Failure of Amplitude Equations

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Erklärung

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Failure of Amplitude Equations

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Contents

List of Spaces, Operators and Symbols

Spaces

H^s	Sobolev space of order s
H^s_{per}	periodic Sobolev space of order s
$L^p(s)$	weighted L^p-space
$\mathbb{R}$	Real numbers
$\mathbb{R}^2$	Real valued vector space in 2-D
$\mathbb{R}^+$	positive real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{C}^2$	Complex valued vector space in 2-D

Operators

$\widehat{(\cdot)}$	Fourier transform
∇	Nabla-Operator
$\\|u\\|_{H^s}$	$:=\sum_{j=0}^{s}(\int_\Omega \|\partial_x^j u(x)\|^2 dx)^{1/2}$
$\\|u\\|_{C_b^n}$	$:=\sum_{j=0}^{n} \\|\partial_x^j u\\|_{C_b^0}$
$\\|u\\|_{C_b^0}$	$:=\sup_{x\in\mathbb{R}} \|u(x)\|$

Symbols

Ω	fixed domain
$\Omega(t)$	time dependent domain
$\Gamma(t)$	time dependent surface
ϕ	potential
ρ	weight function
σ	surface tension parameter
τ	scaled time variable
ξ	scaled space variable
ω	temporal eigen value curve
η	surface elevation
φ	eigenvector in C^2
φ^*	adjoint eigenvector to φ
γ	TWI coefficients
Ω_j	$:= \|\omega_j\|$

Zusammenfassung

Die nichtlineare Schrödinger-Gleichung (NLS-Gleichung) ist ein Beispiel für ein universelles nichtlineares Modell, das viele nichtlineare physikalische Systeme beschreibt. Die Gleichung findet Anwendung in der Hydrodynamic, der nichtlinearen Optik, der nichtlinearen Akustik, Quantenkondensaten, Wärmepulsen in Feststoffen und bei zahlreichen weiteren nichtlinearen Instabilitätsphänomenen. Sie beschreibt kleine zeitliche und örtliche Modulationen eines sowohl bezüglich der Zeit als auch des Ortes oszillierenden Wellenpakets. Die NLS-Gleichung wurde erstmals im Jahre 1968 formal für das sogenannte Wasserwellenproblem hergeleitet. Der Nachweis, dass die NLS-Gleichung korrekte Vorhersagen für dieses System macht, ist gegenwärtig Gegenstand intensiver Forschungen.

In der vorliegenden Arbeit zeigen wir, dass diese Approximation im Allgemeinen ungültig ist, wenn nicht zusätzliche Annahmen gefordert werden. Bereits erbracht wurde der Nachweis, dass eine Approximationseigenschaft (APP) für die NLS-Gleichung gilt, wenn die Approximation im mit den Resonanzen assoziierten Dreiwellensystem (TWI) stabil ist. Im instabilen Fall ist die Situation nicht so klar. Wir zeigen, dass die NLS-Approximation auf einer Zeitskala von $\mathcal{O}(\varepsilon^{-1}|\ln(\varepsilon)|)$ versagt, das heißt auf einer Zeitskala, die kleiner ist als die natürliche Zeitskala der NLS-Approximation, $\mathcal{O}(\varepsilon^{-2})$. Wir konstruieren ein Gegenbeispiel, das zeigt, dass die NLS-Approximation falsche Vorhersagen für das Wasserwellenproblem mit kleiner Oberflächenspannung und periodischen Randbedingungen macht.

Das resonante Vierwellensystem (FWI) ergibt sich als Amplitudengleichung für die Beschreibung von Schwerewellen in Wasser und anderen dispersiven Wellensystemen. Im Falle von nichtresonanten quadratischen Termen im ursprünglichen System ist es möglich, mit Hilfe von Normalformtransformationen und der Grönwallschen Ungleichung Fehlerabschätzungen zu beweisen. In dieser Arbeit zeigen wir, dass das Vierwellensystem falsche Vorhersagen trifft, falls zusätzliche instabile quadratische Resonanzen auftreten. Dies ist zum Beispiel für Kapillar-Schwerewell-en der Fall.

Das Fermi-Pasta-Ulam System (FPU) ist ein Beispiel für makroskopische Wel-le-npakete in unendlich langen Ketten miteinander gekoppelter Oszillatoren. Es kann beschreiben werden durch die bekannten Modulationsgleichungen wie die Korteweg-de-Vries-Gleichung (KdV) oder die NLS-Gleichung. In der Literatur existiert bereits ein Beweis dafür, dass die NLS-Gleichung die Evolution der langsam variierenden Einhüllenden eines zugrundeliegenden Wellenpakets mit kleiner Amplitude approximiert. Wir betrachten ein diatomisches FPU-System und konstruieren ein Gegenbeispiel, das zeigt, dass die NLS-Approximation falsche Vorhersagen für dieses FPU-System macht. Die Grundlage für dieses Gegenbeispiel ist wieder die Existenz eines instabilen Unterraumes.

Abstract

The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena. It describes small modulations in time and space of a spatially and temporally oscillating wave packet advancing in a laboratory frame. It has first been derived for the so called water wave problem in 1968 and the proof that it makes correct predictions has been recently the subject of intensive research.

The present work shows that in general this approximation is not valid without further assumptions. The proof that an approximation property (APP) for the NLS equation holds if the approximation is stable in the system for the three wave interaction (TWI) associated with the resonances already exists. However the situation is less clear when an instability arises. We prove that the NLS approximation breaks down after a time scale $O(\varepsilon^{-1}|\ln\varepsilon|)$ which is much smaller then the natural time scale $O(\varepsilon^{-2})$ of the NLS approximation. We construct a counter example showing that the NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions.

The resonant four wave interaction (FWI) system appears as an amplitude equation in the description of gravity driven surface water waves and other dispersive wave systems. In case of non-resonant quadratic terms in the original system error estimates justifying this approximation can be established with the help of normal form transformations and Gronwall's inequality. In this thesis we explain that the four wave interaction system makes wrong predictions in case of additional unstable quadratic resonances as they appear for instance for capillary-gravity surface water waves.

The Fermi-Pasta-Ulam (FPU) system is an example of macroscopic wave packets in infinite chains of coupled oscillators and it can be described by well known modulation equations like the Korteweg-de Vries (KdV) or the NLS equation. The proof that the NLS equation approximates the evolution of a slowly varying envelope of small amplitude of an underlying oscillating wave packet in the FPU system exists in the literature. We consider a diatomic FPU system and construct a counter example to show how the NLS equation fails for the FPU system. The counter example is again based on the existence of an unstable subspace.

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Chapter 1

Introduction

Many complications in ordinary or partial differential equations arise with the inclusion of nonlinearities and in many cases this is unavoidable. For many years, linear models provided a good description of many physical phenomenas but at the same time these models also failed to provide completely accurate description. The three very famous and basic linear partial differential equations include the Laplace equation, the heat equation and the linear wave equation. These models have been studied extensively and they behave nicely. On the other hand, we have nonlinear models, e.g. the Navier-Stokes equations, the Maxwell's equations or the Einstein's equations. These equations describe the respective physical situations in a much better way but at the same time are much more difficult to study. For example, the Navier-Stokes equations describe the motion of fluids but to date, no one has been able to prove the global existence and uniqueness of smooth solutions.

Computer simulations have been helpful in solving many complicated PDEs but at much higher computational costs because of the number of the variables involved. As a result, approximate models are developed which are much simpler and computationally less expensive. These models, which are called amplitude, envelope, or modulation equations, are among the most well studied nonlinear PDEs.

For example, by perturbation analysis the so-called nonlinear Schrödinger (NLS) equation

$$\partial_\tau A = i\nu_1 \partial_\xi^2 A + i\nu_2 A|A|^2,$$

with $\tau, \xi, \nu_1, \nu_2 \in \mathbb{R}$ and $A(\xi, \tau) \in \mathbb{C}$, has been derived as an amplitude equation for the description of temporally and spatially oscillating wave packets. It describes small modulations in time and space of a spatially and temporarily oscillating wave packet advancing in a laboratory frame. It has first been derived for the so called water wave problem in 1968 and the proof that it makes correct predictions has been recently the subject of intensive research. However the formal derivation does not guarantee that the solutions of the water wave problem actually behave in the same way as the NLS equation predicts. This is evident from the fact that in certain cases original systems tend to behave differently as opposed to predicted by formally derived approximate equations. The motivation comes from [Schn05], where it has been pointed out that with periodic boundary conditions and a carefully chosen unstable quadratic resonance, a counter example can be constructed showing that NLS equation fails to approximate the original system correctly.

In Chapter 2, we show that in general this approximation is not valid without further assumptions. We construct a counter example showing that the NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions.

The main idea is to use the three wave interaction (TWI) system in such a way that the predictions made by the NLS approximation differ from those of the TWI system for the water wave problem. We prove an approximation theorem between the solutions of the water wave problem and an approximation obtained via a TWI system associated to the resonances. We estimate solutions of $\mathcal{O}(\varepsilon)$ on an $\mathcal{O}(1/\varepsilon)$ time scale by applying Gronwall's inequality. We achieve the extension in the error estimates from $t \in [0, T_0/\varepsilon]$ to $t \in [0, |\ln(\varepsilon)|/\varepsilon]$ by considering an extended TWI system and an extended TWI approximation $\varepsilon\psi$. The use of the extended TWI approximation is necessary to make the residual of order $\mathcal{O}(\varepsilon^{\beta+\delta})$ for any given $\beta + \delta \geq 0$. As a result, we prove that the NLS approximation breaks down on a very small time scale $O(\varepsilon^{-1}|\ln\varepsilon|)$ as compared to the the natural time scale $O(\varepsilon^{-2})$ of the NLS approximation.

Another important class of amplitude equations arises when several dominant wave modes are present and they interact significantly. The simplest and most important one is three wave interaction (TWI) system. But not all systems exhibit three wave resonance and the resonant quartets of waves are responsible for the most significant interactions, called as the four wave interaction (FWI) system. In wave amplitude, the nonlinearities which are then of cubic order are much weaker than the quadratic nonlinearities of three wave resonance. The FWI system describes a number of physical phenomenas. Examples include multi wave nonlinear couplings in elastic structures, nonlinear optical waves, pattern formation in vertically oscillated convection and others.

The (resonant) FWI system is given by

$$\partial_\tau A_j = c_j \partial_X A_j + \sum_{l \in I_N} d_{jl} |A_l|^2 A_j + \sum_{(j_1, j_2, j_3) \in R_j} d^j_{j_1 j_2 j_3} \overline{A_{j_1} A_{j_2} A_{j_3}},$$

where $\tau, \xi, c_j, d_{jl}, d^j_{j_1 j_2 j_3} \in \mathbb{R}$, $A(X, \tau) \in \mathbb{C}$ and R_j is called the set of resonances.

The FWI system appears as an amplitude equation in the description of gravity driven surface water waves and other dispersive wave systems. In case of non resonant quadratic terms in the original system error estimates justifying this approximation can be established with the help of normal form transformations and Gronwall's inequality. The question however, if such models, really describe the evolution of dispersive wave systems in case of resonant quadratic terms is a difficult task. In Chapter 3, we extend the idea and methodology from Chapter 2 to prove that the FWI system also makes wrong predictions for the water wave problem in case of spatially periodic boundary conditions and small surface tension.

The next model is concerned with excitations in lattices. These lattices can be viewed as strings of mutually interacting oscillators. The class of Fermi-Pasta-Ulam lattices can be considered as an example of such a string. The following second order differential equation

$$\partial_t^2 q_n(t) = W'(q_{n+1}(t) - q_n(t)) - W'(q_n(t) - q_{n-1}(t)),$$

with $n \in \mathbb{Z}$, $q_n(t) \in \mathbb{R}$ and W is the interaction potential, describes the mutual interaction of each oscillator with the neighboring oscillators. The above differential equation can be described by simple and well known partial differential equations, like the Korteweg-de Vries (KdV) or the NLS equation.

In the real world, we can think of this phenomena as system of masses connected with springs leading to nonlinear interactions. The famous FPU experiment is the foundation for this model and is very well studied for its connections to the soliton theory and interesting dynamics.

These interactions which are nonlinear in nature lead to complex behavior, including periodic oscillations that are localized in space and solitary waves.

In Chapter 4 we consider a special case where we have an infinite chain of two alternating masses connected with a spring so-called as diatomic FPU system. We prove that the NLS system makes wrong predictions for the FPU system. To do so, we develop a counter example which is based on the existence of an unstable subspace corresponding to a chosen wavenumber.

Chapter 2

NLS fails for the Water Wave Problem

The water wave problem as considered in this chapter[1]consists in finding the irrotational flow of an inviscid, incompressible fluid in an infinitely long canal of constant finite or infinite depth under the influence of gravity and surface tension. We denote with $x \in \mathbb{R}$ the coordinate in horizontal direction and with $y \in \mathbb{R}$ the coordinate in vertical direction. The flat surface at rest is given by $\{y = 0\}$. The velocity field of the fluid satisfies Euler's equations and the fluid fills the unknown time-dependent domain $\Omega(t) \subset \mathbb{R}^2$ below the free unknown time-dependent top surface $\Gamma(t) = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$ with $\eta : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$.

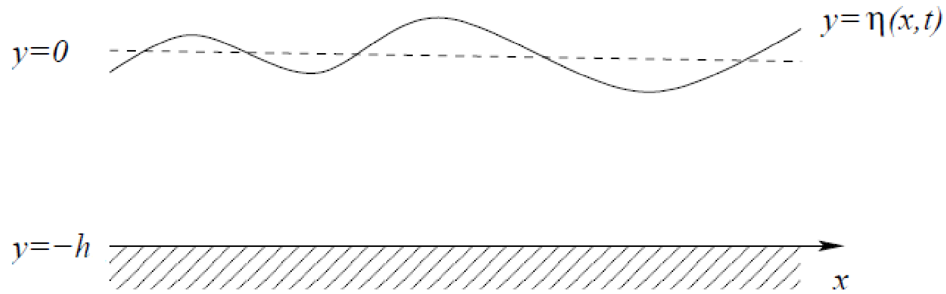

Figure 2.1: The water wave problem.

Under these assumptions there exists a potential $\phi : \Omega(t) \times \mathbb{R}^+ \to \mathbb{R}$ such that the velocity field $(u, v) = (u, v)(x, y, t)$ satisfies $u = \partial_x \phi$ and $v = \partial_y \phi$. In case of a

[1]The contents of this chapter are published as: G. Schneider, D.A. Sunny, D. Zimmermann. The NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions, *Journal of Dynamics and Differential Equations*, 27(3), 1077-1099, (2014).

canal of depth h the potential ϕ and the elevation η of the top surface satisfy

$$\partial_x^2\phi + \partial_y^2\phi = 0, \quad \text{in} \quad \Omega(t), \tag{2.1}$$
$$\partial_y\phi = 0, \quad \text{for} \quad y = -h, \tag{2.2}$$
$$\partial_t\eta = \partial_y\phi - (\partial_x\eta)\partial_x\phi, \quad \text{on} \quad \Gamma(t), \tag{2.3}$$
$$\partial_t\phi = -\frac{1}{2}((\partial_x\phi)^2 + (\partial_y\phi)^2) + \sigma\partial_x\left[\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}\right] - g\eta, \quad \text{on} \quad \Gamma(t), \tag{2.4}$$

where σ is a parameter proportional to surface tension. This formulation is called the Eulerian formulation of the water wave problem. In case of finite depth w.l.o.g. we set the gravitational constant g and the depth h of the canal to one in the following. In case of infinite depth equation (2.2) has to be replaced by

$$\nabla\phi \to 0, \quad \text{for} \quad y \to -\infty. \tag{2.5}$$

We will come back to this case in Section A.2.1. It is well known that the water wave problem is completely described by the evolution of the elevation $\eta = \eta(x,t)$ of the top surface and the horizontal velocity component $w = w(x,t) = u(x,\eta(x,t),t)$ at the top surface. Equation (2.1) can be solved uniquely up to a constant under the boundary conditions (2.2) and $\partial_x\phi|_{\Gamma(t)} = w$. Thus, $\partial_y\phi$ and so all terms on the r.h.s. of (2.3) and (2.4) are known with the knowledge η and w.

The nonlinear Schrödinger (NLS) equation

$$\partial_\tau A = i\nu_1\partial_\xi^2 A + i\nu_2 A|A|^2,$$

with $\tau, \xi, \nu_1, \nu_2 \in \mathbb{R}$ and $A(\xi,\tau) \in \mathbb{C}$, has first been derived via formal perturbation analysis as an amplitude equation for the water wave problem for the description of temporarely and spatially oscillating wave packets in 1968 by Zakharov [Za68] (in case $\sigma = 0$ and infinite depth). See Section 2.1 for details. However, the formal derivation of the NLS equation does not imply that solutions of the water wave problem really behave as predicted by the NLS equation for small values of the perturbation parameter which will be denoted by $0 < \varepsilon \ll 1$ in the following.

In fact there are counter examples where an original system behaves differently than predicted by the formally derived amplitude equation, cf. [Schn95]. Recently, there has been a lot of progress in showing that the water wave problem really behaves as predicted by the NLS equation. Approximation theorems have been shown in [TW12] for the water wave problem in case of infinite depth and no surface tension, in [SW11] for a quasilinear toy problem, and in [DSW16] for the water wave problem in case of finite depth and no surface tension.

