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The Regularity of Boundary Value Problems for the Lamé Equations in a Polygonal Domain

1. Introduction

In the linear theory of elasticity, where the usual relations are satisfied, the displacements $\underline{u}(\underline{x}) = (u_1(\underline{x}), u_2(\underline{x}))^T$ of a twodimensional body Ω with the boundary $\partial\Omega$ are given by the following system:

$$\mu\Delta\underline{u}(\underline{x}) + (\lambda+\mu)\operatorname{grad}\operatorname{div}\underline{u}(\underline{x}) = -\underline{f}(\underline{x}) \quad \text{for } \underline{x} \doteq (x_1,x_2)^T \in \Omega, \tag{1.1}$$

$$\underline{u}(\underline{x}) = \underline{g}(\underline{x}) \quad \text{for } \underline{x} \in \Gamma \subseteq \partial\Omega, \tag{1.2}$$

$$\sigma(\underline{u}(\underline{x}))\varkappa(\underline{x}) + R(\underline{x})\underline{u}(\underline{x}) = \underline{t}(\underline{x}) \quad \text{for } \underline{x} \in \partial\Omega\setminus\Gamma, \tag{1.3}$$

where $\underline{f}(\underline{x}) = (f_1(\underline{x}), f_2(\underline{x}))^T$ is the vector of the volume forces, $\underline{t}(\underline{x}) = (t_1(\underline{x}), t_2(\underline{x}))^T$ is the vector of the surface forces, λ and μ are the Lamé's constants, which are independent of $\underline{x}$ (for simplicity), $\varkappa(\underline{x}) = (n_1(\underline{x}), n_2(\underline{x}))^T$ is the unit vector of the outward normal to $\partial\Omega\setminus\Gamma$ at the point $\underline{x}$, $\sigma(\underline{u}(\underline{x})) = (\sigma_{ij}(\underline{u}(\underline{x})))_{i,j=1,2}$ is the stress tensor with the components

$$\sigma_{ij}(\underline{u}(\underline{x})) = \mu\left(\frac{\partial u_i(\underline{x})}{\partial x_j} + \frac{\partial u_j(\underline{x})}{\partial x_i}\right) + \lambda \operatorname{div}\underline{u}(\underline{x})\delta_{ij},$$

where δ_{ij} denotes the Kronecker symbol, $R(\underline{x}) = (R_{ik}(\underline{x}))_{i,k=1,2}$ is a matrix with nonnegative elements which describes some kind of elastic resistance to the movement of $\underline{x}$ and $\underline{g}(\underline{x}) = (g_1(\underline{x}), g_2(\underline{x}))^T$ are the prescribed displacements on Γ.

Ω is a polygonal domain with the boundary $\partial\Omega = \bigcup_{j \in J_1} \Gamma_j \bigcup_{j \in J_2} \Gamma_j$,

where Γ_j are the sides or pieces of the sides of the polygon $\partial\Omega$ such that $\Gamma = \bigcup_{j \in J_1} \Gamma_j$, $\partial\Omega \setminus \Gamma = \bigcup_{j \in J_2} \Gamma_j$

(see Fig. 1 where $J_1 = \{1,2\}$, $J_2 = \{3,4,5\}$).

Besides that plane problems are of their own interest, their study is also very important if Ω is a threedimensional domain with edges, e.g., if Ω is a polyhedral domain (see [9]).

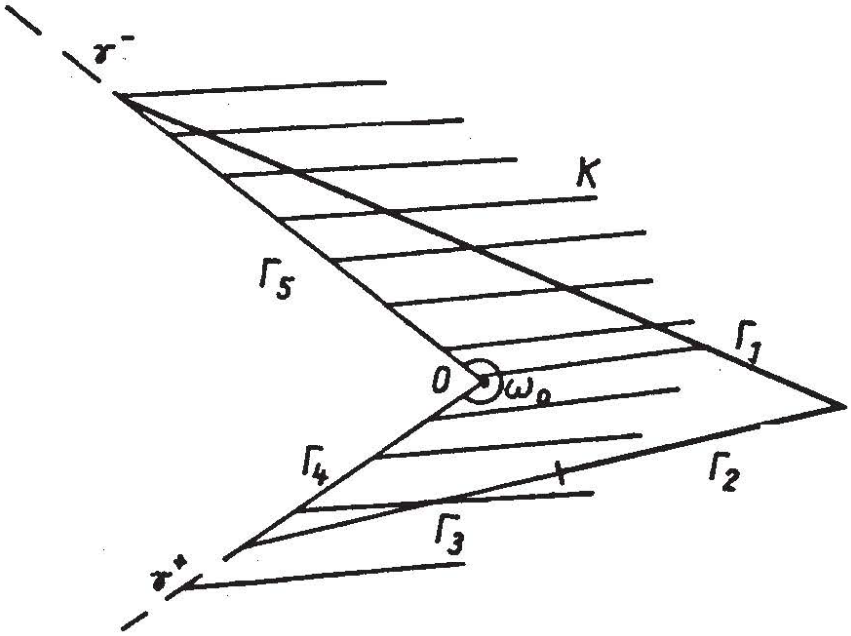

Fig. 1

P. Grisvard [1] already studied the behavior of a solution $\underline{u}$ of (1.1),(1.2),(1.3) in a neighborhood of a corner point or of a point, where the boundary conditions change. He formulated his results without proofs in form of expansions near these "bad" boundary points. In this paper we prove the regularity results using the theory of V.A. Kondrat'ev [4],[3] and of V.G. Maz'ja and B.A. Plamenevskij [8],[9]. Moreover we calculate numerically the generalized eigenvalues of a parameter problem, which determine the regularity properties of the solution.

In particular we are interested in the investigation of the regularity of the weak solution of problem (1.1),(1.2),(1.3). In order to introduce the definition of a weak solution we restrict ourselves to the case that $\underline{g}(\underline{x}) = \underline{0}$ on Γ. We consider the classical Sobolev space

$$W^{1,2}(\Omega) = \{u: \|u\| = (\sum_{|\gamma|\leq 1} \int_\Omega |D^\gamma u|^2 d\underline{x})^{1/2} < \infty\},$$

where $\gamma = (\gamma_1,\gamma_2)$ is a multiindex of the length $|\gamma| = \gamma_1 + \gamma_2$, $\gamma_i \geq 0$ are integers and $D^\gamma = \dfrac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2}}$ denotes the derivative in the distribution sense. Let be

$$V = \text{closure of } \{\underline{v} \in C^\infty(\bar{\Omega})\times C^\infty(\bar{\Omega}),\ \underline{v}|_\Gamma = \underline{0}\} \quad \text{in} \quad W^{1,2}(\Omega) \times W^{1,2}(\Omega). \tag{1.4}$$

We introduce a symmetric bilinear form on $V\times V$ for the system (1.1) considering the scalar product of (1.1) and of an arbitrary element $\underline{v} \in V$ in $L^2(\Omega)\times L^2(\Omega)$ and integrating by parts:

$$a(\underline{u},\underline{v}) = \int_\Omega \sum_{i,j=1,2} [\tfrac{1}{2}\mu(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \operatorname{div}\underline{u}\ \operatorname{div}\underline{v}] d\underline{x} \tag{1.5}$$

<u>Definition 1:</u> The vector function $\underline{u} \in V$ is a weak solution of the problem (1.1),(1.2),(1.3) if

$$a(\underline{u},\underline{v}) = (\underline{H},\underline{v}) = \int_\Omega \sum_{i=1,2} f_i(\underline{x}) v_i(\underline{x}) d\underline{x} + \int_{\partial\Omega\setminus\Gamma} \sum_{i=1,2} t_i(\underline{x}) v_i(\underline{x}) ds$$

$$- \int_{\partial\Omega\setminus\Gamma} \sum_{i,k=1,2} R_{ik}(\underline{x}) u_k(\underline{x}) v_i(\underline{x}) ds \quad \text{for all } \underline{v} \in V, \tag{1.6}$$

provided f_i, t_i and R_{ik} are such functions that $\underline{H}$ is from the dual space of V.

It is well known [12] that an uniquely determined weak solution $\underline{u} \in V$ exists, if the bilinear form $a(\underline{u},\underline{v})$ is bounded on $V\times V$ and if it is V-coercive. These assumptions are satisfied if $\operatorname{mes}\Gamma > 0$. If $\operatorname{mes}\Gamma = 0$ then we consider instead of V the factor space V/N, where N consists of all possible rigid movements of the body Ω,

$$N = \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-x_2\\x_1\end{pmatrix}\right\}. \tag{1.7}$$

In this case there exists an uniquely determined solution $\underline{u} \in V/N$ if the solvability condition

$\int_{\Omega}(\underline{f}\ \underline{v})d\underline{x} + \int_{\partial\Omega}[(\underline{t}\ \underline{v})-(R\underline{u}\ \underline{v})]ds = 0$ is satisfied for all $\underline{v} \in N$.

The regularity problem is now: How smooth is the weak solution $\underline{u} \in V$ of (1.6), if f, t and R are sufficiently smooth ?

We will give an answer in the following.

2. A special problem in an infinite cone

The analysis of the existence, uniqueness and regularity of the boundary value problem (1.1),(1.2),(1.3) is well developed, if the domain Ω is sufficiently smooth. Results in this direction are related to the work of many authors, in particular, to the publications of A.G. Fichera [2], A.I. Koševel [5], O.A. Ladyženskaja and N.N. Ural'tseva [7]. The investigation of the regularity is a local problem. If Ω is a polygonal domain, a regularity principle works in the interior of the domain and on $\partial\Omega \setminus \bigcup_{j=1}^{J} U(O_j)$, where $U(O_j)$ is a neighborhood of a corner point or of a point O_j, where the boundary condition changes. Let us say that these points $\{O_j\}_{j=1,\ldots,J}$ are singular points of $\partial\Omega$.

Thus we have only to investigate the behavior of the solution of (1.1),(1.2),(1.3) near the singular points and to transfer the results to weak solutions. To this aim we choose one of the points $O = O_{jo} \in \{O_j\}_{j=1,\ldots,J}$ with the interior angle ω_o, and we multiply the solution $\underline{u}$ with a cut-off function $\eta(|x|) = \eta(r)$, where $0 \leq \eta(r) \leq 1$,

$$\eta(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq \delta, \\ 0 & \text{for } r \geq 2\delta, \end{cases} \tag{2.1}$$

and $\eta(r) \in C^{\infty}(0,\infty)$. The number δ is so small that no other singular point on $\partial\Omega$ lies in the circle $\{\underline{x}: |\underline{x}| \leq 3\delta\}$. We denote $\underline{w} = \eta\underline{u} = (\eta u_1, \eta u_2)^T$. Let K be the infinite plane cone with the vertex O, the angle ω_o and the sides γ^+ and γ^- (see Fig. 1). Then we have

$$\mu\Delta\underline{w}(\underline{x}) + (\lambda+\mu)\text{grad div } \underline{w}(\underline{x}) = \underline{F}(\underline{x},\underline{u}(\underline{x})) \quad \text{in } K, \tag{2.2}$$

$$\underline{w}(\underline{x}) = \underline{G}(\underline{x},\underline{u}(\underline{x})) \quad \text{on } \gamma^+ \cup \gamma^-, \tag{2.3}$$

or

$$\underline{w}(\underline{x}) = \underline{G}(\underline{x},\underline{u}(\underline{x})) \quad \text{on} \quad \gamma^{+},$$

$$\sigma(\underline{w}(\underline{x}))n(\underline{x}) = \underline{T}(\underline{x},\underline{u}(\underline{x})) - R(\underline{x})\underline{w}(\underline{x}) \quad \text{on} \quad \gamma^{-}, \tag{2.4}$$

$$\sigma(\underline{w}(\underline{x}))n(\underline{x}) = \underline{T}(\underline{x},\underline{u}(\underline{x})) - R(\underline{x})\underline{w}(\underline{x}) \quad \text{on} \quad \gamma^{+} \cup \gamma^{-}. \tag{2.5}$$

right hand sides $\underline{F}$, $\underline{G}$ and $\underline{T}$ are given by $\underline{f}$, $\underline{g}$ and $\underline{t}$ by terms depending on the unknown functions u_1 and u_2;

$$F_1 = \mu(\eta\Delta u_1 + u_1\Delta\eta + 2\frac{\partial\eta}{\partial x_1}\frac{\partial u_1}{\partial x_1} + 2\frac{\partial\eta}{\partial x_2}\frac{\partial u_1}{\partial x_2}) + (\lambda+\mu)(\eta\frac{\partial^2 u_1}{\partial x_1^2}$$

$$+ \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + u_1\frac{\partial^2 \eta}{\partial x_1^2} + 2\frac{\partial u_1}{\partial x_1}\frac{\partial\eta}{\partial x_1} + u_2\frac{\partial^2\eta}{\partial x_1\partial x_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial\eta}{\partial x_1}$$

$$+ \frac{\partial\eta}{\partial x_2}\frac{\partial u_2}{\partial x_1})$$

$$= \eta f_1 + C(u_i\frac{\partial^2\eta}{\partial x_j \partial x_i}, \frac{\partial u_i}{\partial x_k}\frac{\partial\eta}{\partial x_l}).$$

