c_{30}(\alpha)\begin{pmatrix} 0 \\ 0 \\ (1-\nu)P_\alpha^{-1} \end{pmatrix}
\end{aligned}
\tag{15}
$$

and for $i > 0$

$$
\begin{aligned}
\mathbf{A}_i(\theta,\alpha) &= c_{1i}(\alpha)\begin{pmatrix} (3-4\nu-\alpha)P_\alpha^{-i}\cos\theta \\ -(3-4\nu)P_\alpha^{-i}\sin\theta + (P_\alpha^{-i})'\sin\theta\cos\theta \\ -iP_\alpha^{-i}\cos\theta/\sin\theta \end{pmatrix} + \\
&\quad c_{2i}(\alpha)\begin{pmatrix} (\alpha+1)P_{\alpha+1}^{-i} \\ -(P_{\alpha+1}^{-i})'\sin\theta \\ iP_{\alpha+1}^{-i}/\sin\theta \end{pmatrix} + \\
&\quad c_{3i}(\alpha)\begin{pmatrix} (3-4\nu-\alpha)P_\alpha^{-i+1}\sin\theta \\ (3-4\nu)P_\alpha^{-i+1}\cos\theta + (P_\alpha^{-i+1})'\sin^2\theta \\ (4-4\nu-i)P_\alpha^{-i+1} \end{pmatrix}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\mathbf{B}_i(\theta,\alpha) &= d_{1i}(\alpha)\begin{pmatrix} (3-4\nu-\alpha)P_\alpha^{-i}\cos\theta \\ -(3-4\nu)P_\alpha^{-i}\sin\theta + (P_\alpha^{-i})'\sin\theta\cos\theta \\ iP_\alpha^{-i}\cos\theta/\sin\theta \end{pmatrix} + \\
&\quad d_{2i}(\alpha)\begin{pmatrix} (\alpha+1)P_{\alpha+1}^{-i} \\ -(P_{\alpha+1}^{-i})'\sin\theta \\ -iP_{\alpha+1}^{-i}/\sin\theta \end{pmatrix} + \\
&\quad d_{3i}(\alpha)\begin{pmatrix} (3-4\nu-\alpha)P_\alpha^{-i+1}\sin\theta \\ (3-4\nu)P_\alpha^{-i+1}\cos\theta + (P_\alpha^{-i+1})'\sin^2\theta \\ -(4-4\nu-i)P_\alpha^{-i+1} \end{pmatrix}
\end{aligned}
\tag{17}
$$

Here we have denoted $(P_\alpha^{-i})' = (P_\alpha^{-i}(\cos\theta))' = dP_\alpha^{-i}(\cos\theta)/d\cos\theta$. We remark that in the rotationally symmetric case [1, 2], where $u_\phi = 0$, $u_r = u_r(r,\theta)$ and $u_\theta = u_\theta(r,\theta)$ only $\mathbf{A}_0(\theta,\alpha)$ with $c_{30}(\alpha) = 0$ occurs.

4 Dirichlet conditions

In section 3 we have constructed the solution eq.(12) of the Lamé equation system in an infinite cone K. Now we have to determine the complex eigenvalues α in such a way, that the Dirichlet conditions $\mathbf{u} = \mathbf{0}$ on ∂K are satisfied. That means, we demand $\mathbf{u}(r, \theta_0, \phi) = \mathbf{0}$ and therefore we have to calculate those eigenvalues α_i $(i = 0, 1, \ldots)$, for which $\mathbf{A}_0(\theta_0, \alpha_0) = \mathbf{0}$ or $\mathbf{A}_i(\theta_0, \alpha_i) = \mathbf{B}_i(\theta_0, \alpha_i) = \mathbf{0}\,(i = 1, \ldots)$.

Let us begin with the case $i = 0$:
$\mathbf{A}_0(\theta, \alpha)$ consists of three linearly independent solutions (see eq.(15)). A nontrivial solution $\mathbf{u}_0 = r^{\alpha_0}\,\mathbf{A}_0(\theta, \alpha_0)$ with $\mathbf{u}_0(\theta_0, \alpha_0) = \mathbf{0}$ exists, if the determinate $\tilde{D}_0(\alpha, \theta_0)$ of the corresponding linear system of equations for the unknowns $c_{i0}(\alpha)$ $(i = 1, 2, 3)$ vanishes for $\alpha = \alpha_0$.

Using the recurrence formulae for the Legendre functions we get from $\tilde{D}_0$ the following transcendental equation

$$\begin{vmatrix} (3 - 4\nu - \alpha)\, P_\alpha \cos\theta_0 & P_{\alpha+1} & 0 \\ \begin{array}{l} P_\alpha\,(-3 + 4\nu \\ +(4 - 4\nu + \alpha)\cos^2\theta_0) \\ -(\alpha + 1)\, P_{\alpha+1} \cos\theta_0 \end{array} & \begin{array}{l} P_{\alpha+1} \cos\theta_0 \\ -P_\alpha \\ \end{array} & 0 \\ 0 & 0 & \begin{array}{l} P_\alpha \cos\theta_0 \\ -P_{\alpha+1} \end{array} \end{vmatrix} = 0 \tag{18}$$

