Spectral and Hardy Inequalities for the Heisenberg Laplacian

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The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way.

Godfrey Harold Hardy (1877-1947)

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Preface

“In a very dark Chamber, at a round Hole, about one third Part of an Inch broad, made in the Shut of a Window, I placed a Glass Prism, whereby the Beam of the Sun’s Light, which came in at that Hole, might be refracted upwards toward the opposite Wall of the Chamber, and there form a colour’d Image of the Sun.” This is an excerpt of Sir Isaac Newton’s famous work “Opticks”, published in 1704, in which he proves that a ray of sunlight decomposes into its respective wavelengths. In his experiment white light passes through a glass prism such that on a white sheet of paper a small rainbow emerges. Newton called that rainbow “spectrum”, which originates from the Latin word “spectre”, meaning image. His empirical discovery is a foretaste of what spectral analysis is useful for even if Newton’s definition of the word “spectrum” still varies a lot from the one used in modern mathematical physics. An experiment which comes closer to the modern definition of spectrum was performed by the German scientist Robert Wilhelm Bunsen. In the 19th century he repeated Newton’s experiment in which he replaced the sunlight by the burning of an old rag which had been soaked in a sodium chloride solution. The image in Bunsen’s experiment consisted only of a few narrow lines and a yellow bright one. In those days physicists could not explain that phenomenon since the theories of classical mechanics predicted a continuous band of light such as in Newton’s experiment.

In the 1920s the works of Werner Heisenberg and Erwin Schrödinger explained the theory of quantum mechanics. In contrast to classical mechanics, quantum mechanics describes the behavior of matter on the microscopic scale such as of atoms and particles. In Bunsen’s experiment the thermal energy was converted into radiation energy. This process is explained by quantum mechanics, which says that the valence electrons jump from a stable state into a higher one. If the valence electrons fall back into their stable state, energy diminishes, and light emerges. The discrete states of the electrons were the explanation for Bunsen’s discrete picture. In mathematical physics partial differential operators are used to describe such phenomena. The difference of the eigenvalues of the differential operator characterizes the wavelength of the yellow light in Bunsen’s experiment, which is nowadays known as the spectral line of sodium. Thus, in its simplest form spectrum in modern mathematical physics denotes the set of eigenvalues of a given differential operator.

In the last centuries the analysis of differential operators and their spectra became one of the main objectives in mathematical physics since they describe numerous physical phenomena such as mechanical vibrations, sounds, sonar signals, motion of fluids or particles and many more. For instance, in the 18th and 19th century it was discovered that in oscillating systems such as the one of a drum the frequencies of normal modes, also called stationary states or eigenstates, correspond to the eigenfunctions of a differential operator. A normal mode is geometrically speaking a function whose amplitude under time evolution changes but not its shape. These standing waves are mathematically described by the orthonormal basis of eigenfunctions of a differential operator, yielding a canonical decomposition of the underlying Hilbert space on which the self-adjoint differential operator acts, which is nowadays known as spectral representation theorem, abbreviated by spectral theorem[1]. This theorem is the fundament of spectral theory, a branch in mathematics which mainly deals with the analysis of self-adjoint differential operators and properties of their corresponding spectral decomposition. In general it is not possible to compute the spectrum of a differential operator explicitly. Therefore a lot of effort has been done during the last decades to develop analytical and numerical methods for spectrum estimation.

In the beginning of the 20th century a breakthrough was made by Hermann Weyl, who analyzed the eigenvalues of the Dirichlet Laplacian on a bounded domain. He found out that the asymptotic behavior of the eigenvalue counting function is proportional to the volume of the underlying domain, leading to one of the first connections between classical theories and quantum mechanics, being the hour of birth of spectral analysis. Weyl's groundbreaking result, known as Weyl's law, was the starting point of many beautiful problems which arose during the last century. One of the most famous ones was postulated in Mark Kac's article "Can one hear the shape of a drum?" from 1966. It was unclear to Kac whether two drums with different shapes would give the same set of frequencies. However, it took approximately 20 years to provide a response that you can not hear its shape. Another problem which came into the mind of Weyl was whether his result could be further improved by an extra term. He conjectured that the next term in his asymptotic identity would reflect the surface area of the underlying domain, which after a long time was finally proved by V. Ivrii in the year 1980. In 1961, G. Pólya showed that the asymptotic identity of Weyl's result for the eigenvalue counting function is not just a limit, indeed it yields a uniform inequality under the restrictive condition that congruent and pairwise disjoint copies of the underlying domain can be used to cover the whole space up to a set of measure zero. Pólya conjectured that for all bounded domains this result should hold but up to now there is neither a proof nor a counterexample to this conjecture, and the problem remains open. Pólya's conjecture shows the intricacies in deriving uniform bounds for eigenvalues which reflect the correct constants and growing orders in Weyl's law.

[1] For a detailed historical survey on spectrum estimation and the development on the spectral theorem, we refer the interested reader to E. A. Robinson's work [Rob82].

With this thesis I hope to contribute a small step to the mathematical problem of deriving uniform bounds for the eigenvalues of differential operators. This work was carried out from 2013 until 2016 at the university of Stuttgart.

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First of all I would like to express my deep gratitude to my supervisor, Prof. Timo Weidl, for the continuous support of my PhD study and research, for his patience, motivation, trust, enthusiasm and immense knowledge. It was a great pleasure to work with him on problems in spectral theory and mathematical physics. I profited a lot from his guidance in mathematical and especially in personal matters, which I really appreciated.

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Last but not least I dedicate this thesis to the love of my life, Alona. I thank her for her affection, her constant support, her patience and her endless love.

Abstract

In this thesis we consider the first Heisenberg group and study spectral properties of the Dirichlet sub-Laplacian, also known as Heisenberg Laplacian, which is a sum-of-squares differential operator of left-invariant vector fields on the first Heisenberg group. In particular, we consider the bound for the trace of the eigenvalues which reflects the correct geometrical constant and order of growth in Weyl's law and improve this inequality by adding an additional negative lower order term. In addition we investigate on a Hardy-type inequality for the gradient of the Heisenberg Laplacian on bounded domains since an application of such inequalities improves the growing order of the additional lower order term.

Let $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots$ denote the eigenvalues of the Heisenberg Laplacian

$$-\Delta_{\mathbb{H}} := -X_1^2 - X_2^2, \quad X_1 := (\partial_{x_1} + \frac{1}{2}x_2\partial_{x_3}), \quad X_2 := (\partial_{x_2} - \frac{1}{2}x_1\partial_{x_3})$$

for $(x_1, x_2, x_3) \in \mathbb{R}^3$ with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^3$. In this thesis we improve the result in [HL08] by A.M. Hansson and A. Laptev

$$\sum_{k\in\mathbb{N}} (\lambda - \lambda_k(\Omega))_+ \;\leq\; \frac{|\Omega|}{96}\,\lambda^3, \quad \lambda \geq 0.$$

We stress that the geometrical constant and order of growth in λ cannot be improved further. Therefore we add an additional negative lower order term to the right-hand side of that inequality. Such inequalities yield immediately bounds for the eigenvalue sum. In addition we show that the growing order of the additional lower order term in our result can be further improved if there exists a constant $c(\Omega) > 0$ independent of $u \in C_0^\infty(\Omega)$ such that the following Hardy-type inequality holds

$$\frac{1}{c(\Omega)} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\, \mathrm{d}x \;\leq\; \int_\Omega |X_1 u(x)|^2 + |X_2 u(x)|^2 \, \mathrm{d}x, \quad u \in C_0^\infty(\Omega).$$

The Hardy weight δ_C is the distance function to the boundary of Ω measured with respect to the Carnot-Carathéodory metric generated by the span of X_1 and X_2. In this thesis we show that for open bounded convex polytopes this inequality holds and give explicit estimates on the constant $c(\Omega)$.

Zusammenfassung

In dieser Arbeit untersuchen wir spektrale Eigenschaften des Dirichlet-sub-Laplace-Operators auf der Heisenberggruppe, auch bekannt als Heisenberg-Laplace-Operator. Dieser subelliptische Differentialoperator ist die Summe links-invarianten Vektorfelder auf der Heisenberggruppe. Wir untersuchen die Ungleichung der Spur der Eigenwerte, welche den Grenzwert und die Wachstumsordnung in der weylschen Asymptotik widerspiegelt und verbessern diese Ungleichung, indem wir einen zusätzlichen negativen Term addieren. Zusätzlich erforschen wir Hardy-Ungleichungen bezüglich des Gradienten des Heisenberg-Laplace-Operators auf beschränkten Gebieten, deren Anwendung die Wachstumsordnung des zusätzlichen negativen Terms erhöht.

Sei $\Omega \subset \mathbb{R}^3$ ein beschränktes Gebiet und bezeichne $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots$ die Eigenwerte des Heisenberg-Laplace-Operators

$$-\Delta_{\mathbb{H}} := -X_1^2 - X_2^2, \quad X_1 := (\partial_{x_1} + \frac{1}{2}x_2\partial_{x_3}), \quad X_2 := (\partial_{x_2} - \frac{1}{2}x_1\partial_{x_3})$$

mit Dirichlet-Randbedingungen, wobei $(x_1, x_2, x_3) \in \mathbb{R}^3$. In dieser Arbeit untersuchen wir das Resultat [HL08] von A.M. Hansson and A. Laptev

$$\sum_{k\in\mathbb{N}} (\lambda - \lambda_k(\Omega))_+ \;\leq\; \frac{|\Omega|}{96}\,\lambda^3, \quad \lambda \geq 0,$$

wobei in dieser Ungleichung die geometrische Konstante und die Wachstumsordnung in λ die Größen in der weylschen Asymptotik widerspiegelen. Wir verbessern diese Ungleichung, indem wir einen weiteren negativen Term geringeren Wachstums auf die rechte Seite addieren. Diese Art von Ungleichung liefert sofort Abschätzungen an die Eigenwertsumme. Zusätzlich zeigen wir, dass die Wachstumsordnung des negativen Terms erhöht wird, sofern es eine Konstante $c(\Omega) > 0$ unabhängig von $u \in C_0^\infty(\Omega)$, so dass die folgende Hardy-Ungleichung gelte

$$\frac{1}{c(\Omega)} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \;\leq\; \int_\Omega |X_1 u(x)|^2 + |X_2 u(x)|^2 \;\mathrm{d}x, \quad u \in C_0^\infty(\Omega).$$

Das Hardy-Gewicht δ_C ist die Abstandsfunktion bezüglich der Carnot-Carathéodory-Metrik auf der Heisenberggruppe zum Rand des Gebietes Ω. Wir beweisen diese Ungleichung auf offenen, beschränkten, konvexen Polytopen mit explizitem $c(\Omega)$.

Chapter 1

Introduction

In this thesis we study spectral properties of a certain class of differential operators, describing particles which are subject to Heisenberg's uncertainty principle. Heisenberg's uncertainty principle says that it is not possible to measure the exact position and velocity of a particle simultaneously. In mathematical terms the Lie bracket of the position and the momentum operator never vanishes. The Lie group associated to the Lie algebra generated by the commutation relation of the position and momentum operator yields the Heisenberg group, which plays an important role in the representation theory of nilpotent Lie groups, the structure theory of finite groups, geometric optics, the theory of partial differential equations and sub-Riemannian geometry, see for instance [How80]. The latter will be of huge importance for this thesis. We consider the left-invariant tangent vector fields[1] at the identity element of the Heisenberg group and study the subelliptic sum-of-squares differential operator, also referred as sub-Laplacian, given by those vector fields. This operator is called the Heisenberg Laplacian since the vector fields satisfy the same commutation relation as the position and momentum operator. The main objective in this thesis is to analyze the spectrum of that differential operator subject to Dirichlet boundary condition on a bounded domain on the Heisenberg group.

In quantum mechanics the eigenvalues of the Heisenberg Laplacian describe the kinetic energy of a particle trapped in its corresponding domain with respect to a sub-Riemannian system. The interpretation of particles characterized by subelliptic differential operators differs strongly from the case of elliptic ones. Let us consider a given physical system: the configuration space is defined by coordinates of a manifold describing the position of a particle, and the phase space consists of positions and momenta of the particles, given by the configuration space and its tangent space. In the elliptic case the particles are allowed to move in any direction of the configuration space, which is called a Riemannian system. In the subelliptic case there is a restric-

[1]Throughout the thesis we do not distinguish between vector fields on $\mathbb{R}^n$ and first order partial differential operators on $\mathbb{R}^n$ since both can be identified canonically with each other.

tion of these directions, called sub-Riemannian system. However, for a given particle in a sub-Riemannian system there is always at least one possibility to travel between two given positions, which does not have to be the direct path. Thus, let us discuss some well-known facts and important works about subelliptic sum-of-squares differential operators.

In 1967 L. Hörmander studied regularity properties of real-valued sum-of-squares differential operators on open sets. In his groundbreaking work [Hör67] he showed that if the vector space spanned by given vector fields and their commutators of sufficiently high order have full rank at any given point, called Hörmander finite rank condition, then the sum-of-squares of these vector fields is a hypoelliptic operator; especially the Green's kernel of the corresponding fundamental solution, in case it exists, is a smooth function. Hörmander's fundamental result was the starting point of an extensive and fruitful research about mathematical properties of sum-of-squares differential operators.

A special class of sum-of-squares differential operators are subelliptic operators. The first time these operators were classified was in L. Hörmander's work [Hör66]. A differential operator is said to be subelliptic if a Sobolev norm of in most cases fractional order is locally bounded from above by another Sobolev norm with respect to the given differential operator and Dirichlet boundary conditions. Such an estimate exists if the Hörmander finite rank condition of a real-valued sum-of-squares differential operator is satisfied. We stress that this fact is not necessarily true in the case of complex-valued sum-of-squares differential operators, which was proved by J. J. Kohn in [Koh05]. The corresponding metric spaces of subelliptic differential operators are sub-Riemannian spaces, also called Carnot-Carathéodory spaces, which are metric spaces endowed with the Carnot-Carathéodory metric generated by the corresponding vector fields. Carnot-Carathéodory spaces, in the sequel abbreviated as C-C spaces, are the basic geometrical framework for the analysis of hypoelliptic, degenerate elliptic equations, analysis of nilpotent Lie groups, singular integrals, harmonic analysis, geometric control theory and sub-Riemannian geometry [Gro96, FL83, RS76, RS86, Ste76, VSCC92]. The geometric properties of the Carnot-Carathéodory metric were extensively studied in the famous work [NSW85] of A. Nagel, E. M. Stein and S. Waigner. In the last decades many theories from elliptic or Euclidean problems have been systematically worked out in the setting of C-C spaces like the differentiation along vector fields, Sobolev embedding theorems including compactness and extension theorems, Poincaré inequalties, isoperimetric inequalities, heat kernel estimates for subelliptic operators, surface measures, quasiconformal mappings, and much more [CCFI11, GN96, GN98, HK00, KR95, MSC01, VSCC92, VG95]. For a detailed historical survey on subelliptic operators we refer the interested reader to Y. B. Egorov's work [Ego75].

A special class of C-C spaces are Carnot groups. A Carnot group is a simply connected nilpotent Lie group whose associated Lie algebra, being the vector space spanned by the left-invariant tangent vector fields at the identity element endowed with the Lie bracket of vector fields as binary operation, admits a decomposition such that a given linear subspace of the Lie algebra generates the whole algebra by the iteration of its Lie brackets, called stratification. In comparison to a general C-C space we have a priori more geometrical structure on Carnot groups: it is possible to define a homogeneous dimension, (left) translations and a natural family of dilations at the identity element. In particular, the dilation allows us to extend the classical derivation to the derivation of Lipschitz functions between two Carnot groups, which is called Pansu derivative in honor of P. Pansu's work [Pan89]. Vice versa one could start with the commutation relation of the associated Lie algebra of a given Carnot group instead of its Lie group. Then the group structure is explicitly determined by the Baker-Campbell-Hausdorff formula. A detailed introduction of Carnot groups can be found for instance in [BLU07, CG90].

A prime example of a Carnot group is the first Heisenberg group. Let us briefly discuss an example to show how analysis on the first Heisenberg group, denoted in the sequel by $\mathbb{H}$, differs from the one in Euclidean space. A subtle problem on $\mathbb{H}$ is the isoperimetric inequality. Let us endow $\mathbb{H}$ with its Carnot-Carathéodory metric and its Haar measure, being in that case the Lebesgue measure. P. Pansu showed in [Pan83] that the Haar measure of any domain in $\mathbb{H}$ to the power of $3/4$ is bounded by its horizontal perimeter multiplied by a constant independent of the domain. It was proved in [LR03] that the horizontal perimeter divided by its Haar measure to the power of $3/4$ has a minimizer in the class of bounded sets with finite horizontal perimeter. However, it is still unclear what the minimizer looks like. P. Pansu conjectured that the corresponding set should be a bubble set, which up to dilation and translation on $\mathbb{H}$, is obtained by rotating around the x_3-axis a geodesic connecting the points $(0,0,-a)$ with $(0,0,a)$ with $a > 0$. In comparison to the Euclidean case the C-C ball on $\mathbb{H}$ is not the minimizer [Mon00], giving an impression of the intricacy of analyzing mathematical problems on $\mathbb{H}$. Various authors contributed many results concerning the analytic and geometrical properties of the minimizer if one restricts the isoperimetric inequality to a smaller class of sets [LM05, DGN08, Rit12] but a complete answer to Pansu's conjecture is still missing. A detailed introduction to that problem can be found in the following work [CDPT07].

This thesis is organized as follows: in the first chapter we give an overview over spectral estimates for the Dirichlet Laplacian on bounded domains and its corresponding Hardy inequalities since in this work we explain how an application of such an inequality improves certain spectral estimates. These results will serve us as comparison models to the subelliptic case because during the last decades these elliptic problems have been studied thoroughly. In addition we introduce the term Melas-type bound, which is a special kind of inequality for the eigenvalue sum and its trace. We also discuss the subelliptic equivalent to these problems and the objectives for this thesis.

In the second chapter we introduce the first Heisenberg group and its corresponding sub-Laplacian, which is a sum-of-squares differential operator of left-invariant vector fields on that group. We discuss important analytic properties of that differential operator especially the ones related to the Carnot-Carathéodory metric, being fundamental for this thesis. At the end we give a summary of the main results.

In the third chapter we study the eigenvalue sum of the Dirichlet Laplacian in the presence of a constant magnetic field on a domain $\Omega \subset \mathbb{R}^2$ with finite volume. We prove a Melas-type bound for the eigenvalue sum without the assumption of a Hardy inequality, which is a generalization of a result in [KW15], done by H. Kovařík and T. Weidl. For this thesis the Dirichlet Laplacian with constant magnetic field is of huge importance since the two-dimensional Laplacian with constant magnetic field is unitary equivalent to the Heiseberg Laplacian, being discussed in Section 4.4.

In the fourth chapter we study the trace of the eigenvalues and the eigenvalue sum of the Heisenberg Laplacian with Dirichlet boundary conditions on bounded domains on the Heisenberg group. We obtain an inequality with a sharp leading term and an additional lower order term, improving a result of Hansson and Laptev in [HL08].

In the last chapter we prove a Hardy-type inequality for the gradient of the Heisenberg Laplacian on open bounded convex polytopes on the first Heisenberg group. The integral weight of the Hardy inequality is given by the distance function to the boundary measured with respect to the Carnot-Carathéodory metric. The constant depends on the number of hyperplanes given by the boundary of the convex polytope which are not orthogonal to the hyperplane $x_3 = 0$.

This thesis is based upon the following articles:

1. H. Kovařík, B. Ruszkowski and T. Weidl, *Melas-type bounds for the Heisenberg Laplacian on bounded domains*, to appear in the Journal of Spectral Theory (2016).

2. H. Kovařík, B. Ruszkowski and T. Weidl, *Spectral estimates for the Heisenberg Laplacian on cylinders*, to appear in Proceedings of the European Mathematical Society Publishing House in honor of 70th birthday of Pavel Exner (2016).

3. B. Ruszkowski, *Hardy Inequalities for the Heisenberg Laplacian on convex bounded polytopes*, submitted to Mathematica Scandinavica (2016).

1.1 Spectral estimates for the Dirichlet Laplacian

Before we study the spectrum of the Heisenberg Laplacian on the Heisenberg group, we first consider the spectrum of the Laplacian, which is the natural counterpart of the Heisenberg Laplacian in the Euclidean space. In the literature there do not exist many results on the spectrum of subelliptic operators. Therefore we give an overview of spectral estimates for the Laplacian, which will serve us as comparison model to the subelliptic case since during the last decades the elliptic problem has been studied thoroughly. In particular, we focus on spectral estimates reflecting the leading term in the Weyl asymptotics, which can be further refined by an additional lower order term.

The Laplacian has become one of the main objects in mathematical physics during the last centuries. This operator appears in several differential equations, describing various physical phenomena, such as heat flow, the propagation of waves, the motion of viscous fluid substances and phenomena in quantum mechanics [HS96].

For $n \in \mathbb{N}$ let us consider the Laplacian in Cartesian coordinates, which is the following second-order differential operator

$$-\Delta := -\sum_{j=1}^{n} \partial_{x_j}^2,$$

where ∂_{x_j} is the partial derivative in the j-th direction. For a domain $\Omega \subset \mathbb{R}^n$ with finite volume, we consider the self-adjoint operator, denoted by $-\Delta_\Omega$, which is associated to the semi-bounded quadratic form

$$a[u] := \int_\Omega |\nabla u(x)|^2 \; \mathrm{d}x = \sum_{j=1}^{n} \int_\Omega |\partial_{x_j} u(x)|^2 \; \mathrm{d}x$$

with form domain given by the Sobolev space $H_0^1(\Omega)$, see [BS87]. The operator $-\Delta_\Omega$ is called **Dirichlet Laplacian**.

The object of interest in this section are the eigenvalues of the Dirchlet Laplacian. The compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, see [AF03], yields a nondecreasing, positive sequence of eigenvalues $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$, which accumulates only at infinity. The german mathematician Hermann Weyl (1885-1955) studied the eigenvalue counting function

$$N(\lambda, \Omega) := \{j \in \mathbb{N} | \lambda_j(\Omega) < \lambda\}$$

and proved in [Wey12] the following fundamental result

$$\lim_{\lambda \to \infty} N(\lambda, \Omega) \lambda^{-n/2} = \frac{\tau_n}{(2\pi)^d} |\Omega|, \tag{1.1}$$

where $|\Omega|$ is the n-dimensional Lebesgue measure of Ω and τ_n is the volume of the unit ball in $\mathbb{R}^n$. The limit in (1.1) is called **Weyl's law** or **Weyl asymptotitcs**. We stress that the Weyl asymptotics are determined by the phase space volume of a particle trapped in Ω, which is an important quantity in physics. H. Weyl conjectured that there exists a lower order term depending on the surface area of $\partial\Omega$ such that (1.1) can be further improved.

In 1980, V. Ivrii proved this conjecture under strong assumptions on the geometry of Ω in [Ivr80, Ivr98]; in particular it holds

$$N(\lambda, \Omega) = \frac{\tau_n}{(2\pi)^n} |\Omega| \lambda^{n/2} - \frac{1}{4} \frac{\tau_{n-1}}{(2\pi)^{n-1}} |\partial\Omega| \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}), \tag{1.2}$$

as $\lambda \to +\infty$, where $|\partial\Omega|$ is the surface area of the boundary. We refer to that formula as the **refined Weyl asymptotics**.

The question arose whether the limits in the Weyl asymptotics give uniform bounds on the counting function as well. In 1961, G. Pólya showed that if Ω is a tiling domain it holds

$$N(\lambda,\Omega) \le \frac{\tau_n}{(2\pi)^n}|\Omega|\lambda^{n/2} \tag{1.3}$$

for all $\lambda \ge 0$. We stress that the constant on the right-hand side cannot be improved further because of (1.1). Pólya's hypothesis suggests that this inequality remains true for all open domains with finite volume but neither a proof nor a counterexample have been found yet. The only known generalization was done by A. Laptev in [Lap97]. He considered domains with finite measure of the form $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $n = n_1 + n_2$ for $n_1 \ge 2$ and $n_2 \ge 1$. Under the assumption that $N(\lambda,\Omega_1)$ fulfills Pólya's hypothesis, we know then that $N(\lambda,\Omega)$ satisfies that hypothesis as well.

During the last decades it became apparent that things get easier if one does not consider the counting function directly but averaged or smoothed versions. Therefore we concentrate on estimates for the **Riesz means**, which are defined as

$$R_\gamma(\lambda,\Omega) = \mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_-^\gamma := \sum_{k\in\mathbb{N}:\lambda_k(\Omega)<\lambda} (\lambda-\lambda_k(\Omega))^\gamma$$

for $\gamma \ge 0$; for $\gamma = 0$ we obtain the counting function and for $\gamma = 1$ the trace. The identity[2] in [AL78]

$$R_{\gamma+\delta}(\lambda,\Omega) = \frac{1}{\beta(\delta,\gamma+1)}\int_0^\infty t^{\delta-1}R_\gamma(t-\lambda,\Omega)\,\mathrm{d}t, \quad \delta>0, \tag{1.4}$$

where $\beta(\cdot,\cdot)$ is the beta function, shows that uniform bounds or asymptotical results for Riesz means with higher powers can be obtained by using results for lower order Riesz means. Hence for $\sigma = 0$ and suitable Ω we use (1.2) to get the corresponding Weyl asymptotics for the Riesz means

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_-^\gamma = L^{cl}_{\gamma,n}|\Omega|\lambda^{\sigma+n/2} - \frac{1}{4}L^{cl}_{\gamma,n-1}|\partial\Omega|\lambda^{\sigma+(n-1)/2} + o(\lambda^{\gamma+(n-1)/2}) \tag{1.5}$$

as $\lambda \to +\infty$, where the classical Lieb-Thirring constant is denoted by

$$L^{cl}_{\gamma,n} := \frac{\Gamma(\gamma+1)}{(4\pi)^{n/2}\Gamma(\gamma+n/2+1)}$$

and $\Gamma(\cdot)$ is the gamma function. Recently R. Frank and L. Geisinger proved in [FG11] that (1.5) holds for all $\gamma \ge 1$ if $\partial\Omega$ is a $C^{1,\alpha}$ boundary for some $0 < \alpha \le 1$. The condition on $\partial\Omega$ reduces the strong geometrical assumptions of V. Ivrii.

[2]Equation (1.4) also holds for the Riesz means of any positive operator with discrete spectrum, like the Heisenberg Laplacian or the Laplacian with magnetic field on bounded domains subject to Dirichlet boundary conditions since the proof of (1.4) is based on a simple scaling argument.

The first estimate for the Riesz means were done by F.A. Berezin in 1972. He proved in [Ber72] that the semiclassical limit of the leading term in (1.5) gives a uniform bound for some Riesz means as well. In particluar, for a domain $\Omega \subset \mathbb{R}^n$ with finite volume and for all $\gamma \geq 1$ holds

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_-^\gamma \leq L_{\gamma,n}^{cl} |\Omega| \lambda^{\sigma+n/2} \tag{1.6}$$

for any $\lambda \geq 0$. This inequality is called **Berezin inequality** or also **Berezin-Lieb inequality** because of E. H. Lieb's work in [Lie73].

For $\gamma = 1$ one can immediately deduce bounds for the eigenvalue sum, which is discussed explicitly in Section A.2 of the appendix. The Legendre transform transforms then (1.6) for $\gamma = 1$ into

$$\sum_{j=1}^{k} \lambda_j(\Omega) \geq C_n |\Omega|^{-\frac{2}{n}} k^{1+2/n}, \qquad C_n := (2\pi)^2 \tau_n^{-2/n} \frac{n}{n+2} \tag{1.7}$$

for all $k \in \mathbb{N}$. That inequality was first proved in [LY83] by P. Li and S.-T. Yau with a minimization technique in the Fourier space without using the duality of the Legendre transform. Hence, inequality (1.7) is called **Li-Yau inequality**. The constant and the order of growth cannot be improved further, which will be discussed in the next section. We stress that in the literature inequalities of the form (1.6) and (1.7) are also called Berezin-Lieb-Li-Yau inequalities because of the duality given by the Legendre transform.

At that point the question arises whether one can improve Berezin-Lieb inequalities by adding the lower order term of the refined Weyl asymptotics; in the general case this is not possible. Therefore, let us assume that the left-hand side of the Riesz means are less equal than the refined Weyl asymptotics in (1.5) for all $\lambda \geq 0$. Then, we consider the sequence of sets which are used for the construction of Koch's snowflake; these sets are piecewice smooth. The perimeter of that sequence tends to infinity while the volume of it converges, whereas the min-max-principle for the Riesz means of the corresponding construction step are bounded from below by the Riesz means on a circle, which was chosen such that it lies inside of all construction steps of Koch's snowflake, yielding a contradiction.

However, it is possible to add a negative lower order term which reflects the correct growing order of the lower order term in the refined Weyl asymptotics for some Riesz means. In [GLW11] L. Geisinger, A Laptev and T. Weidl proved that for a given convex bounded domain $\Omega \subset \mathbb{R}^n$ and for all $\gamma \geq 3/2$ there exists a constant $C(\gamma, n, \Omega) > 0$ such that for all $\lambda > 0$ it holds

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_-^\gamma \leq L_{\gamma,n}^{cl} |\Omega| \lambda^{\sigma+n/2} - C(\gamma, n, \Omega) \lambda^{\sigma+(n-1)/2}. \tag{1.8}$$

This result was recently improved by S. Larson in [Lar16] by showing that $C(\gamma, n, \Omega)$ can be chosen as a multiple of the constant appearing in the refined Weyl asymptotics

1.2 The Melas-type bound

In this section we discuss improvements for the trace and the eigenvalue sum since the aim of this thesis is to prove similar results for the eigenvalues of the Heisenberg Laplacian. First of all we compute the refined Weyl asymptotics for the eigenvalue sum

$$\sum_{j=1}^{k} \lambda_j(\Omega) = C_n |\Omega|^{-\frac{2}{n}} k^{1+2/n} + \tilde{C}_n \frac{|\partial\Omega|}{|\Omega|^{1+1/n}} k^{1+1/n} + o\left(k^{1+1/n}\right) \tag{1.9}$$

as $k \to \infty$, which can be deduced by the refined Weyl asymptotics for the trace and the counting function under suitable conditions on Ω. The constants are given by

$$C_n := (2\pi)^2 \tau_n^{-2/n} \frac{n}{n+2} \quad \text{and} \quad \tilde{C}_n := \frac{\sqrt{\pi}\Gamma(2+\frac{n}{2})^{1+1/n}}{(n+1)\Gamma(\frac{3}{2}+\frac{n}{2})\Gamma(2)^{1/n}}.$$

In 2003 A. D. Melas showed the first improvement of the Li-Yau inequality. He took an additional restriction into account for the Li-Yau minimization technique and proved in [Mel03] that for any open bounded set Ω holds

$$\sum_{j=1}^{k} \lambda_j(\Omega) \geq C_n |\Omega|^{-\frac{2}{n}} k^{\frac{n+2}{n}} + M_n \frac{|\Omega|}{I(\Omega)} k, \quad k \in \mathbb{N}, \tag{1.10}$$

where

$$I(\Omega) := \min_{a \in \mathbb{R}^n} \int_\Omega |x-a|^2 \, \mathrm{d}x$$

is the second moment of the set Ω, and $M_n > 0$ depends only on the dimension. We observe that this inequality for the eigenvalue sum satisfies the following two properties:

- The leading term of that inequality reflects the order of growth and the geometrical constant in asymptotic identity in (1.9).
- The additional lower order term is of growth order one.

Hence, we call an inequality fulfilling the last two properties a **Melas-type bound**. As mentioned in the last section the geometrical constant for the lower order term can not be achieved without any further assumptions on Ω. By the duality of the Legendre transform we get an improved Berezin-Lieb inequality, meaning that for all $\gamma \geq 1$ and $\lambda \geq 0$ holds

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_-^\gamma \leq L_{\gamma,n}^{cl} |\Omega| \left(\lambda - M_n \frac{|\Omega|}{I(\Omega)}\right)^{\sigma+n/2}. \tag{1.11}$$

We call such inequalities also **Melas-type bounds** since they are equivalent to (1.10). For related results we refer to [Ber72, LY83, Mel03, KW15, KVW09, Yol10, YY13] and also [Str96] for a generalization to Riemannian manifolds.

The best known improvement of (1.10) was done by H. Kovařík, S. Vugalter and T. Weidl. They showed in [KVW09] that the growth order of the lower order term in (1.10) can be approximated arbitrarily close to the one given by the refined Weyl asymptotics; especially for high energies this result is extremely good. However, the proving technique is quite difficult and cannot be adapted to the setting of the Heisenberg Laplacian.

The most important work for this thesis is [KW15] by H. Kovařík and T. Weidl. Assuming the validity of a Hardy inequality, the authors prove a Melas-type bound for the Dirichlet Laplacian and the Dirichlet Laplacian in the presence of a constant magnetic field. The technique is extremely powerful and can be used for the Heisenberg Laplacian, too. In this work we modify that technique and show that for a Melas-type bound there is no need for the validity of a Hardy inequality at all. This is discussed in detail in Chapter 3 for the eigenvalue sum of the Dirichlet Laplacian with constant magnetic field. Only if one is interested to achieve higher growth orders in the lower order term, one has to assume the validity of that Hardy inequality. The same is true for the eigenvalue sum of the Heisenberg Laplacian in which the Hardy inequality has to be adjusted to the setting of the Heisenberg group.

1.3 The Hardy inequality

In this section we introduce briefly some well-known Hardy inequalities since in the last section we mentioned that Hardy inequalities improve Melas-type bounds. In its simplest form a Hardy inequality allows to control weighted norms by derivative norms with respect to Dirichlet boundary conditions. During the last century the study of Hardy inequalities or also called Hardy-type inequalties has received a strong impulse since they are powerful tools in real-variable harmonic analysis, partial differential equations, mathematical physics and spectral theory. In 1920 G. H. Hardy proved in [Har20] a weaker form of Hilbert's inequality, which is known as discrete Hardy inequality. His source of motivation was to find a simple and elementary proof of Hilbert's result. Some years later in [Har25] he proved for any $p > 1$ and any positive function $f \in L^p((0, \infty))$ the following

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\ \mathrm{d}t \right)^p \ \mathrm{d}x \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \ \mathrm{d}x. \tag{1.12}$$

This inequality is called **Hardy inequality** and generalizes his weak version of Hilbert's inequality. The constant on the right-hand side of (1.12) cannot be improved further. For a detailed historical development of that inequality we refer to [KMP06].