In finite depth in case of small surface tension and in infinite depth for arbitrary strictly positive surface tension additional resonances are present in the problem, i.e., there are spatial and temporal wave numbers $k = k_j$ and $\omega = \omega_j$ which satisfy the resonance condition

$$k_1 + k_2 + k_3 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0, \tag{2.6}$$

and which are related by the linear dispersion relation, cf. (2.9), of the water wave problem. In [DS06] a method has been developed how to handle this situation in case of stable quadratic resonances. In [Schn05] it has been pointed out that for general dispersive wave systems with a suitable chosen unstable quadratic resonance and periodic boundary conditions a counter example can be cooked up showing that the NLS approximation fails to make correct predictions. See Section 2.4 for an explanation what is meant by a stable and an unstable resonance. Our goal is to construct such a counter example for the water wave problem in case of finite depth and small surface tension rigorously. Following [Schn05] the idea is to prove an approximation theorem between the solutions of the water wave problem and an approximation obtained via a three wave interaction (TWI) system associated to the resonances, cf. [AS81, Crk85]. Since the NLS equation makes different predictions than the TWI approximation the water wave problem cannot be described correctly by the NLS approximation.

In case of unstable resonances it is not clear if the approximation property holds or not if the spatial periodicity condition is given up. In this case the different group velocities of the resonant wave packets have to be taken into account, cf. Equation (2.12). See [Schn05, Section 4] for a more detailed discussion of this situation. See [MN13] for a promising approach to get rid of this question.

The present work is not self-contained in the sense that we left out big parts of the proof of the approximation result of the TWI approximation theorem 2.5.2 since it follows almost until the end of the proof line for line the proof given in [SW03]. We mainly explain how the proof has to be improved for our purposes, i.e., how to extend the approximation time of the TWI approximation from $\mathcal{O}(1/\varepsilon)$ to $\mathcal{O}(|\ln(\varepsilon)|/\varepsilon)$.

Notation. The Sobolev space H^s over a domain Ω is equipped with the norm $\|u\|_{H^s} = \sum_{j=0}^{s}(\int_\Omega |\partial_x^j u(x)|^2 dx)^{1/2}$. In case of $\Omega = \mathbb{R}$ with periodic boundary conditions we write H^s_{per}. We do not distinguish in our notation between scalar- and vector-valued Sobolev spaces. Moreover, let $\|u\|_{C_b^n} = \sum_{j=0}^{n} \|\partial_x^j u\|_{C_b^0}$, where $\|u\|_{C_b^0} = \sup_{x\in\mathbb{R}} |u(x)|$.

2.1 The formal NLS approximation

The NLS equation describes small modulations in time and space of a spatially and temporally oscillating wave packet advancing in a laboratory frame. By making the ansatz

$$\begin{pmatrix} \eta \\ w \end{pmatrix} = \varepsilon A(\varepsilon(x-c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} \varphi + c.c., \tag{2.7}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter, we obtain that in lowest order the amplitude function A has to satisfy a NLS equation

$$\partial_\tau A = i\nu_1 \partial_\xi^2 A + i\nu_2 A|A|^2, \tag{2.8}$$

with $\tau, \xi, \nu_1, \nu_2 \in \mathbb{R}$ and $A(\xi, \tau) \in \mathbb{C}$. The basic spatial wave number $k = k_0 > 0$ and the basic temporal wave number $\omega = \omega_0$ of the underlying wave packet $e^{i(k_0 x - \omega_0 t)}\varphi$ satisfy the linear dispersion relation

$$\omega^2 = (k + \sigma k^3)\tanh(k) \tag{2.9}$$

of the water wave problems in case of finite depth $h = 1$. Moreover, c_g is the group velocity of the wave packet and given by $d\omega/dk|_{k=k_0,\omega=\omega_0}$. The coefficients ν_1 and ν_2 and the unit vector $\varphi \in \mathbb{C}^2$ depend on the wave numbers k_0 and ω_0.

As explained in the introduction the answer to the question whether an **approximation property** holds or not is an important topic. A typical formulation is as follows.

(APP) *Let $A \in C([0, \tau_0], H^s)$ be a solution of the NLS equation* (2.8) *with $s \geq 0$ sufficiently large. Then for all $C_1 > 0$ there exist $\varepsilon_0 > 0$ and $C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. If the initial conditions of the water wave problem* (2.1)-(2.4) *satisfy*

$$\left\| \begin{pmatrix} \eta \\ w \end{pmatrix} (\cdot_x, 0) - (\varepsilon A(\varepsilon \cdot_x, 0) e^{ik_0 \cdot x}\varphi + c.c.) \right\|_{H^s} \leq C_1 \varepsilon^{3/2},$$

then the associated solutions of the water wave problem (2.1)-(2.4) *satisfy*

$$\sup_{t\in[0,\tau_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} \left| \begin{pmatrix} \eta \\ w \end{pmatrix} (x,t) - (\varepsilon A(\varepsilon(x-c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)}\varphi + c.c.) \right| \leq C_2 \varepsilon^{3/2}.$$

The choice of the exponent 3/2 follows existing approximation results which are presented now. For very general quasilinear hyperbolic systems such an approximation theorem has been established some time ago in [Kal88]. For the water wave problem which is not covered by the assumptions of [Kal88] without surface tension such an approximation theorem has been established very recently in [TW12, DSW16] in case of infinite, respectively finite depth. Estimates for the residual terms, the terms which do not cancel after inserting the NLS approximation into the water wave problem, have already been shown [CSS92]. In [IK90] time-harmonic solutions of the NLS equation have shown to persist in the water wave problem as waves of permanent form in case of finite depth and strictly positive surface tension using spatial dynamics and center manifold theory.

As already said in case of small surface tension additional resonances are present in the problem. In [Schn05] it has been pointed out that for general dispersive wave systems with suitably chosen unstable quadratic resonances and periodic boundary conditions a counter example can be cooked up showing that the NLS approximation fails to make correct predictions.

Spatially periodic boundary conditions with period $2\pi m/k_0$, ($m \in \mathbb{N}$), for the water wave problem are satisfied by the NLS approximation (2.7) if X-independent solutions of the NLS equation are considered. In this case the NLS equation degenerates into the ODE

$$\partial_\tau A = i\nu_2 A|A|^2 \tag{2.10}$$

with $A = A(\tau) \in \mathbb{C}$. Hence it is the purpose to show that the following approximation property is not true in general.

(APP') *Let $A \in C([0, \tau_0], \mathbb{C})$ be a solution of the (NLS) ODE* (2.10). *Then for all $C_1 > 0$ there exist $\varepsilon_0 > 0$ and $C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. If the $2\pi m/k_0$-spatially periodic initial conditions with $m \in \mathbb{N}$ of the water wave problem* (2.1)-(2.4) *satisfy*

$$\left\| \begin{pmatrix} \eta \\ w \end{pmatrix} (\cdot_x, 0) - (\varepsilon A(0) e^{ik_0 \cdot x} \varphi + c.c.) \right\|_{H^s_{per}} \leq C_1 \varepsilon^{3/2} \tag{2.11}$$

for a $s \geq 0$ sufficiently big, then the associated solutions of the water wave problem (2.1)-(2.4) *satisfy*

$$\sup_{t \in [0, \tau_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| \begin{pmatrix} \eta \\ w \end{pmatrix} (x, t) - (\varepsilon A(\varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} \varphi + c.c.) \right| \leq C_2 \varepsilon^{3/2}.$$

2.2 The strategy

In order to prove that **(APP')** is not true in general we prove an approximation theorem which shows that solutions of the water wave problem are correctly described by a formal approximation obtained via an extended TWI system which is associated to the resonances. Since the NLS equation makes different predictions than this extended TWI approximation it then cannot describe the water wave problem correctly. In lowest order the extended TWI system is given by the classical TWI system

$$\begin{aligned} \partial_T A_1 = & \quad c_g(k_1)\partial_X A_1 + i\gamma_1 \overline{A_2 A_3}, \\ \partial_T A_2 = & \quad c_g(k_2)\partial_X A_2 + i\gamma_2 \overline{A_1 A_3}, \\ \partial_T A_3 = & \quad c_g(k_3)\partial_X A_3 + i\gamma_3 \overline{A_1 A_2}, \end{aligned} \tag{2.12}$$

with $T, X, c_g(k_j), \gamma_j \in \mathbb{R}$, and $A_j(X,T) \in \mathbb{C}$. It can be derived by the multiple scaling ansatz

$$\begin{aligned} \begin{pmatrix} \eta \\ w \end{pmatrix} \approx \varepsilon\psi_{twi}(x,t) = & \varepsilon A_1(\varepsilon x, \varepsilon t) e^{i(k_1 x - \omega_1 t)} \varphi_1 \\ & + \varepsilon A_2(\varepsilon x, \varepsilon t) e^{i(k_2 x - \omega_2 t)} \varphi_2 \\ & + \varepsilon A_3(\varepsilon x, \varepsilon t) e^{i(k_3 x - \omega_3 t)} \varphi_3 + c.c., \end{aligned} \tag{2.13}$$

with unit vectors $\varphi_j \in \mathbb{C}^2$ (which depend only on k_j, ω_j and can be computed explicitly), with $0 < \varepsilon \ll 1$ a small perturbation parameter, and the spatial and temporal wave numbers $k = k_j$ and $\omega = \omega_j$ which have to satisfy the resonance condition

$$k_1 + k_2 + k_3 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0, \tag{2.14}$$

and are related by the linear dispersion relation (2.9) of the water wave problem.

In [SW03, Theorem 1.1] the following approximation result has been shown for the water wave problem in case of finite depth and $\sigma > 0$.

Theorem 2.2.1
Let $s \geq 6$ and choose $k = k_j$ and $\omega = \omega_j$ to satisfy (2.9) *and* (2.14)*. Then for all $C_1, T_0 > 0$ there exist $C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following is true. Let $A_1, A_2, A_3 \in C([0,T_0], (H^{s+2}(\mathbb{R}, \mathbb{C}))^3)$ be solutions of* (2.12) *with*

$$\sup_{T \in [0,T_0]} \|A_j(\cdot_X, T)\|_{H^{s+2}} \leq C_1$$

for $j = 1, 2, 3$. Then there are solutions of the water wave problem (2.1)-(2.4) *satisfying*

$$\sup_{t \in [0,T_0/\varepsilon]} \sup_{x \in \mathbb{R}} \left| \begin{pmatrix} \eta \\ w \end{pmatrix}(x,t) - \varepsilon\psi_{twi}(x,t) \right| \leq C_2 \varepsilon^{3/2}.$$

For our purposes in principle we have to prove the same theorem, but with periodic boundary conditions which is rather straightforward. In order to have the TWI approximation, resp. extended TWI approximation, to be an approximation of the water wave problem with periodic boundary conditions we first have to find integers n_1, n_2, $n_3 \in \mathbb{Z}$, and a wave number $k_* > 0$, such that $k = k_j = n_j k_*$ and $\omega = \omega_j$ satisfy

$$k_1 + k_2 + k_3 = k_*(n_1 + n_2 + n_3) = 0, \qquad \text{and} \qquad \omega_1 + \omega_2 + \omega_3 = 0.$$

Moreover, they are related by the linear dispersion relation (2.9) of the water wave problem. We will do so in Section 2.3. The $2\pi m/k_0$-spatially periodic boundary conditions of the water wave problem can then be satisfied by the TWI ansatz (2.13), resp. by the subsequent extended TWI ansatz (2.18), for X-independent solutions. In this case the TWI system degenerates into a system of ODEs, namely

$$\begin{aligned} \partial_T A_1 &= i\gamma_1 \overline{A_2 A_3}, \\ \partial_T A_2 &= i\gamma_2 \overline{A_1 A_3}, \\ \partial_T A_3 &= i\gamma_3 \overline{A_1 A_2}, \end{aligned} \tag{2.15}$$

with $A_j = A_j(T) \in \mathbb{C}$. In order to keep the notation in the following on a reasonable level we assume for the rest of this paper the special situation for which we will construct the counter example. We choose $n_1 = 1$, $n_2 = 2$, $n_3 = -3$, and $k_0 = -3k_*$. An initial condition

$$(\eta, w)|_{t=0} = \sum_{n \in \mathbb{Z}} (\eta, w)_n|_{t=0} e^{ink_* x}$$

of the water wave problem determines the initial conditions of the NLS equation (2.10), namely

$$A|_{\tau=0} \quad = \quad \varepsilon^{-1} < \varphi^*, (\eta, w)_3|_{t=0} >$$

and the initial conditions of the TWI system (2.15), respectively of the subsequent extended TWI system (2.19), namely

$$\begin{aligned} A_1|_{T=0} &= \varepsilon^{-1} < \varphi_1^*, (\eta, w)_1|_{t=0} >, \\ A_2|_{T=0} &= \varepsilon^{-1} < \varphi_2^*, (\eta, w)_2|_{t=0} >, \\ A_3|_{T=0} &= \varepsilon^{-1} < \varphi_3^*, (\eta, w)_{-3}|_{t=0} >, \end{aligned}$$

where the φ_j^* are the adjoint eigenvectors to the φ_j. By the derivation of the NLS equation (2.10) and of the associated TWI system (2.15) in Fourier space it is obvious, cf. [Schn05], that $\varphi = \varphi_3$ such that $A|_{\tau=0} = A_3|_{T=0}$.

In order to have the estimate in **(APP')** to be correct at $t = 0$ we need $(\eta, w)_{\pm 3}|_{t=0} = \mathcal{O}(\varepsilon)$ and for all other $n \in \mathbb{Z} \setminus \{-3, 3\}$ that $|(\eta, w)_n|_{t=0}| \leq \mathcal{O}(\varepsilon^{3/2})$. Hence the order of $A_1|_{T=0}$ and $A_2|_{T=0}$ lies in between 0 and $\mathcal{O}(\varepsilon^{1/2})$. We are done if we show that at least for one $n \in \mathbb{Z} \setminus \{-3, 3\}$ we have $|(\eta, w)_n(t)| \geq \mathcal{O}(\varepsilon)$ for a $t \leq \tau_0/\varepsilon^2$.

Our strategy to show such a result is to use the subsequent extended TWI system (2.19) whose solutions will be denoted with $\mathcal{A}_j = \mathcal{A}_j(T)$. We first prove that the extended TWI-system (2.19) makes correct predictions for the water wave problem and secondly that $|\mathcal{A}_1(\widehat{T})| + |\mathcal{A}_2(\widehat{T})| \geq \mathcal{O}(1)$ for a $\widehat{T} < \tau_0/\varepsilon$ such that as a consequence $|(\eta, w)_{1,2}(\widehat{T}/\varepsilon)| \geq \mathcal{O}(\varepsilon)$. Since $\widehat{T} \gg \mathcal{O}(1)$ an extended TWI system has to be used instead of the classical TWI system for the approximation.

Hence, four things remain to be done in the next sections.

- To find in Section 2.3 integers n_1, n_2, $n_3 \in \mathbb{Z}$, and a wave number $k_* > 0$ such that $k = k_j = n_j k_*$ and $\omega = \omega_j$ satisfy

$$k_1 + k_2 + k_3 = k_*(n_1 + n_2 + n_3) = 0, \qquad \text{and} \qquad \omega_1 + \omega_2 + \omega_3 = 0,$$

  and are related by the linear dispersion relation (2.9) of the water wave problem.

- To analyse in Section 2.4 the dynamics of (2.15) and to compute a time $\widetilde{T} < \tau_0/\varepsilon$ such that $|A_1(\widetilde{T})| + |A_2(\widetilde{T})| \geq \mathcal{O}(1)$. The time $\widetilde{T}$ will be a first guess for $\widehat{T}$.

- To transfer in Section 2.5 Theorem 2.2.1 to the spatially periodic situation and more important since $\widehat{T} \gg \mathcal{O}(1)$ to extend the time scale for the error estimates for the extended TWI approximation up to $2\widetilde{t} = 2\widetilde{T}/\varepsilon$.

- Finally, to check in Section A.1.3 that the required behavior predicted by the classical TWI system can be found in the extended TWI system, too, i.e., to prove $|\mathcal{A}_1(\widehat{T})| + |\mathcal{A}_2(\widehat{T})| \geq \mathcal{O}(1)$ for a $\widehat{T}$ close to $\widetilde{T}$.

2.3 Computation of the resonant wave numbers

In order to use the extended TWI approximation to construct a counter example that the NLS approximation can fail in making correct predictions we have to find wave numbers k_1, k_2, and k_3 such that the following holds:

There exist integers n_1, n_2, and n_3, a wave number $k_* > 0$ such that $k = k_j = n_j k_*$ and $\omega = \omega_j$ satisfy

$$k_1 + k_2 + k_3 = k_*(n_1 + n_2 + n_3) = 0, \qquad \text{and} \qquad \omega_1 + \omega_2 + \omega_3 = 0,$$

and are related by the linear dispersion relation (2.9) of the water wave problem.