The real parts of the zeros $\alpha_0 = \alpha_0(\theta_0, \lambda, \mu)$, where $\alpha_0 \neq 0$ and $\neq 1$, are shown in fig.2 for $\nu = 0.3$ and $0 < \theta_0 < \pi$. Real α_0's are drawn as solid lines, while the real part of complex α_0's is given by dashed ones. The "eigenfunctions" $\mathbf{F}_0(\theta, \alpha_0) = \mathbf{A}_0(\theta, \alpha_0)$ (see eq.(8) and eq.(12)) can be calculated inserting the nontrivial solutions $c_{i0}(\alpha_0)\,(i = 1, 2, 3)$ in eq.(15) . Comparing with the rotationally symmetric case ([2] fig.2) we find in fig.2 new solutions coming from the factor $P_\alpha \cos\theta_0 - P_{\alpha+1}$.
Now we consider the case $i > 0$:
Since $\mathbf{A}_i(\theta, \alpha)$ and $\mathbf{B}_i(\theta, \alpha)$ consist of three linearly independent solutions as in the case $i = 0$, we get a linear system of equations for the unknowns $c_{ij}(\alpha)$ and $d_{ij}(\alpha)$ in eqs.(16, 17). The zeros α_i of the corresponding determinates $\tilde{D}_i(\alpha, \theta_0)$ for $\mathbf{A}_i(\theta_0, \alpha) = \mathbf{0}$ or $\mathbf{B}_i(\theta_0, \alpha) = \mathbf{0}$ coincide. Now the following transcendental equation for α may be derived from $\tilde{D}_i$

$$\begin{vmatrix} (3 - 4\nu - \alpha)\, P_\alpha^{-i} \cos\theta_0 & -(\alpha + 1)\, P_{\alpha+1}^{-i} & i P_{\alpha+1}^{-i} \\ \begin{array}{l} P_\alpha^{-i}\,(-3 + 4\nu + \\ (4 - 4\nu + \alpha)\cos^2\theta_0) - \\ (\alpha + 1 + i)\, P_{\alpha+1}^{-i} \cos\theta_0 \end{array} & \begin{array}{l} (\alpha + 1 - i)\, P_\alpha^{-i} - \\ (\alpha + 1)\, P_{\alpha+1}^{-i} \cos\theta_0 \\ \end{array} & i\, P_{\alpha+1}^{-i} \cos\theta_0 \\ -i\, P_\alpha^{-i} \cos\theta_0 & -i\, P_{\alpha+1}^{-i} & \begin{array}{l} (\alpha + 1)\, P_{\alpha+1}^{-i} - \\ (\alpha + 1 - i)\, P_\alpha^{-i} \cos\theta_0 \end{array} \end{vmatrix}$$

