Available online at www.sciencedirect.com

ScienceDirect

Procedia IUTAM 13 (2015) 127 – 137

Procedia IUTAM

www.elsevier.com/locate/procedia

IUTAM Symposium on "Dynamical Analysis of Multibody Systems with Design Uncertainties"

Sensitivity computation for uncertain dynamical systems using high-dimensional model representation and hierarchical grids

Nico-Philipp Walz*, Markus Burkhardt, Michael Hanss, Peter Eberhard

Institute of Engineering and Computational Mechanics, University of Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany

Abstract

Global sensitivity analysis is an important tool for uncertainty analysis of systems with uncertain model parameters. A general framework for the determination of sensitivity measures for fuzzy uncertainty analysis is presented. The derivation is founded on the high-dimensional model representation, which provides a common basis with Sobol indices, illustrating the similarities and differences of fuzzy and stochastic uncertainty analysis. For the numerical calculation, a sparse-grid approach is suggested, providing an efficient realization due to the direct relationship between hierarchical grids and the sensitivity measures.

© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Peer-review under responsibility of organizing committee of Institute of Engineering and Computational Mechanics University of Stuttgart.

Keywords: Fuzzy Uncertainty Analysis; Global Sensitivity; High Dimensional Model Representation; Sparse-Grid Methods

1. Introduction

Uncertainties are almost always present in every stage of engineering practice. For mechanical design, controller design or parameter estimation, the consideration of inexact data and uncertain parameters due to a lack of information or due to simplifying assumptions may enhance the understanding of the system under investigation and its potential discrepancies or deficiencies with respect to the real system. An important part of these investigations is the quantification of the individual contributions of the uncertain model parameters to the overall uncertainty arising in the result of the uncertainty analysis, i.e. the calculation of sensitivity measures.

The knowledge of sensitivities of specific parameters may greatly improve the initial design and provides information about where additional effort is most effectively concentrated on: for instance, in order to improve the robustness of the design with respect to the uncertainties or to eliminate the source of the uncertainty as far as possible.

For parameters which are uncertain due to a lack of knowledge, i.e for model parameters in the presence of idealizations or simplifications during the modeling process, fuzzy numbers provide a convenient way of modeling the possible non-determinism. As the assumption of infinitesimal deviations from the nominal system is not valid in this case, classical sensitivity analysis is not recommended. The purpose of this contribution is to develop a gen-

* Corresponding author. Tel.: +49-711-685-66388 ; fax: +49-711-685-66400.
E-mail address: nico.walz@itm.uni-stutttgart.de

2210-9838 © 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Peer-review under responsibility of organizing committee of Institute of Engineering and Computational Mechanics University of Stuttgart.
doi:10.1016/j.piutam.2015.01.010

eral framework that provides the possibility of quantifying sensitivities of fuzzy-parameterized dynamical systems, including a computational scheme for its numerical calculation. In contrast to the existing influence measures defined within the transformation method [1,2] the definition is independent from a specific implementation of the fuzzy analysis. Nevertheless, the basic concept for the derivation has a strong relation to those.

At first, the concept of the high-dimensional model representation (HDMR) will be reviewed, which is an additive but finite decomposition of functions into component functions of increasing dimensionality [3]. Such decompositions are often used to simplify the solution or analysis process by exploiting a special structure or formulation. In this sense, the HDMR is perfectly suited for the derivation of sensitivity measures. This has already been pointed out by Sobol for a special type of the HDMR, where the components are designed to have zero mean and, accordingly, the sensitivities are related to variances [4]. These sensitivities are known as global sensitivity indices, sometimes also called Sobol indices.

Here, it will be shown that the HDMR is applicable to sensitivity analysis for fuzzy-parameterized systems as well, by a proper choice of the construction of the component functions. Hence, the derivation of influence measures is straight forward.

After this theoretical definition, the question arises how this sensitivity or influence measure can practically be approximated in an efficient way. The connection of hierarchical grids to the structure of the HDMR [5] makes sparse-grid methods a natural candidate for the numerical solution. The choice of the basis functions for the construction of the approximation space of the sparse grids is directly related to the definition of the HDMR component functions.

Since sparse grids are constructed from tensor products, they can be used as a basis for any linear operation like interpolation and integration [6]. Therefore, by exploiting the connection of sparse grids and the HDMR, the decomposition, i.e. the component functions, are given by certain subspaces of the sparse grid approximation space. Consequently, the sensitivity measures, which are basically given as integrals of these components, may be computed by applying sparse-grid integration [7].

Finally, in order to conclude the paper a brief description of an exemplary application of the sensitivity measure is given.

1.1. Notation

Vectors and multi-indices are printed in bold face, e.g. $\boldsymbol{x} \in \mathbb{R}^n$ is a real vector and $\boldsymbol{l} \in \mathbb{N}_0^m$ a multi-index. The scalar elements are denoted as $x_i \in \mathbb{R}$ and $l_j \in \mathbb{N}_0$, respectively. Additionally, ordered tuples $\mathfrak{U}$ are used to name a selection of elements of vectors or multi-indices, so that $\boldsymbol{x}_\mathfrak{U}$ describes a subset of $\boldsymbol{x}$ where $\mathfrak{U} \subseteq \{1, \ldots, n\}$ with $\boldsymbol{x}_\mathfrak{U} = (x_i \mid i \in \mathfrak{U})$ and the complementary entries are denoted $\boldsymbol{x}_{\not\mathfrak{U}} = (x_i \mid i \in \not\mathfrak{U} = \{1, \ldots, n\} \setminus \mathfrak{U})$.

Binary operations on multi-indices $\boldsymbol{l}, \boldsymbol{k}$ are performed element-wise, for instance

$$\boldsymbol{l} < \boldsymbol{k} \qquad \text{if} \quad l_i < k_i \quad \text{for} \quad i = 1, \ldots, m \,,$$

and the commonly used definitions $|\boldsymbol{l}| = |\boldsymbol{l}|_1 = l_1 + \ldots + l_m$ and $|\boldsymbol{l}|_\infty = \max_i l_i$ as well as $|\boldsymbol{l}|_0 = m$ will be used.

2. Sensitivity measures for fuzzy uncertainty analysis

Fuzzy numbers are a special type of fuzzy sets, namely normal and convex fuzzy sets. The fuzzy parameters $\widetilde{\boldsymbol{p}} \subseteq \mathbb{R}^n$ are defined by their upper-semi-continuous membership functions $\mu_{\widetilde{p}_i} \in [0, 1]$, specifying the membership of elements of the underlying universal set $x \in \mathbb{R}$ to the fuzzy set or fuzzy number, respectively. The normality condition enforces that there is only one specific value $\bar{p}_i$ with $\mu_{\widetilde{p}_i}(\bar{p}_i) = 1$. The support of a fuzzy parameter is consequently given by the set that satisfies the condition $\mu_{\widetilde{p}_i}(x) > 0$. For a detailed introduction into fuzzy set theory and fuzzy arithmetic, see e.g. [8,1].

