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General Interface Problems—II

Serge Nicaise

Université de Valenciennes et du Hainaut Cambrésis, L M^2 I and U.R.A. D 751 CNRS 'GAT', Institut des Sciences et Techniques de Valenciennes, B.P. 311, F-59304-Valenciennes Cedex, France

and

Anna-Margerete Sändig

Mathematisches Institut a Universität Stuttgart, Postfach 80 11 40, D-70511 Stuttgart, Germany

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We continue the study of general interface problems. We prove regularity and asymptotics of solutions in usual Sobolev spaces for non-constant coefficients operators. We also give the stabilization procedure when unstable decompositions appear near a critical angle.

6. Introduction

This paper is the second of a work concerning general interface problems. In Part I, we stated the general framework. We showed that the variational solution admits a decomposition into regular and singular parts in weighted Sobolev spaces for homogeneous operators with constant coefficients. The first aim of Part II is to get similar results for arbitrary operators and in usual Sobolev spaces. The second goal is to restore a stable asymptotics for unstable decomposition occurring near critical angles.

We have numbered the sections continuously with the first part: sections 1–5 form Part I, sections 6–9 form Part II. We obviously use the notations and results of the first part without any comment and refer to them only by quotation of their numbers.

This paper is organized as follows. In section 7, we show that the variational solution of an interface problem for homogeneous operators with constant coefficients admits a decomposition into singular and regular parts. This is made in a more or less usual way, using the previous results in weighted Sobolev spaces (of section 4) and the so-called polynomial resolution, which means that we solve the problem with polynomial right-hand sides. In dimension 2 and for data in H^k, k a positive integer, due to the limit case of the Sobolev imbedding theorem, we use an interpolation argument (see [20] for a particular situation).

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We extend the previous results for general operators in section 8. This is proved in two steps: firstly for a fixed vertex, we show that the principal part (frozen at this vertex) of the operator is Fredholm (between usual Sobolev spaces) in the infinite cone which coincides with Ω in a neighbourhood of this vertex. Here we use a localization technique (instead of the Mellin transformation argument), taking into account the previous results on bounded domains. Secondly, we use Dauge's perturbation arguments (see [4, section 10]).

In the above-mentioned decomposition of the weak solution, the singular part comes from two different considerations: one comes from the comparison results in weighted Sobolev spaces, while the other comes from the polynomial resolution. In both cases (as the numerical examples show), some singular exponents give rise to unstable decomposition with respect to the variation of the angles of the conical points. In section 9, we show how to restore a stable asymptotics in both cases using differential equations in Hilbert spaces, perturbation theory for linear operators and divided differences (see [4]). We give two applications: a boundary value problem for operator of order $2m$ with Dirichlet boundary conditions in a cone of $\mathbb{R}^2$ and a particular transmission problem.

Let us finally say that throughout this paper we impose the assumptions of section 4, except that the operators A_i, F_{iqj} are arbitrary as in section 2.3. Conforming to the notations of section 3, for a differential operator B, defined either in Ω_i or on a side γ_{iq}, we denote by B^S its principal part frozen at S when S is a fixed vertex of Ω.

7. Constant coefficient operators

As usual, to pass from weighted Sobolev spaces to usual ones, we need to solve explicitly the considered problem with right-hand sides which are polynomials. This will be our first goal. Before doing so, we define some spaces of polynomials: If θ is an open set of $\mathbb{R}^n$, $l \in \mathbb{Z}$ and $S \in \mathscr{S}$, we set:

$\mathbf{P}_l^H(\theta) = \{q\colon q$ a homogeneous polynomial of degree l defined on $\theta\}$ if $l \geqslant 0$,
$\mathbf{P}_l^H(\theta) = \{0\}$ if $l < 0$,
$\mathbf{P}_l^H(C_S) = \{q$ defined on $C_S\colon q_i \in \mathbf{P}_l^H(C_i)\}$,

$$X_l^H(C_S) = \mathbf{P}_{l-2m}^H(C_S) \times \prod_{i \in \mathcal{N}_S} \left\{ \prod_{\gamma_{iq} \in \mathscr{E}_S} \prod_{j=0}^{m-1} \mathbf{P}_{l-m_{iqj}}^H(\Gamma_{iq}) \times \prod_{\gamma_{iq} \in \mathscr{I}_S} \prod_{j=0}^{2m-1} \mathbf{P}_{l-m_{iqj}}^H(\Gamma_{iq}) \right\},$$

where $\mathbf{P}_l^H(\Gamma_{iq})$ is simply the set of the restriction to Γ_{iq} of the elements of $\mathbf{P}_l^H(\mathbb{R}^n)$.

Lemma 7.1. *Let $S \in \mathscr{S}$ and $l \in \mathbb{N}$ be fixed. Then for all $\{p, \{q_{iqj}\}, \{q_{ii'j}\}\} \in X_l^H(C_S)$, there exists a (non-unique) solution $w_S \in L^2_{\mathrm{loc}}(C_S)$ of*

$$A_i w_{Si} = p_i \quad \text{in } C_i, \forall i \in \mathcal{N}_S, \tag{7.1}$$

$$B_{iqj} w_{Si} = q_{iqj} \quad \text{on } \Gamma_{iq}, \forall j \in \{0, \ldots, m-1\},\ \gamma_{iq} \in \mathscr{E}_S, \tag{7.2}$$

$$B_{iqj} w_{Si} - B_{i'q'j} w_{Si'} = q_{ii'j} \quad \text{on } \Gamma_{iq}, \forall j \in \{0, 1, \ldots, 2m-1\},\ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S. \tag{7.3}$$

Proof. We use the Kondratiev ansatz [8], looking for w_{Si} in the form

$$w_{Si}(r, \omega) = r^l \sum_{\nu=0}^{N_l} \varphi_{\nu i}(\omega)(\ln r)^\nu, \tag{7.4}$$

where (r, ω) are spherical coordinates centred at S. If l is not an eigenvalue of $\mathscr{A}_S(\zeta)$, then we may take $N_l = 0$; otherwise we have to take N_l as the higher length of the Jordan blocks associated with the eigenvalue l (see also [19, section 4] or section 9.1 hereafter. ∎

Definition 7.2. *Let $S \in \mathscr{S}$ and $l \in \mathbb{N}$ be fixed.*

(a) *We say that the operator $\{A_i, B_{iqj}\}$ is injective modulo the polynomials on $S^l(C_S)$ iff any solution w_S in the form* (7.4) *of* (7.1)–(7.3) *with a right-hand side in $X_l^H(C_S)$, is in $\mathbf{P}_l^H(C_S)$* (*cf.* [4 *Definition* 3.8]).

(b) *If $\{A_i, B_{iqj}\}$ is not injective modulo the polynomials on $S^l(C_S)$, then we take a basis $\{q_\nu\}_{\nu=1}^{M_l}$ of the space*

$$X_l^H(C_S) \backslash Y_l^H(C_S),$$

where we have set

$$Y_l^H(C_S) = \{\{(A_i p_i)_{i \in \mathscr{N}_S}, \{B_{iqj} p_i\}_{\gamma_{iq} \in \mathscr{E}_S}, \{B_{iqj} p_i - B_{i'q'j} p_{i'}\}_{\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S}\}$$
$$\text{such that } p \in \mathbf{P}_l^H(C_S)\}.$$

From Lemma 7.1, *for $\nu \in \{1, \ldots, M_l\}$, we may fix once and for all a solution $e^{S,l,\nu}$ of problem* (7.1)–(7.3) *with data q_ν. The functions $\{e^{S,l,\nu}\}_{\nu=1}^{M_l}$ are clearly linearly independent and cannot be polynomials. For simplicity, if $\{A_i, B_{iqj}\}$ is injective modulo the polynomials of $S^l(C_S)$, we set $M_l = 0$.*

For $k \in \mathbb{N}$, $p \in]1, +\infty[$ in dimension 2 and $p = 2$ in dimension 3, we introduce the Banach space, where we shall take the data

$$X^{k,p}(\Omega) := \mathscr{W}^{k,p}(\Omega) \times \prod_{\gamma_{iq} \in \mathscr{E}} \prod_{j=0}^{m-1} W^{k+2m-m_{iqj}-1/p,p}(\gamma_{iq}) \cap \mathring{H}^{m-m_{iqj}-1/2}(\gamma_{iq})$$
$$\times \prod_{\gamma_{iq} \in \mathscr{I}} \prod_{j=0}^{2m-1} W^{k+2m-m_{iqj}-1/p,p}(\gamma_{iq}) \cap \mathring{H}^{m-m_{iqj}-1/2}(\gamma_{iq}),$$

with the agreement that we take the intersection if $m_{iqj} \leqslant m - 1$, otherwise we simply take $W^{k+2m-m_{iqj}-1/p,p}(\gamma_{iq})$. Let $N = [k + 2m - n/p]$. For a fixed $(f, \{g_{iqj}\}, \{g_{ii'j}\}) \in X^{k,p}(\Omega)$, we denote by $\Pi_{N-2m}^S f_i$ the limited Taylor expansion of f_i at S of order $N - 2m$, i.e.

