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Sylow Numbers in Character Tables and Integral Group Rings

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Summary

Chapter 2

In Chapter 2 we analyze Theorem 1.3 and try to simplify Marshall Hall's formula. If G has two minimal normal subgroups we give a method of how to compute Sylow numbers by Sylow numbers of factor groups.

Proposition 2.5 *Let G be a group with two non-trivial normal subgroups M and N such that $M \cap N = \{1\}$, then*

$$n_p(G) = \frac{n_p(G/M)n_p(G/N)}{n_p(G/MN)}.$$

If $n_p(G/M)$, $n_p(G/N)$ and $n_p(G/MN)$ are known then $n_p(G)$ is known as well.

In search for a possible candidate for a counterexample we analyzed the situation if G has one minimal normal non-solvable subgroup:

Corollary 2.6 *Let G be a group with a minimal normal non-solvable subgroup $N \trianglelefteq G$. Further assume that G is of minimal order such that $\mathcal{X}(G)$ or $\mathbb{Z}G$ does not determine $\operatorname{sn}(G)$. Then $N \leq G \leq \operatorname{Aut}(N)$.*

At the end of this chapter we consider Sylow numbers of Frobenius groups and central extensions of groups.

Chapter 3

In this chapter we consider Question (Q1) whether $\mathcal{X}(G)$ determines $n_p(G)$. Therefore we give an extensive overview of the properties of character tables and apply the methods developed in Chapter 2.

We get an affirmative answer to Question (Q1) for the following classes of groups:

- nilpotent-by-nilpotent groups (Theorem 3.4),
- supersolvable groups (Proposition 3.5),
- groups with cyclic Sylow p-subgroup (Proposition 3.7),
- groups with one non-cyclic Sylow p-subgroup (Proposition 3.9) and
- Frobenius and 2-Frobenius groups (Corollary 3.12).

We give the following improvement of the already established Theorem 1.22:

Proposition 3.13 *Let G be a group such that $G/O_{p'}(G)$ is determined up to isomorphism by $\operatorname{Spec}(G)$. Then $n_p(G)$ is determined by $\operatorname{Spec}(G)$.*

Chapter 4

In Chapter 4 we examine the relation between Sylow numbers and integral group rings and consider Question (Q2) whether Sylow numbers are determined by $\mathbb{Z}G$. In the first part of this chapter we give a summary of properties of $\mathbb{Z}G$ and examine p-constrained groups. We prove that $n_p(G)$ is determined by $\mathbb{Z}G$ provided that G is p-constrained.

> **Theorem 4.6** *Suppose that G is q-constrained. Then $\mathbb{Z}G$ determines $n_p(G)$ for each $p \notin \pi(O_{q'}(G))$. In particular $n_p(G)$ is given by $\mathbb{Z}G$.*

Furthermore we consider groups with abelian Sylow p-subgroups:

> **Theorem 4.8** *Let G be a group with abelian Sylow p-subgroup and assume that $O_{p'}(G)$ is solvable. Then $\mathbb{Z}G$ determines $n_p(G)$.*

For $p = 2$ the integral group ring $\mathbb{Z}G$ determines $n_2(G)$ if G has an abelian Sylow 2-subgroup.

In the last part of Chapter 4 we give evidence that for certain groups with disconnected prime graph the answer to Question (Q2) is affirmative, see Proposition 4.10. If G has a dihedral Sylow 2-subgroup then $\mathbb{Z}G$ determines $\mathrm{sn}(G)$, see Proposition 4.11.

Chapter 5

In the last chapter we give a survey of the definitions and properties of class structures. If G and H are in class correspondence of type JH and G has a cyclic Sylow p-subgroup we give proof that H has a cyclic Sylow p-subgroup as well. We analyze groups with cyclic Sylow p-subgroup, nilpotent-by-nilpotent and supersolvable groups and we prove that class structures of type JHS determine $\mathrm{sn}(G)$ provided

- G has a cyclic Sylow p-subgroup (Theorem 5.5),
- G is nilpotent-by-nilpotent (Theorem 5.6) and
- G is supersolvable (Proposition 5.7).

Zusammenfassung (German summary)

Kapitel 2

In Kapitel 2 werden einige Spezialfälle von Satz 1.3 untersucht und die dort angegebene Formel für Sylowzahlen vereinfacht. Wir geben eine Formel für Sylowzahlen mit zwei verschiedenen minimalen Normalteilern an, die sich aus den Sylowzahlen von Quotientengruppen zusammensetzt:

Proposition 2.5 *Sei G eine Gruppe mit zwei nicht-trivialen minimalen Normalteilern M und N, sodass $M \cap N = \{1\}$ ist. Dann gilt*

$$n_p(G) = \frac{n_p(G/M)n_p(G/N)}{n_p(G/MN)}.$$

Falls $n_p(G/M)$, $n_p(G/N)$ und $n_p(G/MN)$ gegeben sind, ist $n_p(G)$ bestimmt.

Auf der Suche nach einem Gegenbeispiel haben wir folgende Vereinfachung mit Gruppen mit minimalem nicht-auflösbaren Normalteiler entdeckt:

Korollar 2.6 *Sei G eine Gruppe mit minimalem nicht-auflösbaren Normalteiler $N \trianglelefteq G$. Sei G ein minimales Gegenbeispiel, sodass die Sylowzahlen von G nicht durch $\mathcal{X}(G)$ oder $\mathbb{Z}G$ bestimmt sind. Dann ist $N \leq G \leq \mathrm{Aut}(N)$.*

Abschließend betrachten wir Sylowzahlen von Frobeniusgruppen und zentralen Erweiterungen.

Kapitel 3

In Kapitel 3 wird die Frage (Q1) untersucht, ob die Charaktertafel einer Gruppe die Sylowzahlen der Gruppe bestimmt. Dazu werden zunächst einige nützliche Eigenschaften der Charaktertafel zusammengefasst und die in Kapitel 2 entwickelten Methoden angewandt.

Desweiteren wird bewiesen, dass Frage (Q1) für folgende Gruppen eine positive Antwort hat:

- Gruppen mit zyklischer p-Sylowgruppe (Satz 3.4),
- überauflösbare Gruppen (Proposition 3.5),
- Gruppen mit nilpotentem Normalteiler N und nilpotenter Faktorgruppe G/N (Proposition 3.7),
- Gruppen mit einer nicht-zyklischen p-Sylowgruppe (Proposition 3.9) und
- Frobenius- und 2-Frobenius-Gruppen (Korollar 3.12).

Zudem wird eine Erweiterung von Satz 1.22 gezeigt.

Proposition 3.13 *Sei G eine Gruppe, so dass $G/O_{p'}(G)$ bis auf Isomorphie durch* $\mathrm{Spec}(G)$ *bestimmt ist. Dann bestimmt* $\mathrm{Spec}(G)$ *die p-Sylowzahl.*

Kapitel 4

In Kapitel 4 untersuchen wir den Zusammenhang zwischen Sylowzahlen und ganzzahligen Gruppenringen und beschäftigen uns mit der Frage, ob $\mathbb{Z}G$ die Sylowzahlen von G bestimmt (Frage (Q2)). Im ersten Teil fassen wir einige Eigenschaften von ganzzahligen Gruppenringen zusammen und untersuchen Gruppen, die p-beschränkt sind. Wir zeigen, dass für diese Gruppen die p-Sylowzahl bestimmt ist.

Satz 4.6 *Angenommen, G sei q-beschränkt. Dann bestimmt $\mathbb{Z}G$ die p-Sylowzahl für jede Primzahl $p \notin \pi(O_{q'}(G))$. Insbesondere ist $n_p(G)$ durch $\mathbb{Z}G$ gegeben.*

Im Anschluss werden Gruppen mit abelscher p-Sylowgruppe betrachtet:

Satz 1.26 *Sei G eine Gruppe mit abelscher p-Sylowgruppe, sodass $O_{p'}(G)$ auflösbar ist. Dann bestimmt der ganzzahlige Gruppenring die p-Sylowzahl von G.*

Für $p = 2$ ist die Bedingung, dass $O_{p'}(G)$ auflösbar sein muss, wegen dem Satz von Feit–Thompson immer erfüllt und somit obsolet.

Im nächsten Abschnitt befassen wir uns mit Gruppen, dessen Primgraph nicht zusammenhängend ist und geben für einige dieser Klassen von Gruppen einen Beweis, dass $\mathbb{Z}G$ die Sylowzahlen bestimmt. Die Antwort auf Frage (Q2) fällt für Gruppen, dessen 2-Sylowgruppe eine Diedergruppe ist, ebenfalls positiv aus:

Satz 4.8 *Sei G eine Gruppe, sodass die 2-Sylowgruppe eine Diedergruppe ist. Dann bestimmt $\mathbb{Z}G$ alle Sylowzahlen von G.*

Kapitel 5

Im letzten Kapitel gehen wir auf Klassenstrukturen ein und analysieren zunächst die Eigenschaften dieser Strukturen. Falls G und H in Klassenkorrespondenz vom Typ JH sind und G eine zyklische p-Sylowgruppe hat, dann hat auch H eine zyklische p-Sylowgruppe. Analog zu Kapitel 2 gehen wir auf Gruppen mit zusätzlichen Anforderungen an die Gruppenstruktur ein und beweisen, dass die Frage, ob eine Klassenstruktur vom Typ JHS die Sylowzahlen bestimmt, in den folgenden Fällen positiv beantwortet werden kann:

- G hat zyklische p-Sylowgruppen (Satz 5.5),
- G ist metanilpotent (Satz 5.6) und
- G ist überauflösbar (Proposition 5.7).

Contents

Introduction

Groups naturally occur as symmetries of objects and were already known by Pythagoras and Euklid. Even though the abstract definition was not known until the 19th century the requirements of Platonic solids fulfilled the axioms of a (modern) group. The theoretical concept of a group is based on Évariste Galois's research of polynomial equations conducted in the 1830s, cf. [Mas89, p. 108]. The abstract term 'group' was coined by Galois in his last letter but he didn't provide a definition of the objects which are now called Galois groups.

The systematic study of finite groups began about 40 years later. The first formal definition was given by Camille Jordan in 1868 and only named closure as a necessary property of a group, see [Jor07]. The associativity, the identity element and the inverse element were (indirectly) implied in his definition as he only thought of symmetric groups. For a better understanding and easier access to finite groups the theories of subgroups, quotient groups and simple groups were developed and analyzed as well.

In 1872 Ludwig Sylow stated some theorems – based on the definition of Jordan – giving the existence and the number of maximal p-subgroups:

Theorem 1 ([Asc00], p. 19): *Let G be a finite group and p a prime. Then*

(i) $\mathrm{Syl}_p(G)$ *is nonempty.*

(ii) G *acts transitively on* $\mathrm{Syl}_p(G)$ *via conjugation.*

(iii) $|\mathrm{Syl}_p(G)| = |G : \mathrm{N}_G(G_p)| \equiv 1 \bmod p$ *for some* $G_p \in \mathrm{Syl}_p(G)$.

(iv) Every p-subgroup of G is contained in some Sylow p-subgroup of G.

About ten years later the common definition for a group was introduced. As Jordan's definition of a group already included all axioms, Sylow's Theorems remained valid. Simultaneously representations of groups were studied. Group theorists started to analyze properties of groups and developed different methods of representing groups via group homomorphisms into general linear groups of vector spaces. As a result the modern representation theory was born and soon extended. At the beginning of the 20th century many mathematicians continued to consider infinite groups and other algebraic objects while Ferdinand G. Frobenius, William Burnside and Issai Schur started to develop the character theory.

A generalization of Sylow's Theorems given by Philip Hall and the Schur–Zassenhaus Theorem were other famous results accomplished in this period. In 1967 the composition of the so-called Sylow p-numbers $n_p(G) = |\mathrm{Syl}_p(G)|$ was first analysed by Marshall Hall, see [Hal67]. He also presented a formula for Sylow p-numbers as a product of Sylow p-numbers of smaller quotient groups and subgroups. Until then not much had been known about Sylow p-numbers and their structure. The classification of finite simple groups started in 1955 and was completed after almost 50 years in 2004.

At the end of the 20th century the influence of Sylow p-numbers on the properties of the given group was studied. Florian Luca and Wenbin Guo proved that for certain Sylow numbers the given group has to be solvable [Guo96; Luc98], see also [Kös14; Mor13]. Jiping Zhang and Naoki Chigira found an equivalent condition for p-nilpotency [Chi98; Zha95]. In 2006 Xianhua Li proved in [Li06] the uniqueness of Sylow numbers for finite simple groups (excluding $B_n(q)$ and $C_n(q)$).

In 2003 Gabriel Navarro gave in [Nav04] an overview of open problems on characters and Sylow subgroups. He included the following question in his talk at the Gainesville Conference:

$$\text{Does } X(G) \text{ determine } |\mathrm{N}_G(G_p)| \text{ for } G_p \in \mathrm{Syl}_p(G)? \tag{Q1}$$

Ten years later Alexander Moréto named criteria for the existence of nilpotent Hall π-subgroups as a function of Sylow numbers [Mor13]. Due to the fact that nilpotent Hall π-subgroups are determined by the character table (see [KS95]), this again lead to the question, whether Sylow numbers are determined by character tables or not.

So far there has only been an affirmative answer to this question if additional assumptions on the group are fulfilled. In 2016 Gabriel Navarro and Noelia Rizo collected evidence that the character table together with the p-power map determines $|\mathrm{N}_G(G_p)|$ for p-solvable groups[NR16]. In particular for p-solvable groups the Sylow p-number is given by $X(G)$ provided the Sylow p-subgroup is either abelian or has exponent p. For cyclic Sylow p-subgroups Martin Isaacs and Navarro managed to prove that $X(G)$ determines $n_p(G)$ for all finite groups, see [IN02]. For solvable groups the prime divisors of $|\mathrm{N}_G(G_p)|$ are known for each prime p and Sylow p-subgroup G_p of G [IN02].

For arbitrary groups the answer remains unknown.

Instead of character tables one could ask whether other representations of groups providing more information of the structure and properties of the underlying group give a postive (or negative) answer. For Burnside Rings the answer is positive, see

[RCVE04, Theorem 5.2]. As the integral group ring of a group determines the character table this naturally leads to the question if $\mathbb{Z}G$ determines $|\mathrm{N}_G(G_p)|$:

$$\text{Does } \mathbb{Z}G \text{ determine } |\mathrm{N}_G(G_p)| \text{ for } G_p \in \mathrm{Syl}_p(G)? \tag{Q2}$$

On the other hand, for some classes of groups it seems possible to weaken some of the assumptions with respect to the number of information given by the representation.

$$\text{Does a class structure of } G \text{ determine } |\mathrm{N}_G(G_p)| \text{ for } G_p \in \mathrm{Syl}_p(G)? \tag{Q3}$$

In this thesis we want to introduce a new method for obtaining of Sylow numbers by means of the character table, of $\mathbb{Z}G$ or of class structures of type Jordan–Hölder–Sylow (JHS). This approach is based on the following formula for normal subgroups (see [Hal67], Theorem 2.1) and will be discussed in Chapter 2: Assume that $K \trianglelefteq G$ and $G_p \in \mathrm{Syl}_p(G)$, then

$$n_p(G) = n_p(G/K)n_p(K)n_p(\mathrm{N}_{G_pK}(G_p \cap K)/(G_p \cap K)).$$

In addition we want to prove that we can reduce Question (Q1) to groups with exactly one minimal normal subgroup. In Chapter 3 we give a positive answer to Question (Q1) for some classes of groups. This includes groups with cyclic Sylow p-subgroups, nilpotent-by-nilpotent and supersolvable groups, groups with one non-cyclic Sylow p-subgroup and (2-)Frobenius Groups. In particular we give an improvement of the result of Navarro and Rizo.

In Chapter 4 we analyze the situation for integral group rings. The F^*-Theorem (see [Her16]) indicates that we get better results in $\mathbb{Z}G$. Using a generalized version of [NR16, Theorem C] we obtain a positive answer for Question (Q2) for groups with abelian or dihedral Sylow 2-subgroups. The Sylow p-number of p-constrained groups is also determined by $\mathbb{Z}G$.

In 1995 Wolfgang Kimmerle and Robert Sandling introduced in [KS95] the concept of so-called class structures which are motivated by the properties given by character tables. In Chapter 5 we analyze the situation for class structures of type JHS. Question (Q3) has a positive answer for these class structures in the case that the underlying group is either nilpotent-by-nilpotent or supersolvable.

Chapter 1

Basic Concepts and Preliminary Results

A mathematician would rather use the toothbrush of his colleague than his notation.

(unknown)

First we want to introduce some notations and conventions which will be used throughout this thesis. The reader is assumed to be roughly familiar with basic group theoretical concepts, recommended books are *Finite Group Theory* of Michael Aschbacher [Asc00] or *Group Theory I* of Michio Suzuki [Suz82]. During this work we will always consider finite groups, as Sylow's theorems only make sense for primes dividing a given group order.