$$= 0 \tag{19}$$

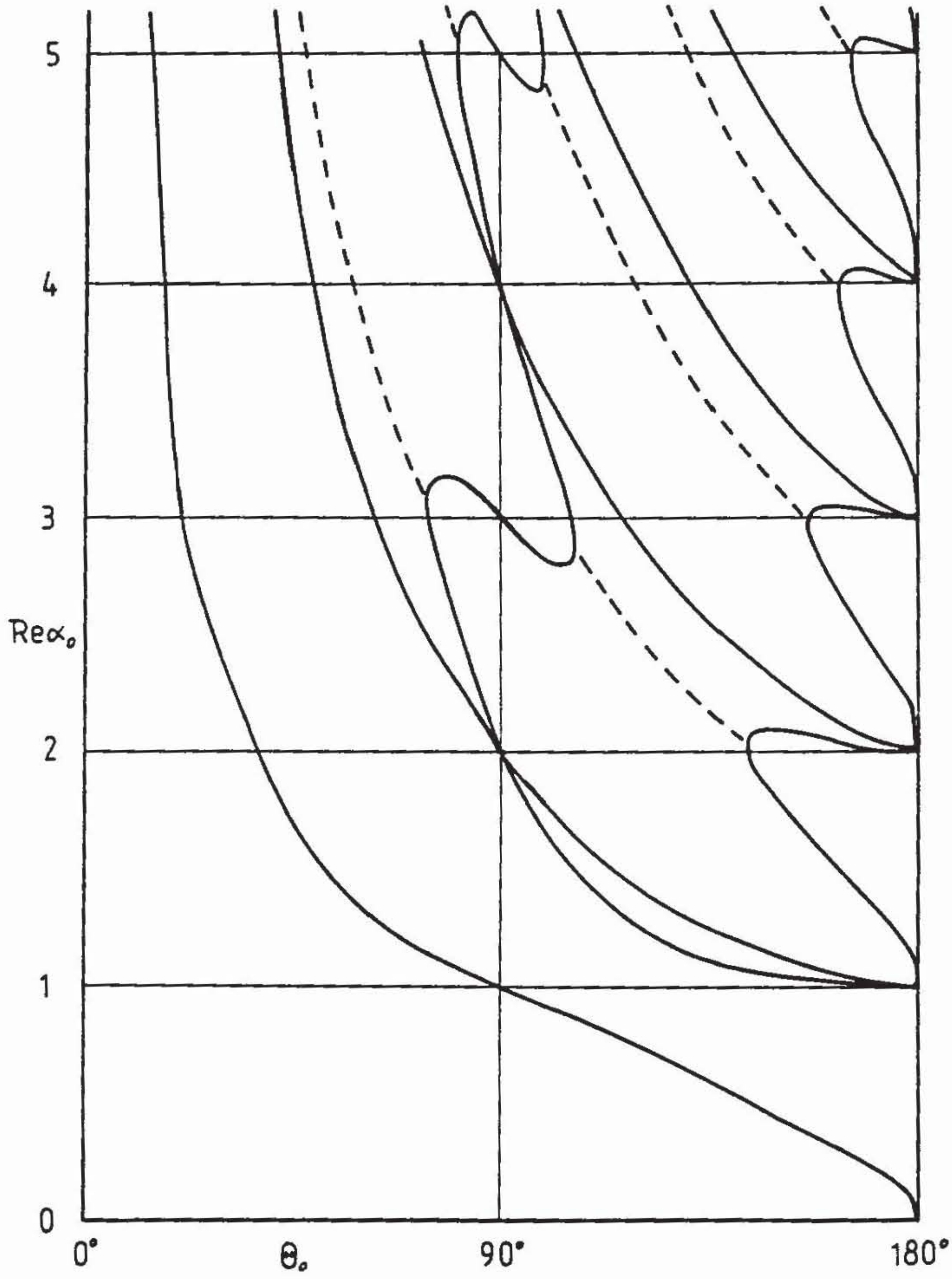

Figure 2: Dependence of the eigenvalues α_0 on θ_0 for Dirichlet conditions at $\nu = 0.3$ (—— $\Im\alpha = 0$, - - - - $\Im\alpha \neq 0$)

For the numerical calculation of the Legendre functions we used the Mehler - Dirichlet representation and the Hermite quadrature approximation

$$
\begin{aligned}
P_\alpha(\cos\theta) &= \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(\alpha+\frac{1}{2})t}{\sqrt{\cos t - \cos\theta}}\quad dt \\
&\approx \frac{2\sqrt{2}}{n\cdot}\sum_{j=1}^{[\frac{n+1}{2}]} G(x_j) \\
G(x) &= \frac{\sqrt{t}\cos(\alpha+\frac{1}{2})t}{\sqrt{\sin\frac{t+\theta}{2}\,\frac{2}{t-\theta}\,\sin\frac{t-\theta}{2}}}
\end{aligned}
\tag{20}
$$

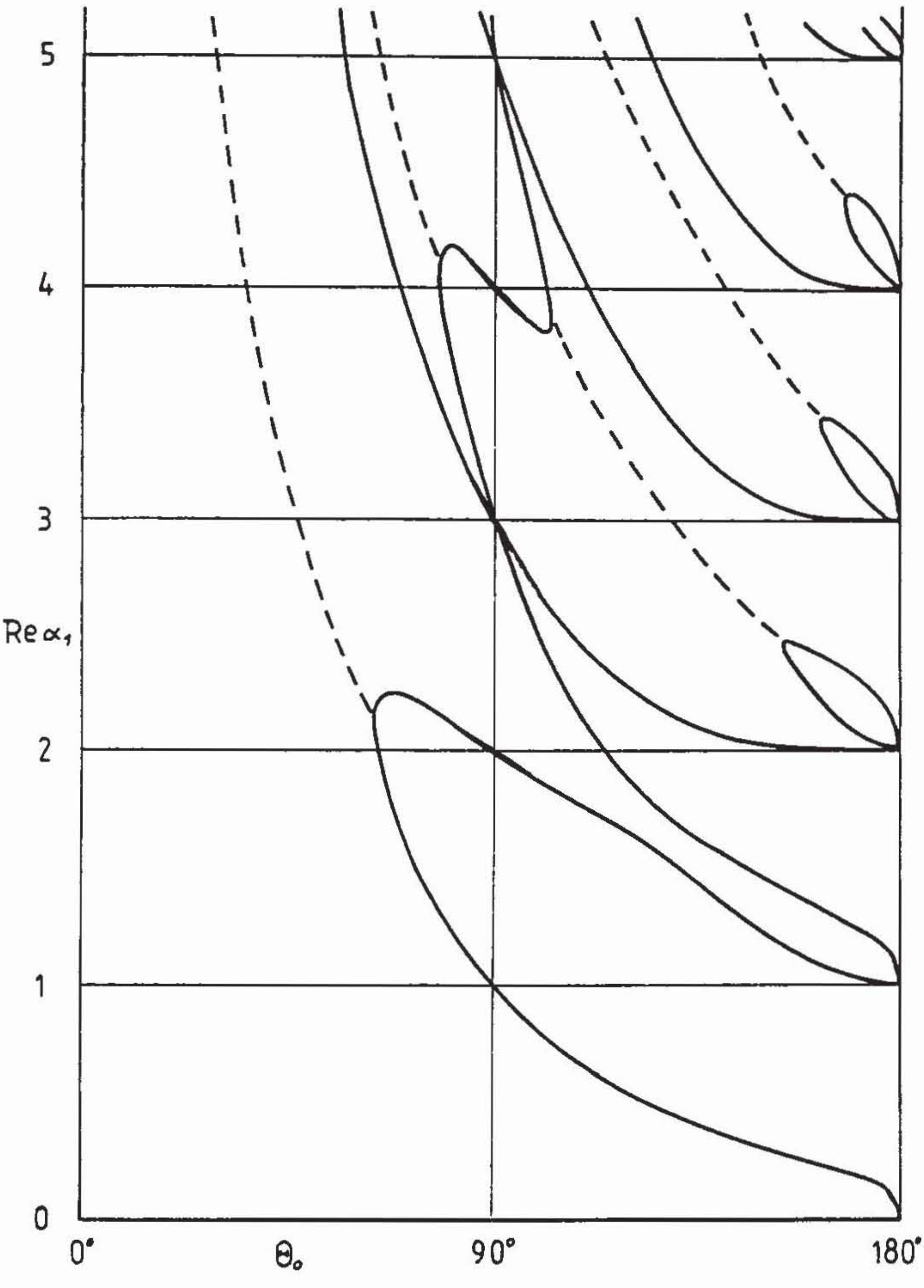

Figure 3: Dependence of the eigenvalues α_1 on θ_0 for Dirichlet conditions at $\nu = 0.3$ (—— $\Im\alpha = 0$, - - - - $\Im\alpha \neq 0$)