In order to do so we choose integers n_1, n_2, and n_3 and look for the zeroes of

$$\begin{aligned} f(k_*,\sigma) &= \omega_1+\omega_2-\omega_3 \\ &= \sqrt{(n_1k_*+\sigma(n_1k_*)^3)\tanh(n_1k_*)}+\sqrt{(n_2k_*+\sigma(n_2k_*)^3)\tanh(n_2k_*)} \\ &\quad -\sqrt{(n_3k_*+\sigma(n_3k_*)^3)\tanh(n_3k_*)}, \end{aligned}$$

i.e., we have to find a basic wave number k_* and a surface tension σ such that $f(k_*,\sigma)=0$. It is easy to see that there are non-trivial zeroes due to the intermediate value theorem. As an example we have for $n_1=1$, $n_2=2$, and $n_3=-3$ that

$$f(10,0)\approx\sqrt{10}+\sqrt{20}-\sqrt{30}>0$$

and

$$f(10,1)\approx\sqrt{1000}+\sqrt{8000}-\sqrt{27000}<0.$$

However, such zeroes can only exist for $\sigma\in(0,1/3)$ if $h=1$ and $g=1$.

2.4 Analysis of the TWI system

For the understanding of the subsequent extended TWI system (2.19) we first have a look at its lowest order approximation, namely the classical TWI system (2.15). The TWI system (2.15) possesses three invariant subspaces. The first is spanned by the vector $(1,0,0)$, the second by $(0,1,0)$, and the third one by $(0,0,1)$. It is well known that for instance the subspace $M_3=\{(0,0,A_3):A_3\in\mathbb{C}\}$ is stable if $\gamma_1\gamma_2<0$ and that it is unstable if $\gamma_1\gamma_2>0$. This can be seen easily by computing

$$\frac{d}{dt}(\varrho_1|A_1|^2+\varrho_2|A_2|^2)=-i(\varrho_1\gamma_1+\varrho_2\gamma_2)(A_2A_3A_1-\overline{A_2A_3A_1}).$$

The energy on the left hand side is positive definite and then measures the distance to M_3 if $\varrho_1>0$ and $\varrho_2>0$. It is conserved if $\varrho_1\gamma_1+\varrho_2\gamma_2=0$ which can be achieved if $\gamma_1\gamma_2<0$ by choosing $\varrho_1>0$ and $\varrho_2>0$ properly.

There are essentially two cases, in the following called Case I and Case II, namely all γ_j have the same sign or not. Case II occurs for the water wave problem and is discussed in this section. There we must have two positive γ_j and one negative γ_j or vice versa. In Case I where all γ_j have the same sign all subspaces are unstable. It is called explosive instability, cf. [Se08]. In Case II we can find some $\rho_j>0$ for $j=1,2,3$ such that $\varrho_1\gamma_1+\varrho_2\gamma_2+\varrho_3\gamma_3=0$. Since then

$$\frac{d}{dt}(\varrho_1|A_1|^2+\varrho_2|A_2|^2+\varrho_3|A_3|^2)=-i(\varrho_1\gamma_1+\varrho_2\gamma_2+\varrho_3\gamma_3)(A_2A_3A_1-\overline{A_2A_3A_1})=0$$

in Case II the energy surface is an ellipsoid and in Case I a hyperboloid.

For the water wave problem it turns out that the subspace associated to the largest wave number, here k_3, is unstable, i.e., $\gamma_1\gamma_2 > 0$. See Section A.2.3. W.l.o.g. we assume $\gamma_1 < 0$, $\gamma_2 < 0$, and $\gamma_3 > 0$. For the following discussion it is advantageous to introduce variables B_j by

$$A_1 = -iB_1/\sqrt{|\gamma_2\gamma_3|}, \qquad A_2 = -iB_2/\sqrt{|\gamma_1\gamma_3|}, \qquad \text{and} \qquad A_3 = -iB_3/\sqrt{|\gamma_1\gamma_2|}$$

and restrict to $B_j \in \mathbb{R}$ for $j = 1, 2, 3$ which yields

$$\partial_T B_1 = B_2 B_3, \qquad \partial_T B_2 = B_1 B_3, \qquad \partial_T B_3 = -B_1 B_2. \tag{2.16}$$

To construct our counter example we start the TWI system (2.16) with

$$B_1|_{T=0} = \varepsilon^{1/2}, \qquad B_2|_{T=0} = \varepsilon^{1/2}, \qquad \text{and} \qquad B_3|_{T=0} = 1.$$

The associated initial condition for the original water wave problem satisfies the required estimate (2.11) in (APP'). We have chosen purely imaginary initial conditions since $i\mathbb{R}^3$ is an invariant subspace for (2.15). Restricting to the surface of constant energy finally reduces the analysis to a two-dimensional system whose dynamics can be understood completely with the help of a phase portrait.

Since $\partial_T B_3 = \mathcal{O}(\varepsilon)$ the variable B_3 will be considered first as a constant. Then we find $\partial_T^2 B_1 = B_1$ such that $B_1(T) = \varepsilon^{1/2} e^T$ and $B_2(T) = \varepsilon^{1/2} e^T$. Hence, we have $B_2(T) = 1/2$ for

$$T = \widetilde{T} := -\ln(2) - \frac{1}{2}\ln(\varepsilon). \tag{2.17}$$

A look a the phase portrait plotted in Figure 2.2 shows that this heuristic argument gives a good impression what happened in the TWI system (2.15). There, we start close to the heteroclinic orbit and leave the neighborhood of the unstable fixed point. The trajectories are a little bit slower than in the linear case, but we still have $|B_1(\widetilde{T})| > 1/3$ and $|B_2(\widetilde{T})| > 1/3$.

A similar behavior can be found in the extended TWI system (2.19), too. For similarly rescaled variables $\mathcal{B}_j$ we can guarantee $|\mathcal{B}_1(\widehat{T})| > 1/4$ and $|\mathcal{B}_2(\widehat{T})| > 1/4$ for $\varepsilon > 0$ sufficiently small and a time $\widehat{T}$ close to $\widetilde{T}$.

2.5 The extended TWI approximation result

Despite functional analytic difficulties it is rather easy to prove a TWI approximation result. Solutions of order $\mathcal{O}(\varepsilon)$ have to be estimated on an $\mathcal{O}(1/\varepsilon)$ time scale which easily can be achieved by an application of Gronwall's inequality, cf. [SW03]. Therefore, the extension of the error estimates from $t \in [0, T_0/\varepsilon]$ to $t \in [0, 2\widetilde{T}/\varepsilon]$, with $\widetilde{T}$ defined in (2.17) is the major achievement in the following. However, in

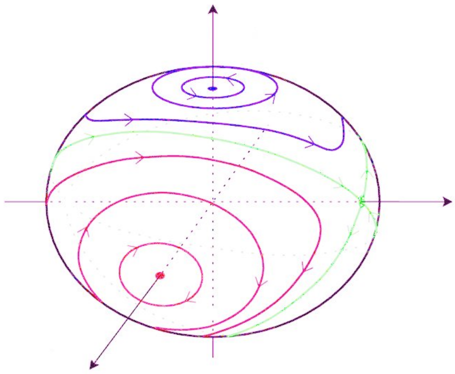

Figure 2.2: The phase portrait of the TWI system in the compact energy surface which occurs if not all γ_j have the same sign.

order to obtain this much longer time scale an extended TWI system and extended TWI approximation $\varepsilon\psi$ have to be considered. The extended TWI approximation is given by

$$\begin{aligned} \begin{pmatrix} \eta \\ w \end{pmatrix}(x,t) = \varepsilon\psi &= \varepsilon\mathcal{A}_1(\varepsilon t)e^{i(k_* x-\omega_1 t)}\varphi_1 + \varepsilon\mathcal{A}_2(\varepsilon t)e^{i(2k_* x-\omega_2 t)}\varphi_2 \\ &\quad +\varepsilon\mathcal{A}_3(\varepsilon t)e^{i(3k_* x-\omega_3 t)}\varphi_3 + \varepsilon^{1+\theta}h_{ext}(x,t), \end{aligned} \tag{2.18}$$

with $\theta \in (0, 1/2)$ a chosen fixed number arbitrarily close to $1/2$. The amplitudes $\mathcal{A}_j$ satisfy a perturbed TWI system of the form

$$\begin{aligned} \partial_T\mathcal{A}_1 &= i\gamma_1\mathcal{A}_{-2}\mathcal{A}_3 + \ldots, \\ \partial_T\mathcal{A}_2 &= i\gamma_2\mathcal{A}_3\mathcal{A}_{-1} + \ldots, \\ \partial_T\mathcal{A}_3 &= -i\gamma_3\mathcal{A}_1\mathcal{A}_2 + \ldots. \end{aligned} \tag{2.19}$$

The lowest order terms in the first three equations give the TWI system (2.15), where $A_1 = \mathcal{A}_1$, $A_2 = \mathcal{A}_2$, and $A_3 = \mathcal{A}_{-3} = \overline{\mathcal{A}_3}$. Therefore, the minus sign in the third equation. The use of the extended TWI approximation $\varepsilon\psi$ is necessary to make the residual of order $\mathcal{O}(\varepsilon^{\beta+\delta})$ for any given $\beta + \delta \geq 0$. The residual $\mathrm{Res}(\varepsilon\psi)$ contains the terms which do not cancel after inserting the approximation $\varepsilon\psi$ into the equations of the water wave problem. However, for our argument these estimates are not only necessary on the natural TWI time scale $\mathcal{O}(1/\varepsilon)$, but on the much longer $\mathcal{O}(|\ln(\varepsilon)|/\varepsilon)$ time scale. Interestingly, due to some conserved quantity, it turns out that the solutions of the extended TWI system (2.19) and the residual can be controlled on an even longer time scale. Hence, in Section A.1.1 we derive the extended TWI system (2.19) and in Section A.1.2 we prove the $\mathcal{O}(1)$-boundedness of the functions $\mathcal{A}_j(T)$ for all $T \in [0, \varepsilon^{-\widetilde{\mu}}]$ and the $\mathcal{O}(1)$-boundedness of $h_{ext}(\cdot, t)$ in every Sobolev space H^s for all $t \in [0, \varepsilon^{-1-\widetilde{\mu}}]$ for a $\widetilde{\mu} \in [0,1)$ arbitrary, but fixed.

Lemma 2.5.1
Let $\widetilde{\mu} \in [0,1)$ arbitrary, but fixed. Then for all $\beta + \delta > 0$ there exists a constant $C > 0$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists an extended TWI approximation $\varepsilon\psi$ of the water wave problem with

$$\sup_{t \in [0, \varepsilon^{-1-\widetilde{\mu}}]} \|\mathrm{Res}(\varepsilon\psi)\|_{H^s_{per}} \leq C\varepsilon^{\beta+\delta}$$

and

$$\sup_{t \in [0, \varepsilon^{-1-\widetilde{\mu}}]} \|\psi\|_{C^s_{b,per}} \leq C.$$

Proof: See Section A.1.

In order to be on the safe side later on we not only prove estimates for the error made by the extended TWI approximation for $t \in [0, \widetilde{T}/\varepsilon]$, but for $t \in [0, 2\widetilde{T}/\varepsilon]$. In contrast to the estimates for the residual the extension to $t \in [0, \varepsilon^{-1-\widetilde{\mu}}]$ is not possible due to some exponential growth of the linear modes. However, by making the residual smaller a logarithmic factor w.r.t. ε can be gained.

Theorem 2.5.2
For the extended TWI approximation $\varepsilon\psi$ from Lemma 2.5.1 there exist $C_0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions $\begin{pmatrix} \eta \\ w \end{pmatrix}$ of the water wave problem (2.1)-(2.4) *satisfying*

$$\sup_{t \in [0, 2\widetilde{t}]} \left\| \begin{pmatrix} \eta \\ w \end{pmatrix}(x,t) - \varepsilon\psi(x,t) \right\|_{(C^{s-2}_b)^2} \leq C_0 \varepsilon^{3/2}$$

with $\widetilde{t} = \widetilde{T}/\varepsilon$.

Proof:

In order to prove Theorem 2.5.2 we use [SW03, Theorem 4.1] and its proof which are formulated for the Lagrangian formulation of the water wave problem. In this formulation for fixed time t the free surface of the fluid is written as a Jordan-curve

$$\Gamma(t) = \{(\widetilde{X}_1(\alpha,t), \widetilde{X}_2(\alpha,t)) = (\alpha + X_1(\alpha,t), X_2(\alpha,t)) | \alpha \in \mathbb{R}\} .$$

The water wave problem is then completely determined through the evolution of the variables $\partial_t X_1(\alpha,t)$, $\partial_\alpha X_1(\alpha,t)$ and $X_2(\alpha,t)$. The Eulerian variables $w = w(x,t)$ and $\eta = \eta(x,t)$ and the Lagrangian variables are related through

$$w(\widetilde{X}_1(\alpha,t),t) = \partial_t X_1(\alpha,t) \qquad \text{and} \qquad \eta(\widetilde{X}_1(\alpha,t),t) = X_2(\alpha,t).$$

Following [SW03, Remark 4.3] from the TWI approximation result [SW03, Theorem 4.1] in Lagrangian formulation follows immediately the TWI approximation [SW03, Theorem 1.1] in Eulerian formulation. Instead of deriving and estimating the extended TWI approximation in Lagrangian formulation we oversimplify our presentation and do not give the details for this step and stay at the Eulerian formulation in deriving the approximation.

Nevertheless, suppose now that we have the extended TWI approximation in Lagrangian formulation with the required estimates. Then we follow the proof of [SW03, Theorem 4.1] line for line replacing the function spaces over the real line by the same function spaces but with periodic boundary conditions until [SW03, last page of Section 4]). There we find an inequality

$$\partial_t E \leq C_1 \varepsilon E + C_2(E)\varepsilon^\beta E^{3/2} + C_3 \varepsilon E^{1/2} \tag{2.20}$$

for an energy E which is a bound for the square of the H^s norm of the error $\varepsilon^\beta R = \binom{\eta}{w} - \varepsilon\psi_{twi}$ made by the TWI approximation. The constants C_1 and C_3 are independent of E, and $\beta = 3/2$ is chosen throughout the proof [SW03, Theorem 4.1]. The function $C_2(E)$ depends smoothly on E, but is independent of ε. The constant C_3 comes from the residual, the terms which do not cancel after inserting the TWI ansatz into the equations of the water wave problem. In [SW03] the residual has been proved to be of order $\mathcal{O}(\varepsilon^{5/2})$.

With the extended TWI approximation $\varepsilon\psi$ from the previous Lemma 2.5.1 the residual will be of order $\mathcal{O}(\varepsilon^{\beta+\delta})$ in H^s for every $\delta, \beta \geq 0$ for all $t \in [0, 2\widetilde{t}]$ and the inequality (2.20) can be improved to

$$\begin{aligned} \partial_t E &= C_1 \varepsilon E + C_2(E)\varepsilon^\beta E^{3/2} + C_3 \varepsilon^\delta E^{1/2} \\ &\leq C_1 \varepsilon E + C_2(E)\varepsilon^\beta E^{3/2} + C_3 \varepsilon^{2\delta-1} + C_3 \varepsilon E \end{aligned}$$

where we used $\varepsilon^{1/2} E^{1/2} \varepsilon^{\delta-1/2} \leq \varepsilon E + \varepsilon^{2\delta-1}$ to come from the first to the second line. For $C_2(E)\varepsilon^\beta E^{1/2} \leq (C_1 + C_3)\varepsilon$ the inequality becomes

$$\partial_t E \leq 2(C_1 + C_3)\varepsilon E + C_3 \varepsilon^{2\delta-1}$$

and so with $E(0) = 0$ we obtain

$$E(t) \;\leq\; e^{2(C_1+C_3)\varepsilon t} E(0) + C_3 \int_0^t e^{2(C_1+C_3)\varepsilon(t-s)} \varepsilon^{2\delta-1} ds$$

$$
\begin{aligned}
&= C_3\varepsilon^{2\delta-1}\frac{e^{2(C_1+C_3)\varepsilon(t-s)}}{-2(C_1+C_3)\varepsilon}\Big|_0^t = \frac{C_3\varepsilon^{2\delta-2}}{2(C_1+C_3)}(e^{2(C_1+C_3)\varepsilon t}-1)\\
&\leq \frac{C_3\varepsilon^{2\delta-2}}{2(C_1+C_3)}(e^{2\varepsilon\widetilde{t}})^{2(C_1+C_3)} \leq \frac{C_3\varepsilon^{2\delta-2}}{2(C_1+C_3)}\left(\frac{1}{2\varepsilon}\right)^{4(C_1+C_3)}\\
&\leq \frac{C_3}{2(C_1+C_3)}\left(\frac{1}{2}\right)^{4(C_1+C_3)}\varepsilon^{2\delta-2-4(C_1+C_3)} \leq M
\end{aligned}
$$

for all $t \in [0, 2\widetilde{t}]$ if we set

$$
M = \frac{C_3}{2(C_1+C_3)}\left(\frac{1}{2}\right)^{4(C_1+C_3)}
$$

and choose

$$
\delta \geq 1 + 2(C_1 + C_3).
$$

We are done if we choose $\varepsilon_0 > 0$ smaller than the one of Lemma 2.5.1 and such that $C_2(M)\varepsilon_0^{\beta-1}M^{1/2} \leq (C_1 + C_3)$ is satisfied.

2.6 The counter example

In this section we finally construct the counter example that the NLS approximation makes wrong predictions. Following the calculations of Section 2.3 we take $k_* = 10$, $n_1 = 1$, $n_2 = 2$, $n_3 = -3$, and a σ with $f(k^*, \sigma) = 0$. We choose the unstable subspace for the NLS approximation, i.e., we choose $k_0 = -3k_*$ as basic wave number for the NLS approximation.