The second part of this chapter contains an overview of some results concerning Sylow numbers, which should be known by the reader. In addition some theorems which are necessary for the results in this thesis are mentioned. Note that most of the results use the classification of finite simple groups.

1.1 Notations and conventions

For a natural number n we denote by $\pi(n)$ the set of prime divisors of n. For a finite group G with order $|G| = p^a m$, where $\gcd(m,p) = 1$, we write $\pi(G) := \pi(|G|)$. The centralizer resp. the normalizer of a subset $X \subset G$ is denoted by $C_G(X)$ resp. $N_G(X)$. The center $Z(G)$ is the subgroup of all commuting elements. By $\langle X \rangle$ we denote the generated subgroup of $X \subset G$. The order of an element $g \in G$ is written as $|x| := |\langle x \rangle|$.

For $\pi \subset \mathbb{P} := \{ p \in \mathbb{N} : p \text{ prime} \}$ and a natural number $n \in \mathbb{N}$ the π-part of n is $n_\pi = \prod_{p \in \pi} p^{e_p}$, where $\prod_{p \in \mathbb{P}} p^{e_p} = n$ is the unique prime factorization of n. The π'-part of n is written as $n_{\pi'} := \prod_{p \in \mathbb{P} \setminus \pi} p^{e_p}$. A subgroup H of G is called π-subgroup (resp. π'-subgroup) if $\pi(H) \subset \pi$ (resp. $\pi(H) \cap \pi = \emptyset$). A π-subgroup is called Hall π-subgroup G_π of G, if $|G : H|_\pi = 1$. Similarly a Hall π'-subgroup $G_{\pi'}$ is a π'-subgroup such that $\gcd(|H|, |G : H|) = 1$. For $\pi = p$ the group $G_p := G_{\{p\}}$ is a Sylow p-subgroup of G. Similarly write $G_{p'} := G_{\{p'\}}$. Note that the existence of G_p is always guaranteed by Sylow's theorem.

The number of Sylow p-subgroups $\mathrm{Syl}_p(G)$ is denoted by $n_p(G)$. The set of all Sylow numbers $\mathrm{sn}(G) = \{ n_p(G) : p \in \pi(G) \}$ is called Sylow numbers of G. By Sylow's theorem we know that $n_p(G) = |G : N_G(G_p)|$ for any Sylow subgroup $G_p \in \mathrm{Syl}_p(G)$ and that $n_p(G) \equiv 1 \bmod p$. Moreover G acts transitively on $\mathrm{Syl}_p(G)$ by conjugation.

For a set π of primes $O_\pi(G)$ is the largest normal π-subgroup of G. Write $O^\pi(G)$ for the smallest normal subgroup of G such that $G/O^\pi(G)$ is a π-group. Note again, that we abbreviate $O_{\{p\}}(G)$ by $O_p(G)$ (resp. $O^p(G) := O^{\{p\}}(G)$). For $\pi \subset \mathbb{P}$ denote by $O_{\pi'}(G)$ the largest normal π'-subgroup of G. Similarly $O^{\pi'}(G)$ is defined as smallest normal subgroup such that $G/O^{\pi'}(G)$ is a π'-group. Again we write $O_{p'}(G) := O_{\{p'\}}(G)$ respectively $O^{p'}(G) := O^{\{p'\}}(G)$ in the case that $\pi = \{p\}$ simply is a prime.

For $x, y \in G$, set $x^y = y^{-1}xy$. For a subset $X \subset G$ set $X^y = \{ x^y : x \in X \}$. The subset $X^G = \{ X^g : g \in G \}$ is the set of conjugates of X under G. The set $x^G := \{x\}^G$ for some $x \in G$ is called conjugacy class of x in G. For a subset X and Y define $XY = \{ xy : x \in X, y \in Y \}$. If X and Y are subgroups of G where $X^Y \subset X$, then XY is a subgroup of G. By $[X, Y]$ we will denote the subgroup of G generated by all commutators $x^{-1}y^{-1}xy$ where $x \in X$, $y \in Y$. A subgroup $H \leq G$ of G is called normal in G, i.e. $H \trianglelefteq G$, if $H^G \leq H$. The unique largest normal nilpotent subgroup of G is called Fitting subgroup $F(G)$ of G. The generalized Fitting subgroup $F^*(G)$ of G is defined as the product of $F(G)$ and the layer $E(G)$ of G where $E(G)$ consists of the components of G. Note that for solvable groups we have $C_G(F(G)) \leq F(G)$ and for non-solvable groups we have $C_G(F^*(G)) \leq F^*(G)$ (see [Hup67, §4, Kapitel III, Satz 4.2 b)] and [HB82, Chapter X, §13, Theorem 13]).

The Frattini subgroup $\Phi(G)$ is the intersection of all maximal subgroups of G. Analogous to $F(G)$, $E(G)$ and $F^*(G)$ the subgroup is characteristic and similar to $F(G)$ nilpotent.

For a given G we denote by $\mathrm{Aut}(G)$ the automorphism group of G. The inner resp. the outer automorphism group is written as $\mathrm{Inn}(G)$ resp. $\mathrm{Out}(G) \cong \mathrm{Aut}(G)/\mathrm{Inn}(G)$. For a given homomorphism $\alpha : G \to H$ between finite groups G and H we write $\ker(\alpha)$ for the kernel and $\alpha(G)$ for the image of G under α. A inner automorphism is always given by a fixed element $g \in G$ and will be written as $\sigma_g : G \to G, x \mapsto x^g$.

A central extension (H, Z) of a group G is an short exact sequence

$$1 \to Z \to H \to G \to 1,$$

such that the image of Z belongs to the center of H.

If $G = HK$ for subgroups $H, K \leq G$, where $K \cap H = \{1\}$ and $K \trianglelefteq G$, we call G semidirect product of H and K and G is written as $G = K \rtimes H$. A complement of $H < G$ in G is a subgroup $K \leq G$ such that $G = KH$ and $H \cap K = \{1\}$. We say that H has a normal complement in G, if the complement K is normal in G, i.e. $G = K \rtimes H$. Assume that G and $\tilde{G}$ are groups, then the wreath product is written as $G \wr \tilde{G}$.

A p-complement is a complement of G_p in G. We say that G is p-nilpotent, if G has a normal p-complement. The following result guarantees the existence of complements for normal Hall π-subgroups:

Theorem 1.1 (Schur-Zassenhaus Theorem, [Asc00], §6, 18.1)
Any normal Hall π-subgroup $G_\pi \trianglelefteq G$ has a complement in G. In particular, any two complements of G_π are conjugate in N.

By $\{1\} = G_{(0)} \trianglelefteq G_{(1)} \trianglelefteq \ldots \trianglelefteq G_{(n)} = G$ we will denote a composition series of G. We write $H \lll G$ for a subnormal subgroup of G, i.e. for a member H of a composition series. The chief series is written as $\{1\} = G^{(0)} \trianglelefteq G^{(1)} \trianglelefteq \ldots \trianglelefteq G^{(k)} = G$. We say that G is solvable if every composition factor $G_{(i)}/G_{(i-1)}$ is abelian. A group G is called supersolvable if it has an invariant chief series such that every chief factor $G^{(i)}/G^{(i-1)}$ is cyclic. If there exists a composition series where all composition factors are either p-groups or have order relative prime to p, then G is called p-solvable.

If G is a group and M an arbitrary set then a (left) group action φ on G of M is a function $\varphi : M \times G \to M$, where we denote $\varphi(m, g) := m \cdot g$. In the same way we define a right group action where we have $\varphi : G \times M \to M$. If M is also a group and G has a left group action φ and a right group action $\tilde{\varphi}$, then we define the centralizer of G in M as $\mathrm{C}_M(G) = \{\, m \in M \;:\; \forall g \in G :\; \varphi(g, m) = \tilde{\varphi}(m, g) \,\}$.

By $\mathcal{X}(G)$ we denote the character table of G. If in addition the power map is given we write $\mathrm{Spec}(G)$ for the spectral table of G. The integral group ring is written as $\mathbb{Z}G$. The set of all irreducible (complex) characters of G is denoted by $\mathrm{Irr}(G)$.

The Sylow graph $\Gamma_s(G)$ of G is defined in the following way: the vertices V are the prime divisors of any Sylow p-number, i.e. $V = \{\, p \in \mathbb{P} \;:\; \exists r \in \pi(G) : p | n_r(G) \,\}$ and two vertices p and q are connected if pq divides $n_r(G)$ for some $r \in \pi(G)$. The prime graph $\Gamma(G)$ of G is defined similarly: the vertices V are the prime divisors $\pi(G)$ of $|G|$ and two vertices are p and q are connected if there exists an element of order pq. A graph is called connected if for each pair $p = a_1, q = a_n \in V$ of vertices there exist $a_2, \dots, a_{n-1} \in V$ such that a_i and a_{i+1} are connected for $1 \leq i \leq n-1$. Otherwise the graph is called non-connected or disconnected. If $\Gamma(G)$ is non-connected then $\Gamma(G)$ consists of a disjoint union of connected subgraphs, called components of $\Gamma(G)$.

Lemma 1.2

Let $N \trianglelefteq G$ then $\mathrm{C}_G(N) \trianglelefteq G$.

Proof: Let $x = \mathrm{C}_G(N)$ then for $n \in N$ and $g \in G$

$$(g^{-1}x^{-1}g)n(g^{-1}xg) = g^{-1}x^{-1}\underbrace{n^{g^{-1}}}_{n^{g^{-1}} \in N} \overset{x \in \mathrm{C}_G(N)}{=} g^{-1}n^{g^{-1}}g = n$$

and $x^g \in \mathrm{C}_G(N)$. □

Note that the list of all finite simple groups is given in Appendix A.

1.2 Preliminary results

This section will give a brief summary of results concerning Sylow numbers. The systematical study of Sylow numbers was initiated in 1967 by Marshall Hall when he analysed the composition of Sylow numbers. In the following 30 years colloquial little work was spent on Sylow numbers. At the end of the 20th century the question arose as to how Sylow p-numbers and properties of the given group were related.

The following theorem of Marshall Hall provides the main tool for most of our results:

Theorem 1.3 ([Hal67], Theorem 2.1)

Let G have a normal subgroup K and let G_p be a Sylow p-subgroup of G. Then

$$n_p(G) = n_p(K)n_p(G/K)n_p(\mathrm{N}_{G_pK}(G_p \cap K)/(G_p \cap K)).$$

He also gave an exact description of the factors of Sylow p-numbers in any arbitrary finite group:

Theorem 1.4 ([Hal67], Theorem 2.2)

The number $n_p(G)$ of Sylow p-subgroups is the product of factors of the following two kinds:

(i) the number $n_p(S)$ of Sylow p-subgroups of a simple group S and

(ii) a prime power q^t where $q^t \equiv 1 \bmod p$.

For $p = 2$ every odd number q satisfies $q \equiv 1 \bmod p$. In fact the dihedral group of order $2q$ has q as Sylow 2-number. For every odd prime p Marshall Hall could prove that there exist numbers n satisfying $n \equiv 1 \bmod p$ that are not Sylow p-numbers for any arbitrary group.

Theorem 1.5 ([Hal67], Theorem 3.1, Theorem 3.2)
If $n = 1 + rp$ with $1 < r < (p + 3)/2$ there exists no group G with $n_p(G) = n$ unless $n = q^t$ for a prime q, or $r = (p - 3)/2$ and $p > 3$ is a Fermat prime.

There is no group G with $n_3(G) = 22$, $n_5(G) = 21$ or with $n_p(G) = 1 + 3p$ for $p \geq 7$.

In 1961 Bertram Huppert assumed that Sylow numbers and p-nilpotency were connected:

Conjecture 1.6 ([Hup61], Hilfssatz 1.3)
G is p-nilpotent if and only if $\gcd(p, n_r(G)) = 1$ and $\mathrm{N}_G(G_r)$ is p-nilpotent for all $r \in \pi(G)$.

Almost forty years later the first attempt to prove the conjecture was given by Jiping Zhang in [Zha95, Theorem 2] and was finalized by Naoki Chigira in [Chi98, Main Theorem]. The proof gave an equivalent criteria for the p-nilpotency of a group.

Theorem 1.7 ([Chi98], Main Theorem)
Let G be a finite group.

(i) Suppose that $p \neq 3$ is a prime. Then G is p-nilpotent if and only if $p \nmid n_r(G)$ for every $r \in \pi(G)$.

(ii) Suppose that G does not have a composition factor isomorphic with ${}^2A_2(2^f)$, where f is even and not divisible by 3. Then G is 3-nilpotent if and only if $\gcd(3, n_r(G)) = 1$ for every $r \in \pi(G)$.

Note that Conjecture 1.6 is a direct corollary of the previous theorem, as ${}^2A_2(2^f)$ does not fulfill the condition of normalizers.

The case for $p = 2$ was already considered by Anatoly S. Kondratev in [Kon88].

At the same time finite groups with given Sylow numbers were studied. Some results concerning groups with less then three Sylow numbers given by Florian Luca in [Luc98] were based on the imperfect proof of Zhang and had to be corrected. In 2012 Alexander Moréto gave evidence that groups with two Sylow numbers are solvable, using a different method.

Theorem 1.8 ([Mor13], Main Theorem)
Let G be a finite group with $\mathrm{sn}(G) = \{a, b\}$. Then G is the product of two nilpotent Hall subgroups. In particular, G is solvable.

If $a = 1$ then one can easily see that G has a nilpotent Hall $\pi(b)$-subgroup N such that G/N is nilpotent. If $a, b > 1$, then $\gcd(a, b) = 1$ and G has nilpotent Hall π- resp. Hall τ-subgroups where π is the set of all primes p with $n_p(G) = a$ resp. τ is the set of all primes q with $n_q(G) = b$.

The result is based on the following theorem, establishing a relation between Sylow numbers and the existence of nilpotent Hall subgroups:

Theorem 1.9 ([Mor13], Theorem A)

Let G be a finite group and π a set of prime numbers. Then G has a nilpotent Hall π-subgroup if and only if the following three conditions hold:

(i) Given any two different primes $p, q \in \pi$ then p does not divide $n_q(G)$.

(ii) If $\{2,3\} \subset \pi$ then G does not have any composition factors isomorphic to $A_2(p^f)$ with $((p^f)^2 - 1)_{\{2,3\}} = 24$.

(iii) If $\{2,7\} \subset \pi$ then G does not have any composition factors isomorphic to ${}^2G_2(3^{2n+1})$ with $n \not\equiv 3 \bmod 7$.

A similar result was given for groups with three Sylow numbers:

Lemma 1.10 ([Kös14; Luc98])

A finite group is solvable if $\mathrm{sn}(G) = \{1, a, b\}$ or $\mathrm{sn}(G) = \{q^x, a, b\}$, where q is prime and either $\gcd(a, b) = 1$ or $q \nmid ab$.

In order to prove the lemma we need the following result for non-connected Sylow graphs:

Theorem 1.11 ([Zha95], Theorem 3, [Chi98], Theorem 2)

If G is a finite group with a non-connected Sylow graph then G is not simple.

Proof: Assume that G is a finite simple group not isomorphic to ${}^2A_2(2^f)$, where f is even and not divisible by 3 and suppose that $\Gamma_s(G)$ is not connected. Then the proof given in [Zha95] holds. We need to consider the case that $G \cong {}^2A_2(2^f)$. Let $r = 2^f$. In [Chi98, Proof of Theorem 1] the Sylow numbers of H are given as

$$sn(H) = \left\{(r+1)(r^2-r+1), \frac{r^3(r^3+1)}{2}, \frac{r^3(r-1)(r^2-r+1)}{6}, \frac{r^3(r+1)^2(r-1)}{3}\right\}.$$

It is easy to see that the Sylow graph is connected, as $\gcd(r^3+1, \frac{r^3(r^3+1)}{2}) = r^3+1$ and 2 is prime divisor of $n_p(G)$ for $p \in \pi(G) \backslash \{2\}$. □

The proof of Lemma 1.10 is only based on Theorem 1.11 and hence remains valid. We want to provide the reader with a more direct proof showing that a group containing the three Sylow numbers mentioned above doesn't have any composition factor isomorphic to ${}^2A_2(2^f)$:

Proof (Lemma 1.10): First consider the case that $\mathrm{sn}(G) = \{q^x, a, b\}$. We want to prove that G doesn't have a composition factor isomorphic to ${}^2A_2(2^f)$ with f even and $3 \nmid f$. Assume that $H \cong {}^2A_2(r)$, $r = 2^f$, $3 \nmid f$ and f even, is a composition factor of G. The Sylow numbers of H are

$$sn(H) = \left\{r^3+1, \frac{r^3(r^3+1)}{2}, \frac{r^3(r-1)(r^2-r+1)}{6}, \frac{r^3(r+1)^2(r-1)}{3}\right\}.$$

In order to prove this assumption we want to show that every Sylow number $n_p(H)$ is divided by at least two different primes. For $p \neq 2$ it is easy to see that $n_p(H)$ is divided by 2 and at least one odd prime. The greatest common divisor of $(r+1)$ and (r^2-r+1) is either 1 or 3. If $\gcd(r+1, r^2-r+1) = 1$ the claim follows immediately. If $\gcd(r+1, r^2-r+1) = 3$, then either $r+1$ or r^2-r+1 is divided by 3 but not by 9. As both $r+1$ and r^2-r+1 are greater than 5, so one of the factors owns a prime divisor $s \neq 3$.