In the literature there exist several inequalities, called Hardy inequalities or Hardy-type inequalities. A well-known Hardy inequality in higher dimensions is the following one: for all $u \in C_0^\infty(\mathbb{R}^n)$ holds

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{||x||_e^2} \ \mathrm{d}x \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \ \mathrm{d}x \tag{1.13}$$

if $n \geq 3$, where $||x||_e$ denotes the Euclidean length of $x \in \mathbb{R}^n$. The constant on the left-hand side cannot be improved further. In mathematical physics this inequality plays an important role because the weight function $|x|_e^{-1}$ on the left-hand side of (1.13) is the Coulomb potential, which describes the force between two point charges. We stress that for the cases $n \in \{1, 2\}$ inequality (1.13) can not hold, see for instance [BS87]. The L^p-version of (1.13) for $1 < p < \infty$, which is less important for this work, can be found in [OK90].

In this thesis we focus on Hardy inequalities whose weight function is given by the distance function to the boundary of a given bounded domain. In some sense such inequalities feel the boundary of that domain, which in regard to the refined Weyl asymptotics is very useful since the geometrical constant of the lower order term depends on the volume and the boundary of the domain. Therefore let us consider a bounded domain $\Omega \subset \mathbb{R}^n$, where $c(\Omega) > 0$ denotes the smallest constant such that for all $u \in C_0^\infty(\Omega)$ holds

$$\int_\Omega \frac{|u(x)|^2}{\delta_e(x)^2} \,\mathrm{d}x \leq c_e(\Omega)^2 \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x, \tag{1.14}$$

where $\delta_e(x) = \operatorname{dist}(x, \partial\Omega)$ in the Euclidean sense. For convex Ω we know that $c_e(\Omega) = 2$, see [Dav99], which is a sharp result since for any bounded domain holds

$$\frac{1}{4} \geq \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x}{\int_\Omega |u(x)|^2 \delta_e(x)^{-2} \,\mathrm{d}x}.$$

In general it is not possible to prove a uniform constant without any additional assumptions on Ω; this is not even possible in the class of smooth domains, see [MMP98]. However, A. Ancona proved in [Anc86] that for any simply connected bounded domain in $\mathbb{R}^2$ that $c_e(\Omega) \leq 4$ holds, which is the best generalizations of the result for convex domains. It is still an open problem to extend this result to higher dimensions since A. Ancona uses powerful tools in complex analysis, namely the Koebe quarter theorem in combination with the Riemann mapping theorem. For the sake of completeness we mention that $c_e(\Omega) < \infty$ if Ω is a bounded Lipschitz domain, which is discussed in [Anc86]. For more information and recent improvements on that inequality we refer the reader to the following book [BEL15].

One of the aims in this thesis is to prove inequalities of the form (1.14) with respect to the setting of the Heisenberg group, see Chapter 5. The weight function will be given by the Carnot-Carathéodory metric generated by the left-invariant vector fields of the Heisenberg group, which is the natural counterpart of the Euclidean distance in the Euclidean setting. As mentioned in the beginning such an inequality allows us to improve the growing order in the lower order term of a Melas-type bound. We discuss this in detail in the summary of the main results of the upcoming chapter and in Section 4.6.

Chapter 2

The Heisenberg group

The Heisenberg group appears in several ways, which depends on the mathematical point of view. In the literature one does not differentiate between these apparently different objects since they are all equivalent to each other in some sense. The Heisenberg group plays an important role in the representation theory of nilpotent Lie groups, the structure theory of finite groups, geometric optics, the theory of partial differential equations, quantum mechanics and sub-Riemannian geometry, see for instance [How80]. The later will be of huge importance for our purposes, which will be explained in this chapter.

The Heisenberg group is named after the german scientist Werner Heisenberg (1901-1976) who established the fundament of quantum mechanics in the beginning of the 20th century. One of his main contributions in quantum mechanics was his uncertainty principle. Heisenberg's uncertainty principle says that it is not possible to measure the exact position and velocity of a particle simultaneously. In mathematical terms this is expressed by the non-vanishing commutation relation of the position and momentum operator with respect to the Lie bracket. Indeed, we will see in the next section that the Lie group associated to the Lie algebra generated by the commutation relation of the position and the momentum operator is exactly the Heisenberg group.

In that chapter we introduce the Heisenberg group and its corresponding sub-Laplacian, being the subelliptic sum-of-squares differential operator of left-invariant vector fields on the Heisenberg group. This differential operator is then the Heisenberg Laplacian, which is the object of interest in that thesis. In addition, we discuss the analytic properties of that differential operator, especially the ones connected to the Carnot-Carathéodory metric on the Heisenberg group. Then we present the corresponding geodesics on the Heisenberg group and the Kaplan metric, which is equivalent to the Carnot-Carathéodory metric and easier to handle in certain situations. At the end of this chapter we give an overview of the main results of that thesis.

2.1 The construction of the first Heisenberg group

There are many motivations and different constructions to introduce the Heisenberg group, depending on the point of view of its applications. In complex function theory for instance it can be identified with the boundary of the Siegel upper half-space in $\mathbb{C}^2$. The group law arises from a subgroup of the automorphism group from the complex unit disc with the Siegel upper half-space. To the interested reader we refer to [Kra09].

However, for our purposes we prefer the point of view in quantum mechanics and follow [Fol89] to introduce the Heisenberg group. For $x := (x_1, x_2, x_3) \in \mathbb{R}^3$ we consider the momentum operator $Q_j f(x) := x_j f(x)$ and the position operator $P_j f(x) := -\mathrm{i}\partial_{x_j} f(x)$, where f denotes a Schwartz function on $\mathbb{R}^3$. We use the Lie Bracket to obtain

$$[Q_j, P_k]f(x) = \mathrm{i}\delta_{j,k} f(x),$$

where $\delta_{\cdot,\cdot}$ is Kronecker's delta. This commutation relation is called the Heisenberg canonical commutation relation. In physics observables satisfying non-vanishing commutation relation play an important role because they are subject to the Heisenberg uncertainty pinciple, see [GS11]. Motivated by the Heisenberg canonical commutation relation, we take $\mathfrak{h}$ to be the 3-dimensional real **Heisenberg Algebra** with basis $\{X_1, X_2, X_3\}$ which satisfies the only non-vanishing commutation relation

$$[X_1, X_2]_{\mathfrak{h}} = -X_3,$$

where $[\cdot,\cdot]_{\mathfrak{h}}$ denotes the Lie bracket of $\mathfrak{h}$. An immediate consequence of the commutation relation is that $\mathfrak{h}$ is 2-step nilpotent and obviously a Carnot group. Lie's third theorem says that $\mathfrak{h}$ is associated to a Lie group. Therefore we consider two points $(x, y, t) \in \mathbb{R}^3$ and $(x', y', t') \in \mathbb{R}^3$ such that

$$X := xX_1 + yX_2 + tX_3, \quad \text{and} \quad Y := x'X_1 + y'X_2 + t'X_3,$$

and compute

$$[X, Y]_{\mathfrak{h}} = (-xy' + yx')X_3. \tag{2.1}$$

At that point we use Ado's Theorem [Hal15, Thm. 2.40] and know then that every finite dimensional real Lie algebra is isomorphic to a real Lie algebra of square matrices with the matrix commutator as Lie bracket. Therefore we set

$$m(X) := m(x, y, t) := \begin{pmatrix} 0 & y & t \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix},$$

and from $m(X)m(Y) = m(x, y, t)m(x', y', t') = m(0, 0, yx')$ we obtain for the matrix commutator, denoted by $[\cdot,\cdot]$, the following

$$[m(X), m(Y)] = [m(x, y, t), m(x', y', t')] = m(0, 0, -xy' + yx') = m([X, Y]_{\mathfrak{h}}).$$

Hence the mapping $X \to m(X)$ is a Lie algebra isomorphism from $\mathfrak{h}$ to the subspace $\{m(x,y,t) \mid (x,y,t) \in \mathbb{R}\}$. For a square matrix M we use the exponential map

$$\exp(M) := \sum_{j=0}^{\infty} \frac{1}{j!} M^j,$$

which is an analytic diffeomorphism between a Lie algebra and its corresponding Lie group, see [CG90, Thm. 1.2.1]. The group structure of that Lie group is then given by the Baker-Campbell-Hausdorff formula. In our case we have to use that the commutators of orders higher than two vanish, which yields

$$\begin{aligned}
&\exp(m(X))\exp(m(Y)) = \exp(m(X) + m(Y) + \frac{1}{2} m([X,Y]_{\mathfrak{h}})) \\
&= \exp\left(m(x+x', y+y', t+t' - \frac{1}{2}xy' + \frac{1}{2}yx') \right) \\
&= \exp\left(m((x+x')X_1 + (y+y')X_2 + (t+t' - \frac{1}{2}xy' + \frac{1}{2}yx')X_3) \right).
\end{aligned}$$

Thus the first **Heisenberg group**, denoted by $\mathbb{H}$, is then defined as the $\mathbb{R}^3$ equipped with the following group law

$$(x_1, x_2, x_3) \boxplus (y_1, y_2, y_3) \ := \ \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - \tfrac{1}{2}(x_1 y_2 - x_2 y_1) \Big). \tag{2.2}$$

In the literature the factor $1/2$ is sometimes replaced by 2 or -2, which yields an isomorphic group; from the analytical point of view there is no difference between these objects. In our construction we omitted Heisenberg groups in higher dimensions for the sake of simplification though all results in that thesis can be extended to the case in higher dimensions.

We recall that the object of interest is the sub-Laplacian, which is the sum of squares of the left-invariant tangent vector fields at the identity element of $\mathbb{H}$. Thus the differential of the Lie group at the identity element gives the representation of $\mathfrak{h}$ in terms of vector fields. A simple computation yields then

$$X_1 = \partial_{x_1} + \frac{1}{2} x_2 \partial_{x_3}, \quad X_2 = \partial_{x_2} - \frac{1}{2} x_1 \partial_{x_3}, \quad X_3 = \partial_{x_3} \tag{2.3}$$

for a given point $x =: (x_1, x_2, x_3) \in \mathbb{H}$. Note that these first order partial differential operators, in the sequel called vector fields because they can be identified canonically with vector fields on $\mathbb{H}$, fulfill the commutation relation of $\mathfrak{h}$, where the Lie bracket in that case is considered as the differential operator $[X_1, X_2] := X_1 X_2 - X_2 X_1$ defined on $C^\infty(\mathbb{H})$. In particular X_1, X_2, X_3 form a left-invariant basis in $\mathfrak{h}$. A vector field $Y : \mathbb{H} \to \mathbb{H}$ is called left-invariant if for all $x, g \in \mathbb{H}$ holds

$$dl_x(g) Y(g) = Y(x \boxplus g),$$

where the mapping $l_x(g) : \mathbb{H} \to \mathbb{H}$ is given by $l_x(g) := x \boxplus g$ and its differential by

$$dl_x(g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}x_2 & -\frac{1}{2}x_1 & 1 \end{pmatrix}.$$

We stress that the Lebesgue measure is preserved under the left-translation l_x because the determinant of dl_x is one. Hence the left Haar measure on $\mathbb{H}$ is the Lebesgue measure. Since the Lebesgue measure is also right invariant, we know that $\mathbb{H}$ is unimodular.

2.2 The Heisenberg Laplacian

In this section we give a brief introduction and a survey of some well-known analytic properties of the Heisenberg Laplacian, which is the natural counterpart of the Laplacian in the Euclidean setting. For the introduction we follow S. Thangavelu in [Tha98]. Let us briefly recall the properties which characterize the Laplacian:

- invariant under translations,
- invariant under rotations,
- homogeneous of degree 2.

It is clear that we have to adapt these properties to the setting of the Heisenberg group to obtain an operator, which acts as natural on $\mathbb{H}$ as the Laplacian on $\mathbb{R}^n$. Therefore we consider the following properties:

a differential operator P on $\mathbb{H}$ is called **left-invariant** if it commutes for all $g \in \mathbb{H}$ with

$$L_g f(x) := f(g^{-1}x),$$

where $x \in \mathbb{H}$ and f denotes a Schwartz function on $\mathbb{R}^3$. The differential operator P on $\mathbb{H}$ is called **rotation invariant** if it commutes for any σ with

$$R_\sigma f(x) := f(\sigma(x_1, x_2), x_3),$$

where $x := (x_1, x_2, x_3)$, and $\sigma \in \mathrm{SO}(2)$. We introduce a family of non-isotropic dilations on $\mathbb{H}$, i.e. for $h > 0$ we define

$$h(x) := (hx_1, hx_2, h^2x_3).$$

It is easy to verify that $h : \mathbb{H} \to \mathbb{H}$ is a group isomorphism. The differential operator P on $\mathbb{H}$ is called **homogeneous** of degree 2 if for all $h > 0$ and $x \in \mathbb{H}$ holds

$$P\left(f(h(x))\right) = h^2(Pf)(h(x)).$$

From [Tha98, Kor83] it is known that a differential operator satisfying the last three properties on $\mathbb{H}$ must be a multiple of

$$-X_1^2 - X_2^2 + \alpha X_3, \quad \alpha \in \mathbb{R}.$$

Hence the only sum-of-squares differential operator satisfying these properties is then given by

$$-\Delta_{\mathbb{H}} := -X_1^2 - X_2^2.$$

This sub-Laplacian is called the **Heisenberg Laplacian**, in the literature also referred as **Kohn Laplacian**.

As an immediate result of the fundamental work of L. Hörmander in [Hör67], we see that $-\Delta_{\mathbb{H}}$ is a second order hypoelliptic differential operator because the vector fields $X_1, X_2, [X_1, X_2]$ form a basis at any point in $\mathbb{H}$. We recall that an operator P is called **hypoelliptic** if for any open set $\Omega \subset \mathbb{H}$ such that $Pu \in C^\infty(\Omega)$, it must follow that $u \in C^\infty(\Omega)$.

In comparison to the Laplacian this operator is not elliptic but **subelliptic** at any point of $\mathbb{H}$. Let $\Omega \subset \mathbb{R}^n$ be a domain and $\mathcal{L}$ be a differential operator of order 2, which is symmetric on $C_0^\infty(\Omega)$. For $0 < \varepsilon < 1$ the differential operator $\mathcal{L}$ is said to be **subelliptic** of order ε at $x \in \Omega$ if there exists a neighborhood K of x and a constant $C_K > 0$ such that for all $u \in C_0^\infty(K)$ holds

$$\|u\|_\varepsilon^2 \leq C_K \left(|\langle \mathcal{L}u, u \rangle| + \|u\|_0^2 \right),$$

where

$$\|u\|_s := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \tag{2.4}$$

denotes the Sobolev norm of order for ε, $\mathcal{F}u$ the Fourier transform of u and $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega)$. Form [Fol73] we know that for any $x \in \mathbb{H}$ there exists a neighborhood $K \subset \mathbb{H}$ of x and a constant $c_K > 0$ such that for all $u \in C_0^\infty(K)$ we have

$$\|u\|_{1/2}^2 \leq c_K \left(\int_K |X_1 \, u(x)|^2 + |X_2 \, u(x)|^2 + |u(x)|^2 \, \mathrm{d}x \right),$$

yielding the subellipticity of the Heisenberg Laplacian.

2.3 The Carnot-Carathéodory metric and the geodesics

In this section we give a detailed description of the sub-Riemannian geometry on $\mathbb{H}$. In particular, we introduce the Carnot-Carathéodory metric. This metric measures the distance between points on $\mathbb{H}$ in a natural way using curves whose derivative lies pointwise in the span of the vector fields X_1 and X_2. We will see that the analytic properties of the Heisenberg Laplacian with the Carnot-Carathéodory metric are as natural as the ones of the Euclidean metric with respect to the Laplacian.

We call a Lipschitz curve $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{H}$ **horizontal** if the curve $\gamma(t) := (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ fulfills for any $t \in (a,b)$ the following differential equation

$$\gamma_3'(t) = \frac{1}{2} \left(\gamma_2(t)\, \gamma_1'(t) - \gamma_1(t)\, \gamma_2'(t) \right). \tag{2.5}$$

This is equivalent to the condition that $\gamma'(t) \in \operatorname{span}\{X_1(\gamma(t)), X_2(\gamma(t))\}$ for all $t \in (a,b)$. By an application of Chow's theorem, see e.g. [Mon02], we know that horizontal curves exist because X_1 and X_2 satisfy Hörmander's finite rank condition. Therefore for a given pair $x, y \in \mathbb{H}$, we consider the family of curves

$$\mathcal{F}(x,y) := \{\gamma : [a,b] \to \mathbb{H} \,:\, \gamma \text{ is horizontal and connects } x \text{ with } y\}. \tag{2.6}$$

Furthermore, we set

$$l_{\mathbb{H}}(\gamma) := \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} \;\; \mathrm{d}t. \tag{2.7}$$

Given $x, y \in \mathbb{H}$, the **Carnot-Carathéodory metric** (C-C metric in the sequel) is then defined as follows;

$$d_C(x,y) := \inf_{\gamma \in \mathcal{F}(x,y)} l_{\mathbb{H}}(\gamma). \tag{2.8}$$

From the geometric point of view we must compute the smallest two-dimensional Euclidean length of a projected horizontal curve onto the x_1-x_2 hyperplane, see Figure 2.1. For further infromation on the C-C metric we refer the interested reader to [CDPT07], [Mon02] and [CCG07].

The arc joining **geodesics** starting from the origin were computed in [Mon00] and [Mar97]. The parametrization of these arcs is given by

$$\gamma_{k,\theta}(t) := \begin{cases} x_1(t,k,\theta) = \dfrac{\cos(\theta) - \cos(kt+\theta)}{k}, \\ x_2(t,k,\theta) = \dfrac{\sin(kt+\theta) - \sin(\theta)}{k}, \\ x_3(t,k,\theta) = \dfrac{kt - \sin(kt)}{2k^2}, \end{cases} \tag{2.9}$$

where $t \in [0, \frac{2\pi}{|k|}]$, $\theta \in [0, 2\pi)$ and $k \in \mathbb{R} \setminus \{0\}$. We stress that the projection of these curves onto the x_1-x_2 hyperplane are arcs of circles which go through the origin. For the computation of the distance from the origin to the point $(0,0,x_3)$ one can transform the condition (2.5) by an application of Stoke's theorem into a slightly modified isoperimetric problem on $\mathbb{R}^2$ whose solution matches then with (2.9), see [CDPT07].

Hence for a given point $\gamma_{k,\theta}(t) \in \mathbb{H}$, it holds $d_C(\gamma_{k,\theta}(t), 0) = t$ if $t \in [0, \frac{2\pi}{|k|}]$. We extend this formula to the case $k = 0$ by taking the limit for $k \to 0$, yielding

$$\gamma_{0,\theta}(t) := \begin{cases} x_1(t, 0, \theta) = t\sin(\theta), \\ x_2(t, 0, \theta) = t\cos(\theta), \\ x_3(t, 0, \theta) = 0. \end{cases} \tag{2.10}$$

Thus we obtain the arcs connecting the origin with points lying in $\{(x_1, x_2, x_3) \in \mathbb{H} \mid x_3 = 0\}$. Next we define the map

$$\Phi(t, k, \theta) := \big(x_1(t, k, \theta),\, x_2(t, k, \theta),\, x_3(t, k, \theta)\big), \tag{2.11}$$

for $t \in [0, \frac{2\pi}{|k|}]$, $\theta \in [0, 2\pi)$, $k \in \mathbb{R}$. The determinant of the Jacobian of Φ, denoted by $J\Phi$, is given by

$$\det\left(J\Phi(t, k, \theta)\right) = \frac{kt\sin(kt) - 2(1 - \cos(kt))}{k^4}\,, \tag{2.12}$$

see [Mon00, S.161].

Throughout the work we will need the following well-known properties of the C-C metric.

Proposition 2.1. *The following statements hold true:*

a) Any two points in $\mathbb{H}$ can be connected by a (not necessarily unique) geodesic.

b) The C-C metric is invariant under left translation with respect to the group law on $\mathbb{H}$, meaning

$$d_C(x, y) = d_C(z \boxplus x, z \boxplus y) \tag{2.13}$$

for $x, y, z \in \mathbb{H}$.

c) The mapping

$$\Phi : \left\{ (t, k, \theta) \in \mathbb{R}^3 \;\middle|\; \theta \in \mathbb{R}/2\pi\mathbb{Z},\ k \in \mathbb{R},\ t \in \left(0, \frac{2\pi}{|k|}\right) \right\} \to \mathbb{H} \setminus \mathbb{P}, \tag{2.14}$$

where Φ is given in (2.11), is a C^1-diffeomorphism, where $\mathbb{P} := \{(x_1, x_2, x_3) \in \mathbb{H} \mid x_1 = 0, x_2 = 0\}$.

d) For a fixed compact set $K \subset \mathbb{H}$ there exists a constant $M > 0$ such that for all $x, y \in K$ holds

$$M\|x - y\|_e \leq d_C(x, y) \leq M^{-1}\|x - y\|_e^{1/2}, \tag{2.15}$$

where $\|x\|_e$ is the Euclidean length in $\mathbb{R}^3$ of x.

e) We recall the family of dilations $h(x) := (hx_1, hx_2, h^2x_3)$ for $x \in \mathbb{H}$ and $h > 0$. Then

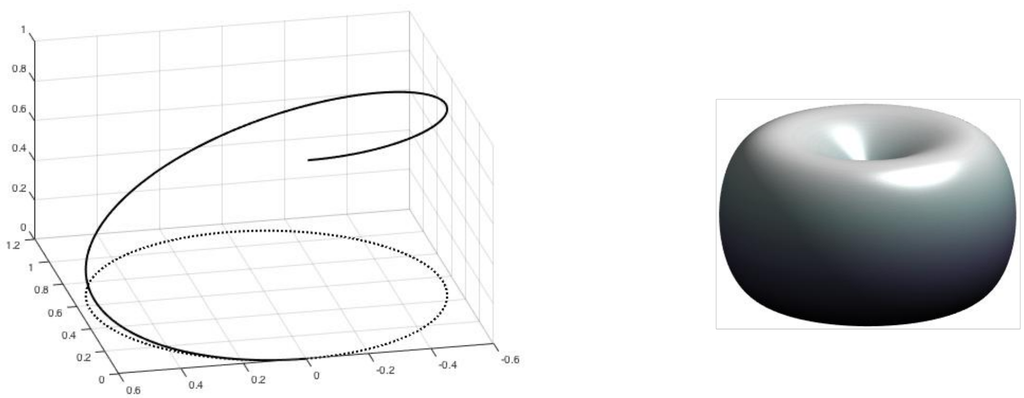
$$h^4\, C_1(0) = C_h(0) := \{x \in \mathbb{H} \,|\, d_C(x,0) < h\}, \tag{2.16}$$

and

$$d_C(h(x), h(y)) = h d_C(x,y).$$

Proof. We refer to [Mon00], [NSW85] and [MR05]. □

An immediate consequence of the last property is that the **homogeneous dimension** of $\mathbb{H}$ is 4. From the analytical point of view the Heisenberg Laplacian behaves in some sense like the Laplacian on $\mathbb{R}^4$ although the underlying topological dimension of $\mathbb{H}$ is 3. We will observe this difficultly throughout the proofs of the Hardy and spectral inequalities on $\mathbb{H}$.



(a) A geodesic connecting the origin with $(0,0,1)$, and its projection onto the x_1-x_2 hyperplane.

(b) The C-C ball with radius one, centered at the origin.

Figure 2.1: Geodesics and C-C balls.

2.4 The Kaplan metric

In that section we discuss another distance on the Heisenberg group, which is equivalent to the C-C metric. We introduce the **Korányi-Folland metric** or also called **Kaplan metric**

$$d_{\mathbb{H}}(x,y) := \|(-y) \boxplus x\|_{\mathbb{H}},$$

where

$$\|x\|_{\mathbb{H}}^4 := (x_1^2 + x_2^2)^2 + 16x_3^2,$$

is the **Korányi-Folland gauge** or **Kaplan gauge**. For the sake of brevity we will use the latter notation and call it Kaplan gauge. Indeed $d_{\mathbb{H}}(\cdot,\cdot)$ is a metric [Kra09], and in [Fol73] G.B. Folland showed that the function $f(x) := \|x\|_{\mathbb{H}}^{-2}$ up to a multiple constant is the fundamental solution of $-\Delta_{\mathbb{H}}$.

It is easy to show that $\|x\|_{\mathbb{H}}$ and $d_C(x,0)$ are equivalent since both are homogeneous of order 1 with respect to the dilations $h(\cdot)$ for any $h>0$. However, the Kaplan metric and the C-C metric do not share the same analytic properties. For instance the distance function to the boundary with respect to the C-C metric fulfills the Eikonal equation, which is of huge importance for the spectral estimates discussed in Section 4.6; the Kaplan metric does not satisfy that property. In this thesis it will be convenient to switch between those two distances since for explicit computations it is more comfortable to work with the Kaplan metric than with the C-C metric. Therefore we need the following result:

Lemma 2.2. *For all $x, y \in \mathbb{H}$ it holds*

$$\frac{1}{\pi^2}\, d_C(x,y)^4 \leq \|(-y) \boxplus x\|_{\mathbb{H}}^4 \leq d_C(x,y)^4. \tag{2.17}$$

Moreover, both inequalities are sharp.

Proof. Using the left-invariance of $d_C(x,y)$ with respect to the group law on $\mathbb{H}$ we transform (2.17) into

$$\frac{1}{\pi^2}\, d_C(y^{-1} \boxplus x, 0)^4 \leq \|(-y) \boxplus x\|_{\mathbb{H}}^4 \leq d_C(y^{-1} \boxplus x, 0)^4. \tag{2.18}$$

We know that $y^{-1} = -y$. Therefore it is sufficient to prove

$$\frac{1}{\pi^2}\, d_C(z,0)^4 \leq \|z \boxplus 0\|_{\mathbb{H}}^4 \leq d_C(z,0)^4 \qquad \forall\ z \in \mathbb{H}.$$

At that point we use the arc joining geodesics starting from the origin in (2.9). Thus, we have to calculate the supremum and the infimum of

$$\frac{\|\gamma_{k,\theta}(t) \boxplus 0\|_{\mathbb{H}}^4}{d_C(z,0)^4} = \frac{4\,(1-\cos(kt))^2 + 4\,(kt - \sin(kt))^2}{(tk)^4}.$$

Hence the aim is to give upper and lower bounds for the function

$$g(\tau) := \frac{4}{\tau^4}\left((1-\cos(\tau))^2 + (\tau - \sin(\tau))^2\right)$$

for $0 \leq \tau \leq 2\pi$ because $t \in [0, \frac{2\pi}{|k|}]$. To proceed we show that the function $g(\tau)$ is non-increasing on $[0, 2\pi]$. By differentiating the function $g(\tau)$ several times we find that the latter is non-increasing on $[0, 2\pi]$, which implies that the same is true for g. Hence

$$\frac{1}{\pi^2} = g(2\pi) \leq g(\tau) \leq \lim_{\tau \to 0+} g(\tau) = 1. \tag{2.19}$$

The sharpness of that inequality is an immediate consequence. $\square$

2.5 Summary of the main results

In this section we give a summary of the main results of this thesis.

2.5.1 Spectral estimates for the Heisenberg Laplacian

In Chapter 4 we consider a bounded domain $\Omega \subset \mathbb{H}$ and study the sequence of positive nondecreasing eigenvalues $\{\lambda_k(\Omega\}_{k\in\mathbb{N}}$ of the Heisenberg Laplacian

$$-\Delta_{\mathbb{H}} := -X_1^2 - X_2^2$$

with Dirichlet boundary condition. For the Riesz means of order one, we obtain

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \;\leq\; \max\left\{0,\; \frac{|\Omega|}{96}\,\lambda^3 - \lambda^2 \frac{\mathrm{R}(\Omega)^8}{150|\Omega|\,\mathrm{D}(\Omega)^2\pi^4}\right\} \tag{2.20}$$

for all $\lambda > 0$, where $|\Omega|$ is the three-dimensional Lebesgue measure of Ω, $\mathrm{D}(\Omega)$ is the diameter and $\mathrm{R}(\Omega)$ the inradius of Ω with respect to the Carnot-Carathéodory metric on $\mathbb{H}$. This inequality improves a recent result in [HL08], proved by A. M. Hanson and A. Laptev. The Li-Yau equivalent of (2.20) is then given by

$$\sum_{k=1}^{n} \lambda_k(\Omega) \;\geq\; \frac{8\sqrt{2}}{3}|\Omega|^{-\frac{1}{2}}\; n^{\frac{3}{2}} + \frac{16\mathrm{R}(\Omega)^8}{75|\Omega|^2\mathrm{D}(\Omega)^2\pi^4}\, n, \quad n \in \mathbb{N}.$$

This result is a Melas-type bound since the leading term reflects the geometrical constant and the order of growth in the Weyl asymptotics and the additional positive term is of order one.

For domains of the type $\Omega = \omega \times (a, b)$, where $\omega \subset \mathbb{R}^2$ is a bounded domain and $a, b \in \mathbb{R}$ are such that $a < b$, we improve (2.20) for large eigenvalues. For convex cross-section ω we show that for all $\lambda \geq 0$ holds

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \;\leq\; \max\left\{0, \frac{|\Omega|}{96}\,\lambda^3 - \frac{\lambda^{2+\frac{1}{4}}}{2^7 \cdot 3^{5/2}}\,\frac{|\Omega|}{\mathrm{R}_e(\omega)^{3/2}}\right\}, \tag{2.21}$$

where $\mathrm{R}_e(\omega)$ is the Euclidean inradius of ω in $\mathbb{R}^2$. This improvement is also possible for general domains but then we have to assume the validity of a Hardy inequality, discussed in the upcoming subsection.

At last we consider the eigenvalue counting function $N(\lambda, \Omega)$ and construct domains such that for all $\lambda \geq 0$ holds

$$N(\lambda, \Omega) \leq \lambda^2 \frac{|\Omega|}{32},$$

which is a Pólya-type inequality in the spirit of G. Pólya's result in [Pól61] for the counting function of the Dirichlet Laplacian.

2.5.2 Hardy inequalities for the Heisenberg Laplacian on convex bounded polytopes

In Chapter 5 we study a Hardy-type inequality for the gradient of the Heisenberg Laplacian. Let $\Omega \subset \mathbb{H}$ be a bounded domain, and let us denote by $c(\Omega) > 0$ the smallest constant independent of $u \in C_0^\infty(\Omega)$ such that

$$\int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\, \mathrm{d}x \ \le\ c(\Omega) \int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\ \mathrm{d}x. \tag{2.22}$$

The sub-gradient is given by $\nabla_{\mathbb{H}} := (X_1, X_2)$, and the distance function δ_C is the distance function to the boundary measured with respect to the Carnot-Carathéodory metric on $\mathbb{H}$.

We prove for an open bounded convex polytope $\Omega \subset \mathbb{H}$ the following

$$c(\Omega) \le 5\left(m^{8/9}\pi^{8/9}3^{5/2} \cdot 2^{47/18}\sqrt{2^{-4/3}\pi^{-2/3} + 16}\left(1 + \frac{1}{3^{3/2}2^{7/6}\pi^{1/3}}\right)^{2/3} + 1\right)^{4/3},$$

where $m \in \mathbb{N}$ denotes the number of hyperplanes of $\partial\Omega$ which are not orthogonal to the hyperplane $x_3 = 0$. Under an additional geometrical assumption on Ω, the estimate for $c(\Omega)$ can be further improved. It is then even possible to show that for any $\varepsilon > 0$ there exists an open bounded convex polytope such that

$$c(\Omega) \le 4 + \varepsilon.$$

This shows that there exist convex domains which are more compatible with the Heisenberg group structure than we expect them to be since we prove $4 \le c(\Omega)$.

2.5.3 Melas-type bounds for the Laplacian with magnetic field

In Chapter 3 we consider a domain $\Omega \subset \mathbb{R}^2$ with finite volume and study the sequence of positive nondecreasing eigenvalues $\{\lambda_j(\Omega, \mathcal{A})\}_{j\in\mathbb{N}}$ of the Dirichlet Laplacian with constant magnetic field, given by

$$H(\mathcal{A}) := (\mathrm{i}\nabla + \mathcal{A}(x))^2, \tag{2.23}$$

where $\mathcal{A}(x) := B/2(-x_2, x_1)$ for $x := (x_1, x_2) \in \mathbb{R}^2$, $B > 0$ and $\nabla := (\partial_{x_1}, \partial_{x_2})$. For the eigenvalue sum we obtain

$$\sum_{j=1}^{n} \lambda_j(\Omega, \mathcal{A}) \ge \frac{2\pi}{|\Omega|}n^2 + \frac{\mathrm{R}_e(\Omega)^2}{32}\frac{\pi^2}{|\Omega|^2}n, \qquad n \in \mathbb{N},$$

where $|\Omega|$ is the two-dimensional Lebesgue measure of Ω and $\mathrm{R}_e(\Omega)$ the Euclidean inradius of Ω. This inequality generalizes a result in [KW15] by H. Kovařík and T. Weidl since the authors assumed the validity of a Hardy inequality, which is not necessary for our result. In particular, we obtain a Melas-type bound because the leading term reflects the semi-classical limit in the Weyl asymptotics and the additional lower order term is of order one.

Chapter 3

Melas-type bounds for the Laplacian with magnetic field

In this chapter we study the eigenvalue sum of the Dirichlet Laplacian in the presence of a constant magnetic field on a two-dimensional domain with finite volume. We prove a Melas-type bound for the eigenvalue sum without the assumption of a Hardy inequality, which is a generalization of a result in [KW15] by H. Kovařík and T. Weidl. We will see later that the Dirichlet Laplacian with constant magnetic field is of huge importance for this thesis since this differential operator is unitary equivalent to the Heiseberg Laplacian, being discussed in Section 4.4.