$$\Pi_{N-2m}^S f_i(x) = \sum_{|\alpha| \leqslant N-2m} D^\alpha f_i(S) \frac{x^\alpha}{\alpha!},$$

in the Euclidean co-ordinates x centred at S. Obviously, if $N < 2m$, we take $\Pi_{N-2m}^S f_i = 0$. Analogously, we denote by $\prod_{N-m_{iqj}}^S g_{iqj}$, the limited Taylor expansion of g_{iqj} at S of order $N - m_{iqj}$ (in dimension 3, this is the restriction to γ_{iq} of $\prod_{N-m_{iqj}}^S h_{iqj}$ when $h_{iqj} \in W^{k+2m-m_{iqj},p}(\Omega_i)$ is such that $h_{iqj} = g_{iqj}$ on γ_{iq}). By Lemma 7.1 and Definition 7.2, there exists a solution $w_S \in L^2_{\text{loc}}(C_S)$ of

$$A_i w_{S,i} = \Pi_{N-2m}^S f_i \text{ in } C_i, \quad \forall i \in \mathscr{N}_S, \tag{7.5}$$

$$B_{iqj}w_{S,i} = \Pi^S_{N-m_{iqj}} g_{iqj} \text{ on } \gamma_{iq}, \quad \forall j \in \{0, \ldots, m-1\}, \ \gamma_{iq} \in \mathscr{E}_S, \tag{7.6}$$

$$B_{iqj}w_{S,i} - B_{i'q'j}w_{S,i'} = \Pi^S_{N-m_{iqj}} g_{ii'j} \text{ on } \gamma_{iq},$$
$$\forall j \in \{0, \ldots, 2m-1\}, \ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S. \tag{7.7}$$

This solution admits the expansion

$$w_S = \sum_{l=0}^{N} \sum_{v=1}^{M_l} d_{S,l,v} e^{S,l,v} + q_S, \tag{7.8}$$

where q_S is a polynomial on each C_i, $i \in \mathscr{N}_S$, and $d_{S,l,v} \in \mathbb{C}$ is a linear combination of $D^\alpha f_i(S)$ for $|\alpha| \leqslant l - 2m$, of $D^\alpha g_{iqj}(S)$ for $|\alpha| \leqslant l - m_{iqj}$ and of $D^\alpha g_{ii'j}(S)$ for $|\alpha| \leqslant l - m_{iqj}$.

Theorem 7.3. *Let $p \in]1, +\infty[\setminus\{2\}$ in dimension 2 and $p = 2$ in dimension 3. Assume that the line $\operatorname{Re}\zeta = k + 2m - n/p$ contains no eigenvalue of $\mathscr{A}_S(\zeta)$ for all $S \in \mathscr{S}$. Take $(f, \{g_{iqj}\}, \{g_{ii'j}\}) \in X^{k,p}(\Omega)$ satisfying*

$$d_{S,l,v} = 0, \quad \forall l \in \mathbb{Z} \cap \left[0, \ldots, m - \frac{n}{2}\right]: v = 1, \ldots, M_l: M_l \neq 0. \tag{7.9}$$

Then there exists a unique weak solution $u \in \mathscr{H}^m_v(\Omega)$ of problems (2.6)–(2.8), in the sense that there exists $v \in \mathscr{H}^m_v(\Omega)$ satisfying (4.12) and (4.13) when $m_{iqj} \leqslant m - 1$ and $u \in V_0 = v + V$ satisfies (4.17). This solution admits the expansion

$$u = u_0 + \sum_{\substack{S \in \mathscr{S} \\ (\lambda,\mu,k) \in \Lambda^1_S(k,p,0)}} c_{S,\lambda,\mu,k} \Phi_S \sigma^{S,\lambda,\mu,k} + \sum_{\substack{S \in \mathscr{S} \\ (l,v) \in \Lambda^2_S(k,p)}} d_{S,l,v} \Phi_S e^{S,l,v}, \tag{7.10}$$

where $u_0 \in \mathscr{W}^{k+2m,p}(\Omega)$, $c_{S,\lambda,\mu,k} \in \mathbb{C}$ and we define

$$\Lambda^2_S(k,p) = \Big\{(l,v): l \in \mathbb{Z} \cap \left]m - \frac{n}{2}, k + 2m - \frac{n}{p}\right[$$
$$\text{such that } M_l \neq 0; v = 1, \ldots, M_l\Big\}. \tag{7.11}$$

Finally, we have the estimate

$$\|u_0\|_{\mathscr{W}^{k+2m,p}(\Omega)} + \sum_{\substack{S \in \mathscr{S} \\ (\lambda,\mu,k) \in \Lambda^1_S(k,p,0)}} |c_{S,\lambda,\mu,k}| \leqslant C \, \|(f, \{g_{iqj}\}, \{g_{ii'j}\})\|_{X^{k,p}(\Omega)}, \tag{7.12}$$

where C does not depend on the data.

Proof. Let us set

$$w = \sum_{S \in \mathscr{S}} \Phi_S w_S.$$

In view of (7.5)–(7.7), and using the Hardy inequalities, we have

$$\tilde{f}_i := f_i - A_i w_i \in \mathscr{V}^{k,p}_0(\Omega_i), \quad \forall i \in \mathscr{N},$$
$$\tilde{g}_{iqj} := g_{iqj} - B_{iqj} w_i \in V^{k+2m-m_{iqj}-1/p,p}_0(\gamma_{iq})$$

for all $j \in \{0, \ldots, m-1\}$, $\gamma_{iq} \in \mathscr{E}$, and

$$\tilde{g}_{ii'j} := g_{ii'j} - B_{iqj} w_i + B_{i'q'j} w_{i'} \in V^{k+2m-m_{iqj}-1/p,p}_0(\gamma_{iq})$$

for all $j \in \{0, \ldots, m-1\}$, $\gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}$. Therefore, by Corollary 4.4, with $\underset{\sim}{\gamma} = 0$, there exists a unique weak solution $\tilde{u} \in \mathscr{H}_v^m(\Omega)$ of problem (2.6)–(2.8) with data $\tilde{f}_i$, $\tilde{g}_{iqj}$, $\tilde{g}_{ii'j}$, which admits the expansion

$$\tilde{u} = \tilde{u}_0 + \sum_{\substack{S \in \mathscr{S} \\ (\lambda, \mu, k) \in \Lambda_s^1(k, p, 0)}} c_{S, \lambda, \mu, k}(\tilde{u}) \Phi_S \sigma^{S, \lambda, \mu, k}, \tag{7.13}$$

where $\tilde{u}_0 \in \mathscr{V}_0^{k+2m, p}(\Omega)$ and $c_{S, \lambda, \mu, k}(\tilde{u})$ means that it is a continuous linear form on the data $\tilde{u}$. Setting $u = \tilde{u} + w$, we deduce that $u \in \mathscr{H}_v^m(\Omega)$ is the unique weak solution of problem (2.6)–(2.8), since the assumption (7.9) implies that

$$w_S \in \mathscr{H}^{2m}(\Omega)$$

(using the expansion (7.8) of w_S and Remark AA.3 of [4]).

By the continuous dependence of w with respect to f, g_{iqj} and $g_{ii'j}$ and using the estimate (4.11) for $\tilde{u}_0$, we obtain (7.12). ■

For $p = 2$ in dimension 2, we shall use an interpolation argument as in [18].

Theorem 7.4. *Let $n = 2$ and $p = 2$. Assume that $\{A_i, B_{iqj}\}$ is injective modulo the polynomials on $S^{k+2m-1}(C_S)$ and the line $\operatorname{Re} \zeta = k + 2m - 1$ contains no eigenvalue of $\mathscr{A}_S(\zeta)$ except possibly at $\zeta = k + 2m - 1$, for all $S \in \mathscr{S}$.*

Under the same assumptions on f, g_{iqj} and $g_{ii'j}$ as in Theorem 7.3, then the conclusion of Theorem 7.3 still holds.

Proof. We set $p(0) = 2 - \varepsilon$ and $p(1) = 2 + \varepsilon$, where $\varepsilon > 0$ is fixed sufficiently small so that no eigenvalue of $\mathscr{A}_S(\zeta)$ belongs to the strip $\operatorname{Re} \zeta \in [k + 2m - 2/p(0), k + 2m - 2/p(1)]$ except eventually $k + 2m - 1$ (see [4, section 4]).

For $j = 0$ or 1, by Theorem 7.3, for all $(f, \{g_{iqj}\}, \{g_{ii'j}\}) \in X^{k, p(j)}(\Omega)$ satisfying (7.9), there exists a unique solution $u^{(j)}$ of (2.6)–(2.8), which admits the expansion

$$u^{(j)} = u_0^{(j)} + \sum_{\substack{S \in \mathscr{S} \\ (\lambda, \mu, k) \in \Lambda_s^1(k, p(j), (0)}} c_{S, \lambda, \mu, k}^j \Phi_S \sigma^{S, \lambda, \mu, k} + \sum_{\substack{S \in \mathscr{S} \\ (l, \nu) \in \Lambda_s^2(k, 2)}} d_{S, l, \nu} \Phi_S e^{S, l, \nu}, \tag{7.14}$$

where $u_0^{(j)} \in \mathscr{W}^{k+2m, p(j)}(\Omega)$. The coefficient $c_{S, \lambda, \mu, k}^j \in \mathbb{C}$ is given by

$$c_{S, \lambda, \mu, k}^j = c_{S, \lambda, \mu, k}(u - \Sigma_{S \in \mathscr{S}} \Phi_S w_S^j), \tag{7.15}$$

where w_S^j is the solution of (7.5)–(7.7) with data $\Pi_{N^j - 2m}^S f_i$, $\Pi_{N^j - m_{iq}}^S g_{iqj}$, $\Pi_{N^j - m_{iqj}}^S g_{ii'j}$, when $N^j = k + 2m - 2$ if $j = 0$ and $k + 2m - 1$ if $j = 1$.