By Theorem 1.3 $n_p(H)$ divides $n_p(G)$. As none of the Sylow numbers of H are of prime power order this yields $n_p(H)|ab$. Note, there exist primes $p_1, p_2 \in \pi(H)$ with $n_{p_1}(H)|a$ and $n_{p_2}(H)|b$. Assume that $p_2 \in \pi(n_{p_1}(H))$ for some $p_1 \in \pi(H)$ and $n_{p_1}(H)|a$. Then $n_{p_1}(G) = a$. Therefore $n_{p_2}(G) \equiv 1 \bmod p_2$ doesn't divide a. Due to the fact that $n_{p_2}(H)|n_{p_2}(G)$ is no prime power, $n_{p_2}(G)$ is not a prime power and $n_{p_2}(G) = b$. As $n_{p_2}(H)|n_{p_2}(G) = b$ the Sylow number $n_{p_1}(H)$ has to divide b.

Assume $\gcd(a, b) = 1$. Without loss of generality we suppose $n_p(H) = (r^3(r^3+1)/2)|a$ for one $p \in \pi(H)$. There exists a Sylow number $n_s(H)$ in H with $n_s(H)|b$ for one $s \in \pi(H)$.

We consider $n_s(H) = (r+1)(r^2-r+1)$. Then $\gcd((r+1)(r^2-r+1), r^3(r^3+1)/2) = r^3+1 \neq 1$. Choose $k \in \pi(r^3+1)$. k divides $n_p(H)$ and $n_s(H)$ and it follows that $k|\gcd(n_p(H) \text{ and } n_s(H))|\gcd(a,b) = 1$. So r^3-1 can't divide b.

Suppose $n_s(H) = r^3(r-1)(r^2-r+1)/6$. Then $r^3(r^2-r+1)/6 \neq 1$ divides $\gcd(r^3(r-1)(r^2-r+1)/6, r^3(r^3+1)/2)$. Choose $k \in \pi(r^3(r^2-r+1)/6)$. Then k divides $\gcd(n_p(H)|n_s(H))|\gcd(a,b) = 1$ and $k = 1$. It follows $n_s(H) \neq r^3(r-1)(r^2-r+1)/6$.

Assume $n_s(H) = r^3(r+1)^2(r-1)/3$. Then $r^3(r+1)/3$ is a divisor of $\gcd(r^3(r+1)^2(r-1)/3, r^3(r^3+1)/2)$. Choose $k \in \pi(r^3(r+1)/3)$. As before it is $k = 1$. We conclude, all Sylow numbers of H divide a and do not divide b, a contradiction to the fact, there has to be one Sylow number which divides b.

Suppose $q \nmid ab$. There exist Sylow numbers $n_{p_1}(H), n_{p_2}(H)$ with $n_{p_1}(H)|a$ and $n_{p_2}(H)|b$. Note, that $\gcd(n_{p_1}(H), n_{p_2}(H)) \neq 1$ for any Sylow numbers of H. Now choose some $k \in \pi(\gcd(n_{p_1}(H), n_{p_2}(H)))$. It is $n_k(H) \equiv 1 \bmod k$. As $k|\gcd(n_{p_1}(H), n_{p_2}(H))|\gcd(a,b)$ and $n_k(H)|n_k(G) \in \{q^x, a, b\}$ we conclude that $n_k(H)|q^x$, a contradiction to the fact, that $n_k(H) \in sn(H)$ can't be of prime power order.

The conclusion of the proposition holds as well provided $sn(G) = \{1, a, b\}$. □

There are groups with three Sylow numbers that are not solvable, for example consider the alternating A_5: The Sylow numbers are $\mathrm{sn}(\mathrm{A}_5) = \{5, 6, 10\}$ and A_5 is the smallest nonsolvable group.

For groups where all Sylow numbers are odd or of prime power, we also obtain solvability of the corresponding group G.

Theorem 1.12 ([Guo96], Theorem 2)
In a finite group G the Sylow numbers $n_p(G)$ are all prime powers or of odd order if and only if G is solvable and $G = KH$, where K and H are Hall subgroups of G, K is nilpotent and normal in some $\{2\}'$-Hall subgroup and H is 2-nilpotent.

It is even sufficient to consider only Sylow 2- and Sylow 3-numbers of G:

Theorem 1.13 ([GS05])
A group G is solvable provided that the normalizer of every Sylow 2- and 3-subgroup is of primary index in G.

In [KG09] Anatoly S. Kondratev und Wenbin Guo studied the composition factors of finite groups where $n_3(G)$ is odd or equal to some prime p.

Theorem 1.14 ([KG09], Theorem 1)
Let $q = p^k$ be a power of a prime $p \in \mathbb{P}$. If $n_3(G)$ is odd then the nonabelian composition factors are isomorphic to one of the following groups:

(i) $A_1(q)$ *for* $q \equiv \pm 1 \bmod 12$,

(ii) $A_{n-1}(q)$ *for* $n \in \{3, 4, 5\}$ *and* $q \equiv -1 \bmod 12$,

(iii) ${}^2A_n(q)$ *for* $n \in \{3, 4, 5\}$ *and* $q \equiv 1 \bmod 12$,

(iv) $C_2(q)$ *for* $q \equiv \pm 1 \bmod 12$,

(v) ${}^2B_2(2^{2n-1})$, $n \geq 1$ *or*

(vi) M_{11}.

In addition the case that $n_p(G) \in \mathbb{P}$ was considered:

Theorem 1.15 ([KG09], Theorem 2)
Let p be a prime and suppose that $n_p(G) = t$ for some $t \in \mathbb{P}$. Then:

(i) if $p = 2$ then either all nonabelian composition factors of G are isomorphic to A_5 or all nonabelian composition factors of G are isomorphic to $C_2(3)$;

(ii) if $p = 3$ then every nonabelian composition factor is isomorphic to ${}^2B_2(2^{2n-1})$ for $n \geq 1$;

(iii) if $p > 3$ then all nonabelian composition factors of G are p'-groups (i.e. G is p-solvable).

In 2006, Xianhua Li proved the uniqueness of Sylow p-numbers for almost all finite simple groups (excluding $B_n(q)$ and $C_n(q)$):

Theorem 1.16 ([Li06], Theorem 3.3.1)
Let G be a finite group and T a simple group such that $|G| = |T|$. If $n_p(G) = n_p(T)$ for all $p \in \pi(G)$, then G is isomorphic to T or $\{G, T\} = \{B_n(q), C_n(q)\}$, where $n \geq 3$ and q is odd prime power.

Instead of considering $n_p(G)$ and the order of the given group G one can also consider $|\mathrm{N}_G(G_p)|$ in some cases (without the need of the group order):

Theorem 1.17
Let $q = p^k$ be a power of a prime $p \in \mathbb{P}$. Assume that X is isomorphic to one of the following simple groups:

(i) $X \cong A_1(q)$,

(ii) $X \cong A_{n-1}(q)$,

(iii) $X \cong C_2(q)$,

(iv) $X \cong \mathrm{A}_n$,

(v) $X \cong {}^2A_{n-1}(q)$,

(vi) $X \cong {}^2D_n(q)$ *or*

(vii) *X is isomorphic to one of the sporadic simple groups.*

Further assume that G is a finite group and $|\mathrm{N}_G(G_p)| = |\mathrm{N}_S(S_p)|$ for every prime p. Then $G \cong S$.

Proof: For (i)-(v) proof has been given by Jianxing Bi in [Bi92; Bi95; Bi01a; Bi01b; Bi04]. If $S \cong {}^2D_n(p^k)$ see [IA08] and if S is a sporadic simple group see [KK05]. □

We don't know whether $\mathrm{sn}(G)$ and $|G|$ determine a group up to isomorphism. It seems possible for simple groups (with the exception mentioned above). In general it is not true that $|\mathrm{N}_G(G_p)|$ determines G.

The following results given bei Alireza Khalili Asboei indicate that at least for some simple groups the Sylow numbers and order of a group are sufficient to determine the group up to isomorphism:

Theorem 1.18 ([Asb15], Main Theorem)
Let G be a finite group. Let $q = p^k$ be a power of a prime $p \in \mathbb{P}$. Assume that $\mathrm{sn}(G) = \mathrm{sn}(S)$, *where S is one of the following groups:*

(i) $S \cong A_2(q)$, $5 \nmid (q-1)$,

(ii) $S \cong {}^2A_2(q)$, $q \neq 4$,

(iii) $S \cong \mathrm{A}_n$, $n \geq 5$,

(iv) $S \cong \mathrm{S}_r$, $r \geq 5$ *and* $r \in \mathbb{P}$ *or*

(v) $S \cong X$ *and X is a simple group.*

Then $G \cong S$.

A similar result is given for projective special linear groups of dimension 2:

Theorem 1.19 ([Asb14], Theorem 3.2)
Let G be a finite group such that $\mathrm{sn}(G) = \mathrm{sn}(\mathrm{A}_1(q))$ *and* $|G| = |A_1(q)|$, *where* $q = p^k > 2$. *Then* $G \cong \mathrm{A}_1(q)$.

In Theorem 1.17 we didn't need the assumption with respect to the order of the given group. Naturally, the question arises whether this condition is necessary at all. In general the claim won't hold: In Proposition 2.7 we will prove that Sylow numbers are invariant under central extensions. But even for finite groups with trivial center we have an example that it can not be true:

Theorem 1.20 ([Asb15], Theorem 6)
Let G be a finite centerless group and $\mathrm{sn}(G) = \mathrm{sn}(A_1(17)) = \{18, 136, 153\}$. *Then* $G \cong A_1(17)$ *or* $G \cong \mathrm{Aut}(A_1(17))$.

In this thesis we want to study the relation between Sylow numbers and character tables. As mentioned in the introduction there are only few results concerning the influence of the character table on Sylow numbers. The following result for cyclic Sylow p-subgroups was given by Gabriel Navarro:

Theorem 1.21 ([Nav04], Theorem 4, Theorem 8, Corollary 5)
Assume G is finite and G_p is cyclic. Then $\chi(G)$ determines $|\mathrm{N}_G(G_p)|$.

In the case that G is p-solvable and G_p is either abelian or has exponent p, Gabriel Navarro and Noelia Rizo could prove that $\chi(G)$ determines $n_p(G)$:

Theorem 1.22 ([NR16], Theorem B, Theorem C)
Let p be a prime and G a finite p-solvable group. If $G_p \in \mathrm{Syl}_p(G)$ *is abelian or has exponent p, then the character table of G determines $n_p(G)$.*

If G is a finite p-solvable group, then the character table and the p-power map determine $n_p(G)$ for arbitrary G_p.

1.3 Groups with abelian or dihedral Sylow p-subgroups

At the end of this chapter we want to recall some results to be used in following chapters. Most of the results constrain the structure of a group with additional assumptions:

Theorem 1.23 (N/C-Theorem)
Let $H \leq G$ then $\mathrm{N}_G(H)/\,\mathrm{C}_G(H)$ *is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

Proof: Let $h \in H$ and $g \in \mathrm{N}_G(H)$ then $\sigma_g : H \to H, h \mapsto h^g$ is an automorphism of H. The map $\Phi : \mathrm{N}_G(H) \to \mathrm{Aut}(H), g \mapsto \sigma_g$ is well-defined with kernel $\ker(\Phi) = \mathrm{C}_G(H)$.□

Theorem 1.24 ([Wal69], Theorem I)
Let G be a group with abelian Sylow 2-subgroups. Then $O^{2'}(G/O_{2'}(G))$ is the direct product of an abelian 2-group and simple groups of one of the following types:

(i) $A_1(2^n)$, $n \geq 1$,

(ii) $A_1(p^f)$, $p^f \equiv \pm 3 \bmod 8$,

(iii) J_1 *or*

(iv) ${}^2G_2(q)$, $q = 3^{2n+1}$ *and* $q \equiv 3 \bmod 8$.

Note that $O_{2'}(G)$ is solvable by the Feit-Thompson Theorem.

Based on the previous results Aviad M. Broshi studied groups where the Sylow p-subgroups are abelian for each $p \in \pi(G)$:

Theorem 1.25 ([Bro71], Theorem)
Let G be a finite group where all Sylow p-subgroups are abelian. Then there exist subgroups H, S and K of G satisfying

(i) $G = HSK$, $|G| = |H||S||K|$,

(ii) $H \trianglelefteq G$, $K \leq \mathrm{N}_G(S)$,

(iii) *H and K are solvable and S is semi-simple, i.e. the direct product of simple groups.*

Particulary the simple groups, which are non-abelian and whose Sylow p-subgroups are abelian for each $p \in \pi(G)$, are isomorphic to the Janko group J_1 or $\mathrm{A}_1(q)$ for a prime power $q > 3$ and $q \equiv 0 \bmod 8$ respectively $q \equiv \pm 3 \bmod 8$, see [Bro71, Theorem 3.2].

A stronger result for abelian Sylow p-subgroups, where p is odd, was given by Wolfgang Kimmerle and Robert Sandling:

Theorem 1.26 ([KS95], Theorem 2.1)
A finite group G has abelian Sylow p-subgroups if and only if $O^{p'}(G/O_{p'}(G))$ is the direct product of simple groups each having a nontrivial abelian Sylow p-subgroup and of an abelian p-group.

Note that $O_{p'}(G)$ is not necessarily solvable for odd primes p.

A similar classification has been done for groups with dihedral Sylow 2-subgroup:

Theorem 1.27 ([GW65], Theorem 1)
Let G be a finite group with dihedral Sylow 2-subgroups. Then G satisfies one of the following conditions:

(i) *$G/O_{2'}(G)$ is isomorphic to A_7,*

(ii) *$G/O_{2'}(G)$ is isomorphic to a Sylow 2-subgroup of G or*

(iii) *$G/O_{2'}(G)$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}_2(q)$ containing $A_1(q)$ for q odd. By $\mathrm{P\Gamma L}_2(q)$ we denote the projective semilinear group.*

In order to analyze automorphism groups of direct products of simple groups the following proposition can be called to mind:

Proposition 1.28 ([BBE06], Proposition 1.1.20)
Let X be a non-abelian simple group and write $X^n = X_1 \times \ldots \times X_n$ for the direct product of n copies $X_1, \ldots, X_n$ of X. Then $\mathrm{Aut}(X^n) \cong \mathrm{Aut}(X) \wr \mathrm{S}_n$.

Chapter 2

Reduction Methods for Sylow Numbers

The theory of groups is a branch of mathematics in which one does something to something and then compares the results with the result of doing the same thing to something else, or something else to the same thing.

(James R. Newman)

Sylow numbers have a strong influence on the structure of a given group and vice versa. While most of the results in Chapter 1 determine the structure of a group with given Sylow numbers, the question arises as to how the structure of a group affects the set of Sylow p-numbers.

Marshall Hall's studies taught us that Sylow numbers are a product of Sylow numbers of smaller groups, see Theorem 1.3. By using his formula and carefully analyzing the properties of the subsets obtained by this method, we try to simplify the computation of Sylow numbers in special cases. If G has two minimal normal subgroups we give a formula of how to compute $n_p(G)$ by the Sylow numbers of certain factor groups. In addition we want to discuss the structure of Sylow numbers of Frobenius groups and of finite central extensions of finite groups.

Some of the results already appeared as preprint in [KK15].

2.1 Simplification Techniques

Our first approach to compute $n_p(G)$ by means of the character table is the application of the formula of Theorem 1.3 by Marshall Hall. By choosing suitable normal subgroups, some simplifications can be done.

Proposition 2.1
Assume that G has a normal q-subgroup $Q \trianglelefteq G$.