3.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a domain with finite volume. We consider the Dirichlet Laplacian with the following vector potential

$$H(\mathcal{A}) := (\mathrm{i}\nabla + \mathcal{A}(x))^2, \tag{3.1}$$

where $\mathcal{A}(x) := B/2(-x_2, x_1)$ for $x := (x_1, x_2) \in \mathbb{R}^2$, $\nabla := (\partial_{x_1}, \partial_{x_2})$ and $B > 0$. We stress that the potential $\mathcal{A}$ fulfills $\operatorname{curl} \mathcal{A} = B$, yielding a constant magnetic field[1]. We denote by $H(\mathcal{A})$ the Friedrichs extension which is associated to the closure of the semi-bounded quadratic form

$$\int_\Omega |(\mathrm{i}\nabla + \mathcal{A})u|^2 \,\mathrm{d}x, \tag{3.2}$$

initially given on all $u \in C_0^\infty(\Omega)$. Thus $H(\mathcal{A})$ is a positive and self-adjoint operator in $L^2(\Omega)$. The well-known compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ yields in combination with

[1] One could take any other vector potential such that $\operatorname{curl} \mathcal{A} = B$ is fulfilled since all of these differential operators are unitarily equivalent to each other, see [FH10].

the diamagnetic inequality, see (3.6), that the associated quadratic form of $H(\mathcal{A})$ is also compactly embedded into $L^2(\Omega)$. Hence, we obtain an unbounded and nondecreasing sequence of positive eigenvalues of $H(\mathcal{A})$, denoted by $\lambda_j(\Omega, \mathcal{A})$ for $j \in \mathbb{N}$, repeating the eigenvalues according to their finite multiplicities.

In this chapter we consider the Riesz means of these eigenvalues, given by

$$\operatorname{Tr}(H(\mathcal{A}) - \lambda)_-^\gamma := \sum_{j \in \mathbb{N}} (\lambda_j(\Omega, \mathcal{A}) - \lambda)_-^\gamma \tag{3.3}$$

for $\gamma \geq 0$, where $y_\pm := (|y| \pm y)/2$ for $y \in \mathbb{R}$. For the case $\gamma = 0$ we obtain the eigenvalue counting function $N(\lambda, H(\mathcal{A})) := \#\{j \in \mathbb{N} \,|\, \lambda_j(\Omega, \mathcal{A}) < \lambda\}$. From [ELV00] and (1.4) one can deduce that the Riesz means satisfy the same Weyl asymptotics as the Dirichlet Laplacian for $d = 2$, meaning

$$\lim_{\lambda \to \infty} \lambda^{-1-\gamma} \operatorname{Tr}(H(\mathcal{A}) - \lambda)_-^\gamma = L_{\gamma,2}^{cl} |\Omega|, \qquad L_{\gamma,2}^{cl} := (4\pi(\gamma + 1))^{-1}, \tag{3.4}$$

where $|\Omega|$ is the two-dimensional Lebesgue measure of Ω. We stress that the constant in the Weyl asymptotics describes the phase space volume of a particle trapped in Ω in the presence of a constant magnetic field, which is an important quantity in physics. In fact, the leading term of the Weyl asymptotics can be used to give a uniform bound

$$\operatorname{Tr}(H(\mathcal{A}) - \lambda)_-^\gamma \leq L_{\gamma,2}^{cl} |\Omega| \lambda^{1+\gamma} \tag{3.5}$$

if $\gamma \geq 1$, see [ELV00]. We mention that for $\gamma \geq 3/2$ A. Laptev and T. Weidl proved in [LW00] that for any magnetic field equation (3.5) holds true, as well. For the first eigenvalue we can apply the diamagnetic inequality [LL01],

$$\big|\nabla |u(x)|\big| \;\leq\; |(\mathrm{i}\nabla + \mathcal{A}(x))u(x)| \quad \text{a.e.} \quad x \in \Omega, \tag{3.6}$$

where $u \in H_0^1(\Omega)$. This inequality holds true for all real-valued vector potentials from $L^2_{loc}(\Omega)$ and appropriate u. The conclusion that the eigenvalues of the Dirichlet Laplacian are always smaller than the ones in the presence of a magnetic field is in general wrong, which was discussed in [ELV00]. Therefore it is not possible to use known spectral estimates of the Dirichlet Laplacian to obtain results for the case in the presence of a constant magnetic field. Nevertheless, L. Erdös, M. Loss, and V. Vougalter proved the following Li-Yau estimate

$$\sum_{j=1}^{n} \lambda_j(\Omega, \mathcal{A}) \geq \frac{2\pi}{|\Omega|} n^2 \tag{3.7}$$

for all $n \in \mathbb{N}$, which is the same bound as the one for the eigenvalue sum of the Dirichlet Laplacian. This result is optimal in the sense that the leading term in the Weyl asymptotics of the magnetic operator matches with the bound in (3.7), see [ELV00]. We can use that result to obtain an estimate for the counting function

$$N(\lambda, H(\mathcal{A})) \leq \frac{1}{2\pi}\lambda|\Omega| = 2L_{0,2}^{cl}\lambda|\Omega|, \quad \lambda \geq 0. \tag{3.8}$$

R. Frank, M. Loss and T. Weidl proved [FLW09] that in the class of bounded domains the constant on the right-hand side of (3.8) cannot be improved further, which disproves Pólya's conjecture for $H(\mathcal{A})$. They also showed that even in the class of tiling domains Pólya's conjecture for the counting function of $H(\mathcal{A})$ is false. In particular the Li-Yau inequality (3.7) yields an optimal bound for the counting function as well.

H. Kovařík and T. Weidl improved (3.7) by showing that under the assumption of a Hardy inequality there exists $c(\Omega) > 0$ such that

$$\sum_{j=1}^{n} \lambda_j(\Omega, \mathcal{A}) \geq \frac{2\pi}{|\Omega|}n^2 + c(\Omega)n, \quad n \in \mathbb{N}, \tag{3.9}$$

which is a Melas-type bound. We discuss that result in more detail in the next section and prove that this result still holds for any domain with finite measure without any further assumption.

3.2 Melas-type bounds and main results

The goal in this chapter is to give an improvement of (3.7) in the sense that there exists a constant $C(\Omega) > 0$ such that

$$\sum_{j=1}^{n} \lambda_j(\Omega, \mathcal{A}) \geq \frac{2\pi}{|\Omega|}n^2 + C(\Omega)n^\alpha, \quad n \in \mathbb{N}, \tag{3.10}$$

where $0 < \alpha < 2$. Since we know that the leading term in the Weyl asymptotics matches with the one of the Dirichlet Laplacian, one could guess that the lower order term matches with the second order term in the refined Weyl asymptotics of the Dirichlet Laplacian as well, see (1.5). However, there is no proof for this conclusion, which means that the optimal $C(\Omega)$ and the correct growth order α remain unknown.

From now on we consider estimates for the Riesz means for the case $\gamma = 1$ because an application of the Legendre transform immediately gives estimates of the form (3.10), see Corollary A.3 in the appendix. As mentioned in the end of the last section H. Kovařík and T. Weidl were the first ones who improved the Li-Yau inequality in the magnetic case. In particular, they showed in [KW15] that for a bounded domain $\Omega \subset \mathbb{R}^2$ we have

$$\operatorname{Tr}(H(\mathcal{A}) - \lambda)_- \ \leq \ \max\left\{0,\ \frac{|\Omega|}{8\pi}\lambda^2 - \frac{1}{128c_h^2(\Omega)\pi}\frac{\sigma(\Omega)^2}{|\Omega|}\lambda\right\}, \quad \lambda \geq 0, \tag{3.11}$$

where the following quantities are given by

$$\sigma(\Omega) = \inf_{0<\beta\leq \mathrm{R}(\Omega)} \frac{|\{x \in \Omega \mid \delta_e(x) < \beta\}|}{\beta}, \quad \delta_e(x) = \inf_{x\in\partial\Omega} \|x-y\|_e, \quad \mathrm{R}_e(\Omega) = \sup_{x\in\Omega} \delta_e(x).$$

The constant $c_h(\Omega) \geq 2$ is the smallest possible such that for all $u \in C_0^\infty(\Omega)$ the following Hardy inequality is valid

$$\int_\Omega \frac{|u(x)|^2}{\delta_e(x)^2}\,\mathrm{d}x \leq c_h^2(\Omega) \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x. \tag{3.12}$$

It is known that for all bounded domains with Lipschitz boundary (3.12) is fulfilled, which is discussed in [Anc86]. We recall that (3.11) yields a Melas-type bound since the order of the additional lower order term is one less than the order of the leading term. In the same work the authors improved that result for convex bounded domains Ω, meaning

$$\operatorname{Tr}(H(\mathcal{A})-\lambda)_- \;\leq\; \max\left\{0,\; \frac{|\Omega|}{8\pi}\left(\lambda^2 - \frac{\lambda^{5/4}}{36\mathrm{R}_e(\Omega)^{3/2}}\right)\right\}, \quad \lambda \geq 0. \tag{3.13}$$

The improvement in the order of the additional negative term is based on an application of (3.12). For convex Ω we know that (3.12) holds true with the optimal constant $c_h(\Omega) = 2$, [Dav99]. In general the computation of the optimal constant fulfilling (3.12) is quite difficult because it depends on the geometry of Ω and not even necessarily on its regularity [MMP98]. For instance, A. Ancona showed in [Anc86] that (3.12) holds for $c_h(\Omega) \leq 4$ if Ω is a simply connected bounded domain. Although we cannot control $c_h(\Omega)$ and do not know whether (3.12) holds for any domain, we prove the following:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain such that $|\Omega| < \infty$. Then holds*

$$\operatorname{Tr}(H(\mathcal{A})-\lambda)_- \;\leq\; \max\left\{0,\; \frac{|\Omega|}{8\pi}\lambda^2 - \frac{\pi\mathrm{R}_e(\Omega)^2}{128|\Omega|}\lambda\right\} \tag{3.14}$$

for all $\lambda > 0$. The Li-Yau type equivalent is given by

$$\sum_{j=1}^n \lambda_j(\Omega,\mathcal{A}) \geq \frac{2\pi}{|\Omega|}n^2 + \frac{\mathrm{R}_e(\Omega)^2}{32}\frac{\pi^2}{|\Omega|^2}n \quad \textit{for } n \in \mathbb{N}. \tag{3.15}$$

Comparing the stated result with the one in (3.11), we see that we do not need the validity of (3.12) any more, and the quantity $\sigma(\Omega)$ does not appear in that result. It is even possible to consider unbounded domains with finite volume. In particular, the assumptions of Theorem 3.1 are the weakest possible since (3.14) always holds if $|\Omega| = \infty$.

The proof of Theorem 3.1 starts in the same way as the proof of (3.11), stated in [KW15]. First, we need the spectral decomposition of the free Dirichlet Laplacian with constant magnetic field to derive a uniform bound for the trace. Thus we obtain the leading term of the Weyl asymptotics and an additional negative term. The latter has to be estimated from below. This is done by an application of a one-dimensional Hardy inequality, which holds true for any $|\Omega|$.

3.3 The spectral decomposition

Here we state the spectral decomposition of the free Laplacian with constant magnetic field on $L^2(\mathbb{R}^2)$ with operator domain $C_0^\infty(\mathbb{R}^2)$. This makes sense since this operator is essentially self-adjoint, see [Hel09, FH10], and its closure yields a unique self-adjoint extension $H(\mathcal{A},\mathbb{R}^2)$ which is associated to the closure of the semi-bounded quadratic form (3.2) initially given on $C_0^\infty(\mathbb{R}^2)$. We stress that the form domain of $H(\mathcal{A},\mathbb{R}^2)$ is not the $H^1(\mathbb{R}^2)$ because the functions lying in its domain need to decay sufficiently strong for $|x| \to \infty$. Let us consider a function u lying in the domain of $H(\mathcal{A},\mathbb{R}^2)$. The spectral theorem yields then the following decomposition

$$H(\mathcal{A},\mathbb{R}^2)u(x) = \sum_{k=1}^{\infty} B(2\mathrm{k}-1)\int_{\mathbb{R}^2} \mathrm{P}_{k,B}(x,y)u(y)\ \mathrm{d}y, \tag{3.16}$$

where $\mathrm{P}_{k,B}$ is the integral kernel of the orthogonal projection[2] in $L^2(\mathbb{R}^2)$ onto the k-th Landau level $B(2k-1)$ of the Landau Hamiltonian with constant magnetic field for $B > 0$ and $k \in \mathbb{N}$. In addtion, we need the following well-known characteristics

$$\begin{aligned}
\mathrm{P}_{k,B}(y,y) &= \frac{1}{2\pi}B, \qquad \text{where } y \in \mathbb{R}^2, \\
\int_{\mathbb{R}^2}\left(\int_{\Omega} |\mathrm{P}_{k,B}(x,y)|^2\ \mathrm{d}y\right)\ \mathrm{d}x &= \int_{\Omega}\left(\int_{\mathbb{R}^2} \mathrm{P}_{k,B}(x,y)\mathrm{P}_{k,B}(y,x)\ \mathrm{d}x\right)\ \mathrm{d}y \\
&= \int_{\Omega} \mathrm{P}_{k,B}(y,y)\ \mathrm{d}y = \frac{B}{2\pi}|\Omega|,
\end{aligned} \tag{3.17}$$

see for instance [Hel09] and [FH10].

3.4 Proof of the main results

In this section we give the proof of Theorem 3.1. As preliminaries we define the Euclidean Ball with radius $r > 0$ centered at $x \in \mathbb{R}^2$ by

$$B_r(x) := \{y \in \mathbb{R}^2 \mid \|x-y\| < r\} \tag{3.18}$$

and prove following two lemmata:

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a domain with finite measure. Then holds*

$$|\Omega^\beta| \geq |B_{\mathrm{R}_e(\Omega)}(0)| - |B_{\mathrm{R}_e(\Omega)-\beta}(0)| \geq \beta \mathrm{R}_e(\Omega)\pi \tag{3.19}$$

for all $\beta \in (0, \mathrm{R}_e(\Omega)]$, where $\Omega^\beta := \{x \in \Omega \mid \delta_e(x) < \beta\}$. The first inequality becomes an equality if $\Omega = B_r(0)$ for any $r > 0$.

[2]These projections have infinite dimension and in the literature one can find explicit formula for the integral kernels $\mathrm{P}_{k,B}$, see [Pou15].

Proof. First of all we know that $\mathrm{R}_e(\Omega)$ is finite and that there exists a point $p \in \Omega$ such that $B_{\mathrm{R}_e(\Omega)}(p) \subseteq \Omega$, which is discussed in Section A.3.

Without loss of generality we can assume that $B_{\mathrm{R}_e(\Omega)}(0) \subseteq \Omega$ because the Lebesgue measure is translation invariant with respect to the Euclidean distance. Let us change into polar coordinates and consider $(x_1, x_2) = r(\cos(\varphi), \sin(\varphi)) := \Phi(r, \varphi) \in \mathbb{R}^2$ for $r > 0, \varphi \in [0, 2\pi)$. For each points (x_1, x_2) converted to polar coordinates, we define for a fixed angle φ the following

$$b_\varphi := \inf\{t > 0 \,|\, t(\cos(\varphi), \sin(\varphi)) \notin \Omega\}.$$

Since $|\Omega|$ has finite volume, the set $\{\varphi \in [0, 2\pi) \,|\, b_\varphi = \infty\}$ is a null set, which means that b_φ exists almost everywhere. Because of $B_{\mathrm{R}_e(\Omega)}(0) \subseteq \Omega$, we immediately get

$$\mathrm{R}_e(\Omega) \leq b_\varphi \quad \text{for all } \varphi \in [0, 2\pi). \tag{3.20}$$

Now we put

$$\Omega(\Phi) := \{(x_1, x_2) \in \Omega^\beta \,|\, \exists (r, \varphi) \in (0, \infty) \times [0, 2\pi) \text{ such that } (x_1, x_2) = \Phi(r, \varphi)\}.$$

Obviously we have $\Omega^\beta \supseteq \Omega(\Phi)$. Now we define the set

$$A^\beta := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, \exists (r, \varphi) \in E(\beta) \text{ such that } (x_1, x_2) = \Phi(r, \varphi)\}, \tag{3.21}$$

where

$$E(\beta) := (b_\varphi - \beta, b_\varphi) \times [0, 2\pi).$$

For a geometrical interpretation of the construction of A^β we refer to Figure 3.1. Since $b_\varphi(\cos(\varphi), \sin(\varphi)) \in \partial\Omega$ and $\delta_e(x) \leq \|x - y\|$ for all $y \in \partial\Omega$, it is easy to check that the following holds

$$\Omega(\Phi) \supseteq A^\beta.$$

Now we compute the volume of the set on the right-hand side by changing into polar coordinates and use (3.20) to obtain

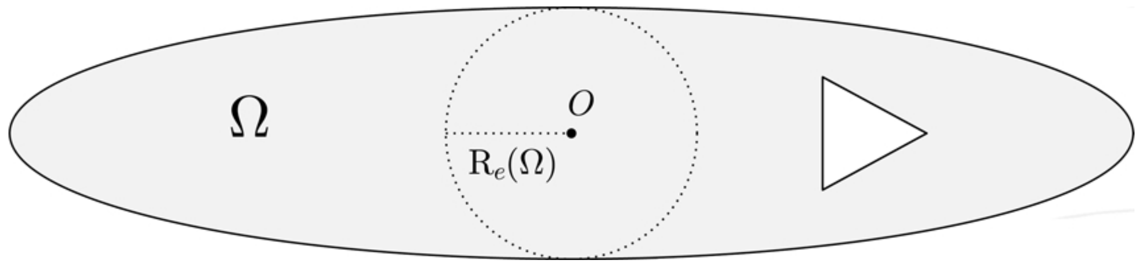
$$|A^\beta| = \int_0^{2\pi} \int_{b_\varphi - \beta}^{b_\varphi} r \,\mathrm{d}r \,\mathrm{d}\varphi \geq \int_0^{2\pi} \int_{\mathrm{R}_e(\Omega) - \beta}^{\mathrm{R}_e(\Omega)} r \,\mathrm{d}r \,\mathrm{d}\varphi = |B_{\mathrm{R}_e(\Omega)}(0)| - |B_{\mathrm{R}_e(\Omega) - \beta}(0)|. \tag{3.22}$$

The right-hand side becomes $\beta\pi(2\mathrm{R}_e(\Omega) - \beta)$, which immediately gives the lower bound on $(0, \mathrm{R}_e(\Omega)]$ and the result. $\square$

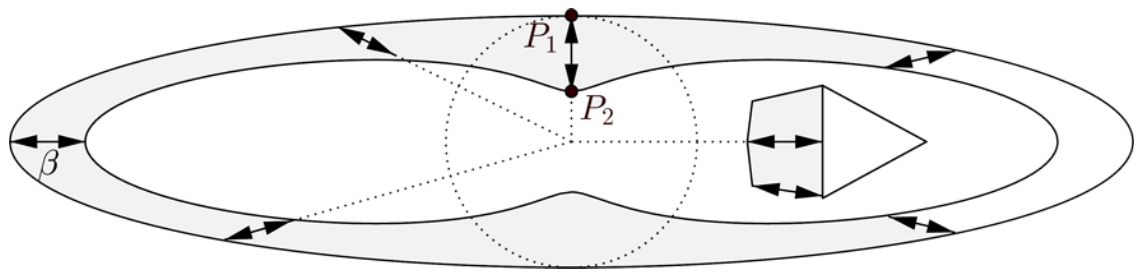
For the next Lemma we define the translation of Ω with respect to the point $p \in \mathbb{R}^2$ by

$$\Omega + p := \{x \in \mathbb{R}^2 \,|\, \exists y \in \Omega \text{ such that } x = y + p\}.$$

Figure 3.1: On the construction of A^β.



(a) Let Ω be given by the grey-colored area. At the origin O we center the largest Euclidean ball which still fits into Ω. Its radius is given by $\mathrm{R}_e(\Omega)$.



(b) The set A^β is described here by the grey-colored area. For the construction we consider all lines emanating from the origin. Let us consider for a moment the line going through the origin, P_1 and P_2. On this line there exists a closest point to the origin such that this point lies on $\partial\Omega$, in the image described by P_1. We parametrize then the convex combination of P_1 and P_2, where P_2 is chosen such that $\|P_1 - P_2\|_e = \beta$. We do this procedure for any line emanating from the origin and take A^β to be the union of all these convex combinations, which obviously is a subset of Ω^β. We stress that the double-headed arrows in that image are all of the length β.

Lemma 3.3. *Let $0 < \beta \leq \mathrm{R}_e(\Omega)$. Under the assumptions of Lemma 3.2 it holds*

$$\int_{A^\beta + p} |u(x)|^2 \, \mathrm{d}x \leq \beta^2 \int_\Omega |(\mathrm{i}\nabla + \mathcal{A}(x))u(x)|^2 \, \mathrm{d}x,$$

for all $u \in C_0^\infty(\Omega)$, where A^β is defined in (3.21) and $p \in \mathbb{R}^2$ has to be chosen such that $B_{\mathrm{R}_e(\Omega)}(p) \subseteq \Omega$.

Proof. Let $u \in C_0^\infty(\Omega)$ and assume first of all that $p = (0,0)$. We take the integral on the left-hand side and change into polar coordinates. With regard to the definition of A^β in (3.21), we arrive at

$$\int_{A^\beta} |u(x)|^2 \, \mathrm{d}x = \int_0^{2\pi} \int_{b_\varphi - \beta}^{b_\varphi} |u(r,\varphi)|^2 r \, \mathrm{d}r \, \mathrm{d}\varphi.$$

In the proof of Lemma 3.2 we discussed that the set $\{\varphi \in [0, 2\pi) \,|\, b_\varphi = \infty\}$ is a null set since $|\Omega|$ has finite volume. Hence b_φ is almost everywhere finite. For $u \in C_0^\infty(\Omega)$

we have $u(b_\varphi, \varphi) = 0$ for almost every $\varphi \in [0, 2\pi)$ and apply then the following Hardy inequality

$$\int_{b_\varphi-\beta}^{b_\varphi} |u(r,\varphi)|^2 r \, \mathrm{d}r \leq \beta^2 \int_{b_\varphi-\beta}^{b_\varphi} |\partial_r u(r,\varphi)|^2 r \, \mathrm{d}r,$$

which is valid if

$$\sup_{b_\varphi-\beta\leq\tau\leq b_\varphi} \left(\int_{b_\varphi-\beta}^{\tau} s \, \mathrm{d}s\right)\left(\int_{\tau}^{b_\varphi} \frac{1}{s} \, \mathrm{d}s\right) \leq \frac{\beta^2}{4} \tag{3.23}$$

holds true, see [OK90, Theorem 1.14]. This is an easy computation if we use the monotonicity of the identity function. Afterwards we change back into our former coordinates to obtain

$$\int_{A^\beta} |u(x)|^2 \, \mathrm{d}x \leq \beta^2 \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x.$$

The inequality above is translation invariant. Thus, we get rid of the assumption $p = (0,0)$ and obtain

$$\int_{A^\beta+p} |u(x)|^2 \, \mathrm{d}x \leq \beta^2 \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x.$$

Since we know that $|u| \in H_0^1(\Omega)$, see [FH10, Prop. 2.1.2], we get

$$\int_{A^\beta+p} |u(x)|^2 \, \mathrm{d}x \leq \beta^2 \int_\Omega (\nabla |u(x)|)^2 \, \mathrm{d}x.$$

An application of the diamagnetic inequality stated in (3.6) yields the result. □

Proof of Theorem 3.1*:* We will follow the same notation as in [KW15]. Let $H(\mathcal{A})$ be the Friedrichs extension of the quadratic form (3.2) with

$$H(\mathcal{A})\phi_j = \lambda_j(\Omega, \mathcal{A})\phi_j$$

for $j \in \mathbb{N}$. The functions ϕ_j are assumed to be an orthonormal basis in $L^2(\Omega)$. We put

$$f_{k,j}(x) := \int_\Omega P_{k,B}(x,y)\phi_j(y) \, \mathrm{d}y$$

and consider

$$\operatorname{Tr}(H(\mathcal{A}) - \lambda)_- = \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} \left(\lambda \|\phi_j\|^2_{L^2(\Omega)} - \|(\mathrm{i}\nabla + \mathcal{A})\phi_j\|^2_{L^2(\Omega)}\right).$$

We extend these functions by $\phi_j(x) = 0$ for $x \in \Omega^c$ to apply the spectral theorem of $H(\mathcal{A}, \mathbb{R}^2)$. Since we do not know whether ϕ_j lies in the domain of $H(\mathcal{A}, \mathbb{R}^2)$ we must approximate ϕ_j by $C_0^\infty(\Omega)$ functions with respect to the quadratic form (3.2). An application of Fatou's lemma yields then

$$\begin{aligned}
\operatorname{Tr}(H(\mathcal{A}) - \lambda)_- &\leq \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} \sum_{k=1}^{\infty} \Big(\lambda \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} - \|(\mathrm{i}\nabla + \mathcal{A}) f_{k,j}\|^2_{L^2(\mathbb{R}^2)} \Big) \\
&\leq \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} \sum_{k=1}^{\infty} (\lambda - B(2k-1))_+ \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} \\
&= \sum_{k=1}^{\infty} (\lambda - B(2k-1))_+ \left(\sum_{j=1}^{\infty} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} - \mathcal{R}(\lambda, k) \right).
\end{aligned} \tag{3.24}$$

We recall that $a_\pm := (|a| \pm a)/2$ for $a \in \mathbb{R}$, and we set

$$\mathcal{R}(\lambda, k) := \sum_{j:\lambda_j(\Omega,\mathcal{A})\geq\lambda} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)}. \tag{3.25}$$

We use Parseval's identity and the properties of the integral kernels of $P_{k,B}$ stated in (3.17) to obtain

$$\begin{aligned}
\sum_{j=1}^{\infty} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \sum_{j=1}^{\infty} \Big| \langle\, P_{k,B}(x,\cdot),\, \phi_j(\cdot) \rangle_{L^2(\Omega)} \Big|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} \int_{\Omega} |P_{k,B}(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x = \int_{\Omega} \mathrm{P}_{k,B}(y,y)\,\mathrm{d}y = \frac{B}{2\pi}|\Omega|.
\end{aligned} \tag{3.26}$$

The next step is to give a lower bound for $R(\lambda, k)$. We see by (3.26) that

$$\begin{aligned}
\mathcal{R}(\lambda, k) &= \frac{B}{2\pi}|\Omega| - \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} \|f_{k,j}\|^2_{L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} \int_{\Omega} \Big| P_{k,B}(x,y) - \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} f_{k,j}(x)\overline{\phi_j}(y) \Big|^2 \,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Let $p \in \mathbb{R}^2$ be chosen such that $B_{\mathrm{R}_e(\Omega)}(p) \subseteq \Omega$. Then, we use the following inclusion $\Omega \supseteq A^\beta + p$, where A^β is defined in (3.21), apply $|a-b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$ for $a, b \in \mathbb{C}$ and use again the properties of the integral kernels $\mathrm{P}_{k,B}$, stated in (3.17). The translation invariance of the Lebesgue measure yields then

$$\begin{aligned}
\mathcal{R}(\lambda, k) &\geq \frac{B}{4\pi}|A^\beta| - \int_{\mathbb{R}^2} \int_{A^\beta + p} \Big| \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} f_{k,j}(x)\overline{\phi_j}(y) \Big|^2 \,\mathrm{d}y\,\mathrm{d}x \\
&\geq \frac{B}{4}\beta \mathrm{R}_e(\Omega) - \int_{\mathbb{R}^2} \int_{A^\beta + p} \Big| \sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} f_{k,j}(x)\overline{\phi_j}(y) \Big|^2 \,\mathrm{d}y\,\mathrm{d}x.
\end{aligned} \tag{3.27}$$

The last estimate is due to (3.22) and Lemma 3.2. The remaining negative term can be estimated by an application of Lemma 3.3 since the linear combination of ϕ_j still lies in the form domain of $H(\mathcal{A})$, yielding

$$\int_{\mathbb{R}^2}\int_{A^\beta+p}\Big|\sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda} f_{k,j}(x)\overline{\phi_j}(y)\Big|^2\,\mathrm{d}y\,\mathrm{d}x \le \beta^2\int_{\mathbb{R}^2}\sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda}\lambda_j(\Omega,\mathcal{A})|f_{k,j}(x)|^2\,\mathrm{d}x.$$

Now we use again (3.26) to obtain

$$\beta^2\int_{\mathbb{R}^2}\sum_{j:\lambda_j(\Omega,\mathcal{A})<\lambda}\lambda_j(\Omega,\mathcal{A})|f_{k,j}(x)|^2\,\mathrm{d}x \le \beta^2\lambda\int_{\mathbb{R}^2}\sum_{j=1}^{\infty}|f_{k,j}(x)|^2\,\mathrm{d}x = \lambda\beta^2\frac{B}{2\pi}|\Omega|.$$

We take this estimate for (3.27), which gives the following lower bound

$$\mathcal{R}(\lambda,k) \ge \frac{B}{4}\beta\mathrm{R}_e(\Omega) - \lambda\beta^2\frac{B}{2\pi}|\Omega| = \frac{B}{4}\beta\left(\mathrm{R}_e(\Omega) - \lambda\beta\frac{2}{\pi}|\Omega|\right). \tag{3.28}$$

We set

$$\beta = \frac{\mathrm{R}_e(\Omega)\pi}{4|\Omega|\lambda}. \tag{3.29}$$

To verify that this is possible, we take (3.7), which yields $\lambda_1(\Omega,\mathcal{A}) \ge 2\pi/|\Omega|$. We use that estimate to obtain for $\lambda \ge \lambda_1(\Omega,\mathcal{A})$

$$\beta = \frac{\mathrm{R}_e(\Omega)\pi}{4|\Omega|\lambda} \le \frac{\mathrm{R}_e(\Omega)\pi}{4|\Omega|\lambda_1(\Omega,\mathcal{A})} \le \frac{\mathrm{R}_e(\Omega)}{8} \le \mathrm{R}_e(\Omega). \tag{3.30}$$

Hence the lower bound on $\mathcal{R}(\lambda,k)$ becomes

$$\mathcal{R}(\lambda,k) \ge \frac{\mathrm{R}_e(\Omega)^2\pi B}{32|\Omega|\lambda}. \tag{3.31}$$

We take (3.31) and (3.26) so that the inequality in (3.24) becomes

$$\mathrm{Tr}(H(\mathcal{A})-\lambda)_- \le \sum_{k=1}^{\infty}(\lambda - B(2k-1))_+\left(\frac{B}{2\pi}|\Omega| - \frac{\mathrm{R}_e(\Omega)^2\pi B}{32|\Omega|\lambda}\right).$$

In [KW15, Proposition 3.3] it was shown that

$$\sum_{k=1}^{\infty}(\lambda - B(2k-1))_+ \le \frac{\lambda^2}{4B}, \quad \lambda \ge 0,$$

which finally yields the result of Theorem 3.1.

Chapter 4

Spectral estimates for the Heisenberg Laplacian

In this chapter we study Riesz means of the eigenvalues of the Heisenberg Laplacian subject to Dirichlet boundary conditions on bounded domains of the first Heisenberg group $\mathbb{H}$. We obtain an inequality with a sharp leading term and an additional lower order term, improving the result of A. M. Hansson and A. Laptev in [HL08].

4.1 Introduction

Let $\Omega \subset \mathbb{H}$ be a bounded domain. We consider the Heisenberg Laplacian on $L^2(\Omega)$ with Dirichlet boundary conditions formally given by

$$\mathrm{A}(\Omega) \;:=\; -X_1^2 - X_2^2\,,$$

where we recall

$$X_1 \;:=\; \partial_{x_1} + \frac{x_2}{2}\,\partial_{x_3}, \quad X_2 \;:=\; \partial_{x_2} - \frac{x_1}{2}\,\partial_{x_3}\,. \tag{4.1}$$

More precisely, $\mathrm{A}(\Omega)$ is the unique self-adjoint operator associated with the closure of the quadratic form

$$\mathrm{a}[u] \;:=\; \int_\Omega \left(|X_1\,u(x)|^2 + |X_2\,u(x)|^2\right)\,\mathrm{d}x, \tag{4.2}$$

initially given on $u \in C_0^\infty(\Omega)$. Note that

$$[X_2, X_1] = \partial_{x_3} =: X_3\,.$$

We recall that the left-invariant vector fields X_1, X_2, X_3 form a basis of the Heisenberg algebra and that the first Heisenberg group $\mathbb{H}$ is given by $\mathbb{R}^3$ equipped with the following group law

$$(x_1, x_2, x_3) \boxplus (y_1, y_2, y_3) \;:=\; \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\Big). \tag{4.3}$$

The subelliptic estimate proved in [Fol73] shows that

$$\|u\|_{1/2}^2 \;\leq\; c\, \left(a[u] + \|u\|^2_{L^2(\Omega)} \right), \qquad u \in C_0^\infty(\Omega) \tag{4.4}$$

holds for some $c > 0$, where the norm on the left-hand side denotes the Sobolev norm of order $1/2$, see (2.4). Hence the domain of the closure of $a[\cdot]$ is continuously embedded in $H_0^{1/2}(\Omega)$. Since the embedding $H_0^{1/2}(\Omega) \to L^2(\Omega)$ is compact, see [DNPV12], it follows that the spectrum of $\mathrm{A}(\Omega)$ is purely discrete. We denote by $\{\lambda_k(\Omega)\}_{k\in\mathbb{N}}$ the nondecreasing sequence of the eigenvalues of $\mathrm{A}(\Omega)$ and by $\{v_k\}_{k\in\mathbb{N}}$ the associated sequence of orthonormalized eigenfunctions;

$$\mathrm{A}(\Omega)\, v_k \;=\; \lambda_k(\Omega)\, v_k, \qquad \|v_k\|_{L^2(\Omega)} = 1. \tag{4.5}$$

Recently A. M. Hanson and A. Laptev proved in [HL08, Thm. 2.1] that

$$\mathrm{Tr}(\, \mathrm{A}(\Omega) - \lambda)_- = \sum_{k\in\mathbb{N}} (\lambda - \lambda_k(\Omega))_+ \;\leq\; \frac{|\Omega|}{96}\, \lambda^3, \qquad \lambda > 0. \tag{4.6}$$

Here the eigenvalues $\lambda_k(\Omega)$ are repeated according to their finite multiplicities and $|\Omega|$ denotes the three-dimensional Lebesgue measure of Ω. Moreover, it is also shown in [HL08] that the constant $\frac{1}{96}$ on the right-hand side of (4.6) is sharp. Indeed, this follows from the asymptotic equation

$$\lim_{\lambda\to\infty} \lambda^{-3}\ \mathrm{Tr}(\, \mathrm{A}(\Omega) - \lambda)_- = \frac{|\Omega|}{96}\,, \tag{4.7}$$

see [HL08, Cor. 2.8].