Now, we introduce the operator

$$T: X^{k, p(0)}(\Omega) \to \mathscr{W}^{k+2m, p(0)}(\Omega) : (f, \{g_{iqj}\}, \{g_{ii'j}\}) \to u_0^{(0)}, \tag{7.16}$$

where $u_0^{(0)}$ is the unique regular part appearing in (7.14) for $j = 0$. The estimate (7.12) shows that T is a bounded operator. Let us now prove that T is also bounded from $X^{k, p(1)}(\Omega)$ into $\mathscr{W}^{k+2m, p(1)}(\Omega)$. To do that let us fix $(f, \{g_{iqj}\}, \{g_{ii'j}\}) \in X^{k, p(1)}(\Omega)$, then the unique solution $u \in \mathscr{H}_v^m(\Omega)$ of (2.6)–(2.8) admits both expansion (7.14) for $j = 0$ and 1.

From the exact expansion (7.8) of w_S^j and the hypothesis of injectivity modulo the polynomials, we deduce that

$$w_S^1 - w_S^0 = q_S^1 - q_S^0 = p \text{ in a neighbourhood of } S,$$

where $p \in \mathbf{P}^H_{k+2m-1}(C_S)$. Moreover, since p behaves like r^{k+2m-1} near S, we clearly have

$$w_S^1 - w_S^0 \in \mathscr{V}_0^{k+2m,p(0)}(\Omega).$$

Owing to Theorem 4.2, we get

$$c_{S,\lambda,\mu,k}(w_S^1 - w_S^0) = 0, \quad \forall \lambda\colon \operatorname{Re}\lambda < k + 2m - 1. \tag{7.17}$$

Joined with (7.15), we obtain

$$c^1_{S,\lambda,\mu,k} = c^0_{S,\lambda,\mu,k}, \quad \forall \lambda\colon \operatorname{Re}\lambda < k + 2m - 1. \tag{7.18}$$

Substracting (7.14) for $j = 0$ from (7.14) for $j = 1$ and using (7.18), we arrive at

$$u_0^{(1)} - u_0^{(0)} = \sum_{\substack{S \in \mathscr{S} \\ \lambda = k+2m-1}} c^1_{S,\lambda,\mu,k} \Phi_S \sigma^{S,\lambda,\mu,k}.$$

From the assumption of injectivity modulo the polynomials, we know that $\sigma^{S,\lambda,\mu,k}$ is a polynomial when $\lambda = k + 2m - 1$. This shows that $u_0^{(0)} \in \mathscr{W}^{k+2m,p(1)}(\Omega)$, while the estimate (7.12) for $j = 1$ proves the boundedness of T as an operator from $X^{k,p(1)}(\Omega)$ into $\mathscr{W}^{k+2m,p(1)}(\Omega)$.

The conclusion follows by interpolation since the Riesz–Thorin theorem implies that T is a bounded operator from $X^{k,2}(\Omega)$ into $\mathscr{W}^{k+2m,2}(\Omega)$.

Let us finally notice that (7.18) shows that the application

$$\mathscr{W}^{k,p(0)}(\Omega) \to \mathbb{C} : (f, \{g_{iqj}\}, \{g_{ii'j}\}) \to c^0_{S,\lambda,\mu,k}$$

is also a bounded operator from $\mathscr{W}^{k,p(1)}(\Omega)$ into $\mathbb{C}$. By interpolation, this proves that (7.12) still holds for $p = 2$. ■

8. Non-constant coefficients operators

In this section, we fix once and for all a vertex $S \in \mathscr{S}$ and define the following domains. For all $i \in \mathscr{N}_S$, we set:

(1) $\Omega_{Si} = C_i \cap B(S, 1)$ in dimension 3,
(2) Ω_{Si} is the triangle in C_i determined by the three vertices of $C_i \cap B(S, 1)$ (see Fig. 19) in dimension 2,
(3) $\Gamma^1_{iq} = \Gamma_{iq} \cap B(S, 1)$,
(4) $\Gamma^2_i = \partial\Omega_{Si} \backslash \bigcup_q \bar{\Gamma}^1_{iq}$.

We now define

$$\Omega_S = \textstyle\bigcup_{i \in \mathscr{N}_S} \Omega_{Si},$$

$$W_S = \{u \in \mathscr{H}^m_v(\Omega_S) \text{ satisfying (8.1)–(8.4)}\},$$

the latter being a Hilbert space with the inner product of $\mathscr{H}^m(\Omega_S)$.

$$F^S_{iqj} u_i = 0 \text{ on } \Gamma^1_{iq}, \quad \forall \gamma_{iq} \in \mathscr{E}_S, \quad j \in \mathscr{S}_{iq}. \tag{8.1}$$

$$F^S_{iqj} u_i = 0 \text{ on } \Gamma^1_{iq},$$

$$F_{i'q'j} u_{i'} = 0 \text{ on } \Gamma^1_{i'q'}, \quad \forall \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S, \quad j \in \mathscr{S}^1_{ii'}. \tag{8.2}$$

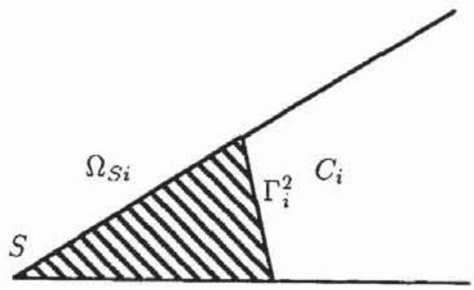

Fig. 18

$$F^S_{iqj}u_i = F^S_{i'q'j}u_{i'} \text{ on } \Gamma^1_{iq}, \quad \forall \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S, \quad j \in \mathscr{S}^2_{ii'}. \tag{8.3}$$

$$\frac{\partial^j u_i}{\partial v^j} = 0 \text{ on } \Gamma_i^2, \quad \forall j = 0, \ldots, m-1, \quad i \in \mathscr{N}_S. \tag{8.4}$$

We finally introduce a sesquilinear form b_S on W_S as follows:

$$b_S(u, v) = \sum_{i \in \mathscr{N}_S} \sum_{|\alpha| = |\beta| = m} a^i_{\alpha\beta}(S) \int_{\Omega_{Si}} D^\alpha u(x) D^\beta \bar{v}(x)\, dx, \quad \forall u, v \in W_S. \tag{8.5}$$

Lemma 8.1. *Under the previous assumptions, b_S is strongly coercive on W_S.*

Proof. By (2.19) and using analogous arguments as in [4, Proposition 8.1], there exists a positive constant α such that

$$\operatorname{Re} \tilde{b}_S(u, u) \geqslant \alpha |u|^2_{\mathscr{H}^m(C_S)}, \quad \forall u \in \tilde{W}_S, \tag{8.6}$$

where $\tilde{b}_S$ and $\tilde{W}_S$ are defined as b_S and W_S replacing Ω_{Si} by C_i; $|\cdot|_{\mathscr{H}^m(C_S)}$ denote, as usual, the semi-norm of $\mathscr{H}^m(C_S)$.

On the other hand, for $u \in W_S$, let us denote by $\tilde{u}$, its extension to C_S by zero outside Ω_S. We easily check that $\tilde{u}$ remains in $\tilde{W}_S$. Therefore, the estimate (8.6) implies that

$$\operatorname{Re} b_S(u, u) \geqslant \alpha |u|^2_{\mathscr{H}^m(\Omega_S)}, \quad \forall u \in W_S.$$

This proves the lemma, since the boundary conditions (8.4) imply that the norm and the semi-norm of $\mathscr{H}^m$ are equivalent on W_S.

Since Ω_S is a domain in the sense of section 2.1 and the sesquilinear form b_S is strongly coercive on W_S, we may apply the results of Theorems 7.3 and 7.4 on Ω_S. This will be useful in the following theorem. In this theorem, we need the following assumptions (H_n):

(H_2) In dimension 2, we suppose that $\{A^S_i, B^S_{iqj}\}$ is injective modulo the polynomials on $S^{k+2m-1}(C_S)$, and the line $\operatorname{Re} \lambda = k + 2m - 1$ contains no eigenvalue of $\mathscr{A}_S(\zeta)$ except possibly at $\zeta = k + 2m - 1$.

(H_3) In dimension 3, we assume that the line $\operatorname{Re} \zeta = k + 2m - \frac{3}{2}$ contains no eigenvalue of $\mathscr{A}_S(\zeta)$.

Theorem 8.2. *If (H_n) holds, then the operator $\mathfrak{U}^S + I$ defined by*

$$\mathfrak{U}^S + I : \mathscr{H}^{2m+k}(C_S) \cap \mathscr{H}^m_v(C_S) \to X^{k,2}(C_S),$$

$$u \to \big(\{A^S_i u_i + u_i\}_{i \in \mathscr{N}_S}, \{B^S_{iqj} u_i\}_{\substack{j=0,\ldots,m-1 \\ \gamma_{iq} \in \mathscr{E}_S}}, \{B^S_{iqj} u_i - B^S_{i'q'j}\}_{\substack{j=0,\ldots,2m-1 \\ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S}}\big) \tag{8.7}$$

is a Fredholm operator of index $\geqslant -N_S$, *where we set* $N_S = \#\Lambda_S^1 + \#\Lambda_S^2$ *with (see* (4.10) *and* (7.11))

$$\Lambda_S^1 = \{(\lambda, \mu, k) \in \Lambda_S^1(k, 2, 0)\colon \sigma^{S,\lambda,\mu,k} \textit{ is not a polynomial}\},$$

$$\Lambda_S^2 = \Lambda_S^2(k, 2).$$

Proof. It consists of four steps. Let us fix $F = (f, \{g_{iqj}\}, \{g_{ii'j}\})$ in $X^{k,2}(C_S)$.