We now forget for some theoretical considerations the special type of the right hand sides of (2.2), (2.3), (2.4) and (2.5) and consider for arbitrary given right hand sides the following special problem in the infinite cone K:

$$\mu\Delta\underline{w}(\underline{x}) + (\lambda+\mu)\,\text{grad div}\,\underline{w}(\underline{x}) = \underline{F}(\underline{x}) \quad \text{in} \quad K, \tag{2.6}$$

$$\underline{w}(\underline{x}) = \underline{G}(\underline{x}) \quad \text{on} \quad \gamma^{+} \cup \gamma^{-}, \tag{2.7}$$

or

$$\left.\begin{aligned} \underline{w}(\underline{x}) &= \underline{G}(\underline{x}) \quad \text{on} \quad \gamma^{+}, \\ \sigma(\underline{w}(\underline{x}))n(\underline{x}) &= \underline{T}(\underline{x}) \quad \text{on} \quad \gamma^{-}, \end{aligned}\right\} \tag{2.8}$$

or

$$\sigma(\underline{w}(\underline{x}))n(\underline{x}) = \underline{T}(\underline{x}) \quad \text{on} \quad \gamma^{+} \cup \gamma^{-} \tag{2.9}$$

We now introduce the polar coordinates $x_1 = r\cos\omega$, $x_2 = r\sin\omega$, set $r = e^{\tau}$ and use the complex Fourier transform

$$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-is\tau} f(\tau)\, d\tau. \tag{2.10}$$

These transforms (joining together we have the Mellin transform) yield $r\frac{\partial}{\partial r} = -\frac{\partial}{\partial \tau} \longrightarrow iz = \alpha$. The transformed boundary value problems (2.6),(2.7),(2.8) and (2.9) have the following form:

$$\mu(\alpha^2 h_1 + h_1'') + (\lambda+\mu)\{h_1[(\tfrac{\alpha^2}{2} - \alpha)\cos 2\omega + \tfrac{\alpha^2}{2}] + h_1'(1-\alpha)\sin 2\omega$$

$$+ h_1''(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\omega) + h_2(\tfrac{\alpha^2}{2} - \alpha)\sin 2\omega$$

$$+ h_2'(\alpha-1)\cos 2\omega + h_2''(-\tfrac{1}{2}\sin 2\omega)\} = K_1(\alpha,\omega), \qquad (2.11)$$

$$\mu(\alpha^2 h_2 + h_2'') + (\lambda+\mu)\{h_2[(-\tfrac{\alpha^2}{2} + \alpha)\cos 2\omega + \tfrac{\alpha^2}{2}] + h_2'(\alpha-1)\sin 2\omega$$

$$+ h_2''(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega) + h_1(\tfrac{\alpha^2}{2} - \alpha)\sin 2\omega + h_1'(\alpha-1)\cos 2\omega - h_1'' \tfrac{1}{2}\sin 2\omega\}$$

$$= K_2(\alpha,\omega) \quad \text{for} \quad 0 < \omega < \omega_o,$$

$$\underline{h}(\alpha,\omega) = \underline{L}^{\pm}(\alpha,\omega) \quad (\text{"}-\text{" for } \omega=0 \text{ and "}+\text{" for } \omega=\omega_o), \qquad (2.12)$$

$$\underline{h}(\alpha,\omega) = \underline{L}^{+}(\alpha,\omega), \quad h_1' + \alpha h_2 = -\tfrac{1}{\mu}L_1^{-}(\alpha,\omega),$$
$$\lambda(\alpha h_1 + h_2') + 2\mu h_2' = -L_2^{-}(\alpha,\omega), \qquad (2.13)$$

$$h_1' + \alpha h_2 = \tfrac{1}{\mu}L_1^{-}(\alpha,\omega), \quad \lambda(\alpha h_1 + h_2') + 2\mu h_2' = -L_2^{-}(\alpha,\omega),$$

$$\alpha h_1(\lambda+\mu)\cos\omega_o\sin\omega_o + \alpha h_2((\lambda+\mu)\sin^2\omega_o - \mu) + h_1'(-(\lambda+\mu)\sin^2\omega_o - \mu)$$

$$+ h_2'(\lambda+\mu)\cos\omega_o\sin\omega_o = L_1^{+}(\alpha,\omega), \qquad (2.14)$$

$$\alpha h_1(\mu - (\lambda+\mu)\cos^2\omega_o) + \alpha h_2(-(\mu+\lambda)\cos\omega_o\sin\omega_o)$$

$$+ h_1'(\lambda+\mu)\sin\omega_o\cos\omega_o + h_2'(-(\lambda+\mu)\cos^2\omega_o - \mu) = L_2^{+}(\alpha,\omega),$$

where $\underline{h} = \underline{h}(\alpha,\omega) = \mathcal{F}(\underline{\hat{w}})$ with $\underline{\hat{w}} = \underline{\hat{w}}(\tau,\omega) = \underline{w}(\underline{x})$,

$\underline{K}(\alpha,\omega) = \mathcal{F}(r^2\underline{\hat{F}})$ with $\underline{\hat{F}} = \underline{\hat{F}}(\tau,\omega) = \underline{F}(\underline{x})$

$\underline{L}^{\pm}(\alpha,\omega) = \mathcal{F}(r\underline{\hat{T}}^{\pm})$ with $\underline{\hat{T}}^{\pm} = \underline{\hat{T}}^{\pm}(\tau,\omega) = \underline{T}^{\pm}(\underline{x})$ in (2.13) or (2.14),

$\underline{L}^{\pm}(\alpha,\omega) = \mathcal{F}(\hat{G}^{\pm})$ with $\hat{G}^{\pm} = \hat{G}^{\pm}(\tau,\omega) = G^{\pm}(\underline{x})$ in (2.12) or (2.13),

$$\underline{T}^{\pm}(\underline{x}) = \underline{T}(\underline{x})\big|_{\gamma^{\pm}},\quad G^{\pm}(\underline{x}) = G(\underline{x})\big|_{\gamma^{\pm}},$$

and $\underline{h}' = \frac{\partial \underline{h}(\alpha,\omega)}{\partial\omega}$, $\underline{h}'' = \frac{\partial^2 \underline{h}(\alpha,\omega)}{\partial\omega^2}$. γ^+ denotes the side of K with $\omega = \omega_0$, γ^- denotes the side of K with $\omega = 0$.

3. The regularity theory in weighted Sobolev spaces

In section 2 the following questions occur: In which spaces is the complex Fourier transform (or the Mellin transform) well defined ? What can we say about the inverse transforms ? Which properties of the transformed problems (2.11), (2.12), (2.13), (2.14) determine such properties as solvability and regularity of the problems (2.6), (2.7), (2.8), (2.9) ?

The introduction of weighted Sobolev spaces is useful for answering these questions (see [3, 4, 8, 9]).

Let be

$$V^{k,p}(K,\beta) \tag{3.1}$$

the closure of the set $C^{\infty}_{M}(K) = \{ v \in C^{\infty}(\bar{K}),$ supp v bounded, supp $v \cap M = \emptyset\}$ for $M = \{0\}$ with respect to the norm

$$\|v;V^{k,p}(K,\beta)\| = \Big(\sum_{|\gamma|\le k} \int_K |D^{\gamma}v(\underline{x})|^p r^{p(\beta-k+|\gamma|)} d\underline{x} \Big)^{1/p} \tag{3.2}$$

and let be

$$V^{k-1/p,p}(\gamma^+,\beta) \quad \text{or} \quad V^{k-1/p,p}(\gamma^-,\beta) \tag{3.3}$$

the spaces of traces, defined as the factor spaces $V^{k,p}(K,\beta)/V^{k,p}(K,\beta,\gamma^{\pm})$, where $V^{k,p}(K,\beta,\gamma^{\pm})$ is the closure of $C^{\infty}_{\gamma^{\pm}}(K)$ with respect to the norm (3.2).

The space $V^{k,p}(\Omega,\beta)$ is defined analogously to $V^{k,p}(K,\beta)$.
Furthermore we use the notation $X \times X = X^2$ for a space X.
We denote by $A(D_x)$ the matrix-differential operator of the system (2.6), that means

$$A(D_x) = \begin{pmatrix} (\lambda+2\mu)\frac{\partial^2}{\partial x_1^2} + \mu\frac{\partial^2}{\partial x_2^2} & (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2} \\ (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2} & (\lambda+2\mu)\frac{\partial^2}{\partial x_2^2} + \mu\frac{\partial^2}{\partial x_1^2} \end{pmatrix} \tag{3.4}$$

and by $B(D_x)$ the boundary operators of (2.7) or (2.8) or (2.9); e.g. for (2.9) we have on γ^+ and γ^-

$$B^{\pm}(D_x) = \begin{pmatrix} (2\mu+\lambda)n_1 \frac{\partial}{\partial x_1} + \mu n_2 \frac{\partial}{\partial x_2} & \lambda\frac{\partial}{\partial x_2}n_1 + \mu\frac{\partial}{\partial x_1}n_2 \\ \mu\frac{\partial}{\partial x_2}n_1 + \lambda\frac{\partial}{\partial x_1}n_2 & \mu\frac{\partial}{\partial x_1}n_1 + (2\mu+\lambda)\frac{\partial}{\partial x_2}n_2 \end{pmatrix}$$

We consider the operator

$$\begin{aligned} \mathfrak{A}(D_x) = \{A(D_x), B^{\pm}(D_x)\} : [V^{l+2,p}(K,\beta)]^2 \longrightarrow [V^{l,p}(K,\beta)]^2 \\ \times [V^{l+2-m^+-1/p,p}(\gamma^+,\beta)]^2 \times [V^{l+2-m^--1/p,p}(\gamma^-,\beta)]^2, \end{aligned} \tag{3.5}$$

here $1 < p < \infty$, $l \geqslant 0$, m^+ and m^- are the orders of the orresponding boundary operators on γ^+ or γ^-. Furthermore we rite for the system (2.11) shortly

$$A(\alpha, D_\omega)\underline{h}(\alpha,\omega) = \underline{K}(\alpha,\omega)$$

nd for the boundary conditions (2.12), (2.13) or (2.14)

$$B^{\pm}(\alpha, D_\omega)\underline{h}(\alpha,\omega) = \underline{L}^{\pm}(\alpha,\omega) \quad \text{for } \omega = 0 \text{ and } \omega = \omega_0 .$$

Ve consider the corresponding operator

$$\begin{aligned} \mathfrak{A}(\alpha, D_\omega) = \{A(\alpha, D_\omega), B^{\pm}(\alpha, D_\omega)\} : [W^{2,2}(I)]^2 \\ \longrightarrow [L^2(I)]^2 \times \mathbb{C}^2 \times \mathbb{C}^2, \quad \text{where } I = (0,\omega_0). \end{aligned} \tag{3.6}$$

The inverse Fourier transform is given by

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+ih}^{\infty+ih} e^{iz\tau} \mathcal{F}(f)(z)\, dz, \tag{3.7}$$

where $h = \operatorname{Im} z = -\operatorname{Re} \alpha$.

