COEFFICIENT FORMULAE FOR ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS NEAR CONICAL POINTS

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Abstract
It is well known that singularities are present in solutions of elliptic boundary value problems in domains with conical boundary points. The solution consists of singular terms, which appear in a neighbourhood of a conical point, and a more regular term. The coefficients of the singular terms, the so-called stress intensity factors, are especially of interest for applications. We describe a method, how some of them may be calculated, if the right hand sides are from standard Sobolev spaces. In some cases the coefficients are unstable and a stabilization procedure is necessary. We handle as examples boundary value problems for the Laplace equation in two and three dimensional domains.

I. THE ASYMPTOTIC EXPANSION

Let Ω be an open subset of $\mathbb{R}^n$, whose $n-1$ dimensional boundary $\partial\Omega$ is smooth with exception of one conical point 0. We consider an elliptic boundary value problem with constant coefficients

$$A(D_x)u(x) = \sum_{|\alpha|=2m} a_\alpha\, D_x^\alpha u(x) = f(x) \text{ in } \Omega \tag{1}$$

$$B_j(D_x)u(x) = \sum_{|\alpha|=m_j} b_{j,\alpha}\, D_x^\alpha u(x) = g_j(x) \text{ on } \partial\Omega|0, j = 1,\ldots,m.$$

The right hand sides are from the Sobolev spaces $W^{k,p}(\Omega)$ and the trace spaces $W^{k+2m-m_j-\frac{1}{p},p}(\partial\Omega)$. If $n=2$, the boundary conditions are also considered on pieces of the boundary. In this case we assume, that the functions $g_j(x)$ satisfy compability conditions. Since the right hand sides can be splitted into Taylor polynomials and functions from certain weighted Sobolev spaces, we get e.g. the following asymptotic expansion of a weak solution u from $W^{2,m}(\Omega)$:

$$u = \eta(r)\left(\sum_{l=0}^{k+2m-s-1} r^l v_l + \sum_{Re\alpha_\nu \in I} r^{\alpha_\nu} u_{\alpha_\nu}\right) + w, \tag{2}$$

where

$$v_l = \sum_{i=0}^{M_l} (\ln r)^i \psi_i(l,\underline{\omega}), \tag{3}$$

$$u_{\alpha_\nu} = \sum_{i=0}^{M_{\alpha_\nu}} C_{\alpha_\nu,i} \sum_{j=0}^{i} (\ln r)^i e_{i-j}(\alpha_\nu,\underline{\omega}), \tag{4}$$

$I = \left(-\frac{n}{2}+m, -\frac{n}{p}+k+2m\right)$, and $w \in W^{k+2m,p}(\Omega)$ for $p \neq 2$.

We denote by $(r,\underline{\omega})$ the spherical coordinates, $s = s(n,p)$ is an integer. $M_l > 0$ is the multiplicity of the "eigenvalue" l of a generalized eigenvalue problem, which we get from (1) introducing $(r,\underline{\omega})$ and using the Mellin transform with respect to r. $M_l = 0$ means, that l is not an eigenvalue. In the second term the complex numbers α_ν are eigenvalues of the multiplicity M_{α_ν} and $M'_{\alpha_\nu} = M_{\alpha_\nu} - 1$.
The first term of (2) comes from the polynomial parts of the right hand sides [1], the second term is known from the theory in weighted Sobolev spaces [1], [2] .