Sensitivity analysis, in a very general sense, deals with the rating of the sensitivity of a system with respect to a set of the system parameters. Of course, this may only be accomplished for some specified configuration and output quantity of the system. Since nonlinear behavior in the range of the possible parameter variations may not be negligible, a simple sensitivity calculation based on partial derivatives is not applicable in uncertainty analysis.

In the following, the problem of quantifying the sensitivity of a function $f : [0, 1]^n \to \mathbb{R} : \boldsymbol{x} \mapsto y = f(\boldsymbol{x})$ will be addressed. Here, f is an arbitrary scalar, n-variate mapping of some vector space $\mathcal{V}$, which usually will be assumed to be a product space $\mathcal{V} = V_1 \times \cdots \times V_n$ consisting of spaces of at least continuous functions $V_i = C^0[0, 1]$.

Note that for a function $g : \mathcal{P} \to \mathbb{R}$ with parameter domain $\mathcal{P}$, where $\mathcal{P} = [a_1, b_1] \times \ldots \times [a_n, b_n] \subsetneq \mathbb{R}^n$ with $[a_i, b_i]$ representing closed intervals, which may be the supports of the fuzzy parameters $\widetilde{\boldsymbol{p}}$, the following derivations are still applicable by introducing an affine transformation $\eta : \mathcal{P} \to [0,1]^n : \boldsymbol{\xi} \mapsto \boldsymbol{x}$ such that $g(\xi) = f(\eta(\xi)) = f(\boldsymbol{x})$. This transformation ensures that the calculated sensitivity quantities are scale invariant, i.e. independent of the widths of the intervals $[a_i, b_i]$.

The goal of the sensitivity analysis is to quantify the influence of the individual parameters $\widetilde{p}_i$ or subsets of the parameter vector $\widetilde{\boldsymbol{p}}_{\mathfrak{U}}$ on the result or output $\widetilde{q} = f(\widetilde{\boldsymbol{p}})$. This means that the analysis shall provide a quantity which relates the permissible deviations in $\boldsymbol{x}_{\mathfrak{U}}$ to the resulting deviation of the output y.

2.1. High-dimensional model representation

The high-dimensional model representation (HDMR) is a general method for the decomposition of a function f, as defined above, into functions of increasing dimensionality [3]. This may be written as

$$f(\boldsymbol{x}) = f_\emptyset + \sum_{1 \le i_1 \le n} f_{i_1}(x_{i_1}) + \ldots + \sum_{1 \le i_1 < \ldots < i_k \le d} f_{i_1,\ldots,i_k}(x_{i_1}, \ldots, x_{i_k}) + \cdots + f_{1,2,\ldots,n}(\boldsymbol{x}) = \sum_{\mathfrak{U} \subseteq \{1,\ldots,n\}} f_{\mathfrak{U}}(\boldsymbol{x}_{\mathfrak{U}}) \,. \tag{1}$$

The decomposition has thus a finite number of component functions, amounting to a total of 2^n elements. Note that the latter, compact notation includes $f_\emptyset$ for the empty tuple $\mathfrak{U} = \emptyset$, which in other publications often is termed f_0, see e.g. [3]. Hereby, $f_\emptyset$ is a constant, while the terms f_{i_1} and $f_{i_1,\ldots,i_k}$ are univariate and k-variate functions, respectively.

The single components may be interpreted as the projections of $f \in \mathcal{V}$ onto orthogonal subspaces of the vector space $\mathcal{V}$, i.e $f_{\mathfrak{U}} \in \mathcal{W}_{\mathfrak{U}} \subsetneq \mathcal{V}$ such that $\mathcal{V} = \bigoplus_{\mathfrak{U} \subseteq \{1,\ldots,n\}} \mathcal{W}_{\mathfrak{U}}$. The subspaces $\mathcal{W}_{\mathfrak{U}}$ thereby contain functions that are constant in the directions not contained in $\mathfrak{U}$.

Given a projector $P_{\mathfrak{U}}$, i.e. an idempotent mapping with $P_{\mathfrak{U}} P_{\mathfrak{U}} = P_{\mathfrak{U}}$ and $\ker P_{\mathfrak{U}} = \operatorname{range}(I - P_{\mathfrak{U}})$ with the identity operator I, and the distinct index tuples $\mathfrak{U}, \mathfrak{H} \subseteq \{1, \ldots, d\}$, $\mathfrak{U} \neq \mathfrak{H}$, it holds that $f_{\mathfrak{U}} = P_{\mathfrak{U}} f$ and $P_{\mathfrak{H}} f_{\mathfrak{U}} = 0$, which implies that $\operatorname{range} P_{\mathfrak{U}} P_{\mathfrak{H}} = \operatorname{range} P_{\mathfrak{H}} P_{\mathfrak{U}} = \emptyset$.

The projectors $P_{\mathfrak{U}}$ may be constructed from a one-dimensional projector $P^i_\emptyset(\bullet) = \int (\bullet) \, \mathrm{d}\lambda(x_i)$. For a function $f(x_i) \in V_i$ it is seen that $f_\emptyset = P^i_\emptyset f(x_i)$ and $f_1(x_i) = P^i f(x_i) = (I^i - P^i_\emptyset) f(x_i) = f(x_i) - f_\emptyset$, since $\operatorname{range} P^i = \ker P^i_\emptyset = \{g \in V_i \,|\, P^i_\emptyset g = 0\}$ for univariate functions. The multidimensional projector results from a tensor product by using the one-dimensional operators for every single direction. Thus, the projector of the constant component may be written as

$$P_\emptyset = \bigotimes_{i=1}^{n} P^i_\emptyset = \prod_{i=1}^{n} P^i_\emptyset \,. \tag{2}$$

Analogously, for the higher-order terms the complementary projectors for the directions contained in $\mathfrak{U}$ have to be used, yielding

$$P_{\mathfrak{U}} = \prod_{i \notin \mathfrak{U}} P^i_\emptyset \prod_{j \in \mathfrak{U}} P^j = \prod_{i \notin \mathfrak{U}} P^i_\emptyset \prod_{j \in \mathfrak{U}} (I^j - P^j_\emptyset) \,. \tag{3}$$

Using this definition of $P_{\mathfrak{U}}$, an individual component function is recursively defined by the lower-order components, see e.g. [5], as

$$f_{\mathfrak{U}} = \left(\prod_{i \notin \mathfrak{U}} P^i_\emptyset \prod_{j \in \mathfrak{U}} I^j \right) f - \sum_{\mathfrak{H} \subsetneq \mathfrak{U}} f_{\mathfrak{H}} \,. \tag{4}$$

The projector thereby eliminates the dependence on $x_{\mathfrak{U}}$, and the lower-order components are subtracted to remove the influence of interactions where not all variables in $x_{\mathfrak{U}}$ are involved. Hence, a component function $f_{\mathfrak{U}}$ represents the dependence of f on the group of variables $x_{\mathfrak{U}}$, neglecting all lower-order components $f_{\mathfrak{H}}$ with $\mathfrak{H} \subsetneq \mathfrak{U}$.