The aim of this chapter is to improve (4.6) by adding to its right-hand side a negative term of lower order in λ. In other words, we are going to show for all $\lambda > 0$ that

$$\mathrm{Tr}(\, \mathrm{A}(\Omega) - \lambda)_- \;\leq\; \frac{|\Omega|}{96}\, \lambda^3 - \mathcal{C}(\Omega)\, \lambda^\alpha, \tag{4.8}$$

where $\mathcal{C}(\Omega)$ is a positive constant which depends only on Ω and $\alpha \in (0,3)$. In our main result, see Theorem 4.1, we will prove inequality (4.8) with $\alpha = 2$ and give an explicit expression for the constant $\mathcal{C}(\Omega)$. This is in the spirit of Melas-type improvements, which was discussed in Section 1.2. In particular, our main result improves inequality (4.6) in a similar way in which [KW15] improves inequality (1.6).

However, the method that we employ in the present chapter is different from the one used in [KW15] since it does not rely on a Hardy inequality involving the distance to the boundary. In fact, as far as we know an analog of such an inequality for the Heisenberg Laplacian with explicit constants is not known. Instead we exploit the properties of the Carnot-Carathéodory metric, which is connected to the Heisenberg Laplacian in a natural way, see Section 2.3 for details.

In addition we will prove that the order of the remainder term in (4.8) can be further improved if we consider cylindrical domains of the type $\Omega = \omega \times (a, b)$, where $\omega \subset \mathbb{R}^2$ is a bounded domain and $a, b \in \mathbb{R}$ are such that $a < b$. In particular, Theorem 4.8 implies for cylinders with convex cross-section ω that

$$\operatorname{Tr}(\mathrm{A}(\Omega) - \lambda)_- \;\leq\; \max\left\{0, \frac{|\Omega|}{96}\,\lambda^3 - \frac{\lambda^{2+\frac{1}{4}}}{2^7 \cdot 3^{5/2}}\,\frac{|\Omega|}{\mathrm{R}_e(\omega)^{3/2}}\right\}, \qquad \lambda > 0, \tag{4.9}$$

where $\mathrm{R}_e(\omega)$ is the Euclidean inradius of ω, see Corollary 4.9. We mention that this result and the identity in (1.4) can be used to obtain estimates for Riesz means of order greater than one as well.

At last we consider the counting function of the Heisenberg Laplacian

$$N(\lambda, \Omega) := \sum_{k : \lambda_k(\Omega) < \lambda} 1.$$

In [HL08] A. M. Hansson and A. Laptev proved the following Weyl-type asymptotics for the counting function, meaning

$$\lim_{\lambda \to \infty} \lambda^{-2} N(\lambda, \Omega) = \frac{|\Omega|}{32}. \tag{4.10}$$

The question arises whether there exist domains such that the limit (4.10) yields a uniform bound as well. We say that Ω satisfies Pólya's inequality, in the spirit of [Pól61], if

$$N(\lambda, \Omega) \leq \lambda^2 \frac{|\Omega|}{32}, \quad \lambda \geq 0. \tag{4.11}$$

In Section 1.1 we have already discussed that for the eigenvalue counting function of the Laplacian with Dirichlet boundary conditions tiling domains satisfy an inequality of the form (1.3). However, we mentioned in Section 3.1 that in the class of bounded domains the eigenvalue counting function of the Laplacian in the presence of a constant magnetic field does not fulfill an inequality which reflects the leading term in Weyl's law, even in the class of tiling domains. In this chapter we construct domains such that Pólya's inequality holds true for the counting function of the Heisenberg Laplacian, see Theorem 4.10.

This chapter is organized as follows. The main results are announced in Section 4.2. In Section 4.3 and in particular in Theorem 4.12, we present some auxiliary results concerning the properties of balls with respect to the Carnot-Carathéodory metric. The proof of the Melas-type bound is given in Section 4.4. The construction of domains satisfying Pólya's inequality is given in Subsection 4.2.4. In Section 4.5 we proof the improvement on cylinders under the additional assumption that the Euclidean Hardy inequality on the cross-section ω has to be valid. In the closing section we establish a similar improvement on general domains. However for this result we need the validity a Hardy inequality with respect to the Carnot-Carathéodory metric on Ω, which in comparison to the previous condition is more restrictive.

4.2 Main results

4.2.1 Preliminaries

For a fixed point $x \in \mathbb{H}$ we denote its Euclidean length by $\|x\|_e$. We recall that $d_C(x, y)$ denotes the Carnot-Carathéodory metric (in the sequel C-C metric) between two given points $x, y \in \mathbb{H}$, which was introduced in Section 2.3. Let

$$C_r(0) = \{x \in \mathbb{H} \,:\, d_C(x, 0) < r\}$$

be the ball centered at the origin with radius $r > 0$ with respect to the C-C metric. Let us introduce the distance from a fixed point $x \in \Omega$ to the boundary of Ω with respect to the C-C metric

$$\delta_C(x) := \inf_{y \in \partial\Omega} d_C(x, y). \tag{4.12}$$

When needed, we extend the function $\delta_C(\cdot)$ on $\mathbb{H}$; for points lying in $x \in \Omega^c$ we set $\delta_C(x) = 0$. In addition we denote the inradius of Ω with respect to the C-C metric by

$$\mathrm{R}(\Omega) := \sup_{x \in \Omega} \delta_C(x) \tag{4.13}$$

and the diameter of Ω by

$$\mathrm{D}(\Omega) := \inf\{l > 0 |\, \exists a \in \Omega \text{ such that } \Omega \subseteq C_{l/2}(a)\}.$$

4.2.2 Melas-type bounds

With the above notation at hand we can state our main result.

Theorem 4.1. *Let $\Omega \subset \mathbb{H}$ be a bounded domain. Then*

$$\operatorname{Tr}(\mathrm{A}(\Omega) - \lambda)_- \;\leq\; \max\left\{0,\; \frac{|\Omega|}{96}\,\lambda^3 - \lambda^2 \frac{\mathrm{R}(\Omega)^8}{150|\Omega|\,\mathrm{D}(\Omega)^2\pi^4}\right\} \tag{4.14}$$

holds true for all $\lambda > 0$.

Remark 4.2. Equation (4.7) implies that

$$\operatorname{Tr}(\mathrm{A}(\Omega) - \lambda)_- = \frac{|\Omega|}{96}\,\lambda^3 + o(\lambda^3) \qquad \lambda \to \infty. \tag{4.15}$$

So far the order of the remainder term in (4.15) is not known.

Remark 4.3. The identity in (1.4) and equation (4.14) can be used to obtain estimates for Riesz means of orders greater than one as well.

Remark 4.4. We stress that the term in the asymptotic identity (4.15) cannot be considered as a phase space volume of the Heisenberg Laplacian because a change of variables yields

$$\int_\Omega \int_{\mathbb{R}^3} \left(\lambda - (\xi_1 + \frac{1}{2} x_2 \xi_3)^2 - (\xi_2 - \frac{1}{2} x_1 \xi_3)^2 \right)_+ \, \mathrm{d}\xi \, \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^3} \left(\lambda - \xi_1^2 - \xi_2^2 \right)_+ \, \mathrm{d}\xi \, \mathrm{d}x.$$

Since the right-hand side is unbounded, the phase-space volume does not exist.

Remark 4.5. When $\Omega = \mathbb{R}^3$, then the spectrum of $\mathrm{A}(\Omega)$ is continuous and contains no discrete eigenvalues. The integrated density of states of $\mathrm{A}(\mathbb{R}^3)$ was calculated in [Str96, Thm. 6.1].

The upper bound in (4.6) is equivalent, by means of the Legendre transform, to the Li-Yau type lower bound

$$\sum_{k=1}^{n} \lambda_k(\Omega) \;\geq\; \frac{8\sqrt{2}}{3} |\Omega|^{-\frac{1}{2}} \, n^{\frac{3}{2}} \qquad n \in \mathbb{N}, \tag{4.16}$$

see [HL08, Cor. 2.10]. In the same way Theorem 4.1 implies an improvement of (4.16). We use Corollary A.3, stated in the appendix, to (4.14) and obtain:

Corollary 4.6. *For any $n \in \mathbb{N}$ it holds*

$$\sum_{k=1}^{n} \lambda_k(\Omega) \;\geq\; \frac{8\sqrt{2}}{3} |\Omega|^{-\frac{1}{2}} \, n^{\frac{3}{2}} + \frac{16 \mathrm{R}(\Omega)^8}{75 |\Omega|^2 \mathrm{D}(\Omega)^2 \pi^4} \, n \,. \tag{4.17}$$

4.2.3 Spectral estimates on cylinders

For cylindrical domains of the type $\Omega = \omega \times (a, b)$, where $\omega \subset \mathbb{R}^2$ is a bounded domain and $a, b \in \mathbb{R}$ are such that $a < b$, we can improve the result of Theorem 4.1. In the sequel we will decompose the vector $x = (x', x_3) \in \mathbb{R}^3$ and assume the following:

Assumption 4.7. *Let $\omega \subset \mathbb{R}^2$ be a bounded domain such that the constant*

$$c^{-2} := \inf_{u \in C_0^\infty(\omega) \setminus \{0\}} \frac{\int_\omega |\nabla_{x'} u(x')|^2 \, \mathrm{d}x'}{\int_\omega |u(x') / \delta_e(x')|^2 \, \mathrm{d}x'} \,, \tag{4.18}$$

is positive, where $\nabla_{x'} := (\partial_{x_1}, \partial_{x_2})$ and

$$\delta_e(x') \;:=\; \operatorname{dist}(x', \partial\omega)$$

is the Euclidean distance from $x' \in \omega$ to the boundary of ω.

Clearly, the constant c is the best constant in Hardy's inequality

$$\int_\omega \frac{u(x')^2}{\delta_e(x')^2}\, \mathrm{d}x' \ \leq\ c^2 \int_\omega |\nabla_{x'} u(x')|^2\, \mathrm{d}x', \qquad u \in C_0^\infty(\omega). \tag{4.19}$$

We recall that for simply connected ω holds $2 \leq c \leq 4$, and for convex ω we get the sharp constant $c = 2$, which was already discussed in Section 1.3. To state the main result for cylinders we need the following geometrical quantities: the Euclidean inradius of ω is given by

$$\mathrm{R}_e(\omega) := \sup_{x' \in \omega} \delta_e(x').$$

We define for any $\beta \in (0, \mathrm{R}_e(\omega)]$ the set

$$\omega^\beta \ :=\ \{x' \in \omega \,|\, \delta(x') < \beta\}\ .$$

Then, we introduce the quantity

$$l(\omega) := (b-a) \inf_{0<\beta\leq \mathrm{R}(\omega)} \frac{|\omega^\beta|}{\beta}, \tag{4.20}$$

where $|\omega|$ is the two-dimensional Lebesgue measure. In Lemma 3.2 we have already seen that $l(\omega) > 0$. The main result for cylindrical domains is then given by the following Theorem:

Theorem 4.8. *Let $\Omega := \omega \times (a,b)$ and let c be defined by (4.18). Then*

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \leq \max\left\{0, \frac{|\Omega|}{96}\lambda^3 - \lambda^{\frac{2c+5}{c+2}} \frac{(1+2/c)}{96}\, l(\omega)^{\frac{2c+2}{c+2}}\, |\Omega|^{-\frac{c}{c+2}}\, (4c+4)^{-\frac{2c+2}{c+2}}\right\} \tag{4.21}$$

holds for all $\lambda \geq 0$.

Note that the order of the remainder term is larger than λ^2 for any $c > 0$, which for large eigenvalues is an improvement of Theorem 4.1.

Corollary 4.9. *Let $\Omega := \omega \times (a,b)$. If ω is convex, then*

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \leq \max\left\{0, \frac{1}{96}|\Omega|\lambda^3 - \lambda^{2+\frac{1}{4}} \frac{1}{2^7 \cdot 3^{5/2}} \frac{|\Omega|}{\mathrm{R}_e(\omega)^{3/2}}\right\}$$

holds for all $\lambda \geq 0$.

Proof. In case that ω is convex we have $c = 2$ in (4.18) , see e.g. [Dav99]. In addition, we know that $\frac{|\omega^\beta|}{\beta}$ is uniformly bounded on $(0, \mathrm{R}_e(\omega)]$ by

$$(b-a)\frac{|\omega^\beta|}{\beta} \geq (b-a)\frac{|\omega|}{\mathrm{R}_e(\omega)} = \frac{|\Omega|}{\mathrm{R}_e(\omega)},$$

see Theorem A.6 in the appendix. Hence we simplify the constant in Theorem 4.1 and obtain the result. □

For the improvement on general domains we need the Hardy inequality with respect to the Hardy weight $\delta_C(\cdot)$, which is discussed in Section 4.6.

4.2.4 Pólya's inequality

In this subsection we construct bounded domains such that Pólya's inequality is satisfied for the eigenvalue counting function of the Heisenberg Laplacian.

Theorem 4.10. *Let $\emptyset \neq \Omega \subset \mathbb{H}$ be a bounded domain of the form*

$$(\omega \times \mathbb{R}) \cap H_+ \cap H_-,$$

where $\omega \subset \mathbb{R}^2$ is a bounded tiling domain, H_+, H_- are half-space such that ∂H_+ and ∂H_- are parallel and $\partial H_+ \neq \partial H_-$, then

$$N(\lambda, \Omega) \leq \frac{|\Omega|}{32}\lambda^2, \quad \textit{for all } \lambda \geq 0.$$

Proof. First of all we denote by $h(x) := (hx_1, hx_2, h^2x_3)$ the Heisenberg dilation for any $h > 0$. Since the Heisenberg Lapalcian is left-invariant with respect to the group law of $\mathbb{H}$ and invariant with respect to the dilation $h(\cdot)$, we get:

$$N(\lambda, \Omega) = N(\lambda, v \boxplus \Omega), \quad \text{for all } v \in \mathbb{H}, \tag{4.22}$$
$$N(\lambda, h(\Omega)) = N(h^2\lambda, \Omega), \quad \text{for all } h > 0. \tag{4.23}$$

Without loss of generality we assume that $\Omega := \omega \times (a, b)$ with $a < b$ because of the left-invariance of the Heisenberg Laplacian and the boundedness of Ω. Thus let ω be a tiling domain in $\mathbb{R}^2$. Then we know that there exist congruent copies of ω such that

$$\bigcup_{\beta \in \mathbb{N}} \overline{\omega_\beta} = \mathbb{R}^2, \quad \text{with } \omega_{\beta_1} \cap \omega_{\beta_2} = \emptyset \text{ for } \beta_1 \neq \beta_2. \tag{4.24}$$

We first observe that for fixed $v := (v_1, v_2, v_3)$ we obtain for the set $v \boxplus (\omega \times (a, b))$ the following

$$\begin{aligned}
&\Big\{(x,y,z) \in \mathbb{H} \,|\, \exists(\omega_1,\omega_2,\omega_3) \in \omega \times (a,b) \text{ with} \begin{pmatrix} x = v_1 + \omega_1 \\ y = v_2 + \omega_2 \\ z = \omega_3 + v_3 - 1/2v_1\omega_2 + 1/2v_2\omega_1 \end{pmatrix} \Big\} \\
&= \Big\{(x,y,z) \in \mathbb{H} \,|\, \exists(\omega_1,\omega_2,\omega_3) \in \omega \times (a,b) \text{ with} \begin{pmatrix} x = v_1 + \omega_1 \\ y = v_2 + \omega_2 \\ z = \omega_3 + v_3 - 1/2v_1y + 1/2v_2x \end{pmatrix} \Big\} \\
&= \Big\{(x,y,z) \in \mathbb{H} \,|\, \exists(\omega_1,\omega_2) \in \omega \text{ with } \begin{pmatrix} x = v_1 + \omega_1 \\ y = v_2 + \omega_2 \end{pmatrix} \Big\} \\
&\cap \{(x,y,z) \in \mathbb{H} \,|\, b \geq z - v_3 + 1/2v_1y - 1/2v_2x\} \\
&\cap \{(x,y,z) \in \mathbb{H} \,|\, a \leq z - v_3 + 1/2v_1y - 1/2v_2x\},
\end{aligned} \tag{4.25}$$

Note that the top and the bottom of $v \boxplus (\omega \times (a,b))$ are parallel to each other, and if $v_1 = v_2 = 0$, we have only a translation in the z-direction. At that point we do the following procedure: we take copies of $\omega \times (a,b)$ with respect to the Heisenberg group law such that their projections onto the x_1-x_2 hyperplane are congruent and pairwise disjoint copies of ω to cover $\mathbb{R}^2$ up to a set of measure zero. This is possible since (4.24) is assumed. Then we use (4.25) to translate these domains along the x_3-direction for $v_1 = v_2 = 0$. This yields iso-spectral copies of Ω, denoted by Ω_β, such that

$$\bigcup_{\beta\in\mathbb{N}} \overline{\Omega_\beta} = \mathbb{H}, \quad \text{with } \Omega_{\beta_1} \cap \Omega_{\beta_2} = \emptyset \text{ for } \beta_1 \neq \beta_2. \tag{4.26}$$

At that point we mimic Pólya's proof in [Pól61].

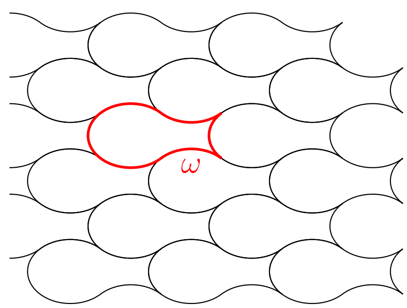


Figure 4.1: An example of a tiling domain.

The Heisenberg cube of length L is denoted by $W_L := \{x = (x_1, x_2, x_3) \in \mathbb{H} \,|\, |x_1| \leq L, |x_2| \leq L, |x_3| \leq L^2\}$. For the domain $h(\Omega)$ we can construct in the same way as above iso-spectral copies of $h(\Omega)$. For $\alpha \in \mathbb{N}$ let Ω_α be the countable set of congruent copies of $h(\Omega)$ covering $\mathbb{H}$ up to a set of measure zero. Further $A = \{\alpha \in \mathbb{N} \,|\, \Omega_\alpha \cap W_{L-2hd} \neq \emptyset\}$. The number d ist the diameter of Ω with respect to the Kaplan gauge, which was defined as

$$\|x\|_{\mathbb{H}}^4 := (x_1^2 + x_2^2)^2 + 16x_3^2$$

for $x := (x_1, x_2, x_3) \in \mathbb{H}$, and we set

$$d := \sup_{x,y\in\Omega} \|(-x) \boxplus y\|_{\mathbb{H}} \;\Rightarrow\; \sup_{x,y\in\Omega_\alpha} \|(-x) \boxplus y\|_{\mathbb{H}} = hd.$$

Thus we obtain

$$W_{L-2hd} \subseteq \cup_{\alpha\in A}\overline{\Omega_\alpha} \subseteq W_L, \quad \text{for } 2hd < L.$$

Let $\tilde{\Omega}$ be $\cup_{\alpha\in A}\Omega_\alpha$. Hence we get

$$|W_{L-2hd}| \leq |\tilde{\Omega}| = \#\{\alpha \in A\}|h(\Omega)| = \#\{\alpha \in A\}h^4|\Omega|\,.$$

Passing with Dirichlet boundary conditions from W_L to $\tilde{\Omega}$ and adding then Dirichlet boundary conditions to all $\partial\Omega_\alpha$, we claim by the variational principle

$$\begin{aligned} N(\lambda, W_L) \geq N(\lambda, \tilde{\Omega}) &\geq N(\oplus_{\alpha \in A}(\lambda, \Omega_\alpha)) \\ &= \#\{\alpha \in A\} N(\lambda, h(\Omega)) = \#\{\alpha \in A\} N(h^2\lambda, \Omega). \end{aligned}$$

Thus we get

$$\begin{aligned} \lambda^{-2} N(\lambda, W_L) &\geq \left(\#\{\alpha \in A\} h^4\right) \left(h^2\lambda\right)^{-2} N(h^2\lambda, \Omega) \\ &\geq \frac{|W_{L-2hd}|}{|\Omega|} \left(h^2\lambda\right)^{-2} N(h^2\lambda, \Omega). \end{aligned}$$

If we keep $\gamma = h^2\lambda > 0$ fixed and pass to $\lambda \to +\infty$ with respect to the number $h = \sqrt{\gamma\lambda^{-1}} \to 0$, Weyl's law, see (4.10), for the Heisenberg Laplacian on the cube W_L implies then

$$\frac{|W_L|}{32} \geq \frac{|W_L|}{|\Omega|} \gamma^{-2} N(\gamma, \Omega),$$

which yields the desired result. $\square$

4.3 The volume near the boundary

The goal of this section is to prove a bound on the Lebesgue measure of the set

$$\Omega^\beta := \left\{ x \in \Omega \;\middle|\; \delta_C(x) < \beta \right\}, \tag{4.27}$$

for a given $\beta \in (0, \mathrm{R}(\Omega)]$. We start by proving that $\delta_C(\cdot)$ is continuous with respect to the Euclidean metric.

Lemma 4.11. *Let Ω be a bounded domain in $\mathbb{H}$. The function $\delta_C(\cdot)$ is continuous with respect to the Euclidean distance on $\mathbb{H}$.*

Proof. We show that

$$|\delta_C(x) - \delta_C(y)| \leq d_C(x, y) \tag{4.28}$$

holds for $x, y \in \mathbb{H}$. Once the above inequality is established, the continuity of $\delta_C(\cdot)$ with respect to the Euclidean distance will follow by (2.15). We recall that we set $\delta_C(x) = 0$ for $x \in \Omega^c$. Let $x \neq y$. The case $x, y \in \Omega^c$ is trivial. For the case $x \in \Omega^c$ and $y \in \Omega$, we know that $\delta_C(x) = 0$. Let use denote by $\phi(t)$ the arc of a geodesic connecting x and y, which exists in view of Proposition 2.1. This curve is continuous and must intersect the boundary of Ω. Therefore exists $b \in \mathrm{Dom}(\phi)$ such that $\phi(b) \in \partial\Omega$. This gives

$$d_C(x, y) \geq d_C(\phi(b), y) \geq \delta_C(y) = |\delta_C(y) - \delta_C(x)|. \tag{4.29}$$

It remains to prove the claim in the case $x, y \in \Omega$. Without loss of generality we assume that $\delta_C(x) > \delta_C(y)$. Since d is continuous, see (2.15) and $\partial\Omega$ compact, there exists a $z \in \partial\Omega$ such that $\delta_C(y) = d_C(z, y)$. Thus we get

$$|\delta_C(x) - \delta_C(y)| = \delta_C(x) - \delta_C(y) \leq d_C(x, z) - d_C(y, z) \leq d_C(x, y). \tag{4.30}$$

The last inequality follows by the triangle inequality. Then we use (2.15) again and know that for any compact set $K \subset \mathbb{H}$ there exists a constant $M > 0$ such that

$$|\delta_C(x) - \delta_C(y)| \leq M^{-1} \|x - y\|_e^{1/2}$$

for all $x, y \in K$, yielding the result. $\square$

After these prerequisites we need the following result, which will be needed for the proof of Theorem 4.1.

Theorem 4.12. *Let $\Omega \subset \mathbb{H}$ be a bounded domain. Then the inequality*

$$|\Omega^\beta| \geq \beta \, \frac{16 \, \mathrm{R}(\Omega)^4}{5\pi^2 \mathrm{D}(\Omega)}. \tag{4.31}$$

holds true for all $\beta \in (0, \mathrm{R}(\Omega)]$.

Proof. First of all let us fix the parameter β with $0 < \beta \leq \mathrm{R}(\Omega)$. Because $\overline{\Omega}$ is compact and $\delta_C(\cdot)$ is continuous on $\mathbb{H}$, see Lemma 4.11, there exists an $x \in \Omega$ such that $C_{\mathrm{R}(\Omega)}(x) \subseteq \Omega$. We know that the Lebesgue measure and $d_C(\cdot, \cdot)$ are left-invariant with respect to the group law of the Heisenberg group. Hence we translate Ω in such a way that x is the origin of its translated copy. This means that

$$C_{\mathrm{R}(\Omega)}(0) \subseteq \Omega. \tag{4.32}$$

Now we construct a subset of Ω^β using the formula of the geodesics in (2.9). Therefore we consider the map

$$x = \left(\frac{\cos(\theta) - \cos(kt + \theta)}{k}, \frac{\sin(kt + \theta) - \sin(\theta)}{k}, \frac{tk - \sin(kt)}{2k^2} \right) = \Phi(t, k, \theta) \tag{4.33}$$

with $t \in [0, \frac{2\pi}{|k|}]$, $\theta \in [0, 2\pi)$ and $k \in [-\pi/\mathrm{D}(\Omega), \pi/\mathrm{D}(\Omega)]$. We stress that the set, parametrized by Φ, is geometrically speaking a subset of the C-C ball $C_{2\mathrm{D}(\Omega)}(0)$. For fixed k, θ we define

$$a := \inf \left\{ t > 0 \;\middle|\; \varphi(t) \notin \Omega \right\}, \tag{4.34}$$

which is well-defined since $\varphi(0) = 0 \in C_{\mathrm{R}(\Omega)}(0) \subseteq \Omega$ and $\Omega \subseteq C_{\mathrm{D}(\Omega)}(0)$. It follows that $\varphi(a) \in \partial\Omega$. From $C_{\mathrm{R}(\Omega)} \subseteq \Omega$ we obtain

$$\mathrm{R}(\Omega) \leq a. \tag{4.35}$$

Now we define the set

$$\Omega(\Phi) := \left\{ x \in \Omega^\beta \;\middle|\; \exists\,(t,k,\theta) \in E(\alpha,\beta) \text{ such that } x = \Phi(t,k,\theta) \right\}, \tag{4.36}$$

where

$$E(a,\beta) := (a-\beta,a) \times \left[-\frac{\pi}{\mathrm{D}(\Omega)}, \frac{\pi}{\mathrm{D}(\Omega)}\right] \times [0,2\pi).$$

Note that the map $\Phi : E(a,\beta) \to \mathbb{H}$ is injective, see Proposition 2.1. That means that $\Omega^\beta \supseteq \Omega(\Phi)$.

Now we consider the restriction of the curve φ on the interval $[a-\beta,a]$. Notice that this curve connects the point $\varphi(a-\beta)$ with $\varphi(a) \in \partial\Omega$. Moreover, in view of the definition of a, this curve is still a horizontal curve, lying in Ω. Therefore by the definition of the C-C metric the following estimate holds

$$d_C(\varphi(t),\varphi(a)) \leq a - t < \beta \qquad \forall\, t \in (a-\beta,a).$$

From $\varphi(a) \in \partial\Omega$ we obtain $\delta_C(\varphi(t)) \leq d_C(\varphi(t),\varphi(a)) < \beta$ for all $t \in (a-\beta,a)$, which means that $\varphi(t) \in \Omega^\beta$ for any $t \in (a-\beta,a)$. It follows then

$$\Omega^\beta \supseteq \Omega(\Phi) = \left\{ \Phi(t,k,\theta) \in \mathbb{H} \;\middle|\; (t,k,\theta) \in E(a,\beta) \right\} =: E^\beta. \tag{4.37}$$

The set E^β is interpreted geometrically in Figure 4.2. The inclusion in (4.37) and the formula (2.12) imply

$$|E^\beta| \geq \int_0^{2\pi} \int_{-\pi/\mathrm{D}(\Omega)}^{\pi/\mathrm{D}(\Omega)} \int_{a-\beta}^{a} \frac{2 - kt\sin(kt) - 2\cos(kt)}{k^4}\, \mathrm{d}t\, \mathrm{d}k\, \mathrm{d}\theta. \tag{4.38}$$

Since $a \leq \mathrm{D}(\Omega)$ because of $\Omega \subseteq C_{\mathrm{D}(\Omega)}(0)$, we have $|tk| \leq \pi$. In order to obtain a suitable lower bound on the integral on the right-hand side of (4.38) we notice that

$$f(\tau) = 2 - \tau\sin(\tau) - 2\cos(\tau) \tag{4.39}$$

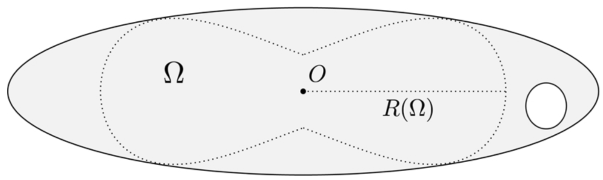
is nondecreasing on $[0,\pi]$. Indeed, this follows from the fact that $f(0) = f'(0) = 0$ and $f''(\tau) \geq 0$ on $[0,\pi]$. Since $|kt| \leq \pi$, and f is nondecreasing, we use (4.35) to get

$$\begin{aligned} |E^\beta| &\geq \int_0^{2\pi} \int_{-\pi/\mathrm{D}(\Omega)}^{\pi/\mathrm{D}(\Omega)} \int_{\mathrm{R}(\Omega)-\beta}^{\mathrm{R}(\Omega)} \frac{2 - kt\sin(kt) - 2\cos(kt)}{k^4}\, \mathrm{d}t\, \mathrm{d}k\, \mathrm{d}\theta \\ &\geq 2 \int_0^{2\pi} \int_0^{\pi/\mathrm{D}(\Omega)} \int_{\mathrm{R}(\Omega)-\beta}^{\mathrm{R}(\Omega)} \frac{2 - kt\sin(kt) - 2\cos(kt)}{(tk)^4} t^4\, \mathrm{d}t\, \mathrm{d}k\, \mathrm{d}\theta. \end{aligned} \tag{4.40}$$

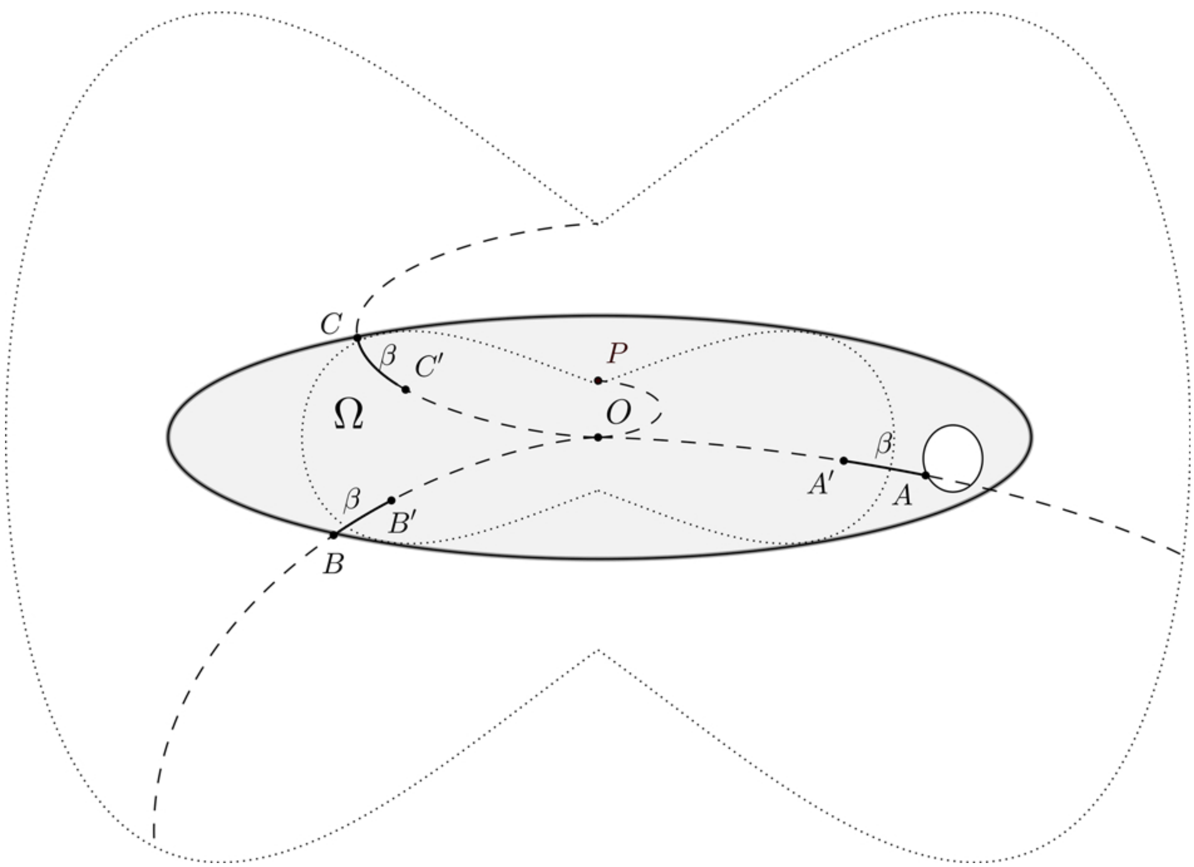
In the same way as we have proved the monotonicity of f it follows that the function

$$\frac{2 - x\sin(x) - 2\cos(x)}{x^4} \tag{4.41}$$

Figure 4.2: On the construction of E^{β}.



(a) Let Ω be given by the grey-colored area. At the origin O we center the largest C-C ball which still fits into Ω. Its radius is given by $\mathrm{R}(\Omega)$.



(b) In this picture we constructed a larger C-C ball with radius $2\mathrm{D}(\Omega)$ centered at the origin, which is described by the outer dotted line. We take the parametrization of that ball and obtain geodesics emanating from the origin, depicted in that image by dashed lines. The problem in that parametrization is that there exist geodesics which do not cross the boundary of Ω, see for example the geodesic connecting the origin with P. In view of the proof of Theorem 4.1 , we need the intersection points with the boundary since in Subsection 4.4.3 we establish a Hardy-type inequality on E^{β}. Hence we take only those geodesics which cross $\partial\Omega$ and compute then the first intersection point of that geodesic with $\partial\Omega$, see for instance the points A, B, C. Let us fix for a moment a geodesic which intersects $\partial\Omega$, for instance the one containing A in our picture, and compute the point A' which lies on that geodesic connecting the origin with A and which has the following property: $d_C(A, A') = \beta$. The part of the geodesic between A and A' is an arc. We then obtain for every geodesic emanating from the origin which reaches $\partial\Omega$, such a piece of arc, and take E^{β} to be the union of them.

is nonincreasing on $(0, \pi]$. Hence

$$|E^\beta| \geq \frac{16}{\mathrm{D}(\Omega)\pi^2} \int_{\mathrm{R}(\Omega)-\beta}^{\mathrm{R}(\Omega)} t^4 \, \mathrm{d}t = \beta \, \frac{16}{5\pi^2 \mathrm{D}(\Omega)} \frac{\mathrm{R}(\Omega)^5 - (\mathrm{R}(\Omega) - \beta)^5}{\beta}. \tag{4.42}$$

The quotient $(\mathrm{R}(\Omega)^5 - (\mathrm{R}(\Omega) - \beta)^5)\beta^{-1}$ on the right-hand side is nonincreasing in β on $(0, \mathrm{R}(\Omega)]$. This together with (4.37) yields then

$$|\Omega^\beta| \geq |E^\beta| \geq \beta \, \frac{16 \, \mathrm{R}(\Omega)^4}{5\pi^2 \mathrm{D}(\Omega)}.$$

□

Remark 4.13. We stress that from geometrical point of view the construction of E^β in Theorem 4.12 and A^β in Lemma 3.2 are based on the same idea. The difference is that in the Euclidean case it is always possible to extend a geodesic such that it becomes an infinite ray which still is a geodesic. This is not possible with geodesics on $\mathbb{H}$.