It is well defined for solutions of the boundary value problem $\mathfrak{A}(\alpha, D_\omega)\underline{h}(\alpha,\omega) = \{\underline{K}(\alpha,\omega), \underline{L}^{\pm}(\alpha,\omega)\}$ provided no "eigenvalue" of

$\mathfrak{A}(\alpha,D_\omega)$ is situated on the line $\mathrm{Re} = -h = -\beta-2/p+2+l$ (see [6] for one equation or [8] for a system). Let us give the definition of an eigenvalue of $\mathfrak{A}(\alpha,D_\omega)$:

<u>Definition 2:</u> The complex number $\alpha = \alpha_0$ is an eigenvalue of $\mathfrak{A}(\alpha,D_\omega)$ if there exists a nontrivial solution $\underline{e}^0(\alpha_0,\omega) \in [W^{2,2}(I)]^2$ of $\mathfrak{A}(\alpha,D_\omega)\underline{e}(\alpha,\omega)|_{\alpha=\alpha_0} = \underline{O}$; $\underline{e}^0(\alpha_0,\omega)$ is an eigenvector function of $\mathfrak{A}(\alpha,D_\omega)$ with respect to α_0. The vector function $\underline{e}^1(\alpha_0,\omega)$ is an associate vector function to α_0 and $\underline{e}^0$ if

$$\frac{d\mathfrak{A}(\alpha_0,D_\omega)}{d\alpha}\,\underline{e}^0(\alpha_0,\omega) + \mathfrak{A}(\alpha_0,D_\omega)\underline{e}^1(\alpha_0,\omega) = \underline{O}. \tag{3.8}$$

The following solvability and regularity theorems are formulated in [8].

<u>Theorem 1</u> (Solvability): The operator (3.5) is an isomorphism if and only if no eigenvalue of $\mathfrak{A}(\alpha,D_\omega)$ lies on the line $\mathrm{Re}\,\alpha = -\beta-2/p+2+l$.

<u>Theorem 2</u> (Regularity): Assume that the right hand side of (2.6) $\underline{F}(\underline{x})$ from $[V^{l,p}(K,\beta)]^2 \cap [V^{l',p'}(K,\beta')]^2$, the right hand side of (2.7) or (2.8) $\underline{G}(\underline{x})$ from $[V^{l+2-1/p,p}(\gamma^\pm,\beta)]^2 \cap [V^{l'+2-1/p',p'}(\gamma^\pm,\beta')]^2$ and the right hand side of (2.8) or (2.9) $\underline{T}(\underline{x})$ from $[V^{l+1-1/p,p}(\gamma^\pm,\beta)]^2 \cap [V^{l'+1-1/p',p'}(\gamma^\pm,\beta')]^2$. If no eigenvalues of $\mathfrak{A}(\alpha,D_\omega)$ lie on the lines $\mathrm{Re}\,\alpha = -h = -\beta-2/p+l+2$ and $\mathrm{Re}\,\alpha = -h' = -\beta'-2/p'+l'+2$ and if the eigenvalues $\alpha_1,\ldots,\alpha_N$ are situated in the strip $-h < \mathrm{Re}\,\alpha < -h'$, then the solution of (2.6), (2.7), (2.8), (2.9) $\underline{w} \in [V^{l+2,p}(K,\beta)]^2$ allows the following expansion:

$$\underline{w}(r,\omega) = \sum_{\nu=1}^{N}\sum_{6=1}^{I_\nu}\sum_{k=0}^{\mathfrak{S}_{6\nu}} c_{6k\nu}\underline{u}^{(6)}_{k,\nu}(r,\omega) + \underline{v}(r,\omega), \tag{3.9}$$

where $\underline{v}(r,\omega) \in [V^{l'+2,p'}(K,\beta')]^2$, $I_\nu = \dim N(\mathfrak{A}(\alpha_\nu,D_\omega)) = \dim \mathrm{span}\{\underline{e}^0_1(\alpha_\nu,\omega),\ldots,\underline{e}^0_{I_\nu}(\alpha_\nu,\omega)\}$ is the number of the linearly independent eigenvector functions to α_ν,

$$\delta_{6\nu} = \begin{cases} 1 & \text{if an associate vector function exists} \\ & \text{for } \alpha_\nu \text{ and } \underline{e}^{0}(\alpha_\nu,\omega), \\ 0 & \text{else,} \end{cases}$$

$c_{6k\nu}$ are constants and

$$\underline{u}^{(6)}_{k,\nu}(r,\omega) = r^{\alpha_\nu} \sum_{s=0}^{k} (\log r)^s \underline{e}_6^{k-s}(\alpha_\nu,\omega) \tag{3.10}$$

are the so-called "singular" vector functions.

Remarks to the proof of Theorem 2: The structure of the singular vector functions can be explained by the following consideration ([4, 6]): The inverse Fourier transform of the right hand sides of (2.11), (2.12), (2.13), (2.14) or shortly of (3.6) can be written as

$$\hat{\underline{w}}(\tau,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+ih}^{\infty+ih} e^{iz\tau}\mathfrak{a}^{-1}(\alpha,D_\omega)[\underline{K}(\alpha,\omega),\underline{L}^{\pm}(\alpha,\omega)]\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{M\to\infty} \Big\{ \int_{-M+ih}^{-M+ih'} e^{iz\tau}\mathfrak{a}^{-1}(\alpha,D_\omega)[\underline{K}(\alpha,\omega),\underline{L}^{\pm}(\alpha,\omega)]\, dz$$

$$+ \int_{-M+ih'}^{M+ih'} e^{iz\tau}\mathfrak{a}^{-1}(\alpha,D_\omega)[\underline{K}(\alpha,\omega),\underline{L}^{\pm}(\alpha,\omega)]\, dz$$

$$+ \int_{M+ih'}^{M+ih} e^{iz\tau}\mathfrak{a}^{-1}(\alpha,D_\omega)[\underline{K}(\alpha,\omega),\underline{L}^{\pm}(\alpha,\omega)]\, dz\Big\}$$

$$+ \frac{1}{\sqrt{2\pi}}\, 2\pi i \sum_{\nu=1}^{N} \operatorname{Res}(e^{\alpha\tau}\mathfrak{a}^{-1}(\alpha,D_\omega)[\underline{K}(\alpha,\omega),\underline{L}^{\pm}(\alpha,\omega)]\Big|_{\alpha=\alpha_\nu}.$$

The first and third integrals tend to $\underline{O}$ for $M\to\infty$; the second integral yields $\underline{v}(r,\omega)$ and the calculation of the residuae yields the singular terms. ■

We now go back to the problem (1.1), (1.2), (1.3). The following lemma can be proved analogously to that which is formulated in [6] for one equation:

Lemma 1: Let $\underline{u}$ be a solution of (1.1), (1.2), (1.3) for the right hand sides $\underline{f}$, $\underline{g}$ and $\underline{t}$. Assume $\underline{w} = \eta\underline{u} \in [V^{l+2,p}(K,\beta)]^2$, $\eta\underline{f} \in [V^{l,p}(K,\beta)]^2 \cap [V^{l',p'}(K,\beta')]^2$, $\eta\underline{g} \in [V^{l+2-1/p,p}(\gamma^{\pm},\beta)]^2 \cap [V^{l'+2-1/p',p'}(\gamma^{\pm},\beta')]^2$, $\eta\underline{t} \in [V^{l+1-1/p,p}(\gamma^{\pm},\beta)]^2 \cap [V^{l'+1-1/p',p'}(\gamma^{\pm},\beta')]^2$. If

$$0 \leqslant h - h' \leqslant 1 \qquad (3.11)$$

or

$$h - h' > 1 \text{ and } R(\underline{x}) \text{ is the O-matrix,} \qquad (3.12)$$

then the right hand sides of (2.2), (2.3), (2.4), (2.5) satisfy the suppositions of Theorem 2, provided no eigenvalue of $\mathcal{O}'(\cdot, D_\omega)$ lies on the lines $\operatorname{Re}\alpha = -h$ and $\operatorname{Re}\alpha = -h'$.

We get as corollary from Lemma 1

Theorem 3: Let $\underline{u}$ be a solution of the boundary value problem (1.1), (1.2), (1.3) for which the suppositions of Lemma 1 are satisfied. Then the expansion (3.9) near the singular point $\underline{O}$ holds:

$$\eta^2\underline{u}(r,\omega) = \sum_{\nu=1}^{N}\sum_{\sigma=1}^{I_\nu}\sum_{k=0}^{s_{\sigma\nu}} c_{\sigma k\nu}\, \underline{u}_{k,\nu}^{(\sigma)}(r,\omega) + \eta\underline{v}(r,\omega), \qquad (3.13)$$

where $\eta\underline{v} \in [V^{l+2,p}(K,\beta)]^2$.

4. The regularity of the weak solutions

Let us consider the weak solution $\underline{u} \in V$ defined by (1.6). Again let be $\underline{O}$ a singular point of $\partial\Omega$ and η the corresponding cut-off function defined by (2.1). We can use the results of section 3, if we can show that $\eta\underline{u} \in V \cap [V^{l+2,p}(K,\beta)]^2$ for appropriate right hand sides $\underline{f}$, $\underline{g}$, $\underline{t}$ and for some l, p and β.

Lemma 2: Let be $\underline{f} \in [L^2(\Omega,1+\varepsilon)]^2$, $\underline{g} = \underline{O}$ and $\underline{t} = \underline{O}$ on $\partial\Omega$ for a small real number $\varepsilon > 0$. Then it holds for a weak solution $\underline{u} \in V$ of (1.6) that $\eta\underline{u} \in [V^{2,2}(K,1+\varepsilon)]^2$.

Proof: We follow the ideas of V.A. Kondrat'ev [3]. We consider a sequence of domains Ω_k, $k = 1,2,\dots$, where $\Omega_k = \Omega \cap R_k$, $R_k = \{\underline{x}: \hat{\delta}/2^{k+1} \leqslant |\underline{x}| \leqslant \hat{\delta}/2^k\}$. For the real number $\hat{\delta} = 2\delta$ we consider a cut-off function $\hat{\eta}$ defined as in (2.1). We have $\bigcup_k \Omega_k = K_o \subset K$. The usual regularity theorems yield for $|\gamma| = 2$:

$$\iint_{\Omega_k} |D^\gamma \underline{u}|^2 d\underline{x} \leqslant C\Big[\iint_{\Omega_{k-1}\cup\Omega_k\cup\Omega_{k+1}} |\underline{f}|^2 d\underline{x} + \iint_{\Omega_{k-1}\cup\Omega_k\cup\Omega_{k+1}} r^{-4}|\underline{u}|^2 d\underline{x}\Big], \qquad (4.1)$$

where $|D^\gamma \underline{u}|^2 = |D^\gamma u_1|^2 + |D^\gamma u_2|^2$. We multiply (4.1) by $(\frac{\hat{\delta}}{2^k})^{2(1+\varepsilon)}$ and estimate (4.1)

$$\int\int_{\Omega_k} r^{2(1+\varepsilon)} |D^\gamma \underline{u}|^2 d\underline{x} \leqslant C\Big(\int\int_{\Omega_{k-1}\cup\Omega_k\cup\Omega_{k+1}} r^{2(1+\varepsilon)} |\underline{f}|^2 d\underline{x} + \int\int_{\Omega_{k-1}\cup\Omega_k\cup\Omega_{k+1}} r^{2(-1+\varepsilon)} |\underline{u}|^2 d\underline{x}\Big). \tag{4.2}$$

Summing with respect to k we get

$$\int\int_{K_o} r^{2(1+\varepsilon)} |D^\gamma \underline{u}|^2 d\underline{x} \leqslant C\Big(\int\int_{K_o} r^{2(1+\varepsilon)} |\underline{f}|^2 d\underline{x} + \int\int_{K_o} r^{2(-1+\varepsilon)} |\underline{u}|^2 d\underline{x}\Big). \tag{4.3}$$

Let us consider the term

$$\int\int_{K_o} r^{2(-1+\varepsilon)} |\underline{u}|^2 d\underline{x} = \int\int_{K_o} r^{2(-1+\varepsilon)} |\eta \underline{u}|^2 d\underline{x}. \tag{4.4}$$

We write (4.4) in polar coordinates and use the Hardy inequality:

$$\int_o^\infty |f(t)|^2 t^{\varepsilon'-2} dt \leqslant (2/|\varepsilon'-1|)^2 \int_o^\infty |f'(t)|^2 t^{\varepsilon'} dt$$

for $\varepsilon' > 1$ and $f(\infty) = 0$.