According to the computations in Section 2.4 we start the extended TWI system (2.19) with $\mathcal{A}_1|_{T=0} = -i\varepsilon^{1/2}\sqrt{|\gamma_2\gamma_3|}$, $\mathcal{A}_2|_{T=0} = -i\varepsilon^{1/2}/\sqrt{|\gamma_1\gamma_3|}$, and $\mathcal{A}_{-3}|_{T=0} = -i/\sqrt{|\gamma_1\gamma_2|}$. In Section A.1.3 we prove that for the associated solution of the extended TWI system we have $|\mathcal{A}_1(\widehat{T})| > 1/4\sqrt{|\gamma_2\gamma_3|}$ and $|\mathcal{A}_2(\widehat{T})| > 1/4\sqrt{|\gamma_1\gamma_3|}$ for $\widehat{T} \in [0, 2\widetilde{T}]$, with $\widetilde{T} = -\ln(2) - \frac{1}{2}\ln(\varepsilon)$. The associated initial conditions of the water wave problem satisfy the assumptions of (APP'). Hence, if the NLS approximation property holds then all modes except of the ones associated to $k = \pm 3k_*$ should be of order $\mathcal{O}(\varepsilon^{3/2})$ for all $t \in [0, \tau_0/\varepsilon^2]$, especially we should have that the coefficients $(\eta, w)_1|_{t=\widetilde{T}/\varepsilon}$ of e^{ik_*x} and $(\eta, w)_2|_{t=\widetilde{T}/\varepsilon}$ of e^{2ik_*x} should be of order $\mathcal{O}(\varepsilon^{3/2})$.

On the other hand if we apply Theorem 2.5.2 we immediately find that $(\eta, w)_1|_{t=\widehat{T}/\varepsilon} = \mathcal{O}(\varepsilon)$ and $(\eta, w)_2|_{t=\widehat{T}/\varepsilon} = \mathcal{O}(\varepsilon)$ contradicting (APP'). Hence the NLS equation (2.10) makes wrong predictions at $t = \widehat{T}/\varepsilon \ll 1/\varepsilon^2$. See Figure 2.3.

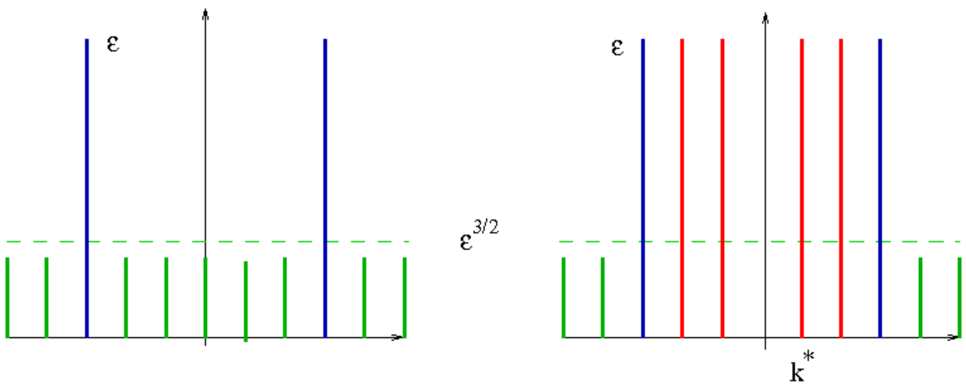

Figure 2.3: The initial mode distribution in the left panel and the mode distribution at time $t = \widehat{T}/\varepsilon$ in the right panel.

Remark 2.6.1
The phenomenon is not restricted to the initial condition and to the chosen wave number from above. There is an open neighborhood around this initial condition and a continuum of wave numbers for which the NLS approximation makes wrong predictions. Hence, making wrong predictions is not an exception. Moreover, the initial size of $\mathcal{A}_{1,2}$ can be chosen much smaller w.r.t. powers of ε. However, the analysis in Section A.1.3 would be much more involved.

Chapter 3

FWI fails too

When several dominant wave-modes are present, their mutual interaction is significant. This is specially so when some of these modes resonate. Systems for the (resonant) four wave interaction (FWI) can be derived via multiple scaling analysis for the approximate description of the interaction of N modulated wave packets in a number of physical situations. Examples are pattern formation in vertically oscillated convection [RPS03], multi wave non-linear couplings in elastic structures [KMP02], nonlinear optical waves [MB99], or the four-wave interactions in plasmas [Ver82]. They are also used as a model, cf. [AS81, Crk85], for the description of gravity driven surface water waves and in the description of so called freak waves in deep sea, cf. [Ja03]. It is the goal of this chapter to show that the (resonant) FWI system makes wrong predictions on the natural time scale of the approximation if unstable quadratic resonances are present in the original system. In detail, we explain that counter example constructed for the NLS approximation can be transferred line to line to the FWI approximation.

3.1 The FWI approximation

The (resonant) four wave interaction system appears as an amplitude equation in the description of gravity driven surface water waves and other dispersive wave systems. In case of non resonant quadratic terms in the original system error estimates justifying this approximation can be established with the help of normal form transformations and Gronwall's inequality. The following ideas apply to the water wave problem too.

Here, as an additional example consider the Boussinesq equation

$$\partial_t^2 u = \partial_x^2 u + \partial_x^2(u^2) + \partial_t^2\partial_x^2 u + \mu\partial_x^6 u, \tag{3.1}$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$ as original system. Unrelated to its derivation as an approximation equation (in case $\mu = 0$) for the water wave problem we use this equation as a model for a dispersive wave system which possesses for $\mu > 0$ nontrivial resonances for nonzero wave numbers.

In order to derive the N wave interaction system we make the ansatz

$$\varepsilon\psi_{\mathrm{FWI}}(\varepsilon, x, t) = \sum_{j\in I_N} \varepsilon A_j(\varepsilon^2 x, \varepsilon^2 t)e^{i(k_j x+\omega_j t)} + \mathcal{O}(\varepsilon^2),$$

with index set $I_N = \{-N, \ldots, -1, 1, \ldots, N\}$, a small perturbation parameter $0 < \varepsilon \ll 1$, amplitude functions $A_j(X,\tau) \in \mathbb{C}$ with $A_j = \overline{A_{-j}}$, the spatial wave number $k_j \in \mathbb{R}$ with $k_j = -k_{-j}$, and the temporal wave number $\omega_j \in \mathbb{R}$ with $\omega_j = -\omega_{-j}$. We plug in this ansatz $u = \varepsilon\psi_{\mathrm{FWI}}$ into (3.1). Equating the coefficients of $\varepsilon e^{i(k_j x+\omega_j t)}$ to zero yields the linear dispersion relation

$$\omega^2 = k^2 - \omega^2 k^2 + \mu k^6. \tag{3.2}$$

Equating the coefficients of $\varepsilon^3 e^{i(k_j x+\omega_j t)}$ to zero yields the N wave interaction system. The (resonant) FWI system is given by

$$\partial_\tau A_j = c_j\partial_X A_j + \sum_{l\in I_N} d_{jl}|A_l|^2 A_j + \sum_{(j_1,j_2,j_3)\in R_j} d^j_{j_1 j_2 j_3}\overline{A_{j_1}A_{j_2}A_{j_3}}, \tag{3.3}$$

where R_j is called the set of resonances. It is defined by

$$R_j = \{(j_1, j_2, j_3) \in I_N^3 \;\mid\; k_{j_1} + k_{j_2} + k_{j_3} + k_j = 0, \quad \omega_{j_1} + \omega_{j_2} + \omega_{j_3} + \omega_j = 0\}.$$

The coefficients appearing in (3.3) are given by

$$c_j = \omega'(k_j), \quad d_{jl} = -\frac{4k_l^2}{\iota\omega_j(1+k_j^2)}, \quad \text{and} \quad d^j_{j_1 j_2 j_3} = -\frac{6k_j^2}{\iota\omega_j(1+k_j^2)}.$$

in the resonant case, and $d^j_{j_1 j_2 j_3} = 0$ in the non resonant case, i.e., in case $R_j = \emptyset$ for $j = 1, \ldots N$.

However, the formal derivation of the FWI system does not imply that the Boussinesq model (3.1) really behaves as predicted by the FWI system for small values of the perturbation parameter $0 < \varepsilon \ll 1$ on the natural time scale $\mathcal{O}(1/\varepsilon^2)$ of the FWI approximation.

For notational simplicity we restrict ourselves to the case $N = 4$. Then in order to construct a proper counter example we choose periodic boundary conditions

$$u(x,t) = u(x + \frac{2\pi}{k_*}) \tag{3.4}$$

for (3.1) with k_* a chosen wave number. Next we choose $k_1 = 3k_*$, $k_2 = 4k_*$, $k_3 = 5k_*$ and $k_4 = 6k_*$. Then in case $R_j = \emptyset$, the ansatz degenerates into

$$\varepsilon\psi_{FWI,per} \quad = \quad \varepsilon A_1(\varepsilon^2 t)e^{i(k_1 x+\omega_1 t)} + \ldots + \varepsilon A_4(\varepsilon^2 t)e^{i(k_4 x+\omega_4 t)} + c.c.,$$

where the A_j now satisfy

$$\partial_\tau A_j = \sum_{l\in I_N} d_{jl}|A_l|^2 A_j, \tag{3.5}$$

and d_{jl} is given above.

3.2 The TWI system and the unstable subspaces

We adopt the strategy from last chapter of using the estimates from the TWI system to prove that the FWI system makes wrong prediction too. It has already been established that the TWI system provides correct estimates for the description of surface water waves [SW03]. And since the FWI system gives different predictions so it does not describe the original system correctly.

In detail, the equations

$$\begin{aligned}
\partial_T A_1 &= c_g(\widetilde{k_1})\partial_X A_1 + i\gamma_1 \overline{A_2 A_3}, \\
\partial_T A_2 &= c_g(\widetilde{k_2})\partial_X A_2 + i\gamma_2 \overline{A_1 A_3}, \\
\partial_T A_3 &= c_g(\widetilde{k_3})\partial_X A_3 + i\gamma_3 \overline{A_1 A_2},
\end{aligned} \tag{3.6}$$

with $T, X, c_g(\widetilde{k_j}), \gamma_j \in \mathbb{R}$, and $A_j(X,T) \in \mathbb{C}$ for the three wave interaction can be derived by the multiple scaling ansatz

$$\begin{aligned} \varepsilon\psi_1(x,t) &= \varepsilon A_1(\varepsilon x, \varepsilon t)e^{i(\widetilde{k_1}x-\omega_1 t)} + \varepsilon A_2(\varepsilon x, \varepsilon t)e^{i(\widetilde{k_2}x-\omega_2 t)} \\ &\quad + \varepsilon A_3(\varepsilon x, \varepsilon t)e^{i(\widetilde{k_3}x-\omega_3 t)} + c.c., \end{aligned} \tag{3.7}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter, and the spatial and temporal wave numbers $\widetilde{k} = \widetilde{k_j}$ and $\widetilde{\omega} = \widetilde{\omega_j}$ which have to satisfy the resonance condition

$$\widetilde{k_1} + \widetilde{k_2} + \widetilde{k_3} = 0, \qquad \widetilde{\omega_1} + \widetilde{\omega_2} + \widetilde{\omega_3} = 0, \tag{3.8}$$

and are related by the linear dispersion relation (3.2).

The gamma's for the TWI system are calculated as:

$$\gamma_1 = \frac{1}{\sqrt{(\widetilde{k_1^2}+1)(1+\hat{\mu}\widetilde{k_1^4})}}$$

$$\gamma_2 = \frac{1}{\sqrt{(4\widetilde{k_2^2}+1)(1+8\hat{\mu}\widetilde{k_2^4})}}$$

$$\gamma_3 = -\frac{1}{\sqrt{(9\widetilde{k_3^2}+1)(1+16\hat{\mu}\widetilde{k_3^4})}}$$

with $\hat{\mu} \in \{\mu : f(\widetilde{k_*}, \mu) = 0\}$. Clearly, γ_1 and γ_2 have same signs and $\gamma_1\gamma_2 > 0$ which implies that the subspace associated to A_3 is unstable.

In Figure 3.1 we have plotted the products of $\gamma_1\gamma_2$, $\gamma_1\gamma_3$, and $\gamma_2\gamma_3$ to observe the stability of subspaces associated with modes A_3, A_2 and A_1 respectively.

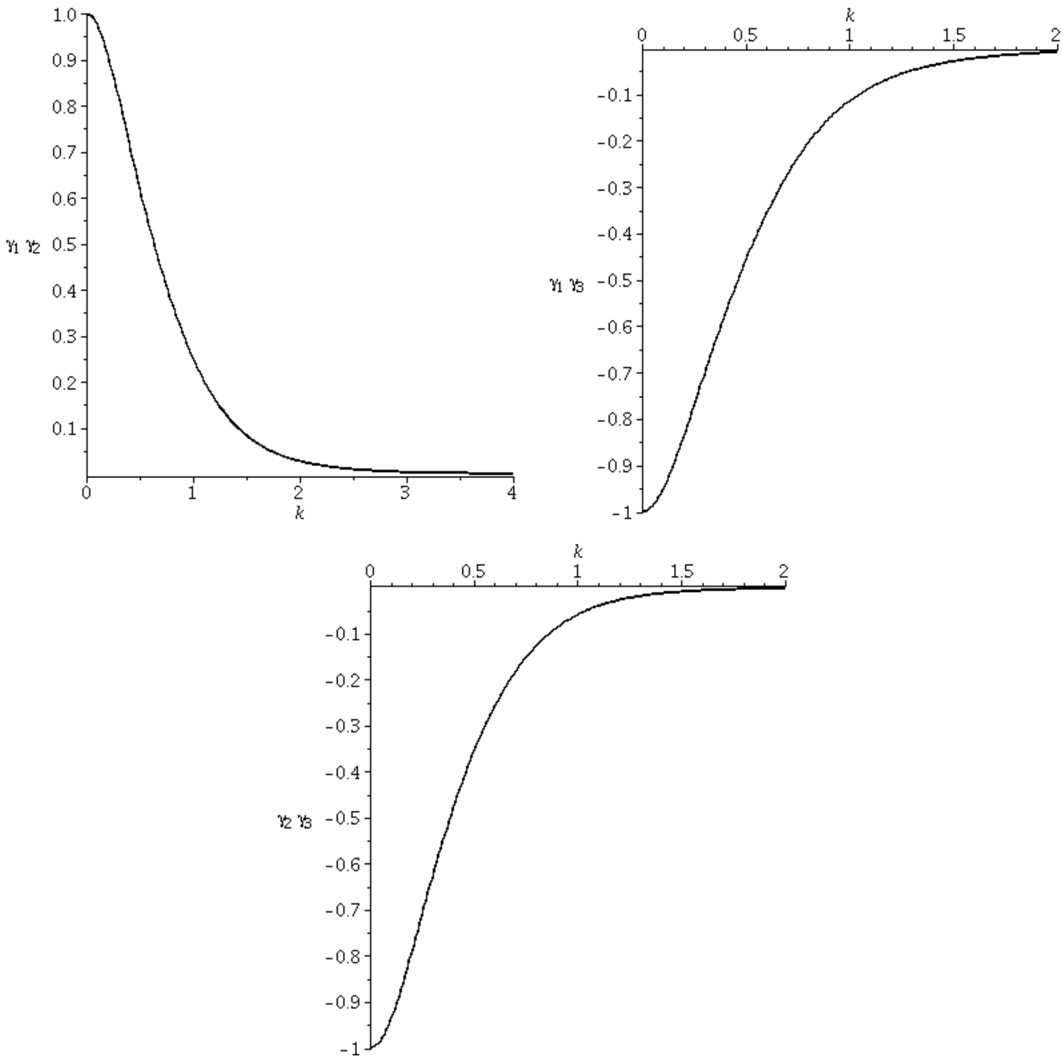

Figure 3.1: The products $\gamma_1\gamma_2$, $\gamma_1\gamma_3$, and $\gamma_2\gamma_3$.

3.3 Counter example

In order to use the TWI approximation to construct a counter example that the FWI approximation can fail in making correct predictions we have to find wave numbers $\widetilde{k_1}$, $\widetilde{k_2}$, and $\widetilde{k_3}$ such that the following holds:

There exist integers n_1, n_2, and n_3, a wave number $\widetilde{k_*} > 0$ such that $\widetilde{k} = \widetilde{k_j} =$

$n_j \widetilde{k}_*$ and $\widetilde{\omega} = \widetilde{\omega_j}$ satisfy

$$\widetilde{k_1} + \widetilde{k_2} + \widetilde{k_3} = \widetilde{k}_*(n_1 + n_2 + n_3) = 0, \qquad \text{and} \qquad \widetilde{\omega_1} + \widetilde{\omega_2} + \widetilde{\omega_3} = 0,$$

and are related by the linear dispersion relation (2.9).