Step 1. Suppose that F satisfies

$$d_{S,l,\nu}(F) = 0, \quad \forall (l, \nu) \in \Lambda_S^2. \tag{8.8}$$

Owing to Lemma 7.1, there exists a polynomial p, depending linearly on F, solution of

$$A_i^S p_i = \Pi_{N-2m}^S f_i \text{ in } C_i, \quad \forall i \in \mathcal{N}_S, \tag{8.9}$$

$$B_{iqj}^S p_i = \Pi_{N-m_{iqj}}^S g_{iqj} \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, m-1, \quad \gamma_{iq} \in \mathscr{E}_S. \tag{8.10}$$

$$B_{iqj}^S p_i - B_{i'q'j}^S p_{i'} = \Pi_{N-m_{iqj}}^S g_{ii'j} \quad \text{on } \Gamma_{iq},$$

$$\forall j = 0, \ldots, 2m-1, \quad \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S. \tag{8.11}$$

Moreover, from the constructive method of this lemma, we know that

$$p_i(x) = \sum_{|\alpha| \geqslant m - n/2} a_{i\alpha} x^\alpha$$

for some $a_{i\alpha} \in \mathbb{C}$, so that

$$p_i \in \mathscr{H}_{v,\mathrm{loc}}^m(C_S). \tag{8.12}$$

In dimension 2, we notice that $\{A_i^S, B_{iqj}^S\}$ is injective modulo the polynomials on $S^{m-1}(C_S)$ due to (2.19) and using analogous arguments as in [4, Section 9.D].

Step 2. We set

$$h_{iqj} = g_{iqj} - B_{iqj}^S(\Phi_S p_i) \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, m-1, \quad \gamma_{iq} \in \mathscr{E}_S,$$

$$h_{ii'j} = g_{ii'j} - B_{iqj}^S(\Phi_S p_i) + B_{i'q'j}^S(\Phi_S p_{i'}) \text{ on } \Gamma_{iq},$$

$$\forall j = 0, \ldots, 2m-1, \quad \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S.$$

Owing to Proposition AA.29 of [4], $h_{iqj}, h_{ii'j} \in V_\gamma^{k+2m-m_{iqj}-1/2,2}$ near S, with $\gamma = 0$ in dimension 3 and $\gamma > 0$ in dimension 2. By Lemma 4.3, there exists $v \in \mathscr{V}_\gamma^{k+2m,2}(C_S) \cup \mathscr{H}^{k+2m}(C_S)$ solution of

$$B_{iqj}^S v_i = h_{iqj} \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, m-1, \quad \gamma_{iq} \in \mathscr{E}_S. \tag{8.13}$$

$$B_{iqj}^S v_i - B_{i'q'j}^S v_{i'} = h_{ii'j} \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, 2m-1, \quad \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S. \tag{8.14}$$

Step 3. We introduce

$$\hat{f} = \{f_i - (A_i^S + (1 - \Phi_S))(v_i + \Phi_S p_i)\}_{i \in \mathcal{N}_S}.$$

From the regularity of v, we see that $\hat{f} \in \widetilde{W}_S'$. By the estimate (8.6) and a perturbation argument, if $\operatorname{supp} \Phi_S$ is sufficiently small (what we suppose from now on), then there exists a unique solution $\hat{u} \in \widetilde{W}_S$ of

$$\tilde{b}_S(\hat{u}, v) + \sum_{i \in \mathcal{N}_S} \int_{C_i} (1 - \Phi_S) \hat{u}_i \bar{v}_i \, dx = \langle \hat{f}, v \rangle, \quad \forall u \in \widetilde{W}_S. \tag{8.15}$$

Setting $u = \hat{u} + v + \Phi_S p$, by (8.12)–(8.15), it is clear that u belongs to $\mathscr{H}^m_v(C_S)$ and satisfies.

$$
\begin{aligned}
&A_i^S u_i + (1 - \Phi_S)u_i = f_i \text{ in } C_i, \quad \forall i \in \mathscr{I}_S,\\
&B_{iqj}^S u_i = g_{iqj} \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, m-1, \quad \gamma_{iq} \in \mathscr{E}_S, \qquad (8.16)\\
&B_{iqj}^S u_i - B_{i'q'j}^S u_{i'} = g_{ii'j} \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, 2m-1, \ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S.
\end{aligned}
$$

From the regularity results for systems of PDE in smooth domains, we can say that

$$u \in \mathscr{H}^{k+2m} \text{ far from } S. \qquad (8.17)$$

Step 4. In view of (8.16), (8.17) and the fact that $(1 - \Phi_S)\Phi_S = 0$ near S, $\Phi_S u \in \mathscr{H}^m_v(\Omega_S)$ may be seen as a solution of

$$
\begin{aligned}
&A_i^S(\Phi_S u_i) = f_i^S \text{ in } \Omega_{Si}, \quad \forall i \in \mathscr{N}_S,\\
&B_{iqj}^S(\Phi_S u_i) = g_{iqj}^S \text{ on } \Gamma_{iq}^1, \quad \forall j = 0, \ldots, m-1, \ \gamma_{iq} \in \mathscr{E}_S,\\
&B_{iqj}^S(\Phi_S u_i) - B_{i'q'j}^S(\Phi_S u_{i'}) = g_{ii'j}^S \text{ on } \Gamma_{iq}^1,\\
&\forall j = 0, \ldots, 2m-1, \ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S, \qquad (8.18)\\
&\frac{\partial^j}{\partial v^j}(\Phi_S u_i) = 0 \quad \text{on } \Gamma_i^2, \ \forall j = 0, \ldots, m-1, \ i \in \mathscr{N}_S.
\end{aligned}
$$

The function f^S has the following properties:

(i) $f_i^S \equiv f_i$ in a neighbourhood of S,
(ii) there exists a constant $C > 0$ (independent of f) such that

$$\|f^S\|_{\mathscr{H}^k(\Omega_S)} \leqslant C\,\|F\|_{X^k(C_S)}. \qquad (8.19)$$

Analogous properties hold for g_{iqj}^S and $g_{ii'j}^S$.

Applying Theorem 7.3 or 7.4 to the problem (8.18), we can say that if

$$c_{S,\lambda,\mu,k}((f^S, \{g_{iqj}^S\}, \{g_{ii'j}^S\})) = 0, \quad \forall\,(\lambda, \mu, k) \in \Lambda_S^1, \qquad (8.20)$$

then $\Phi_S u \in \mathscr{H}^{k+2m}(\Omega_S)$.

Conclusion. We have shown that if F satisfies (8.8) and (8.20), then there exists $u \in \mathscr{H}^{k+2m}(C_S)$ solution of (8.16). Since (8.8) and (8.20) define continuous linear forms on $X^k(C_S)$, the range of $\mathfrak{U}^S + (1 - \Phi_S)$ is closed and has a finite codimension $\leqslant N_S$. Finally, its kernel is reduced to $\{0\}$ by the third step.

We are now in a position to give the decomposition results for operators with variable coefficients before we need to define the singular functions of our problem (see [4, equation (5.9)]).

Definition 8.3. *For an arbitrary* $l \in \mathbb{N} \cup \{0\}$, *we set*

$$A_{i,l}^S(x, D_x) = \sum_{2m-l \leqslant |\alpha| \leqslant 2m} \ \sum_{|\beta| = l+|\alpha|-2m} D^\beta a_\alpha^i(S)\frac{x^\beta}{\beta!}D_x^\alpha,$$

$$B_{iqj,l}^S(x, D_x) = \sum_{m_{iqj}-l \leqslant |\alpha| \leqslant m_{iqj}} \ \sum_{|\beta| = l+|\alpha|-m_{iqj}} D^\beta b_{iqj\alpha}(S)\frac{x^\beta}{\beta!}D_x^\alpha.$$

We remark that $A_{i,0}^S = A_i^S$, $B_{iqj,0}^S = B_{iqj}^S$.

If $\sigma^{S,\lambda,\mu,k}$ is a singular function of $\mathfrak{U}^S$, we define $\sigma_p^{S,\lambda,\mu,k}$ by recurrence over $p \in \mathbb{N} \cup \{0\}$ as follows:

$$\sigma_0^{S,\lambda,\mu,k} = \sigma^{S,\lambda,\mu,k},$$

$\sigma_p^{S,\lambda,\mu,k}$ is a solution of

$$A_i^S \sigma_{p,i}^{S,\lambda,\mu,k} = -\sum_{l=0}^{p-1} A_{i,p-l}^S \sigma_{p,i}^{S,\lambda,\mu,k} \text{ in } C_i, \quad \forall i \in \mathcal{N}_S,$$

$$B_{iqj}^S \sigma_{p,i}^{S,\lambda,\mu,k} = -\sum_{l=0}^{p-1} B_{iqj,p-l}^S \sigma_{p,i}^{S,\lambda,\mu,k} \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, m-1, \quad \gamma_{iq} \in \mathscr{E}_S,$$

$$B_{iqj}^S \sigma_{p,i}^{S,\lambda,\mu,k} - B_{i'q'j}^S \sigma_{p,i'}^{S,\lambda,\mu,k} = -\sum_{l=0}^{p-1} (B_{iqj,p-l}^S \sigma_{l,i}^{S,\lambda,\mu,k} - B_{i'q'j,p-l}^S \sigma_{l,i'}^{S,\lambda,\mu,k})$$

$$\text{on } \Gamma_{iq}, \quad \forall j = 0, \ldots, 2m-1, \ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S.$$

Its existence follows from Theorem 9.2. We define analogously $e_p^{S,l,\nu}$ for $(l, \nu) \in \Lambda_S^2$.