II. THE PLANE CASE

Due to the compatibility conditions we restrict to the problem

$$Au = 0 \text{ in } \Omega \tag{5}$$

$$B_i^{(q)}u = g_i^{(q)} \text{ on } \Gamma_q\,, \quad \bigcup_{q=1}^{Q} \overline{\Gamma_q} = \partial\Omega,\ j = 1,\ldots,m.$$

Assume that the domain Ω coincides in a neighbourhood of a corner point $\overline{\Gamma_q} \cap \overline{\Gamma_{q+1}} = O_q = O$ with the infinite cone $K = \{(r,\omega),\ 0 < r < \infty, \omega_0^+ = 0 < \omega < \omega_0 = \omega_0^-\}$ with the sides $\Gamma^\pm$. Introducing polar coordinates we write the differential operators as

$$A(D_x) = r^{-2m} L(rDr,\omega,D\omega)$$

$$B_j^\pm(D_x) = r^{-m_j^\pm} M_j^\pm(r,Dr,\omega,D\omega)\Big|_{\Gamma^\pm} \quad .$$

We say, the complex number α is an eigenvalue of the operator $\mathcal{A}(\lambda) = \left\{L(\lambda,\omega,D_\omega),\ M_j^\pm(\lambda,\omega,D_\omega)\right\}$ if there is a nontrivial solution $e_0(\lambda,\omega)$ of

$$\mathcal{A}(\lambda)e(\lambda,\omega) = \underline{0} \quad .$$

We now split the right hand sides of (5) :

$$g_j^\pm(r,\omega_0^\pm) = \sum_{l=0}^{k+2m-s-1} G_{l,j}^\pm(0,\omega_0^\pm)r^l + \tilde{g}_j^\pm(r,\omega_0^\pm),$$

where $G_{l,j}^\pm(0,\omega_0^\pm) = \frac{1}{l!}\ \frac{\partial^l g_j^\pm}{\partial r^l}(0,\omega_0^\pm)$,
$s = 0$ for $2 < p$ and $s = 1$ for $1 < p < 2$.
The functions v_l in the expansion (2) are solutions of

$$Ar^l v = 0 \quad \text{in } K$$

$$B_j^\pm r^l v = G^\pm_{l-m_j^\pm,j}(0,\omega_0^\pm)r^{l-m_j^\pm} \text{ on } \Gamma^\pm.$$

They can be calculated "easily", starting from the general solution of the ordinary differential equation $L(\lambda,\omega,D_\omega)e(l,\omega) = 0$ and using the ansatz (3).

Thus we get for the Dirichlet problem for the Laplace equation

$$v_l = \begin{cases} G_l(0,0)\left[\cos l\omega + \frac{1-\cos l\omega_0}{\sin l\omega_0}\sin l\omega\right] & \text{for } l\omega_0 \neq \nu\pi \\ G_l(0,0)\left[\cos l\omega + \frac{1-\cos l\omega_0}{\omega_0 \cos l\omega_0}(\ln r \sin l\omega + \omega\cos l\omega)\right] & \text{for } l\omega_0 = \nu\pi \end{cases} \tag{6}$$

It is evident that one coefficient in the first row of v_l is unbounded (unstable), if ω_0 is from a neighbourhood of $\frac{\nu\pi}{l}$. This behavior influences also the coefficients c_{α_ν} of u_{α_ν} (in our example is $\alpha_\nu = \frac{\nu\pi}{\omega_0}$, $u_{\alpha_\nu} = c_{\alpha_\nu}\sin\alpha_\nu\omega$). Following an idea of V.G.Maz'ya [3] we get a stable asymptotics, organizing the sums of (2) as follows (here for our example): In a neighbourhood of a critical angle $\frac{\nu\pi}{l}$ we write

$$
\begin{aligned}
& r^l G_l(0,0)\left(\frac{1-\cos l\omega_0}{\sin l\omega_0}\right)\sin l\omega + r^{\frac{\nu\pi}{\omega_0}} c_{\alpha_\nu}\sin\alpha_\nu\omega \\
&= G_l(0,0)(1-\cos l\omega_0)\left[\frac{r^l\sin l\omega - r^{\frac{\nu\pi}{\omega_0}}\sin\alpha_\nu\omega}{\sin l\omega_0}\right] \\
&\quad + r^{\frac{\nu\pi}{\omega_0}}\left[c_{\alpha_\nu} + \frac{G_l(0,0)(1-\cos l\omega_0)}{\sin l\omega_0}\right]\sin\alpha_\nu\omega \qquad (7)
\end{aligned}
$$

The new coefficients $c_l = G_l(0,0)(1-\cos l\omega_0)$ and $c'_{\alpha_\nu} = c_{\alpha_\nu} + \frac{G_l(0,0)(1-\cos l\omega_0)}{\sin l\omega_0}$ are bounded, if ω_0 is from a neighbourhood of $\frac{\nu\pi}{l}$ and the first term of (7) converges for $\omega_0 \to \frac{\nu\pi}{l}$ to the term of the second row of (6).
A general stabilization procedure is given in [3] and [4].