In order to do so we choose integers n_1, n_2, and n_3 and look for the zeroes of

$$\begin{aligned} f(\widetilde{k}_*, \mu) &= \widetilde{\omega_1} + \widetilde{\omega_2} - \widetilde{\omega_3} \\ &= \sqrt{\frac{(n_1 \widetilde{k}_*)^2 + \mu (n_1 \widetilde{k}_*)^6}{1 + (n_1 k_*)^2}} + \sqrt{\frac{(n_2 \widetilde{k}_*)^2 + \mu (n_2 \widetilde{k}_*)^6}{1 + (n_2 \widetilde{k}_*)^2}} \\ &\quad - \sqrt{\frac{(n_3 \widetilde{k}_*)^2 + \mu (n_3 \widetilde{k}_*)^6}{1 + (n_3 \widetilde{k}_*)^2}}, \end{aligned}$$

i.e., we have to find a basic wave number $\widetilde{k}_*$ and $\mu \in \mathbb{R}$ such that $f(\tilde{k}_*, \mu) = 0$. It is easy to see that there are non trivial zeroes due to the intermediate value theorem. As an example we have for $n_1 = 1$, $n_2 = 2$, and $n_3 = -3$ that

$$f(10, 0) \approx \sqrt{\frac{100}{101}} + \sqrt{\frac{400}{401}} - \sqrt{\frac{900}{901}} > 0$$

and

$$f(10, 1) \approx \sqrt{\frac{10^2 + 10^6}{101}} + \sqrt{\frac{20^2 + 20^6}{401}} - \sqrt{\frac{30^2 + 30^6}{901}} < 0.$$

In Figure 3.2 we have plotted the zeroes of $f(\widetilde{k}_*, \mu)$.

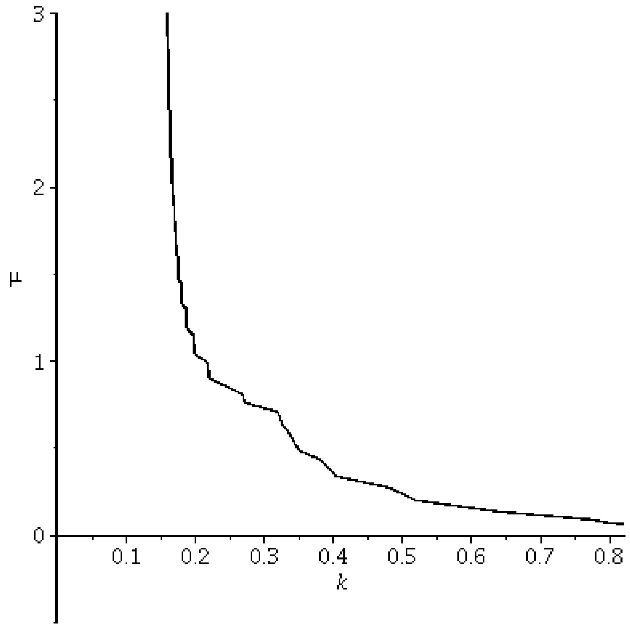

Figure 3.2: The set $\{(\widetilde{k_*}, \mu) \in \mathbb{R}^2 : f(\widetilde{k}, \mu) = 0\}$.

The NLS ansatz in case of periodic boundary conditions is given by:

$$\varepsilon\psi_{NLS,per} = \varepsilon A_1(\varepsilon^2 t)e^{i(\widetilde{k_1}x+\widetilde{\omega_1}t)} + c.c., \tag{3.9}$$

whereas the ansatz for periodic FWI system is given by:

$$\varepsilon\psi_{FWI,per} = \varepsilon A_1(\varepsilon^2 t)e^{i(\widetilde{k_1}x+\widetilde{\omega_1}t)} + \ldots + \varepsilon A_4(\varepsilon^2 t)e^{i(\widetilde{k_4}x+\widetilde{\omega_4}t)} + c.c., \tag{3.10}$$

We choose an initial condition for the FWI system such that

$$A_1 \neq 0 \quad , \quad A_2 = A_3 = A_4 = 0.$$

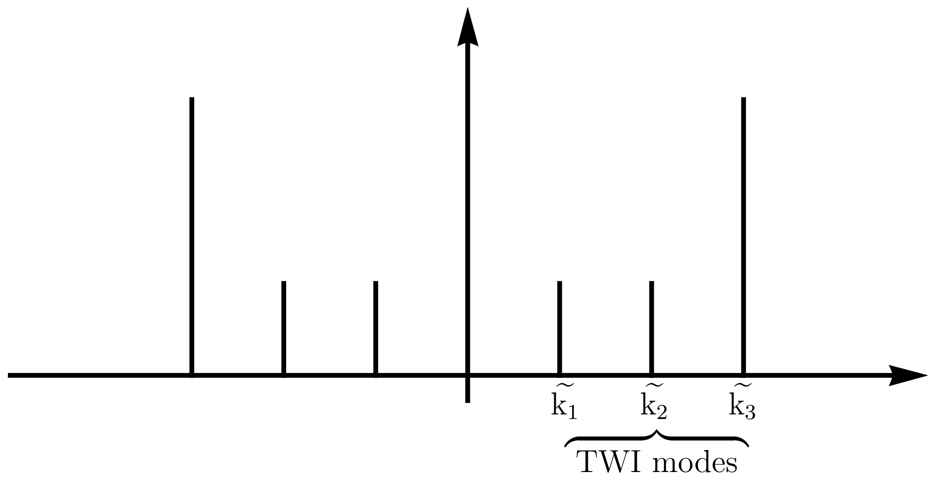

Figure 3.3: The TWI modes.

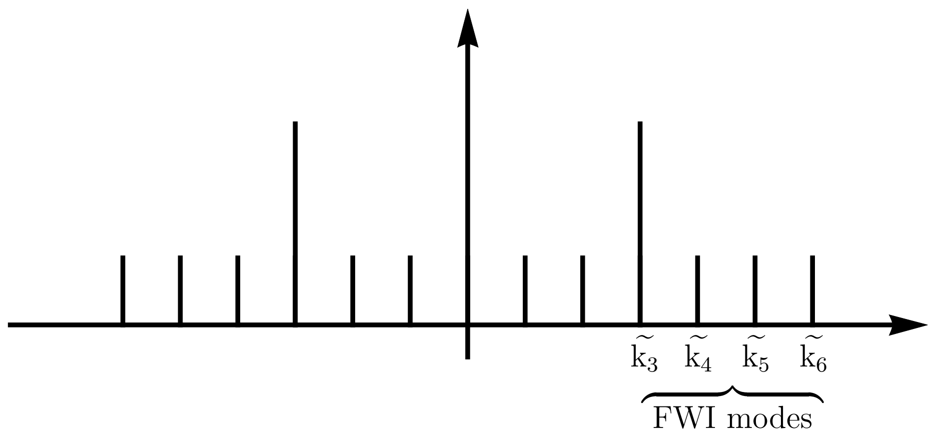

Figure 3.4: The FWI modes.

Thus clearly NLS approximation equals the FWI approximation exhibiting the same dynamics. The ideas of the last chapter apply 1-to-1 from water wave problem to the Boussinesq model. Since we proved in the last chapter that NLS approximation makes wrong predictions for the water wave problem, therefore the FWI approximation fails to approximate the water wave problem as well.

Chapter 4

NLS fails for the FPU system

A FPU system consists of a 1-D chain of masses connected by anharmonic springs. We consider an infinite diatomic FPU chain with two alternating different masses m_1 and m_2 such that $m_1 \leq m_2$.

The original Fermi-Pasta-Ulam(FPU) system was first numerically studied by Fermi et al. [FPU55] for a finite set of oscillators in order to see how energy was spread through the various modes of the system by the nonlinear coupling via the inter-particle forces which are described by the potential function $W : \mathbb{R} \to \mathbb{R}$. They found out that most trajectories did not thermalize as expected but rather exhibited a regular motion. This observation has been explained in [ZK65], where Zabusky and Kruskal derived the KdV equation as a formal approximation to the FPU system, and in studying the KdV equation numerically they found soliton dynamics. Our interest lies in developing a counter example showing that the NLS equation makes wrong predictions for the special FPU systems, too.

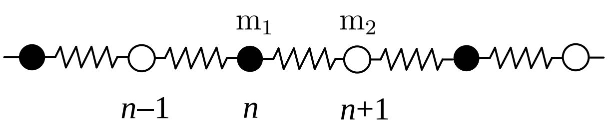

Figure 4.1: Diatomic Fermi-Pasta-Ulam chain.

4.1 The FPU system

The Fermi-Pasta-Ulam(FPU) system,

$$\partial_t^2 q_n(t) = W'(q_{n+1}(t) - q_n(t)) - W'(q_n(t) - q_{n-1}(t)), \tag{4.1}$$

with $n \in \mathbb{Z}$, $q_n(t) \in \mathbb{R}$ can be described by simple and well known partial differential equations, like the Korteweg-de Vries (KdV) or the nonlinear Schröndinger (NLS) equation. The nonlinear couplings via the intra particle forces are described by the potential function $W : \mathbb{R} \to \mathbb{R}$ which satisfies

$$W'(u) = a_1 u + a_2 u^2 + a_3 u^3 + ...,$$

where $a_1 > 0, a_n \in \mathbb{R}$ for $n \in \mathbb{R}$.

We rewrite (4.1) in terms of the difference variables

$$u(n,t) = q_{n+1}(t) - q_n(t),$$

so that (4.1) becomes

$$\partial_t^2 u(n,t) = W'(u(n+1,t)) - 2W'(u(n,t)) + W'(u(n-1,t)), n \in \mathbb{Z} \tag{4.2}$$

In the long wave limit

$$u(n,t) = \varepsilon^2 A(\varepsilon(n - c_g t)) + O(\varepsilon^3) \tag{4.3}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter and c_g the group velocity, the amplitude function $A = A(X,T) \in \mathbb{R}$ satisfies in lowest order a KdV equation

$$\partial_T A = \nu_1 \partial_X^3 A + \nu_2 \partial_X (A^2) \tag{4.4}$$

with $\nu_1, \nu_2, X, T \in \mathbb{R}$. The error between the approximation (4.3) and the actual solutions of the FPU system (4.2) is $O(\varepsilon^3)$ for all $t \in [0, \frac{T_0}{\varepsilon^3}]$, cf. [SW00].

For oscillatory wave packets of the form

$$u(n,t) = \varepsilon A(\varepsilon(n - c_g t), \varepsilon^2 t) e^{\iota(k_0 n - \omega_0 t)} + c.c \tag{4.5}$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter,c_g the group velocity and k_0 and ω_0 the basic and spatial temporal numbers. The amplitude function $A = A(X,T) \in \mathbb{C}$ satisfies in lowest order a NLS equation

$$\partial_T A = \iota \nu_3 \partial_X^2 A + \iota \nu_4 A |A|^2 \tag{4.6}$$

with $\nu_3, \nu_4, X, T \in \mathbb{R}$. The error between the approximation (4.5) and the actual solutions of the FPU system (4.2) is $O(\varepsilon^2)$ for all $t \in [0, \frac{T_0}{\varepsilon^2}]$, cf. [Schn10].

We first consider a polyatomic FPU model where the potential W acts in a periodic manner, i.e.

$$m_n \partial_t^2 q_n(t) = W_n'(q_{n+1}(t) - q_n(t)) - W_{n-1}'(q_n(t) - q_{n-1}(t)), \tag{4.7}$$

where m_n is the mass of the nth particle, $m_n > 0$ with the periodicity

$$m_n = m_{n+N} \text{ and } W_n = W_{n+N}$$

for a fixed $N \in \mathbb{N}$. The nonlinear couplings for the polyatomic FPU system via the intra particle forces are now described by the following potential function $W : \mathbb{R} \to \mathbb{R}$ which satisfies

$$W_n'(u) = a_{n1}u + a_{n2}u^2 + a_{n3}u^3 + ..., \tag{4.8}$$

where $a_{n1} > 0, a_{nn'} \in \mathbb{R}$ for $n' \in \mathbb{N}$. Again we make use of the difference of variables

$$u(n,t) = q_{n+1}(t) - q_n(t),$$

such that (4.7) becomes

$$\begin{aligned} \partial_t^2 u(n,t) = & \frac{1}{m_{n+1}} W_{n+1}'(u(n+1,t)) - \left(\frac{1}{m_n} + \frac{1}{m_{n+1}}\right) W_n'(u(n,t)) \\ & + \frac{1}{m_n} W_{n-1}'(u(n-1,t)), \end{aligned} \tag{4.9}$$

for all $n \in \mathbb{Z}$.

By using (4.8) the linearized system is given by

$$\begin{aligned} \partial_t^2 u(n,t) = & \frac{1}{m_{n+1}} a_{(n+1)1} u(n+1,t) - \left(\frac{1}{m_n} + \frac{1}{m_{n+1}}\right) a_{(n)1}(u(n,t)) \\ & + \frac{1}{m_n} a_{(n-1)1} u(n-1,t). \end{aligned} \tag{4.10}$$

Our interest lies in the diatomic FPU model. We set the periodicity at $N = 2$ and use the ansatz

$$u(n,t) = e^{\iota(kn - \omega t)} g_n(k) \tag{4.11}$$

where

$$g_n(k) = g_{n+N}(k) \in \mathbb{C}$$

for all $k \in \mathbb{R}$, $g_n = g_n(k)$ and ω and k are related by the linearized dispersion relation

$$-\omega^2 g_n = \frac{1}{m_{n+1}} a_{(n+1)1} e^{\iota k} g_{n+1} - \left(\frac{1}{m_n} + \frac{1}{m_{n+1}}\right) a_{(n)1} g_n + \frac{1}{m_n} a_{(n-1)1} e^{-\iota k} g_{n-1}. \tag{4.12}$$

For our purposes we have $a_{(n)1} = a_{(n+2)1}$, $m_n = m_{n+2}$ and $g_n(k) = g_{n+2}(k)$ so that we get the following eigenvalue problem

$$C(k)g(k) = -\omega^2 g(k) \tag{4.13}$$

where $g = (g_1, g_2)$ and

$$C(k) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

with

$$\begin{aligned} c_{11} &= -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) a_{11}, \\ c_{12} &= \left(\frac{e^{-\iota k}}{m_1} + \frac{e^{\iota k}}{m_2}\right) a_{21}, \\ c_{21} &= \left(\frac{e^{\iota k}}{m_1} + \frac{e^{-\iota k}}{m_2}\right) a_{11}, \\ c_{22} &= -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) a_{21}. \end{aligned}$$

The system (4.13) has a non-trivial solution if and only if $C - \omega^2 I$ is singular which results in the dispersion relation

$$0 = \omega^4 - \left(\frac{1}{m_1} + \frac{1}{m_2}\right)(a_{11} + a_{21})\omega^2 + \left(\frac{2}{m_1 m_2}\right)(1 - \cos(2k))(a_{11}a_{21}). \quad (4.14)$$

The four solutions $\pm\omega_+(k)$ and $\pm\omega_-(k)$ are explicitly given by

$$\omega_+(k) = \sqrt{\frac{KL}{2} + \frac{1}{2}\sqrt{K^2L^2 - \frac{8a_{11}a_{21}}{m_1 m_2}\big(1 - \cos(2k)\big)}} \quad (4.15)$$

and

$$\omega_-(k) = \sqrt{\frac{KL}{2} - \frac{1}{2}\sqrt{K^2L^2 - \frac{8a_{11}a_{21}}{m_1 m_2}\big(1 - \cos(2k)\big)}} \quad (4.16)$$

where $K = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)$ and $L = (a_{11} + a_{21})$. The curves $\omega_\pm$ represent the so called optical and acoustic bands respectively.

4.2 Unstable resonances and the counter example

We adopt the strategy from last chapter and the chapter before of using the estimates from the TWI system to prove that the NLS equation respectively FWI system makes wrong prediction for special FPU systems. As before it can be established that the TWI system provides correct estimates for the diatomic FPU system. And since the NLS equation respectively the FWI system gives different predictions so they do not describe the original system correctly.

The equations

$$\begin{aligned} \partial_T A_1 &= c_g(k_1)\partial_X A_1 + i\gamma_1 \overline{A_2 A_3}, \\ \partial_T A_2 &= c_g(k_2)\partial_X A_2 + i\gamma_2 \overline{A_1 A_3}, \\ \partial_T A_3 &= c_g(k_3)\partial_X A_3 + i\gamma_3 \overline{A_1 A_2}, \end{aligned} \tag{4.17}$$

with $T, X, c_g(k_j), \gamma_j \in \mathbb{R}$, and $A_j(X,T) \in \mathbb{C}$ for the three wave interaction can be derived by the multiple scaling ansatz

$$\begin{aligned} u_n &= \varepsilon A_1(\varepsilon n, \varepsilon t) e^{i(k_1 n - \omega_1 t)} g_{1n} + \varepsilon A_2(\varepsilon n, \varepsilon t) e^{i(k_2 n - \omega_2 t)} g_{2n} \\ &\quad + \varepsilon A_3(\varepsilon n, \varepsilon t) e^{i(k_3 n - \omega_3 t)} g_{3n} + c.c., \end{aligned} \tag{4.18}$$

with vectors $g_{j,n} \in \mathbb{C}$ from above, with $0 < \varepsilon \ll 1$ a small perturbation parameter, and the spatial and temporal wave numbers $k = k_j$ and $\omega = \omega_j$ which have to satisfy the resonance condition

$$k_1 + k_2 + k_3 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0, \tag{4.19}$$

and are related by the linear dispersion relation of the polyatomic FPU system.

In order to use the TWI approximation to construct a counter example that the NLS approximation can fail in making correct predictions about the diatomic FPU system, we have to find wave numbers k_1, k_2, and k_3 such that the following holds:

There exist integers n_1, n_2, and n_3, a wave number $k_* > 0$ such that $k = k_j = n_j k_*$ and $\omega = \omega_j$ satisfy

$$k_1 + k_2 + k_3 = k_*(n_1 + n_2 + n_3) = 0, \qquad \text{and} \qquad \omega_1 + \omega_2 + \omega_3 = 0,$$

and are related by the linear dispersion relation of the diatomic FPU system.