In contrast to stochastic sensitivity measures, where the projector is chosen such that the component functions are mean-value free – by choosing $\mathrm{d}\lambda(x_j) = \mathrm{d}x_j$, i.e. the standard Lebesgue measure – leading to the well-known

ANOVA decomposition, in fuzzy uncertainty analysis the component functions have to have zero nominal value in order to describe the deviations from the nominal value, and hence, the influences of the parameters.

The projection of f onto the nominal value is achieved by using the Dirac measure δ centered at the nominal parameter values $\bar{\boldsymbol{p}}$. This type of HDMR is also known as cut-HDMR or anchor-ANOVA, since f is split along hyperplanes of increasing dimensionality and is anchored at a certain point. The cut-HDMR projector is thus given by

$$P_{\emptyset}^{j}(\bullet) = \int (\bullet)\, \delta(x_j - \bar{p}_j)\, \mathrm{d}x_j\,, \tag{5}$$

so that indeed the nominal value is recovered at the anchor point $\bar{\boldsymbol{p}}$ by applying $P_{\emptyset}$, i.e.

$$f_{\emptyset} = P_{\emptyset} f(\boldsymbol{x}) = \prod_{i=1}^{n} P_{\emptyset}^{i} f(\boldsymbol{x}) = \int \cdots \int f(x_1, \ldots, x_n)\, \delta(x_1 - \bar{p}_1) \cdots \delta(x_n - \bar{p}_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = f(\bar{\boldsymbol{p}})\,. \tag{6}$$

Using Eqs. (4) and (5), for an arbitrary $\mathfrak{U}$, the component function results in

$$f_{\mathfrak{U}}(\boldsymbol{x}_{\mathfrak{U}}) = P_{\mathfrak{U}} f(\boldsymbol{x}) = f(\boldsymbol{x}_{\mathfrak{U}}, \bar{\boldsymbol{p}}_{\mathfrak{U}}) - \sum_{\mathfrak{H} \subsetneq \mathfrak{U}} f(\boldsymbol{x}_{\mathfrak{H}}, \bar{\boldsymbol{p}}_{\mathfrak{H}})\,. \tag{7}$$

In simple terms, the component functions $f_{\mathfrak{U}}$ of the cut-HDMR are computed by fixing the variables $x_{\mathfrak{U}}$ at the nominal values $\bar{p}_{\mathfrak{U}}$ and subtracting the lower-order components.

2.2. Measuring the influence of fuzzy parameters

As described, the HDMR components contain the variation of f caused exclusively by the interaction of a group of variables. For analyses using cut-HDMR, this deviation is measured relative to the anchor point, i.e. to the nominal values of the fuzzy parameters, and the hyperplanes cutting this anchor point.

There are two obvious choices for the definition of sensitivities in this setting: on the one side, the variation of f given by the interaction of variables that are contained in the group of variables $x_{\mathfrak{U}}$, and on the other side, the variation of f which is encoded in all components where variables of the group $\mathfrak{U}$ are encompassed, i.e. all components $f_{\mathfrak{H}}$ with $\mathfrak{H} \cap \mathfrak{U} \neq \emptyset$ are considered. For the latter case, higher-order components are additionally included, and in both cases, $f_{\emptyset}$ is of course excluded.

A sensitivity measure for the exclusive variability of f due to the variables $x_{\mathfrak{U}}$ may thus be calculated as the integral

$$K_{\mathfrak{U}} = \int \sum_{\substack{\mathfrak{H} \subseteq \mathfrak{U} \\ \mathfrak{H} \neq \emptyset}} f_{\mathfrak{H}}^{2}(\boldsymbol{x}_{\mathfrak{H}})\, \mathrm{d}M(\boldsymbol{x}_{\mathfrak{U}})\,. \tag{8}$$

The second measure which expresses all non-trivial interactions of $x_{\mathfrak{U}}$ may be defined by

$$K_{\mathfrak{U}}^{\mathrm{T}} = \int \sum_{\mathfrak{H} \cap \mathfrak{U} \neq \emptyset} f_{\mathfrak{H}}^{2}(\boldsymbol{x}_{\mathfrak{H}})\, \mathrm{d}M(\boldsymbol{x}_{\mathfrak{U}})\,. \tag{9}$$

In correspondence with the influence measures defined in the Transformation Method, these quantities are called absolute sensitivity measures.

A normalization of the defined measures using the total squared variation of f from $f_{\emptyset}$ yields two sensitivity measures $S_{\mathfrak{U}}$ and $S_{\mathfrak{U}}^{\mathrm{T}}$, providing a parameter-independent and dimensionless scale. These measures are defined as

$$S_{\mathfrak{U}} = \frac{K_{\mathfrak{U}}}{K} \qquad \text{and} \qquad S_{\mathfrak{U}}^{\mathrm{T}} = \frac{K_{\mathfrak{U}}^{\mathrm{T}}}{K} \tag{10}$$

$$\text{with} \quad K = \int_{[0,1]^n} (f(\boldsymbol{x}) - f_{\emptyset})^2\, \mathrm{d}M(\boldsymbol{x})\,. \tag{11}$$

Note that these definitions are equal to Sobol's global sensitivity indices and global total sensitivity indices when applying the ANOVA decomposition instead of the cut-HDMR[4].

Both absolute and normalized measures offer a distinct view on the problem. If, for instance, a trajectory is analyzed, the measure is time-dependent, and hence, the absolute measure gives information on the total variability at each time instant. The normalization improves the interpretability of the resulting values, since values significantly smaller than unity now indicate unimportant parameters. A comparison between the sensitivity measure $S_{\mathfrak{u}}$ and the total sensitivity measure $S_{\mathfrak{u}}^{\mathrm{T}}$ indirectly gives information on the importance of the higher-order interactions of $\boldsymbol{x}_{\mathfrak{u}}$.