(i) *If $p = q$, then $n_p(G) = n_p(G/Q)$.*

(ii) *If $p \neq q$, then $n_p(G) = n_p(G/Q)n_p(G_pQ)$.*

Proof: This follows immediately from Theorem 1.3. □

As a direct conclusion of Proposition 2.1 Hall's theorem 1.3 can be rewritten in the following way:

Corollary 2.2
Let G be finite and $M \trianglelefteq G$, then

$$n_p(G) = n_p(M)n_p(G/M)n_p(\mathrm{N}_{G_pM}(G_p \cap M)).$$

It seems easier to calculate $\mathrm{N}_{G_pM}(G_p \cap M)$ instead of $\mathrm{N}_{G_pM}(G_p \cap M)/(G_p \cap M)$. If $G_p \cap M = \{1\}$ we obtain an easier formula for the last term:

Proposition 2.3
Let $K \trianglelefteq G$ and $K \cap G_p = \{1\}$. Then $n_p(G) = n_p(G/K)n_p(G_pK)$ and

$$n_p(G_pK) = |K : \mathrm{C}_K(G_p)|.$$

Proof: By Sylow's theorem we have

$$\begin{aligned} n_p(G_pK) &= |G_pK : \mathrm{N}_{G_pK}(G_p)| \\ &= |G_pK : G_p(\mathrm{N}_{G_pK}(G_p) \cap K)| \\ &= |K : \mathrm{N}_{G_pK}(G_p) \cap K|. \end{aligned}$$

Proposition 1C of [Bra76] completes the proof. For convenience' sake we want to familiarize the reader with a more direct proof of the last part:

Let $G_p \in \mathrm{Syl}_p(G)$. Suppose that $G_p^g \in \mathrm{Syl}_p(G_pK)$. As $\gcd(|K|, |G_p|) = 1$ Theorem 1.1 yields an element $k \in K$ such that $G_p^g = G_p^k$.

Assume for $k_1, k_2 \in K$ that $G_p^{k_1} = G_p^{k_2}$. Then $k_1k_2^{-1} \in \mathrm{N}_G(G_p) \cap K$ and $n_p(G_pK) = |K : \mathrm{N}_G(G_p) \cap K|$.

Now assume that $k \in \mathrm{N}_G(G_p) \cap K$. It follows that $p^{-1}k^{-1}pk \in K \cap G_p = \{1\}$ and therefore $pk = kp$ for all $k \in K, p \in G_p$. □

Note that every solvable group has at least one normal q-subgroup $Q \leq G$ for some prime q dividing $|G|$. In order to use induction the main problem that remains is to determine the Sylow number of G_pQ or the size of $\mathrm{C}_Q(G_p)$. So far there is no general approach to determine $n_p(G_pQ)$ by means of the character table. The following equivalent propositions are easily to see.

Corollary 2.4

Assume that $K \trianglelefteq G$, where $p \notin \pi(K)$. Then the following assertions are equivalent:

(i) $n_p(G) = n_p(G/K)$,

(ii) $K \subset \mathrm{N}_G(G_p)$,

(iii) $K \subset \mathrm{C}_G(G_p)$.

In order to use induction the following result for normal subgroups with trivial intersection is helpful:

Proposition 2.5

Let G be a finite group with non-trivial normal subgroups M and N where $M \cap N = \{1\}$.

(i) The Sylow p-number of $MN = M \times N$ is the product of the Sylow p-numbers of M and N, i.e. $n_p(MN) = n_p(M) \cdot n_p(N)$.

(ii) Suppose that $n_p(G/M)$, $n_p(G/N)$ and $n_p(G/MN)$ are known, then $n_p(G)$ is given as

$$n_p(G) = \frac{n_p(G/M)n_p(G/N)}{n_p(G/MN)}.$$

In particular $n_p(\mathrm{N}_{G_pM}(G_p \cap M))$ and $n_p(\mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N))$ coincide.

Proof: (i) For the first part consider $\mathrm{N}_{MN}((MN)_p)$ with $(MN)_p = M_pN_p \in \mathrm{Syl}_p(MN)$. Then we have

$$\begin{aligned}
\mathrm{N}_{MN}(M_pN_p) &= \{\, m \in M, n \in N \;:\; n^{-1}m^{-1}\tilde{m}\tilde{n}mn \in M_pN_p \text{ for } \tilde{m} \in M_p, \tilde{n} \in N_p \,\} \\
&= \{\, m \in M, n \in N \;:\; m^{-1}\tilde{m}mn^{-1}\tilde{n}n \in M_pN_p \text{ for } \tilde{m} \in M_p, \tilde{n} \in N_p \,\} \\
&= \{\, m \in M \;:\; \tilde{m}^m \in M_p \,\} \cdot \{\, n \in N \;:\; \tilde{n}^n \in N_p \,\} \\
&= \mathrm{N}_M(M_p)\,\mathrm{N}_N(N_p)
\end{aligned}$$

and the first assertion holds.

(ii) Let $G_p \in \mathrm{Syl}_p(G)$. By Theorem 1.3 we obtain

$$n_p(G) = n_p(G/M)n_p(M)n_p(\mathrm{N}_{G_pM}(G_p \cap M))$$

and

$$\begin{aligned}
n_p(G/N) &= n_p(G/MN)n_p(MN/N)n_p(\mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N)) \\
&= n_p(G/MN)n_p(M)n_p(\mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N))
\end{aligned}$$

After rearranging the given equations we need to prove that

$$\begin{aligned} n_p(G) &\overset{!}{=} \frac{n_p(G/M)n_p(G/N)}{n_p(G/MN)} \\ \Leftrightarrow n_p(G/M)n_p(M)n_p(\mathrm{N}_{G_pM}(G_p \cap M)) &= n_p(G/M)n_p(M) \\ &\quad \cdot n_p(\mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N)) \\ \Leftrightarrow n_p(\mathrm{N}_{G_pM}(G_p \cap M)) &= n_p(\mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N)). \end{aligned}$$

Now consider the natural surjective group homomorphism $\phi : G \to G/N$, $g \mapsto gN$ and the restriction $\tilde{\phi} : \mathrm{N}_{G_pM}(G_p \cap M) \to \mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N)$.

The map $\tilde{\phi}$ is well-defined: Assume that $x \in G_p \cap M$, then obviously $xN \in G_pN/N \cap MN/N$. By [Hal67, Theorem 2.1] $G_p \cap M$ is a Sylow p-subgroup of M and G_pN/N is a Sylow p-subgroup of G/N. In particular $G_pN/N \cap MN/N$ is Sylow p-subgroup of $MN/N \cong M$. As $MN/N \cong M$ we have that the restriction of ϕ to M is injective. Thus there is a bijection (even a isomorphism) between $G_p \cap M$ and $G_pN/N \cap MN/N$. Now for each $yN \in G_pN/N \cap MN/N$ there exists a unique representative $x \in G_p \cap M$ such that $xN = yN$. Let $s \in \mathrm{N}_{G_pM}(G_p \cap M)$ and $xN \in G_pN/N \cap MN/N$ with $x \in G_p \cap M$, then

$$\begin{aligned} (xN)^{-1}sNxN &= \underbrace{(x^{-1}sx)}_{x \in \mathrm{N}_{G_pM}(G_p \cap M)} N \\ &= s^x N \in G_pN/N \cap MN/N. \end{aligned}$$

Thus the map is well-defined.

We want to study the kernel $\ker(\tilde{\phi})$. Let $x \in G_p$ and $m \in M$ and assume that $xm \in \ker(\tilde{\phi})$, i.e. $xm \in N$. Consider the equation $xm = n$, where $n \in N$. This yields

$$\begin{aligned} & x = nm^{-1} \\ \overset{x \in G_p \cap M \times N}{\Leftrightarrow} \quad & x_1x_2 = m^{-1}n,\ x_1 \in G_p \cap M,\ x_2 \in G_p \cap N \\ \overset{M \cap N = \{1\}}{\Leftrightarrow} \quad & x_1m = nx_2^{-1} = \{1\}. \end{aligned}$$

Thus m is already an element in G_p and $x_1x_2m \in G_p$. Therefore we have that $\ker(\tilde{\phi})$ is a normal p-group in G.

In the last part we need to prove that $\tilde{\phi}$ is surjective. Let $amN \in \mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N)$, i.e for each $xN \in G_pN/N \cap MN/N$ there exists $yN \in G_pN/N \cap MN/N$ such that

$$(am)^{-1}NxNamN = m^{-1}a^{-1}xamN = yN.$$

As above let $yN \in G_pN/N \cap MN/N$ such that $y \in G_p \cap M$. In particular we have

$$\begin{aligned} &\exists n \in N : m^{-1}a^{-1}xam = yn \\ &\overset{M \trianglelefteq G}{\Leftrightarrow} \underbrace{m^{-1}x^a m}_{\in M} \underbrace{n^{-1}}_{\in N} = y \in G_p \cap M. \end{aligned}$$

As $M \cap N = \{1\}$ we have that $n = 1$ and thus this yields that $am \in \mathrm{N}_{G_pM}(G_p \cap M)$. Therefore $\tilde{\phi}$ is surjective and

$$\mathrm{N}_{G_pM}(G_p \cap M)/\ker(\tilde{\phi}) \cong \tilde{\phi}(\mathrm{N}_{G_pM}(G_p \cap M)) = \mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N).$$

This yields that

$$\begin{aligned} n_p(\mathrm{N}_{G_pM}(G_p \cap M)) &= n_p(\mathrm{N}_{G_pM}(G_p \cap M)/\ker(\tilde{\phi})) \\ &= n_p(\mathrm{N}_{G_pMN/N}(G_pN/N \cap MN/N)) \end{aligned}$$

and the result holds. □

We want to give a more direct proof for the case that $\gcd(|M|, |N|) = 1$:

Proof: Without loss of generality assume $p \notin \pi(M)$. By Proposition 2.3 and $p \notin \pi(M)$ we get

$$n_p(G) = n_p(G/M)|M : \mathrm{C}_G(G_p) \cap M|,$$

and similarly, because $p \notin \pi(N)$

$$n_p(G/N) = n_p(G/MN)n_p(M)|MN/N : \mathrm{C}_{G/N}(G_pN/N) \cap MN/N|.$$

Consider the restriction $\tilde{\kappa} := \kappa|_{\mathrm{C}_G(G_p)\cap M} : \mathrm{C}_G(G_p) \cap M \to \mathrm{C}_{G/N}(G_pN/N) \cap MN/N$ of the canocical quotient map $\kappa : G \mapsto G/N$. The map $\tilde{\kappa}$ is injective and we need to prove that it is surjective as well. Consider $x \in M$ such that $xN \in \mathrm{C}_{G/N}(G_pN/N) \cap MN/N$. For every $y \in G_p$ we obtain $y \cdot x = x \cdot y \cdot n$ for some $n \in N$. But as $n = [x, y] = x^{-1}(y^{-1}xy) \in M$ and $M \cap N = 1$, then $x \in \mathrm{C}_G(G_p) \cap M$ and therefore

$$|M : \mathrm{C}_G(G_p) \cap M| = |MN/N : \mathrm{C}_{G/N}(G_pN/N) \cap MN/N|.$$ □

Proposition 2.5 confines the problem to the so-called monolithic groups, i.e. groups with a unique minimal subgroup.

Note that character tables (respectively integral group rings) determine the character tables of factor groups and thus Proposition 2.5 gives us a possibility to argue by induction in the case that the group has at least two minimal normal subgroups. How to handle groups with exactly one minimal normal subgroup in general remains unsolved. In order to find a candidate for a counterexample Proposition 2.5 yields the following simplification in the case that G has a unique minimal normal subgroup which is not solvable:

Corollary 2.6

Let G be a group with minimal normal non-solvable subgroup $N \trianglelefteq G$. Further assume that G is a of minimal order such that $\mathcal{X}(G)$ or $\mathbb{Z}G$ does not determine $\mathrm{sn}(G)$. Then $N \leq G \leq \mathrm{Aut}(N)$.

Proof: By Proposition 2.5 we can assume that G has exactly one minimal normal subgroup which is not solvable. As $N \cong X^k$ with X simple non-abelian we see that $\mathrm{C}_G(N) \cap N = \{1\}$. If $\mathrm{C}_G(N) \neq \{1\}$ then there exists a minimal normal subgroup M of G which is contained in $\mathrm{C}_G(N)$, a contradiction to the fact that G has only N as minimal normal subgroup. Thus $\mathrm{C}_G(N) = \{1\}$. Consider $\varphi : \mathrm{Inn}(G) \to \mathrm{Aut}(N)$, $\sigma_g \mapsto \sigma_g|_N$. As φ is group homomorphism the fundamental homomorphism theorem yields

$$G \cong G/\,\mathrm{Z}(G) \cong \mathrm{Inn}(G) \cong \varphi(G) \leq \mathrm{Aut}(N). \qquad \square$$

Proposition 1.28 shows that G is a subgroup of $\mathrm{Aut}(X^k) \cong \mathrm{Aut}(X) \wr \mathrm{S}_k$.

2.2 Frobenius Groups and Central Extensions

For certain classes of groups it is possible to reduce the problem of computing the Sylow numbers to the problem of calculating the Sylow numbers of some underlying subgroups. It is clear, that these results won't hold for any arbitrary group. Note that we don't need the character tables for these reductions.

Proposition 2.7

Let E be a finite central extension of G with kernel K. Then $n_p(G) = n_p(E)$ for each prime $p \in \pi(E)$.

Proof: Let $p \in \pi(E)$ be a prime. If $G_p \cap K = \{1\}$, then by Proposition 2.1(i) we obtain $n_p(E) = n_p(G/K)n_p(G_pK)$ and $n_p(G_pK) = 1$, as K is central in G. So assume that $G_p \cap K \neq \{1\}$. Then $G_p \cap K \trianglelefteq G$ and Proposition 2.1(ii) finishes the proof. $\square$

For Frobenius groups it is possible to compute the Sylow numbers provided that the Sylow numbers of the Frobenius complement are given:

Proposition 2.8

Let G be a finite Frobenius group with Frobenius kernel K and Frobenius complement H. Let $\mathrm{sn}(H) = \{a_1, \ldots, a_n\}$. Then

$$\mathrm{sn}(G) = \{1, |K| \cdot a_1, \ldots, |K| \cdot a_n\}.$$

In particular if H is nilpotent then $\mathrm{sn}(G) = \{1, |K|\}$.

Proof: If $p \in \pi(K)$, then $n_p(G) = 1$ because the Frobenius kernel is nilpotent and $\gcd(|K|, |H|) = 1$. Assume $p \notin \pi(K)$, then we get by Proposition 2.3 that

$$n_p(G) = n_p(G/K) \cdot |K : \mathrm{C}_K(G_p)|.$$

Because H acts fixpointfreely on K we always have $\mathrm{C}_K(G_p) = \{1\}$. This completes the proof. □

Chapter 3

Sylow Numbers in Character Tables

Tu as voulu de l'algèbre, et tu
en auras jusqu'au menton!

(Jules Verne)

In 2003 Gabriel Navarro presented a summary of open questions concerning characters and Sylow subgroups [Nav04]. Among this unsolved problems he asked whether the character table determines $n_p(G)$. In this proceeding he proved that $n_p(G)$ is determined by the character table provided G_p is cyclic, see Theorem 1.21. In 2013 Alexander Moréto published criteria for the existence of nilpotent Hall π-subgroups as a function of Sylow numbers (see Theorem 1.9). As nilpotent Hall π-subgroups are determined by the character table (see Remark 3.1) this (again) led to the question, whether Sylow numbers are determined by character tables.

In 2016 Navarro and Noelia Rizo proved that $\mathcal{X}(G)$ determines $n_p(G)$ provided G is p-solvable and the Sylow p-subgroups are either abelian or have exponent p. If additionally the power map is given then $\mathcal{X}(G)$ determines $n_p(G)$.

By using the methods developed in Chapter 2 we are able to give a positive answer to other classes of groups. For supersolvable and nilpotent-by-nilpotent groups the Sylow numbers can be computed from their character tables. If G is a group where only one Sylow p-subgroup is non-cyclic for some p then $\mathcal{X}(G)$ determines $n_p(G)$ for all G (and not only for the cyclic Sylow subgroups). Note that these groups are not necessarily p-solvable.

Finally, we want to provide a generalization of the result given by Navarro and Rizo as well as some further applications.

The main problem remains the calculation of $|x^G \cap G_p|$. It is not clear which elements of a conjugacy class of a p-element are contained in G_p. So far there is no general approach for arbitrary groups.

First recapitulate some of the known results on ordinary character tables concerning properties of the underlying groups.

Remark 3.1: Let G be a finite group and denote by $\mathcal{X}(G)$ its character table .

(i) The order $|G|$ is given by $\mathcal{X}(G)$ [Isa76, Theorem (2.13)].

(ii) The length of conjugacy classes $|x^G| = |G : \mathrm{C}_G(x)|$ of some $x \in G$ is determined by $\mathcal{X}(G)$ (see [Isa76, Theorem (2.18)]).

(iii) The lattice of normal subgroups can be calculated by $\mathcal{X}(G)$, and for each normal subgroup N of G the conjugacy classes of G which lie in N are determined (see [Isa76, p. 23]).

(iv) Let $C = x^G$ and assume $g \in C$. Then $\pi(|g|)$ is determined by $\mathcal{X}(G)$ (see [PH71, p. 206]).

(v) The derived subgroup can be calculated by $\mathcal{X}(G)$ [Isa76, Corollary (2.23)].

(vi) For $N \trianglelefteq G$ the character table $\mathcal{X}(G/N)$ can be computed by $\mathcal{X}(G)$ (see [Isa76, Lemma (2.22)]).

(vii) It is possible to decide whether G has cyclic or abelian Sylow p-subgroups (see [KS95, Theorem 6] or [NT14]).