Theorem 8.4. *If (H_n) holds, then there exist operators $\tilde{A}_i$, $\tilde{B}_{iqj}$, which, respectively, coincide with A_i and B_{iqj} in a neighbourhood of S such that any solution $u \in \mathscr{H}_v^m(C_S)$ of*

$$\tilde{A}_i u_i = \tilde{f}_i \text{ in } C_i, \quad \forall i \in \mathcal{N}_S, \tag{8.21}$$

$$\tilde{B}_{iqj} u_i = 0 \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, m-1, \quad \gamma_{iq} \in \mathscr{E}_S, \tag{8.22}$$

$$\tilde{B}_{iqj} u_i - \tilde{B}_{i'q'j} u_{i'} = 0 \text{ on } \Gamma_{iq}, \quad \forall j = 0, \ldots, 2m-1, \ \gamma_{iq} = \gamma_{i'q'} \in \mathscr{I}_S, \tag{8.23}$$

with $\tilde{f} \in \mathscr{H}^k(C_S)$, admits the following decomposition:

$$u = u_0 + \sum_{(\lambda,\mu,k) \in \Lambda_s^1} c_{S,\lambda,\mu,k} \Phi_S \tau^{S,\lambda,\mu,k} + \sum_{(l,\nu) \in \Lambda_s^2} d_{S,l,\nu} \Phi_S E^{S,l,\nu}, \tag{8.24}$$

where $u_0 \in \mathscr{H}^{k+2m}(C_S)$; $c_{S,\lambda,\mu,k}$, $d_{S,l,\nu} \in \mathbb{C}$ and the singular functions are given by

$$\tau^{S,\lambda,\mu,k} = \sum_{\operatorname{Re}\lambda + p \leq k+2m-n/2} \sigma_p^{S,\lambda,\mu,k},$$

$$E^{S,l,\nu} = \sum_{l+p \leq k+2m-n/2} e_p^{S,l,\nu}.$$

Proof. We follow the arguments of [4, section 10.C] using Theorem 8.2 instead of Theorem 10.2″ of [4] and an estimate analogous to (8.6) with non-homogeneous interface conditions (as in (2.19)). ■

In order to get the final result in the domain Ω, we need the following localized procedure based on usual trace theorems (as Lemma 4.3).

Proposition 8.5. *Denote by $\tilde{W}_S$ the Hilbert space defined as W_S, replacing B_{iqj}^S by $\tilde{B}_{iqj}$ (where the $\tilde{B}_{iqj}$'s are the operators appearing in the previous theorem). Given $v \in \tilde{W}_S$, there exists $w \in V$ such that*

(i) *$w = v$ in a neighbourhood $\mathscr{V}_S$ of S independent of v,*
(ii) *$w = 0$ outside $B(S, \eta)$ for some $\eta > 0$ (independent of v).*

Theorem 8.6. *Assume that (H_n) holds. Let $u \in V$ be a solution of (4.6)–(4.8) with a datum $f \in \mathscr{H}^k(\Omega)$. Then u admits the following decomposition in a neighbourhood of S:*

$$u = u_0 + \sum_{(\lambda,\mu,k)\in\Lambda_s^1} c_{S,\lambda,\mu,k}\,\tau^{S,\lambda,\mu,k} + \sum_{(l,v)\in\Lambda_s^2} d_{S,l,v}E^{S,l,v}, \tag{8.25}$$

where $u_0 \in \mathscr{H}^{k+2m}(\Omega)$; $c_{S,\lambda,\mu,k}$, $d_{S,l,v} \in \mathbb{C}$.

Proof. Fix η_S, a cut-off function analogous to Φ_S, such that

(i) supp $\eta_S \subset \mathscr{V}_S$,
(ii) supp η_S is included in the set where the operators A_i, B_{ipj} *and* $\tilde{A}_i$, $\tilde{B}_{ipj}$ coincide.

We denote by $\tilde{b}_S$, the sesquilinear form defined on $\tilde{W}_S$ associated with $\tilde{A}_i$.

Fix $v \in \tilde{W}_S$ and denote by $w \in V$, the function constructed in the previous proposition. It is clear that

$$\tilde{b}_S(\eta_S u, v) = a(u, w) + a((\eta_S - 1)u, w). \tag{8.26}$$

As $(\eta_S - 1)u$ belongs to $\mathscr{H}^{k+2m}$ far from the vertices of Ω, we may apply Green's identity in the second term on the right-hand side of (8.26). This allows one to show that $\eta_S u$ satisfies

$$\tilde{b}_S(\eta_S u, v) = \sum_{i\in\mathscr{N}_S}\int_{C_i}\tilde{f}_i\bar{v}_i\,\mathrm{d}x + \sum_{i,q,j}\int_{\Gamma_{iq}} h_{iqj}F_{iqj}\bar{v}\,\mathrm{d}\sigma,$$

where $\tilde{f}_i = A_i(\eta_S u_i)$, $h_{iqj} = \Phi_{iqj}(\eta_S u_i)$. When h_{iqj} appears, then $h_{iqj} \equiv 0$ in a neighbourhood of S; so modulo a trace lifting theorem (Lemma 4.3), we may apply Theorem 8.4 to $\eta_S u$. Therefore, the expansion (8.24) for $\eta_S u$ proves (8.25). ∎

9. Stabilization procedure

Following the ideas of [3, 14] and using the abstract setting of [19], we give the stabilization procedure for the polynomial resolution and the comparison theorem in weighted Sobolev spaces for differential equations in a Hilbert space. As an application, we consider the Dirichlet problem for elliptic operators of order $2m$ with smooth coefficients in a plane cone. By an example, we also show that our theory can be applied to interface problems.

9.1. *The polynomial resolution*

Let X be a Hilbert space with a norm denoted by $\|\cdot\|$. Further, let A_α be a family of closed operators on X, for $\alpha \in I$, where I is a fixed neighbourhood of a point α_0 of a topological space V. For convenience, we write $A_{\alpha_0} = A$. We suppose that the A_α's satisfy the following assumptions:

$$D(A_\alpha) = D(A), \quad \forall \alpha \in I. \tag{9.1}$$

$$A_\alpha \text{ has a compact resolvent } R(\lambda, A_\alpha) = (\lambda - A_\alpha)^{-1}, \quad \forall \alpha \in I. \tag{9.2}$$

There exists a continuous function c on I such that $c(\alpha_0) = 0$ and

$$\|A_\alpha u - Au\| \leqslant c(\alpha)\|u\|_{D(A)}, \quad \forall u \in D(A), \alpha \in I, \tag{9.3}$$

where we recall that $\|u\|_{D(A)} = \|u\| + \|Au\|$ for all $u \in D(A)$.

With these assumptions, we are able to show the continuity of the eigenvalues of A_α with respect to α.

Theorem 9.1. *Under the previous assumptions, A_α tends to A in the generalized sense of Kato (cf. [7, section IV.2.6]) or equivalently*

$$R(\lambda, A_\alpha) \to R(\lambda, A) \text{ in norm as } \alpha \to \alpha_0, \quad \forall \lambda \in \rho(A). \tag{9.4}$$

Consequently, if λ is an eigenvalue of A with algebraic multiplicity k (i.e. the sum of the lengths of every Jordan block associated with λ; in the sequel, we only speak about algebraic multiplicity and write simply multiplicity), then for all sufficiently small $\varepsilon > 0$, there exists a neighbourhood I_ε of α_0 such that A_α has exactly k eigenvalues (repeated according to their multiplicity) in the open ball $B(\lambda, \varepsilon)$ (see Figs. 11–17 for some illustrations).

Proof. Using a Neumann series, we can show that every $\lambda \in \rho(A)$ belongs to $\rho(A_\alpha)$ too if α is close enough to α_0; moreover, $R(\lambda, A_\alpha)$ will be uniformly bounded with respect to α. Now, (9.4) follows directly from (9.3) and the following easily checked identity:

$$R(\lambda, A_\alpha) - R(\lambda, A) = R(\lambda, A_\alpha)(A_\alpha - A)R(\lambda, A), \quad \forall \lambda \in \rho(A).$$

The remainder follows from Theorems IV.2.25 and IV.3.16 of [7]. ■

Let us pass to the differential equation associated with A_α: for a given function f_α (from $\mathbb{R}$ into X), we look for a solution u_α of

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}t} - A_\alpha u_\alpha = f_\alpha \text{ on } \mathbb{R}. \tag{9.5}$$

This problem was studied in detail in [19], where Nicaise gave some comparison results in weighted Sobolev spaces and the polynomial resolution but using the classical singular functions (which lead to some instabilities). In order to define the stable singular functions, we need the notion of divided differences (see [3, section 8]). Let $\mu_1, \ldots, \mu_K$ be arbitrary complex numbers; then the divided difference of a holomorphic function w at $\mu_1, \ldots, \mu_K$ is defined by

$$w[\mu_1, \ldots, \mu_K] = \frac{1}{2i\pi} \int_\gamma \frac{w(\lambda)}{\prod_{i=1}^K (\lambda - \mu_j)} \mathrm{d}\lambda, \tag{9.6}$$

where γ is a simple curve surrounding all the μ_j's. We also recall Leibniz's formula for the divided difference of the product $v \cdot w$ of two holomorphic functions v and w, proved in [3, Lemma 8.1]:

$$(v \cdot w)[\mu_1, \ldots, \mu_K] = \sum_{j=1}^{K} v[\mu_1, \ldots, \mu_j] w[\mu_j, \ldots, \mu_K]. \tag{9.7}$$

Let us finally define

$$\mathscr{S}[\mu_1, \ldots, \mu_K; t] = w_t[\mu_1, \ldots, \mu_K], \tag{9.8}$$

where w_t is the holomorphic function $w_t : z \to e^{zt}$.