As deviations of f from $f_{\emptyset}$ in the vicinity of $\bar{\boldsymbol{p}}$, i.e. with a high degree of membership, are considered more important, a weighting of the integrand is introduced, using a measure that is proportional to the membership value. In order to ensure scale invariance, the measure is defined such that it has unity mass $\int_{[0,1]^k} \mathrm{d}M(\boldsymbol{x}) = 1$ for any $1 \le k \le n$. Since the problem involves several fuzzy parameters, its density m with $\mathrm{d}M(\boldsymbol{x}) = m(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is given by an implication operation applied to the membership functions of the fuzzy parameters, i.e. some t-norm operator $t(\boldsymbol{x}) = t\left(\mu_{\widetilde{p}_1}(x_1), \ldots, \mu_{\widetilde{p}_n}(x_n)\right)$. Under these conditions, the density m of M can be written as

$$m(\boldsymbol{x}) = \frac{t(\boldsymbol{x})}{\int_{[0,1]^n} t(\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}}\,. \tag{12}$$

The most obvious choices for the t-norm are the minimum $t_{\min}(\boldsymbol{x}) = \min\left\{\mu_{\widetilde{p}_1}(x_1), \ldots, \mu_{\widetilde{p}_n}(x_n)\right\}$ and the algebraic product $t_{\mathrm{prod}}(\boldsymbol{x}) = \prod_{i=1}^{n} \mu_{\widetilde{p}_i}(x_i)$. The first one corresponds to the implication of the extension principle, while the latter one, due to the fact that it is a simple product measure, yields smoother integrands and is hence better suited for numerical computations.

3. Numerical approximation using sparse grids

The HDMR decomposition (1) is an exact representation of a mapping f in terms of its component functions. However, for many relevant applications the mapping f is given as a numerical solution procedure. Since the decomposition can thus not be conducted analytically, for the practical computation of the sensitivity measures as defined in Eqs. (8) and (9) an efficient approximation scheme is necessary. Especially the solution of multi-dimensional integrals is a computationally demanding task.

Sparse-grid methods, which rely on a tensor-product construction, are getting increasing attention as a general numerical approximation scheme in higher dimensions, since arbitrary one-dimensional linear operations may be extended to multiple dimensions effectively, while reducing the curse of dimensionality significantly compared to conventional tensor-product methods[6]. The applications of sparse grids are becoming more diverse, including of course the basic operations of interpolation and integration , which will be shortly reviewed in this section as well. As already pointed out in[5], sparse grid decompositions may be interpreted as discrete approximations of HDMRs where the type of the HDMR is specified by the choice of the basis functions. Hence, sparse grids provide a natural and efficient approximation scheme for the sensitivity computations, as will be shown in the following.

3.1. Basic theory of hierarchical grids

In order to find a reasonable approximation $\hat{f}$ of f, a suitable approximation space has to be employed. The approximation is based on point evaluations of f, so for this purpose, basis functions and a grid of corresponding support nodes, i.e. a discrete set of points $X \subsetneq [0,1]^n$ are specified, where f is to be evaluated. The knowledge of the function values at the grid points in conjunction with the basis functions allows for an estimation of f in the whole unit box. Note that the terms basis and grid may often be used interchangeably, since each basis is associated with an ensemble of grid points.

A sparse grid may be derived from a conventional full-grid discretization $\mathcal{V}_{\boldsymbol{l}}$, which is a Cartesian grid or product space $\mathcal{V}_{\boldsymbol{l}} = V_{l_1} \times \cdots \times V_{l_n}$, where the multi-index $\boldsymbol{l}$ specifies the discretization level for each direction. For the derivations made here, an equidistant discretization in each direction is assumed, although other schemes are possible as well as homogeneous boundary conditions so that the boundary nodes may be omitted. A one-dimensional grid of level l is thus given by its grid points $x_{l,i}$ as a set $X_l = \{x_{l,i} = ih \mid i = 1, \ldots, 2^l - 1\}$ with mesh size $h = 2^{-l}$. In the same manner a sequence of bases $\Phi_l = \{\phi_{l,i}\}_{i=1}^{2^l-1}$ for each level of the one-dimensional spaces $V_l = \mathrm{span}\{\Phi_l\}$ is specified.

The tensor-product construction yields the approximation space $\mathcal{V}_{\boldsymbol{l}} = \text{span}\{\boldsymbol{\Phi}_{\boldsymbol{l}}\} = \text{span}\{\Phi_{l_1} \otimes \cdots \otimes \Phi_{l_n}\}$ with n-dimensional basis functions $\phi_{\boldsymbol{l},\boldsymbol{i}} \colon [0,1]^n \mapsto \mathbb{R}$ as

$$\phi_{\boldsymbol{l},\boldsymbol{i}} = \bigotimes_{j=1}^{n} \phi_{l_j,i_j} = \prod_{j=1}^{d} \phi_{l_j,i_j} \,. \tag{13}$$

The associated grid points are obtained by the Cartesian product of the point sets, i.e. $x_{\boldsymbol{l},\boldsymbol{i}} \in \mathcal{X}_{\boldsymbol{l}} = X_{l_1} \times \cdots \times X_{l_n}$. So, the number of basis functions and grid points accumulates to $2^{|\boldsymbol{l}|_1}$ for the full grid, which illustrates the curse of dimensionality for a full-grid discretization.

For the sparse-grid representation, a construction of a hierarchical basis is required. In contrast to conventional discretization schemes, where a higher resolution demands a complete change of basis and often a recalculation of the grid values, a hierarchical basis allows for a higher resolution by the additional contribution of basis functions from a higher hierarchical level, while keeping already considered basis functions and information of the lower hierarchical grids.

Therefore, the space $\mathcal{V}_{\boldsymbol{l}}$ is now decomposed into hierarchical subspaces, where each subspace itself is of tensor-product structure, so that tensor-product operations can be applied individually in the subspaces. Hence, although tensor-product operators are applied, the resulting grid does not have to have Cartesian structure, which is the basic idea of sparse grids, and thus gives the opportunity to optimize the convergence rates.

Defining the incremental complement spaces $W_{\boldsymbol{k}}$ such that $W_{\boldsymbol{k}} = \{x_{\boldsymbol{k},\boldsymbol{i}} \in \mathcal{X}_{\boldsymbol{k}} \mid \boldsymbol{i} \text{ odd}\}$, i.e. the grid only contains the support nodes of the full grid $\mathcal{V}_{\boldsymbol{k}}$ with odd indices $\boldsymbol{i}$, the complement spaces are disjunctive, and it holds that $V_{\boldsymbol{l}} = \bigoplus_{\boldsymbol{k} \le \boldsymbol{l}} W_{\boldsymbol{k}}$.