Note that $\mathcal{X}(G)$ does not determine G up to isomorphism. As smallest counterexample consider the quaternion group and the dihedral group of order 8. Both groups have the same character table but obviously these groups are not isomorphic. Yet the Sylow numbers coincide.

In general the character table does not provide the order of the representatives of all classes. By Remark 3.1(ii) the prime divisors of the order are known. Thus for each prime p the conjugacy classes of the p-elements are recognizable.

If $\mathcal{X}(G)$ determines G up to isomorphism then the Sylow numbers of G are determined by the character table. By [Kim91, Satz 6.3] this holds especially for simple groups. If G is a semisimple group, i.e. the direct product of non-abelian simple groups, the character table determines the Sylow numbers as well (see [Kim91, Satz 6.3]).

We want to consider the case for G quasinilpotent:

Proposition 3.2

Suppose that $G/\mathrm{Z}(G)$ is a semisimple group. Then $\mathcal{X}(G)$ determines $\mathrm{sn}(G)$. In particular the character table determines $\mathrm{sn}(G)$ in the case that G is quasinilpotent.

Proof: By Proposition 2.7 Sylow numbers remain unchanged under central extensions and hence by Remark 3.1 and [Kim91, Satz 6.3] the first part of the assumption is proven. As quasinilpotent groups are central extensions of semisimple groups, the assertion holds. □

In Proposition 2.3 we analysed the situation if G has a normal subgroup K with $G_p \cap K = \{1\}$. The following proposition yields a correlation between $|K : \mathrm{C}_K(G_p)|$ and character tables $\mathcal{X}(G)$:

Proposition 3.3

Let $K \trianglelefteq G$ and $G_p \in \mathrm{Syl}_p(G)$, then it is possible to decide by the character table whether the intersection $\mathrm{C}_K(G_p) = K \cap \mathrm{C}_G(G_p)$ is trivial or not.

Especially, if K is cyclic of prime order q, then $n_p(G)$ may be calculated from $\mathcal{X}(G)$ provided $n_p(G/K)$ is known.

Proof: Regarding the character table we obtain by Remark 3.1 the conjugacy classes which are contained in K. By Remark 3.1 we can compute $|\mathrm{C}_G(h)|$ for each $h \in K$. If $|G_p|$ divides $|\mathrm{C}_G(h)|$ then $h \in \mathrm{C}_G(G_p^g)$ for some $g \in G$. Consequently $h^{g^{-1}}$ centralizes G_p. Conversely, if a non-trivial element h of K centralizes G_p, then $|G_p|$ divides the order of its centralizer.

If $q = p$ then $n_p(G) = n_p(G/K)$. In the other case we use Proposition 2.3. As K has prime order the index $|K : \mathrm{C}_K(G_p)|$ equals q if and only if there is a non-trivial conjugacy class $k^G \in K$ such that $|G_p|$ divides $|\mathrm{C}_G(k)|$. □

Note that Proposition 3.3 can be generalized to the case that K is cyclic (and not necessarily simple). As every subgroup of K is characteristic and therefore normal in G it is sufficient to assume that K is simple.

In general there is no method known for the computation of $|K : \mathrm{C}_K(G_p)|$. We can not decide by means of the character table whether the whole conjugacy class g^G of g is contained in $\mathrm{C}_K(G_p)$.

3.1 Nilpotent-by-nilpotent and supersolvable Groups

In this section we want to offer a method of how to calculate $n_p(G)$ out of $\mathcal{X}(G)$ in the case that G has a nilpotent normal subgroup N such that G/N is nilpotent or G is supersolvable. No further assumptions on the structure of Sylow p-subgroups are necessary in this case.

Theorem 3.4 ([KK15], Theorem 3.4.)

Suppose that the finite group G has a nilpotent normal subgroup N such that G/N is nilpotent then $\mathcal{X}(G)$ determines $\mathrm{sn}(G)$.

Proof: By Remark 3.1 we may assume that the Sylow numbers of all proper quotients of G are given. If $p \in \pi(N)$ then the intersection $G_p \cap N$ is normal in G, as $G_p \cap N$ is a characteristic subgroup of N. By Proposition 2.1 the Sylow p-number remains unchanged if we consider $n_p(G/(G_p \cap N))$ instead of $n_p(G)$.

So assume that $p \notin \pi(N)$. Then $G_p \cap N = 1$ and we obtain with Proposition 2.3 that

$$n_p(G) = \underbrace{n_p(G/N)}_{=1} n_p(G_pN) = n_p(G_pN).$$

If G has two minimal normal subgroups we apply Proposition 2.5. Now assume that G has a unique minimal normal subgroup of order q^t.

Consider $G_pN/\operatorname{Z}(N)$, where $\operatorname{Z}(N) \trianglelefteq G$ denotes the center of N. Note that $\operatorname{Z}(N) \neq \{1\}$. The center $\operatorname{Z}(N)$ is normal in G as for each $n \in \operatorname{Z}(N)$, $\tilde{n} \in N$ and $x \in G$ we have

$$(n^{-1})^x \tilde{n} n^x = x^{-1} n^{-1} \underbrace{x \tilde{n} x^{-1}}_{\in N} nx \overset{n \in \operatorname{Z}(N)}{=} x^{-1} \tilde{n}^{x^{-1}} x = \tilde{n}.$$

As normal subgroup $\operatorname{Z}(N)$ is the disjoint union of conjugacy classes. We want to prove the following: if $a \in \operatorname{C}_{\operatorname{Z}(N)}(G_p)$ then $a^G \in \operatorname{C}_{\operatorname{Z}(N)}(G_p)$.

By Theorem 1.3 we get

$$n_p(G_pN) = n_p(G_pN/\operatorname{Z}(N)) \cdot n_p(G_p \operatorname{Z}(N)).$$

Proposition 2.3 yields $n_p(G_p \operatorname{Z}(N)) = |\operatorname{Z}(N) : \operatorname{C}_{\operatorname{Z}(N)}(G_p)|$. Suppose $a \in \operatorname{C}_{\operatorname{Z}(N)}(G_p)$. By assumption G/N is nilpotent hence for each $\tilde{G_p} \in \operatorname{Syl}_p(G)$ there exists $n \in N$ such that $\tilde{G_p} = G_p^n$. It follows that $a \in \operatorname{C}_G(\tilde{G_p})$, as

$$\tilde{x}^a = (x^n)^a \overset{a \in \operatorname{Z}(N)}{=} (x^a)^n \overset{a \in \operatorname{C}_{\operatorname{Z}(N)}(G_p)}{=} G_p^n = \tilde{x},$$

where $\tilde{x} = x^n \in \tilde{G_p}$. Let $a^g \in a^G$. Since $(G_p)^{a^g} = G_p$ we see that a^G is contained in $\operatorname{C}_G(G_p)$. Therefore $|n_p(G_p \operatorname{Z}(N))|$ may be computed by the character table. Note that the Sylow number $n_p(G_pN/\operatorname{Z}(N)) = n_p(G/\operatorname{Z}(N))$ is given by induction and thus $n_p(G)$ is determined. □

In the next proposition we want to consider supersolvable groups, i.e. finite solvable groups where each chief factor is simple:

Proposition 3.5

Suppose that G has a supersolvable normal subgroup N such that each chief factor of N is also a chief factor of G. Assume further that $n_p(G/N)$ is known for each $p \in \pi(G/N)$. Then $n_p(G)$ may be calculated from $\mathcal{X}(G)$.

We immediately obtain the following result for $N = G$:

Corollary 3.6

Let G be a supersolvable group. Then the Sylow numbers $\operatorname{sn}(G)$ *are known.*

Proof (Proposition 3.5): We use induction: Let K be a minimal normal subgroup of N, i.e. $K \cong C_q$ is cyclic of order q. If $p = q$, then $n_p(G) = n_p(G/K)$. Otherwise if $p \neq q$ we can use Proposition 3.3 to compute $|\operatorname{C}_K(G_p)|$. Thus Theorem 1.3 yields

$$n_p(G) = \underbrace{n_p(K)}_{=1} n_p(G/K) \underbrace{|K : \operatorname{C}_K(G_p)|}_{\in\{1,q\}}$$

and as $n_p(G/K)$ is given by induction the claim holds. □

3.2 Groups with cyclic Sylow p-subgroups

The following result was already given by Gabriel Navarro in [Nav04]. We want to lay open a more direct proof that $\mathcal{X}(G)$ determines $n_p(G)$ provided that G_p is cyclic:

Proposition 3.7

Suppose that G has a cyclic Sylow p-subgroup then $n_p(G)$ may be calculated by means of the character table.

The following corollary gives a correlation between orders of centralizers and normalizers:

Corollary 3.8

Let G be a group with cyclic Sylow p-subgroup. If $U \subset G_p$ then

$$\mathrm{N}_G(G_p) \subset \mathrm{N}_G(U) \Leftrightarrow \mathrm{C}_G(G_p) \subset \mathrm{C}_G(U).$$

Proof: In order to prove the statement assume that $\mathrm{N}_G(G_p) < \mathrm{N}_G(U)$. Let $y \in \mathrm{N}_G(U) \setminus \mathrm{N}_G(P)$. If $x \in G_p$ we have that $x^y \in \mathrm{C}_G(U)$, as

$$(x^{-1})^y u x^y = y^{-1} x^{-1} \underbrace{y u y^{-1}}_{\in U} x y \overset{G_p \text{ abelian}}{=} y^{-1} u^{y^{-1}} y = u \text{ for } u \in U.$$

Consider $\mathrm{C}_G(G_p)$. As G_p is central in $\mathrm{C}_G(G_p)$ there exists at least one element $x^y \in G_p^y$ which is not contained in $\mathrm{C}_G(G_p)$ (otherwise $G_p = G_p^y$, a contradiction to the fact that $y \notin \mathrm{N}_G(G_p)$). Thus $\mathrm{C}_G(G_p) < \mathrm{C}_G(U)$.

On the other hand let $\mathrm{C}_G(G_p) < \mathrm{C}_G(U)$, i.e. there exists an element $y \in \mathrm{C}_G(U) \setminus \mathrm{C}_G(G_p)$ such that $y^{-1} h y \neq h$. Assume that $y \in \mathrm{N}_G(G_p)$. By Theorem 1.23 $\mathrm{N}_G(G_p)/\mathrm{C}_G(G_p)$ is isomorphic to a subgroup of $\mathrm{Aut}(G_p)$. The homomorphism $\mathrm{N}_G(G_p) \to \mathrm{Aut}(G_p)$ with kernel $\mathrm{C}_G(G_p)$ is given as $z \mapsto \sigma_z$, where σ_z denotes the conjugation with $z \in \mathrm{N}_G(G_p)$ in G_p. Obviously $\sigma_y \neq \mathrm{id}_{\mathrm{Aut}(G_p)}$.

By [Hup67, I §13, Satz 13.9] we obtain that $\mathrm{Aut}(G_p)$ is cyclic of order $p^{a-1}(p-1)$ if p is odd or $|\mathrm{Aut}(G_2)| = 2^{a-1}$.

If $p = 2$ the factor group $\mathrm{N}_G(G_2)/\mathrm{C}_G(G_2)$ is isomorphic to a subgroup of $\mathrm{Aut}(G_2)$ and thus a 2-group. As G_2 is a maximal 2-group of G and contained in $\mathrm{C}_G(G_2)$ this yields that $\mathrm{N}_G(G_2)/\mathrm{C}_G(G_2) \cong \{1\}$ and $\sigma_y = \mathrm{id}_{\mathrm{Aut}(G_p)}$, a contradiction. Thus the claim holds for $p = 2$.

Now let p be odd. As $G_p \subset \mathrm{C}_G(G_p)$ the factor group $\mathrm{N}_G(G_p)/\mathrm{C}_G(G_p)$ is a p'-group and moreover a subgroup of $\mathrm{Aut}(G_p) \cong Aut(C_p) \times X$ where X is a p-group and C_p the cyclic group of order p. Every element of $\mathrm{N}_G(G_p)/\mathrm{C}_G(G_p)$ has order dividing $p-1$

and is isomorphic to an element of the form $\{\phi : G_p \to G_p, x \mapsto x^l \ : \ 1 \leq l < p\}$, see [May98, Proposition 1.17].

For $y \in \mathrm{N}_G(G_p) \setminus \mathrm{C}_G(G_p)$ the element $y\,\mathrm{C}_G(G_p)$ is non-trivial in $\mathrm{N}_G(G_p)/\,\mathrm{C}_G(G_p)$, so $y\,\mathrm{C}_G(G_p) \mapsto \sigma_y \neq 1$ and $\sigma_y(x) = x^l$ for some $1 \leq l < p$. But

$$u \overset{y \in \mathrm{C}_G(U)}{=} \sigma_y(u) = u^l \neq u,$$

as $l < p$ and u is a p-element. Thus y can not be contained in $\mathrm{N}_G(G_p)$ and the claim holds. □

Proof (of Proposition 3.7): The result is clear when the Sylow p-subgroup is central in G. Thus we assume that $|\mathrm{C}_G(G_p)| < |G|$. Let h be a generator of a cyclic Sylow p-subgroup G_p and p^a the order of G_p.

By Corollary 3.8 subgroups generated by p-elements with minimal centralizer order have normalizers of minimal order as well. Let M be the number of conjugacy classes of p-elements g such that $|\mathrm{C}_G(g)|$ is minimal. Because Sylow p-subgroups are cyclic we see that there is a certain $n \in \mathbb{N}$ such that all p-elements of order bigger than p^n have centralizers of minimal length. The number of such p-elements contained in G_p is $p^a - p^n$. As $|\mathrm{C}_G(G_p)| = |\mathrm{C}_G(g)|$ for all $g \in G_p$ with $|g| > p^n$ the size of conjugacy classes of elements with order bigger than p^n coincide. Thus we get

$$p^a - p^n = M \cdot \frac{|G|}{|\mathrm{C}_G(G_p)|} \Big/ \frac{|G|}{|\mathrm{N}_G(G_p)|},$$

as the number of all such p-elements in G is $M \cdot |G/\,\mathrm{C}_G(G_p)|$. Note that a p-element with centralizer of minimal order is not contained in different Sylow p-subgroups otherwise a conjugated Sylow p-subgroup would be contained in $\mathrm{C}_G(G_p)$ which is not possible. Consequently

$$n_p(G) = M \cdot \frac{L}{p^a - p^n},$$

where $L = |G|/|\mathrm{C}_G(g)|$ and g is a p-element with minimal centralizer order. M, L and p^a are given by $\mathcal{X}(G)$. By Sylow's Theorem $n_p(G) \equiv 1 \bmod p$ and there is precisely one $n \in \mathbb{N}_0$ such that this congruence holds and $n_p(G)$ is determined by $\mathcal{X}(G)$. □

Note that this proof is based on the knowledge of orders of centralizers and does not involve the values of the character table. Thus a generalization of the results is possible, see Chapter 5.

Another proof of this result is given by the use of block theory: The order of the centralizer $|\mathrm{C}_G(G_p)|$ coincides with the smallest order of the centralizer of a p-element. The inertia group of the principal p-block of $\mathrm{C}_G(G_p)$ in $\mathrm{N}_G(G_p)$ is $\mathrm{N}_G(G_p)$. Denote by B_0 the principal p-block of G. Then the number $|\mathrm{N}_G(G_p) : \mathrm{C}_G(G_p)|$ coincides with the number of non-exceptional characters in B_0, see [Dad66]. Similarly to the proof of Gabriel Navarro we need the character values given in the character table.

3.3 Groups with one non-cyclic Sylow p-subgroup

As we know that $n_p(G)$ is given by the character table for cyclic Sylow p-subgroups, we analysed the situation that G has exactly one non-cyclic Sylow p-subgroup:

Proposition 3.9

Let G be a group of order $|G| = q^a \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$, where p_i are pairwise different primes and q is a prime different from all p_i. Assume further that all Sylow p_i-subgroups of G are cyclic. Then $X(G)$ determines $\mathrm{sn}(G)$.

Proof: First assume that G is solvable. Let G be a counterexample of minimal order. Note that the result holds if G is a p-group respectively if G is simple. If N is a normal subgroup of G then G/N suffices the hypothesis of the theorem as well. So we may assume that the result holds for G/N. Assume that N is a minimal normal subgroup of G and that G is not simple. If N is not a q-group then N is cyclic and $|N| = p_i^k$ for some i and $k \leq a_i$. By Lemma 3.3 we get that $\mathrm{sn}(G)$ is determined by $X(G)$. If N is a q-group then $n_q(G) = n_q(G/N)$ is known by induction and $n_{p_i}(G)$ is determined by Theorem 2.1.

Now suppose that G is not solvable. Hence 2 divides $|G|$ and it follows by [Hup67, IV, Satz 2.8] that $q = 2$. For each cyclic Sylow p-subgroup the Sylow p-number is determined by Proposition 3.7 and it remains the calculation of $n_2(G)$. Similarly to the solvable case we obtain the insight that a minimal counterexample does not have minimal normal subgroups which are cyclic or of order 2^m. Therefore G has no minimal solvable subgroup.