In order to do so we choose n_1, n_2, and n_3 and look for the zeroes of

$$\begin{aligned} f(k_*, a_{11}, a_{21}, m_1, m_2) &= \omega_1 + \omega_2 - \omega_3 \\ &= \sqrt{\frac{KL}{2} + \frac{1}{2}\sqrt{K^2L^2 - \frac{8a_{11}a_{21}}{m_1 m_2}\big(1 - \cos(2n_1 k_*)\big)}} \\ &\quad + \sqrt{\frac{KL}{2} + \frac{1}{2}\sqrt{K^2L^2 - \frac{8a_{11}a_{21}}{m_1 m_2}\big(1 - \cos(2n_2 k_*)\big)}} \\ &\quad - \sqrt{\frac{KL}{2} + \frac{1}{2}\sqrt{K^2L^2 - \frac{8a_{11}a_{21}}{m_1 m_2}\big(1 - \cos(2n_3 k_*)\big)}}, \end{aligned}$$

i.e., we have to find a basic wave number k_*, masses m_1 m_2 and the coefficients a_{11} and a_{21} such that $f(k_*, a_{11}, a_{21}, m_1, m_2) = 0$. It is easy to see that there are non trivial zeroes due to the intermediate value theorem. As an example we set $n_1 = 1$, $n_2 = 2$, $n_3 = -3$, $k_* = \pi/6$, $a_{11} = 1$, $m_1 = 1$, $a_{21} = 1/5$ and $m_2 = 1/17$ such that

$$f(\pi/6, 1, 1/5, 1, 1/17) \approx \sqrt{\frac{18000}{101}} + \sqrt{\frac{34400}{5467}} - \sqrt{\frac{900}{901}} > 0$$

and

$$f(\pi/6, 1, 1/3, 1, 1/8) \approx \sqrt{\frac{70^3 + 10^4}{1836}} + \sqrt{\frac{30^4 + 60^2}{401}} - \sqrt{\frac{560^6 + 480^7}{901}} < 0.$$

In Figure 4.2 we have plotted the zeroes of $f(k_*, 1, a_{21}, 1, m_2)$.

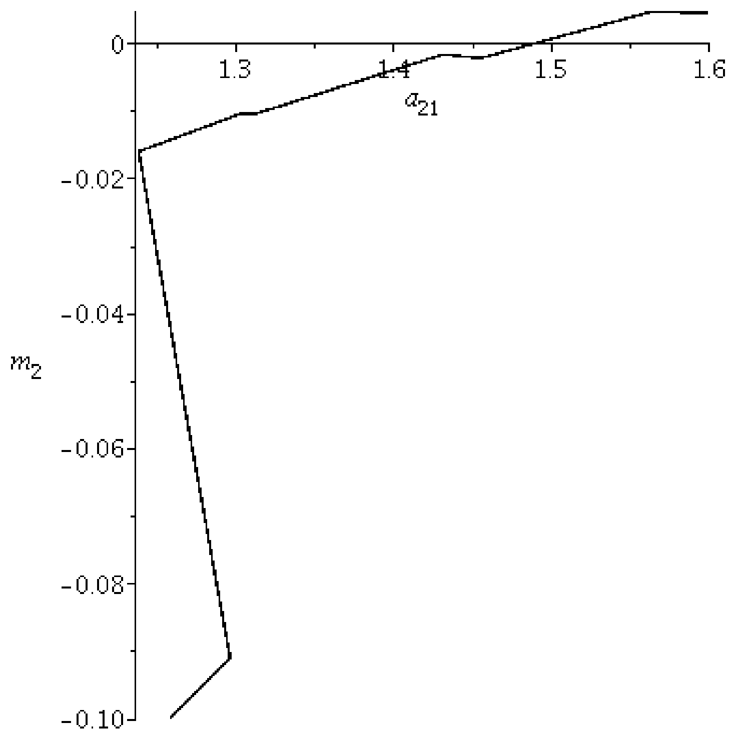

Figure 4.2: The set $\{(a_{21}, m_2) \in \mathbb{R}^2 : f(\pi/6, 1, a_{21}, 1, m_2) = 0\}$.

We have seen in Section 2.4. that for instance the A_1 subspace spanned by $(1, 0, 0)$ is unstable if $\gamma_2\gamma_3 > 0$ Moreover, we have

$$\frac{d}{dt}(\varrho_2|A_2|^2 + \varrho_3|A_3|^2) = -i(\varrho_2\gamma_2 + \varrho_3\gamma_3)(A_2A_3A_1 - \overline{A_2A_3A_1})$$

such that in systems which conserve the energy on the right hand side, i.e., $\rho_j > 0$ for $j = 1, 2, 3$, we have $\varrho_2\gamma_2 + \varrho_3\gamma_3 = 0$. Therefore, we must have two positive γ_j and one negative γ_j or vice versa.

The energy E_{FPU} of the FPU system, which can be written as sum of energy densities e_i of each of the particles in the chain,

$$E_{FPU} = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{u_i}^2 + \frac{1}{2}[W(u_{i+1} - u_i) + W(u_i - u_{i-1})]$$

is positive definite and conserved for (4.9) in the TWI system and the products $\gamma_1\gamma_2$, $\gamma_1\gamma_3$ and $\gamma_2\gamma_3$ do not have the same signs simultaneously ensuring the existence of at least one unstable subspace. We do not need to compute the gamma's again. As a consequence we always have one unstable subspace and hence we have our counter example.

Appendix A

Appendix Chapter 2

A.1 The extended approximation

A.1.1 Derivation of the extended TWI system

The water wave problem can be written as a first order system in the variables η and w. The solutions of the water wave problem can be represented as infinite Fourier series for the two components. The linear system possesses solutions $e^{i(jk_*x-\omega_{j,\pm}t)}\varphi_{j,\pm}$ with $\varphi_{j,\pm} \in \mathbb{C}^2$ for $j \in \mathbb{Z}$. In principle there are infinitely many possibilities for the three resonant wave numbers. However, to keep the notation on a reasonable level we choose like in the previous counter example k_*, $2k_*$, and $-3k_*$ as resonant wave numbers. Hence, the wave numbers and the eigenvectors are ordered in such a way that the resonant wave numbers are given by $\omega_j = \omega_{j,+}$ for $j = 1, 2, -3$ and that the associated eigenvectors are given by $\varphi_j = \varphi_{j,+}$ for $j = 1, 2, -3$.

For the derivation of the higher order TWI approximation we make the ansatz

$$\begin{aligned}
\begin{pmatrix} \eta \\ w \end{pmatrix}(x,t) \;=\;& \varepsilon\mathcal{A}_1(\varepsilon t)e^{i(k_*x-\omega_1 t)}\varphi_1 + \varepsilon\mathcal{A}_2(\varepsilon t)e^{i(2k_*x-\omega_2 t)}\varphi_2 \\
&\qquad +\varepsilon\mathcal{A}_3(\varepsilon t)e^{i(3k_*x-\omega_3 t)}\varphi_3 \\
&+\varepsilon^{1+\theta}\mathcal{A}_{1,-}(\varepsilon t)e^{i(k_*x-\omega_{1,-}t)}\varphi_{1,-} + \varepsilon^{1+\theta}\mathcal{A}_{2,-}(\varepsilon t)e^{i(2k_*x-\omega_{2,-}t)}\varphi_{2,-} \\
&\qquad +\varepsilon^{1+\theta}\mathcal{A}_{3,-}(\varepsilon t)e^{i(3k_*x-\omega_{3,-}t)}\varphi_{3,-} \\
&+\sum_{j=\pm}(\varepsilon^{1+\theta}\mathcal{A}_{4,j}(\varepsilon t)e^{i(4k_*x-\omega_{4,j}t)}\varphi_{4,j} + \varepsilon^{1+\theta}\mathcal{A}_{5,j}(\varepsilon t)e^{i(5k_*x-\omega_{5,j}t)}\varphi_{4,j} \\
&\qquad +\varepsilon^{1+\theta}\mathcal{A}_{6,j}(\varepsilon t)e^{i(6k_*x-\omega_{6,j}t)}\varphi_{6,j}) \\
&+\sum_{n=2}^{N}\sum_{p=1}^{3}\sum_{j=\pm}\varepsilon^{1+n\theta}\mathcal{A}_{3n+p,j}(\varepsilon t)e^{i((3n+p)k_*x-\omega_{3n+p,j}t)}\varphi_{3n+p,j}
\end{aligned}$$

$$+c.c.$$

where $\theta \in [0, 1/2)$ is chosen arbitrarily close to $1/2$ and $N \in \mathbb{N}$ is a later on suitably chosen number. If only estimates for all $T \in [0, T_0]$ are necessary then $\theta = 1$ would be the optimal choice. By equating the coefficients in front of $e^{i(lk_* x - \omega_{l,j} t)}\varphi_{l,j}$ to zero we find the following system of amplitude equations

$$\begin{aligned}
\partial_T \mathcal{A}_1 &= i\gamma_1 \mathcal{A}_{-2}\mathcal{A}_3 + \sum_{j=\pm} \varepsilon^{\theta} e^{-i(\omega_{4,j}+\omega_{-3}-\omega_1)T/\varepsilon} c_* \mathcal{A}_{-3}\mathcal{A}_{4,j} + \ldots, \\
\partial_T \mathcal{A}_2 &= i\gamma_2 \mathcal{A}_3\mathcal{A}_{-1} + \sum_{j=\pm} e^{-i(\omega_1+\omega_1-\omega_2)T/\varepsilon} c_* \mathcal{A}_1\mathcal{A}_1 \\
&\qquad + \varepsilon^{\theta} e^{-i(\omega_{4,j}+\omega_{-2}-\omega_2)T/\varepsilon} c_* \mathcal{A}_{-2}\mathcal{A}_{4,j} + \ldots, \\
\partial_T \mathcal{A}_3 &= -i\gamma_3 \mathcal{A}_1\mathcal{A}_2 + \sum_{j=\pm} \varepsilon^{\theta} e^{-i(\omega_{4,j}+\omega_{-1}-\omega_3)T/\varepsilon} c_* \mathcal{A}_{-1}\mathcal{A}_{4,j} + \ldots, \\
\partial_T \mathcal{A}_{4,j} &= \varepsilon^{-\theta} e^{-i(\omega_2+\omega_2-\omega_{4,j})T/\varepsilon} c_* \mathcal{A}_2\mathcal{A}_2 + \varepsilon^{-\theta} e^{-i(\omega_1+\omega_3-\omega_{4,j})T/\varepsilon} c_* \mathcal{A}_1\mathcal{A}_3 + \ldots, \\
&\vdots \\
\partial_T \mathcal{A}_{7,j} &= \sum_{l=\pm} (\varepsilon^{-\theta} e^{-i(\omega_3+\omega_{4,l}-\omega_{7,j})T/\varepsilon} c_* \mathcal{A}_3\mathcal{A}_{4,l} + \varepsilon^{-\theta} e^{-i(\omega_2+\omega_{5,l}-\omega_{7,})T/\varepsilon} c_* \mathcal{A}_2\mathcal{A}_{5,l} \\
&\qquad + \varepsilon^{-\theta} e^{-i(\omega_1+\omega_{6,l}-\omega_{7,j})T/\varepsilon} c_* \mathcal{A}_1\mathcal{A}_{6,l}) + \ldots, \\
&\vdots
\end{aligned}$$

where the many occurring coefficients are simply denoted with the same symbol c_*.

A.1.2 Estimates for the residual

The terms that remain at $e^{i(lk_* x + \omega_{l,j} t)}\varphi_{l,j}$ with $|l| > N$ form the so called residual. We choose N so large that formally $\mathrm{Res}(\varepsilon\psi) = \mathcal{O}(\varepsilon^{\beta+\delta})$. In order to prove the required estimates we have to show that the solutions of the system of amplitude equations stay $\mathcal{O}(1)$ bounded for all $T \in [0, \varepsilon^{-\widetilde{\mu}}]$ and not only for $T \in [0, T_0]$. In order to do so we define suitable energies, namely

$$\begin{aligned}
E &= E_{twi} + E_{rest}, \\
E_{twi} &= \sum_{k \in I_3} \rho_k |\mathcal{A}_k|^2, \\
E_{rest} &= \sum_{k \notin I_m \setminus I_3} \sum_{j=\pm} |\mathcal{A}_{k,j}|^2 + \sum_{k \in I_3} |\mathcal{A}_{k,-}|^2,
\end{aligned}$$

where $I_\nu = \{-\nu, -\nu+1, \cdots, -1, 1, \ldots, \nu-1, \nu\}$ and $m = 3(N+1)$. We find

$$\partial_T E_{twi} = \sum_{k \in I_3} \rho_k (\mathcal{A}_k \overline{\partial_T \mathcal{A}_k} + \overline{\mathcal{A}_k} \partial_T \mathcal{A}_k)$$

$$
\begin{aligned}
&= (\rho_1\gamma_1+\rho_2\gamma_2+\rho_3\gamma_3)(\mathcal{A}_1\mathcal{A}_2\mathcal{A}_{-3}+\overline{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_{-3}}) \\
&\quad +(e^{-i(\omega_1+\omega_1-\omega_2)T/\varepsilon}c_*\mathcal{A}_1\mathcal{A}_1\mathcal{A}_{-2}+\dots \\
&\quad +\sum_{l=\pm}\varepsilon^{\theta}(e^{-i(\omega_3+\omega_{-4,l}-\omega_{-1})T/\varepsilon}c_*\mathcal{A}_1\mathcal{A}_3\mathcal{A}_{-4,l} \\
&\qquad\qquad +e^{-i(\omega_{-3}+\omega_{4,l}-\omega_1)T/\varepsilon}c_*\mathcal{A}_{-1}\mathcal{A}_{-3}\mathcal{A}_{4,l})+\dots, \\
\partial_T E_{rest} &= \sum_{k\notin I_m\setminus I_3}\sum_{l=\pm}(\mathcal{A}_{k,l}\overline{\partial_T\mathcal{A}_{k,l}}+\overline{\mathcal{A}_{k,l}}\partial_T\mathcal{A}_{k,l})+\dots \\
&= \sum_{l=\pm}(\mathcal{A}_{4,l}\overline{\partial_T\mathcal{A}_{4,l}}+\overline{\mathcal{A}_{4,l}}\partial_T\mathcal{A}_{4,l})+\dots+(\mathcal{A}_{7,l}\overline{\partial_T\mathcal{A}_{7,l}}+\overline{\mathcal{A}_{7,l}}\partial_T\mathcal{A}_{7,l})+\dots \\
&= \sum_{l=\pm}\varepsilon^{-\theta}(e^{-i(\omega_{-2}+\omega_{-2}-\omega_{4,l})T/\varepsilon}c_*\mathcal{A}_{-2}\mathcal{A}_{-2}\mathcal{A}_{-4,l} \\
&\qquad\qquad +e^{-i(\omega_2+\omega_2-\omega_{-4,l})T/\varepsilon}c_*\mathcal{A}_2\mathcal{A}_2\mathcal{A}_{4,l})+\dots
\end{aligned}
$$