The interpolation of f in terms of these hierarchical subspaces may now be written as

$$\hat{f}(\boldsymbol{x}) = I(f)(\boldsymbol{x}) = \sum_{(\boldsymbol{l},\boldsymbol{i}) \in G} v_{\boldsymbol{l},\boldsymbol{i}} \phi_{\boldsymbol{l},\boldsymbol{i}}(\boldsymbol{x}) \,, \tag{14}$$

where the coefficients $v_{\boldsymbol{l},\boldsymbol{i}}$ are the so-called *hierarchical surpluses*. The grid G specifies which subspaces $W_{\boldsymbol{k}}$ are to be used for the interpolation. Setting $G = G^{\infty}(N) = \{(\boldsymbol{l},\boldsymbol{i}) \mid 1 \le |\boldsymbol{l}|_{\infty} \le N, \boldsymbol{1} \le \boldsymbol{i} \le 2^{\boldsymbol{l}} - 1, \boldsymbol{i} \quad \text{odd}\}$, Eq. (14) is equal to the interpolation on the full grid $\mathcal{V}_N = \mathcal{V}_{(N,\ldots,N)}$ of equal mesh size $h = 2^{-N}$ in every direction.

The quadrature of f can simply be stated as the integration of the interpolation formula (14), see also[7], which yields

$$\int_{[0,1]} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \approx Q(f) = \sum_{(\boldsymbol{l},\boldsymbol{i}) \in G} v_{\boldsymbol{l},\boldsymbol{i}} \int_{[0,1]} \phi_{\boldsymbol{l}(\boldsymbol{x})\boldsymbol{i}} \mathrm{d}\boldsymbol{x} = \sum_{(\boldsymbol{l},\boldsymbol{i}) \in G} v_{\boldsymbol{l},\boldsymbol{i}} w_{\boldsymbol{l},\boldsymbol{i}} \,. \tag{15}$$

The weights $w_{\boldsymbol{l},\boldsymbol{i}}$ can be computed in advance if the basis functions are known beforehand, which is usually the case.

3.2. Hierarchical basis functions

In general, arbitrary bases can be constructed for a sparse-grid discretization as long as the hierarchical structure is assured. For certain applications, more specific requirements may be important. In particular for space-adaptive sparse grids, basis functions with only local support are advantageous from the implementational point of view.

Here, only piecewise linear bases will be considered, which is, of course, one of the most simple basis type. Still, it suffices to derive both cut-HDMR and ANOVA discretizations. In addition, the weighting with the composition of the membership functions of the parameters enforces homogeneous boundary conditions for the sensitivity measures. Thus, all basis functions should vanish on the boundary as well.

The probably most common piecewise linear basis is the hat basis defined as

$$\phi_i(x) = \begin{cases} 1 - \left|\frac{x_i - x}{h}\right| & \text{for} \quad x \in [x_i - h, x_i + h] \\ 0 & \text{otherwise} \end{cases} \,. \tag{16}$$

A major advantage of this basis is that it has compact support such that the supports of basis functions of the same level are only connected via the boundaries. This property of non-overlapping supports leads to simplified algorithms, in particular in the case of adaptive refinement strategies.

A basis which spans the same space as the hat basis is the linear pre-wavelet basis[5]. The pre-wavelet basis is semi-orthogonal. Hence, basis functions of different levels are orthogonal with respect to the L_2-product, i.e. $\int_{[0,1]} \phi_{l,i}\, \phi_{k,j}\, \mathrm{d}x = 0$ for $l \neq k$. Setting the lowest-order basis as the constant unity function, it is immediately clear that in addition it holds that $\int_{[0,1]} \phi_{1,1}\, \mathrm{d}x = 1$ and $\int_{[0,1]} \phi_{l,i}\, \phi_{1,1}\, \mathrm{d}x = \int_{[0,1]} \phi_{l,i}\, \mathrm{d}x = 0$ for $l \neq 1$ by definition. This means that the integral of all basis functions with $l > 1$ vanishes and that they have zero mean.

Observe that, for non-vanishing boundary values, which may be the case if the component functions shall be interpolated without the membership weighting, boundary nodes have to be inserted or modifications of basis functions whose support touch the boundary have to be conducted. This can, however, be done in a straightforward manner[9].

3.3. Optimal subspace selection – sparse grids

Using the hierarchical grids with the hierarchical hat basis for interpolation, it is possible to show that the interpolation error with respect to the amount of grid points, for functions with bounded second-order mixed derivatives, is optimal in an L_2- and L_∞-sense, for a subgrid $G^1(N) = \{(\boldsymbol{l}, \boldsymbol{i}) \in G^\infty(N) \,|\, |\boldsymbol{l}|_1 \leq N\}$ of $G^\infty(N)$. That is a grid that contains the subspaces $W_{\boldsymbol{l}}$ where the sum of the level indices is at maximum N which results in $2^{|\boldsymbol{l}|_1 - n}$ grid points and is significantly less nodes than for the full grid. Compared to the full grid G^∞, especially the subspaces with the highest amount of grid nodes are omitted, since they contribute the least to improving the approximation quality. This grid structure is known as the regular sparse grid.

The refinement may of course also be accomplished by an adaptive procedure, both in space, i.e. a local refinement, and in dimension, i.e. certain dimensions are allowed to have higher maximum levels than others. The refinement is naturally implemented due to the hierarchical structure of the grid and can easily be controlled by the hierarchical surpluses, for details see e.g.[9,5,10].

3.4. The relation of hierarchical subspaces and HDMR components

A hierarchical basis as defined above may now be used to discretize the projectors $P_{\mathfrak{U}}$, which results in an approximation $\hat{P}_{\mathfrak{U}}$. The projection thus performs an approximation of the component functions so that

$$f \approx \hat{f} = \sum_{\mathfrak{U}} \hat{f}_{\mathfrak{U}} = \sum_{\mathfrak{U}} \hat{P}_{\mathfrak{U}} f\,. \tag{17}$$

This is may as well be interpreted as performing the interpolation of f as given in Eq. (14) on a restricted subset of subspaces $W_{\boldsymbol{l}}$, i.e. on subsets of basis functions which correspond to the range of $\hat{P}_{\mathfrak{U}}$.

By using the same discretization for the projector and the interpolation of f, it is easily concluded that $\hat{P}_{\mathfrak{U}}(f) = \hat{P}_{\mathfrak{U}}(\hat{f}) = P_{\mathfrak{U}}(\hat{f})$. This fact is useful for the identification of the subspaces of the sparse grid discretization that construct the component functions. Assuming that an approximation $\hat{f}$ of f has been calculated, resulting in a sparse grid G as in Eq. (14), approximations $\hat{f}_{\mathfrak{U}}$ of the component functions $f_{\mathfrak{U}}$ may be calculated by

$$\hat{f}_{\mathfrak{U}} = P_{\mathfrak{U}} \hat{f} = P_{\mathfrak{U}} \sum_{(\boldsymbol{l},\boldsymbol{i}) \in G} v_{\boldsymbol{l},\boldsymbol{i}}\, \phi_{\boldsymbol{l},\boldsymbol{i}}(\boldsymbol{x}) = \sum_{(\boldsymbol{l},\boldsymbol{i}) \in G} v_{\boldsymbol{l},\boldsymbol{i}}\, P_{\mathfrak{U}}\, \phi_{\boldsymbol{l},\boldsymbol{i}}(\boldsymbol{x}) \tag{18}$$

$$\overset{(3)}{=} \sum_{(\boldsymbol{l},\boldsymbol{i}) \in G} v_{\boldsymbol{l},\boldsymbol{i}} \prod_{k \in \mathfrak{U}} P^k_{\emptyset}\, \phi_{l_k,i_k} \prod_{j \in \mathfrak{U}} (I^j - P^j_{\emptyset})\, \phi_{l_j,i_j}\,. \tag{19}$$