III. THE THREE DIMENSIONAL CASE

We consider the problem

$$
\begin{aligned}
Au &= 0 \quad \text{in } \Omega \\
B_j u &= g_j \quad \text{on } \partial\Omega|0,
\end{aligned}
$$

where the domain Ω coincides in a neighbourhood of 0 with a circle cone $K = \{(r,\varphi,\vartheta) = (r,\underline{\omega}) : 0 < r < \infty,\ 0 \le \varphi < 2\pi,$

$0 < \vartheta < \vartheta_0\}$. Analogously to the plane case we define $\mathcal{A}(\lambda)$ and consider a decomposition of g_j:

$$
g_j(r,\varphi,\vartheta_0) = \sum_{l=0}^{k+2m-m_j-s-1} r^l G_{j,l}(0,\varphi,\vartheta_0) + \tilde{g}_j(r,\varphi,\vartheta_0),
$$

where

$$
s = \begin{cases} 0 & \text{for } p > 3 \\ 1 & \text{for } \frac{3}{2} < p < 3 \\ 2 & \text{for } 1 < p < \frac{3}{2} \end{cases}
$$

and

$$
G_{j,l}(0,\varphi,\vartheta_0) = \frac{\partial^l g_j(0,\varphi,\vartheta_0)}{\partial r^l}\frac{1}{l!}
$$

It is meaningful to assume that $g_j(r,\varphi,\vartheta_0) = g_j(r,\varphi+2\pi,\vartheta_0)$ and to consider a Fourier expansion of $G_{j,l}$:

$$
G_{j,l}(0,\varphi,\vartheta_0) = \sum_{h=0}^{\infty} A_h^j(l,\vartheta_0)\cos h\varphi + B_h^j(l,\vartheta_0)\sin h\varphi
$$

The functions v_l of (2), which satisfy the equations

$$
\begin{aligned}
L(l,\omega,D_\omega)v_l &= 0 \quad \text{in } K \\
M_j(l,\omega,D_\omega)v_l &= G_{j,l-m_j}(0,\varphi,\vartheta_0)
\end{aligned}
$$

can be calculated, writing v_l as Fourier series with respect to φ. Thus we get for the Neumann problem for the Laplace equation (the index j is cancelled)

$$
\begin{aligned}
v_l = \sum_{h=0}^{\infty}\Bigg(& \frac{-A_h(l-1,\vartheta_0)}{\frac{\partial}{\partial\vartheta}P_l^{-h}(\cos\vartheta_0)} P_l^{-h}(\cos\vartheta)\cos h\varphi \\
& - \frac{B_h(l-1,\vartheta_0)}{\frac{\partial}{\partial\vartheta}P_l^{-h}(\cos\vartheta_0)} P_l^{-h}(\cos\vartheta)\sin h\varphi\Bigg),
\end{aligned}
$$

if $\frac{\partial}{\partial\vartheta}P_l^{-h}(\cos\vartheta_0) \ne 0$ for all h. $P_l^{-h}(\cos\vartheta)$ are Legendre functions of the first kind.

If there is for a given ϑ_0 an index h_0 with $\frac{\partial}{\partial\vartheta}P_l^{-h_0}(\cos\vartheta_0) = 0$ (see Figure), then l is an eigenvalue and terms with $\ln r$ as in (3) occur.
The asymptotic expansion is unstable in this case and we can apply a stabilization procedure analogously to the plane case.

References

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