where $x_j = \cos((2j-1)\pi/2n)$ and $t = \theta x^2$. Numerical tests showed, that with $n = 100$ (i.e. 50 integration points) the relative accuracy of $\Re P_\alpha$ is about 10^{-4} and of $\Im P_\alpha$ about 10^{-8} in the interesting region for α.

The recurrence formulae

$$P_\alpha^{-i}(\cos\theta) = \left[\cos\theta P_\alpha^{-i+1} - P_{\alpha+1}^{-i+1}\right]/(\alpha - i + 1)\sin\theta \qquad (21)$$
$$P_{\alpha+1}^{-i}(\cos\theta) = \left[P_\alpha^{-i+1} - \cos\theta P_{\alpha+1}^{-i+1}\right]/(\alpha + i + 1)\sin\theta \qquad (22)$$

for $\alpha - i + 1 \neq 0$ or $\alpha + i + 1 \neq 0$ are sufficient stable for $i = 1, 2$. To calculate the zeros $\alpha_i(\theta_0)$ of the above given nonlinear equations we used a downhill simplex method [10], which searches for all

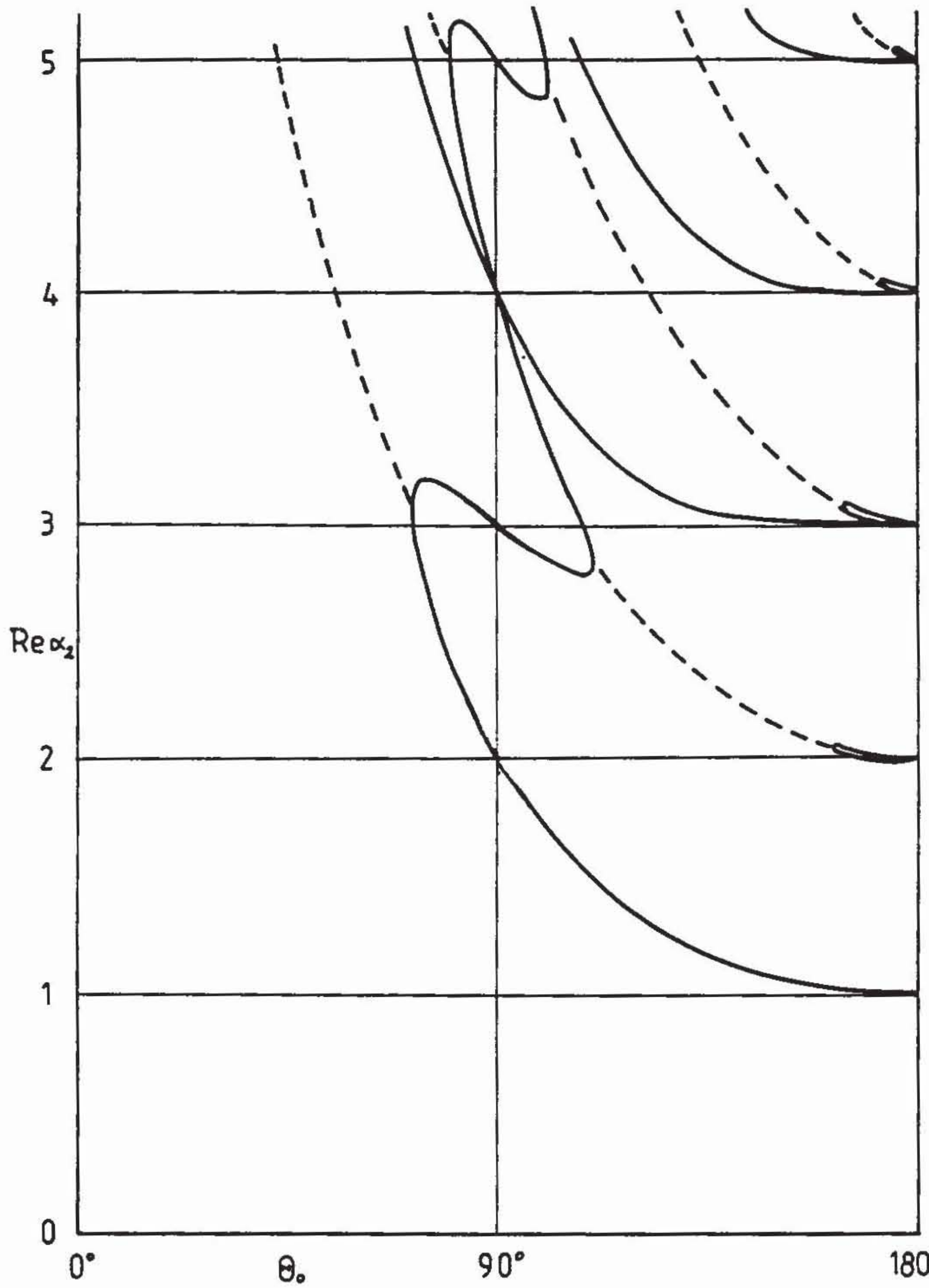

Figure 4: Dependence of the eigenvalues α_2 on θ_0 for Dirichlet conditions at $\nu = 0.3$ (——— $\Im\alpha = 0$, - - - - $\Im\alpha \neq 0$)

zeros in a rectangle in the α-plane. The figures 3 and 4 show the distribution of the real parts of the eigenvalues α_i $(i = 1, 2)$ for $\nu = 0.3$ and $0 < \theta_0 < \pi$.