With these notions, we can solve, in a stable way, problem (9.5) with some particular right-hand sides corresponding to polynomials.

Theorem 9.2. *Let μ be a complex number and $f_\alpha \in X$, for all $\alpha \in I$, satisfying*

$$f_\alpha \to f \textit{ in } X, \text{ as } \alpha \to \alpha_0. \tag{9.9}$$

Then there exists a solution u_α (from $\mathbb{R}$ into $D(A)$) of

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}t} - A_\alpha u_\alpha(t) = \mathrm{e}^{\mu t} f_\alpha \text{ on } \mathbb{R}, \ \forall \alpha \in I. \tag{9.10}$$

This solution admits the following expansion:

(i) *if μ is not an eigenvalue of A, then there exists a sufficiently small neighbourhood J of α_0 such that*

$$u_\alpha(t) = c_\alpha \mathrm{e}^{\mu t}, \quad \forall \alpha \in J. \tag{9.11}$$

(ii) *if μ is an eigenvalue of A of multiplicity k, then denoting by $\lambda_{1\alpha}, \ldots, \lambda_{k\alpha}$ the k eigenvalues of A_α, for α in a sufficiently small neighbourhood J of α_0, we have*

$$u_\alpha(t) = c_\alpha \mathrm{e}^{\mu t} + \sum_{j=1}^{k} d_{\alpha j} \mathscr{S}[\mu, \lambda_{1\alpha}, \ldots, \lambda_{j\alpha}; t]. \tag{9.12}$$

In both cases, c_α and $d_{\alpha j}$ are continuous in J with values in $D(A)$, and fulfil

$$A_\alpha c_\alpha \to A c_{\alpha_0} \quad \text{in } X,$$

$$A_\alpha d_{\alpha j} \to A d_{\alpha_0 j} \quad \text{in } X, \quad \forall j = 1, \ldots, k \quad \text{as } \alpha \to \alpha_0.$$

Proof. Using [5, Lemma 1.3.4], we deduce that u_α given by

$$u_\alpha(t) = \frac{1}{2\mathrm{i}\pi} \int_\gamma \frac{\mathrm{e}^{zt} R(z, A_\alpha) f_\alpha}{(z - \mu)} \mathrm{d}z \tag{9.13}$$

is a solution of (9.10) (where γ is a closed curve surrounding μ). Therefore, it remains to show (9.11) or (9.12).

Since the only poles of $R(z, A_\alpha)$ inside γ are the $\lambda_{j\alpha}$'s, for $j \in \{1, \ldots, k\}$, we deduce that

$$\chi(z, \alpha) = R(z, A_\alpha) \prod_{j=1}^{k} (x - \lambda_{j\alpha}) \tag{9.14}$$

is holomorphic in a neighbourhood of $\overline{\operatorname{int} \gamma}$ (in the first case, we simply have $\chi(z, \alpha) = R(z, A_\alpha)$). Moreover, owing to (9.4) and the continuity of the eigenvalues, we see that

$$\chi(z, \alpha) \to \chi(z, \alpha_0) \text{ in norm}, \quad \text{as } \alpha \to \alpha_0, \quad \forall z \in \gamma. \tag{9.15}$$

Setting

$$v_\alpha : z \to \chi(z, \alpha) f_\alpha,$$

and comparing with (9.13), we obtain

$$u_\alpha(t) = (w_t \cdot v_\alpha)[\mu, \lambda_{1\alpha}, \ldots, \lambda_{k\alpha}].$$

By the Leibniz formula (9.7), we get (9.11) or (9.12), with

$$c_\alpha = v_\alpha[\mu, \lambda_{1\alpha}, \ldots, \lambda_{k\alpha}],$$

$$d_{\alpha j} = v_\alpha[\lambda_{j\alpha}, \ldots, \lambda_{k\alpha}], \quad \forall j = 1, \ldots, k.$$

The continuity results on the c_α's and the $d_{\alpha j}$'s can easily be deduced from (9.15) and (9.9). ∎

When μ is an eigenvalue of A, the previous theorem provides a stable solution of problem (9.10), since any term of the expansion (9.12) of u_α tends to its respective term of the expansion (9.12) of u_{α_0} as α goes to α_0. It will be used in sections 9.3 and 9.4 in the polynomial resolution for elliptic operators of order $2m$ and for a regular elliptic transmission problem. Let us note that Theorem 9.2 also restores the results of Sändig [20], who studied a particular case, corresponding to the case when μ is a simple eigenvalue of A. This will be checked in section 9.4 for a particular transmission problem.

We now pass on to the stable asymptotics for the comparison result between two different weighted Sobolev spaces.

9.2. *Stabilities in weighted Sobolev spaces*

According to [19], we need more assumptions on the A_α's: we suppose that there exist two positive real numbers δ and N and a neighbourhood $\mathscr{U}$ of α_0 such that

$$\rho(A_\alpha) \supset \Sigma_{\delta,N} = \{\lambda \in \mathbf{C} : |\arg \lambda \pm \pi/2| \leqslant \delta \text{ and } |\lambda| \geqslant N\}, \quad \forall \alpha \in \mathscr{U}.$$

Moreover, we suppose that the assumption (H1) of [19] is satisfied by A_α uniformly in $\mathscr{U}$, i.e. there exists a closed subspace S of X and a constant $C > 0$ (independent of α) such that

$$\|\lambda R(\lambda, A_\alpha) f\|_X + \|R(\lambda, A_\alpha) f\|_{D(A)} \leqslant C \cdot \|f\|_X, \quad \forall f \in S,\ \lambda \in \Sigma_{\delta,N},\ \alpha \in \mathscr{U}. \tag{9.16}$$

Using the definition of the weighted spaces $H_\alpha^k(\mathbb{R}, X)$ of [19], we can give the following theorem.

Theorem 9.3. *Let $\alpha(1)$, $\alpha(2)$ be two real numbers such that $\alpha(1) < \alpha(2)$. For $j = 1$ and 2, assume that the line* Re $\lambda = -\alpha(j)$ *contains no eigenvalue of A and let $f_\alpha \in L^2_{\alpha(j)}(\mathbb{R}, S)$, for all $\alpha \in \mathscr{U}$, satisfying*

$$f_\alpha \to f \text{ in } L^2_{\alpha(j)}(\mathbb{R}, S), \text{ as } \alpha \to \alpha_0. \tag{9.17}$$

Then there exists a neighbourhood $\mathscr{U}'$ of α_0 such that, for all $\alpha \in \mathscr{U}'$, (9.5) has a solution $u_\alpha^{(j)} H^1_{\alpha(j)}(\mathbb{R}, X) \cap L^2_{\alpha(j)}(\mathbb{R}, D(A))$; moreover, their difference is given by

$$u_\alpha^{(1)}(t) - u_\alpha^{(2)}(t) = \sum_{\lambda \in Sp(A),\, -\alpha(2) < \mathrm{Re}\,\lambda < -\alpha(1)} R_{\lambda\alpha}, \quad \forall \alpha \in \mathscr{U}'. \tag{9.18}$$

where for all eigenvalues λ of A of multiplicity k, we set

$$R_{\lambda\alpha} = \sum_{j=1}^{k} c_{j\alpha} \mathscr{S}[\lambda_{1\alpha}, \ldots, \lambda_{j\alpha}; t], \quad \forall \alpha \in \mathscr{U}'; \tag{9.19}$$

$\{\lambda_{j\alpha}\}_{j=1}^k$ denotes the k eigenvalues of A_α in a neighbourhood of λ satisfying

$$\lambda_{j\alpha} \to \lambda, \text{ as } \alpha \to \alpha_0, \quad \forall j = 1, \ldots, k.$$

The $c_{j\alpha}$'s are continuous from $\mathscr{U}'$ with values in $D(A)$ and

$$A_\alpha c_{j\alpha} \to A c_{j\alpha_0} \text{ in } X, \text{ as } \alpha \to \alpha_0.$$

Finally, there exists a constant $C_1 > 0$ (independent of α) such that

$$\|u_\alpha^{(j)}\|_{1,\alpha(j),X} + \|u_\alpha^{(j)}\|_{0,\alpha(j),D(A)} \leqslant C_1 \|f\|_{0,\alpha(j),S}, \quad \forall j = 1, 2. \tag{9.20}$$

Proof. The existence of $u_\alpha^{(j)}$ and the estimate (9.20) follow from [19, Theorem 2.7] and from the uniform estimate (9.16). Theorem 2.7 of [19] also shows (9.18) with $R_{\lambda\alpha}$ given by (see [19, (2.25)])

$$R_{\lambda\alpha} = \frac{\sqrt{2\pi}}{2i\pi} \int_\gamma e^{tz} R(z, A_\alpha)(\mathfrak{F} f_\alpha)(-iz)\, dz,$$

where γ is a fixed curve surrounding $\lambda \in Sp(A)$ (this actually holds for all α in a sufficiently small neighbourhood of α_0). In order to obtain (9.18), we use the same arguments as in Theorem 9.2; indeed using $\chi(z, \alpha)$ given by (9.14), we remark that

$$R_{\lambda\alpha} = (w_t V_\alpha)[\lambda_{1\alpha}, \ldots, \lambda_{k\alpha}],$$

where the function V_α is defined by

$$V_\alpha : z \to \sqrt{2\pi}\chi(z, \alpha)(\mathfrak{F} f_\alpha)(-iz),$$

which is a holomorphic function in a neighbourhood of $\overline{\operatorname{int}\gamma}$ (see the proof of Theorem 2.7 of [19]). We conclude by Leibniz's formula (9.7). The continuity of the coefficients follows from (9.17). ■

Again the expansion (9.18) is stable, because each term of the decomposition (9.19) of $R_{\lambda\alpha}$ tends to the respective term of the decomposition (9.19) of $R_{\lambda\alpha_0}$, as α goes to α_0. Here, this stabilization procedure is necessary when the multiplicity changes with respect to α, i.e. when λ is of multiplicity $k > 1$ for $\alpha = \alpha_0$, while A_α has at least two different eigenvalues for $\alpha \neq \alpha_0$. The figures of section 5 show that this phenomenon is not exceptional.