We have for $\tilde{\underline{u}}(r,\omega) = \underline{u}(\underline{x})$

$$\begin{aligned}
\int\int_{K_o} r^{-2+2\varepsilon+1} |\hat{\eta}\tilde{\underline{u}}|^2 dr d\omega &\leqslant \int_o^{\omega_o}\int_o^\infty r^{-2+2\varepsilon+1} |\hat{\eta}\tilde{\underline{u}}|^2 dr d\omega \\
&\leqslant \int_o^{\omega_o} (2/2\varepsilon)^2 \int_o^\infty r^{2\varepsilon} \Big|\frac{\partial}{\partial r} \hat{\eta}\tilde{\underline{u}}\Big|^2 r\, dr d\omega \\
&\leqslant C \int\int_{\Omega\cap \operatorname{supp}\hat{\eta}} r^{2\varepsilon} |\underline{u}|^2 + r^{2\varepsilon}(|\operatorname{grad} u_1|^2 + |\operatorname{grad} u_2|^2 d\underline{x} \\
&\leqslant C \, \|\underline{u};\ [W^{1,2}(\Omega)]^2\|^2
\end{aligned}$$

We now consider the cut-off function η for $\delta = \hat{\delta}/2$. Then for $|\gamma| = 2$ it holds

$$\iint_{K_0} r^{2(1+\varepsilon)} |D^{\gamma} \eta \underline{u}|^2 d\underline{x} \leq C \sum_{|\gamma'| \leq 2} \iint_{K_0} r^{2(1+\varepsilon)} |D^{\gamma'} \underline{u}|^2 d\underline{x} ,$$

and therefore $\eta \underline{u} \in [V^{2,2}(K, 1+\varepsilon)]^2$. ■

Remark: Lemma 2 works too if $\eta \underline{g} \in [V^{2-1/2,2}(\gamma^{\pm}, 1+\varepsilon)]^2$ and $\eta \underline{t} \in [V^{1/2,2}(\gamma^{\pm}, 1+\varepsilon)]^2$.

The following theorem follows immediately from Lemma 2.

Theorem 4: Let $\underline{u} \in \acute{V}$ be a weak solution of (1.6). Let be $\varepsilon > 0$ such a small real number that no eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ lie on the line $\mathrm{Re}\,\alpha = -\varepsilon$. Furthermore we assume that no eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ lie on the line $\mathrm{Re}\,\alpha = 1$. We assume for the right hand sides of (1.1), (1.2), (1.3) that $\eta \underline{f} \in [L^2(K)]^2$, $\eta \underline{g} \in [V^{2-1/2,2}(\gamma^{\pm}, 0)]^2$, and $\eta \underline{t} \in [V^{1/2,2}(\gamma^{\pm}, 0)]^2$. Let $R(\underline{x})$ be the 0-matrix. Then $\underline{u}$ allows the expansion (3.13) with $\eta \underline{v} \in [W^{2,2}(\Omega)]^2$.

Remarks to the proof: Theorem 3 yields that $\eta \underline{v} \in [V^{2,2}(K,0)]^2$ and this means that $\eta \underline{v} \in [W^{2,2}(\Omega)]^2$. The condition "$R(\underline{x})$ is the 0-matrix" is unnecessary if the line $\mathrm{Re}\,\alpha = 0$ is free of eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$. In this case we have $\eta \underline{u} \in [V^{2,2}(K,1)]^2$ and $0 \leq h-h' \leq 1$. ■

5. The calculation of the singular functions

Our goal is to calculate the functions $\underline{u}^{(6)}_{k,\nu}(r,\omega)$ in the expansion (3.13). Formula (3.10) shows that we need the knowledge of the eigenvalues α_ν and of the corresponding eigenvector functions and associate vector functions of $\mathfrak{A}(\alpha, D_\omega)$. We take the following actions: We derive the general solution of the system (2.11) with $K_1(\alpha,\omega) = K_2(\alpha,\omega) = 0$. The fundamental system consists of four linearly independent solutions, consequently, we determine the four arbitrary constants in the general solution in such a way that nontrivial solutions exist which satisfy the homogeneous boundary conditions (2.12), (2.13) or (2.14).

This leads to the calculation of the zeros of some determinates.

Let us start: Every vector function $\underline{w} = (w_1, w_2)^T$ which is solution of the system (2.6) with $\underline{F}(\underline{x}) = \underline{0}$ satisfies the biharmonic equation in the following manner: $\Delta^2 w_1 = 0$, $\Delta^2 w_2 = 0$ in K. Therefore we can use the investigations for the biharmonic operator [6]. Thus we get for the solution $\underline{h}$ of the homogeneous system (2.11):

<u>for $\alpha \neq 0$</u>

$$\underline{h}(\alpha,\omega) = C_1(\alpha)\begin{pmatrix}\cos\alpha\omega\\ -\sin\alpha\omega\end{pmatrix} + C_2(\alpha)\begin{pmatrix}\sin\alpha\omega\\ \cos\alpha\omega\end{pmatrix}$$
$$+ C_3(\alpha)\begin{pmatrix}\cos(\alpha-2)\omega\\ -\sin(\alpha-2)\omega - A\sin\alpha\omega\end{pmatrix} + C_4(\alpha)\begin{pmatrix}\sin(\alpha-2)\omega\\ \cos(\alpha-2)\omega + A\cos\alpha\omega\end{pmatrix}, \tag{5.1}$$

where $A = \frac{2(\lambda+3\mu)}{(\lambda+\mu)\alpha}$,

<u>for $\alpha = 0$</u>

$$\underline{h}(0,\omega) = C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix} + C_3\begin{pmatrix}2C+\sin 2\omega\\ -\cos 2\omega\end{pmatrix} + C_4\begin{pmatrix}\cos 2\omega\\ -2C\omega+\sin 2\omega\end{pmatrix}, \tag{5.2}$$

$$\text{where} \quad C = \frac{\lambda+3\mu}{\lambda+\mu} \tag{5.3}$$

5.1 The Dirichlet conditions

We consider the Dirichlet condition $\underline{h}(\alpha,\omega) = 0$ for $\omega = 0$ and $\omega = \omega_0$. There are nontrivial solutions (5.1) or (5.2) if the following determinates vanish:

<u>for $\alpha \neq 0$</u>

$$D(\alpha) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1+A \\ \cos\alpha\omega_0 & \sin\alpha\omega_0 & \cos(\alpha-2)\omega_0 & \sin(\alpha-2)\omega_0 \\ -\sin\alpha\omega_0 & \cos\alpha\omega_0 & -\sin(\alpha-2)\omega_0 - A\sin\alpha\omega_0 & \cos(\alpha-2)\omega_0 + A\cos\omega_0 \end{vmatrix}$$

$$= 4\sin^2\alpha\omega_0 - A^2\sin^2\omega_0 = 0. \tag{5.4}$$

Consequently, the eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ are the zeros of the equation

$$\sin^2\alpha\omega_0 = \frac{\alpha^2 G^2}{(G-2)^2}\sin^2\omega_0, \quad \text{where} \quad G = \frac{-(\lambda+\mu)}{\mu} < 0 \quad \text{(compare [1])},$$

for $\alpha = 0$

$$\hat{D}(0) = \begin{vmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2C\omega_0+\sin 2\omega_0 & \cos 2\omega_0 & 1 & 0 \\ -\cos 2\omega_0 & -2C\omega_0+\sin 2\omega_0 & 0 & 1 \end{vmatrix}$$

$$= 2 - 2\cos 2\omega_0 - 4C^2\omega_0^2 = 0.$$

Consequently, $\sin^2\omega_0 = C^2\omega_0^2$ and this equation is only satisfied if $\omega_0 = 0$. Therefore $\alpha = 0$ is no eigenvalue of $\mathfrak{A}(\alpha, D_\omega)$.

Figure 2 and 3 show the distribution of the eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ for the materials lead $(G \approx -10)$ and concrete $(G \approx -1.5)$ for $0 \leqslant \mathrm{Re}\,\alpha \leqslant 3$. The dotted lines indicate real eigenvalues, the full lines indicate the real parts of the complex eigenvalues $\alpha_\nu = \mathrm{Re}\,\alpha_\nu + i\mathrm{Im}\,\alpha_\nu$ and $\bar{\alpha}_\nu = \mathrm{Re}\,\alpha_\nu - i\mathrm{Im}\,\alpha_\nu$. The corresponding eigenvector functions $\underline{e}_1^0(\alpha_\nu,\omega),\ldots,\underline{e}_I^0(\alpha_\nu,\omega)$ are described in the following lemma.

<u>Lemma 3:</u> If α_ν is a zero of (5.4) for the angle ω_0, $\omega_0 \neq \pi$, $\omega_0 \neq 2\pi$, then $I_\nu = 1$ and $\underline{e}_1^0(\alpha_\nu,\omega) = C_3(\alpha_\nu)\underline{y}_3(\alpha_\nu,\omega) + C_4(\alpha_\nu)\underline{y}_4(\alpha_\nu,\omega)$ is an eigenvector function, where

$$\underline{y}_3(\alpha_\nu,\omega) = \begin{pmatrix} -\cos\alpha_\nu\omega + \cos(\alpha_\nu-2)\omega \\ (1-A_\nu)\sin\alpha_\nu\omega - \sin(\alpha_\nu-2)\omega \end{pmatrix} = \begin{pmatrix} y_{31}(\alpha_\nu,\omega) \\ y_{32}(\alpha_\nu,\omega) \end{pmatrix},$$

$$\underline{y}_4(\alpha_\nu,\omega) = \begin{pmatrix} -(1+A_\nu)\sin\alpha_\nu\omega + \sin(\alpha_\nu-2)\omega \\ -\cos\alpha_\nu\omega + \cos(\alpha_\nu-2)\omega \end{pmatrix} = \begin{pmatrix} y_{41}(\alpha_\nu,\omega) \\ y_{42}(\alpha_\nu,\omega) \end{pmatrix}, \qquad (5.6)$$

$$C_3(\alpha_\nu) = -\cos\alpha_\nu\omega_0 + \cos(\alpha_\nu-2)\omega_0,$$

$$C_4(\alpha_\nu) = -(1-A_\nu)\sin\alpha_\nu\omega_0 + \sin(\alpha_\nu-2)\omega_0 \quad \text{and} \qquad (5.7)$$

$$A_\nu = \frac{2(\lambda+3\mu)}{(\lambda+\mu)\alpha_\nu}.$$

If $\omega_0 = \pi$ or $\omega_0 = 2\pi$ then $\alpha_\nu = \alpha_\nu(\pi) = \nu$ or $\alpha_\nu = \alpha_\nu(2\pi) = \nu/2$, $\nu = 1,2,\ldots$. In this case we have $I_\nu = 2$ and

$$\underline{e}_1^o(\alpha_\nu,\omega) = \underline{y}_3(\alpha_\nu,\omega),\ \ \underline{e}_2^o(\alpha_\nu,\omega) = \underline{y}_4(\alpha_\nu,\omega) \tag{5.8}$$

are two linearly independent eigenvector functions.

Proof: We calculate a nontrivial solution $\underline{e}_1^o(\alpha_\nu,\omega)$ of $\mathfrak{A}(\alpha_\nu,D_\omega)\underline{h}(\alpha_\nu,\omega) = \underline{0}$, where $\underline{h}(\alpha_\nu,\omega)$ is given by (5.1). $(B^-(\alpha_\nu,D_\omega)\underline{h}(\alpha_\nu,\omega) = \underline{0}$ $(\omega = 0)$ implies that $C_1(\alpha_\nu) = -C_3(\alpha_\nu)$ and $C_2(\alpha_\nu) = -C_4(1+A_\nu)$. The condition $(B^+(\alpha_\nu,D_\omega)\underline{h}(\alpha_\nu,\omega) = \underline{0}$ $(\omega = \omega_o)$ leads to the system of equations for $\underline{C}(\alpha_\nu)$

$$M(\alpha_\nu,\omega_o)\underline{C}(\alpha_\nu) = \underline{0} \tag{5.9}$$

where

$$M(\alpha_\nu,\omega_o) = \begin{pmatrix} y_{31}(\alpha_\nu,\omega_o) & y_{41}(\alpha_\nu,\omega_o) \\ y_{32}(\alpha_\nu,\omega_o) & y_{42}(\alpha_\nu,\omega_o) \end{pmatrix}$$

and $\underline{C}(\alpha_\nu) = (C_3(\alpha_\nu),\ C_4(\alpha_\nu))^T$. Since the determinate of $M(\alpha_\nu,\omega_o)$ is equal to $D(\alpha_\nu)$in(5.4) and therefore vanishes, we can choose $C_3(\alpha_\nu)$ and $C_4(\alpha_\nu)$ as in (5.7). Thus we get for $\underline{e}_1^o(\alpha,\omega)$, given by (5.6), (consider α instead of α_ν) that

$$B^+(\alpha,D_\omega)\underline{e}_1^o(\alpha,\omega) = M(\alpha,\omega_o)\underline{C}(\alpha) = \binom{D(\alpha)}{0} = \underline{0} \quad \text{for} \quad \alpha = \alpha_\nu. \tag{5.10}$$

If $\omega_o = \pi$ or $\omega_o = 2\pi$, then the rank of the matrix belonging to the determinate of (5.4) is equal to two and consequently we have $I_\nu = 4 - 2 = 2$. We can choose $C_3(\alpha_\nu) = 1$, $C_4(\alpha_\nu) = 0$ and $C_3(\alpha_\nu) = 0$, $C_4(\alpha_\nu) = 1$ and get (5.8). ■

We now have to investigate, whether associate functions occur. M.A. Najmark [10] has proved some results about the connection between the multiplicity of the zeros of $D(\alpha)$ and the existence of associate functions for boundary value problems for ordinary differential equations. These results are also valid for our boundary value problem (3.6), namely for $\mathfrak{A}(\alpha,D_\omega)\underline{h}(\alpha,\omega) = \underline{0}$. Let $m(\alpha_\nu)$ be the multiplicity of the zero α_ν of $D(\alpha)$ (formula (5.4)).