Remark 4.14. The same technique in Theorem 4.12 yields also a lower bound in the Euclidean case $\mathbb{R}^n$ with respect to the Euclidean metric and inradius. In that case the lower bound does not depend on the diameter, and the proof is much easier because the determinant of the spherical coordinates is obviously monotonically increasing in the radial part for any fixed angle. We have proved this result for the case $n = 2$ in Lemma 3.2.

4.4 Proof of the Melas-type bound

In order to find a representation of the spectral decomposition of the Heisenberg Laplacian, we introduce the Fourier transform in the x_3-direction;

$$\mathcal{F}_3 \, u(x', \xi_3) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i} x_3 \xi_3} \, u(x', x_3) \, \mathrm{d}x_3, \tag{4.43}$$

where $x' := (x_1, x_2)$ and $x := (x', x_3) \in \mathbb{H}$. Then

$$\mathcal{F}_3 \, \mathrm{A}(\Omega) \, \mathcal{F}_3^* \;=\; \left(\mathrm{i}\partial_{x_1} - \frac{x_2}{2}\,\xi_3\right)^2 + \left(\mathrm{i}\partial_{x_2} + \frac{x_1}{2}\,\xi_3\right)^2 \;=\; \left(\mathrm{i}\nabla_{x'} + \xi_3 \mathbf{A}(x')\right)^2, \tag{4.44}$$

where $\mathbf{A}(x') := \frac{1}{2}(-x_2, x_1)$. Hence for each fixed $\xi_3 \in \mathbb{R}$ the right-hand side is the Landau Hamiltonian in $L^2(\mathbb{R}^2)$ associated with the constant magnetic field ξ_3. Its eigenvalues are given by the Landau levels $\{|\xi_3|(2\mathrm{k} - 1)\}_{k \in \mathbb{N}}$. We denote by $\mathrm{P}_{\mathrm{k},\xi_3}$ the orthogonal projection in $L^2(\mathbb{R}^2)$ onto the Landau level $|\xi_3|(2\mathrm{k} - 1)$. In Section 3.3 we already discussed well-known properties of these orthogonal projections.

Hence for any u such that $\mathcal{F}_3\, u(\cdot,\xi_3)$ belongs to the domain of $(\mathrm{i}\nabla_{x'}+\xi_3\mathbf{A}(x'))^2$ we have

$$\mathcal{F}_3\,\mathrm{A}(\Omega)\,u(x',\xi_3)=\sum_{\mathrm{k}=1}^{\infty}|\xi_3|(2\mathrm{k}-1)\int_{\mathbb{R}^2}\mathrm{P}_{\mathrm{k},\xi_3}(x',y')\mathcal{F}_3\,u(y',\xi_3)\ \mathrm{d}y'\,. \tag{4.45}$$

Proof of Theorem 4.1**.** We split the proof into three steps. In the first one we derive the sharp leading term with an additional negative term. The appearing negative term will be treated in the second part of the proof. The last part of the proof is dedicated to the proof of an auxiliary result needed in step two.

4.4.1 The sharp leading term

In the sequel we will decompose a vector $x\in\mathbb{H}$ as

$$x=(x',x_3)=(x_1,x_2,x_3). \tag{4.46}$$

We extend the eigenfunctions $v_j(x)$ of $\mathrm{A}(\Omega)$ by zero for $x\in\Omega^c$ and write

$$\begin{aligned}
&\operatorname{Tr}(\mathrm{A}(\Omega)-\lambda)_-\\
&=\sum_{j:\lambda_j(\Omega)<\lambda}\lambda\|v_j\|^2_{L^2(\Omega)}-\left\|\left(\partial_{x_1}+\tfrac{1}{2}x_2\partial_{x_3}\right)v_j\right\|^2_{L^2(\mathbb{R}^3)}-\left\|\left(\partial_{x_2}-\tfrac{1}{2}x_1\partial_{x_3}\right)v_j\right\|^2_{L^2(\mathbb{R}^3)}\\
&=\int_{\mathbb{R}}\sum_{j:\lambda_j(\Omega)<\lambda}\lambda\left(\|\mathcal{F}_3\,v_j(\cdot,\xi_3)\|^2_{L^2(\mathbb{R}^2)}-\left\|\left(\mathrm{i}\partial_{x_1}-\tfrac{1}{2}x_2\xi_3\right)\mathcal{F}_3\,v_j(\cdot,\xi_3)\right\|^2_{L^2(\mathbb{R}^2)}\right)\mathrm{d}\xi_3\\
&\quad-\int_{\mathbb{R}}\sum_{j:\lambda_j(\Omega)<\lambda}\left\|\left(\mathrm{i}\partial_{x_2}+\tfrac{1}{2}x_1\xi_3\right)\mathcal{F}_3\,v_j(\cdot,\xi_3)\right\|^2_{L^2(\mathbb{R}^2)}\ \mathrm{d}\xi_3.
\end{aligned}$$

We apply the spectral decomposition in (4.45) and use Fatou's lemma to obtain the following estimate for the trace:

$$\operatorname{Tr}(\mathrm{A}(\Omega)-\lambda)_-\ \le\ \int_{\mathbb{R}}\sum_{j:\lambda_j(\Omega)<\lambda}\sum_{\mathrm{k}=1}^{\infty}(\lambda-|\xi_3|(2\mathrm{k}-1))\|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\ \mathrm{d}\xi_3, \tag{4.47}$$

where

$$\begin{aligned}
f_{j,\mathrm{k},\xi_3}(x')\ &:=\ \int_{\mathbb{R}^2}\mathrm{P}_{\mathrm{k},\xi_3}(x',y')\mathcal{F}_3v_j(y',\xi_3)\ \mathrm{d}y'\\
&=\frac{1}{\sqrt{2\pi}}\int_{\Omega}\mathrm{P}_{\mathrm{k},\xi_3}(x',y')\,\mathrm{e}^{-\mathrm{i}y_3\xi_3}v_j(y',y_3)\ \mathrm{d}y'\ \mathrm{d}y_3\\
&=\frac{1}{\sqrt{2\pi}}\left\langle\,\mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\ v_j(\cdot)\right\rangle_{L^2(\Omega)}.
\end{aligned}$$

Next we estimate the right-hand side of (4.47) further by considering the positive part of $(\lambda - |\xi_3|(2\mathrm{k}-1))$. This gives

$$\begin{aligned}\operatorname{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;\leq\; &\int_{\mathbb{R}} \sum_{k=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+ \sum_{j=1}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}\xi_3 \\ &- \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+ \sum_{j:\lambda_j(\Omega)\geq\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}\xi_3.\end{aligned} \tag{4.48}$$

Since the sequence $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$ we can use Parseval's identity to evaluate the sum over j. Taking into account (3.17) we obtain

$$\sum_{j=1}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sum_{j=1}^{\infty} \left| \left\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\, \mathrm{e}^{-\mathrm{i}\xi_3 \cdot},\, v_j(\cdot) \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}x' = \frac{|\xi_3|}{4\pi^2} |\Omega|. \tag{4.49}$$

This allows us to calculate the first term on the right-hand side of (4.48). Then we have

$$\begin{aligned}&\int_{\mathbb{R}} \sum_{k=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+ \sum_{j=1}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}\xi_3 \\ &= \frac{|\Omega|}{2\pi^2} \sum_{k=1}^{\infty} \int_0^{\infty} (\lambda - \xi_3(2\mathrm{k}-1))_+\, \xi_3\, \mathrm{d}\xi_3 = \frac{|\Omega|\, \lambda^3}{12\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{|\Omega|}{96}\, \lambda^3,\end{aligned}$$

where we have used the identity

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}. \tag{4.50}$$

Putting together the above estimates and using (4.48) we get

$$\operatorname{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;\leq\; \frac{|\Omega|}{96}\, \lambda^3 - \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+ \sum_{j:\lambda_j(\Omega)\geq\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}\xi_3. \tag{4.51}$$

On the right-hand side we thus have the sharp leading term and an additional negative term. The latter will be treated in the next step.

4.4.2 The negative lower order term

The next step is to establish a suitable lower bound on

$$\mathrm{Q}(\lambda, k, \xi_3) := \sum_{j:\lambda_j(\Omega)\geq\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}. \tag{4.52}$$

Using equation (4.49) we rewrite the series as follows;

$$\begin{aligned} Q(\lambda, k, \xi_3) = {} & \frac{|\xi_3|}{4\pi^2}\,|\Omega| - \sum_{j:\lambda_j(\Omega)<\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)} \\ = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\Omega} & \Big|\,\mathrm{P}_{\mathrm{k},\xi_3}(x',y')\,\mathrm{e}^{-\mathrm{i}y_3\xi_3} - \sum_{j:\lambda_j(\Omega)<\lambda} \big\langle\, \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y)\Big|^2 \,\mathrm{d}y\,\mathrm{d}x'. \end{aligned} \tag{4.53}$$

To estimate the right-hand side form below we consider the set

$$E^\beta := \Big\{\, \Phi(t,k,\theta) \in \mathbb{H} \;\Big|\; (t,k,\theta) \in (a-\beta, a) \times (-\pi/\mathrm{D}(\Omega), \pi/\mathrm{D}(\Omega)) \times [0, 2\pi) \,\Big\}. \tag{4.54}$$

Note that in view of (4.37) we have

$$\Omega \supseteq \Omega^\beta \supseteq E^\beta\,.$$

We use the following inequality

$$|z-w|^2 \;\geq\; \frac{1}{2}|z|^2 - |w|^2, \qquad z, w \in \mathbb{C} \tag{4.55}$$

and equation (3.17) to obtain

$$\begin{aligned} Q(\lambda, k, \xi_3) \geq {} & \frac{|\xi_3|}{8\pi^2}\Big|\,E^\beta\Big| \\ & - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{E^\beta}\Big|\sum_{j:\lambda_j(\Omega)<\lambda} \big\langle\, \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y)\Big|^2 \,\mathrm{d}y\,\mathrm{d}x'. \end{aligned}$$

In the end of the proof of Theorem 4.12 we have shown that

$$|E^\beta| \geq \beta\frac{16}{5\mathrm{D}(\Omega)\pi^2}\mathrm{R}(\Omega)^4 \tag{4.56}$$

for all $\beta \in (0, \mathrm{R}(\Omega)]$. Hence

$$\begin{aligned} Q(\lambda, k, \xi_3) \;\geq\; {} & \beta\;\frac{2|\xi_3|}{5\mathrm{D}(\Omega)\pi^4}\mathrm{R}(\Omega)^4 \\ & - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{E^\beta}\Big|\sum_{j:\lambda_j(\Omega)<\lambda} \big\langle\, \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y)\Big|^2 \,\mathrm{d}y\,\mathrm{d}x'. \end{aligned}$$

At this point we have to estimate the negative integral from above. Note that the linear combination of v_j lies in $d[a]$. Therefore we can use the inequality

$$\int_{E^\beta}|u|^2\,\mathrm{d}x \;\leq\; \beta^2\int_\Omega|\nabla_{\mathbb{H}}\,u|^2\,\mathrm{d}x, \qquad u \in d[a], \tag{4.57}$$

which is proved in Subsection 4.4.3. Assuming for the moment that (4.57) holds true we get

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{E^\beta}\Big|\sum_{j:\lambda_j(\Omega)<\lambda}\big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y)\Big|^2\,\mathrm{d}y\,\mathrm{d}x'$$
$$\leq \frac{\beta^2}{2\pi}\int_{\mathbb{R}^2}\int_{\Omega}\Big|\sum_{j:\lambda_j(\Omega)<\lambda}\big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)}\nabla_{\mathbb{H}}\, v_j(y)\Big|^2\,\mathrm{d}y\,\mathrm{d}x'.$$

Integration by parts and the fact that the eigenfunctions v_j are mutually orthogonal yield

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{E^\beta}\Big|\sum_{j:\lambda_j(\Omega)<\lambda}\big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y)\Big|^2\,\mathrm{d}y\,\mathrm{d}x'$$
$$\leq \frac{\beta^2}{2\pi}\int_{\mathbb{R}^2}\int_{\Omega}\sum_{j:\lambda_j(\Omega)<\lambda}\lambda_j(\Omega)\Big|\big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y)\Big|^2\,\mathrm{d}y\,\mathrm{d}x'$$
$$\leq \frac{\beta^2\,\lambda}{2\pi}\int_{\mathbb{R}^2}\sum_{j:\lambda_j(\Omega)<\lambda}\Big|\big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)}\Big|^2\,\mathrm{d}x'\,.$$

Finally we sum over all j and use (4.49) to obtain

$$\int_{\mathbb{R}^2}\sum_{j:\lambda_j(\Omega)<\lambda}\Big|\big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\,\mathrm{e}^{-\mathrm{i}\xi_3\cdot},\, v_j(\cdot)\big\rangle_{L^2(\Omega)}\Big|^2\,\mathrm{d}x' \;\leq\; \frac{|\xi_3||\Omega|}{2\pi}\,. \tag{4.58}$$

Summarizing these estimates we arrive at the following lower bound on Q:

$$\mathrm{Q}(\lambda,k,\xi_3)\;\geq\beta\;\frac{2|\xi_3|}{5\mathrm{D}(\Omega)\pi^4}\mathrm{R}(\Omega)^4-\beta^2\frac{|\xi_3|}{4\pi^2}|\Omega|\lambda=\beta\frac{|\xi_3|}{8\pi^2}\left(\frac{16\mathrm{R}(\Omega)^4}{5\mathrm{D}(\Omega)\pi^2}-2\beta|\Omega|\lambda\right). \tag{4.59}$$

Now we set

$$\beta\;:=\;\frac{4\mathrm{R}(\Omega)^4}{5|\Omega|\,\mathrm{D}(\Omega)\pi^2}\;\lambda^{-1}. \tag{4.60}$$

We have to show that with this choice $\beta\leq\mathrm{R}(\Omega)$ holds true. By (4.16) we get

$$\frac{1}{\lambda_1(\Omega)}\leq\frac{3}{8\sqrt{2}}\,|\Omega|^{1/2}\leq|\Omega|^{1/2}, \tag{4.61}$$

and Theorem 4.12 yields for $\beta=\mathrm{R}(\Omega)$ and $\Omega=C_1(0)$ the following

$$\frac{8}{5\pi^2}\leq|C_1(0)|. \tag{4.62}$$

Both inequalities in combination with Proposition 2.1(e) yield

$$\beta \;\leq\; \frac{4\mathrm{R}(\Omega)^4}{5|\Omega|^{1/2}\,\mathrm{D}(\Omega)\pi^2} = \frac{4|C_{\mathrm{R}(\Omega)}(0)|}{5|\Omega|^{1/2}\,\mathrm{D}(\Omega)\pi^2|C_1(0)|} \leq \frac{|C_{\mathrm{R}(\Omega)}(0)|^{1/2}}{2\,\mathrm{D}(\Omega)}. \tag{4.63}$$

From the fact that $|B_1(0)| \leq 1$, see e.g. [Sac11], we thus deduce that

$$\beta \;\leq \frac{|C_{\mathrm{R}(\Omega)}(0)|^{1/2}}{\mathrm{D}(\Omega)} = \frac{\mathrm{R}(\Omega)^2|B_1(0)|^{1/2}}{\mathrm{D}(\Omega)} \leq \mathrm{R}(\Omega) \tag{4.64}$$

as required. Hence we may insert (4.60) into (4.59), which yields

$$\mathrm{Q}(\lambda, k, \xi_3) \;\geq |\xi_3|\lambda^{-1}\frac{4\mathrm{R}(\Omega)^8}{25|\Omega|\,\mathrm{D}(\Omega)^2\pi^6}. \tag{4.65}$$

Finally we estimate the sum of the negative integral of (4.51)

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \leq \frac{|\Omega|}{96}\,\lambda^3 - \lambda^{-1}\frac{4\mathrm{R}(\Omega)^8}{25|\Omega|\,\mathrm{D}(\Omega)^2\pi^6}\int_{\mathbb{R}}\sum_{k=1}^{\infty}(\lambda - |\xi_3|(2\mathrm{k}-1))_+\,|\xi_3|\;\mathrm{d}\xi_3 \tag{4.66}$$

and calculate the integral on the right-hand side by using the following substitution $\xi_3(2k-1) = s$ and (4.50):

$$2\sum_{\mathrm{k}=1}^{\infty}\int_0^{\infty}(\lambda - \xi_3(2\mathrm{k}-1))_+\xi_3\;\mathrm{d}\xi_3 \;=\; \sum_{\mathrm{k}=1}^{\infty}\frac{2}{(2\mathrm{k}-1)^2}\int_0^{\infty}s(\lambda - s)_+\;\mathrm{d}s \;=\; \frac{\pi^2\lambda^3}{24}.$$

This yields then inequality (4.14). It thus remains to prove (4.57).

4.4.3 Proof of inequality (4.57)

Without loss of generality we assume that $u \in C_0^\infty(\Omega)$. With the help of (2.12) and (4.54) we change the coordinates and obtain

$$\int_{E^\beta}|u(x)|^2\;\mathrm{d}x = \int_0^{2\pi}\int_{-\pi/\mathrm{D}(\Omega)}^{\pi/\mathrm{D}(\Omega)}\int_{a-\beta}^{a}|u(t,k,\theta)|^2\,\frac{f(tk)}{k^4}\;\mathrm{d}\theta, \tag{4.67}$$

where f is defined in (4.39). We can assume that k is positive. Otherwise we substitute k by $-k$ and use that $f(\cdot)$ is even. We know that $u(a,k,\theta) = 0$ by definition of a, see (4.34), for all $k \in [-\pi/\mathrm{D}(\Omega), \pi/\mathrm{D}(\Omega)]$ and $\theta \in [0, 2\pi)$. We recall that $a \leq \mathrm{D}(\Omega)$, $f(\cdot)$ is increasing on $[0,\pi]$ and $|tk| \leq \pi$, see the proof of Theorem 4.12. Then we easily show that

$$\sup_{a-\beta\leq\tau\leq a}\int_{a-\beta}^{\tau}f(sk)\;\mathrm{d}s\int_{\tau}^{a}\frac{1}{f(sk)}\;\mathrm{d}s \;\leq\; \sup_{a-\beta\leq\tau\leq a}(\tau - a + \beta)(a-\tau) = \frac{\beta^2}{4}\,. \tag{4.68}$$

In view of [OK90, Theorem 1.14] we thus conclude that

$$\begin{aligned}
&\int_0^{2\pi}\int_{-\pi/\mathrm{D}(\Omega)}^{\pi/\mathrm{D}(\Omega)}\int_{a-\beta}^{a} |u(t,k,\theta)|^2 \frac{f(tk)}{k^4}\;\mathrm{d}t\,\mathrm{d}k\,\mathrm{d}\theta \\
&\le \beta^2 \int_0^{2\pi}\int_{-\pi/\mathrm{D}(\Omega)}^{\pi/\mathrm{D}(\Omega)}\int_{a-\beta}^{a} |\partial_t u(t,k,\theta)|^2 \frac{f(tk)}{k^4}\;\mathrm{d}t\,\mathrm{d}k\,\mathrm{d}\theta.
\end{aligned} \tag{4.69}$$

Let us now turn to the coordinate system (x_1, x_2, x_3). Keeping in mind the parametrization (2.9) we get

$$\int_{E^\beta} |\partial_t u|^2\;\mathrm{d}x \le \int_\Omega |\partial_{x_1} u\,\partial_t x_1 + \partial_{x_2} u\,\partial_t x_2 + \partial_{x_3} u\,\partial_t x_3|^2\;\mathrm{d}x. \tag{4.70}$$

From the differential equation (2.5) of the geodesics; $2\partial_t x_3(t) = x_2(t)\partial_t x_1(t) - \partial_t x_2(t) x_1(t)$, it further follows that

$$\int_{E^\beta} |\partial_t u|^2\;\mathrm{d}x \le \int_\Omega |\partial_t x_1 X_1 u + \partial_t x_2 X_2\, u|^2\;\mathrm{d}x. \tag{4.71}$$

The cross terms will be estimated with the help of the Cauchy-Schwarz inequality and $2ab \le a^2 + b^2,\; a, b, \in \mathbb{R}$. This gives

$$2|\langle\, \partial_t x_1\, X_1 u,\, \partial_t x_2\, X_2 u\rangle_{L^2(\Omega)}| \le \|\partial_t x_2 X_1 u\|^2_{L^2(\Omega)} + \|\partial_t x_1 X_2 u\|^2_{L^2(\Omega)}. \tag{4.72}$$

Now we collect all the above estimates to arrive at

$$\begin{aligned}
\int_{E^\beta} |u|^2\;\mathrm{d}x &\le \beta^2 \left(\|\partial_t x_1 X_1 u\|^2_{L^2(\Omega)} + \|\partial_t x_2 X_1 u\|^2_{L^2(\Omega)}\right) \\
&\quad + \beta^2 \left(\|\partial_t x_2 X_2 u\|^2_{L^2(\Omega)} + \|\partial_t x_1 X_2 u\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{4.73}$$

From (2.9) we see that $\partial_t x_1 = \sin(kt+\theta)$ and $\partial_t x_2 = \cos(kt+\theta)$, which implies inequality (4.57) and completes the proof of Theorem 4.1. □

4.5 Proof of spectral estimates on cylinders

We have seen in Theorem 4.1 that for a bounded domain $\Omega \subset \mathbb{H}$ we improved the sharp bound for the eigenvalue sum by adding a term of the form $-\lambda^2 \mathcal{C}(\Omega)$, where $\mathcal{C}(\Omega)$ is a positive constant only depending on the geometry of Ω. However, Theorem 4.8 shows that the growth order of the negative remainder term can be further improved if we assume additional conditions on Ω .

In this section we give the proof of Theorem 4.8. First of all we need the following Lemma, which holds since Assumption 4.7 is assumed to be true, and $\Omega \subset \mathbb{H}$ is of the form $\Omega = \omega \times (a, b)$ with a bounded domain $\omega \subset \mathbb{R}^2$ and $a < b$.

Lemma 4.15. *Let $\Omega := \omega \times (a,b) \subset \mathbb{H}$ and let c be given by (4.18). Then*

$$\int_a^b \int_{\omega^\beta} |u(x',x_3)|^2 \,\mathrm{d}x' \,\mathrm{d}x_3 \;\leq\; c^{2+\frac{2}{c}} \beta^{2+\frac{2}{c}} \left\|\mathrm{A}(\Omega)\, u\right\|_{L^2(\Omega)} \left\|\mathrm{A}(\Omega)^{1/c}\, u\right\|_{L^2(\Omega)} \tag{4.74}$$

holds for all $u \in \mathrm{Dom}(\mathrm{A}(\Omega))$ and any $\beta > 0$, where $\omega^\beta := \{x' \in \omega \mid \delta_e(x') < \beta\}$.

Proof. This inequality is true because of Theorem A.1, which can be found in the appendix. We check briefly the assumptions. For this purpose we note that the Eikonal equation holds true

$$|X_1 \delta_e(x')|^2 + |X_2 \delta_e(x')|^2 = |\partial_{x_1} \delta_e(x')|^2 + |\partial_{x_2} \delta_e(x')|^2 = 1$$

for almost every $x' \in \omega$. We can not prove that $\delta_e \in d[a]$, where $d[a]$ is the form domain of the closure of (4.2) but taking a closer look on the proof of Theorem A.1, we only have to use the following: if $u \in d[a]$, then $u\delta_e \in d[a]$ for all $u \in d[a]$. This holds since $\delta_e \in H_0^1(\omega)$.

Still we have to proof that (A.5) holds; in particular, we have to show that for all $u \in d[a]$ holds

$$\int_a^b \int_\omega \frac{|u(x',x_3)|^2}{\delta_e(x')^2} \,\mathrm{d}x' \,\mathrm{d}x_3 \leq c^2 \int_\Omega \left(|X_1\, u(x)|^2 + |X_2\, u(x)|^2\right) \,\mathrm{d}x. \tag{4.75}$$

Let u be in $C_0^\infty(\Omega)$. In addition let us denote by $\mathcal{F}_3$ the Fourier transform in x_3-direction, which is a unitary map in $L^2(\mathbb{R})$. Because Ω is a cylinder, the function $|\mathcal{F}_3 u(x',\xi_3)|$ lies in $H_0^1(\omega)$ for fixed $\xi_3 \in \mathbb{R}$. Therefore we can apply inequality (4.19) to get

$$\begin{aligned}\int_a^b \int_\omega \frac{|u(x',x_3)|^2}{\delta_e(x')^2} \,\mathrm{d}x' \,\mathrm{d}x_3 &= \int_{\mathbb{R}} \int_\omega \left(\frac{|\mathcal{F}_3 u(x',\xi_3)|}{\delta_e(x')}\right)^2 \,\mathrm{d}x' \,\mathrm{d}\xi_3 \\ &\leq c^2 \int_{\mathbb{R}} \int_\omega \left(\nabla_{x'} |\mathcal{F}_3 u(x',\xi_3)|\right)^2 \,\mathrm{d}x' \,\mathrm{d}\xi_3,\end{aligned}$$

where $\nabla_{x'} := (\partial_{x_1}, \partial_{x_2})$. Now we set

$$\mathbf{A}(x') := \frac{1}{2}(-x_2, x_1), \tag{4.76}$$

and apply the diamagnetic inequality which states that

$$|\nabla|\psi|| \;\leq\; |(\mathrm{i}\nabla + \mathbf{A})\psi| \qquad \text{a. e.} \tag{4.77}$$

holds for all $\psi \in H^1(\omega)$, see e.g. [LL01]. This gives

$$\int_{\mathbb{R}} \int_\omega \left(\nabla_{x'} |\mathcal{F}_3 u(x',\xi_3)|\right)^2 \,\mathrm{d}x' \,\mathrm{d}\xi_3 \leq \int_{\mathbb{R}} \int_\omega \left|(\mathrm{i}\nabla_{x'} + \xi_3 \mathbf{A}(x'))\, \mathcal{F}_3 u(x',\xi_3)\right|^2 \,\mathrm{d}x' \,\mathrm{d}\xi_3.$$

Integration by parts in the x_3-direction yields the inequality for $u \in C_0^\infty(\Omega)$. A density argument completes the proof of inequality (4.75) and yields the result of the Lemma. □

Proof of Theorem 4.8*:* Since Ω satisfies the assumptions of Theorem 4.1, we can follow the proof of the latter. From Subsection 4.4.1, in particular from equation (4.51), we infer that

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \;\leq\; \frac{|\Omega|}{96}\,\lambda^3 - \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+\, \mathrm{Q}(\lambda, k, \xi_3)\, \mathrm{d}\xi_3. \tag{4.78}$$

with $\mathrm{Q}(\lambda, k, \xi_3) =: Q_\lambda$ given by (4.52). We consider then for $\beta \in (0, \mathrm{R}_e(\Omega))$ the set

$$\Omega^\beta := \{x := (x', x_3) \in \Omega \,|\, \delta_e(x') < \beta\}. \tag{4.79}$$

Obviously it holds $\Omega^\beta \subseteq \Omega$. Hence equations (4.53) and (4.55) imply that

$$Q_\lambda \geq \frac{|\xi_3|\,|\Omega^\beta|}{8\pi^2} - \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\Omega^\beta} \Big| \sum_{j:\lambda_j(\Omega)<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x', \cdot)\, \mathrm{e}^{-\mathrm{i}\xi_3 \cdot},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} v_j(y) \Big|^2\, \mathrm{d}y\; \mathrm{d}x'.$$

Since $\Omega = \omega \times (a, b)$ we get further

$$\begin{aligned} Q_\lambda \;\geq\;& \frac{|\xi_3|}{8\pi^2}(b-a)\,|\omega^\beta| \\ &- \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_a^b \int_{\omega^\beta} \left| \sum_{j:\lambda_j(\Omega)<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x', \cdot) \mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} v_j(y', y_3) \right|^2\, \mathrm{d}y'\; \mathrm{d}y_3\; \mathrm{d}x'. \end{aligned}$$

Next we estimate the negative integral. Note that the linear combinations of v_j lies in $\mathrm{Dom}(\mathrm{A}(\Omega))$. Therefore we may apply Lemma 4.15 and obtain

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_a^b \int_{\omega^\beta} \left| \sum_{j:\lambda_j(\Omega)<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x', \cdot) \mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} v_j(y', y_3) \right|^2\, \mathrm{d}y'\; \mathrm{d}y_3 \right)\, \mathrm{d}x' \\ &\leq\; c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\sum_{j:\lambda_j(\Omega)<\lambda} \left| \big\langle P_{k,\xi_3}(x', \cdot) \mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} \right|^2 \right)\, \mathrm{d}x' \\ &\leq\; c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_\Omega |P_{k,\xi_3}(x', y')|^2\, \mathrm{d}y'\; \mathrm{d}x_3 \right)\, \mathrm{d}x' \\ &=\; c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \frac{|\Omega|}{4\pi^2} |\xi_3|, \end{aligned}$$

which yields the following lower bound on Q_λ:

$$\begin{aligned} Q_\lambda \;&\geq \frac{|\xi_3|}{8\pi^2}(b-a)\,\left|\omega^\beta\right| - c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \frac{|\Omega|}{4\pi^2} |\xi_3| \\ &\geq \frac{|\xi_3|}{8\pi^2} \beta \left(l(\omega) - 2c^{2+2/c} \beta^{1+2/c} \lambda^{1+1/c} |\Omega| \right). \end{aligned}$$

Now we set

$$\beta^{1+2/c} = \frac{l(\omega)}{c^{2+2/c}\lambda^{1+1/c}(4+4/c)|\Omega|},$$

see (4.20) for $l(\omega)$, which is possible for $\lambda \geq \lambda_1(\Omega)$ because of

$$\beta^{1+2/c} \leq \frac{1}{c^{2+2/c}\lambda_1(\Omega)^{1+1/c}(4+4/c)\mathrm{R}_e(\omega)} \leq \frac{\mathrm{R}_e(\omega)^{1+2/c}}{4}.$$

The last inequality was obtained by applying Lemma 4.15 to $u = v_1$ and $\beta = \mathrm{R}_e(\omega)$. Summing up we thus arrive at

$$Q_\lambda \geq \frac{|\xi_3|}{4\pi^2}\lambda^{-\frac{c+1}{c+2}} l(\omega)^{\frac{2c+2}{c+2}} |\Omega|^{-\frac{c}{c+2}}(1+2/c)(4c+4)^{-\frac{2c+2}{c+2}} = \lambda^{-\frac{c+1}{c+2}}\, G(\Omega)\, |\xi_3|,$$

where

$$G(\Omega) := \frac{l(\omega)^{\frac{2c+2}{c+2}}}{4\pi^2} |\Omega|^{-\frac{c}{c+2}}(1+2/c)(4c+4)^{-\frac{2c+2}{c+2}}.$$

This in combination with (4.78) gives

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \leq \frac{|\Omega|}{96}\lambda^3 - G(\Omega)\lambda^{-\frac{c+1}{c+2}} \int_{\mathbb{R}} \sum_{\mathrm{k}:\lambda>|\xi_3|(2k-1)} (\lambda - |\xi_3|(2\mathrm{k}-1))|\xi_3| \; \mathrm{d}\xi_3.$$

To finish the proof we calculate in the same way as in the beginning of the proof:

$$\sum_{\mathrm{k}=1}^{\infty} \int_0^\infty (\lambda - \xi_3(2\mathrm{k}-1))_+ \xi_3 \; \mathrm{d}\xi_3 = \sum_{\mathrm{k}=1}^{\infty} \frac{1}{(2\mathrm{k}-1)^2} \int_0^\infty s(\lambda - s)_+ \; \mathrm{d}s = \frac{\pi^2\,\lambda^3}{48}.$$

This gives the estimate stated in Theorem 4.8. □

4.6 Improved spectral estimates on general domains

We have seen in Theorem 4.8 that for domains of the form $\Omega = \omega \times (a,b)$ with convex cross-section ω the growth order of the negative lower order term in Theorem 4.1 can be further improved. Indeed, we can extend this to general bounded domains assuming a more restrictive condition; we have to assume the validity of a Hardy inequality with respect to the C-C metric:

Assumption 4.16. *Let $\Omega \subset \mathbb{H}$ be a bounded domain. We assume that there exists a constant $c \in [2,\infty)$ such that*

$$\int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2} \; \mathrm{d}x \leq c^2 \int_\Omega \left(|X_1\, u(x)|^2 + |X_2\, u(x)|^2\right) \; \mathrm{d}x \tag{4.80}$$

holds for all $u \in C_0^\infty(\Omega)$.

We will discuss this assumption in the next chapter in detail and state only the most important properties. If Ω has a $C^{1,1}$ regular boundary, then Assumption 4.16 holds true, even though an explicit constant is not known, see [DGP09, Thm.4.1 and p.120]. We will see later that if $c = 2$, then (4.80) is sharp. Now, we can state the improved version of Theorem 4.1 for general domains.

Theorem 4.17. *Let $\Omega \subset \mathbb{H}$ be a bounded domain and let*

$$\sigma(\Omega) := \inf_{0<\beta\leq \mathrm{R}(\Omega)} \frac{|\Omega^\beta|}{\beta}.$$

Under Assumption 4.16 *we have*

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;\leq\; \max\left\{0, \frac{|\Omega|}{96}\,\lambda^3 - \frac{1+2/c}{96}\sigma(\Omega)^{\frac{2c+2}{c+2}}(4c+4)^{\frac{-2c-2}{c+2}}|\Omega|^{\frac{-1}{1+2/c}}\lambda^{2+\frac{1}{c+2}}\right\}.$$

Note that the quantity $\sigma(\Omega)$ is strictly positive because of Theorem 4.12. The proof of Theorem 4.17 is verbatim the same as the proof of Theorem 4.8; we only have to replace δ_e by δ_C. For the sake of completeness we give the proof and therefore need the following Lemma:

Lemma 4.18. *Under Assumption* 4.16 *it holds*

$$\int_{\Omega^\beta} |u(x)|^2 \, \mathrm{d}x \leq c^{2+2/c}\beta^{2+2/c}\|\mathrm{A}(\Omega)u\|_{L^2(\Omega)}\left\|\mathrm{A}(\Omega)^{1/c}u\right\|_{L^2(\Omega)} \tag{4.81}$$

for all $u \in \mathrm{Dom}(A(\Omega))$, where $\Omega^\beta := \{x \in \Omega \,|\, \delta_C(x) < \beta\}$.