The application of $P^j_{\emptyset}$ to the basis function results in

$$P^j_{\emptyset}\, \phi_{l_j,i_j} = \begin{cases} 1 & \text{if } l_j = 1 \\ 0 & \text{if } l_j > 1 \end{cases} \tag{20}$$

for every $j \in \{1, \ldots, n\}$ and for any realization of $P^j_{\emptyset}$. Hence, this is true for each basis that provides a suitable discretization of an HDMR, which is the case if $W_{\boldsymbol{1}}$ contains the n-dimensional constant function and the basis fulfills condition (20).

As already mentioned, the type of the HDMR is specified by the definition of the basis functions $\phi_{\boldsymbol{l},\boldsymbol{i}}$. For instance, by applying the presented hat basis, a discretized cut-HDMR is obtained, while using a pre-wavelet basis results in a discretized ANOVA decomposition, respectively[5]. When the spans of the basis of the discretized cut-HDMR and ANOVA decomposition are equivalent, they even may be transformed into each other.

In particular, it is obvious that it holds for the cut-HDMR and the corresponding hat basis, i.e. the point evaluation of the basis functions at the anchor point $P_{\emptyset}^{j}\phi_{l_j,i_j} = \phi_{l_j,i_j}(\bar{p}_j)$, and for the ANOVA and the pre-wavelet basis, i.e. the integration of the basis function $P_{\emptyset}^{j}\phi_{l_j,i_j} = \int_{[0,1]} \phi_{l_j,i_j}(x_j)\,\mathrm{d}x_j$.

This observation yields a simplified representation of the discrete component function, since with Eq. (19) and (20) it holds that

$$P_{\mathfrak{U}}\phi_{\boldsymbol{l},\boldsymbol{i}} = 0 \quad \text{if } l_k > 1 \text{ for } k \in \mathfrak{A} \text{ and } l_j = 1 \text{ for } j \in \mathfrak{U}\,. \tag{21}$$

Hence, subgrids of the grid G can be constructed, containing the basis for describing the component functions. This is the set for which Eq. (21) is not satisfied, thus amounting to the subgrid

$$G_{\mathfrak{U}} = \left\{(l_1,\ldots,l_n) \in G \mid l_i > 1 \text{ for } i \in \mathfrak{U} \text{ and } l_j = 1 \text{ for } j \in \mathfrak{A}\right\} \tag{22}$$

so that Eq. (18) may be written as

$$\hat{f}_{\mathfrak{U}} = \sum_{(\boldsymbol{l},\boldsymbol{i})\in G_{\mathfrak{U}}} v_{\boldsymbol{l},\boldsymbol{i}} \prod_{j\in\mathfrak{U}} \phi_{l_j,i_j}\,. \tag{23}$$

Here, the product is restricted to directions contained in $\mathfrak{U}$, so the interpolation is indeed also reduced to the dimension of $\mathfrak{U}$, which is due to the fact that the subspaces considered in $G_{\mathfrak{U}}$ only contain the constant function in directions $x \notin \mathfrak{U}$. Hence, $G_{\mathfrak{U}}$ can be represented by a lower-dimensional sparse grid.

Now, the correspondence between the component functions and the sparse-grid subspaces are obviously

$$\hat{f}_0 \in W_{\mathbf{1}} \qquad \text{,and in general,} \qquad \hat{f}_{\mathfrak{U}} \in \bigcup_{\boldsymbol{l}\in G_{\mathfrak{U}}} W_{\boldsymbol{l}}\,.$$

Observe that for this connection to be valid, the evaluation point of the subspace $W_{\mathbf{1}}$ defining the constant component has to be placed accordingly for the cut-HDMR. In particular, the anchor point of the cut-HDMR has to be used as root node for $W_{\boldsymbol{l}}$, which is the nominal point $\bar{\boldsymbol{p}}$.

Hence, for the one-dimensional case, the constant function spans the image of $\hat{P}_{\emptyset}^{i}$ while the remaining basis functions span the image of $\hat{P}^{i}$. The application of these one-dimensional approximate projectors $\hat{P}_{\emptyset}$ and $\hat{P}$ is illustrated in Fig. 1 for a generic function $f(x)$. The projectors execute the splitting of f by the split basis, displayed by the hierarchical hat basis. The resulting approximated component functions $\hat{f}_{\emptyset}$ and $\hat{f}_1$ are shown in the lower plots together with the original function f and the exact components $f_{\emptyset}$ and f_1 (dashed lines), respectively, for both the hierarchical hat and the pre-wavelet basis. Hence, approximations of the cut-HDMR as well as the ANOVA decomposition are shown.

Figure 2 shows the process of hierarchization of the full grid $\mathcal{V}_{\boldsymbol{l}}$ into the hierarchical subspaces $W_{\boldsymbol{k}}$ in a two-dimensional setting. In addition, the correspondence between the hierarchical subspaces of the sparse-grid decomposition to the HDMR components for a two-dimensional function is illustrated. Note that the tensor-product structure can be seen also for this connection.

3.5. Approximate sensitivity measures

As can be seen from Eq. (23), the projection operation is encoded in the hierarchical surpluses $v_{\boldsymbol{l},\boldsymbol{i}}$. Thus, all the components may directly be extracted from a single sparse-grid approximation of f. This is of course also true for the evaluation of the sensitivity measures, as this connection is equally true for integration.

For the calculation of the sensitivity measures, the component functions have to be squared before integration. The integration may be applied directly using squared grid-point values $f^2_{\mathfrak{U}}(\boldsymbol{x}_{\boldsymbol{l},\boldsymbol{i}})$ by a successive dehierarchization and hierarchization. Observe that this approach does result in the exact square of the interpolant $\hat{f}_{\mathfrak{U}}$ only in the grid points $\boldsymbol{x} \in G$. For an arbitrary point $\boldsymbol{x} \in [0,1]^n$ it holds for the interpolation of the squared component that $I(f^2_{\mathfrak{U}})(\boldsymbol{x}) = I(\hat{f}^2_{\mathfrak{U}})(\boldsymbol{x}) \neq \left(I(\hat{f}_{\mathfrak{U}})(\boldsymbol{x})\right)^2$ for $\boldsymbol{x} \notin G$.