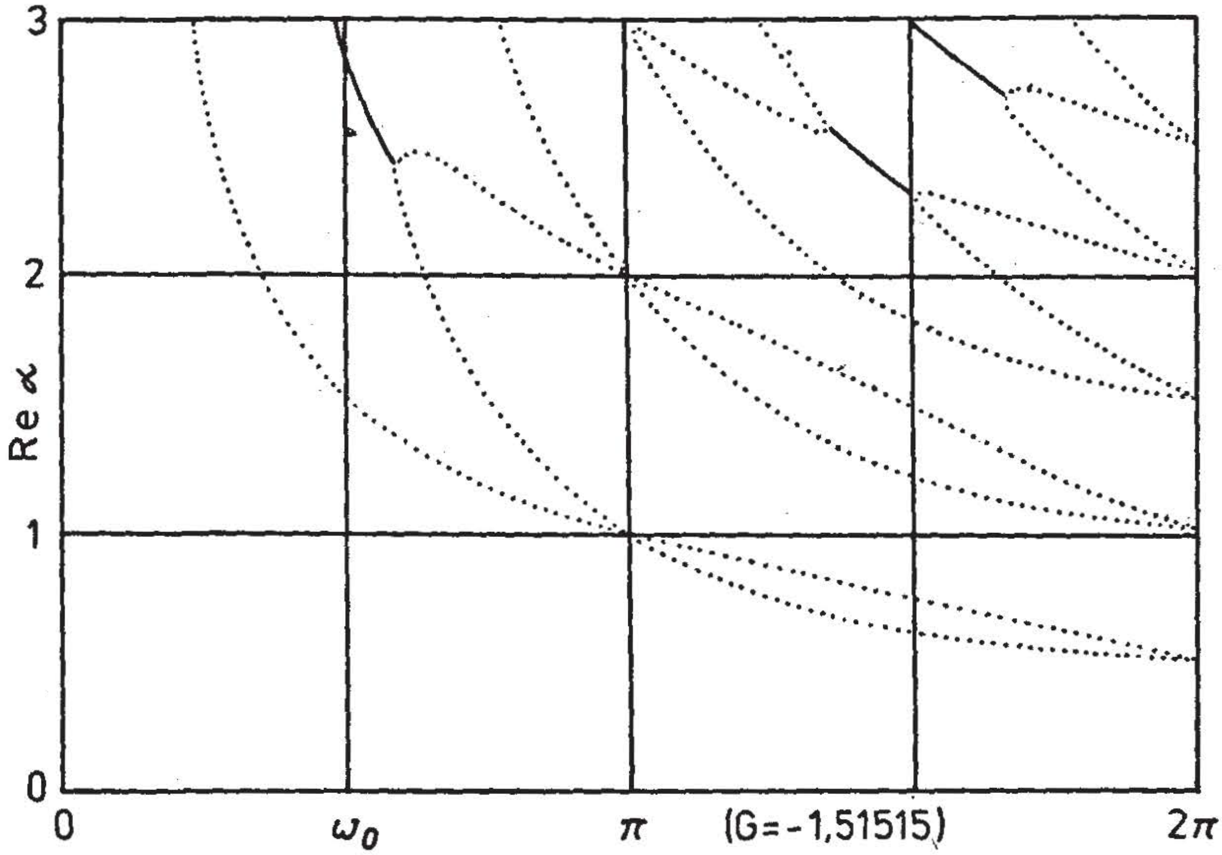

Fig. 2

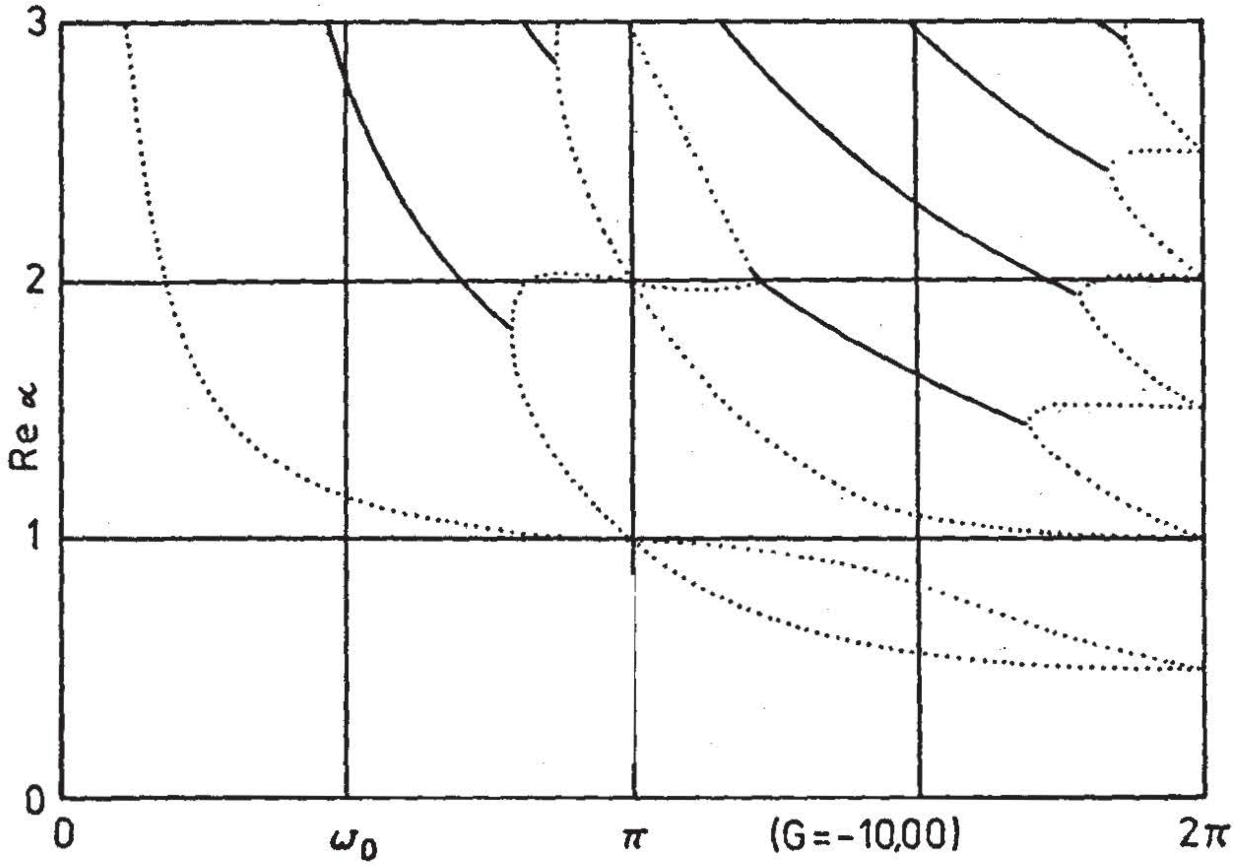

Fig. 3

<u>Lemma 4:</u> It holds that $I_\nu \leq \sum_{6=1}^{I_\nu} (\delta_{6\nu}+1) = m(\alpha_\nu)$.

<u>Corollary:</u> There are only associate functions for the eigenvalue α_ν, if $m(\alpha_\nu) = 2$ and $I_\nu = 1$. If $\omega_o = \pi$ or $\omega_o = 2\pi$ then no associate functions exist.

Lemma 5 describes, when $m(\alpha_\nu) = 2$ and how the corresponding associate functions are to calculate.

<u>Lemma 5:</u> If

$$\alpha_\nu\omega_o = \tan\alpha_\nu\omega_o \quad \text{and} \quad \left(\frac{\sin\omega_o}{\omega_o}\right)^2 = \left(\frac{G-2}{G}\cos\alpha_\nu\omega_o\right)^2, \tag{5.11}$$

$\sin\alpha_\nu\omega_o \neq 0$, $\cos\alpha_\nu\omega_o \neq 0$, then $m(\alpha_\nu) = 2$. The associate functions to the eigenvalue α_ν are

$$\underline{e}_1^1(\alpha_\nu,\omega) = \frac{d}{d\alpha}\underline{e}_1^o(\alpha,\omega)\Big|_{\alpha=\alpha_\nu} ,$$

where $\underline{e}_1^o(\alpha_\nu,\omega)$ is given by (5.6) with α instead of α_ν.

<u>Proof:</u> The equations (5.11) are valid if and only if $D(\alpha_\nu) = 0$ and $\frac{dD(\alpha)}{d\alpha}\Big|_{\alpha=\alpha_\nu} = 0$ for the angle ω_o. The associate functions $\underline{e}_1^1(\alpha_\nu,\omega)$ are solutions of the boundary value problem

$$\mathfrak{A}(\alpha_\nu, D_\omega)\underline{e}_1^1(\alpha_\nu,\omega) + \frac{d\mathfrak{A}(\alpha_\nu, D_\omega)}{d\alpha}\underline{e}_1^o(\alpha_\nu,\omega) = \underline{0},$$

that means for $\alpha = \alpha_\nu$

$$A(\alpha, D_\omega)\underline{e}_1^1(\alpha,\omega) + \frac{dA(\alpha, D_\omega)}{d\alpha}\underline{e}_1^o(\alpha,\omega) = \underline{0} \quad \text{for} \quad 0 < \omega < \omega_o \tag{5.13}$$

and

$$B^\pm(\alpha, D_\omega)\underline{e}_1^1(\alpha,\omega) + \underline{0} = \underline{0} \qquad \text{"+" for } \omega = \omega_o, \quad \text{"−" for } \omega = 0. \tag{5.14}$$

Since $A(\alpha, D_\omega)\underline{e}_1^o(\alpha,\omega) = \underline{0}$ for all α from a neighborhood of α_ν, we get there

$$\frac{d}{d\alpha}[A(\alpha, D_\omega)\underline{e}_1^o(\alpha,\omega)] = \frac{dA(\alpha, D_\omega)}{d\alpha}\underline{e}_1^o(\alpha,\omega) + A(\alpha, D_\omega)\frac{d\underline{e}_1^o(\alpha,\omega)}{d\alpha} = \underline{0}.$$

Therefore the equation (5.13) is satisfied for

$$\underline{e}_1^1(\alpha,\omega) = \frac{d\underline{e}_1^o(\alpha,\omega)}{d\alpha}$$ at the point $\alpha = \alpha_\nu$ especially.

Now we consider the equation (5.14). We have again $B^-(\alpha,D_\omega)\underline{e}_1^o(\alpha,\omega) = \underline{0}$ for all α from a neighbourhood of α_ν and consequently

$$\frac{dB^-(\alpha,D_\omega)}{d\alpha}\underline{e}_1^o(\alpha,\omega) + B^-(\alpha,D_\omega)\frac{d\underline{e}_1^o(\alpha,\omega)}{d\alpha} = \underline{0} \quad \text{for } \alpha = \alpha_\nu.$$

Furthermore (5.10) implies

$$\frac{d}{d\alpha}[B^+(\alpha,D_\omega)\underline{e}_1^o(\alpha,\omega)]\Big|_{\alpha=\alpha_\nu} = \frac{dB^+(\alpha,D_\omega)}{d\alpha}\underline{e}_1^o(\alpha,\omega)\Big|_{\alpha=\alpha_\nu}$$
$$+ B^+(\alpha,D_\omega)\frac{d\underline{e}_1^o(\alpha,\omega)}{d\alpha}\Big|_{\alpha=\alpha_\nu}$$
$$= \frac{d}{d\alpha}[M(\alpha,\omega_o)\underline{C}(\alpha)]\Big|_{\alpha=\alpha_\nu}$$
$$= \frac{d}{d\alpha}(D(\alpha),0)^T\Big|_{\alpha=\alpha_\nu}$$
$$= \left(\frac{dD(\alpha)}{d\alpha},0\right)^T\Big|_{\alpha=\alpha_\nu} = \underline{0}.$$

■

Remarks: (i) The eigenvalues α_ν with $m(\alpha_\nu) = 2$ and $I_\nu = 1$ are those points in Fig. 2 and 3, where the full curve-pieces of the complex eigenvalues start or end. In Fig. 4 there is demonstrated how these eigenvalues (we have $\mathrm{Re}\,\alpha = \alpha_\nu$ in this case) depend on different materials, $-1 \geq G > -\infty$, and on the angle ω_o, that means $\alpha_\nu = \alpha_\nu(G,\omega_o)$. The numbering and the marked direction of the curves describe this connection.