9.3. *Some boundary value problems in infinite cones of the plane*

Let L_0 be a properly elliptic operator of order $2m$, homogeneous with constant coefficients in $\mathbb{R}^2$. We consider the Dirichlet problem in the cone $C_\omega = \{re^{i\theta}: r > 0, 0 < \theta < \omega\}$, $\omega \in]0, 2\pi]$:

$$L_0 u = f \quad \text{in } C_\omega, \tag{9.21}$$

$$u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 \text{ on } \partial C_\omega. \tag{9.22}$$

Reducing (9.21) and (9.22) into a differential equation in a Hilbert space, we shall show that the results of the previous section allow us to give a stable asymptotics for a weak solution of (9.21) and (9.22).

Using polar co-ordinates (r, θ) and the Euler change of variable $r = e^t$, (9.21), (9.22) are equivalent to

$$\sum_{l=0}^{2m} A_l(\theta, D_\theta) D_t^l v = g \quad \text{in } B_\omega, \tag{9.23}$$

$$u = \frac{\partial v}{\partial \nu} = \cdots = \frac{\partial^{m-1} v}{\partial \nu^{m-1}} = 0 \text{ on } \partial B_\omega, \tag{9.24}$$

where $B_\omega = \{(t, \theta): t \in \mathbb{R}, 0 < \theta < \omega\}$ and

$$A_l(\theta, D_\theta) = \sum_{k=0}^{2m-l} a_{l,k}(\theta) \frac{\partial^k}{\partial \theta^k}, \tag{9.25}$$

when $a_{l,k}$ are infinitely differentiable functions (see [9], for instance). Due to the ellipticity assumption, $a_{2m,0}(\theta)$ is different from 0 for every θ; therefore, without loss of generality, we may suppose that $a_{2m,0}(\theta) \equiv 1$.

We now use the argument of reduction of order in D_t (see [2, sections 16 and 17]). If we introduce the vectors

$$V = (v, D_t v, \ldots, D_t^{2m-1} v), \tag{9.26}$$

$$F = (0, 0, \ldots, 0, g), \tag{9.27}$$

then (9.23) can be written as

$$D_t V - \mathscr{A} V = F \quad \text{in } \mathbb{R}, \tag{9.28}$$

where

$$\mathscr{A}(u_0, u_1, \ldots, u_{2m-1}) = \left(u_1, u_2, \ldots, u_{2m-1}, -\sum_{l=0}^{2m-1} A_l u_l\right). \tag{9.29}$$

Since (9.23) is set on B_ω and in order to take into account the boundary conditions (9.24), we introduce

$$X_\omega = \prod_{l=0}^{2m-1} H^{2m-1-l}(]0, \omega[),$$

$$D(\mathscr{A}) = \{(v_l)_{l=0,\ldots,2m-1} \in X_\omega : v_l \in H^{2m-l}(]0, \omega[),$$
$$\forall l = 0, \ldots, 2m-1 \text{ and } v_0 \in \overset{\circ}{H}{}^m(]0, \omega[)\}.$$

We have just proved that $v \in H^{2m}_{\mathrm{loc}}(B_\omega)$ is a solution of (9.23), (9.24) iff V given by (9.26) belongs to $D(\mathscr{A})$ and fulfils (9.28) (see [21, Lemma 3.1]). Similarly, ([21, Lemma 3.2]), λ_0 is an eigenvalue of the operator $\mathscr{A}$ in X_ω iff the operator $\mathscr{L}_\omega(\lambda_0)$ defined hereafter is not invertible (as usual, we say that λ_0 is an eigenvalue of $\mathscr{L}_\omega(\lambda)$). Analogous equivalence holds for the associated Jordan chains as explained in [21, section 3].

$$\mathscr{L}_\omega(\lambda) : H^{2m}([0, \omega[) \cap \overset{\circ}{H}{}^m(]0, \omega[) \to L^2(]0, \omega[: u \mapsto \sum_{l=0}^{2m} \lambda^l A_l(\theta, D_\theta) u.$$

The most important problem in (9.28) is that it is set in a Hilbert space X_ω depending on ω. Therefore, in order to reduce it to a fixed one, we make the change of variable

$$\Psi_\alpha :]0, \omega_0[\to]0, \omega[: \theta \to \alpha\theta,$$

where $\alpha = \omega/\omega_0$, with ω_0 supposed to be a critical angle. This change of variable induces an isomorphism between X_ω and X_{ω_0}; moreover, the differential equation (9.28) set in X_ω becomes

$$D_t V_\alpha - \mathscr{A}_\alpha V_\alpha = F_\alpha \quad \text{in } X_{\omega_0}, \tag{9.30}$$

where $\mathscr{A}_\alpha$ is given by (9.29) replacing θ by $\alpha\theta$ (in the operators A_l) and $F_\alpha = F \circ \Psi_\alpha$.

In order to apply the results of section 9.1 to (9.30), we need to check the assumptions made on the operators $\mathscr{A}_\alpha$. Equations (9.1) and (9.2) are clearly satisfied; let us consider (9.3).

Lemma 9.4. *There exists a neighbourhood I of α_0 and a constant $C > 0$ such that*

$$\|(\mathscr{A}_\alpha - \mathscr{A})u\|_{X\omega_0} \leqslant C|1-\alpha|\|V\|_{D(\mathscr{A})}, \quad \forall V \in D(\mathscr{A}).$$

Proof. Fix $V \in D(\mathscr{A})$, from the definition of the $\mathscr{A}_\alpha$'s, we have

$$\|(\mathscr{A}_\alpha - \mathscr{A})u\|_{X\omega_0} = \left\| \sum_{l=0}^{2m-1} (A_{l\alpha} - A_l)v_l \right\|_{L^2(]0,\omega_0[)}$$
$$\leqslant \sum_{l=0}^{2m-1} c_l(\alpha)\|v_l\|_{H^{2m-l}(]0,\omega_0[)},$$

where we have set

$$c_l(\alpha) = \sum_{l=0}^{2m-l} \sup_{0 \leqslant \theta \leqslant \omega_0} |a_{lk}(\alpha\theta)/\alpha^k - a_{lk}(\theta)|.$$

Due to the smooth properties of the a_{lk}'s, we have

$$c_l(\alpha) \leqslant C|1-\alpha|.$$

Therefore, it remains to show that

$$\sum_{l=0}^{2m-1} \|v_l\|_{H^{2m-l}(]0,\omega_0[)} \leqslant C\|V\|_{D(\mathscr{A})}.$$

This last one follows from the ellipticity of A_0, since Agmon–Douglis–Nirenberg *a priori* estimates imply that

$$\|v_0\|_{H^{2m}(]0,\omega_0[)} \leqslant C\{\|A_0 v_0\|_{L^2(]0,\omega_0[)} + \|v_0\|_{L^2(]0,\omega_0[)}\}. \qquad \blacksquare$$

In the same way, using the estimate (3.14) of [19] fulfilled by $\mathscr{A}$ in X_{ω_0} and a perturbation argument, we can show that the $\mathscr{A}_\alpha$'s satisfy (9.16) uniformly in a neighbourhood of α_0, with $S = \{F \in X_{\omega_0}$ in the form (9.27)$\}$.

We are now ready to give a stable decomposition for a weak solution of (9.21) and (9.22). Applying Theorems 9.2 and 9.3 to problems (9.30) and going back to the original problem, we have (see [4, Theorem 5.11 and Lemma 10.4]) the following theorem.