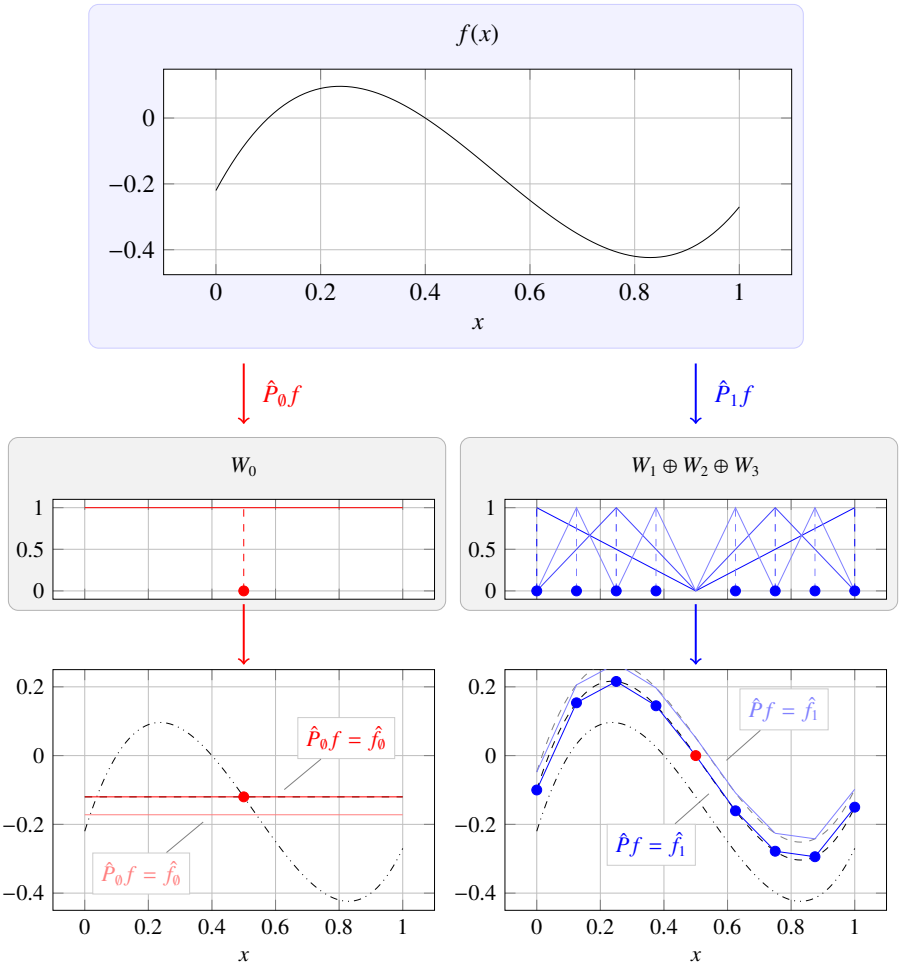

Fig. 1: Exemplary application of the (one-dimensional) approximate projectors $\hat{P}_\emptyset$ and $\hat{P}_1$ based on a hierarchical hat basis. The top plot shows the original function f, the gray boxes contain the span of the image of the corresponding projector, and the lower two plots display the resulting approximated component functions $\hat{f}_\emptyset$ and $\hat{f}_1$. The lower plots also show the function f (dash-dotted) and the exact components $f_\emptyset$ and f_1 (dashed), respectively. Observe that the constant component is exact, i.e. $f_\emptyset = \hat{f}_\emptyset$, for the cut-HDMR. For comparison, the results for the application of the pre-wavelet basis, i.e. the ANOVA decomposition, is illustrated in light colors in the lower plots as well.

4. Example

The presented methods are used to analyze the influence of uncertain system parameters on the feed-forward control of a parallel manipulator with flexible links, whose end effector is intended to follow an ellipse. Figure 3 shows the topology of the parallel manipulator and the desired trajectory of the end-effector point. The generalized coordinates, which consist of the slider positions s_1 and s_2, the angles α_1 and α_2, and the set of elastic coordinates $\boldsymbol{q}_e$, are used to describe the kinematics of the system. All bodies of the system, except the long link, are assumed to be rigid. The forces of the direct drives, which act on the sliders, are used as the control inputs. By using the concept of servo constraints [11], the model-based feed-forward control problem is stated as a set of differential-algebraic equations. For the chosen input-output combination, the problem is non-minimum phase, i.e. the internal dynamics of the DAE is

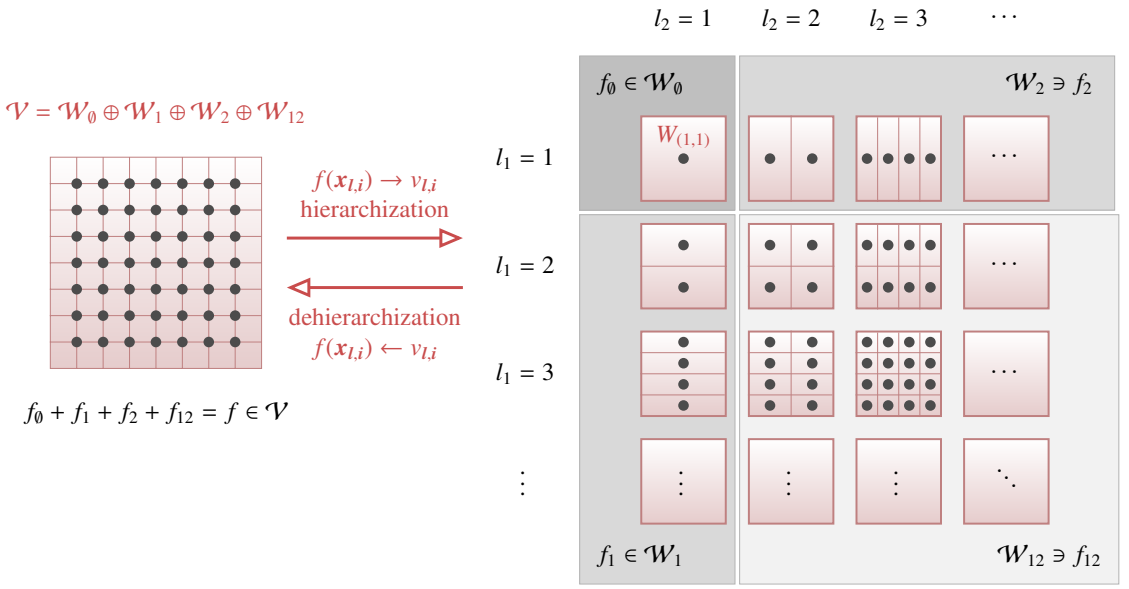

Fig. 2: Hierarchization of the approximation space $\mathcal{V}$ for a two-dimensional function $f(x_1, x_2)$ into hierarchical subspaces with illustration of the component spaces in $\mathcal{W}_{\mathfrak{U}}$, and the relation of the HDMR components $f_\emptyset$, f_1, f_2 and f_{12} to the hierarchical subspaces $W_{(l_1,l_2)}$.

unstable. The solution of this problem, which has to be obtained by a two-sided boundary value problem[12], yields bounded trajectories of the states as well as of the control inputs. These trajectories can be used to provide a collocated feedback control for the implementation in hardware.