(ii) The associate functions, which arise if $m(\alpha_\nu) = 2$ and $I_\nu = 1$, are not uniquely determined. The multiplication by constants and the addition of an eigenvector-function (5.6) lead to other associate functions. In the following we will see that this fact does not play a role.

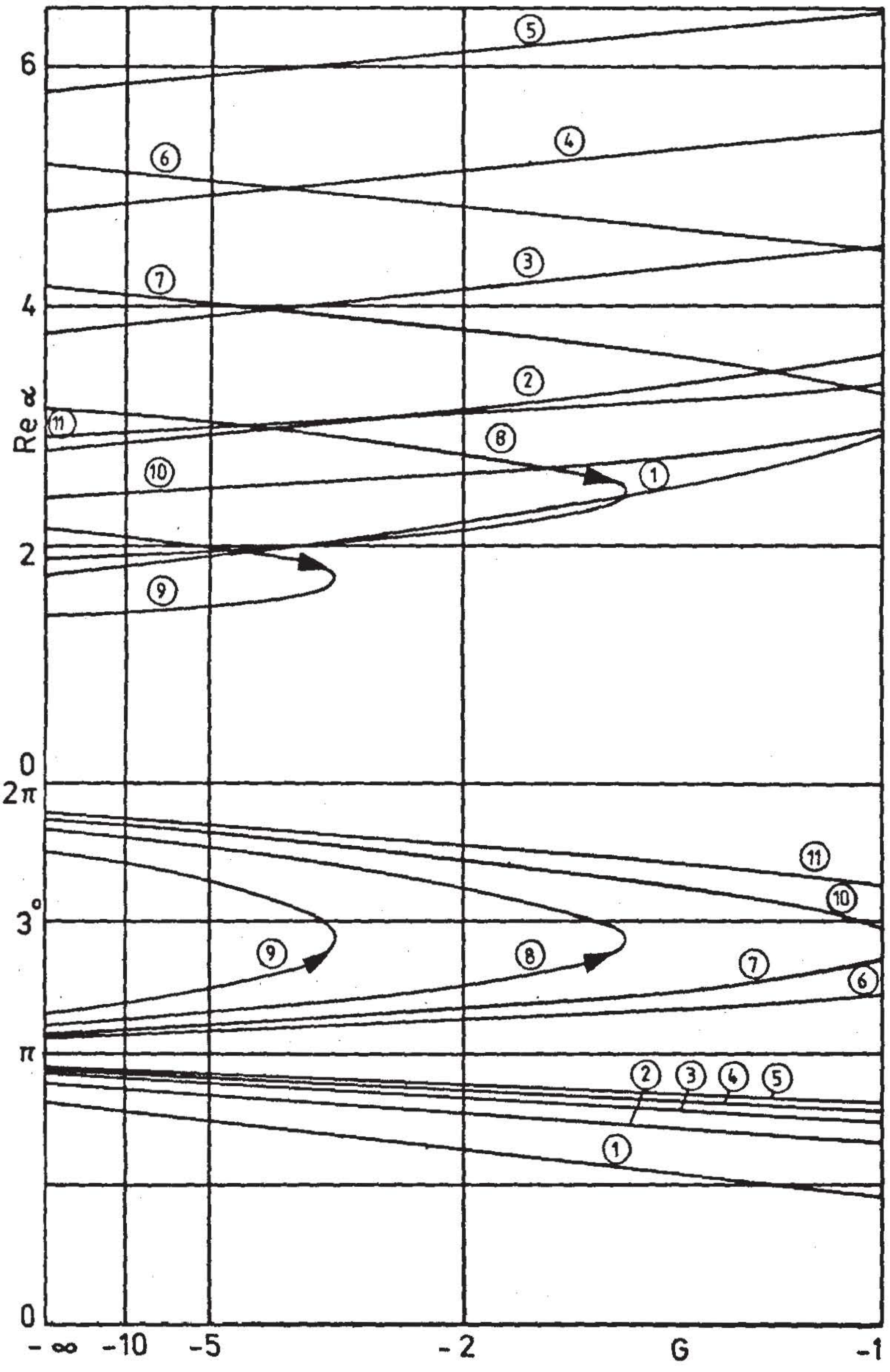

Fig. 4

Let us summarize the results:

Theorem 5: The singular functions (3.10) of the weak solution $\underline{u} \in V$ of the Dirichlet problem have the following form:

(i) If $\omega_o \neq \pi$, $\omega_o \neq 2\pi$ and $\alpha_\nu = \alpha_\nu(G,\omega_o) \neq 0$ is a simple zero of $D(\alpha)$ (formula (5.4)), then only the singular function

$$\underline{u}^{(1)}_{o,\nu}(r,\omega) = r^{\alpha_\nu}\underline{e}^o_1(\alpha_\nu,\omega) \tag{5.15}$$

arises, where $\underline{e}^o_1(\alpha_\nu,\omega)$ is given by (5.6).

(ii) If $\omega_o \neq \pi$, $\omega_o \neq 2\pi$ and $\alpha_\nu = \alpha_\nu(G,\omega_o) \neq 0$ is a double zero of $D(\alpha)$, then the singular functions

$$\underline{u}^{(1)}_{o,\nu}(r,\omega) = r^{\alpha_\nu}\underline{e}^o_1(\alpha_\nu,\omega) \quad \text{and}$$

$$\underline{u}^{(1)}_{1,\nu}(r,\omega) = r^{\alpha_\nu}(\underline{e}^1_1(\alpha_\nu,\omega) + (\log r)\underline{e}^o_1(\alpha_\nu,\omega)) \tag{5.16}$$

occur, where $\underline{e}^o_1(\alpha_\nu,\omega)$ is given by (5.6) and $\underline{e}^1_1(\alpha_\nu,\omega)$ by(5.12).

(iii) If $\omega_o = \pi$, then $\alpha_\nu(G,\pi) = \alpha_\nu(\pi) = \nu$, $\nu = 1,2,\ldots,$ and no "proper" singular functions exist.

(iv) If $\omega_o = 2\pi$ then $\alpha_\nu(G,2\pi) = \alpha_\nu(2\pi) = \nu/2$, $\nu = 1,2,\ldots,$ and

$$\begin{aligned} \underline{u}^{(1)}_{o,\nu}(r,\omega) &= r^{\nu/2}\underline{e}^o_1(\alpha_\nu,\omega), \\ \underline{u}^{(2)}_{o,\nu}(r,\omega) &= r^{\nu/2}\underline{e}^o_2(\alpha_\nu,\omega) \end{aligned} \tag{5.17}$$

are the singular functions. Here $\underline{e}^o_1(\alpha_\nu,\omega)$ and $\underline{e}^o_2(\alpha_\nu,\omega)$ are given by (5.8).

Proof: We have only to look for $\omega_o = \pi$. In this case formula (5.8) yields the eigenvector functions $\underline{e}^o_i(\alpha_\nu,\omega)$, $i = 1,2$. Since $\underline{u}^{(i)}_{o,\nu}(r,\omega) = r\underline{e}^o_i(\alpha_\nu,\omega)$ are smooth vector functions, we can say no "proper" singular functions occur. ■

5.2 The mixed boundary conditions

The solutions $\underline{h}(\alpha,\omega)$, given by formula (5.1), have to satisfy the boundary conditions (2.12) and (2.13), namely

$\underline{h}(\alpha,\omega) = \underline{0}$ for $\omega = \omega_o$ and

$h_1' + \alpha h_2 = 0$ and $\lambda(\alpha h_1 + h_2') + 2\mu h_2' = 0$ for $\omega = 0$.

If $\alpha \neq 0$, then there are nontrivial solutions if

$$D(\alpha) = \begin{vmatrix} \cos\alpha\omega_o & \sin\alpha\omega_o & \cos(\alpha-2)\omega_o & \sin(\alpha-2)\omega_o \\ -\sin\alpha\omega_o & \cos\alpha\omega_o & -\sin(\alpha-2)\omega_o - A\sin\alpha\omega_o & \cos(\alpha-2)\omega_o + A\cos\alpha\omega_o \\ 0 & 2\alpha & 0 & \alpha A + 2\alpha - 2 \\ -2\mu\alpha & 0 & (\lambda+2\mu)(2-\alpha A) - 2\mu\alpha & 0 \end{vmatrix}$$

$$= 16\mu\alpha^2\sin^2\omega_o - 16\mu\frac{(\lambda+2\mu)^2}{(\lambda+\mu)^2} + \sin^2\alpha\omega_o\frac{16\mu(\lambda+3\mu)}{(\lambda+\mu)} = 0. \qquad (5.18)$$

The eigenvalues α_ν of $\mathfrak{A}(\alpha, D_\omega)$ are the zeros of the equation

$$\sin^2\alpha\omega_o = \frac{-\alpha^2\sin^2\omega_o(\lambda+\mu)^2 + (\lambda+2\mu)^2}{(\lambda+\mu)(\lambda+3\mu)}$$

For $\alpha = 0$ the solution (5.2) yields

$$\hat{D}(0) = \begin{vmatrix} 1 & 0 & 2C\omega_o + \sin 2\omega_o & \cos 2\omega_o \\ 0 & 1 & -\cos 2\omega_o & -2C\omega_o + \sin 2\omega_o \\ 0 & 0 & 2C + 2 & 0 \\ 0 & 0 & 0 & 2(1-C)(\lambda+2\mu) \end{vmatrix}$$

$$= 4(1-C^2)(\lambda+2\mu) \neq 0.$$

Therefore $\alpha = 0$ is no eigenvalue of $\mathfrak{A}(\alpha, D_\omega)$.

Fig. 5 and 6 show the distribution of the eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ for $C = -10$ and $C \approx -1.5$ for $0 \leq \mathrm{Re}\,\alpha \leq 3$. Again the dotted lines indicate real eigenvalues, the full lines indicate the real parts of the complex eigenvalues α_ν.