Consequently the generalized Fitting subgroup $\mathrm{F}^*(G)$ only consists of the layer $\mathrm{E}(G)$. As direct products of isomorphic groups do not have cyclic Sylow p-subgroups the layer $\mathrm{E}(G)$ is a simple non-abelian group S. Then G is isomorphic to an almost simple group of type S. By [Asc00], see also [HB82, p. 190], the only simple groups where all Sylow subgroups of odd order are cyclic are $A_1(2^f)$, $f > 1$, $A_1(p)$, $p > 3$, ${}^2B_2(2^{2n+1})$ and the Janko group J_1 of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.

All these simple groups have a cyclic outer automorphism group. Thus for each divisor m of $|\mathrm{Out}(S)|$ there is precisely one almost simple group of type S of order $m \cdot |S|$. By Remark 3.1 and [Kim90] $X(G)$ determines G up to isomorphism and in particular $\mathrm{sn}(G)$ is given by the character table. □

Note that Proposition 3.9 does not provide a method how one can compute the Sylow q-number.

Remark 3.10: (i) In [Nav04, Corollary 5] Martin Isaacs and Gabriel Navarro proved that if G has abelian Sylow p-subgroups then $X(G)$ determines the order of the p-elements.

(ii) Note that for an application of [Nav04, Theorem 8] for p-elements $g \in G$ the field $F_g := \mathbb{Q}(\chi(g); \chi \in \mathrm{Irr}\, G)$ has to be computed. Thus the knowledge of the character values of the ordinary irreducible characters is essential.

If G has Sylow p-subgroups with trivial intersection then $n_p(G)$ can be determined as well:

Proposition 3.11

Suppose that G has Sylow p-subgroups where $G_p \cap G_p^y = \{1\}$ for each $y \notin \mathrm{N}_G(G_p)$. Then $\mathcal{X}(G)$ determines $n_p(G)$.

Proof: Let $L_1, \ldots, L_k$ be the length of all conjugacy classes of p-elements. If G has Sylow p-subgroups with trivial intersection we see that

$$L_1 + \ldots + L_k = n_p(G) \cdot (|G_p| - 1)$$

and thus $\mathcal{X}(G)$ determines $n_p(G)$. □

3.4 Frobenius and 2-Frobenius Groups

At the end of Chapter 2 we mentioned the Sylow numbers of Frobenius groups. If G is solvable with more than one prime graph component then G is either a Frobenius group or 2-Frobenius group and G has exactly two components, where one consists of the primes dividing the order of the lower Frobenius complement.

A group G is called 2-Frobenius group if there exists a normal series $1 < M < T < G$ such that T is a Frobenius group with kernel M and G/M is a Frobenius group with kernel T/M.

Corollary 3.12

(i) Let G be a Frobenius group. Then $\mathcal{X}(G)$ determines $\mathrm{sn}(G)$.

(ii) Let G be a 2-Frobenius group. Then $\mathcal{X}(G)$ determines $\mathrm{sn}(G)$.

Proof: (i) By Proposition 2.8 it suffices to show that the Sylow numbers of a Frobenius complement H are determined. Denote by K the Frobenius kernel then $\mathcal{X}(G)$ determines $\mathcal{X}(G/K) = \mathcal{X}(H)$. But Sylow subgroups of odd order of H are cyclic, see [Hup67, II, §8]. Thus H satisfies the hypothesis of Proposition 3.9 and the result follows.

(ii) Let G be a 2-Frobenius group then it has a normal series

$$1 < M < T < G$$

such that T is a Frobenius group with kernel M and G/M is a Frobenius group with kernel T/M. By the structure theorems of Frobenius groups (see [Hup67, II, §8] it follows that M as Frobenius kernel is nilpotent. The quotient T/M is both nilpotent (as Frobenius kernel of G/M) and has cyclic Sylow p-subgroups of odd order (as Frobenius complement of T). If 2 divides $|T/M|$ then the Sylow 2-subgroup of T/M is cyclic as well, otherwise $(T/M)_2$ would be a generalized quaternion group with a unique involution, a contradiction to the fixpointfree operation of G/T on T/M. Thus T/M

is cylic. Moreover because automorphism groups of cyclic groups are abelian we see that G/T has to be cyclic again as direct product of cyclic Sylow p-subgroups.

Now let G be a counterexample of minimal order. By Proposition 2.5 we see that M is not a direct product of a p-group and a q-group where $p \neq q$ are primes. Thus M is a p-group. For $q \neq p$ all Sylow q-subgroups of G are cyclic as $\gcd(|T/M|, |G/T|) = 1$. But by Proposition 3.9 we see that G is also in this case not a counterexample. This completes the proof. □

Note that we are not able to compute the Sylow numbers of non-cyclic Sylow p-subgroup by only the character table.

3.5 A Generalization of Theorem 1.22

In the first chapter we mentioned in Theorem 1.22 a result of Navarro and Rizo for p-solvable groups. We want to give a similar proof for non p-solvable groups where $G/O_{p'}(G)$ is determined up to isomorphism by the spectral table (respectively by $\mathbb{Z}G$):

Proposition 3.13

Let G be a group such that $G/O_{p'}(G)$ is determined up to isomorphism by $\mathrm{Spec}(G)$*. Then $n_p(G)$ is determined by* $\mathrm{Spec}(G)$*.*

In order to prove Proposition 3.13 we need to analyse the proof given in [NR16]. The theorem is based on the following formula of Helmut Wielandt:

Theorem 3.14 ([Wie60], Hauptsatz 2.2)

Suppose that G_p is a p-group acting on a p'-group G. Then

$$|\mathrm{C}_G(G_p)| = \left(\prod_{x \in G_p} \frac{|\mathrm{C}_G(x)|}{|\mathrm{C}_G(x^p)|^{1/p}}\right)^{\frac{p}{(p-1)|G_p|}}.$$

A different proof of the previous theorem can also be found in [NR16, Theorem 1.1].

Proof (Proof of Proposition 3.13): Assume that G is a group such that $G/O_{p'}(G)$ is determined up to isomorphism. Let $O := O_{p'}(G)$. Using Proposition 2.3 we obtain

$$n_p(G) = n_p(G/O)|O : \mathrm{C}_O(G_p)|.$$

Both $n_p(G/O)$ and $|O|$ are given by $\mathrm{Spec}(G)$, thus we need to calculate $|\mathrm{C}_O(G_p)|$. By applying Theorem 3.14 we get

$$|\mathrm{C}_O(G_p)| = \left(\prod_{x \in G_p} \frac{|\mathrm{C}_O(x)|}{|\mathrm{C}_O(x^p)|^{1/p}}\right)^{\frac{p}{(p-1)|G_p|}}.$$

Now consider $|x^G \cap G_p|$ for some $x \in G$. The epimorphism $\varphi : G \to G/O$ yields a natural bijection $G_p \Leftrightarrow G_pO/O$, as $\gcd(G_p, O) = 1$. Denote by $\bar{x}$, $\bar{G}$ respectively $\bar{G_p}$ the images of x, G and G_p in G/O. We want to prove that $|x^G \cap G_p| = |\bar{x}^{\bar{G}} \cap \bar{G_p}|$. As $\varphi|_{G_p}$ is a injective map from G_p to $\bar{G_p}$ for each Sylow p-subgroup of G we see that $x \in G$ is a p-element if and only if xO is a p-element.

If $x^G \cap G_p = \emptyset$ for some Sylow p-subgroup then x is not a p-element (note that x^G can not be contained in any other Sylow p-subgroup by Sylow's Theorem) and $|x^G \cap G_p| = |\bar{x}^{\bar{G}} \cap \bar{G_p}| = 0$. Thus assume that x is a p-element which lies in G_p.

Let $yO \in \bar{x}^{\bar{G}} \cap \bar{G_p}$ and assume that $y \in G_p$ is a preimage of yO in G. We want to show that $\varphi|_{G_p}$ is surjective, i.e. $y \in x^G$. Let $z \in x^G \cap y \cdot O$. Note that z is a p-element, as $z = x^g$ for some $g \in G$. Then we have $\langle y \rangle \cdot O = \langle z \rangle \cdot O$ and $\langle y \rangle$ and $\langle z \rangle$ are Sylow p-subgroups in $\langle y \rangle O$. By Theorem 1.1 $\langle y \rangle$ and $\langle z \rangle$ are conjugated via an element h in O. As $y \cdot O = z \cdot O$ this yields that $z^h = y$ and $y \in z^G \cap G_p = x^G \cap G_p$. It follows that $\varphi|_{G_p}$ is a bijection and $|x^G \cap G_p| = |\bar{x}^{\bar{G}} \cap \bar{G_p}|$.

Next we need to prove that for $x \in G_p$ the centralizer $\mathrm{C}_O(x)$ is given by $\mathcal{X}(G)$. The map $\mathrm{C}_{G/O}(xO) \to \mathrm{C}_G(x)O/O$ is bijective for each p-element $x \in G$. In particular this yields

$$|\mathrm{C}_{G/O}(xO)| = |\mathrm{C}_G(x)O/O| = |\mathrm{C}_G(x)|/|\mathrm{C}_O(x)|$$

and both $|\mathrm{C}_{G/O}(xO)|$ and $|\mathrm{C}_G(x)|$ are given by $\mathrm{Spec}(G)$ (even by $\mathcal{X}(G)$). Now denote by $\{x_1, \ldots, x_l\}$ the representatives of conjugacy classes of p-elements in G. The Sylow p-subgroup G_p is given as disjoint union $|G_p| = k_1 \cup \ldots \cup k_l$ where $k_i = |x_i^G \cap G_p|$ for $1 \leq i \leq k$.

Applying Theorem 3.14 we see that

$$\begin{aligned} |\mathrm{C}_O(G_p)| &= \left(\prod_{x \in G_p} \frac{|\mathrm{C}_O(x)|}{|\mathrm{C}_O(x^p)|^{1/p}} \right)^{\frac{p}{(p-1)|G_p|}} \\ &= \left(\prod_{i=1}^{l} \frac{|\mathrm{C}_O(x_i)|^{k_i}}{|\mathrm{C}_O(x_i^p)|^{k_i/p}} \right)^{\frac{p}{(p-1)|G_p|}} . \end{aligned}$$

Now $k_i = |x_i^G \cap G_p| = |\bar{x}_i{}^{\bar{G}} \cap \bar{G_p}|$ is given as $\bar{G} := G/O_{p'}(G)$ is determined up to isomorphism. Both $|\mathrm{C}_O(x_i)|$ and $|\mathrm{C}_O(x_i^p)|$ are given by the spectral table and therefore $|\mathrm{C}_O(G_p)|$ is determined by $\mathrm{Spec}(G)$. □

Chapter 4

Sylow Numbers in Integral Group Rings

> In real life, I assure you, there is no such thing as algebra.
>
> *(Fran Lebowitz)*

In the previous chapter we analyzed the situation for character tables. So far we don't know whether there are non-isomorphic groups with isomorphic character tables where the Sylow numbers don't coincide. It seems that character tables don't provide sufficient information about the underlying groups.

The integral group ring $\mathbb{Z}G$ is defined as the free $\mathbb{Z}$-module with basis G. If G and H are isomorphic then their group rings are isomorphic as well. The reverse is not true in general: Martin Hertweck gave in [Her01] a counterexample of order $2^{21} \cdot 97^{28}$. As $\mathbb{Z}G$ determines the character table and the power map Theorem 1.22 implies that $n_p(G)$ is determined by $\mathbb{Z}G$ for p-solvable groups.

For finite groups (and in some sense for infinite groups) Sylow's Theorems (or for infinite groups Sylow-like theorems) give us the existence of maximal p-groups. Furthermore these maximal p-groups are conjugated in G. For $\mathbb{Z}G$ there are the following open questions:

$$\text{Is every } p\text{-subgroup of } V(\mathbb{Z}G) \text{ conjugate in } \mathbb{Q}G \text{ to a } p\text{-subgroup of } G? \tag{Q4}$$

$$\text{Is every } p\text{-subgroup of } V(\mathbb{Z}G) \text{ isomorphic to a } p\text{-subgroup of } G? \tag{Q5}$$

By $V(\mathbb{Z}G)$ we denote the normalized units in $\mathbb{Z}G$. Question (Q4) refers to the so-called strong and Question (Q5) to the so-called weak Sylow-like theorem.

In finite groups each p-subgroup is contained in a Sylow p-subgroup. It is still an open problem whether Question (Q4) resp. Question (Q5) have a positive answer for arbitrary groups. So far there are only partial results to both questions. In order to establish Sylow-like theorems in $\mathbb{Z}G$ one should first ask whether $\mathbb{Z}G \cong \mathbb{Z}H$ yields $n_p(G) = n_p(H)$ for each $p \in \pi(G)$ (see Question (Q2)).

In Chapter 3 we were able to improve the result of Gabriel Navarro and Noelia Rizo provided $G/O_{p'}(G)$ is given up to isomorphism by the spectral table. Due to the so-called F^*-Theorem 4.2 isomorphic group rings yield isomorphic groups provided the generalized Fitting subgroup $\mathrm{F}^*(G)$ is a p-group. This again leads to Question (Q2).

In this chapter we want to analyze the situation for groups with abelian Sylow p-subgroups respectively for groups with dihedral Sylow 2-subgroups. These groups are not necessarily solvable. Note that these results do not hold for character tables or spectral tables as our proofs rely on the F^*-Theorem.

First, we want to reconsider some known facts about $\mathbb{Z}G$.

Remark 4.1: Let G be a finite group.

(i) If $\mathbb{Z}G$ and $\mathbb{Z}H$ coincide then $X(G)$ and $X(H)$ are isomorphic. Thus, they share all properties mentioned in Remark 3.1 [Isa76, p. 3.17].

(ii) If $\mathbb{Z}G \cong \mathbb{Z}H$ then there exists a bijection $\sigma : G \to H$ such that the conjugacy classes of $g \in G$ and $\sigma(g)$ have representatives of same order and $|g^G| = |\sigma(g)^H|$. The power map on the classes is determined.

(iii) Let $N \trianglelefteq G$ and $\mathbb{Z}G \cong \mathbb{Z}H$. Then exists $M \trianglelefteq H$ such that $\mathbb{Z}G/N \cong \mathbb{Z}H/M$. If $s \in N$ then $\sigma(g) \in M$ where σ is the bijection mentioned in (ii).

(iv) A group is called p-constrained provided $G/O_{p'}(G)$ has a normal p-subgroup P such that $\mathrm{C}_{G/O_{p'}(G)}(P) \subset P$. If $\mathbb{Z}G \cong \mathbb{Z}H$ then G is p-constrained if and only if H is p-constrained.

The following theorem was obtained by L. L. Scott and K. W. Roggenkamp in 1987. A published account has been completed in [Her16].

Theorem 4.2 ([Her01], F*-Theorem)

Let G be a finite group which has a normal p-subgroup N containing its own centralizer, i.e. $\mathrm{C}_G(N) \subset N$. Let R be a p-adic ring. Then for any augmented automorphism α of RG which stabilizes the kernel of the natural map $RG \to RG/N$, the groups $G\alpha$ and G are conjugate in the units of G.

An equivalent version of the F^*-Theorem is the following for $R = \mathbb{Z}$:

Theorem 4.3 ([Sco87])

Assume that $\mathrm{F}^*(G)$ is a p-group then $\mathbb{Z}G \cong \mathbb{Z}H$ implies that $G \cong H$.

If G is a p-constrained group then $\mathbb{Z}G$ determines $G/O_{p'}(G)$ up to isomorphism. Each p-solvable group is p-constrained.

4.1 p-constrained Groups

A group G is called p-constrained if $\overline{G} := G/O_{p'}(G)$ has a normal p-subgroup $P \subset \overline{G}$ such that $\mathrm{C}_{\overline{G}}(P) \subset P$. In particular $\overline{G}$ is determined by $\mathbb{Z}G$. For that reason we will consider in this section mainly p-constrained groups.

Proposition 4.4
Suppose that G is p-constrained and $O_p(G) \neq \{1\}$. Then $\mathbb{Z}G$ determines $n_q(G)$ for each prime where $n_q(G/O_p(G))$ is known.

Proof: We use induction. By Theorem 4.2 $G/O_{p'}(G)$ is determined up to isomorphism. If $O_{p'}(G)$ is trivial then G is already given by $\mathbb{Z}G$. If $O_{p'}(G) \neq \{1\}$ then apply Proposition 2.5 with $M = O_p(G)$ and $N = O_{p'}(G)$. □

Corollary 4.5
Assume that G is solvable. Then $\mathbb{Z}G \cong \mathbb{Z}(H)$ yields $n_p(G) = n_p(H)$ for each $p \in \pi(G)$.