Now we illustrate by an example, how the expansion eq.(8) looks like. Let $\mathbf{f}$ be from $[L_2(\Omega)]^3$. Then only those singular vector functions $r^{\alpha_i}\,\mathbf{F}_i(\phi, \theta, \alpha_i)$ occur in eq.(8), for which $-1/2 < \Re\alpha_i < 1/2$, whilst the singular vector functions $\mathbf{H}_i, \mathbf{G}_i$ and $\mathbf{J}_i$ do not occur.

Let us denote by θ_0^i those angles with $\alpha_i(\theta_0^i) = 1/2$, $i = 0, 1$. The numerical calculation yields $\theta_0^1 \approx 124.22^\circ$ and $\theta_0^0 \approx 143.66^\circ$ for $\nu = 0.3$.

Lemma 1 : Let $\mathbf{u}$ be a weak solution from $[\overset{\circ}{W}{}^1_2(\Omega)]^3$ of the Dirichlet problem

$$\mathbf{L}\,\mathbf{u} = -\mathbf{f} \quad \text{in } \Omega \tag{23}$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{24}$$

where $\Omega \subset R^3$ is a bounded domain with only one circular conical point $\mathbf{0}$ on the boundary, $\mathbf{f} \in [L_2(\Omega)]^3$ and $\nu = 0.3$. Then the following expansions near the conical point $\mathbf{0}$ hold:
For $0 < \theta_0 < \theta_0^1$ we have $\mathbf{u} = \mathbf{w} \in [W_2^2(\Omega)]^3$.
For $\theta_0^1 < \theta_0 < \theta_0^0$ we have $\mathbf{u} = r^{\alpha_1}(c_1\mathbf{A}_1(\theta,\alpha_1)\cos\phi + c_2\mathbf{B}_1(\theta,\alpha_1)\sin\phi) + \mathbf{w}$.
For $\theta_0^0 < \theta_0 < \pi$ we have $\mathbf{u} = r^{\alpha_1}(d_1\mathbf{A}_1(\theta,\alpha_1)\cos\phi + d_2\mathbf{B}_1(\theta,\alpha_1)\sin\phi) + r^{\alpha_0}c_0\mathbf{A}_0(\theta,\alpha_0) + \mathbf{w}$.

The eigenvalues $\alpha_0 = \alpha_0(\theta_0,\nu)$ and $\alpha_1 = \alpha_1(\theta_0,\nu)$ are given by the lowest lines of fig.(2) and fig.(3), respectively.

5 Stress boundary conditions

We start again from the general solutions eq.(12) of the Lamé equation system $\mathbf{L}\,\mathbf{u} = \mathbf{0}$ in the infinite cone and determine the complex numbers α in such a way, that $\mathbf{S}(\mathbf{u}) = \mathbf{0}$ on ∂K. Since $\mathbf{n} = \mathbf{e}_\theta$, the normal stresses are $\mathbf{S}(\mathbf{u}) \cdot \mathbf{n} = (S_{r\theta}, S_{\theta\theta}, S_{\phi\theta})^T$. After some calculations we get a linear system of equations for the unknowns $c_{ij}(\alpha)$ and $d_{ij}(\alpha)$ $(i = 0,1,\ldots; j = 1,2,\ldots)$ in eqs.(15, 16, 17). Nontrivial solutions exist, if the corresponding determinates vanish. We get for the calculation of the zeros α_i $(i = 0,1,\ldots)$ of the determinates the following transcendental equations

$$\begin{aligned}
N_i(\alpha,\theta_0) &\equiv det(\mathrm{N}) = 0 \\
N_{11} &= P_\alpha^{-i}(-\alpha(1-2\nu) + (\alpha^2 - 2 + 2\nu)\cos^2\theta_0) + \\
&\quad P_{\alpha+1}^{-i}(\alpha + i + 1)(2 - 2\nu - \alpha)\cos\theta_0 \\
N_{12} &= (\alpha - 1)iP_\alpha^{-i}/2 \\
N_{13} &= P_{\alpha+1}^{-i}\,\alpha\cos\theta_0 + P_\alpha^{-i}(-\alpha + i/2) \\
N_{21} &= P_\alpha^{-i}\cos\theta_0(2\alpha + 2 - 2\nu + \alpha^2 - i^2 + \\
&\quad \cos^2\theta_0(-3\alpha - 3 + 2\nu - \alpha^2)) + \\
&\quad P_{\alpha+1}^{-i}((\alpha + i + 1)((-3 + 2\nu)\sin^2\theta_0 + 1)) \\
N_{22} &= i(P_\alpha^{-i}\cos\theta_0(-\alpha - 2) + (\alpha + i + 1)P_{\alpha+1}^{-i}) \\
N_{23} &= P_\alpha^{-i}\cos\theta_0(i + 1) + P_{\alpha+1}^{-i}(-\alpha\sin^2\theta_0 - i - 1) \\
N_{31} &= i(P_\alpha^{-i}((-1 + 2\nu)(\alpha + 3 - 2\nu)\cos^2\theta_0) + \\
&\quad P_{\alpha+1}^{-i}\cos\theta_0(-\alpha - 1 - i)) \\
N_{32} &= P_\alpha^{-i}(\alpha + 1 + i^2 - \sin^2\theta_0(\alpha + 1)(\alpha + 2)/2) - \\
&\quad P_{\alpha+1}^{-i}(\alpha + 1 + i)\cos\theta_0 \\
N_{33} &= P_{\alpha+1}^{-i}\cos\theta_0(i + 1) + P_\alpha^{-i}((-i - 1) + \sin^2\theta_0(1 + \alpha/2))
\end{aligned} \tag{25}$$