Proof. We check the assumptions of Theorem A.1, stated in the appendix, which implies then the result. In [MC01, Thm. 3.1] was shown that the Eikonal equation holds

$$|X_1\,\delta_C(x)|^2 + |X_2\,\delta_C(x)|^2 = 1 \qquad \text{a.e.} \quad x \in \Omega. \tag{4.82}$$

In the proof of Theorem 5.20 we show that $\delta_C \in d[a]$. Assumption 4.16 yields then the validity of (A.5), yielding the result. □

Proof of Theorem 4.17. Since Ω satisfies the assumptions of Theorem 4.1, we can follow the proof of the latter. From Subsection 4.4.1, in particular from equation (4.51), we infer that

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;\leq\; \frac{|\Omega|}{96}\,\lambda^3 - \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+\, \mathrm{Q}(\lambda,k,\xi_3)\, \mathrm{d}\xi_3. \tag{4.83}$$

with $\mathrm{Q}(\lambda,k,\xi_3)$ given by (4.52). For $\beta \in (0, \mathrm{R}(\Omega))$ we consider the set

$$\Omega^\beta := \{x \in \Omega \,|\, \delta_C(x) < \beta\}. \tag{4.84}$$

Obviously it holds $\Omega^\beta \subseteq \Omega$. Hence equations (4.53) and (4.55) imply that

$$\begin{aligned} \mathrm{Q}(\lambda, k, \xi_3) \geq{}& \frac{|\xi_3|\, |\Omega^\beta|}{8\pi^2} \\ & - \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\Omega^\beta} \Big| \sum_{j:\lambda_j(\Omega)<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\, \mathrm{e}^{-\mathrm{i}\xi_3 \cdot},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} v_j(y) \Big|^2 \,\mathrm{d}y\, \mathrm{d}x'. \end{aligned} \tag{4.85}$$

An application of Lemma 4.18 yields

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\Omega^\beta} \Big| \sum_{j:\lambda_j(\Omega)<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\, \mathrm{e}^{-\mathrm{i}\xi_3 \cdot},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} v_j(y) \Big|^2 \mathrm{d}y\, \mathrm{d}x' \\ & \qquad \leq c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \int_{\mathbb{R}^2} \sum_{j:\lambda_j(\Omega)<\lambda} \Big| \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\, \mathrm{e}^{-\mathrm{i}\xi_3 \cdot},\, v_j(\cdot) \big\rangle_{L^2(\Omega)} \Big|^2 \,\mathrm{d}x' \\ & \qquad \leq c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \frac{|\xi_3|}{2\pi} |\Omega|. \end{aligned} \tag{4.86}$$

For the last inequality we used (4.49). Inserting this result into (4.85) and using the definition of $|\Omega^\beta|$, we find that

$$\begin{aligned} \mathrm{Q}(\lambda, k, \xi_3) &\geq \frac{|\xi_3|\, \sigma(\Omega)}{8\pi^2} \beta - c^{2+2/c} \beta^{2+2/c} \lambda^{1+1/c} \frac{|\xi_3|}{4\pi^2} |\Omega| \\ &= \frac{|\xi_3|}{8\pi^2} \beta \left(\sigma(\Omega) - 2c^{2+2/c} \beta^{1+2/c} \lambda^{1+1/c} |\Omega| \right) \end{aligned}$$

holds true uniformly in k, where

$$\sigma(\Omega) = \inf_{0<\beta\leq \mathrm{R}(\Omega)} \frac{|\Omega^\beta|}{\beta}.$$

From Theorem 4.12 we know that $\sigma(\Omega) > 0$. Hence upon setting

$$\beta := \sigma(\Omega)^{\frac{1}{1+2/c}} (4+4/c)^{\frac{-1}{1+2/c}} c^{-1-\frac{1}{1+2/c}} |\Omega|^{\frac{-1}{1+2/c}} \lambda^{\frac{-1-1/c}{1+2/c}}, \tag{4.87}$$

we obtain

$$\mathrm{Q}(\lambda, k, \xi_3) \;\geq\; \frac{|\xi_3|}{4\pi^2} (1+2/c) \sigma(\Omega)^{\frac{2c+2}{c+2}} (4c+4)^{\frac{-2c-2}{c+2}} |\Omega|^{\frac{-1}{1+2/c}} \lambda^{-1+\frac{1}{c+2}} \qquad \forall\, k \in \mathbb{N}. \tag{4.88}$$

However, as in the proof of Theorem 4.1 we have to verify that β given by (4.87) with $\lambda \geq \lambda_1(\Omega)$ satisfies $\beta \leq \mathcal{R}(\omega)$. An application of Lemma 4.18 for $u = v_1$ and $\beta = \mathcal{R}(\Omega)$ yields $1 \leq \mathcal{R}(\Omega)^{2+2/c} c^{2+2/c} \lambda_1(\Omega)^{1+1/c}$. The latter inequality shows that

$$\begin{aligned} \beta^{1+2/c} \;&\leq\; \sigma(\Omega)(4+4/c)^{-1} c^{-2-2/c} |\Omega|^{-1} \lambda_1(\Omega)^{-1-1/c} \\ &\leq\; \sigma(\Omega)(4+4/c)^{-1} \mathcal{R}(\Omega)^{2+2/c} |\Omega|^{-1} \\ &\leq\; (4+4/c)^{-1} \mathcal{R}(\Omega)^{1+2/c} \;\leq\; \mathcal{R}(\Omega)^{1+2/c} \end{aligned}$$

holds for all $\lambda \geq \lambda_1(\Omega)^{-1}$ and therefore justifies the choice of β in (4.87). The result in Theorem 4.17 now follows by inserting the lower bound (4.88) in (4.83), evaluating the integral in ξ_3 and then the series in k as in the proof of Theorem 4.1. $\square$

Example 4.19. Let us consider $\Omega = C_h(0) := \{x \in \mathbb{H} \,|\, d_C(x,0) < h\}$ for $h > 0$. In [Yan13] it was shown that Assumption 4.16 holds true for $C_h(0)$ with the constant $c = 2$. Theorem 4.17 and the lower estimate for $\sigma(\Omega)$ in Theorem 4.12 yield

$$\operatorname{Tr}(\mathrm{A}(\Omega) - \lambda)_- \;\leq\; \max\left\{0, \frac{|\Omega|}{96}\,\lambda^3 - \frac{1}{48\pi^3}\left(\frac{2}{15}\right)^{\frac{3}{2}} h^{9/2}|\Omega|^{-\frac{1}{2}}\lambda^{2+\frac{1}{4}}\right\}.$$

Chapter 5

Hardy inequalities for the Heisenberg Laplacian on convex bounded polytopes

In this chapter we prove a Hardy-type inequality for the gradient of the Heisenberg Laplacian on open bounded convex polytopes on the first Heisenberg Group. The integral weight of the Hardy inequality is given by the distance function to the boundary measured with respect to the Carnot-Carathéodory metric. The constant depends on the number of hyperplanes given by the boundary of the convex polytope which are not orthogonal to the hyperplane $x_3 = 0$.

5.1 Introduction

We recall the first Heisenberg group, which is given by $\mathbb{R}^3$ equipped with the group law

$$(x_1, x_2, x_3) \boxplus (y_1, y_2, y_3) \;:=\; \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\Big), \tag{5.1}$$

and the sub-gradient $\nabla_{\mathbb{H}} := (X_1, X_2)$ given by

$$X_1 \;:=\; \partial_{x_1} + \frac{x_2}{2}\partial_{x_3}, \quad X_2 \;:=\; \partial_{x_2} - \frac{x_1}{2}\partial_{x_3},$$

for $x := (x_1, x_2, x_3) \in \mathbb{R}^3$. In addition, we mention that the left-invariant vector fields $X_1, X_2, X_3 := [X_2, X_1] = \partial_{x_3}$ form a basis of the Heisenberg algebra on $\mathbb{H}$ and that the Heisenberg Laplacian is given by

$$\Delta_{\mathbb{H}} := -X_1^2 - X_2^2,$$

also called Kohn Laplacian. There is a considerable amount of literature concerning the Hardy-type inequality

$$\int_{\mathbb{H}} \frac{|u(x)|^2}{\|x\|_{\mathbb{H}}^4}\,(x_1^2+x_2^2)\ \mathrm{d}x \ \leq\ \int_{\mathbb{H}} |\nabla_{\mathbb{H}}\, u(x)|^2\ \mathrm{d}x \qquad u\in C_0^\infty(\mathbb{H}\setminus\{0\}), \tag{5.2}$$

where

$$\|x\|_{\mathbb{H}}^4 := (x_1^2+x_2^2)^2 + 16x_3^2\,.$$

For the proof of (5.2) we refer to [GL90, AL11, NZW01], see also various improvements obtained in [D'A04, Xia11]. The anisotropic norm $\|x\|_{\mathbb{H}}$, which appears in (5.2), is referred to in the literature as Korányi-Folland gauge or Kaplan gauge, see also Section 2.4. For the sake of brevity we will use the latter notation and call it Kaplan gauge.

In this chapter we deal with Hardy inequalities for the Heisenberg Laplacian on bounded domains. In particular we consider the following problem; given a bounded domain $\Omega\subset\mathbb{R}^3$, we would like to find a constant $c>0$ for which the inequality

$$\int_{\Omega} \frac{|u(x)|^2}{\delta_C(x)^2}\ \mathrm{d}x \ \leq\ c^2 \int_{\Omega} |\nabla_{\mathbb{H}}\, u(x)|^2\ \mathrm{d}x, \tag{5.3}$$

holds for all $u\in C_0^\infty(\Omega)$, where $\delta_C(x)$ is the Carnot-Carathéodory distance (C-C distance in the sequel) between x and the boundary of Ω, see Subsection 5.2.1 and Section 2.3 for its definition. With respect to the well-studied inequality (5.2), it is less known about the validity of (5.3) especially if one is interested in explicit constants. In [DGP09] D. Danielli, N. Garofalo and N. C. Phuc proved that for every Ω with a $C^{1,1}$ regular boundary there exists $c>0$ such that (5.3) holds true. Later it was shown by Q.-H. Yang [Yan13] that if Ω is a ball with respect to the C-C distance, then (5.3) holds with $c=2$.

The fundamental problem of deriving inequalities of the form (5.3) lies in the fact that we a priori don't know much about domains which are the most natural ones for a Hardy inequality on $\mathbb{H}$. In comparison to the Euclidean setting it is well-known that if Ω is *convex* then

$$\int_{\Omega} \frac{|u(x)|^2}{\operatorname{dist}(x,\partial\Omega)^2}\ \mathrm{d}x \ \leq\ 4\int_{\Omega} |\nabla u(x)|^2\ \mathrm{d}x \tag{5.4}$$

holds for all $u\in C_0^\infty(\Omega)$, and the constant 4 is sharp independently of Ω, see e.g. [Anc86, OK90, Dav99, MMP98, AW07, Avk10, HOHOL02] and Section 1.3.

In this chapter we prove that for open bounded convex polytopes (5.3) holds true, and we obtain a constant depending only on the number of hyperplanes of $\partial\Omega$ which are not orthogonal to the hyperplane $x_3=0$. Under an additional geometrical assumption, the constant in (5.3) for convex polytopes can be further improved, see Theorem 5.15. It is even possible to show that for any $c>2$ there exists a bounded convex domain such that (5.3) is fulfilled, which is an almost sharp result since we prove that for any bounded domain Ω holds

$$\inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2 \,\mathrm{d}x}{\int_\Omega |u(x)|^2\delta_C(x)^{-2} \,\mathrm{d}x} \leq \frac{1}{4}. \tag{5.5}$$

This shows that at least some convex domains are more compatible with the Heisenberg group structure than we expect them to be.

In [LY08] J.-W. Luan and Q.-H. Yang proved on the half-space $\Omega := \{x \in \mathbb{H} | x_3 > 0\}$ that for any $u \in C_0^\infty(\Omega)$ holds

$$\int_\Omega \frac{x_1^2 + x_2^2}{4x_3^2}|u(x)|^2 \,\mathrm{d}x \leq 4\int_\Omega |\nabla_{\mathbb{H}} u(x)|^2 \,\mathrm{d}x. \tag{5.6}$$

This result was recently generalized by Larson [Lar16] to any bounded convex domain. Under an additional convexity condition, where $H(x)$ denotes the horizontal plane to x, we can replace the weight on the left-hand side by

$$\omega(x) := \inf_{y\in\partial\Omega\cap H(x)} d_C(x, y), \tag{5.7}$$

see Theorem 5.1. This result turns out to be (5.6) for the case of the half-space.

This chapter is organized as follows. In the next section we introduce necessary notations. Main results are formulated in Section 5.2 and the proof of each Theorem is done in a separate section.

5.2 Main results

5.2.1 Preliminaries and notation

The *horizontal plane* to $x := (x_1, x_2, x_3) \in \mathbb{H}$ is given by

$$\begin{aligned} H(x) :=& \left\{ y \in \mathbb{H} \;\middle|\; \left\langle \left(-\tfrac{x_2}{2}, \tfrac{x_1}{2}, 1\right),\, y - x\right\rangle = 0 \right\}, \\ =& \left\{ y \in \mathbb{H} \;\middle|\; x_1y_2 - x_2y_1 = 2(x_3 - y_3) \right\}, \end{aligned} \tag{5.8}$$

where $\langle\cdot,\,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$.

Let us briefly recall the definition of the C-C distance $d_C(x, y)$. We call a Lipschitz curve $\gamma : [a, b] \to \mathbb{H}$ parametrized by $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ *horizontal* if

$$\gamma'(t) \in \operatorname{span}\left\{ \left(1, 0, \frac{\gamma_2(t)}{2}\right),\, \left(0, 1, -\frac{\gamma_1(t)}{2}\right) \right\}, \quad \text{for all } t \in (a, b).$$

The C-C distance between x and y is then defined as

$$d_C(x, y) := \inf_\gamma \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} \,\mathrm{d}t, \tag{5.9}$$

where the infimum is taken over all horizontal curves γ connecting x and y, see also Section 2.3.

We define the C-C and Kaplan distance functions for a bounded domain Ω by

$$\delta_C(x) := \inf_{y\in\partial\Omega} d_C(x,y), \qquad \delta_K(x) := \inf_{y\in\partial\Omega} \|(-y)\boxplus x\|_{\mathbb{H}}. \tag{5.10}$$

If $x \in \Omega^c$, we set $\delta_K(x) := 0$ and $\delta_C(x) := 0$. With these prerequisites we can state the main results.

5.2.2 Main results

Theorem 5.1. *Let $\Omega \subset \mathbb{H}$ be a bounded domain, and let the connected components of $H(x)\cap\Omega$ be convex for all $x \in \Omega$. Then*

$$\int_\Omega \frac{|u(x)|^2}{\omega(x)^2}\,\mathrm{d}x \;\le 4\int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\,\mathrm{d}x \tag{5.11}$$

holds true for all $u \in C_0^\infty(\Omega)$, where $\omega(\cdot)$ is defined in (5.7) and it holds

$$\omega(x) = \inf_{y\in\partial\Omega\cap H(x)} \|(-y)\boxplus x\|_{\mathbb{H}} = \inf_{y\in\partial\Omega\cap H(x)} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}. \tag{5.12}$$

We call the weight $\omega(\cdot)$ the reduced C-C distance. The proof of (5.11) is done in the following way. We proof the Hardy inequality for each separate X_j, where the distance function is given by the C-C metric generated by X_j for $j \in \{1,2\}$. Then we apply the hyperplane separation theorem in the same way as E. B. Davies did for the proof of (5.4) for convex domains, see [Dav99].

Theorem 5.2. *Let $\Omega \subset \mathbb{H}$ be an open bounded convex polytope, and let $m \in \mathbb{N}$ be the number of hyperplanes of $\partial\Omega$ which are not orthogonal to the hyperplane $x_3 = 0$. Then holds*

$$\frac{1}{5}\left(\frac{3^{3/2}\sqrt{2}}{c_m}+1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \;\le \int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\,\mathrm{d}x, \tag{5.13}$$

for all $u \in C_0^\infty(\Omega)$, where c_m is the unique positive number satisfying the following identity

$$\sqrt{c_m^2+16}\left(1+\frac{c_m}{3^{3/2}\sqrt{2}}\right)^{2/3} {c_m}^{4/3} = \frac{1}{2^{7/3}3\pi m}.$$

In addition, we proof that for c_m holds the following

$$\frac{1}{c_m} \leq m^{8/9}\pi^{8/9}3 \cdot 2^{19/9}\sqrt{2^{-4/3}\pi^{-2/3}+16}\left(1+\frac{1}{3^{3/2}2^{7/6}\pi^{1/3}}\right)^{2/3}, \tag{5.14}$$

which yields a result with an explicit constant in (5.13).

The strategy of the proof of Theorem 5.2 consists of two steps. We use the same idea as in the proof of Theorem 5.1 and proof a Hardy inequality on a bounded convex polytope for each separate X_j, where the distance function is given by the C-C metric generated by X_j for $j \in \{1,2\}$. Then we take into account the following Hardy inequality

$$\int_\Omega \frac{|u(x)|^2}{d_C(x,0)^2}\,\mathrm{d}x \;\leq\; \int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\,\mathrm{d}x, \tag{5.15}$$

for all $u \in C_0^\infty(\Omega)$, which was proved in [GK08, RS15, Yan13]. The sum of the weight functions is then comparable to the distance function to the hyperplanes of the given polytope with respect to the Kaplan gauge, which is equivalent to the distance function with respect to the C-C metric.

We can improve the constant in Theorem 5.2 under an additional geometrical assumption, which is discussed in Section 5.5. The main consequence of that result is the following;

Theorem 5.3. *For any $\varepsilon > 0$ there exists a bounded convex domain Ω such that*

$$\int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \;\leq (2+\varepsilon)^2\int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\,\mathrm{d}x$$

for all $u \in C_0^\infty(\Omega)$.

The last result has an almost optimal constant since we prove that for any bounded domain Ω holds

$$\inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\,\mathrm{d}x}{\int_\Omega |u(x)|^2\delta_C(x)^{-2}\,\mathrm{d}x} \leq \frac{1}{4},$$

see Theorem 5.20.

5.3 Restricted C-C distance and its connection to the Euclidean distance

5.3.1 The natural restriction of $\partial\Omega$

In this section we show that the reduced distance $\omega(\cdot)$, defined by (5.7), can be expressed in terms of a simple explicit formula. In particular, we proof the following formulae:

Theorem 5.4. *Let $\Omega \subset \mathbb{H}$ be open bounded, then holds*

$$\omega(x) \;=\; \inf_{y\in\partial\Omega\cap H(x)} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \;=\; \inf_{y\in\partial\Omega\cap H(x)} \|(-y)\boxplus x\|_{\mathbb{H}},$$

for all $x \in \Omega$.

Proof. Let $x \in \Omega$ and let $y \in \partial\Omega \cap H(x)$. Consider the curve $\gamma : [0,1] \to \mathbb{H}$ given by the parametrization $\gamma(t) = (1-t)x + ty,\ t \in [0,1]$. Obviously γ connects x and y. Moreover, since $y \in H(x)$ we obtain that γ is horizontal. Indeed, we verify that the following identity holds

$$\gamma'(t) = (y_1 - x_1)\left(1, 0, \frac{\gamma_2(t)}{2}\right) + (y_2 - x_2)\left(0, 1, -\frac{\gamma_1(t)}{2}\right)$$

for all $t \in (0,1)$. The first two rows are easily verified. For the third one we start with the right-hand side and get for $x := (x_1, x_2, x_3)$ and $y := (y_1, y_2, y_3)$ the following

$$\begin{aligned}
&(y_1-x_1)\frac{(1-t)x_2+ty_2}{2} - (y_2-x_2)\frac{(1-t)x_1+ty_1}{2}\\
=&\frac{1-t}{2}\left((y_1-x_1)x_2-(y_2-x_2)x_1\right)+\frac{t}{2}\left((y_1-x_1)y_2-(y_2-x_2)y_1\right)\\
=&\frac{1-t}{2}\left(y_1x_2-y_2x_1\right)+\frac{t}{2}\left(-x_1y_2+x_2y_1\right) = \frac{y_1x_2-y_2x_1}{2} = y_3-x_3.
\end{aligned}$$

The last equality holds because $y \in H(x)$, see (5.8). By definition of the C-C distance, see equation (5.9), it thus follows that

$$d_C(x,y) \;\leq\; \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}. \tag{5.16}$$

Using $y \in \partial\Omega \cap H(x)$ we see that

$$\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \;=\; \|(-y)\boxplus x\|_{\mathbb{H}}. \tag{5.17}$$

Then we apply Lemma 2.2 to obtain the following chain of inequalities

$$d_C(x,y) \;\leq\; \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \;=\; \|(-y)\boxplus x\|_{\mathbb{H}} \leq\; d_C(x,y). \tag{5.18}$$

Taking the infimum over $y \in \partial\Omega \cap H(x)$ yields the result. □

5.3.2 The Hardy inequality involving ω

We need the following auxiliary result.

Lemma 5.5. *Let Ω be a bounded domain in $\mathbb{H}$. Then holds*

$$\int_\Omega \left(\frac{|u(x)|^2}{d_1(x)^2} + \frac{|u(x)|^2}{d_2(x)^2} \right) \, \mathrm{d}x \ \leq 4 \int_\Omega |\nabla_{\mathbb{H}} u(x)|^2 \, \mathrm{d}x \tag{5.19}$$

for all $u \in C_0^\infty(\Omega)$, where the distances $d_1(x)$ and $d_2(x)$ are given by

$$d_1(x) \ := \ \inf_{s\in\mathbb{R}} \{|s| > 0 \,|\, x + s(1, 0, x_2/2) \notin \Omega\}, \tag{5.20}$$

$$d_2(x) \ := \ \inf_{s\in\mathbb{R}} \{|s| > 0 \,|\, x + s(0, 1, -x_1/2) \notin \Omega\}. \tag{5.21}$$

Proof. Let $u \in C_0^\infty(\Omega)$. First we show that

$$\int_\Omega \frac{|u(x)|^2}{d_1(x)^2} \, \mathrm{d}x \ \leq 4 \int_\Omega |X_1 u(x)|^2 \, \mathrm{d}x. \tag{5.22}$$

To this end we define the following coordinate transformation

$$F(t, \varphi, \theta) \ := \ \begin{cases} x_1(t, \varphi, \theta) = t + \varphi, \\ x_2(t, \varphi, \theta) = \theta, \\ x_3(t, \varphi, \theta) = t\theta/2, \end{cases} \tag{5.23}$$

where $(t, \varphi, \theta) \in A := \{(t, \varphi, \theta) \in \mathbb{R}^3 \mid \theta \neq 0\}$. It can be easily checked that $F : A \mapsto \mathrm{Ran}(A)$ is a diffeomorphism and that the determinant of F is equal to $\theta/2$. For a given $x \in \Omega^c$ we set $u(x) = 0$. If $x = F(t, \varphi, \theta)$ for fixed $\theta \in \mathbb{R} \setminus \{0\}$ and $\varphi \in \mathbb{R}$, we see that there exists a constant $c \in \mathbb{R}$ such that $F(c, \varphi, \theta) = \hat{x} \in \partial\Omega$ and such the following identity is satisfied $d_1(x) = d_C(x, \hat{x})$. By $\{a_j\}_{j\in\mathbb{N}}$ we denote the increasing sequence such that $F(a_j, \varphi, \theta) \in \partial\Omega$. Such a sequence exists since Ω is bounded and open. Thus for a fixed $x = F(t, \varphi, \theta) \in \Omega$ we immediately see that there exists a $k \in \mathbb{N}$ such that

$$\begin{aligned} d_1(F(t, \varphi, \theta)) &= d_C(F(t, \varphi, \theta), F(a_k, \varphi, \theta)) \\ &= d_C(F(t, \varphi, \theta), F(t, \varphi, \theta) + (a_k - t)(1, 0, \theta/2)). \end{aligned}$$

We use Proposition 2.1 b) to arrive at

$$d_1(F(t, \varphi, \theta)) = d_C((0, 0, 0), (-t + a_k, 0, 0)) = |a_k - t|.$$

The last equality holds because of the formula for the geodesics in (2.10). By the last observation we apply then the transformation F to find out that to prove (5.22) it suffices to show that

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{j=1}^\infty \int_{a_j}^{a_{j+1}} \frac{|u(t, \varphi, \theta)|^2}{\delta_j(t)^2} \, \mathrm{d}t \frac{|\theta|}{2} \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &\quad \leq \ 4 \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{j=1}^\infty \int_{a_j}^{a_{j+1}} |\partial_t u(t, \varphi, \theta)|^2 \, \mathrm{d}t \frac{|\theta|}{2} \, \mathrm{d}\theta \, \mathrm{d}\varphi, \end{aligned} \tag{5.24}$$

where $\delta_j(t) := \inf(a_{j+1} - t, t - a_j)$. Hence the one-dimensional Hardy inequality in the t-direction then implies that (5.24) holds true, which in turn yields (5.22). It remains to prove

$$\int_\Omega \frac{|u(x)|^2}{d_2(x)^2}\, \mathrm{d}x \ \leq 4 \int_\Omega |X_2 u(x)|^2\, \mathrm{d}x. \tag{5.25}$$

This is done in the same way as (5.22) replacing the transformation of (5.23) by

$$\tilde{F}(t,\varphi,\theta) \ := \ \begin{cases} x_1(t,\varphi,\theta) = \theta, \\ x_2(t,\varphi,\theta) = t + \varphi, \\ x_3(t,\varphi,\theta) = -t\theta/2, \end{cases} \tag{5.26}$$

for $(t,\varphi,\theta) \in A$. Summing up (5.22) and (5.25) then completes the proof. $\square$

Proof of Theorem 5.1**.** Let us consider the result of Theorem 5.4 and choose $a := (a_1, a_2, a_3) \in \partial\Omega \cap H(x)$ such that

$$\omega(x) = \inf_{y \in \partial\Omega \cap H(x)} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \ = \ \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}. \tag{5.27}$$

The existence of such a is guaranteed by the compactness of $\overline{\Omega}$ and the continuity of the distance. We know that all connected components of $H(x) \cap \Omega$ are convex. For fixed $x \in \Omega$ lying in a given convex component, the corresponding $a \in \partial\Omega \cap H(x)$ must lie in the same convex component as x. Otherwise the identity in (5.27) could not hold. Therefore we assume without loss of generality that $H(x) \cap \Omega$ consists of a single connected convex component. Next we apply the hyperplane separation theorem, which implies that the hyperplane

$$T := \left\{ y \in \mathbb{H} \;\middle|\; \left\langle \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 - a_1 \\ y_2 - a_2 \\ y_3 \end{pmatrix} \right\rangle = 0 \right\} \tag{5.28}$$

separates $H(x) \cap \Omega$ from the point $a \in \partial\Omega \cap H(x)$. We consider Lemma 5.5 and compute the intersection point of the line $c(s) = x + s(1, 0, 1/2x_2)^t$ for $s \in \mathbb{R}$ with the hyperplane (5.28). This yields

$$s = -\frac{(x_1 - a_1)^2 + (x_2 - a_2)^2}{x_1 - a_1}. \tag{5.29}$$

With respect to the distance $d_1(x)$ in Lemma 5.5, we get with the hyperplane separation theorem

$$d_1(x) \leq \frac{(x_1 - a_1)^2 + (x_2 - a_2)^2}{|x_1 - a_1|}.$$

Now we do the same computation for $d_2(x)$ and obtain

$$d_2(x) \leq \frac{(x_1 - a_1)^2 + (x_2 - a_2)^2}{|x_2 - a_2|}. \tag{5.30}$$

Altogether we get

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \geq \frac{(x_2 - a_2)^2}{((x_1 - a_1)^2 + (x_2 - a_2)^2)^2} + \frac{(x_1 - a_1)^2}{((x_1 - a_1)^2 + (x_2 - a_2)^2)^2} = \frac{1}{\omega(x)^2}.$$

We recall that the point $a \in \partial\Omega \cap H(x)$ was chosen such that the following holds true $\omega(x) = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$, which proves inequality (5.11). □

Remark 5.6. For $p \geq 2$ it is possible to get an L^p version of Theorem 5.1 as well. In Lemma 5.5 we use the L^p version of the one-dimensional Hardy inequality, which holds for $p > 1$. Then we mimic the last proof and apply for $p \geq 2$ Jensen's inequality

$$(a^2 + b^2)^{p/2} = 2^{p/2}(a^2/2 + b^2/2)^{p/2} \leq 2^{p/2-1}(a^p + b^p),$$

for $a, b > 0$.

5.3.3 Some properties of ω

Since it seems that $\omega(\cdot)$ is rather natural on $\mathbb{H}$, we check that the assumptions of Theorem 5.1 are preserved by the Heisenberg group structure. Therfore we define

$$a \boxplus \Omega := \{y \in \mathbb{H} \,|\, \exists z \in \Omega \text{ such that } y = a \boxplus z\}.$$

Proposition 5.7. *Let $a \in \mathbb{H}$. Let $\Omega \subset \mathbb{H}$ be such that every connected component of $H(x) \cap \Omega$ is convex for all $x \in \Omega$, then every connected component of $(a \boxplus \Omega) \cap H(y)$ is convex for all $y \in a \boxplus \Omega$.*

Proof. Since hyperplanes are convex we only have to show that $a \boxplus \Omega$ is convex if Ω is convex. This is a simple proof since

$$(1 - \lambda)(a \boxplus z_1) + \lambda(a \boxplus z_2) \;=\; a \boxplus ((1 - \lambda)z_1 + \lambda z_2) \tag{5.31}$$

holds for $z_1, z_2, a \in \mathbb{H}$. □

The distance $\omega(\cdot)$ is also compatible with the natural dilation on the Heisenberg group. We recall that

$$\Phi_h(x) := hx \;=\; (hx_1, hx_2, h^2 x_3) \tag{5.32}$$

for $x := (x_1, x_2, x_3) \in \mathbb{H}$ and $h > 0$. Thus we define

$$h(\Omega) \;:=\; \left\{ x \in \mathbb{H} \,\middle|\, \exists y \in \Omega \text{ such that } x = hy \right\}. \tag{5.33}$$

Proposition 5.8. *Let $\Omega \subset \mathbb{H}$ be open bounded and let the connected components of $H(y) \cap \Omega$ be convex for all $y \in \Omega$. For any $h > 0$ holds then that the connected components of $H(x) \cap h(\Omega)$ are convex for all $x \in h(\Omega)$.*

Proof. Let us assume that there exist x_1 and $x_2 \in h(\Omega) \cap H(x)$ for fixed $x \in h(\Omega)$. Because $H(x)$ is a hyperplane we know that the convex combination of x_1 and x_2 lies in $H(x)$. Thus we only have to show that this convex combination lies in $h(\Omega)$. By definition there exist $y_1, y_2, y \in h(\Omega)$ such that $x_1 = hy_1$, $x_2 = hy_2$ and $x = hy$. We know that $H(x) \cap \Omega$ is convex, thus holds

$$\Phi_{h^{-1}}\left((1-\lambda)x_1 + \lambda x_2\right) \;=\; (1-\lambda)y_1 + \lambda y_2 \in H(y) \cap \Omega$$

for all $\lambda \in [0,1]$. We see that it is sufficient to prove $\Phi_h(H(y) \cap \Omega) \subseteq H(x) \cap h(\Omega)$. Let $z \in \Phi_h(H(y) \cap \Omega)$, thus there exists $q \in H(y) \cap \Omega$ such that $z = hq$. Obviously is then $z \in h(\Omega)$. We use that $h^{-1}z \in H(y)$, which is equivalent to $z \in H(hy)$. Since $hy = x$, we obtain the result. $\square$

Remark 5.9. In Lemma 4.15 we proved for a bounded convex domain of the form $\Omega := D \times (a,b) \subset \mathbb{H}$ that following inequality holds

$$\int_a^b \int_D \frac{|u(x',x_3)|^2}{\delta_e(x')^2}\,\mathrm{d}x'\,\mathrm{d}x_3 \leq 4 \int_\Omega \left(|X_1\,u(x)|^2 + |X_2\,u(x)|^2\right)\,\mathrm{d}x \quad u \in C_0^\infty(\Omega),$$

where $x := (x', x_3) \in D \times (a,b)$ for fixed $a < b$. Although the Hardy weight measures the two-dimensional Euclidean distance to ∂D, it coincides with the the C-C distance to ∂D, which is shown in the next result.

Proposition 5.10. *Let us consider the set $\Omega := D \times \mathbb{R} \subseteq \mathbb{H}$, where $\emptyset \neq D \subseteq \mathbb{R}^2$, then holds*

$$\inf_{y\in\Omega} d_C(x,y) = \inf_{y\in\Omega\cap H(x)} d_C(x,y) = \inf_{(y_1,y_2)\in D} \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

for all $x := (x_1, x_2, x_3) \in \Omega$.