The proposed feed-forward control is based on an estimative nominal model, since exact model parameter values are not available. In order to assess the performance and robustness of the feed-forward control with respect to varying model configurations, different analyses should be taken into account. Here, in order to illustrate the applicability of the method for relevant applications in multibody dynamics, solely the influences of several model parameters are investigated. As the underlying uncertainties are associated with imperfect knowledge and simplifications of the model, fuzzy numbers provide a suitable description thereof.

For the performed analysis, specific properties of the elastic link, namely the mass density ρ (±5%), Young's modulus E (±5%), Poisson's ratio ν (±5%) as well as the Rayleigh damping parameters α_{M} (±20%) and α_{K} (±20%), are modeled as triangular fuzzy numbers with the worst-case deviations given in parentheses. The assumption of significantly larger supports for the damping parameters are based on the fact that these values are difficult to obtain and Rayleigh damping is considered a simplified damping model. Figure 4 shows the absolute influences of these uncertainties on the slider positions s_1 and s_2. For both outputs, the variations of Young's modulus and of mass density have the largest influence. The variations of Poisson's ratio and of the damping parameters, even though they are varied by 20 percent, are negligible.

Thus, the dynamic effects of mass and damping are of minor influence compared to the elasticity, which is an advantageous property for the control design. Additionally, the overall magnitude of the influence measures shows that the total variations of the slider positions are quite small for both sliders and even smaller for the first one. This is due to the fact that, as discussed, the assumed uncertainties mainly effect the elastic behavior, which has to be compensated by the short arm, and hence, by the second slider.

The analysis shows that the designed control concept is robust with respect to the properties of the elastic link of the system. A detailed uncertainty analysis of the manipulator, accounting for different configurations of uncertainties in the system, by performing direct and inverse fuzzy computations as well as the calculation of the presented sensitivity measures has been conducted in[13].

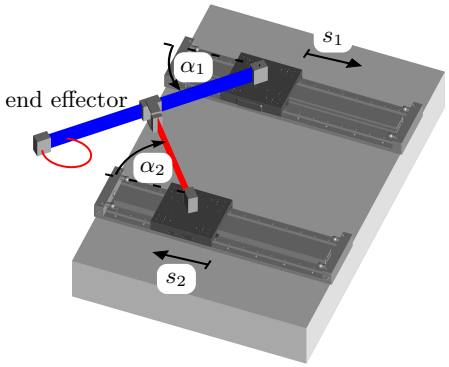

Fig. 3: Topology of the parallel manipulator with flexible links together with the desired elliptic trajectory.

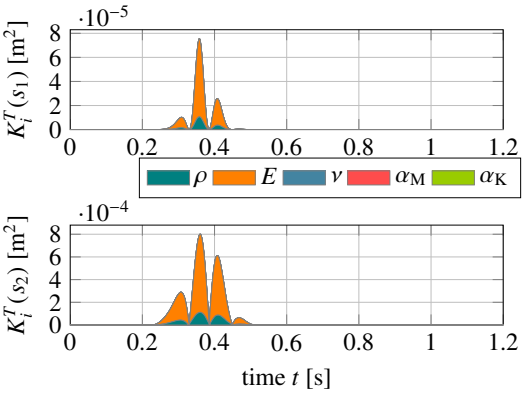

Fig. 4: Total influence measures K_i^T of the drive positions s_1 and s_2 with respect to the uncertain model parameters ρ, E, ν, α_M and α_K.

5. Conclusion

A sensitivity measure for fuzzy uncertainty analysis has been introduced, which is similar to the well-known Sobol indices used in stochastic analyses. The numerical approximation may be performed efficiently by applying sparse-grid integration due to the direct connection of sparse grids to the HDMR. It is shown that, in terms of a hierarchical-grid approximation, the cut-HDMR and the ANOVA decompositions differ mainly in the associated basis type, i.e. in the requirements that the basis has to fulfill. Thus, a combined calculation of fuzzy sensitivity measures and Sobol indices is basically possible by a combination of these bases. Thereby, similarities and differences of modeling uncertainties by fuzzy or random numbers are illustrated in a sensitivity setting. The fuzzy sensitivity measure may be applied in any setting where epistemic uncertainties are present in model parameters. The applicability has been briefly shown for a non-trivial application in multibody dynamics.

References

1. Hanss, M.. *Applied Fuzzy Arithmetic - An Introduction with Engineering Applications*. Berlin: Springer; 2005.
2. Gauger, U., Turrin, S., Hanss, M., Gaul, L.. A new uncertainty analysis for the transformation method. *Fuzzy Sets and Systems* 2008; **159**(11):1273–1291.
3. Li, G., Rosenthal, C., Rabitz, H.. High dimensional model representations. *The Journal of Physical Chemistry A* 2001;**105**(33):7765–7777. doi:10.1021/jp010450t.
4. Sobol, I.. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. *Mathematics and Computers in Simulation* 2001;**55**(1–3):271–280. doi:10.1016/S0378-4754(00)00270-6.
5. Feuersänger, C.. *Sparse Grid Methods for Higher Dimensional Approximation*. Ph.D. thesis; Institut für Numerische Simulation, Universität Bonn; 2010.
6. Bungartz, H.J., Griebel, M.. Sparse grids. *Acta Numerica* 2004;**13**:147–269.
7. Gerstner, T., Griebel, M.. Numerical integration using sparse grids. *Numerical algorithms* 1998;**18**(3-4):209–232.
8. Zadeh, L.A.. Fuzzy sets. *Information and Control* 1965;**8**(3):338–353.
9. Pflüger, D.. *Spatially Adaptive Sparse Grids for High-Dimensional Problems*. München: Verlag Dr. Hut; 2010. ISBN 9783868535556.
10. Gerstner, T., Griebel, M.. Dimension–adaptive tensor–product quadrature. *Computing* 2003;**71**(1):65–87.
11. Blajer, W., Kolodziejczyk, K.. A geometric approach to solving problems of control constraints: Theory and a dae framework. *Multibody System Dynamics* 2004;**11**:343–364.
12. Chen, D., Paden, B.. Stable inversion of nonlinear non-miniumum systems. *International Journal of Control* 1996;**64**(1):81–97.
13. Walz, N.P., Burkhardt, M., Hanss, M., Eberhard, P.. A comprehensive fuzzy uncertainty analysis of a controlled nonlinear system with unstable internal dynamics. *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering* 2015; (to appear).