Figures 5, 6 and 7 show the distribution of the eigenvalues α_i for $i = 0,1,2$, $\nu = 0.3$ and $0 < \theta_0 < \pi$.

Comparing with the results for rotationally symmetric solutions [1] and [2] (fig.3) we have new lines for $i = 0$ coming from the new factor $P_{\alpha+1}\cos\theta_0 + P_\alpha(-1 + \sin^2\theta_0(\alpha + 2)/2)$ in the nonrotationally symmetric case.

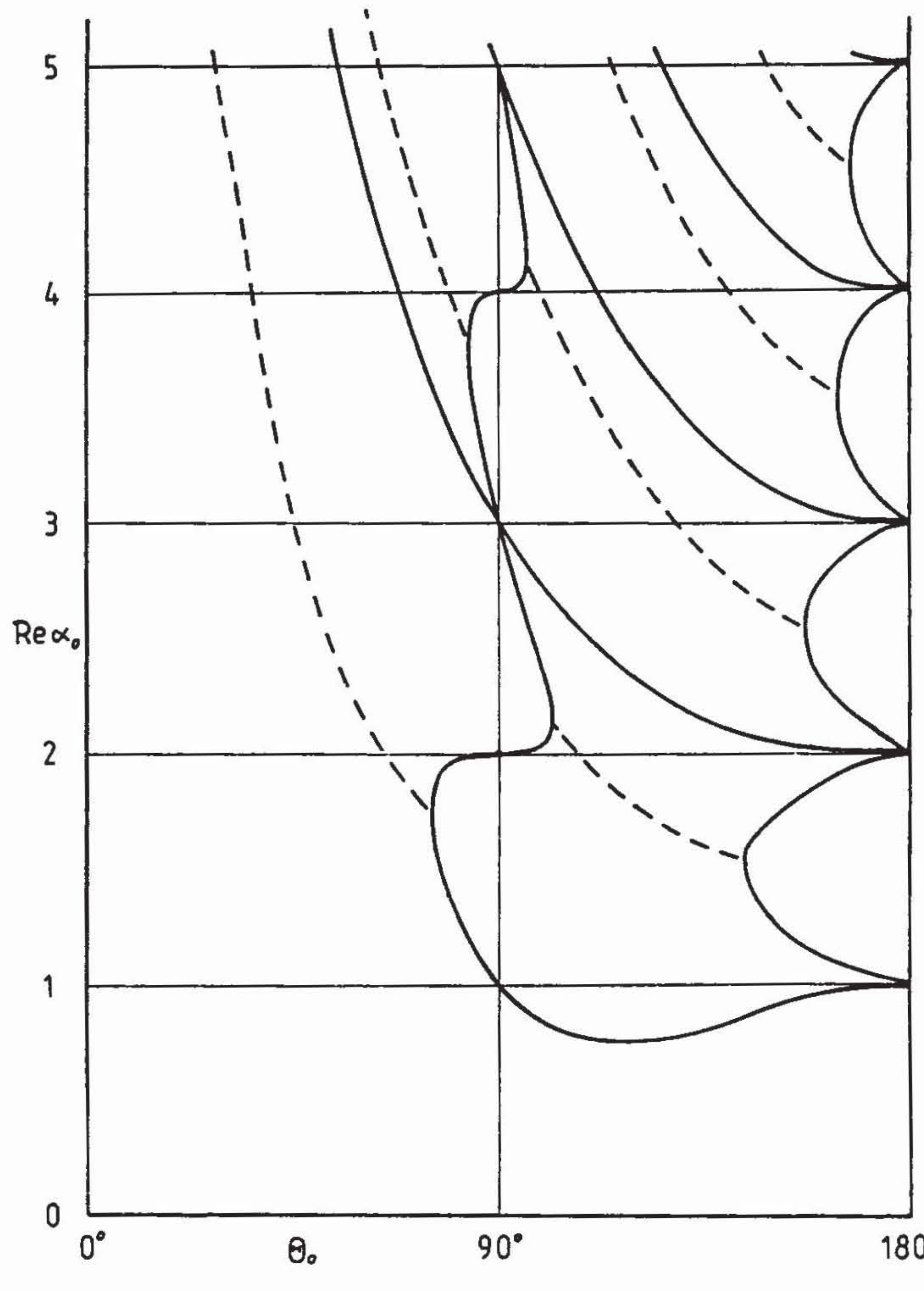

Figure 5: Dependence of the eigenvalues α_0 on θ_0 for Neumann conditions at $\nu = 0.3$ (——— $\Im\alpha = 0$, - - - - $\Im\alpha \neq 0$)