Theorem 9.5. *Let $\omega_0 \in]0, 2\pi]$ and $l \in \mathbb{N} \cup \{0\}$ be fixed. Assume that the line $\operatorname{Re}\lambda = l + 2m - 1$ contains no eigenvalue of $\mathscr{L}_{\omega_0}(\lambda)$. Suppose given f_ω in $H^l(C_\omega)$, for all ω in a neighbourhood $\mathscr{U}$ of ω_0, satisfying*

$$\|f_\omega \circ \Psi_{\omega/\omega_0} - f_{\omega_0}\|_{H^l(C_{\omega_0})} \to 0, \text{ as } \omega \to \omega_0.$$

Then there exists a neighbourhood $\mathscr{U}'$ of ω_0 such that for all $\omega \in \mathscr{U}'$ we have the following results: if $u_\omega \in \overset{\circ}{H}{}^m(C_\omega)$ is a solution of (9.21) and (9.22) with data f_ω, then

$$u_\omega = u_{0\omega} + \Sigma_\lambda R_{\lambda\omega} + \Sigma_n S_{n\omega}, \tag{9.31}$$

where $u_{0\omega} \in H^{2m+l}(C_\omega)$, the first sum extends to all eigenvalues λ of $\mathscr{L}_{\omega_0}(\lambda)$ in the strip $\operatorname{Re}\lambda \in]m-1, l+2m-1[$; if λ is such an eigenvalue of multiplicity k, according to Theorem 9.1, there exist k eigenvalues $\lambda_{1\omega}, \ldots, \lambda_{k\omega}$ of *$\mathscr{L}_\omega(\lambda)$ such that*

$$\lambda_{j\omega} \to \lambda, \quad \text{as } \omega \to \omega_0, \quad \forall j = 1, \ldots, k;$$

finally, we have

$$R_{\lambda\omega} = \sum_{j=1}^{k} c_{j\omega} \mathscr{S}[\lambda_{1\omega}, \ldots, \lambda_{j\omega}; \ln r]. \tag{9.32}$$

The second sum extends to all non-negative integers $n \in [0, k-1]$ and

$$S_{n\omega} = k_{n\omega} \mathscr{S}[n + 2m; \ln r], \tag{9.33}$$

if $n + 2m$ is not an eigenvalue of $\mathscr{L}_{\omega_0}(\lambda)$; otherwise if $n + 2m$ is an eigenvalue of $\mathscr{L}_{\omega_0}(\lambda)$ of multiplicity k', then denoting by $\mu_{1\omega}, \ldots, \mu_{k'\omega}$ the k' eigenvalues of $\mathscr{L}_{\omega}(\lambda)$ satisfying

$$\mu_{j'\omega} \to n + 2m \text{ as } \omega \to \omega_0, \quad \forall j' = 1, \ldots, k',$$

we have

$$S_{n\omega} = k_{n\omega} \mathscr{S}[n + 2m; \ln r] + \sum_{j'=1}^{k'} k_{nj'\omega} \mathscr{S}[n + 2m, \mu_{1\omega}, \ldots, \mu_{j'\omega}; \ln r]. \tag{9.34}$$

In expressions (9.32)–(9.34), the coefficients $c_{j\omega}$, $k_{n\omega}$ and $k_{nj'\omega}$ belong to $H^{2m}(]0, \omega[)$ and depend continuously on ω in the following sense:

$$\|c_{j\omega} \circ \Psi_{\omega/\omega_0} - c_{j\omega_0}\|_{H^{2m}(]0,\omega_0[)} \to 0, \text{ as } \omega \to \omega_0$$

and analogously for $k_{n\omega}$ and $k_{nj'\omega}$.

Remark 9.6. For convenience, we have treated here the Dirichlet problem for an elliptic operator in dimension 2. Nevertheless, it is possible to consider other boundary conditions and higher dimensions (for instance, for rotationally symmetric cones of $\mathbb{R}^3$ as considered in [20] for the Lamé system).

In our example, if the functions $a_{l,k}$ would be analytic functions (which is the case for the usual example as the Laplace operator or the biharmonic one), the family $\mathscr{A}_\alpha$ would be a holomorphic family of type (A) in the sense of Kato (see [8, section VII. 2]); in that case, Lemma 9.4 and Theorem 9.1 are in accordance with Theorem VII.1.8 of [7]. Let us also notice that we cannot hope an analytical dependence if we apply a non-analytic change of variable Ψ_α (as considered by [9] in a non-regular cone of $\mathbb{R}^n$ or for transmission problems, where two (or more) parameters appear, as we shall explain in the following subsection).

9.4. *A transmission problem*

In this section, we firstly study a particular polynomial resolution and solve the stabilization problem using the procedure of [20]. We secondly show that it is in accordance with the results of section 9.1.

For convenience, we consider example 1 with only 2 media; moreover, we take the following characteristic polynomial resolution: for a fixed $l \in \mathbb{N} \cup \{0\}$, find explicitly a solution u of

$$\begin{aligned}
&p_l \Delta u_l = 0 && \text{in } \Omega_l, \quad \forall l = 1, 2,\\
&u_1(r, \omega_1) = u_2(r, \omega_1) && \text{on } \Gamma_1^1,\\
&p_1 \frac{\partial u_1}{\partial \theta}(r, \omega_1) = p_2 \frac{\partial u_2}{\partial \theta}(r, \omega_1) && \text{on } \Gamma_1^1,\\
&u_1(r, 0) = r^l && \text{on } \Gamma_0^1,\\
&u_2(r, \omega_1 + \omega_2) = 0 && \text{on } \Gamma_2^1.
\end{aligned} \tag{9.35}$$

From Lemma 7.1, we find u in the form

$$u_1(r, \theta) = r^l\{\cos(l\theta) + \frac{D_1(l)}{D_2^{\mathrm{D}}(l)}\sin(l\theta)\}, \tag{9.36}$$

$$u_2(r, \theta) = r^l + \frac{p_1}{D_2^{\mathrm{D}}(l)}\sin l(\omega_1 + \omega_2 - \theta), \tag{9.37}$$

if l is not an eigenvalue of $\mathscr{L}_{\omega_1,\omega_2}(\lambda)$, i.e. if $D_2^{\mathrm{D}}(l) \neq 0$, where we have set

$$D_1(l) = -\, p_2 \cos(l\omega_1)\cos(l\omega_2) + p_1 \sin(l\omega_1)\sin(l\omega_2).$$

Conversely, if l is an eigenvalue of $\mathscr{L}_{\omega_1,\omega_2}(\lambda)$, then we have

$$u_1(r, \theta) = r^l\{\cos(l\theta) - \frac{c(l)\sin(l\omega_2)}{2lD_2^{\mathrm{D}'}(l)}\theta\cos(l\theta)\} + c(l)r^l \ln r \sin(l\omega_2)\sin(l\theta). \tag{9.38}$$

$$u_2(r, \theta) = r^l\{d_2(l)\sin l(\omega_1 + \omega_2 - \theta) - \frac{c(l)\sin(l\omega_1)}{2lD_2^{\mathrm{D}'}(l)}(\omega_1 + \omega_2 - \theta)$$

$$\times \cos l(\omega_1 + \omega_2 - \theta) + c(l)\ln r \sin(l\omega_1)\sin l(\omega_1 + \omega_2 - \theta)\} \tag{9.39}$$

in the case $\sin(l\omega_1) \neq 0 \neq \sin(l\omega_2)$ (otherwise, it is rather different), where $c(l)$ and $d_2(l)$ are two constants. It is easy to see that under the previous assumption, l is a simple eigenvalue of $\mathscr{L}_{\omega_1,\omega_2}(\lambda)$.

Let us now pass on to the stabilization procedure: fix $(\omega_{10}, \omega_{20})$ such that l is an eigenvalue of $\mathscr{L}_{\omega_{10},\omega_{20}}(\lambda)$ and, for simplicity, assume that

$$\sin(l\omega_{10}) \neq 0 \neq \sin(l\omega_{20}).$$

From the results of section 9.1 (see also Figs 6–8), for (ω_1, ω_2) near $(\omega_{10}, \omega_{20})$, l is no more an eigenvalue of $\mathscr{L}_{\omega_1,\omega_2}(\lambda)$. Nevertheless, there exists a simple eigenvalue $\lambda(\omega_1, \omega_2)$ of $\mathscr{L}_{\omega_1,\omega_2}(\lambda)$ such that

$$\lambda(\omega_1, \omega_2) \to l, \quad \text{as } (\omega_1, \omega_2) \to (\omega_{10}, \omega_{20}).$$

In that case, the stabilization procedure consists in replacing (9.36) and (9.37) by

$$u_1(r, \theta) = r^l \cos(l\theta) + \frac{D_1(l)}{D_2^{\mathrm{D}}(l)}\{r^l \sin(l\theta) - r^{\lambda(\omega_1,\omega_2)}\sin(\lambda(\omega_1, \omega_2)\theta)\}. \tag{9.40}$$

$$u_2(r, \theta) = \{r^l p_1 \sin l(\omega_1 + \omega_2 - \theta)$$

$$- r^{\lambda(\omega_1,\omega_2)} D_1(l)\frac{\sin(\lambda(\omega_1, \omega_2)\omega_1)}{\sin(\lambda(\omega_1, \omega_2)\omega_2)}\sin(\lambda(\omega_1, \omega_2)(\omega_1 + \omega_2 - \theta))\}/D_2^{\mathrm{D}}(l), \tag{9.41}$$

which is still a solution of (9.35). This one is stable since we can show that the right-hand side of (9.40) (resp. (9.41)) tends to the right-hand side of (9.38) (resp. (9.39)) (with (ω_1, ω_2) replaced by $(\omega_{10}, \omega_{20})$), as (ω_1, ω_2) goes to $(\omega_{10}, \omega_{20})$.

Let us finally note that it coincides with the results of Theorem 9.2; indeed from this theorem, we know that a stable solution exists in the form

$$u_i(r, \theta) = r^l\varphi_{i,\omega_1,\omega_2}(\theta) + (r^l - r^{\lambda(\omega_1,\omega_2)})\psi_{i,\omega_1,\omega_2}(\theta), \quad \forall i = 1, 2,$$

for some stable functions $\varphi_{i,\omega_1,\omega_2}$, $\psi_{i,\omega_1,\omega_2}$. This is exactly what (9.40) and (9.41) make.

For the sake of simplicity, we have only considered the particular problem (9.35), but obviously we may use the results of sections 9.1 and 9.2 to general regular elliptic transmission problems. Again we emphasize on the non-analytical dependence of the obtained operator $\mathscr{A}_\alpha$ with respect to α, since α is no more a real number.

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