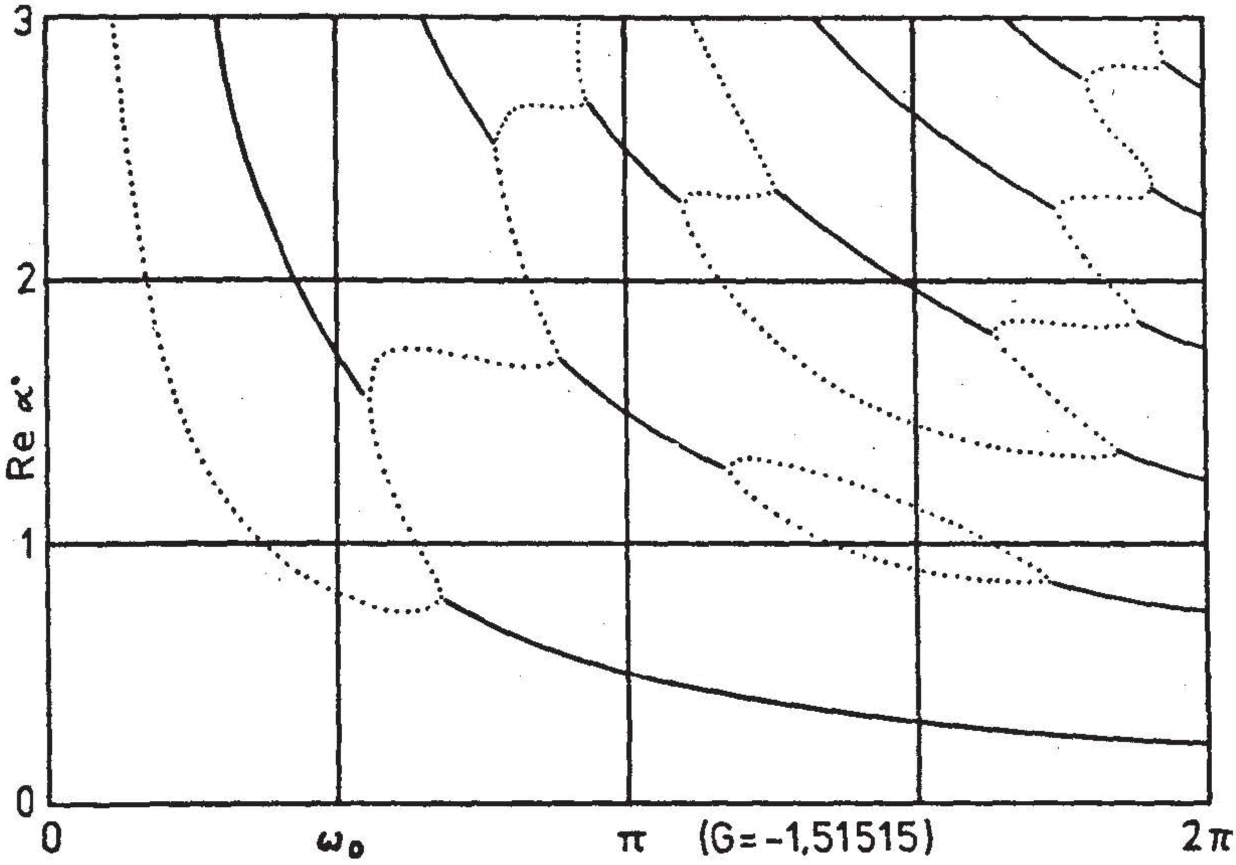

Fig. 5

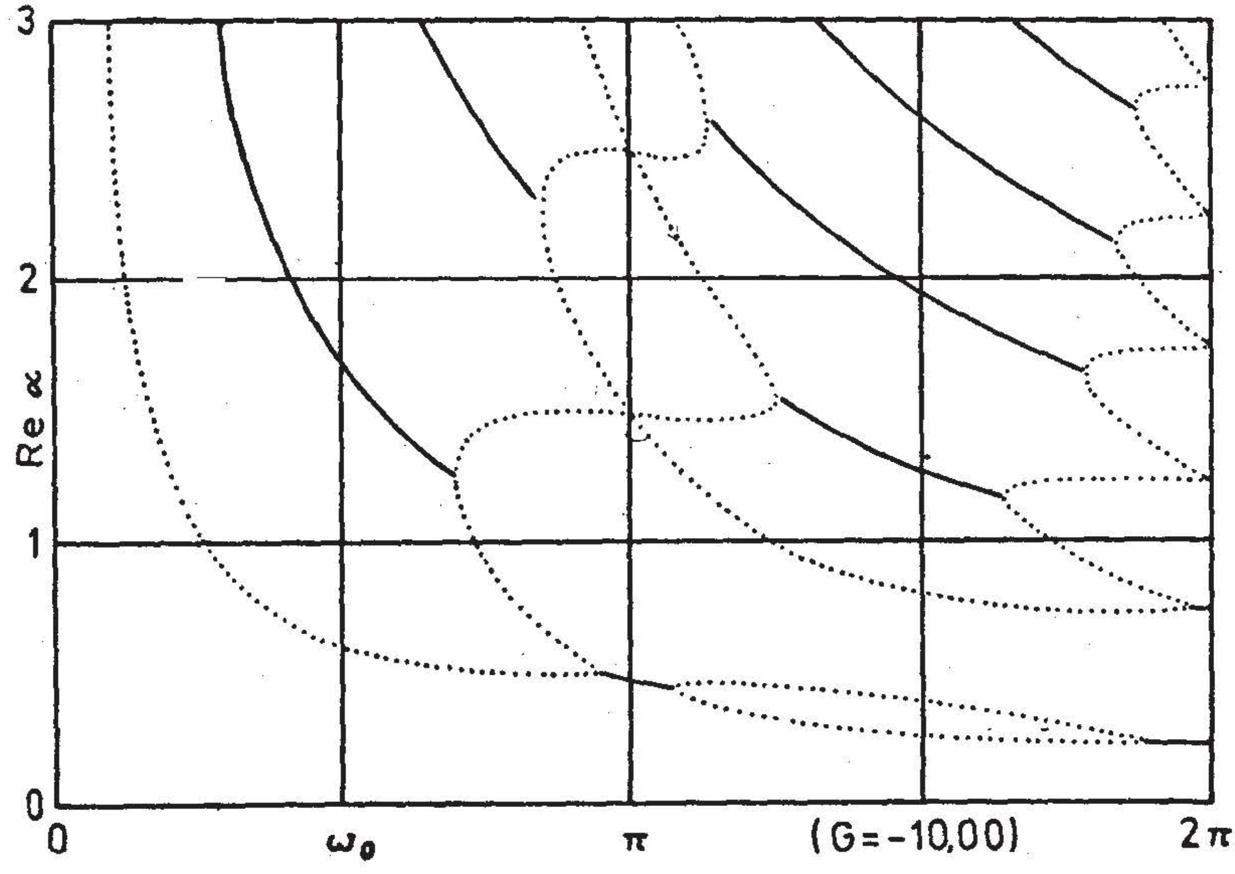

Fig. 6

Lemma 6: If $\alpha_\nu = \alpha_\nu(G,\omega_0)$ is a zero of (5.18) then $I_\nu = 1$ and $\underline{e}_1^0(\alpha_\nu,\omega) = C_3(\alpha_\nu)\underline{y}_3(\alpha_\nu,\omega) + C_4(\alpha_\nu)\underline{y}_4(\alpha_\nu,\omega)$ is an eigenvector function where

$$\underline{y}_3(\alpha_\nu,\omega) = \begin{pmatrix} (\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1)\cos\alpha_\nu\omega_0 + \cos(\alpha_\nu-2)\omega_0 \\ (\frac{2\mu}{(\lambda+\mu)\alpha_\nu} - 1)(-\sin\alpha_\nu\omega_0) - \sin(\alpha_\nu-2)\omega_0 \end{pmatrix},$$

$$\underline{y}_4(\alpha_\nu,\omega) = \begin{pmatrix} (\frac{-2\mu}{\alpha_\nu(\lambda+\mu)} - 1)\sin\alpha_\nu\omega_0 + \sin(\alpha_\nu-2)\omega_0 \\ (\frac{2(\lambda+2\mu)}{\alpha_\nu(\lambda+\mu)} - 1)\cos\alpha_\nu\omega_0 + \cos(\alpha_\nu-2)\omega_0 \end{pmatrix}, \tag{5.19}$$

$$\begin{aligned} C_3(\alpha_\nu) &= (\frac{2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1)\cos\alpha_\nu\omega_0 + \cos(\alpha_\nu-2)\omega_0, \\ C_4(\alpha_\nu) &= (\frac{2\mu}{(\lambda+\mu)\alpha_\nu} - 1)\sin\alpha_\nu\omega_0 + \sin(\alpha_\nu-2)\omega_0. \end{aligned} \tag{5.20}$$

Proof: The rank of the matrix belonging to the determinate (5.18) is equal to three. Consequently $I_\nu = 4 - 3 = 1$. The ideas of the proof are the same ones as of the proof of Lemma 3. We consider $\underline{h}(\alpha_\nu,\omega)$, given by (5.1), and determine the constants $C_i(\alpha_\nu)$, $i = 1,\dots,4$, in such a way that $B^\pm(\alpha_\nu,D_\omega)\underline{h}(\alpha_\nu,\omega) = \underline{0}$. The condition $B^-(\alpha_\nu,D_\omega)$ implies that

$$C_1(\alpha_\nu) = C_3(\alpha_\nu)(\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1) \quad \text{and} \quad C_2(\alpha_\nu) = C_4(\alpha_\nu)(\frac{-2\mu}{(\lambda+\mu)\alpha_\nu} - 1).$$

This leads to (5.19).

The condition $B^+\underline{h}(\alpha_\nu,\omega) = \underline{0}$ yields $C_3(\alpha_\nu)$ and $C_4(\alpha_\nu)$ as in (5.20) and it holds that

$$B^+(\alpha,D_\omega)\underline{e}_1^0(\alpha,\omega) = (D(\alpha),0)^T = \underline{0} \quad \text{for} \quad \alpha = \alpha_\nu \quad \blacksquare \tag{5.21}$$

We now look for the associate functions.

Lemma 7:

(i) If $m(\alpha_\nu) = 2$, then associate functions exist.

(ii) The equations

$$\alpha_\nu\omega_0\sin\alpha_\nu\omega_0\cos\alpha_\nu\omega_0 = \sin^2\alpha_\nu\omega_0 - \frac{(G-1)^2}{G(G-2)}$$

$$\frac{\sin^2\omega_0}{\omega_0^2} = -\frac{\sin\alpha_\nu\omega_0\cos\alpha_\nu\omega_0}{\alpha_\nu\omega_0}(\frac{G-2}{G}), \quad G = \frac{-(\lambda+\mu)}{\mu}, \tag{5.22}$$

are sufficient and necessary for $m(\alpha_\nu) = 2$.

(iii) Associate functions to the eigenvalues of $\mathfrak{A}(\alpha_\nu, D_\omega)$ are

$$\underline{e}_1^1(\alpha_\nu,\omega) = \frac{d}{d\alpha}\, \underline{e}_1^0(\alpha,\omega)\Big|_{\alpha=\alpha_\nu}, \tag{5.23}$$

where $\underline{e}_1^0(\alpha,\omega)$ is given by (5.19) with α instead of α_ν.

Proof: The assertion (i) follows immediately from Lemma 4. The equations (5.22) are satisfied if and only if $D(\alpha_\nu) = D'(\alpha_\nu) = 0$, where $D(\alpha)$ is given by (5.18). The proof of assertion (iii) is analogous to that of Lemma 5. ■

Remark: The eigenvalues α_ν with $m(\alpha_\nu) = 2$ are again those points in Fig. 5 and 6, where the full curve-pieces of the complex eigenvalues meet the dotted curves of the real eigenvalues.

Summarizing our results we get the following theorem.

Theorem 6: The singular functions (3.10) of the weak solution $\underline{u} \in V$ of the mixed boundary value problem have the following form:

(i) If $\alpha_\nu = \alpha_\nu(G,\omega_0)$ is a simple zero of $D(\alpha)$ (formula (5.18)), then only the singular functions

$$\underline{u}_{0,\nu}^{(1)}(r,\omega) = r^{\alpha_\nu}\underline{e}_1^0(\alpha_\nu,\omega) \tag{5.24}$$

arise, where $\underline{e}_1^0(\alpha_\nu,\omega)$ is given by (5.19).

(ii) If $\alpha_\nu = \alpha_\nu(G,\omega_0)$ is a double zero of $D(\alpha)$, then the singular functions

$$\underline{u}_{0,\nu}^{(1)}(r,\omega) = r^{\alpha_\nu}\underline{e}_1^0(\alpha_\nu,\omega) \quad \text{and}$$

$$\underline{u}_{1,\nu}^{(1)}(r,\omega) = r^{\alpha_\nu}(\underline{e}_1^1(\alpha_\nu,\omega) + (\log r)\underline{e}_1^0(\alpha_\nu,\omega)) \tag{5.25}$$

occur, where $\underline{e}_1^0(\alpha_\nu,\omega)$ is given by (5.19) and $\underline{e}_1^1(\alpha_\nu,\omega)$ by (5.23).