Proof. First of all we note that $\Omega \cap H(x) \neq \emptyset$ for all $x \in \mathbb{H}$, which is easy to check by using the definition of $H(x)$, see (5.8). The proposition obviously holds true for $x \in \Omega$. Thus let us assume $x \notin \Omega$. We know that there exists a sequence $y_n := (\alpha_n, \beta_n, \gamma_n) \in \Omega$ such that

$$\begin{aligned} &\inf_{y\in\Omega} \|(-y) \boxplus x\|_{\mathbb{H}} \;=\; \lim_{n\to\infty} \|(-y_n) \boxplus x\|_{\mathbb{H}} \\ &= \lim_{n\to\infty} \sqrt[4]{((x_1-\alpha_n)^2 + (x_2-\beta_n)^2)^2 + 16(x_3 - \gamma_n + \frac{1}{2}(\alpha_n x_2 - \beta_n x_1))^2} \end{aligned}$$

Let us choose $\tilde{\gamma}_n \in \mathbb{R}$ such that

$$x_3 - \tilde{\gamma}_n + \frac{1}{2}(\alpha_n x_2 - \beta_n x_1) = 0 \tag{5.34}$$

is fulfilled. We define $\tilde{y}_n := (\alpha_n, \beta_n, \tilde{\gamma}_n)$ and see that

$$\inf_{y\in\Omega} \|(-y) \boxplus x\|_{\mathbb{H}} = \lim_{n\to\infty} \|(-\tilde{y}_n) \boxplus x\|_{\mathbb{H}} \tag{5.35}$$

holds since obviously $\tilde{y}_n$ lies in Ω. Because (5.34) is fulfilled, we see that $\tilde{y}_n \in H(x)$, see (5.8). Therefore we obtain

$$\inf_{y\in\Omega\cap H(x)} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \inf_{y\in\Omega\cap H(x)} \|(-y) \boxplus x\|_{\mathbb{H}} \leq \inf_{y\in\Omega} \|(-y) \boxplus x\|_{\mathbb{H}}.$$

Then we obtain with Theorem 5.4 the following chain of inequalities

$$\inf_{y\in\Omega\cap H(x)} d_C(x,y) \;=\; \inf_{y\in\Omega\cap H(x)} \|(-y) \boxplus x\|_{\mathbb{H}} \leq \inf_{y\in\Omega} \|(-y) \boxplus x\|_{\mathbb{H}} \leq \inf_{y\in\Omega} d_C(x,y)\ .$$

The last inequality was an application of Lemma 2.2. Obviously holds

$$\inf_{y\in\Omega} d_C(x,y) \leq \inf_{y\in\Omega\cap H(x)} d_C(x,y),$$

which immediately yields

$$\inf_{y\in\Omega\cap H(x)} d_C(x,y) \;=\; \inf_{y\in\Omega} d_C(x,y) \;=\; \inf_{y\in\Omega\cap H(x)} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$

Hence we only have to show that

$$\inf_{y\in\Omega\cap H(x)} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \inf_{(y_1,y_2)\in D} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2},$$

which can be shown in the same way as the proof of the previous identity; the only thing we have to take into account is that Ω is of the form $\Omega = D \times \mathbb{R}$. $\square$

5.4 Proof of the Hardy inequalities for open bounded convex polytopes

In this section we give the proof of Theorem 5.2. First we have to give some lower estimates for the Kaplan distance function to hyperplanes. Therefore we need the following:

Lemma 5.11. *Let $p > 0$ and $q \in \mathbb{R} \setminus \{0\}$. Consider*

$$z^3 + pz = q,$$

for $z \in \mathbb{R}$. Then there exists a unique real solution for which holds

$$|z| \geq \frac{|q^{1/3}|}{3} \left(1 + \frac{p\sqrt{p}}{|q|3\sqrt{3}}\right)^{-2/3}.$$

Proof. First we consider the case $q > 0$. Then Cardano's formula gives the unique real solution

$$\begin{aligned} z &= \left(q/2 + \sqrt{q^2/4 + p^3/27}\right)^{1/3} + \left(q/2 - \sqrt{q^2/4 + p^3/27}\right)^{1/3} \\ &= \frac{1}{3}\int_{-q/2+\sqrt{q^2/4+p^3/27}}^{q/2+\sqrt{q^2/4+p^3/27}} s^{-2/3}\,\mathrm{d}s \geq \frac{q}{3}\left(q/2 + \sqrt{q^2/4 + p^3/27}\right)^{-2/3} \\ &\geq \frac{q}{3}\left(q + \sqrt{p^3/27}\right)^{-2/3} \end{aligned}$$

The case $q < 0$ will be treated in the same way.

$\square$

Proposition 5.12. *Let $x \in \mathbb{H}$ and $a > 0$. We consider*

$$\Pi := \{y \in \mathbb{H} |\ n_1y_1 + n_2y_2 + n_3y_3 = c\},$$

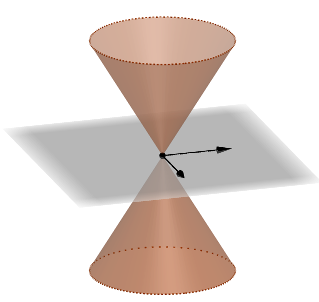
where $n_1, n_2, n_3, c \in \mathbb{R}$ and $n_3 \neq 0$. For the case $(-2n_2/n_3 + x_1)^2 + (2n_1/n_3 + x_2)^2 \leq a|-c/n_3 + x_3 + x_1n_1/n_3 + x_2n_2/n_3|$ it holds

$$\left(\inf_{y\in\Pi} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^2 \geq \frac{4|-c/n_3 + x_3 + x_1n_1/n_3 + x_2n_2/n_3|}{3^3}\left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2},$$

and for $(-2n_2/n_3+x_1)^2+(2n_1/n_3+x_2)^2 \geq a|-c/n_3+x_3+x_1n_1/n_3+x_2n_2/n_3|$ it holds

$$\left(\inf_{y\in\Pi} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^2 \geq \frac{4|-c/n_3 + x_3 + x_1n_1/n_3 + x_2n_2/n_3|^2}{(-2n_2/n_3 + x_1)^2 + (2n_1/n_3 + x_2)^2}\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3}.$$

Remark 5.13. In Proposition 5.10 we already proved that $\delta_C(\cdot)$ can be expressed as $\omega(\cdot)$ on Ω if $\partial\Omega$ is a hyperplane which is orthogonal to the hyperplane $x_3 = 0$. This is not true for other hyperplanes.



Let us assume $\Omega := \{(x_1, x_2, x_3) \in \mathbb{H} \,|\, x_3 \neq 0\}$. The hyperplane in our picture is then $\partial\Omega$. The apexes of the cones are the origin. Proposition 5.12 in combination with Lemma 2.2 yields a constant $c_a > 0$ such that for points lying in the interior of the double cone we have $\delta_C(x) \geq c_a\sqrt{\delta_e(x)}$, where $\delta_e(\cdot)$ is the Euclidean distance function to $\partial\Omega$. For points lying outside of the double cone, we have $\delta_C(x)^2 \geq c_a 4x_3^2(x_1^2 + x_2^2)^{-1} = c_a\omega(x)^2$.

Proof of Proposition 5.12 : First of all we consider the case $n_1 = n_2 = c = 0$ and $n_3 = 1$. Let $y := (y_1, y_2, y_3) \in \mathbb{H}$ such that $y_3 = 0$ and fix $x := (x_1, x_2, x_3) \in \mathbb{H}$ with $x_3 \neq 0$. We set $z_1 := y_1 - x_1$ and $z_2 := y_2 - x_2$ and consider

$$\begin{aligned} \|(-y) \boxplus x\|_{\mathbb{H}}^4 &= ((y_1 - x_1)^2 + (y_2 - x_2)^2)^2 + 16(y_3 - x_3 - 1/2y_1x_2 + 1/2y_2x_1)^2 \\ &= (z_1^2 + z_2^2)^2 + 16(-x_3 - 1/2z_1x_2 + 1/2z_2x_1)^2. \end{aligned} \tag{5.36}$$

Then we compute the minimum of the right-hand side in dependence of x. We assume that $x_1 \neq 0$ since $x_1 = 0$ is a null set, and δ_K is continuous because of (2.15) and Lemma 2.2. The derivatives with respect to y_1 and y_2 yield then

$$\begin{aligned} (z_1^2 + z_2^2)4z_1 - x_2 16(-x_3 - 1/2z_1x_2 + 1/2z_2x_1) = 0, \\ (z_1^2 + z_2^2)4z_2 + x_1 16(-x_3 - 1/2z_1x_2 + 1/2z_2x_1) = 0. \end{aligned} \tag{5.37}$$

Since $x_1 \neq 0$ we easily deduce that $z_1^2 + z_2^2 \neq 0$ and obtain

$$z_1 = \frac{-z_2x_2}{x_1}.$$

Inserting this in (5.36) yields

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 = z_2^4 \frac{(x_2^2 + x_1^2)^2}{x_1^4} + 16 \left(-x_3 + 1/2z_2 \frac{x_2^2 + x_1^2}{x_1} \right)^2. \tag{5.38}$$

We compute the critical points with respect to y_2 and obtain

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 = z_2^4 \frac{(x_2^2 + x_1^2)^2}{x_1^4} + z_2^6 \frac{(x_2^2 + x_1^2)^2}{x_1^6}, \tag{5.39}$$

where z_2 is the unique real solution of

$$z_2^3 + 2z_2x_1^2 = \frac{4x_3x_1^3}{x_2^2 + x_1^2}, \quad p := 2x_1^2, \quad q := \frac{4x_3x_1^3}{x_2^2 + x_1^2}.$$

Using the estimate in the previous Lemma we obtain

$$|z_2| \geq \frac{\sqrt[1/3]{4}|x_3|^{1/3}|x_1|}{3 \sqrt[1/3]{x_1^2 + x_2^2}} \left(1 + \frac{x_1^2 + x_2^2}{|x_3| 3^{3/2} \sqrt{2}} \right)^{-2/3}. \tag{5.40}$$

For the case $x_1^2 + x_2^2 \leq a|x_3|$ we use

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 \geq z_2^6 \frac{(x_2^2 + x_1^2)^2}{x_1^6},$$

and (5.40) to get

$$\left(\inf_{y\in\mathbb{H},y_3=0}\|(-y)\boxplus x\|_{\mathbb{H}}\right)^2 \geq \frac{4|x_3|}{3^3}\left(1+\frac{a}{3^{3/2}\sqrt{2}}\right)^{-2}.$$

For the case $x_1^2+x_2^2\geq a|x_3|$ we use (5.40) again for

$$\|(-y)\boxplus x\|_{\mathbb{H}}^4 \geq z_2^4\frac{(x_2^2+x_1^2)^2}{x_1^4},$$

which yields

$$\left(\inf_{y\in\mathbb{H},y_3=0}\|(-y)\boxplus x\|_{\mathbb{H}}\right)^2 \geq \frac{4|x_3|^2}{(x_2^2+x_1^2)}\left(\frac{3^{3/2}\sqrt{2}}{a}+1\right)^{-4/3},$$

To obtain the result for a general hyperplane we consider

$$\inf_{y\in\Pi}\|(-y)\boxplus x\|_{\mathbb{H}} = \inf_{y\in\Pi}\|(-(v\boxplus y))\boxplus(v\boxplus x)\|_{\mathbb{H}} = \inf_{(-v)\boxplus q\in\Pi}\|(-q)\boxplus(v\boxplus x)\|_{\mathbb{H}},$$

where $q:=(q_1,q_2,q_3)\in\mathbb{H}$, and $v\in\mathbb{H}$ is set

$$v:=\frac{1}{n_3}\left(-2n_2,2n_1,-c\right).$$

Then $(-v)\boxplus q\in\Pi$ is equivalent to $q_3=0$, which yields the result. □

Proof of Theorem 5.2: Let us assume that Ω is an open bounded convex polytope. Let $m\in\mathbb{N}$ be the number of hyperplanes of $\partial\Omega$, which are not orthogonal to the hyperplane $y_3=0$. We denote these hyperplanes by Π_j for $1\leq j\leq m$. Thus there exist $n_{1,j},n_{2,j},n_{3,j},c_j\in\mathbb{R}$ such that

$$\Pi_j:=\{y\in\mathbb{H}|\ n_{1,j}y_1+n_{2,j}y_2+n_{3,j}y_3=c_j\},$$

where $n_{3,j}\neq 0$ for $1\leq j\leq m$. By $n_j\in\mathbb{R}^3$ we denote the unit normal of Π_j. We use Lemma 5.5 and inequality (5.15) to obtain

$$\int_\Omega\left(\frac{1}{d_1(x)^2}+\frac{1}{d_2(x)^2}+\frac{1}{m}\sum_{j=1}^m\frac{1}{d_C(x,a_j)^2}\right)|u(x)|^2\,\mathrm{d}x\ \leq 5\int_\Omega|\nabla_{\mathbb{H}}u(x)|^2\,\mathrm{d}x \tag{5.41}$$

for $u\in C_0^\infty(\Omega)$, where

$$a_j:=\frac{1}{n_{3,j}}\left(2n_{2,j},-2n_{1,j},c_j\right).$$

The aim is to give a pointwise estimate for the weights on the left-hand side from below. We take $b \in \partial\Omega$ such that $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$, which exists since $\partial\Omega$ is compact and δ_K is continuous.

The first case is $b \in \Pi_j$ for a fixed j. Since Ω is convex we compute the intersection point with respect to $d_1(x)$ with Π_j as well as the intersection point with respect to $d_2(x)$ with Π_j. The hyperplane separation theorem yields then

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \geq \frac{1}{4} \frac{(-2n_{2,j} + x_1 n_{3,j})^2 + (2n_{1,j} + x_2 n_{3,j})^2}{|-c_j + \langle x, n_j \rangle|^2}.$$

Let $a > 0$. We use Proposition 5.12 for the case $(-2n_{2,j}/n_{3,j} + x_1)^2 + (2n_{1,j}/n_{3,j} + x_2)^2 \geq a|-c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|$ and get

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \geq \left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2}.$$

For the case $(-2n_{2,j}/n_{3,j} + x_1)^2 + (2n_{1,j}/n_{3,j} + x_2)^2 \leq a|-c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|$ we use Lemma 2.2

$$\begin{aligned}\frac{1}{m}\sum_{k=1}^{m} \frac{1}{d_C(x, a_k)^2} &\geq \frac{1}{\pi m} \frac{1}{\|(-a_j) \boxplus x\|_{\mathbb{H}}^2} \\ &\geq \frac{1}{\pi m \sqrt{a^2 + 16}} |-c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|^{-1},\end{aligned}$$

and then again Proposition 5.12 yields

$$\frac{1}{m}\sum_{k=1}^{m} \frac{1}{d_C(x, a_k)^2} \geq \frac{1}{4 \cdot 3^3 \pi m \sqrt{a^2 + 16}} \left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2}.$$

We choose $a > 0$ such that

$$\frac{1}{4 \cdot 3^3 \pi m \sqrt{a^2 + 16}} \left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2} = \left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3},$$

which obviously exists. The positive constant which fulfills that equation is denoted by c_m. If we summarize our estimates the weight function in (5.41) is then bounded from below by

$$\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2} \geq \left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \|(-b) \boxplus x\|_{\mathbb{H}}^{-2},$$

where we used $b \in \Pi_j$. We recall that b was chosen such that $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$.

The second case is $b := (b_1, b_2, b_3) \in \partial\Omega$ when the hyperplane which contains b, is orthogonal to the hyperplane $x_3 = 0$. We denote that hyperplane by Π. Because of the orthogonality condition, the hyperplane is parametrized by

$$\Pi_j := \{y \in \mathbb{H} | \ (b_1 - x_1)(y_1 - b_1) + (b_2 - x_2)(y_2 - b_2) = 0\}.$$

We use the hyperplane separation theorem again and compute the intersection points of $d_1(x), d_2(x)$ with Π_j obtaining

$$\begin{aligned} \frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} &\geq \frac{(b_1 - x_1)^2}{((b_1 - x_1)^2 + (b_2 - x_2)^2)^2} + \frac{(b_2 - x_2)^2}{((b_1 - x_1)^2 + (b_2 - x_2)^2)^2} \\ &= \frac{1}{(b_1 - x_1)^2 + (b_2 - x_2)^2} \geq \frac{1}{\|(-b) \boxplus x\|_{\mathbb{H}}^2}. \end{aligned}$$

At that point we use that b was chosen, such that $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$ is fulfilled. Summarizing our estimates we arrive at

$$\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_K(x)^2} \, \mathrm{d}x \ \leq 5 \int_\Omega |\nabla_{\mathbb{H}} \, u(x)|^2 \, \mathrm{d}x,$$

where Lemma 2.2 finally yields the result. □

Proof of inequality (5.14)*:* Let us assume that $c_m > 0$ fulfills

$$\sqrt{c_m^2 + 16} \left(1 + \frac{c_m}{3^{3/2}\sqrt{2}}\right)^{2/3} {c_m}^{4/3} = \frac{1}{2^{7/3} 3\pi m}.$$

It can be easily seen that

$$c_m \leq (4m\pi)^{-1/3} \leq (4\pi)^{-1/3}.$$

Thus we get the following estimate

$$\frac{1}{2^{7/3} 3\pi m} \leq \sqrt{(4\pi)^{-2/3} + 16} \left(1 + \frac{(4\pi)^{-1/3}}{3^{3/2}\sqrt{2}}\right)^{2/3} (4m\pi)^{-1/9} c_m,$$

which yields

$${c_m}^{-1} \leq m^{8/9} \pi^{8/9} 3 \cdot 2^{19/9} \sqrt{2^{-4/3}\pi^{-2/3} + 16} \left(1 + \frac{1}{3^{3/2} 2^{7/6} \pi^{1/3}}\right)^{2/3}.$$

□

5.5 Convex polytopes with improved constants

In this section we prove that for some open bounded convex polytopes the constant in Theorem 5.2 can be improved. We discuss that behavior in detail for convex cylinders. At the end we show for the smallest constant $c > 0$ satisfying (5.3) that $2 \leq c$, which is a similar result compared to the setting in the Euclidean case, which was discussed in Section 1.3.

5.5.1 The improved version

Assumption 5.14. *Let Ω be an open bounded convex polytope. Let $m \in \mathbb{N}$ denote the number of hyperplanes of $\partial\Omega$ which are not orthogonal to the hyperplane $x_3 = 0$. We denote these hyperplanes by Π_j for $1 \leq j \leq m$. Thus there exist $n_{1,j}, n_{2,j}, n_{3,j}, c_j \in \mathbb{R}$ such that*

$$\Pi_j := \{y \in \mathbb{H} |\ n_{1,j}y_1 + n_{2,j}y_2 + n_{3,j}y_3 = c_j\},$$

where $n_{3,j} \neq 0$ for $1 \leq j \leq m$. We assume that there exists a constant $a > 0$ such that for all $x \in \Omega$ and all $j \in \{1, \dots, m\}$ holds

$$(-2n_{2,j}/n_{3,j} + x_1)^2 + (2n_{1,j}/n_{3,j} + x_2)^2 \geq a| - c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|. \tag{5.42}$$

Theorem 5.15. *Under Assumption 5.14 it holds*

$$\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\ \mathrm{d}x \ \leq 4 \int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\ \mathrm{d}x \tag{5.43}$$

for all $u \in C_0^\infty(\Omega)$.

Proof. We use Lemma 5.5 to obtain

$$\int_\Omega \left(\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2}\right) |u(x)|^2\ \mathrm{d}x \ \leq 4 \int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\ \mathrm{d}x \tag{5.44}$$

for $u \in C_0^\infty(\Omega)$ and proceed in the same way as in the proof of Theorem 5.2. We treat only the case $b \in \Pi_j$ with $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$ since the other one is verbatim the same. By n_j we denote the unit normal to Π_j. Again we use the hyperplane separation theorem and get

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \geq \frac{1}{4} \frac{(-2n_{2,j} + x_1 n_{3,j})^2 + (2n_{1,j} + x_2 n_{3,j})^2}{| - c_j + \langle x,\, n_j\rangle|^2}.$$

Under Assumption 5.14 we use Proposition 5.12, yielding

$$\frac{1}{4} \frac{(-2n_{2,j} + x_1 n_{3,j})^2 + (2n_{1,j} + x_2 n_{3,j})^2}{| - c_j + \langle x,\, n_j\rangle|^2} \geq \left(\frac{3^{3/2}}{a 2^{-1/2}} + 1\right)^{-4/3} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2}.$$

Since $b \in \Pi_j$ we use Lemma 2.2 and get the result. $\square$

Remark 5.16. The last result can be extended to any convex bounded Ω as long as there exists a constant $a > 0$ such that for any hyperplane which is not orthogonal to $x_3 = 0$ and which separates Ω from points lying on its boundary, inequality (5.42) must hold.

5.5.2 Convex cylinders

We discuss briefly that there are domains, satisfying Assumption 5.14. Therefore we consider domains of the form $\Omega = D \times (\alpha, \beta)$, where $D \subset \mathbb{R}^2$ is a bounded convex domain and $\alpha < \beta$. This domain is not a polytope but the hyperplanes which separate the points lying in $b \in \partial D \times (\alpha, \beta)$ are orthogonal to the hyperplane $x_3 = 0$. Thus the proof of Theorem 5.15 goes through and we get;

Corollary 5.17. *Let $\Omega = D \times (\alpha, \beta)$ such that $\alpha < \beta$ and $D \subset \mathbb{R}^2$ is a bounded convex domain. For fixed $a > 0$ we assume that for all $x \in \Omega$ holds*

$$x_1^2 + x_2^2 \geq a|-\alpha + x_3|, \quad \textit{and} \quad x_1^2 + x_2^2 \geq a|-\beta + x_3|.$$

Then holds for all $u \in C_0^\infty(\Omega)$

$$\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\, \mathrm{d}x \ \leq 4 \int_\Omega |\nabla_{\mathbb{H}}\, u(x)|^2\, \mathrm{d}x. \tag{5.45}$$

Proof of Theorem 5.3*:* Let $a > 0$ be fixed. We consider the following domain $\Omega_a := B_1(p_a) \times (0, 1)$, where $B_1(p_a)$ is the two-dimensional Euclidean ball with radius one centered at $p_a := (\sqrt{a} + 1, 0)$. We check briefly the conditions of Corollary 5.17, where $\alpha = 0$ and $\beta = 1$. Let $(x_1, x_2) \in B_1(p_a)$, then we have $|x_1 - \sqrt{a} - 1| < 1$ and immediately get $x_1 \geq \sqrt{a}$, yielding

$$x_1^2 \geq a \geq a x_3 = a|-\alpha + x_3|, \qquad x_1^2 \geq a \geq a(1 - x_3) = a|-\beta + x_3|$$

since $x_3 \in (0, 1)$. Thus for Ω_a inequality (5.45) holds, where the constant depends only on $a > 0$. $\square$

5.5.3 On the sharp constant

We recall that we set $\delta_C(x) = 0$ for $x \in \Omega^c$ and prove the following:

Lemma 5.18. *Let Ω be a bounded domain in $\mathbb{H}$ and let $\alpha \geq \frac{1}{2}$. Then there exists a constant $C(\alpha) > 0$ only depending on α such that*

$$|\max\{\delta_C(x)^\alpha - \beta, 0\} - \max\{\delta_C(y)^\alpha - \beta, 0\}| \leq C(\alpha) d_C(x, y)^{1/2}$$

for any $x, y \in \mathbb{H}$ and $\beta \geq 0$.

Proof. We only have to prove the inequality for the case $\beta = 0$. For the case $\beta > 0$, we use then

$$|\max\{\delta_C(x)^\alpha - \beta, 0\} - \max\{\delta_C(y)^\alpha - \beta, 0\}| \leq |\delta_C(x)^\alpha - \delta_C(y)^\alpha|,$$

which holds for all $x, y \in \mathbb{H}$. Let $K \subset \mathbb{R}_+$ be a compact set. For the following function $f(u) = u^\alpha$ there exists a constant $M > 0$ such that $|f'(u)| \leq Mu^{-1/2}$ for all $u \in K$. Thus we easily deduce for $b > a$ that

$$|b^\alpha - a^\alpha| \leq \int_a^b |f'(t)| \, \mathrm{d}t \leq 2M|\sqrt{b} - \sqrt{a}| \leq 2M|a-b|^{1/2}, \quad \text{for all } a, b \in K. \tag{5.46}$$

At that point we mimic the proof of Lemma 4.11 for the function $\delta_C^\alpha(\cdot)$. The cases $x, y \in \Omega^c$ and $x \in \Omega^c, y \in \Omega$ are verbatim the same and the trivial ones. Thus let us assume $x, y \in \Omega$. Without loss of generality we assume that $\delta_C(x) > \delta_C(y)$. Since $d(\cdot, \cdot)$ is continuous, see (2.15) and $\partial\Omega$ compact, there exists a $z \in \partial\Omega$ such that $\delta_C(y) = d_C(z, y)$. Thus we get

$$|\delta_C(x)^\alpha - \delta_C(y)^\alpha| = \delta_C(x)^\alpha - \delta_C(y)^\alpha \leq d_C(x,z)^\alpha - d_C(y,z)^\alpha.$$

Then we use (5.46) to arrive at.

$$|\delta_C(x)^\alpha - \delta_C(y)^\alpha| \leq 2M|d_C(x,z) - d_C(y,z)|^{1/2} \leq 2M d_C(x,y)^{1/2}. \tag{5.47}$$

The last inequality follows by the triangle inequality. □

Lemma 5.19. *Let Ω be a bounded domain in $\mathbb{H}$. For any $\alpha \geq \frac{1}{2}$ and $\beta > 0$ the function*

$$f_{(\alpha,\beta)}(x) := \max\{\delta_C(x)^\alpha - \beta, 0\}$$

is weakly differentiable with respect to X_1 and X_2 on $\mathbb{H}$ with

$$X_1 f_{\alpha,\beta}(x) = \alpha\chi_{A_{(\alpha,\beta)}}(x)\delta_C(x)^{\alpha-1} X_1\delta_C(x),$$
$$X_2 f_{\alpha,\beta}(x) = \alpha\chi_{A_{(\alpha,\beta)}}(x)\delta_C(x)^{\alpha-1} X_2\delta_C(x),$$

where $\chi_{A_{(\alpha,\beta)}}(\cdot)$ is the characteristic function of $A_{(\alpha,\beta)} := \{x \in \Omega \mid \delta_C(x)^\alpha \geq \beta\}$.

Proof. Without loss of generality we consider only the case X_1. We must show that there exists a function $v \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega X_1 u(x)\, f_{(\alpha,\beta)}(x) \, \mathrm{d}x = -\int_\Omega u(x) v(x) \, \mathrm{d}x \tag{5.48}$$

holds for any $u \in C_0^\infty(\mathbb{H})$. Since we extended $\delta_C(\cdot)$ by zero to the whole space, we can integrate over $\mathbb{H}$. We know that $f_{(\alpha,\beta)}(\cdot)$ is bounded on any compact set, and an application of the dominated convergence theorem yields then

$$\int_{\mathbb{H}} X_1 u(x) f_{(\alpha,\beta)}(x) \, \mathrm{d}x \;=\; \lim_{h \to 0} \left(\int_{\mathbb{H}} \frac{u(x + h\tilde{x})}{h} f_{(\alpha,\beta)}(x) \, \mathrm{d}x - \int_{\mathbb{R}^3} \frac{u(x)}{h} f_{(\alpha,\beta)}(x) \, \mathrm{d}x \right), \tag{5.49}$$

where $\tilde{x} := (1, 0, x_2/2)$. We make the change of variables $x + h\tilde{x} \mapsto x$ to obtain

$$\begin{aligned}\int_{\mathbb{H}} X_1 u(x) f_{(\alpha,\beta)}(x)\ \mathrm{d}x &= \lim_{h\to 0}\left(\int_{\mathbb{H}} u(x)\frac{f_{(\alpha,\beta)}(x-h\tilde{x})}{h}\ \mathrm{d}x - \int_{\mathbb{R}^3}\frac{u(x)}{h} f_{(\alpha,\beta)}(x)\ \mathrm{d}x\right)\\ &= -\lim_{h\to 0}\left(\int_{\mathbb{H}} u(x)\frac{f_{(\alpha,\beta)}(x-h\tilde{x}) - f_{(\alpha,\beta)}(x)}{-h}\ \mathrm{d}x\right).\end{aligned}$$

In the proof of Lemma 4.11 we showed that

$$|\delta_C(x) - \delta_C(y)| \le d_C(x,y), \quad \text{for all } x \in \mathbb{H}.$$

We take that inequality and use [MC01, Theorem 2.5] to obtain that the following limit

$$\lim_{h\to 0}\frac{\delta_C(x-h\tilde{x}) - \delta_C(x)}{-h} =: X_1\delta_C(x)$$

exists almost everywhere on $\mathbb{H}$ since (2.15) holds. Next we know by Lemma 5.18 in combination with (2.15) that the function $f_{(\alpha,\beta)}$ is absolutely continuous on $\mathbb{H}$. Thus we use the fundamental theorem of calculus to obtain

$$\begin{aligned}|f_{(\alpha,\beta)}(x-h\tilde{x}) - f_{(\alpha,\beta)}(x)| &\le \int_0^h |\partial_t f_{(\alpha,\beta)}(x-t\tilde{x})|\ \mathrm{d}t\\ &\le \alpha\int_0^h \chi_{A(\alpha,\beta)}(x-t\tilde{x})\delta_C^{\alpha-1}(x-t\tilde{x})|X_1\delta_C(x-t\tilde{x})|\ \mathrm{d}t,\end{aligned}$$

which holds almost everywhere on $\mathbb{H}$. At that point we use that $\delta_C(\cdot)$ is bounded since Ω is bounded, and then we take $\chi_{A(\alpha,\beta)}(x-t\tilde{x})\delta_C^{-1}(x-t\tilde{x}) \le \beta^{-1/\alpha}$ into account to obtain a constant $\tilde{M} > 0$ independent of h such that

$$|f_{(\alpha,\beta)}(x-h\tilde{x}) - f_{(\alpha,\beta)}(x)| \le \tilde{M}\int_0^h 1\cdot|X_1\delta_C(x-t\tilde{x})|\ \mathrm{d}t \le \tilde{M}h.$$

For the last inequality we used the Cauchy-Schwarz inequality and the Eikonal equation. We stress that the last inequality holds almost everywhere on $\mathbb{H}$. Hence, with an application of the dominated convergence we arrive at

$$\int_{\mathbb{H}} X_1 u(x) f_{(\alpha,\beta)}(x)\ \mathrm{d}x = -\left(\int_{\mathbb{H}} u(x)\lim_{h\to 0}\frac{f_{(\alpha,\beta)}(x-h\tilde{x}) - f_{(\alpha,\beta)}(x)}{-h}\ \mathrm{d}x\right).$$

Since we know that δ_C is almost everywhere differentiable on $\mathbb{H}$, we compute the derivative and obtain the desired result. The case of X_2 is treated in the same way. □

Theorem 5.20. *Let $\Omega \subset \mathbb{H}$ be a bounded domain. Then holds*

$$\inf_{u\in C_0^\infty(\Omega)\setminus\{0\}}\frac{\int_\Omega |\nabla_{\mathbb{H}} u(x)|^2\ \mathrm{d}x}{\int_\Omega |u(x)|^2\delta_C(x)^{-2}\ \mathrm{d}x} \le \frac{1}{4}. \tag{5.50}$$

Proof. Let $d[a]$ be the set of functions, which are given by the closure of

$$\int_\Omega |X_1\, u(x)|^2 + |X_2\, u(x)|^2 \;\mathrm{d}x$$

initially given on $C_0^\infty(\Omega)$. It suffices to construct a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \frac{\int_\Omega |\nabla_{\mathbb{H}}\, u_n(x)|^2 \;\mathrm{d}x}{\int_\Omega |u_n(x)|^2 \delta_C(x)^{-2} \;\mathrm{d}x} = \frac{1}{4}. \tag{5.51}$$

To this end we consider the sequence

$$\tilde{u}_n(x) = \delta_C(x)^{1/2+1/n}, \qquad n \in \mathbb{N}.$$

and recall that $\delta_C(\cdot)$ satisfies the Eikonal equation

$$|\nabla_{\mathbb{H}}\, \delta_C(x)|^2 = 1, \quad \text{for a.e. } x \in \Omega, \tag{5.52}$$

see [MC01, Thm 3.1]. Moreover, from (2.15) we know that

$$M\|x-y\|_e \leq d_C(x,y) \leq M^{-1}\|x-y\|_e^{1/2}. \tag{5.53}$$

holds for some $M > 0$ and all $x, y \in \overline{\Omega}$. With the coarea formula one can easily prove that

$$\int_\Omega \delta_e(x)^{2/n-1} \;\mathrm{d}x < \infty, \quad \delta_e(x) := \inf_{y\in\partial\Omega} \|x-y\|_e$$

because $\delta_e(\cdot)$ is a Lipschitz function on $\mathbb{R}^3$ even when we set $\delta_e(\cdot) := 0$ on Ω^c. Hence the integral $\int_\Omega \delta_C(x)^{2/n-1} \;\mathrm{d}x < \infty$, and using (5.52) we easily find that

$$\frac{\int_\Omega |\nabla_{\mathbb{H}}\, \tilde{u}_n(x)|^2 \;\mathrm{d}x}{\int_\Omega |\tilde{u}_n(x)|^2 \delta_C(x)^{-2} \;\mathrm{d}x} = \left(\frac{1}{2} + \frac{1}{n}\right)^2 \qquad \forall\, n \in \mathbb{N}. \tag{5.54}$$

At that point let us assume for a moment that $f_{\alpha,\beta} \in d[a]$ for any $\beta > 0$ and $\alpha > \frac{1}{2}$, see Lemma 5.19 for the definition. Then we prove that $f_{(\alpha,\beta)} \to \delta_C^\alpha$ for $\beta \to 0$ in $L^2(\Omega)$. We use Lemma 5.19 and the Eikonal equation to arrive at

$$\left\|\nabla_{\mathbb{H}}(f_{(\alpha,\beta)} - \delta_C^\alpha)\right\|_2 = \alpha^2 \int_\Omega \delta_C(x)^{2\alpha-2}(1 - \chi_{A(\alpha,\beta)}(x)) \;\mathrm{d}x.$$

Since $\alpha > \frac{1}{2}$ we use the dominated convergence theorem and obtain that $\delta_C^\alpha \in d[a]$ for any $\alpha > \frac{1}{2}$.

Thus the only thing left to prove is $f_{\alpha,\beta} \in d[a]$ for any $\beta > 0$ and $\alpha > \frac{1}{2}$. The idea is basically the same as the one to prove that the Euclidean distance function can be approximated by $C_0^\infty(\Omega)$ functions with respect to the gradient of the Laplacian. For the sake of completeness we state the proof here. Let us consider a function $j : \mathbb{R}^3 \to \mathbb{R}$ satisfying the following conditions:

1. $j \in C_0^\infty(\mathbb{R}^3)$,
2. $j(x) \geq 0$ for all $x \in \mathbb{R}^3$,
3. $j(x) = 0$ for $\|x\|_{\mathbb{H}} \geq 1$,
4. $\int_{\mathbb{R}^3} j(x) \, \mathrm{d}x = 1$.

For $\varepsilon > 0$, we set

$$j_\varepsilon(x) := \frac{1}{\varepsilon^4} j\left(\varepsilon^{-1}(x)\right), \quad \varepsilon^{-1}(x) := (\varepsilon^{-1}x_1, \varepsilon^{-1}x_2, \varepsilon^{-2}x_3),$$

and define the convolution operator with respect to $\mathbb{H}$ by

$$J_\varepsilon u(x) := j_\varepsilon * u(x) := \int_{\mathbb{R}^3} j_\varepsilon(x \boxplus (-y))u(y) \, \mathrm{d}y, \quad \text{for } u \in L^1_{\mathrm{loc}}(\mathbb{R}^3).$$

From [Fol16, Prop. 2.39] we know that for any $u \in L^p(\mathbb{R}^3)$ we have

$$\|J_\varepsilon u\|_p \leq \|u\|_p, \quad 1 \leq p \leq \infty$$

where $\|\cdot\|_p$ denotes L^p-norm.