Analogously to the Dirichlet problem in section 4 we consider now a solution $\mathbf{u} \in [W_2^1(\Omega)]^3$ of $\mathbf{L}\,\mathbf{u} = -\mathbf{f}$ in Ω and $\mathbf{S}(\mathbf{u}) \cdot \mathbf{n} = 0$ on $\partial\Omega$ for $\nu = 0.3$ and $\mathbf{f} \in [L_2(\Omega)]^3$. Since only the eigenvalues in the strip $-1/2 < \Re\alpha_i < 1/2$ are of interest in the asymptotic expansion eq.(8), we get the following

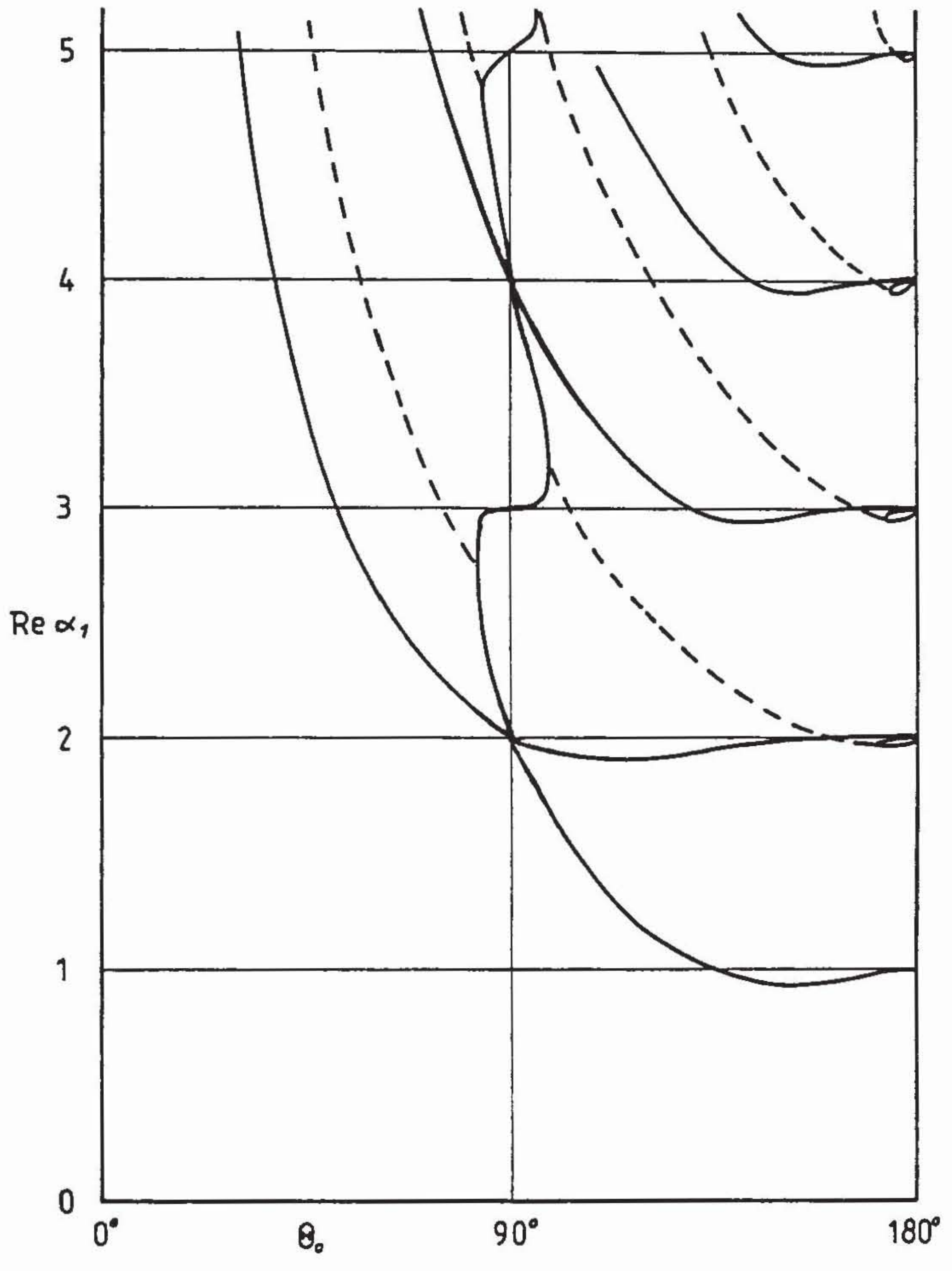

Figure 6: Dependence of the eigenvalues α_1 on θ_0 for Neumann conditions at $\nu = 0.3$ (——— $\Im\alpha = 0$, - - - - $\Im\alpha \neq 0$)

Lemma 2 : Let $\mathbf{u}$ be a solution from $[W_2^1(\Omega)]^3$ of the Neumann problem

$$\begin{aligned} \mathbf{L}\,\mathbf{u} &= -\mathbf{f} \quad \text{in } \Omega \\ \mathbf{S}(\mathbf{u}) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{26}$$

where Ω is a bounded domain in R^3 with circular conical points on the boundary, $\mathbf{f} \in [L_2(\Omega)]^3$ and $\nu = 0.3$. Then $\mathbf{u}$ is from $[W_2^2(\Omega)]^3$.