Proof: We use induction. It is clear if G resp. H is a p-group. Solvable groups are p-constrained for each prime p dividing $|G|$. Using induction we can assume that for each proper factor group of G the Sylow numbers are determined. Let N be a minimal normal subgroup of order q^a then apply Proposition 4.4. □

By Theorem 1.22 character tables and the knowledge of the power map of solvable groups determine the Sylow numbers of the underlying group. As the so-called spectral table is given by its integral group ring this gives us another proof of Corollary 4.5 with different methods.

Combining these results provides the following theorem:

Theorem 4.6
Assume that G is q-constrained. Then $\mathbb{Z}G$ determines $n_p(G)$ for each $p \notin \pi(O_{q'}(G))$. In particular $n_q(G)$ is given by $\mathbb{Z}G$.

Proof: By Theorem 1.3 we obtain that

$$n_p(G) = n_p(O_{q'}(G))n_p(G/O_{q'}(G))n_p(\mathrm{N}_{G_pO_{q'}(G)}(G_p \cap O_{q'}(G))).$$

As $p \nmid |O_{q'}(G)|$ we obtain $n_p(O_{q'}(G)) = 1$. By Theorem 4.2 $G/O_{q'}(G)$ is determined up to isomorphism thus $n_p(G/O_{q'}(G))$ is known. Theorem 3.13 gives an algorithm to compute $n_p(\mathrm{N}_{G_pO_{q'}(G)}(G_p \cap O_{q'}(G))) = |O_{q'}(G) \cap \mathrm{C}_G(G_p)|$ and therefore $n_p(G)$ is determined by $\mathbb{Z}G$. □

4.2 Groups with abelian Sylow p-subgroup

The structure of groups with abelian Sylow p-subgroups has been studied in various papers. John H. Walter proved in [Wal69] that $O^{2'}(G/O_{2'}(G))$ is isomorphic to a direct product of simple non-abelian groups with abelian Sylow 2-subgroups and an abelian 2-group. Wolfgang Kimmerle and Robert Sandling generalized the result for arbitrary primes p, see [KS95]. The case where all Sylow p-subgroups of G are abelian was studied by Aviad M. Broshi in [Bro71].

In this section we want to give proof that $\mathbb{Z}G$ determines $n_p(G)$ for abelian Sylow p-subgroups provided $O_{p'}(G)$ is solvable. The solvability of $O_{2'}(G)$ is always guaranteed by the Feit-Thompson Theorem and therefore is in this case dispensible.

Theorem 4.7
Assume that G has abelian Sylow 2-subgroups. Then $\mathbb{Z}G$ determines $n_2(G)$.

Proof: Suppose that G is a finite group with abelian Sylow 2-subgroup. By Theorem 1.24 G has a normal series

$$\{1\} < M < N < G$$

such that $M = O_{2'}(G)$, G/N has odd order and N/M is a direct product of of simple groups with abelian Sylow 2-subgroups and an abelian 2-group. It is clear that $n_2(G) = n_2(N)$ as $G_2 \subset N$ for every Sylow 2-subgroup.

Now assume that G is a counterexample of minimal order. In particular $\mathrm{Z}(G) = \mathrm{Z}(N) = \{1\}$ as by Proposition 2.7 Sylow numbers remain unchanged under central extensions. If G has a minimal normal subgroup V which is not solvable then V is a normal subgroup of N with $\mathrm{C}_N(V) \cap V = \mathrm{Z}(V) = \{1\}$. Denote by $\sigma_g : N \to N$ the conjugation of $g \in N$ in N. The map σ_g induces an inner automorphism $\sigma_g|_V$ on V via restriction. The homomorphism $\tau : \mathrm{Inn}(N) \to \mathrm{Inn}(V)$ with $\sigma_g \mapsto \sigma_g|_V$ has kernel $\ker(\tau) = \mathrm{C}_N(V)$. Therefore we obtain that $N = \mathrm{C}_N(V) \cdot V$ and as consequence $N = \mathrm{C}_N(V) \times V$.

For the Sylow 2-number we conclude that $n_2(N) = n_2(V) \cdot n_2(\mathrm{C}_N(V))$ by Proposition 2.5 and $n_2(\mathrm{C}_N(V)) = n_2(N/V) = n_2(G/V)$. By Remark 4.1 V is determined up to isomorphism and $n_2(G/V)$ is known as G is a counterexample of minimal order. So G does not have a non-solvable minimal normal subgroup.

If $M = \{1\}$ then N has to be a normal 2-subgroup and particularly a normal Sylow 2-subgroup. So let $M \neq \{1\}$. By Proposition 2.5 G only has one minimal normal subgroup V and V is a q-group for some prime $q \in \pi(G)$. The socle $\mathrm{soc}(G) = V$ is contained in M as $M \neq 1$. The Fitting subgroup $\mathrm{F}(G)$ is a q-group as well.

Consider the generalized Fitting subgroup $F := \mathrm{F}^*(G)$. Assume that F is not solvable. Then the components of the layer $\mathrm{E}(G)$ include simple groups with abelian Sylow 2-subgroups. By [Wal69] these finite simple groups are either $A_1(2^n)$, $A_1(q)$ with

$q = p^f$ and $q \equiv \pm 3 \bmod 8$, ${}^2G_2(q)$ with $q = 3^{2n+1}$ or J_1. The Schur multipliers of these groups do not involve odd primes. As the Schur multiplier coincides with the center of $\mathrm{E}(G)$ this yields $V \subset \mathrm{Z}(\mathrm{E}(G))$. But the order of $\mathrm{Z}(\mathrm{E}(G))$ is a prime power of 2 and thus the center is trivial. The layer is isomorphic to the direct product of simple non-abelian groups and therefore G has a minimal normal non-solvable subgroup, constituting a contradiction.

It follows that $\mathrm{F}^*(G) = \mathrm{F}(G)$. As $O_{q'}(G) = \{1\}$ and $\mathrm{F}(G)$ is a q-group with $\mathrm{C}_G(\mathrm{F}(G)) = \mathrm{C}_G(\mathrm{F}^*(G)) \subset \mathrm{F}^*(G)$ we see that G is q-constrained and determined up to isomorphism by Theorem 4.2. Then G is not a counterexample. □

For groups with abelian Sylow p-subgroups we need an additional assumption on $O_{p'}(G)$:

Theorem 4.8

Let G be a group with abelian Sylow p-subgroup and assume that $O_{p'}(G)$ is solvable. Then $\mathbb{Z}G$ determines $n_p(G)$.

Proof: Let G be a minimal counterexample thus $\mathrm{Z}(G) = O_p(G) = \{1\}$ and G has only one minimal normal subgroup. By Theorem 1.26 G has a normal series

$$\{1\} < O_{p'}(G) < N < G$$

where G/N is a p'-group and $N/O_{p'}(G)$ is the direct product of simple groups with abelian Sylow p-subgroups and an abelian p-group.

By assumption the p'-core $O_{p'}(G)$ is solvable. If G has a unique minimal normal non-solvable subgroup then $O_{p'}(G) = \{1\}$ and $n_p(G) = n_p(N)$ is given by $\mathbb{Z}G$, as N is determined by $\mathbb{Z}G$ as direct product of simple groups with abelian Sylow 2-subgroup and an abelian 2-group. Thus G is not a counterexample.

Let F be a maximal normal subgroup of order q^a, i.e. $\mathrm{F}(G) = F$. If $\mathrm{E}(G) = \{1\}$ then by [HB82, X, Theorem 13.12] we obtain $\mathrm{C}_G(F) \leq F$ and G is q-constrained. Applying Theorem 4.6 yields that $n_p(G)$ is given by $\mathbb{Z}G$.

If $\mathrm{E}(G) \neq \{1\}$ then there exists at least one non-trivial component. Then p is a divisor of $|\mathrm{E}(G)|$ as otherwise $\mathrm{E}(G) \subset O_{p'}(G)$ would be a contradiction to the assumption that $O_{p'}(G)$ is solvable. Thus there exists a component $H \leq G$ where p divides $|H|$. Note that $\mathrm{Z}(H)$ centralizes H_p.

We want to prove that $\mathrm{Z}(\mathrm{E}(G)) \subset \mathrm{C}_G(G_p)$, i.e. $\mathrm{Z}(\mathrm{E}(G))$ centralizes a Sylow p-subgroup and vice versa. Let $G_{(j)} \leq G$ be a subgroup of the subnormal series $\{1\} < O_{p'}(G) < G_{(j)} \leq N < G$ such that $G_{(j)} \neq H$. By [HB82, X, Theorem 13.18] it is either $[H, G_{(j)}] = \{1\}$ or $H \leq [H, G_{(j)}]$. If $H \leq [H, G_{(j)}]$ then

$$\begin{aligned} \{1\} \neq H/\mathrm{Z}(H) &= HO_{p'}(G)/O_{p'}(G) \\ &\leq [HO_{p'}(G)/O_{p'}(G), G_{(j)}/O_{p'}(G)] \\ &= \{1\} \end{aligned}$$

a contradiction. Thus H and $G_{(j)}$ commute and $Z(H) \subset C_G(G_{(j)p})$. In particular this holds for arbitrary components and we have that $Z(E(G)) \subset C_G(G_p)$. Using Theorem 1.3 we see that

$$n_p(G) = n_p(G/O_{p'}(G))|Z(E(G)) : C_{Z(E(G))}(G_p)| = n_p(G/O_{p'}(G))$$

and $n_p(G/O_{p'}(G))$ is known as $|G/O_{p'}(G)| < |G|$. □

4.3 Sylow Numbers of Groups with non-connected Prime Graph

Julian S. Williams studied in [Wil81] groups with non-connected prime graphs.

Theorem 4.9 ([Wil81], Theorem A and Proposition 1)

Let G be a group with non-connected prime graph $\Gamma(G)$. If $2 \in \pi(G)$ denote by π_0 the component of $\Gamma(G)$ which contains 2. Then G has one of the following structures:

(i) G is a Frobenius or 2-Frobenius group,

(ii) G is non-abelian simple,

(iii) G has a nilpotent normal π_0-subgroup $N \trianglelefteq G$ such that G/N is simple,

(iv) G has a simple non-abelian normal subgroup $S \trianglelefteq G$ and G/S is a solvable π_0-subgroup,

(v) G has a normal series $1 < N < T < G$ such that N is a nilpotent π_0-subgroup, T/N is simple non-abelian and G/T is a solvable π_0-group.

The Sylow numbers of Frobenius and 2-Frobenius groups are given by the character table (see Corollary 3.12, the Sylow numbers of non-abelian simple groups are known as well.

Now consider groups such that G has a nilpotent normal subgroup $N \trianglelefteq G$ and G/N is simple:

Proposition 4.10

Suppose that G has a nilpotent normal subgroup N such that G/N is simple. Then $\mathbb{Z}G$ determines $\mathrm{sn}(G)$.

Proof: If G/N is simple abelian then $\mathrm{sn}(G)$ is determined by $\mathcal{X}(G)$ using Theorem 3.4. If G is non-abelian simple then G is determined up to isomorphism by $\mathcal{X}(G)$.

Let G be a counterexample of minimal order. By Proposition 2.5 we can assume that N is a q-group. Consider the centralizer $C_G(N) \trianglelefteq G$. If $C_G(N) \subset N$ then G is q-constrained and thus determined up to isomorphism by $\mathbb{Z}G$. Thus assume that $C_G(N) \cap G \backslash N \neq \emptyset$. As the quotient map $G \to G/N$ is surjective the image $C_G(N)N/N$

is normal in G/N. By assumption G/N is simple and $\mathrm{C}_G(N)N/N$ either equals $\{1\}$ or G/N.

If $\mathrm{C}_G(N)N/N = \{1\}$ then $\mathrm{C}_G(N)$ was already a subset of N, a contradiction. Therefore we have $\mathrm{C}_G(N)N/N = G/N$ and $G = \mathrm{C}_G(N)N$. As q-group N has a non-trivial center $\mathrm{Z}(N) \neq \{1\}$ which is also central in $\mathrm{C}_G(N)$. Thus the intersection $\mathrm{Z}(N) = N \cap \mathrm{C}_G(N)$ is central in G and $n_p(G) = n_p(G/\mathrm{Z}(N))$, as Sylow numbers remain unchanged under central extensions (see Proposition 2.7). Thus G is not a minimal counterexample. □

For the remaining cases we could study the outer automorphism groups of those simple groups which have two or more components. A list of these groups can be found in [IY96; Kon89; Wil81]. It seems possible to obtain results for the remaining cases.

4.4 Groups with dihedral Sylow 2-subgroup

In [GW65] groups with dihedral Sylow 2-subgroups were studied. Similarly to abelian Sylow 2-groups Daniel Gorenstein and John H. Walter gave a very specific description of groups containing dihedral Sylow 2-subgroups.

Proposition 4.11

Let G be a group such that G_2 is a dihedral group. Then $\mathbb{Z}G$ determines $n_p(G)$ for each prime $p \in \pi(G)$.

Proof: Let G be a minimal counterexample to the claim. Thus we can assume that $\mathrm{Z}(G) = \{1\}$ and G has only one minimal normal subgroup.

If $\mathbb{Z}G \cong \mathbb{Z}H$ then H has also a dihedral Sylow 2-subgroup, see [Mar15, Theorem 2]. By Theorem 1.27 and [Ben81, Theorem of Gorenstein and Walter] we have to consider the following three cases:

(i) $G/O_{2'}(G)$ is isomorphic to A_7,

(ii) $G/O_{2'}(G)$ is isomorphic to a Sylow 2-subgroup of G or

(iii) $G/O_{2'}(G)$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}_2(q)$ containing $A_1(q)$ or the projective general linear group $\mathrm{PGL}_2(q)$ as normal subgroup of odd index.

If $G/O_{2'}(G) \cong G_2$ we see that G is solvable and $n_p(G)$ is given by Theorem 1.22. Thus G is not a counterexample.

If $O_{2'}(G) = \{1\}$ then G is isomorphic to A_7 or a subgroup of $\mathrm{P\Gamma L}_2(q)$. As simple groups are determined up to isomorphism by $\mathcal{X}(G)$ this yields that A_7 is not a minimal counterexample. For $A_1(q) \leq G \leq \mathrm{P\Gamma L}_2(q)$ we can decide whether 2 is a divisor of $G/A_1(q)$ or not. If $2 \nmid |G/A_1(q)|$ then $\mathrm{Out}(A_1(q))$ is cyclic by [Wil09, Theorem 3.2b)] and for each divisor of $|G/A_1(q)|$ there is precisely one subgroup of $\mathrm{Out}(A_1(q))$. If 2 divides $|G/A_1(q)|$ then $\mathrm{PGL}_2(q) \subset G$ and $G/\mathrm{PGL}_2(q)$ is cyclic. Again G is determined up to isomorphism by $\mathbb{Z}G$. Thus we can assume that $O_{2'}(G) \neq \{1\}$.

Now consider the generalized Fitting subgroup $\mathrm{F}^*(G)$. Note that $\mathrm{F}(G)$ is a p-group as $\mathrm{O}_{2'}(G) \neq \{1\}$ is solvable and thus contains a minimal normal and solvable subgroup. If $\mathrm{F}^*(G) = \mathrm{F}(G)$ then $\mathbb{Z}G$ determines G (and thus $\mathrm{sn}(G)$) up to isomorphism by Theorem 4.2. Thus assume that $\mathrm{E}(G) \neq \{1\}$.

As G has only a unique non-solvable composition factor we get $\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)) \cong \mathrm{A}_7$ or $\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)) \cong A_1(q)$. The Schur multiplier of A_7 is trivial, thus $\mathrm{E}(G) = A_7$, a contradiction to the claim that G has exactly one minimal normal subgroup which is solvable. If $\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)) \cong A_1(q)$ we see that the Schur multiplier is either trivial or of order two respectively of order six (see [Kar94, Chapter 5.8.D]). As $2 \nmid |O_{2'}(G)|$ the center of $\mathrm{E}(G)$ either is trivial or has order three. If $|\mathrm{Z}(\mathrm{E}(G))| = 1$ then $\mathrm{E}(G) = A_1(q) \trianglelefteq G$ is again a contradiction. If $|\mathrm{Z}(\mathrm{E}(G))| = 3$ then

$$n_p(G) = n_p(G/\mathrm{Z}(\mathrm{E}(G)))|\mathrm{Z}(\mathrm{E}(G)) : \mathrm{C}_{\mathrm{Z}(\mathrm{E}(G))}(G_p)|$$

and $|\mathrm{Z}(\mathrm{E}(G)) : \mathrm{C}_{\mathrm{Z}(\mathrm{E}(G))}(G_2)| \in \{1, 3\}$ is given by $\mathbb{Z}G$ by Proposition 3.3. Thus G is not a minimal counterexample. □

Additional premises on the structure of Sylow p-subgroups yield results for other (not necessarily p-solvable) groups. So far there is no method known to deal with arbitrary groups (respectively with groups without additional assumptions to the Sylow p-subgroup). It seems promising to study the outer automorphism groups of all finite simple groups more precisely.