Now we prove for fixed $1 \leq p < \infty$ that $J_\varepsilon u \to u$ with respect to $\|\cdot\|_p$ for $\varepsilon \to 0$. Let us fix $u \in L^1(\mathbb{R}^3)$. Since $C_0^\infty(\mathbb{R}^3))$ is dense, we take $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\|u - \varphi\|_p < \delta$. Then we get

$$\|J_\varepsilon u - u\|_p \leq \|J_\varepsilon(u - \varphi)\|_p + \|J_\varepsilon \varphi - \varphi\|_p + \|\varphi - u\|_p < 2\delta + \|J_\varepsilon \varphi - \varphi\|_p.$$

For $\varepsilon < 1$ we know that there exists a compact set $K \subset \mathbb{R}^3$ with $\mathrm{supp}(J_\varepsilon \varphi - \varphi) \subseteq K$ since φ has compact support, (5.53) and Lemma 2.2 hold. Thus we get the following

$$\begin{aligned}
\|J_\varepsilon \varphi - \varphi\|_p &\leq |K|^{1/p} \sup_{x \in K} |J_\varepsilon \varphi(x) - \varphi(x)| \\
&= |K|^{1/p} \sup_{x \in K} \left| \int_{\mathbb{R}^3} j_\varepsilon(x \boxplus (-y))(\varphi(y) - \varphi(x)) \, \mathrm{d}y \right| \\
&\leq |K|^{1/p} \sup_{||x \boxplus (-y)||_{\mathbb{H}} < \varepsilon} |(\varphi(y) - \varphi(x))|.
\end{aligned}$$

We know that $\varphi \in C_0^\infty(\mathbb{R}^3)$, and therefore φ is uniform continuous on $\mathbb{R}^3$ because of (5.53) . This yields $\|J_\varepsilon \varphi - \varphi\|_p \to 0$ for $\varepsilon \to 0$, implying that $\|J_\varepsilon u - u\|_p \to 0$ for $\varepsilon \to 0$. The sequence to approximate $f_{\alpha,\beta}$ is then given by

$$v_\varepsilon(x) := J_\varepsilon f_{\alpha,\beta}(x) = \int_{A(\alpha,\beta)} j_\varepsilon(x \boxplus (-y))(\delta_C(y)^\alpha - \beta) \, \mathrm{d}y.$$

We stress that we can integrate over $A(\alpha,\beta)\cap B_{\varepsilon/M}(x)$, where we set $B_{\varepsilon/M}(x) := \{y \in \mathbb{R}^3 \,|\, \|x-y\|_e < \frac{\varepsilon}{M}\}$ since (5.53) holds. Again we apply (5.53) to prove that

$$A(\alpha,\beta) \subseteq \{y \in \Omega \,|\, \delta_e(y) > \beta^{2/\alpha} M^2\},$$

where $\delta_e(y) := \operatorname{dist}(\partial\Omega, y)$ in the Euclidean sense. Hence, we immediately get for $\varepsilon < \frac{M^3\beta^{2/\alpha}}{3}$ that the function $v_\varepsilon \in C_0^\infty(\Omega)$. At that point we use the dominated convergence theorem to obtain

$$\begin{aligned}
X_1 v_\varepsilon(x) &= \int_{A(\alpha,\beta)} X_1 j_\varepsilon(x \boxplus (-y))(\delta_C(y)^\alpha - \beta)\ \mathrm{d}y \\
&= -\int_{A(\alpha,\beta)} Y_1 j_\varepsilon(x \boxplus (-y))(\delta_C(y)^\alpha - \beta)\ \mathrm{d}y = \int_{A(\alpha,\beta)} j_\varepsilon(x \boxplus (-y)) Y_1(\delta_C(y)^\alpha - \beta)\ \mathrm{d}y,
\end{aligned}$$

where $Y_1 := \partial_{y_1} + \frac{1}{2} y_2 \partial_{y_3}$. For the last equality we used Lemma 5.19. Hence we know that $\left\| X_1 v_\varepsilon - X_1 f_{(\alpha,\beta)} \right\|_p \to 0$ for $\varepsilon \to 0$ and show the same for X_2, yielding that $f_{(\alpha,\beta)} \in d[a]$ for $\alpha > \frac{1}{2}$ and $\beta > 0$.

$\square$

Appendix A

Miscellaneous results

For the sake of completeness we give here proofs of theorems which are important for this thesis and appear only in modified versions in the literature. However we do not make any claim to originality. Throughout the appendix we assume that $n \in \mathbb{N}$.

A.1 Boundary estimates for differential operators

We have seen in Section 4.5 and 4.6 that the growing order of the lower order term in a Melas-type bound can be further improved if we assume a Hardy inequality. The key was a straightforward generalization of the result in [Dav00] by E. B. Davies, which was pointed out by R. Frank in [KW15]. In [Dav00] E. B. Davies studied second order elliptic operators with Dirichlet boundary conditions on bounded domains and proved L^2 boundary decay estimates for the corresponding eigenfunctions. Although the Heisenberg Laplacian is a subelliptic operator, Davies' result is valid for this operator, too. Therefore in this section we extend Davies' L^2 boundary decay estimate to a wider class of differential operators. At the end of this section we discuss briefly differential operators with magnetic field as well.

A.1.1 Boundary estimates for vector fields

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A vector field X on Ω is a real-valued first order partial differential operator of the form

$$X = \sum_{j=1}^{n} a_j(x)\partial_{x_j} \tag{A.1}$$

where ∂_{x_j} denote the partial derivatives and $a_j : \Omega \to \mathbb{R}$ are Lipschitz continuous functions. Let $m \in \mathbb{N}$ and $X_1, \dots, X_m$ be vector fields, then we consider the sum-of-squares differential operator with respect to Dirichlet boundary conditions on Ω, denoted by

$$A(\Omega) := -\sum_{j=1}^{m} X_j^2 + V, \tag{A.2}$$

where $V \in \mathcal{L}^1_{loc}(\Omega)$ is a non-negative potential. The vector fields are of the form

$$X_j := \sum_{k=1}^{n} a_{j,k}(x)\partial_{x_k}, \tag{A.3}$$

satisfying the additional condition $\partial_{x_k} a_{j,k}(x) = 0$ for all $k \in \{1, \dots, n\}$. We choose $A(\Omega) =: A$ to be the Friedrichs extension, which is the self-adjoint operator in $L^2(\Omega)$ associated to the closure of the quadratic form

$$a[u] := \int_\Omega |\nabla_A u(x)|^2 + V|u(x)|^2 \, \mathrm{d}x \tag{A.4}$$

initially given on $C_0^\infty(\Omega)$, where the gradient of A is given by $\nabla_A := (X_1, \dots, X_m)$, $d[a]$ is the domain of (A.4) and $D(A)$ is the domain of A.

For a given real-valued function $\delta \in d[a]$ with $|\nabla_A \delta| \leq 1$ almost everywhere on Ω, we assume that there exists a constant $c > 0$ such that

$$\int_\Omega \frac{|u(x)|^2}{\delta(x)^2} \, \mathrm{d}x \leq c^2 \left(\int_\Omega |\nabla_A u(x)|^2 + V|u(x)|^2 \, \mathrm{d}x \right) \tag{A.5}$$

holds for all $u \in C_0^\infty(\Omega)$. In spirit of [Dav00] we say that A satisfies Hardy's inequality with respect to δ. The following result extends E. B. Davies' result [Dav00, Theorem 4] to the operator A:

Theorem A.1. *Let Ω be a bounded domain, $c \geq 2$ be given by (A.5), $\delta \in d[a]$ satisfying $|\nabla_A \delta| \leq 1$ and $\delta > 0$ on Ω, and for fixed $j \in \{1, \dots, m\}$ we assume $\partial_{x_k} a_{j,k}(x) = 0$ for all $k \in \{1, \dots, n\}$. Then*

$$\int_{\Omega^\beta} \frac{|u(x)|^2}{\delta(x)^2} \, \mathrm{d}x \;\leq\; c^{2+\frac{2}{c}} \beta^{\frac{2}{c}} \, \|A\, u\|_{L^2(\Omega)} \big\|A^{1/c}\, u\big\|_{L^2(\Omega)} \tag{A.6}$$

holds for all $u \in \mathrm{Dom}(A)$ and any $\beta > 0$, where $\Omega^\beta := \{x \in \Omega \,|\, \delta(x) < \beta\}$. Hence

$$\int_{\Omega^\beta} |u(x)|^2 \, \mathrm{d}x \;\leq\; c^{2+\frac{2}{c}} \beta^{2+\frac{2}{c}} \, \|A\, u\|_{L^2(\Omega)} \big\|A^{1/c}\, u\big\|_{L^2(\Omega)} \tag{A.7}$$

any $\beta > 0$.

If we choose A to be the Laplacian without potential for $m = n$ and δ as the Euclidean distance function to the boundary, then we obtain the condition of the well-known Hardy inequality for the Laplacian, which was discussed in Section 1.3. From geometrical point of view this kind of inequality is extendable to a huge class of sum-of-squares differential operators.

The vector field (A.1) has a geometrical interpretation. We consider the vector $(a_1(x), \dots, a_n(x))^t$ as an element of the tangential space at the point x. The natural distance function is then given by the Carnot-Carathéodory metric generated by the vector fields of the considered sum-of-squares differential operator. Hence, we assume that the vector fields fulfill the Hörmander finite rank condition, which guarantees the connectivity between any two points by piecewise smooth curves lying in the span of the vector fields, see for instance [Mon02]. We recall that A is then a subelliptic operator if $m < n$, see [Hör66, Ego75]. In [DGP09] D. Danielli, N. Garofalo and N. C. Phuc studied several inequalities of Hardy-Sobolev type in Carnot-Carathéodory spaces and proved for a class of sum-of-squares differential operators the validity of (A.5) for suitable Ω, where δ is the distance function to the boundary of Ω with respect to the Carnot-Carathéodory metric. We stress that under additional conditions on the vector fields the Eikonal equation

$$|\nabla_A \delta| = 1 \tag{A.8}$$

holds almost everywhere on Ω, see [MC01, Thm. 3.1]. We have already seen that the Heisenberg Laplacian is an operator which satisfies (A.5) for certain domains and (A.8), as well.

Proof of Theorem (A.1): We follow the proof of [Dav00, Theorem 4]. Let us fix $u \in \mathrm{Dom}(A)$ and set

$$\varphi(x) := (\max\{\delta(x), \beta\})^{-1/c}.$$

for $x \in \Omega$ and $\beta > 0$. First we check that $\varphi u \in d[a]$. Since $u \in \mathrm{Dom}(A) \subseteq d[a]$, $\varphi \in d[a]$, we get

$$\int_\Omega |\nabla_A(\varphi(x)u(x))|^2 \, \mathrm{d}x \le 2 \int_\Omega |\varphi(x)\nabla_A u(x)|^2 \, \mathrm{d}x + 2 \int_\Omega |\nabla_A \varphi(x)|^2 |u(x)|^2 \, \mathrm{d}x.$$

We use $\varphi \le \beta^{-1/c}$ and $|\nabla_A \delta| \le 1$ to obtain then that $\varphi u \in d[a]$. Thus we may use (A.5) to get

$$\begin{aligned} c^{-2} \int_\Omega \frac{|\varphi(x)u(x)|^2}{\delta(x)^2} \, \mathrm{d}x &\le \int_\Omega |\varphi(x)\nabla_A u(x) + u(x)\nabla_A\varphi(x)|^2 \, \mathrm{d}x + \int_\Omega V|u(x)\varphi(x)|^2 \, \mathrm{d}x \\ &= \langle \varphi^2 \nabla_A u, \nabla_A u\rangle + \langle u, |\nabla_A\varphi|^2 u\rangle + \frac{1}{2}\langle \nabla_A u, u\nabla_A(\varphi^2)\rangle \\ &+ \frac{1}{2}\langle u\nabla_A(\varphi^2), \nabla_A u\rangle + \langle Vu\varphi, u\varphi\rangle, \end{aligned}$$

where we denote by $\langle\,\cdot,\,\cdot\rangle$ the scalar product in $L^2(\Omega)$. An integration by parts and the condition $\partial_{x_k} a_{j,k}(x) = 0$ for all $k \in \{1, \ldots, n\}$ yield

$$c^{-2} \int_\Omega \frac{|\varphi(x)u(x)|^2}{\delta(x)^2}\, \mathrm{d}x \;\leq\; \mathrm{Re}\langle\, \varphi^2 u,\, Au\rangle + \langle\, u,\, |\nabla_A\, \varphi|^2 u\rangle. \tag{A.9}$$

Next we will estimate the first term on the right hand side. To this end we use (A.5), which gives

$$\delta^{-2} \leq c^2 A$$

in the operator sense. Then, by the Heinz inequality [Dav80, Lemma 4.20], which is applicable since $c \geq 2$, we get

$$\varphi^4 \leq (\delta^{-2})^{2/c} \leq \left(c^2 A\right)^{2/c}.$$

Since $A^{-1/c}$ is bounded in $L^2(\Omega)$ we obtain

$$\left\|\varphi^2 A^{-1/c}\right\| \leq c^{2/c},$$

where $\|\cdot\|$ stands for the operator norm in $L^2(\Omega)$. Hence

$$\begin{aligned} |\langle\, Au,\, \varphi^2 u\rangle| &= |\big\langle\, Au,\, \varphi^2 A^{-1/c} A^{1/c} u\big\rangle| \\ &\leq \|Au\|_{L^2(\Omega)} c^{2/c} \big\|A^{1/c} u\big\|_{L^2(\Omega)}. \end{aligned}$$

So we arrive at

$$c^{-2} \int_\Omega \frac{|\varphi(x)u(x)|^2}{\delta(x)^2}\, \mathrm{d}x \leq \|Au\|_{L^2(\Omega)} c^{2/c} \big\|A^{1/c} u\big\|_{L^2(\Omega)} + \langle\, u,\, |\nabla_A \varphi|^2 u\rangle. \tag{A.10}$$

On the other hand, $|\nabla_A \delta| \leq 1$ implies that

$$|\nabla_A \varphi(x)|^2 \leq c^{-2} \delta(x)^{-2/c-2} \chi_{\{\delta(x)\geq\beta\}}(x),$$

where $\chi_{\{\delta(x)>\beta\}}(\cdot)$ is the characteristic function of the set $\{x \in \Omega \,|\, \delta(x) \geq \beta\}$. Inserting the above identity into (A.10), we obtain

$$\int_{\{x\in\Omega|\delta(x)<\beta\}} \frac{|u(x)|^2}{\delta(x)^2}\, \mathrm{d}x \leq \beta^{2/c} \|Au\|_{L^2(\Omega)} c^{2+2/c} \big\|A^{1/c} u\big\|_{L^2(\Omega)}. \tag{A.11}$$

The other result now follows from the estimate

$$\int_{\{x\in\Omega|\delta(x)<\beta\}} |u(x)|^2\, \mathrm{d}x \leq \beta^2 \int_{\{x\in\Omega|\delta(x)<\beta\}} \frac{|u(x)|^2}{\delta(x)^2}\, \mathrm{d}x.$$

$\square$

A.1.2 The magnetic case

Here we briefly discuss that Davies' result holds for vector fields with magnetic fields, too. We discuss here a more general case; we consider partial differential operator of the form

$$X = \sum_{j=1}^{n} a_j(x)\partial_{x_j} + \mathrm{i}b_j(x) \tag{A.12}$$

where $a_j, b_j : \Omega \to \mathbb{R}$ are Lipschitz continuous functions. We call this differential operator a generalized vector field. Let $m \in \mathbb{N}$ and $X_1, \ldots, X_m$ be generalized vector fields, then we consider the sum-of-squares differential operator with respect to Dirichlet boundary conditions on Ω, denoted by

$$A_b(\Omega) := -\sum_{j=1}^{m} X_{j,b}^2 + V, \tag{A.13}$$

where $V \in \mathcal{L}^1_{loc}(\Omega)$ is a non-negative potential. The vector fields are of the form

$$X_{j,b} := \sum_{k=1}^{n} a_{j,k}(x)\partial_{x_k} + \mathrm{i}b_{j,k}(x), \tag{A.14}$$

satisfying the additional condition $\partial_{x_k} a_{j,k}(x) = 0$ for all $k \in \{1, \ldots, n\}$. If all the $b_{j,k}$ do not depend on k for any $1 \leq j \leq m$ and $m = n$, then the vector potential $B(x) := n(b_{1,k}(x), \ldots, b_{n,k})^t$ describes a magnetic field in $\mathbb{R}^n$. The differential operator $X_{j,0}$ is defined as $X_{j,b}$ such that all $b_{j,k} = 0$.

We choose $A_b(\Omega) =: A_b$ to be the Friedrichs extension, which is the the self-adjoint operator in $L^2(\Omega)$ associated with the closure of the quadratic form

$$a_b[u] := \int_{\Omega} |\nabla_{A_b} u(x)|^2 + V|u(x)|^2 \,\mathrm{d}x \tag{A.15}$$

initially given on $C_0^\infty(\Omega)$, where the gradient of A_b is given by $\nabla_{A_b} := (X_{1,b}, \ldots, X_{m,b})$, $d[a_b]$ is the domain of (A.15) and $D(A_b)$ is the domain of A_b. The operator A_0 is defined as A_b such that all $b_{j,k} = 0$ for $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$, which yields an operator of the form (A.2).

We still assume that (A.5) and (A.8) are fulfilled for the operator A_0, not for A_b. We will see in the proof of the next theorem that the diamagnetic inequality extends the result of the previous subsection for A_0 to A_b.

Corollary A.2. *Let Ω be an open bounded domain, $c \geq 2$ be given by* (A.5) *with respect to A_0, $d[a_0] = d[a_b]$, $\delta \in d[a_0]$ satisfying $|\nabla_{A_0}\delta| \leq 1$ and $\delta > 0$ on Ω, and for fixed $j \in \{1, \ldots, m\}$ we assume $\partial_{x_k} a_{j,k}(x) = 0$ for all $k \in \{1, \ldots, n\}$. Then*

$$\int_{\Omega^\beta} \frac{|u(x)|^2}{\delta(x)^2} \,\mathrm{d}x \;\leq\; c^{2+\frac{2}{c}} \beta^{\frac{2}{c}} \left\|A_b\, u\right\|_{L^2(\Omega)} \left\|A_b^{1/c}\, u\right\|_{L^2(\Omega)} \tag{A.16}$$

holds for all $u \in \mathrm{Dom}(A_b)$ *and any* $\beta > 0$*, where* $\Omega^\beta := \{x \in \Omega \,|\, \delta(x) < \beta\}$*. Hence*

$$\int_{\Omega^\beta} |u(x)|^2 \, \mathrm{d}x \;\le\; c^{2+\frac{2}{c}} \beta^{2+\frac{2}{c}} \, \|A_b\, u\|_{L^2(\Omega)} \left\| A_b^{1/c}\, u \right\|_{L^2(\Omega)} \tag{A.17}$$

any $\beta > 0$.

Proof. First of all we consider $u \in d[a_0] = d[a_b]$, then we prove that $|u| \in d[a_0]$. Therefore we recall [LL01, Thm. 6.17]

$$\partial_{x_j} |u(x)| = \begin{cases} \mathrm{Re}\left(\frac{\overline{u(x)}}{|u(x)|} \partial_{x_j} u(x) \right) & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Since all $a_{j,k}(x)$ and $b_{j,k}$ are real-valued functions, we obtain

$$\begin{aligned} \langle\, X_{j,0}|u|,\, X_{j,0}|u| \rangle &= \sum_{k=1}^{n} \sum_{\tilde{k}=1}^{n} \left\langle\, a_{j,k} \partial_{x_k} |u|,\, a_{j,\tilde{k}} \partial_{x_{\tilde{k}}} |u| \right\rangle \\ &= \left\langle\, \mathrm{Re}\left(\tfrac{\overline{u(x)}}{|u(x)|} \textstyle\sum_{k=1}^{n} a_{j,k} \partial_{x_k} u(x) \right),\, \mathrm{Re}\left(\tfrac{\overline{u(x)}}{|u(x)|} \textstyle\sum_{\tilde{k}=1}^{n} a_{j,\tilde{k}} \partial_{x_{\tilde{k}}} u(x) \right) \right\rangle \\ &= \left\langle\, \mathrm{Re}\left(\tfrac{\overline{u(x)}}{|u(x)|} X_{j,b} u \right),\, \mathrm{Re}\left(\tfrac{\overline{u(x)}}{|u(x)|} X_{j,b} u \right) \right\rangle \le \langle\, X_{j,b} u,\, X_{j,b} u \rangle \end{aligned} \tag{A.18}$$

If we sum over all j, we get $|u| \in d[a_0]$. Let us skip back to the proof of Theorem (A.1). Inequality (A.9) and a density argument yield for all $f \in d[a_0]$

$$\begin{aligned} c^{-2} \int_\Omega \frac{|\varphi(x) f(x)|^2}{\delta(x)^2} \, \mathrm{d}x \;\le\; & \mathrm{Re}\langle\, \nabla_{A_0}(\varphi^2 f),\, \nabla_{A_0} f \rangle + \mathrm{Re}\langle\, V \varphi^2 f,\, f \rangle \\ & + \langle\, f,\, |\nabla_{A_0} \varphi|^2 f \rangle, \end{aligned}$$

where $\varphi(x) := (\max\{\delta(x), \beta\})^{-1/c}$ since $\varphi^2 \cdot f \in d[a_0]$. At this point we assume $u \in D(A_b) \subset d[a_b] = d[a]$ and set $f(x) = |u(x)|$. Then we apply (A.18) and use integration by parts to arrive at

$$c^{-2} \int_\Omega \frac{|\varphi(x) u(x)|^2}{\delta(x)^2} \, \mathrm{d}x \;\le\; \mathrm{Re}\langle\, \varphi^2 u,\, A_b u \rangle + \langle\, u,\, |\nabla_{A_0} \varphi|^2 u \rangle.$$

The rest is done in the same way as the proof of Theorem (A.1).

□

A.2 The Legendre transform

We consider the Legendre transform of a function $f : [0, \infty) \to \mathbb{R}$, which is defined for $p \geq 0$, by

$$f^\star(p) := \sup_{x \geq 0} (px - f(x)).$$

We assume that the supremum of $f^\star(p)$ exists for all $p \geq 0$, which is valid for instance if f is a convex, non-negative function.

In the Chapters 3 and 4 we saw that the Legendre tranformation transforms inequalities for Riesz means of order one into inequalities for the eigenvalue sum. Let us assume that we have an inequality of the following form

$$\sum_{j:\,\lambda_j<x} (x-\lambda_j) \leq f(x)$$

for all $x \geq 0$, where $\{\lambda_j\}_{j\in\mathbb{N}}$ is a positive, nondecreasing and unbounded sequence. The Legendre transform turns this inequality into

$$\sup_{x\geq 0} \big(px - \sum_{j:\,\lambda_j<x} x + \sum_{j:\,\lambda_j<x} \lambda_j\big) \geq f^\star(p), \quad p \geq 0. \tag{A.19}$$

We show that the left-hand side is bounded from above by

$$(p-[p])\lambda_1 + \sum_{j=1}^{[p]} \lambda_j,$$

for all $p > 0$, where $[p] := \min\{k \in \mathbb{Z} | k \geq p\}$. Let us fix $p > 0$ and consider $x \geq 0$ such that $\lambda_k \leq x < \lambda_{k+1}$ for $k \in \mathbb{N}$.

First of all we assume that $k < [p]$ and use $\lambda_k \leq x < \lambda_{k+1}$ to obtain

$$\begin{aligned} px - \sum_{j:\,\lambda_j<x} x + \sum_{j:\,\lambda_j<x} \lambda_j &= (p-[p])x + ([p]-k)x - \sum_{j=k+1}^{[p]} \lambda_j \ + \sum_{j=1}^{[p]} \lambda_j \\ &\leq (p-[p])x + \sum_{j=1}^{[p]} \lambda_j \leq (p-[p])\lambda_1 + \sum_{j=1}^{[p]} \lambda_j. \end{aligned}$$

The case $k = [p]$ is trivial. Thus we assume that $k > [p]$ and get

$$\begin{aligned} px - \sum_{j:\,\lambda_j<x} x + \sum_{j:\,\lambda_j<x} \lambda_j &= px - kx + \sum_{j=[p]+1}^{k} \lambda_j \ + \sum_{j=1}^{[p]} \lambda_j \\ &\leq px - kx + (k-[p])\lambda_k \ + \sum_{j=1}^{[p]} \lambda_j \\ &\leq (p-[p])x + \sum_{j=1}^{[p]} \lambda_j \leq (p-[p])\lambda_1 + \sum_{j=1}^{[p]} \lambda_j. \end{aligned}$$

The last case $0 \leq x < \lambda_1$ is trivial. Thus (A.19) yields the following:

Corollary A.3. *Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a positive, nondecreasing and unbounded sequence such that for all $x \geq 0$ holds*

$$\sum_{j\in\mathbb{N}\,:\,\lambda_j<x} (x-\lambda_j) \leq f(x),$$

where $f : [0,\infty) \to \mathbb{R}$. Then holds for all $n \in \mathbb{N}$

$$\sum_{j=1}^{n} \lambda_j \geq \sup_{x\geq 0}(nx - f(x)).$$

A.3 The inradius for domains with infinite volume

For $x \in \Omega$, we consider the following

$$\delta_e(x) := \inf_{y\in\partial\Omega} \|x-y\|_e, \qquad \mathrm{R}_e(\Omega) := \sup_{x\in\Omega} \delta_e(x),$$

where $\|x\|_e$ is the Euclidean length of x in $\mathbb{R}^n$. The quantity $\mathrm{R}_e(\Omega)$ is called the inradius.

Proposition A.4. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain such that $|\Omega| < \infty$. Then there exists a point $p \in \Omega$ such that*

$$B_{\mathrm{R}_e(\Omega)}(p) \subseteq \Omega, \tag{A.20}$$

where $B_{\mathrm{R}_e(\Omega)}(p) := \{y \in \mathbb{R}^n \,:\, \|p-y\|_e < \mathrm{R}_e(\Omega)\}$.

Proof. This proposition is trivial if Ω is bounded because $\overline{\Omega}$ is then compact and $\delta_e(\cdot)$ continuous since one can easyily check that

$$|\delta_e(x) - \delta_e(y)| \leq \|x-y\|_e, \quad y, x \in \mathbb{R}^n,$$

where $\delta_e(x) := 0$ for $x \in \Omega^c$.

Let us assume that Ω is an unbounded domain with $|\Omega| < \infty$. Since Ω is a nonempty domain, one can easily check that $\partial\Omega \neq \emptyset$. First of all we show that

$$\mathrm{R}_e(\Omega) = \sup\{R > 0 \,|\, \exists a \in \Omega \text{ such that } B_R(a) \subseteq \Omega\}. \tag{A.21}$$

Let $R > 0$ and $a \in \Omega$ such that $B_R(a) \subseteq \Omega$. It follows then

$$R \leq \delta_e(a) \leq \mathrm{R}_e(\Omega).$$

Taking the supremum yields the first inequality. For the next one we take the sequence $\delta_e(x_n) \to \mathrm{R}_e(\Omega)$ for $n \to \infty$ with $x_n \in \Omega$. We show that $B_{\delta_e(x_n)}(x_n) \subseteq \Omega$ for all $n \in \mathbb{N}$. Let us assume that there exists an $n \in \mathbb{N}$ such that there exists a point $x \in B_{\delta_e(x_n)}(x_n)$

which does not lie in Ω. Then we consider the convex combination $\varphi(t) = (1-t)x_n + tx$ for $t \in [0,1]$ and know that there exists a $\tilde{t} \in (0,1)$ such that $\varphi(\tilde{t}) \in \partial\Omega$. We obtain then

$$\delta_e(x_n) \leq \left\| x_n - \varphi(\tilde{t}) \right\|_e \leq \| x_n - x \|_e < \delta_e(x_n),$$

yielding a contradiction. Thus $B_{\delta_e(x_n)}(x_n) \subseteq \Omega$ holds for all $n \in \mathbb{N}$ and we know that (A.21) holds.

Since $|\Omega| < \infty$, we see that $\mathrm{R}_e(\Omega) < \infty$ because (A.21) holds. In addition we obtain a sequence of R_n and a_n such that for all $n \in \mathbb{N}$ holds

$$B_{R_n}(a_n) \subseteq \Omega \tag{A.22}$$

with $R_n \to \mathrm{R}_e(\Omega)$ for $n \to \infty$. Without loss of generality we assume that R_n is strictly increasing. We show that a_n is a bounded sequence. Let us assume that a_n is unbounded. Then we obtain

$$\bigcup_{n=1}^{\infty} B_{R_1}(a_n) \subseteq \bigcup_{n=1}^{\infty} B_{R_n}(a_n) \subseteq \Omega.$$

We can find as subsequence n_k of a_n such that $B_{R_1}(a_{n_k})$ are disjoint for all $k \in \mathbb{N}$ because a_n is unbounded, which contradicts $|\Omega| < \infty$. Thus, we know that a_n and R_n are bounded, meaning that

$$A := \bigcup_{n \in \mathbb{N}} B_{R_n}(a_n)$$

is bounded, too. Obviously we get then by $A \subseteq \Omega$ the following

$$R_n \leq \sup_{x \in A} \delta_e(x) \leq \sup_{x \in \Omega} \delta_e(x) = \mathrm{R}_e(\Omega).$$

Taking the limit we obtain that the inradius of Ω is the same as the one of A. Since A is bounded, we know that there exists a point $p \in \mathbb{R}^n$ such that

$$B_{\mathrm{R}_e(\Omega)}(p) \subseteq A \subseteq \Omega,$$

which gives the proof. □

Remark A.5. Proposition A.4 also holds if we replace the Euclidean distance by the C-C metric d_C induced by some vector fields on $\mathbb{R}^n$ under the following assumption: the generated C-C metric must fulfill that any two points lying in $\mathbb{R}^n$ are connected by a (not necessarily unique) geodesic and that for any given compact set $K \subset \mathbb{R}^n$ the identity function $(K, \|\cdot\|_e) \mapsto (K, d_C)$ is Hölder continuous.

A.4 On the volume of convex domains

Let $\Omega \subseteq \mathbb{R}^n$ be open bounded and convex. In this section we give a uniform bound on the volume of points lying in the outer parallel set of Ω. Hence we define the following quantities

$$\Omega^\beta := \{x \in \Omega \,|\, \delta_e(x) < \beta\}, \qquad \delta_e(x) := \inf_{y \in \partial\Omega} \|x - y\|_e$$

for any $\beta \in (0, \mathrm{R}_e(\Omega)]$, where $\mathrm{R}_e(\Omega) := \sup_{x \in \Omega} \delta_e(x)$ denotes the inradius of Ω. The n-dimensional Lebesgue measure of Ω is denoted by $|\Omega|$. In this section we want to prove the following:

Theorem A.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded convex set. Then we have*

$$\frac{|\Omega^\beta|}{\beta} \geq \frac{|\Omega|}{\mathrm{R}_e(\Omega)}$$

for all $\beta \in (0, \mathrm{R}_e(\Omega)]$.

This result was stated the first time in [KW15, Lemma 4.2]. However, it was pointed out by S. Larson that a mistake appeared in the proof. The authors used that if Ω has smooth boundary the inner parallel set $\{x \in \Omega \mid \delta_e(x) \geq \beta\}$ has smooth boundary as well, which generally is not true, not even in the convex case, see [Lar15]. S. Larson also explained an alternative proof, which is presented here:

Proof of Theorem (A.6): First of all let us denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. Let us recall the Eikonal equation

$$|\nabla \delta_e(x)| = 1, \quad \text{for a.e. } x \in \Omega$$

which in combination with the coarea formula yields

$$|\Omega^\beta| = \int_0^\beta \mathcal{H}^{n-1}(\{x \in \Omega | \delta_e(x) = t\}) \, \mathrm{d}t \tag{A.23}$$

Hence, we know that $|\Omega^\beta|$ is absolutely continuous on $[0, \mathrm{R}_e(\Omega)]$ and therefore almost everywhere differentiable on $[0, \mathrm{R}_e(\Omega)]$. Let us consider the function $f(\beta) := |\Omega^\beta|\beta^{-1}$ and compute its derivative

$$f'(\beta) = \frac{\mathcal{H}^{n-1}(\{x \in \Omega | \delta_e(x) = \beta\})\beta - \int_0^\beta \mathcal{H}^{n-1}(\{x \in \Omega | \delta_e(x) = t\}) \, \mathrm{d}t}{\beta^2}.$$

Since Ω is convex, we know that $\{x \in \Omega | \delta_e(x) \geq \beta\}$ is convex, too. Thus for $\beta_1 < \beta_2$ we get

$$\{x \in \Omega | \delta_e(x) \geq \beta_2\} \subseteq \{x \in \Omega | \delta_e(x) \geq \beta_1\}.$$

From [Web94, p. 295] we know that for two convex compact domains with $A \subseteq B$ the surface area of A is less equal than the surface area of B. From [Fed69, p.271] we know that the surface area and the $(n-1)$-dimensional Hausdorff measure of the boundary of convex sets match up to a multiple constant. Thus for $\beta_1 < \beta_2$ we obtain

$$\mathcal{H}^{n-1}(\{x \in \Omega | \delta_e(x) = \beta_2\}) \subseteq \mathcal{H}^{n-1}(\{x \in \Omega | \delta_e(x) = \beta_1\}).$$

An immediate consequence is then that $f'(\beta) \leq 0$ almost everywhere on $[0, \mathrm{R}_e(\omega)]$. Let us fix $a \in (0, \mathrm{R}_e(\Omega)]$. We can easily prove using (A.23) that the function $f(\beta)$ is absolutely continuous on $[a, \mathrm{R}_e(\Omega)]$. Thus, we can apply the fundamental theorem of calculus and use the negativity of $f'(\beta)$ almost everywhere on $[a, \mathrm{R}_e(\Omega)]$ to prove that the function $f(\beta)$ is decreasing on $[a, \mathrm{R}_e(\Omega)]$. Hence, we immediately obtain

$$\frac{|\Omega^a|}{a} \geq \frac{|\Omega|}{\mathrm{R}_e(\Omega)}$$

for any $a \in (0, \mathrm{R}_e(\Omega)]$, yielding the desired result.

$\square$

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