We choose $\rho_j>0$ such that $\rho_1\gamma_1+\rho_2\gamma_2+\rho_3\gamma_3=0$. Integration w.r.t. time and using partial integration then yields

$$
\begin{aligned}
E_{twi}(T) &= E_{twi}(0)+c_*\int_0^T(e^{-i(\omega_1+\omega_1-\omega_2)T/\varepsilon}\mathcal{A}_1(s)\mathcal{A}_1(s)\mathcal{A}_{-2}(s)ds+\dots) \\
&\quad +\sum_{l=\pm}(c_*\varepsilon^{\theta}\int_0^T e^{-i(\omega_3+\omega_{-4,l}-\omega_{-1})s/\varepsilon}\mathcal{A}_1(s)\mathcal{A}_3(s)\mathcal{A}_{-4,l}(s)ds \\
&\qquad +c_*\varepsilon^{\theta}\int_0^T e^{-i(\omega_{-3}+\omega_{4,l}-\omega_1)s/\varepsilon}\mathcal{A}_{-1}(s)\mathcal{A}_{-3}(s)\mathcal{A}_{4,l}(s)ds+\dots) \\
&= E_{twi}(0)-c_*\varepsilon i(\omega_1+\omega_1-\omega_2)^{-1} \\
&\qquad\qquad \times e^{i(\omega_1+\omega_1-\omega_2)s/\varepsilon}\mathcal{A}_1(s)\mathcal{A}_1(s)\mathcal{A}_{-2}(s)|_0^T \\
&\qquad +c_*\varepsilon^1 i(\omega_1+\omega_1-\omega_2)^{-1} \\
&\qquad\qquad \times\int_0^T(e^{-i(\omega_1+\omega_1-\omega_2)s/\varepsilon}\partial_s(\mathcal{A}_1(s)\mathcal{A}_1(s)\mathcal{A}_{-2}(s))ds+\dots) \\
&\quad +\sum_{l=\pm}(-c_*\varepsilon^{1+\theta}i(\omega_3+\omega_{-4,l}-\omega_{-1})^{-1} \\
&\qquad\qquad \times e^{i(\omega_3+\omega_{-4,l}-\omega_{-1})s/\varepsilon}\mathcal{A}_1(s)\mathcal{A}_3(s)\mathcal{A}_{-4,l}(s)|_0^T \\
&\qquad +c_*\varepsilon^{1+\theta}i(\omega_3+\omega_{-4,l}-\omega_{-1})^{-1} \\
&\qquad\qquad \times\int_0^T(e^{-i(\omega_3+\omega_{-4,l}-\omega_{-1})s/\varepsilon}\partial_s(\mathcal{A}_1(s)\mathcal{A}_3(s)\mathcal{A}_{-4,l}(s))ds+\dots) \\
&\leq E_{twi}(0)+C\varepsilon(E_{twi}^{1/2}(T)E_{twi}^{1/2}(T)E_{rest}^{1/2}(T)+\dots)
\end{aligned}
$$

with an ε-independent constant C since for instance $\partial_s \mathcal{A}_{1,2} = \mathcal{O}(1)$ and $\partial_s \mathcal{A}_{-4,l} = \mathcal{O}(\varepsilon^{-\theta})$. Similarly we obtain

$$\begin{aligned}
& E_{rest}(T) \\
= \; & E_{rest}(0) + \sum_{l=\pm} (\varepsilon^{-\theta} \int_0^T e^{-i(\omega_2+\omega_2-\omega_{4,l})s/\varepsilon} c_* \mathcal{A}_2(s)\mathcal{A}_2(s)\mathcal{A}_{-4,l}(s))ds \\
& \varepsilon^{-\theta} \int_0^T e^{-i(\omega_{-2}+\omega_{-2}-\omega_{-4,l})s/\varepsilon} c_* \mathcal{A}_{-2}(s)\mathcal{A}_{-2}(s)\mathcal{A}_{4,l}(s)ds + \ldots) + \ldots \\
= \; & E_{rest}(0) + \sum_{l=\pm} (-\varepsilon^{1-\theta} i(\omega_2+\omega_2-\omega_{4,l})^{-1} e^{-i(\omega_2+\omega_2-\omega_{4,l})s/\varepsilon} c_* \mathcal{A}_2(s)\mathcal{A}_2(s)\mathcal{A}_{-4,l}(s)|_0^T \\
& + \varepsilon^{1-\theta} i(\omega_2+\omega_2-\omega_{4,l})^{-1} \int_0^T (e^{-i(\omega_2+\omega_2-\omega_{4,l})s/\varepsilon} \partial_s (c_* \mathcal{A}_2(s)\mathcal{A}_2(s)\mathcal{A}_{-4,l}(s))ds + \ldots) + \ldots \\
\leq \; & E_{rest}(0) + C(\varepsilon^{1-2\theta} E_{twi}^{1/2}(T) E_{twi}^{1/2}(T) E_{rest}^{1/2}(T) + \ldots)
\end{aligned}$$

with an ε-independent constant C since again $\partial_s \mathcal{A}_{4,l} = \mathcal{O}(\varepsilon^{-\theta})$, etc.. With

$$B_{twi}(T) = \sup_{T' \in [0,T]} E_{twi}(T'), \qquad B_{rest}(T) = \sup_{T' \in [0,T]} E_{rest}(T')$$

we obtain

$$\begin{aligned}
B_{twi}(T) & \leq E_{twi}(0) + C\varepsilon T(B_{twi}(T) + B_{rest}(T))^{3/2}, \\
B_{rest}(T) & \leq E_{rest}(0) + C\varepsilon^{1-2\theta} T(B_{twi}(T) + B_{rest}(T))^{3/2}.
\end{aligned}$$

Adding the two inequalities yields

$$B_{twi}(T) + B_{rest}(T) \leq E_{twi}(0) + E_{rest}(0) + 2C\varepsilon^{1-2\theta} T(B_{twi}(T) + B_{rest}(T))^{3/2}$$

and so

$$B_{twi}(T) + B_{rest}(T) \leq 2(E_{twi}(0) + E_{rest}(0))$$

for $0 \leq T \leq \varepsilon^{2\theta-1+\vartheta}$ for an arbitrary small, but fixed $\vartheta > 0$ if we choose $\varepsilon_0 > 0$ so small that

$$2^{5/2} C \varepsilon_0^{\vartheta} (E_{twi}(0) + E_{rest}(0))^{1/2} < 1.$$

Hence the solutions of the system of approximation equations stay $\mathcal{O}(1)$ bounded for all $T \in [0, \varepsilon^{2\theta-1+\vartheta}]$ and not only for $T \in [0, T_0]$. As a consequence all terms remaining in the residual can be estimated for all $T \in [0, \varepsilon^{2\theta-1+\vartheta}]$ by purely counting powers of ε. Since we have chosen N so large that all terms up to order $\mathcal{O}(\varepsilon^{\beta+\delta})$ cancel, we are done.

Remark: Again we oversimplified the presentation in the sense that [SW03, Theorem 4.1] and its proof are formulated for the Lagrangian formulation of the water wave problem and the residual should have been made small in the Lagrangian formulation as already been said in the proof of Theorem 2.5.2. The transfer is straightforward and we leave it to the reader.

A.1.3 Qualitative behavior of solutions of the extended TWI system

The amplitudes $\mathcal{A}_j$ for $j \in \{1,2,3\}$ satisfy a perturbed TWI system, namely the first three equations of (2.19),

$$\begin{aligned}
\partial_T \mathcal{A}_1 &= i\gamma_1 \mathcal{A}_{-2}\mathcal{A}_3 + \sum_{j=\pm} \varepsilon^\theta e^{-i(\omega_{4,j}+\omega_{-3}-\omega_1)T/\varepsilon} c_* \mathcal{A}_{-3}\mathcal{A}_{4,j} + \dots, \\
\partial_T \mathcal{A}_2 &= i\gamma_2 \mathcal{A}_3\mathcal{A}_{-1} + \sum_{j=\pm} e^{-i(\omega_1+\omega_1-\omega_2)T/\varepsilon} c_* \mathcal{A}_1\mathcal{A}_1 \\
&\qquad + \varepsilon^\theta e^{-i(\omega_{4,j}+\omega_{-2}-\omega_2)T/\varepsilon} c_* \mathcal{A}_{-2}\mathcal{A}_{4,j} + \dots, \\
\partial_T \mathcal{A}_3 &= -i\gamma_3 \mathcal{A}_1\mathcal{A}_2 + \sum_{j=\pm} \varepsilon^\theta e^{-i(\omega_{4,j}+\omega_{-1}-\omega_3)T/\varepsilon} c_* \mathcal{A}_{-1}\mathcal{A}_{4,j} + \dots .
\end{aligned}$$

In general the solutions of the classical TWI system (2.15) and the extended TWI system (2.19) will differ by an $\mathcal{O}(1)$-amount for $T = \widetilde{T}$ and so quantitatively the classical TWI system (2.15) is not a good approximation of the extended TWI system (2.19). However, qualitatively for the solutions we are interested in, the classical TWI system (2.15) and the extended TWI system (2.19) show the same behavior. For showing this, we integrate the first two equations of (2.19) w.r.t. T, and after some partial integration we obtain

$$\begin{aligned}
\mathcal{A}_1(T) &= \mathcal{A}_1(0) + i\gamma_1 \int_0^T \mathcal{A}_{-2}(s)\mathcal{A}_3(s)ds \\
&\quad + \sum_{j=\pm} \int_0^T \varepsilon^{1+\theta}(\omega_{4,j}+\omega_{-3}-\omega_1)^{-1} e^{-i(\omega_{4,j}+\omega_{-3}-\omega_1)s/\varepsilon} c_* \partial_s(\mathcal{A}_{-3}(s)\mathcal{A}_{4,j}(s)) + \dots ds, \\
&\quad - \sum_{j=\pm} \varepsilon^{1+\theta}(\omega_{4,j}+\omega_{-3}-\omega_1)^{-1} e^{-i(\omega_{4,j}+\omega_{-3}-\omega_1)s/\varepsilon} c_* (\mathcal{A}_{-3}(s)\mathcal{A}_{4,j}(s)) + \dots |_{s=0}^T, \\
\mathcal{A}_2(T) &= \mathcal{A}_2(0) + i\gamma_2 \int_0^T \mathcal{A}_3(s)\mathcal{A}_{-1}(s)ds \\
&\quad + \sum_{j=\pm} \int_0^T \varepsilon(\omega_1+\omega_1-\omega_2)^{-1} e^{-i(\omega_1+\omega_1-\omega_2)s/\varepsilon} c_* \partial_s(\mathcal{A}_1(s)\mathcal{A}_1(s))ds \\
&\quad - \sum_{j=\pm} \varepsilon(\omega_1+\omega_1-\omega_2)^{-1} e^{-i(\omega_1+\omega_1-\omega_2)s/\varepsilon} c_* (\mathcal{A}_1(s)\mathcal{A}_1(s))|_{s=0}^T, \\
&\quad + \sum_{j=\pm} \int_0^T \varepsilon^{1+\theta}(\omega_{4,j}+\omega_{-2}-\omega_2)^{-1} e^{-i(\omega_{4,j}+\omega_{-2}-\omega_2)s/\varepsilon} c_* \partial_s(\mathcal{A}_{-2}(s)\mathcal{A}_{4,j}(s)) + \dots ds \\
&\quad - \sum_{j=\pm} \varepsilon^{1+\theta}(\omega_{4,j}+\omega_{-2}-\omega_2)^{-1} e^{-i(\omega_{4,j}+\omega_{-2}-\omega_2)s/\varepsilon} c_* \mathcal{A}_{-2}(s)\mathcal{A}_{4,j}(s) + \dots |_{s=0}^T,
\end{aligned}$$

Since for instance $\partial_s \mathcal{A}_1(s) = \mathcal{O}(1)$ and $\partial_s \mathcal{A}_4(s) = \mathcal{O}(\varepsilon^{-\theta})$ we have

$$\begin{aligned}
\mathcal{A}_1(T) &= \mathcal{A}_1(0) + i\gamma_1 \int_0^T \mathcal{A}_{-2}(s)\mathcal{A}_3(s)ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T), \\
\mathcal{A}_2(T) &= \mathcal{A}_2(0) + i\gamma_2 \int_0^T \mathcal{A}_3(s)\mathcal{A}_{-1}(s)ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T),
\end{aligned}$$

uniformly for all $T \in [0, \varepsilon^{2\theta-1+\mu}]$. As before we introduce variables $\mathcal{B}_j$ by

$$\mathcal{A}_1 = -i\mathcal{B}_1/\sqrt{|\gamma_2\gamma_3|}, \qquad \mathcal{A}_2 = -i\mathcal{B}_2/\sqrt{|\gamma_1\gamma_3|}, \qquad \text{and} \qquad \mathcal{A}_3 = -i\mathcal{B}_3/\sqrt{|\gamma_1\gamma_2|}$$

which satisfy

$$\mathcal{B}_1(T) = \mathcal{B}_1(0) + \int_0^T \mathcal{B}_{-2}(s)\mathcal{B}_3(s)ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T), \tag{A.1}$$

$$\mathcal{B}_2(T) = \mathcal{B}_2(0) + \int_0^T \mathcal{B}_3(s)\mathcal{B}_{-1}(s)ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T). \tag{A.2}$$

To construct our counter example we start the extended TWI system (2.19) with

$$\mathcal{B}_1|_{T=0} = \varepsilon^{1/2}, \qquad \mathcal{B}_2|_{T=0} = \varepsilon^{1/2}, \qquad \text{and} \qquad \mathcal{B}_3|_{T=0} = 1.$$

All other initial conditions we assume to vanish.

Remark A.1.1
Before we go on we would like to remark that the estimates in the last section easily imply

$$|E_{twi}(T) - E_{twi}(0)| \leq 2^{3/2} C\varepsilon T (E_{twi}(0) + E_{rest}(0))^{3/2}. \tag{A.3}$$

As a consequence if we assume that $\mathcal{B}_1(T)$ and $\mathcal{B}_2(T)$ remain of order $\mathcal{O}(\varepsilon^{1/2})$ for all $T \in [0, \varepsilon^{2\theta-1+\mu}]$ then

$$\mathcal{B}_3(T) = 1 + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T)$$

for all $T \in [0, \varepsilon^{2\theta-1+\mu}]$. This immediately leads to a contradiction as can be seen as follows. The quantity $r(T) = \mathcal{B}_1(T) + \mathcal{B}_2(T) + \mathcal{B}_{-1}(T) + \mathcal{B}_{-2}(T)$ then satisfies

$$\begin{aligned}
r(T) &= 4\varepsilon^{1/2} + \int_0^T r(s)ds + \int_0^T (\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T))ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T) \\
&= 4\varepsilon^{1/2} + \int_0^T r(s)ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T) + \mathcal{O}(\varepsilon T^2).
\end{aligned}$$

The first two terms lead to the desired growth. The last three terms in the worst case will stop the solutions to grow. We find

$$r(T) \;=\; 4\varepsilon^{1/2}e^{T} + \int_0^T e^{T-s}(\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon s) + \mathcal{O}(\varepsilon s^2))ds$$

We recall that $\varepsilon^{1/2}e^{\widetilde{T}} = 1/2$, that $\widetilde{T} = \mathcal{O}(|\ln(\varepsilon)|)$, and find

$$|r(\widetilde{T})| \;\geq\; 4\varepsilon^{1/2}e^{\widetilde{T}} - C\varepsilon e^{\widetilde{T}}(1 + \widetilde{T} + \widetilde{T}^2) \geq 2 - C\varepsilon^{1/2}(1 + \widetilde{T} + \widetilde{T}^2) \geq 1.$$

Then either $\mathcal{B}_1(\widetilde{T})$ or $\mathcal{B}_2(\widetilde{T})$ are of order $\mathcal{O}(1)$ contradicting our assumption.

■

The previous remark gives some first insight into the problem and allows to conclude by a contradiction argument that $\mathcal{B}_{1,2}(\widetilde{T}) \gg \mathcal{O}(\varepsilon^{1/2})$, but it does not allow to conclude $\mathcal{B}_{1,2}(\widetilde{T}) = \mathcal{O}(1)$. In order to prove this statement we proceed as follows.

We split $\mathcal{B}_j(T) = \Re\mathcal{B}_j(T) + i\Im\mathcal{B}_j(T)$ in real and imaginary part. The quantity $r(T)$ from the previous remark then satisfies

$$r(T) \;=\; 4\varepsilon^{1/2} + \int_0^T (\Re\mathcal{B}_3(s))r(s)ds + q(T) + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T)$$

with

$$q(T) \;=\; i\int_0^T (\Im\mathcal{B}_3(s))(\mathcal{B}_{-2}(s) + \mathcal{B}_{-1}(s) - \mathcal{B}_1(s) - \mathcal{B}_2(s))ds.$$

The factor $\Re\mathcal{B}_3(s)$ will be close to 1 and will lead to exponential growth rates. The last three terms in the worst case will stop the solutions to grow. In order to bound $q(t)$ we have to bound $\Im\mathcal{B}_3(s)$ which together with all other $\Im\mathcal{B}_j(s)$ is initially zero. Gronwall's inequality then applied to (A.1) and (A.2) and to the imaginary part of (A.1) and (A.2) yields

$$\begin{aligned}
|\mathcal{B}_1(T)| + |\mathcal{B}_2(T)| \;&\leq\; C(\varepsilon^{1/2} + \varepsilon T + \varepsilon T^2)e^{T} \leq C\varepsilon^{1/2}e^{T},\\
|\Im\mathcal{B}_1(T)| + |\Im\mathcal{B}_2(T)| \;&\leq\; C(\varepsilon T + \varepsilon T^2)e^{T},
\end{aligned}$$

where we used $|\mathcal{B}_3(T)| \approx 1 \leq 2$. Here and in the following all constants which can be chosen independently of the small bifurcation parameter $0 < \varepsilon \ll 1$ are denoted with the same symbol C. Similarly as in Remark A.1.1, we find

$$\mathcal{B}_3(T) = \mathcal{B}_3(0) - \int_0^T \mathcal{B}_1(s)\mathcal{B}_2(s)ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T).$$

As a consequence the imaginary part satisfies

$$\Im\mathcal{B}_3(T) = -\int_0^T (\Im\mathcal{B}_1(s))(\Re\mathcal{B}_2(s))ds - \int_0^T (\Re\mathcal{B}_1(s))(\Im\mathcal{B}_2(s))ds + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T).$$

Therefore, it obeys the estimates

$$|\Im\mathcal{B}_3(T)| \leq C(\varepsilon^{3/2}(1+T^2)e^{2T} + \varepsilon(1+T^2)).$$

Assume now that $|\mathcal{B}_3(T) - 1| \leq 1/10$ for all $T \in [0, 2\tilde{T}]$. If this is not satisfied due to (A.3) we would be done. Under this assumption we obtain

$$|q(T)| \leq CT \times \varepsilon^{1/2}e^T \times \varepsilon^{3/2}(1+T^2)e^{2T}.$$

Hence finally

$$r(T) \;=\; 4\varepsilon^{1/2} + \int_0^T (\Re\mathcal{B}_3(s))r(s)ds + \mathcal{O}(\varepsilon^2(T+T^3)e^{3T}) + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon T).$$

The first two terms lead to the desired growth. The last three terms in the worst case will stop the solutions to grow. We find

$$r(T) \;=\; 4\varepsilon^{1/2}e^T + \int_0^T e^{T-s}\mathcal{O}(\varepsilon^2(s+s^3)e^{3s}) + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon s))ds$$

and so since $\Re\mathcal{B}_3(T) > 0.9$ finally

$$\begin{aligned} |r(10\widetilde{T}/9)| &\geq 4\varepsilon^{1/2}e^{\widetilde{T}} - 2\varepsilon^2 e^{30\widetilde{T}/9}(1+\widetilde{T}^2+\widetilde{T}^4) - 2\varepsilon e^{10\widetilde{T}/9}(1+\widetilde{T}+\widetilde{T}^2) \\ &\geq 2 - 2\varepsilon^{3/9}(1+\widetilde{T}^2+\widetilde{T}^4) - 2\varepsilon^{4/9}(1+\widetilde{T}+\widetilde{T}^2) \geq 1 \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. This contradicts the assumption $|\mathcal{B}_3(T) - 1| \leq 1/10$ and so there exists a $\widehat{T} \in [0, 2\tilde{T}]$ where either $\mathcal{B}_1(\widehat{T})$ or $\mathcal{B}_2(\widehat{T})$ are of order $\mathcal{O}(1)$.