Chapter 5

Sylow Numbers in Class Structures

> The art of doing mathematics consists in finding that special case which contains all the germs of generality.
>
> *(David Hilbert)*

Character tables provide information about the group order, the length of conjugacy classes or the poset of normal subgroups under inclusion. Some of the results mentioned in Chapter 3 did not need the entries given by the character table, but rather specific information which can be computed out of $\mathcal{X}(G)$. In 1995 Wolfgang Kimmerle and Robert Sandling developed in [KS95, Section 1] the so-called class structures that are based on the idea of character tables. The premises for groups to be in class correspondence of type JHS are weaker than the assumption that their character tables coincides. In this chapter we want to prove that Sylow numbers of cyclic Sylow p-subgroups are determined by class structures of type JHS. Furthermore the Sylow numbers of nilpotent-by-nilpotent groups are given by class structures of the same type.

In the first part we want to recall the definitions of class structures and summarize the relations between class structures, character tables and spectral tables. Subsequently we want to generalize Lemma 1.21 and Theorem 3.4.

Definition 5.1 (Class Structures): *Let G and H be finite groups.*

(i) A class structure of G is a labelled poset given by its normal subsets such that the label contains at least the size of the corresponding subset.

(ii) A class structure X is called of type $\mathcal{JH}$ if the labels give the information which elements of X form normal subgroups and which are of the form $N \cdot C$ for some conjugacy class C of G and $N \trianglelefteq G$.

X is called of type JHS if it is of type JH and additionally the labels indicate for all primes p which conjugacy classes contain elements of order a power of p.

X is called of type JHB if X is of type JH and the labels contain the power map on the conjugacy classes.

The term JH is an abbreviation for Jordan-Hölder, as class structures of type JH determine composition factors and their multiplicities. Each character table is a class structure of type JHS where S as short form for Sylow indicates the given information for p-elements. Groups with isomorphic spectral tables are often called Brauer pairs and correspond to class structures of type JHB where B is an acronym for Brauer.

To compare given class structures we need to introduce the so-called class correspondences:

Definition 5.2 (Class Correspondence): *Let $\tau : G \to H$ be a bijection.*

(i) τ is called a class correspondence if it gives a bijection on the conjugacy classes of G and H.

(ii) τ is called of type JH if it is a bijection on the normal subgroups of G and H and if $\tau(N \cdot x^G) = \tau(N) \cdot \tau(x^G)$ for each normal subgroup N and each conjugacy class x^G of G (where $x \in G$).

(iii) τ is called of type JHS if it is of type JH and $\tau(x)$ is a p-element if, and only if, x is a p-element for each prime p.

(iv) τ is called of type JHB if it is of type JH and $\tau(C)^n = \tau(C^n)$ for each $n \in \mathrm{N}$ and all conjugacy classes C of G.

In order to use class structures similarly to character tables we need to prove that class structures apply to quotient groups of G:

Proposition 5.3 ([KS95], Proposition 1.5)
Let $N \trianglelefteq G$. Then a class structure of type JH, JHS or resp. JHB defines naturally a class structure of the same type of G/N.

We want to point out that class structures of type JHS have weaker requirements than character tables. For example consider the so-called Rottlaender groups, these are semidirect products of order $11^2 \cdot 5$. They are in class correspondence of type JHS but their character tables do not coincide [KS95, Examples 1.7(c)].

In order to prove that class structures of type JHS determine $n_p(G)$ provided G_p is cyclic we first have to prove the following:

5.1 Groups with cyclic Sylow p-subgroup

Proposition 5.4

Let G be a group with cyclic Sylow p-subgroups for some $p \in \pi(G)$. Assume that G and H are in class correspondence of type JH. Then H has cyclic Sylow p-subgroups.

Proof: Suppose that G and H are in class correspondence of type JH and assume that G is p-solvable. Using [KS95, Lemma 1.8] we can assume that $O_{p'}(G) = O_{p'}(H) = \{1\}$. Thus by [KS95, Theorem 2.1] we see that $O^{p'}(G)$ is the direct product of simple groups with abelian p-group and of an abelian p-group. As G is p-nilpotent we have that $O^{p'}(G)$ is a normal Sylow p-subgroup of G and for each divisor d of G_p there exists exactly one normal subgroup of order d. The normal subgroup correspondence given by a class correspondence of type JH shows that H has a normal Sylow p-subgroup H_p which has for each divisor d of its order precisely one normal subgroup of order d.

We argue by induction that H_p is cyclic. Assume that $|H_p| = p$ then H_p is clearly cyclic. If $|H_p| = p^a > p$ we consider a normal subgroup $M < H$ of order p^{a-1}. Obviously this is a p-group and must have at least (and hence precisely) one subgroup for each divisor of $|M|$. Let $x \in H_p \backslash M$. Then $\langle x \rangle$ is a subgroup of some order not contained in M. Thus $|\langle x \rangle| = p^a$ and H_p is cyclic. This proves the proposition for p-solvable groups.

Now suppose that G is not p-solvable. Again we can assume that $O_{p'}(G) = \{1\}$. By [Bra76, Theorem 3C] there exists a normal subgroup N of G such that $G_p \subset N$ or $p \nmid |N|$. As $O_{p'}(G) = \{1\}$ we see that $G_p < N$ and G has precisely one minimal normal subgroup. Since G is not p-solvable we can use [Bra76, Theorem 3D] and obtain that G has precisely a single simple composition factor S of an order properly divisible by p. In particular we have $S \cong O^{p'}(G)O_{p'}(G)/O_{p'}(G) = O^{p'}(G)$. The generalized Fitting subgroup $\mathrm{F}^*(G)$ is a simple nonabelian group and $G/\,\mathrm{C}_G(\mathrm{F}^*(G)) = G$ is contained in $\mathrm{Aut}(G)$. Thus G is an almost simple group of type S where $p \nmid |S|$. A class structure of type JH determines the chief factors of G resp. of H, see [KS95, Theorem 5]. Consequently H has cyclic Sylow p-subgroups as well. This completes the proof. □

In the following theorem we want to prove that even the data given by a class structure of type JHS is sufficient to compute the Sylow number of a cyclic Sylow p-subgroup:

Theorem 5.5

Suppose that G has a cyclic Sylow p-subgroup and let H be a group such that G and H are in a class correspondence of type JHS. Then $n_p(G) = n_p(H)$ and $n_p(G)$ can be computed.

Proof: This proof is equivalent to the proof of Proposition 3.7.

By Proposition 5.4 we know that H has cyclic Sylow p-subgroups. If $G_p \subset \mathrm{Z}(G)$ the result is clear. So assume that $\mathrm{C}_G(G_p) < G$.

By Corollary 3.8 subgroups which are generated by p-elements with minimal centralizer order have normalizers of minimal order as well. Let M be the number of conjugacy classes of p-elements $g \in G_p$ such that $|\mathrm{C}_G(g)|$ is minimal. As class structures of type JHS determine the size of normal subsets of order p and this properties are invariant under class correspondences of type JHS the size of M is determined by class structures of type JHS.

As the Sylow p-subgroups of G are cyclic we see that there is a certain $n \in \mathbb{N}_0$ such that all p-elements of order $\geq p^n$ have centralizers of minimal length. The number of such p-elements in G_p is $p^a - p^n$, where $|G_p| = p^a$. Thus we have

$$p^a - p^n = M \cdot \mathrm{N}_G(G_p)/\,\mathrm{C}_G(G_p).$$

The total number of p-elements with centralizers of minimal length in G is $M \cdot |G : \mathrm{C}_G(G_p)|$. In particular these elements can not be contained in different Sylow p-subgroups. Consequently

$$n_p(G) = M \cdot \frac{L}{p^a - p^n},$$

where $L = |G : \mathrm{C}_G(g)|$ and g is a p-element with centralizer order of minimal length. M, L and p^a are given by the class structure of type JHS. By Sylow's theorem we have $n_p(G) = 1 \bmod p$ and hence there is precisely one $n \in \mathbb{N}_0$ such this congruence holds. Therefore $n_p(G)$ is determined. □

5.2 Nilpotent-by-nilpotent Groups

In Section 3 we gave a proof that character tables determine $n_p(G)$ provided N and G/N are nilpotent for some normal subgroup N in G, see Theorem 3.4. This proof also holds for class structures of type JHS:

Theorem 5.6

Suppose that G is a finite group with normal subgroup $N \trianglelefteq G$ such that N and G/N are nilpotent. Let H be a group which is in class correspondence of type JHS. Then $n_p(G) = n_p(H)$ for all $p \in \pi(G)$.

Proof: First note that H is also nilpotent-by-nilpotent. By Proposition 5.3 we may assume that the Sylow numbers of all proper quotients of G are given.

If $p \in \pi(N)$ then the intersection $G_p \cap N = N_p$ is normal in G and by Proposition 2.1(i) the Sylow p-number remains unchanged if we consider $n_p(G/N_p)$ instead of $n_p(G)$. Thus assume that $p \notin \pi(N)$. Then by Proposition 2.1(ii) we have

$$n_p(G) = n_p(G_pN).$$

If G has two minimal normal subgroups we can apply Proposition 2.5 and $n_p(G)$ is given as we know the Sylow numbers of proper quotients. Thus N is a q-group.

Consider $\{1\} \neq \mathrm{Z}(N) \trianglelefteq G$. By Proposition 2.1(ii) and Proposition 2.3 we have

$$n_p(G) = n_p(G_pN) = n_p(G_pN/\mathrm{Z}(N)) \cdot |\mathrm{Z}(N) : \mathrm{C}_{Z(N)}(G_p)|.$$

Assume $a \in \mathrm{C}_G(G_p) \cap \mathrm{Z}(N)$. In particular, as N is normal, a^G is contained in $\mathrm{Z}(N)$. By assumption G/N is nilpotent hence for each $G_p^g \in \mathrm{Syl}_p(G)$ there exists $n \in N$ such that $G_p^g = G_p^n$. It follows that $a \in \mathrm{C}_G(G_p^a)$, as

$$(x^n)^a = (x^a)^n \overset{a\in\mathrm{Z}(N)}{=} x^n,$$

where $x^n \in G_p^n$. Let $a^g \in a^G$. Since $G_p^{a^g} = G_p$ we see that a^G is contained in $\mathrm{C}_G(G_p)$. It follows that $a^G \in \mathrm{C}_G(G_p) \cap \mathrm{Z}(N)$ if $|G_p|$ divides $|\mathrm{C}_G(x)|$ for some $x \in a^G$. Thus $|\mathrm{C}_{\mathrm{Z}(N)}(G_p)|$ is determined. By induction $n_p(G_p\,\mathrm{Z}(N)) = n_p(G/\mathrm{Z}(N))$ can be computed and $n_p(G)$ is given by a class structure of type JHS. □

5.3 Supersolvable Groups

In the last part we want to analyze supersolvable groups:

Proposition 5.7

Let G be a supersolvable group. Then $n_p(G)$ is determined by a class structure of type JHS.

Proof: A class structure of type JHS determines whether G is supersolvable. We use induction and can assume that Sylow numbers of proper quotients are known. Let $N \trianglelefteq G$ be a minimal normal subgroup of order q. If $p = q$, then $n_p(G) = n_p(G/K)$. Otherwise if $p \neq q$ we can use Proposition 3.3 to compute $|\mathrm{C}_K(G_p)|$. Thus Theorem 1.3 yields

$$n_p(G) = \underbrace{n_p(K)}_{=1}\, n_p(G/N)\, \underbrace{|N : \mathrm{C}_N(G_p)|}_{\in\{1,q\}}$$

and as $n_p(G/N)$ is given the claim holds. □

Chapter 6

Conclusion

Representations and Sylow numbers are strongly connected. The question whether Sylow numbers are determined by $\mathcal{X}(G)$, $\mathbb{Z}G$ or class structures is still open in general. While we were able to give results for non-p-solvable groups, an answer without additional assumptions on the structure of groups or their Sylow subgroups is not in sight. Proposition 2.5 and Corollary 2.6 reduce the problem to so-called monolithic groups, i.e. groups with unique minimal normal subgroup.

In order to find an (positive or negative) answer to Questions (Q1), (Q2) and (Q3) the study of automorphism groups seems inevitable. If X^k is the direct product of k isomorphic simple groups X then Proposition 1.28 yields that $\mathrm{Aut}(X^k) \cong \mathrm{Aut}(X) \wr \mathrm{S}_k$.

First we should examine almost simple groups, i.e. let $k = 1$. If the outer automorphism group of a simple group X is cyclic then each G with $X \leq G \leq \mathrm{Aut}(X)$ is determined up to isomorphism by $\mathcal{X}(G)$. For example this is the case if X is a sporadic simple group (the outer automorphism group is either trivial or has order two). If $X \cong \mathrm{A}_n$ then $\mathrm{Out}(X)$ is cyclic except for $n = 6$. The alternating group A_6 has an outer automorphism group which is isomorphic to $C_2 \times C_2$. If G is a subgroup of $\mathrm{Aut}(\mathrm{A}_6)$ of index 2 then G is isomorphic to S_6, $\mathrm{PGL}_2(9)$ or the Mathieu Group M_{10}. But the character tables of these three groups are distinct and thus G is determined up to isomorphism. Note that the Sylow numbers of these three groups coincide.

For $k > 1$ we need to study transitive operations of S_k on k copies of a simple group X. For small k we should be able to give a positive answer to Question (Q1).

In Chapter 4 we consider integral group rings instead of character tables. The F^*-Theorem enables us to determine $\mathbb{Z}G/O_{p'}(G)$ up to isomorphism if G is p-constrained. There are indications that $G/O_{p'}(G)$ is always determined up to isomorphism by $\mathbb{Z}G$ and if this conjecture is true we could apply Proposition 3.13 to arbitrary groups.

Appendix A

List of Finite Simple Groups

Family	Order	Class
$A_n(q), n \geq 1$	$q^{n(n+1)/2} \prod_{i=1}^{n}(q^{i+1}-1) \gcd(n+1, q-1)$	Classical Chevalley groups
$B_n(q), n \geq 3$	$\frac{q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)}{\gcd(2,q-1)}$	
$C_n(q), n \geq 3$	$\frac{q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)}{\gcd(2,q-1)}$	
$D_n(q), n \geq 4$	$\frac{q^{n(n-1)}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)}{\gcd(4,q^n-1)}$	
$G_2(q)$	$q^6(q^6-1)(q^2-1)$	Exceptional Chevalley groups
$F_4(q)$	$q^{24}(q^{12}-1)(q^8-1)(q^6)(q^2-1)$	
$E_6(q)$	$\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{\gcd(3,q-1)}$	
$E_7(q)$	$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)$ $\cdot(q^8-1)(q^6-1)(q^2-1)\gcd(2,q-1)$	
$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)$ $\cdot(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	
${}^2A_n(q), n \geq 2$	$\frac{q^{n(n+1)/2} \prod_{i=1}^{n}(q^{i+1}-(-1)^{i+1})}{\gcd(n+1,q+1)}$	Classical Steinberg groups
${}^2D_n(q), n \geq 4$	$\frac{q^{n(n-1)}(q^n+1) \prod_{i=1}^{n-1}(q^{2i}-1)}{\gcd(4,q^n+1)}$	
${}^3D_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	Exceptional Steinberg groups
${}^2E_6(q)$	$\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{\gcd(3,q+1)}$	
${}^2B_2(q), q = 2^{2m+1}$	$q^2(q^2+1)(q-1)$	Suzuki groups
${}^2G_2(q), q = 3^{2m+1}$	$q^3(q^3+1)(q-1)$	Ree groups and Tits group
${}^2F_4(q), q = 2^{2m+1}$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	

Table A.1: List of finite simple groups of Lie type.

G	Order	Name of G
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	
M_{12}	$2^6 \cdot 3^3 \cdot 7 \cdot 11$	
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	Mathieu groups
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	Janko
$J_2 = HJ$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	Janko-Hall
$J_3 = HJM$	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	Janko-Hall-McKay
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	Janko
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	Higman-Sims
Mc	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	McLaughlin
Sz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	Suzuki
Ly=Lys	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	Lyons-Sims
He=HHM	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	Held-Higman-McKay
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	Rudvalis
O'N	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	O'Nan-Sims
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	Conway
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	
$M(22) = F_{22}$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	
$M(23) = F_{23}$	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	Fischer
$M(24) = F_{24}$	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	
$F_3 = E$	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 14 \cdot 19 \cdot 31$	Thompson
$F_5 = D$	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	
$F_2 = B$	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ $\cdot 23 \cdot 31 \cdot 47$	Baby-Monster
$F_1 = M$	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17$ $\cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	Monster

Table A.2: List of sporadic simple groups.

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I hereby declare that the work submitted is my own and that all passages and ideas that are not mine have been fully and properly acknowledged. Furthermore, I declare that this work has not been in parts or wholly published as a submission for another examination procedure and that all copies, both printed and electronic, are the same.

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