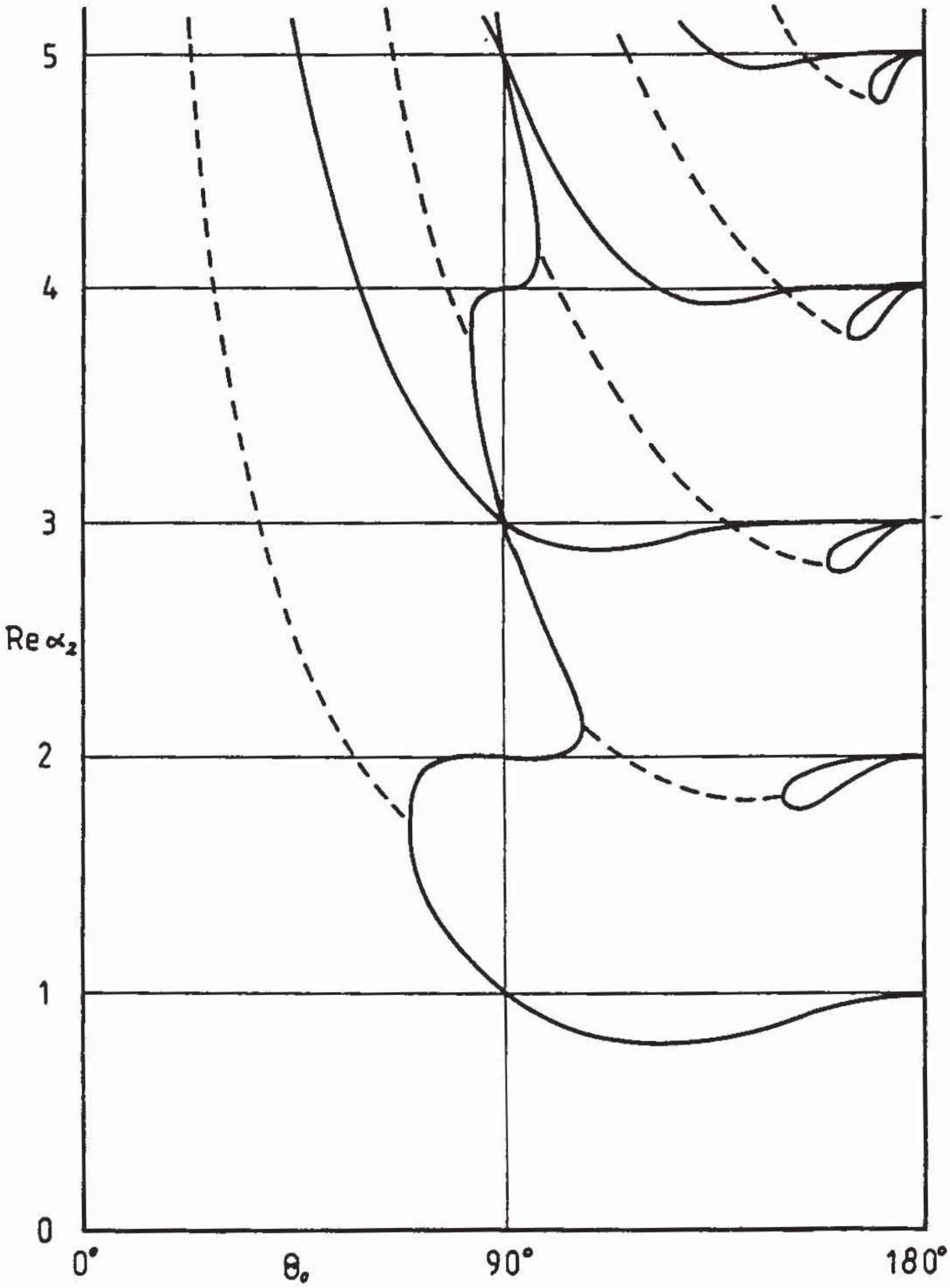

Figure 7: Dependence of the eigenvalues α_2 on θ_0 for Neumann conditions at $\nu = 0.3$ (——— $\Im\alpha = 0$, - - - - $\Im\alpha \neq 0$)

6 Conclusions

In this paper we have considered bounded domains with a circular conical point and calculated the corresponding singular terms. For polyhedral domains the situation is much more complicated. However we can estimate the "eigenvalues" α_k for the polyhedral corner singularities by the "eigenvalues" of axial symmetric conical corners in some cases, using a result of V.A.Kozlov, V.G.Maz'ja and C.Schwab. The corollary 1 of theorem 3.2 in [6] states for Dirichlet conditions:

Let be $K_i = (0,\infty) \times \Omega_i \, (i = 1,2)$ cones in R^3, where $\Omega_i \subset S^2$ and K_i can have edges. Then the "eigenvalues" α_k depend monotonically on Ω in the interval $(-1/2, \Lambda_\gamma(\Omega))$, i.e. if $\alpha_k \in (-1/2, \Lambda_\gamma(\Omega))$ and $\Omega_1 < \Omega_2$ then $\alpha_k(\Omega_2) \geq \alpha_k(\Omega_1)$. Here $\gamma = \mu/(\lambda+\mu)$ and $\Lambda_\gamma(\Omega) \geq 1$ is a real number.

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