A.2 The TWI coefficients

A.2.1 The case of infinite depth

The water wave problem in infinite depth differs from the one in finite depth h only by replacing the symbol of the linearized Dirichlet-Neumann operator $\tanh(hk)$ by $\mathrm{sign}(k)$, cf. [SW03]. This simplifies the equations of water wave problem at various points and so we expect that it is rather straightforward to transfer [SW03, Theorem 1.1] respectively Theorem 2.5.2 from finite to infinite depth.

The linear dispersion relation of the water wave problems with infinite depth is given by

$$\omega^2 = (k + \sigma k^3)\mathrm{sign}(k) \tag{A.4}$$

and so the function f in Section 2.3 has to be replaced by

$$\begin{aligned} f(k_*, \sigma) \quad = \quad & \sqrt{(n_1 k_* + \sigma(n_1 k_*)^3)\mathrm{sign}(n_1 k_*)} + \sqrt{(n_2 k_* + \sigma(n_2 k_*)^3)\mathrm{sign}(n_2 k_*)} \\ & - \sqrt{(n_3 k_* + \sigma(n_3 k_*)^3)\mathrm{sign}(n_3 k_*)}. \end{aligned}$$

Again as an example we have for $n_1 = 1$, $n_2 = 2$, and $n_3 = -3$ that

$$f(10, 0) \approx \sqrt{10} + \sqrt{20} - \sqrt{30} > 0$$

and

$$f(10, 1) \approx \sqrt{1000} + \sqrt{8000} - \sqrt{27000} < 0$$

such that a zero exists due to the intermediate value theorem. In Figure A.1 we have plotted the zeroes of $f(k_0, \sigma)$ for two different cases of n_1, n_2, and n_3.

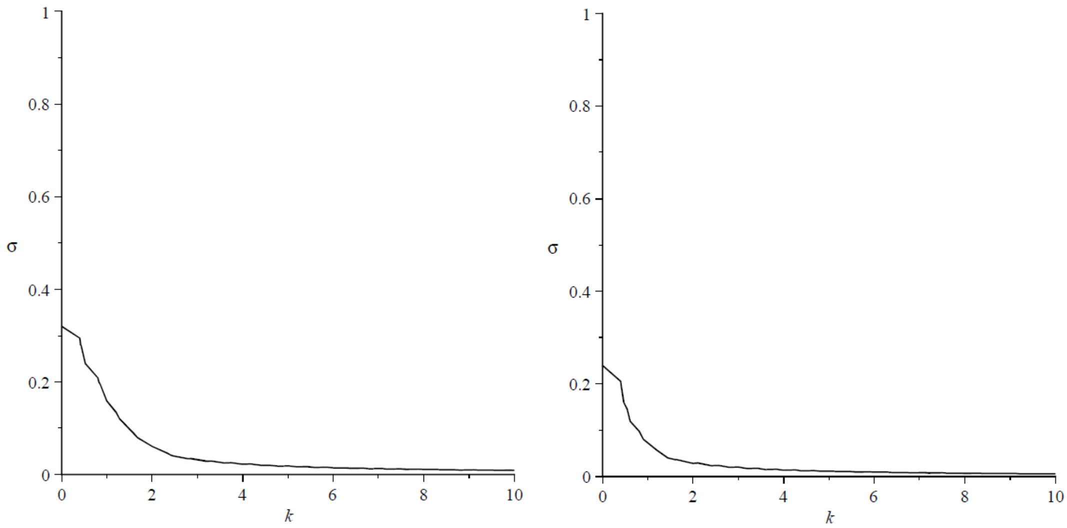

Figure A.1: The set $\{(k_*, \sigma) \in \mathbb{R}^2 : f(k_*, \sigma) = 0\}$ for $n_1 = 1$, $n_2 = 2$, $n_3 = -3$ in the left panel and for $n_1 = 1$, $n_2 = 5$, $n_3 = -6$. in the right panel in case of infinite depth.

These curves are typical and in infinite depth such zeroes exist for all $\sigma > 0$. In case of finite depth qualitatively the same behavior occurs. Since in the definition of f we have either $\mathrm{sign}(\cdot)$ functions or $\tanh(h\cdot)$ functions for hk_* large the functions in finite and infinite depth almost show no difference. For hk_* small the difference is visible.

A.2.2 The TWI coefficients in case of infinite depth

Throughout the next two sections assume $k_j, \omega_j \in \mathbb{R}$ for $j = 1, 2, 3$ so that the linear dispersion relation (A.4) and

$$k_1 + k_2 + k_3 = 0, \quad \omega_1 + \omega_2 + \omega_3 = 0 \tag{A.5}$$

are satisfied with $0 < k_1 \leq k_2 < |k_3|$ and $\omega_1 > 0$, $\omega_2 > 0$, $\omega_3 < 0$. We start with the water wave problem in case of infinite depth where in [CC77] the following explicit formulas for the coefficients γ_j appearing in the TWI system can be found. We have

$$\begin{aligned}\gamma_1 = \frac{1}{2}\{&|k_1|\omega_1^{-1}(\omega_2^2 + \omega_3^2) + (\omega_3|k_3| + \omega_2|k_2| + |k_1|\omega_2\omega_3\omega_1^{-1})\\ &+ k_2k_3[\omega_2|k_2|^{-1} + \omega_3|k_3|^{-1} - \omega_2\omega_3|k_1|((|k_2||k_3|\omega_1)^{-1}]\},\end{aligned}$$

$$\begin{aligned}\gamma_2 = \frac{1}{2}\{&|k_2|\omega_2^{-1}(\omega_1^2 + \omega_3^2) + (\omega_1|k_1| + \omega_3|k_3| + |k_2|\omega_1\omega_3\omega_2^{-1})\\ &+ k_1k_3[\omega_1|k_1|^{-1} + \omega_3|k_3|^{-1} - \omega_1\omega_3|k_2|((|k_1||k_3|\omega_2)^{-1}]\},\end{aligned}$$

$$\begin{aligned}\gamma_3 = \frac{1}{2}\{&|k_3|\omega_3^{-1}(\omega_1^2 + \omega_2^2) + (\omega_1|k_1| + \omega_2|k_2| + |k_3|\omega_1\omega_2\omega_3^{-1})\\ &+ k_1k_2[\omega_2|k_2|^{-1} + \omega_1|k_1|^{-1} - \omega_1\omega_2|k_3|((|k_1||k_2|\omega_3)^{-1}]\}.\end{aligned}$$

As pointed out in Section 2.4 these coefficients determine the stability of the A_3, A_2, and A_1 subspace in the TWI system, respectively. The formulas from [CC77] for the coefficients γ_j can be simplified strongly in the two-dimensional case. To our knowledge these simplifications are not documented in the literature so far, and they will allow us to analyse the signs of the γ_js immediately.

Rather than working with the γ_j we consider $\delta_j = 2|k_1||k_2||k_3|\frac{\omega_j}{|k_j|}\gamma_j$ which allows us to eliminate the denominators in the above expressions. We find

$$\begin{aligned}\delta_1 &= |k_1||k_2||k_3|(\omega_2^2 + \omega_3^2) + |k_2||k_3|^2\omega_1\omega_3 + |k_2|^2|k_3|\omega_1\omega_2 + |k_1||k_2||k_3|\omega_2\omega_3\\ &\quad + k_2 \cdot k_3|k_3|\omega_1\omega_2 + k_2 \cdot k_3|k_2|\omega_1\omega_3 - |k_1|k_2 \cdot k_3\omega_2\omega_3,\\ \delta_2 &= |k_1||k_2||k_3|(\omega_1^2 + \omega_3^2) + |k_1||k_3|^2\omega_2\omega_3 + |k_1|^2|k_3|\omega_1\omega_2 + |k_1||k_2||k_3|\omega_1\omega_3\\ &\quad + k_1 \cdot k_3|k_3|\omega_1\omega_2 + k_1 \cdot k_3|k_1|\omega_2\omega_3 - |k_2|k_1 \cdot k_3\omega_1\omega_3,\\ \delta_3 &= |k_1||k_2||k_3|(\omega_1^2 + \omega_2^2) + |k_1||k_2|^2\omega_2\omega_3 + |k_1|^2|k_2|\omega_1\omega_3 + |k_1||k_2||k_3|\omega_1\omega_2\\ &\quad + k_1 \cdot k_2|k_1|\omega_2\omega_3 + k_1 \cdot k_2|k_2|\omega_1\omega_3 - |k_3|k_1 \cdot k_2\omega_1\omega_2.\end{aligned}$$

Recall that $k_1, k_2 > 0 > k_3$ and $\omega_1, \omega_2 > 0 > \omega_3$. Set $K_j = |k_j|$ and $\Omega_j = |\omega_j|$ for $j = 1, 2, 3$. The resonance condition becomes

$$K_1 + K_2 = K_3 \quad \text{and} \quad \Omega_1 + \Omega_2 = \Omega_3.$$

Then we can write

$$\begin{aligned}\delta_1 &= K_1K_2K_3(\Omega_2^2+\Omega_3^2) - K_2K_3^2\Omega_1\Omega_3 + K_2^2K_3\Omega_1\Omega_2 - K_1K_2K_3\Omega_2\Omega_3 \\ &\quad - K_2K_3^2\Omega_1\Omega_2 + K_2^2K_3\Omega_1\Omega_3 - K_1K_2K_3\Omega_2\Omega_3 \\ &= K_1K_2K_3(\Omega_2^2 - 2\Omega_2\Omega_3 + \Omega_3^2) + K_2K_3\Omega_1(K_2\Omega_2 - K_3\Omega_2 + K_2\Omega_3 - K_3\Omega_3) \\ &= K_1K_2K_3(\Omega_2-\Omega_3)^2 + (K_2-K_3)K_2K_3\Omega_1(\Omega_2+\Omega_3) \\ &= K_1K_2K_3\Omega_1(\Omega_1-\Omega_3-\Omega_2) \\ &= -2K_1K_2K_3\Omega_1\Omega_2.\end{aligned}$$

So that finally we arrive at

$$\gamma_1 = -2|k_1||k_2||k_3||\omega_1||\omega_2| \cdot \frac{1}{2|k_2||k_3|\omega_1} = -|k_1|\mathrm{sign}(\omega_1)|\omega_2|.$$

Since for δ_2 only the roles of k_1 and k_2 as well as those of ω_1 and ω_2 have to be exchanged and $\mathrm{sign}(k_1) = \mathrm{sign}(k_2)$ as well as $\mathrm{sign}(\omega_1) = \mathrm{sign}(\omega_2)$, we have after the same computation that

$$\gamma_2 = -|k_2|\mathrm{sign}(\omega_2)|\omega_1|.$$

The computations for δ_3 are different to the above since the distribution of the signs is different. We have

$$\begin{aligned}\delta_3 &= K_1K_2K_3(\Omega_1^2+\Omega_2^2) - K_1^2K_2\Omega_1\Omega_3 - K_1K_2^2\Omega_2\Omega_3 + K_1K_2K_3\Omega_1\Omega_2 \\ &\quad - K_1^2K_2\Omega_2\Omega_3 - K_1K_2^2\Omega_1\Omega_3 - K_1K_2K_3\Omega_1\Omega_2 \\ &= K_1K_2K_3(\Omega_1^2+\Omega_2^2) - K_1K_2\Omega_3(K_1\Omega_1 + K_2\Omega_2 + K_1\Omega_2 + K_2\Omega_1) \\ &= K_1K_2K_3(\Omega_1^2+\Omega_2^2) - K_1K_2(K_1+K_2)(\Omega_1+\Omega_2)\Omega_3 \\ &= K_1K_2K_3(\Omega_1^2+\Omega_2^2-\Omega_3^2) \\ &= -2K_1K_2K_3\Omega_1\Omega_2\end{aligned}$$

and so $\delta_1 = \delta_2 = \delta_3$. This leads to

$$\gamma_3 = -\frac{|k_3||\omega_1||\omega_2|}{|\omega_3|}\mathrm{sign}(\omega_3).$$

So, in addition to these vastly simplified expressions we get the result that $\mathrm{sign}(\gamma_j) = -\mathrm{sign}(\omega_j)$. This guarantees that the γ_j cannot all have the same sign when the resonance condition $\omega_1+\omega_2+\omega_3 = 0$ is satisfied. Moreover, the A_3 subspace associated to k_3 is unstable.

A.2.3 The TWI coefficients in case of finite depth

Since the water wave problem in infinite depth approximates the water wave problem with large, but finite depth h, the coefficients γ_j that occur in finite depth $h \gg 1$ and infinite depth are close together if hk_* is large. Similar to [CC77] the coefficients γ_j appearing in the TWI system can be computed in case of finite depth h in a straightforward manner, too. Make the ansatz

$$\begin{aligned}\phi(x,y,t) &= \sum_{j=1}^{3} \omega_j |k_j|^{-1} (iP_j(t)e^{\xi_j} - i\overline{P_j(t)}e^{-\xi_j})\psi_j(y), \\ \eta(x,t) &= \sum_{j=1}^{3} (P_j(t)e^{\xi_j} + \overline{P_j(t)}e^{-\xi_j}),\end{aligned}$$

where $\xi_j = i(k_j x + \omega_j t)$ and

$$\psi_j(y) = \frac{\cosh(|k_j|(z+h))}{\sinh(|k_j|h)}.$$

After some lengthy, but straightforward calculation, we obtain

$$\gamma_j = \frac{|k_j|}{\omega_j} \cdot \frac{1}{2|k_1||k_2||k_3|}\delta_j^h,$$

with

$$\begin{aligned}\delta_1^h = &-|k_1|\omega_2\omega_3(|k_2||k_3| + \coth(|k_2|h)\coth(|k_3|h)k_2 \cdot k_3) \\ &- |k_2|\omega_1\omega_3(|k_1||k_3| + \coth(|k_3|h)k_1 \cdot k_3) \\ &- |k_3|\omega_1\omega_2(|k_1||k_2| + \coth(|k_2|h)k_1 \cdot k_2), \\ \delta_2^h = &-|k_1|\omega_2\omega_3(|k_2||k_3| + \coth(|k_3|h)k_2 \cdot k_3) \\ &- |k_2|\omega_1\omega_3(|k_1||k_3| + \coth(|k_1|h)\coth(|k_3|h)k_1 \cdot k_3) \\ &- |k_3|\omega_1\omega_2(|k_1||k_2| + \coth(|k_1|h)k_1 \cdot k_2), \\ \delta_3^h = &-|k_1|\omega_2\omega_3(|k_2||k_3| + \coth(|k_2|h)k_2 \cdot k_3) \\ &- |k_2|\omega_1\omega_3(|k_1||k_3| + \coth(|k_1|h)k_1 \cdot k_3) \\ &- |k_3|\omega_1\omega_2(|k_1||k_2| + \coth(|k_1|h)\coth(|k_2|h)k_1 \cdot k_2).\end{aligned}$$

As pointed out in Section 2.4 these coefficients determine the stability of the A_3, A_2, and A_1 subspace in the TWI system, respectively. In the limit $h \to \infty$ we have $\delta_1^\infty = \delta_2^\infty = \delta_3^\infty = \delta$ with

$$\delta = -|k_1|\omega_2\omega_3(|k_2||k_3| + k_2 \cdot k_3)$$

$$
\begin{aligned}
&- |k_2|\omega_1\omega_3(|k_1||k_3| + k_1 \cdot k_3) \\
&- |k_3|\omega_1\omega_2(|k_1||k_2| + k_1 \cdot k_2)
\end{aligned}
$$

In the scalar case we have $k_1 \cdot k_2 = |k_1||k_2| > 0$. Furthermore, it holds

$$\omega_1\omega_2 > 0, \quad \omega_1\omega_3 < 0, \quad \omega_2\omega_3 < 0, \quad k_1 \cdot k_3 = -|k_1||k_3|, \quad k_2 \cdot k_3 = -|k_2||k_3|,$$

so that

$$\delta = -2|k_1||k_2||k_3|\omega_1\omega_2 < 0.$$

Since $\coth(x) > 1$ for all $x > 0$ and since $k_1 \cdot k_2 > 0$, then we have for $j = 1, 2, 3$ and all $h > 0$ that $\delta_j^h < 0$. As a consequence, the γ_j cannot all have the same sign in the scalar case. Moreover, the A_3 subspace associated to k_3 is unstable.

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