5.3 The Neumann conditions

In this case we consider the boundary conditions (2.14) for $\underline{L}^\pm(\alpha,\omega) = \underline{0}$. If $\alpha \neq 0$, then there are nontrivial solutions $\underline{h}(\alpha,\omega)$ (see (5.1)) under the conditions that

$$D(\alpha) = |a_{ij}(\alpha)| = 32\mu^3\alpha^2(\alpha^2\sin^2\omega_o - \sin^2\alpha\omega_o) = 0. \qquad (5.26)$$

Here there are

$a_{11}(\alpha) = 0,$ $\qquad a_{31}(\alpha) = 2\alpha\mu\sin\alpha\omega_o,$

$a_{12}(\alpha) = 2\alpha,$ $\qquad a_{32}(\alpha) = -2\alpha\mu\cos\alpha\omega_o,$

$a_{13}(\alpha) = 0,$ $\qquad a_{33}(\alpha) = 2\mu\ [\alpha\sin(\alpha-2)\omega_o + \frac{2\mu}{\lambda+\mu}\sin\alpha\omega_o],$

$a_{14}(\alpha) = \alpha A + 2\alpha - 2,$ $\qquad a_{34}(\alpha) = -2\mu\ [\alpha\cos(\alpha-2)\omega_o + \frac{2\mu}{\lambda+\mu}\cos\alpha\omega_o],$

$a_{21}(\alpha) = -2\alpha\mu,$ $\qquad a_{41}(\alpha) = 2\alpha\mu\cos\alpha\omega_o,$

$a_{22}(\alpha) = 0,$ $\qquad a_{42}(\alpha) = 2\alpha\mu\sin\alpha\omega_o,$

$a_{23}(\alpha) = (\lambda+2\mu)(2-\alpha A) - 2\alpha\mu.$

$a_{24}(\alpha) = 0,$

$a_{43}(\alpha) = 2\mu\ [\alpha\cos(\alpha-2)\omega_o + \left(2 + \frac{2\mu}{\lambda+\mu}\right)\cos\alpha\omega_o],$

$a_{44}(\alpha) = 2\mu\ [\alpha\sin(\alpha-2)\omega_o + \left(2 + \frac{2\mu}{\lambda+\mu}\right)\sin\alpha\omega_o],$ and

$A = A(\alpha) = \frac{2(\lambda+3\mu)}{(\lambda+\mu)\alpha}.$

For $\alpha = 0$ the solutions (5.2) yield that the corresponding determinate $D(0) = 0$ for all ω_o. Since $I_\gamma = 2$, we get the linearly independent regular eigenvector functions

$$e_1^o(0,\omega) = (1,0)^T \quad \text{and} \quad \underline{e}_2^o(0,\omega) = (0,1)^T. \qquad (5.27)$$

Fig. 7 shows the distribution of the eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ for $0 \leq \text{Re}\ \alpha \leq 3$. It is taken from the paper [11]. Again the dotted lines indicate the real eigenvalues, the full lines indicate the real parts of the complex eigenvalues.

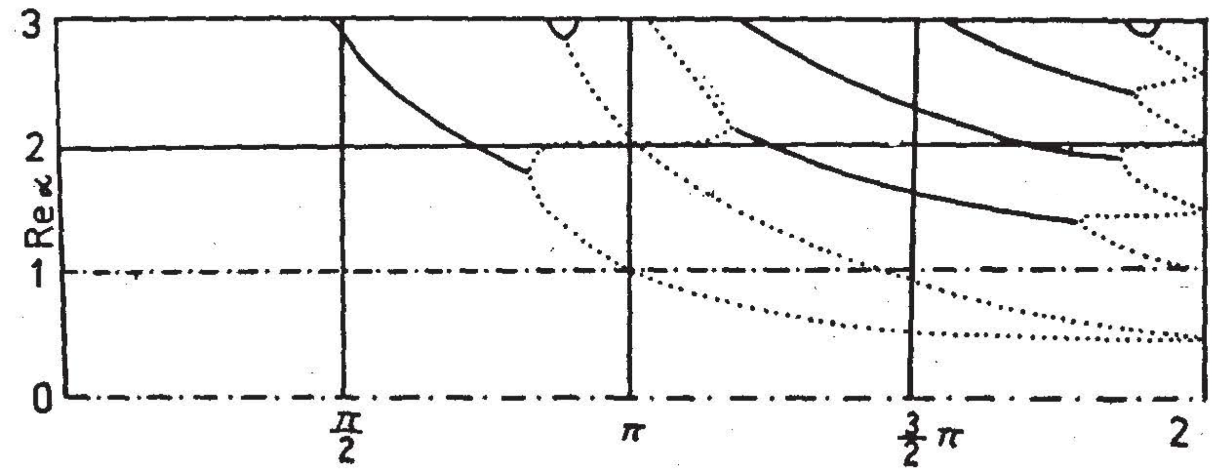

Fig. 7

Lemma 8:

(i) Assume that $\alpha_\nu = \alpha_\nu(\omega_0)$ is a zero of $D(\alpha)$, (5.26), and $\alpha_\nu \neq 0$, $\alpha_\nu \neq 1$, $\omega_0 \neq \pi$, $\omega_0 \neq 2\pi$. Then we have $I_\nu = 1$ and

$$\underline{e}_1^0(\alpha_\nu,\omega) = C_3(\alpha_\nu)\underline{y}_3(\alpha_\nu,\omega) + C_4(\alpha_\nu)\underline{y}_4(\alpha_\nu,\omega) \tag{5.28}$$

is an eigenvector function, where

$$\underline{y}_3(\alpha_\nu,\omega) = \begin{pmatrix} (\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1)\cos\alpha_\nu\omega_0 + \cos(\alpha_\nu-2)\omega_0 \\ (\frac{-2\mu}{(\lambda+\mu)\alpha_\nu} + 1)\sin\alpha_\nu\omega_0 - \sin(\alpha_\nu-2)\omega_0 \end{pmatrix},$$

$$\underline{y}_4(\alpha_\nu,\omega) = \begin{pmatrix} (\frac{-2\mu}{(\lambda+\mu)\alpha_\nu} - 1)\sin\alpha_\nu\omega_0 + \sin(\alpha_\nu-2)\omega_0 \\ (\frac{2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1)\cos\alpha_\nu\omega_0 + \cos(\alpha_\nu-2)\omega_0 \end{pmatrix},$$

and

$$\begin{aligned} C_3(\alpha_\nu) &= \mu[(2-\alpha_\nu)\sin\alpha_\nu\omega_0 + \alpha_\nu\sin(\alpha_\nu-2)\omega_0], \\ C_4(\alpha_\nu) &= \alpha_\nu(\cos\alpha_\nu\omega_0 - \cos(\alpha_\nu-2)\omega_0). \end{aligned} \tag{5.29}$$

(ii) The number $\alpha_\nu = \alpha_\nu(\omega_0) = 1$ is for all $\omega_0 \in (0,2\pi]$ an eigenvalue of $\mathfrak{A}(\alpha,D_\omega)$ and the corresponding eigenvector function is

$$\underline{e}_1^0(1,\omega) = \begin{pmatrix} \sin\omega \\ -\cos\omega \end{pmatrix}. \tag{5.30}$$

(iii) If $\omega_0 = \pi$ or $\omega_0 = 2\pi$, then $\alpha_\nu(\pi) = \nu$, $\alpha_\nu(2\pi) = \nu/2$, $\nu = 1,2,\dots,$ and $I_\nu = 2$. The eigenvector functions are for $\alpha_\nu = 1$ besides (5.30)

$$\underline{e}_2^0(1,\omega) = \underline{y}_3(1,\omega), \tag{5.31}$$

for $\alpha_\nu \neq 1$, $\nu = 1,2,3,\dots,$

$$\underline{e}_1^0(\alpha_\nu,\omega) = \underline{y}_4(\alpha_\nu,\omega),\ \underline{e}_2^0(\alpha_\nu,\omega) = \underline{y}_3(\alpha_\nu,\omega). \tag{5.32}$$

Proof:

(i) We consider $\underline{h}(\alpha_\nu,\omega)$, given by (5.1), and determine the constants $C_i(\alpha_\nu)$, $i = 1,2,\ldots,4$, in such a way that $B^{\pm}(\alpha_\nu, D_\nu)\underline{h} = \underline{O}$. The condition $B^{-}(\alpha_\nu, D_\nu)$ implies that $C_1(\alpha_\nu) = C_3(\alpha_\nu)[\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} -1]$ and $C_2(\alpha_\nu) = C_4(\alpha_\nu)[\frac{-2\mu}{(\lambda+\mu)\alpha_\nu} - 1]$. Thus we get formula (5.28). The condition $B^{+}(\alpha_\nu, D_\omega)\underline{h}(\alpha_\nu,\omega) = \underline{O}$ yields the equations (5.29) and it holds that

$$B^{+}(\alpha,D_\omega)\underline{e}_1^{o}(\alpha,\omega) = (D(\alpha),0)^T = \underline{O} \quad \text{for} \quad \alpha = \alpha_\nu. \tag{5.33}$$

(ii) Analogously to part (i) of the proof we get (5.28) with the unknown coefficients $C_3 = C_3(1)$ and $C_4 = C_4(1)$. The boundary condition $B^{+}(1,\omega)\underline{e}_1^{o}(1,\omega) = \underline{O}$ yields

$$\begin{pmatrix} -4\sin\omega_o & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \underline{O}. \tag{5.34}$$

Choosing $C_3 = 0$ and $C_4 = -\frac{\lambda+\mu}{2(\lambda+2\mu)}$ we get (5.30).

(iii) For $\omega_o = \pi$ or $\omega_o = 2\pi$ the matrix in (5.34) is the O-matrix. Therefore we can take $C_3 = 1$, $C_4 = 0$ or $C_3 = 0$, $C_4 = 1$ in (5:28). ■

Lemma 9:

(i) If $\alpha_\nu \neq 1$, $I_\nu = 1$, and $m(\alpha_\nu) = 2$, then associate functions exist.

(ii) The equations $\alpha_\nu\omega_o = \tan\alpha_\nu\omega_o$ and $\cos^2\alpha_\nu\omega_o = \frac{\sin^2\omega_o}{\omega_o^2}$, $\alpha_\nu \neq 1$, are sufficient and necessary for $m(\alpha_\nu) = 2$.

(iii) Associate functions to these eigenvalues of $\mathfrak{A}(\alpha, D_\omega)$ are

$$\underline{e}_1^{1}(\alpha_\nu,\omega) = \frac{d}{d\alpha}\, \underline{e}_1^{o}(\alpha,\omega)\Big|_{\alpha=\alpha_\nu}, \tag{5.35}$$

where $\underline{e}_1^{o}(\alpha,\omega)$ is given by (5.28) with α instead α_ν.

For the proof compare Lemma 5 and Lemma 7.

Remark: The eigenvalues $\alpha_\nu \neq 1$ with $m(\alpha_\nu) = 2$ and $I_\nu = 1$ are those points in Fig. 7, where the full curve-pieces of the real parts of the complex eigenvalues meet the dotted lines of the real eigenvalues.

We formulate our results.

Theorem 7: The singular functions of the weak solutions $\underline{u} \in V/N$ of the Neumann problem have the following form:

(i) If $\omega_0 \neq \pi$, $\omega_0 \neq 2\pi$, and $\alpha_\nu = \alpha_\nu(\omega_0)$ is a simple zero of $D(\alpha)$ (formula (5.26)), then only the singular functions

$$\underline{u}^{(1)}_{0,\nu}(r,\omega) = r^{\alpha_\nu} \underline{e}^0_1(\alpha_\nu,\omega)$$

arise, where $\underline{e}^0_1(\alpha_\nu,\omega)$ is given by (5.28).

(ii) If $\omega_0 \neq \pi$, $\omega_0 \neq 2\pi$, and α_ν is a double zero of $D(\alpha)$, then the singular functions

$$\underline{u}^{(1)}_{0,\nu}(r,\omega) = r^{\alpha_\nu} \underline{e}^0_1(\alpha_\nu,\omega) \quad \text{and}$$

$$\underline{u}^{(1)}_{1,\nu}(r,\omega) = r^{\alpha_\nu}(\underline{e}^1_1(\alpha_\nu,\omega) + (\log r)\underline{e}^0_1(\alpha_\nu,\omega))$$

occur, where $\underline{e}^0_1(\alpha_\nu,\omega)$ is given by (5.28) and $\underline{e}^1_1(\alpha_\nu,\omega)$ by (5.35).

(iii) If $\omega_0 = \pi$, then no "proper" singular functions exist.

(iv) If $\omega_0 = 2\pi$, then it follows from (5.32) that

$$\underline{u}^{(1)}_{0,\nu}(r,\omega) = r^{\nu/2} \underline{e}^0_1(\alpha_\nu,\omega),$$

$$\underline{u}^{(2)}_{0,\nu}(r,\omega) = r^{\nu/2} \underline{e}^0_2(\alpha_\nu,\omega).$$

Remark: The singular functions $\underline{u}^{(i)}_{0,\nu}(r,\omega) = \underline{e}^0_i(0,\omega)$, $i = 1,2$, for $\alpha_\nu = 0$ (see (5.27)) and $\underline{u}^{(1)}_{0,\nu}(r,\omega) = r(\sin\omega, -\cos\omega)^T = (x_2,-x_1)^T$ for $\alpha_\nu = 1$ (see (5.30)) are a basis of the space N (see (1.7))

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